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On the canonical, fpqc, and finite topologies on affine schemes. The state of the art

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Abstract. This is a systematic study of the behaviour of finite coverings of (affine) schemes with regard to two Grothendieck topologies: the canonical topology and the fpqc topology. The history of the problem takes roots in the foundations of Grothendieck topologies, passes through main strides in Commutative Algebra and leads to new Mathematics up to perfectoids and prisms.

We first review the canonical topology of affine schemes and show, keeping with Olivier's lost work, that it coincides with the effective descent topology. Covering maps are given by universally injective ring maps, which we discuss in detail.

We then give a "catalogue raisonné" of examples of finite coverings which separate the canonical, fpqc and fppf topologies. The key result is that *finite coverings of regular schemes are coverings for the canonical topology, and even for the fpqc topology* (but not necessarily for the fppf topology). We discuss a "weakly functorial" aspect of this result.

"Splinters" are those affine Noetherian schemes for which every finite covering is a covering for the canonical topology. We present in geometric terms the state of the art about them. We also investigate their mysterious fpqc analogs and prove that, in prime characteristic, they are all regular. This leads us to the problem of descent of regularity by (non necessarily flat) morphisms which are coverings for the fpqc topology, which is settled thanks to a recent theorem of Bhatt-Iyengar-Ma.

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