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Smoothing discrete Morse theory

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Abstract. After surveying classical notions of PL topology of the Seventies, we clarify the relation between Morse theory and its discretization by Forman. We show that PL handles theory and discrete Morse theory are equivalent, in the sense that every discrete Morse vector on some PL triangulation is also a PL handle vector, and conversely, every PL handle vector is also a discrete Morse vector on some PL triangulation. It follows that, in dimension up to 7, every discrete Morse vector on some PL triangulation is also a smooth Morse vector; the viceversa is true in all dimensions. This revises and improves a result by Gallais.

Some further consequences of our work are:

(1) For $d \neq 4$, every simply connected smooth d-manifold admits locally constructible triangulations. In contrast, the Mazur 4-manifold has no locally constructible triangulation. (This solves a question by Živaljevič and completes a work by the author and Ziegler.)
(2) The Heegaard genus of 3-manifolds can be characterized as the smallest g for which some triangulation of the manifold has discrete Morse vector $(1, g, g, 1)$. (This allows for heuristics to bound the Heegaard genus of any 3-manifold.)
(3) Some non-PL 5-spheres admit discrete Morse functions with only 2 critical faces. (This result, joint with Adiprasito, completes the Sphere Theorem by Forman.)

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