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The integrability of negative powers of the solution of the Saint Venant problem

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Abstract. We initiate the study of the finiteness condition

$$\int_{\Omega} u(x)^{-\beta}\, dx \leq C(\Omega, \beta) < +\infty$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set and u is the solution of the Saint Venant problem $\Delta u = -1$ in Ω, $u = 0$ on $\partial\Omega$. The central issue which we address is that of determining the range of values of the parameter $\beta > 0$ for which the aforementioned condition holds under various hypotheses on the smoothness of Ω and demands on the nature of the constant $C(\Omega, \beta)$. Classes of domains for which our analysis applies include bounded piecewise C^1 domains in $\mathbb{R}^n$, $n \geq 2$, with conical singularities (in particular polygonal domains in the plane), polyhedra in $\mathbb{R}^3$, and bounded domains which are locally of class C^2 and which have (finitely many) outwardly pointing cusps. For example, we show that if u_N is the solution of the Saint Venant problem in the regular polygon Ω_N with N sides circumscribed by the unit disc in the plane, then for each $\beta \in (0, 1)$ the following asymptotic formula holds:

$$\int_{\Omega_N} u_N(x)^{-\beta}\, dx = \frac{4^{\beta}\pi}{1-\beta} + \mathcal{O}(N^{\beta-1}) \quad \text{as} \quad N \to \infty.$$

One of the original motivations for addressing the aforementioned issues was the study of sublevel set estimates for functions v satisfying $v(0) = 0$, $\nabla v(0) = 0$ and $\Delta v \geq c > 0$.

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