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Higher dimensional Lemniscates: the geometry of r particles in n-space with logarithmic potentials

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Abstract. The main purpose of this paper is to lay down the foundations of the theory of higher dimensional lemniscates and to propose the investigation of several interesting open problems in the field.

The underlying geometrical problem is quite old (see [16, 17, 20, 21]), but, whereas in the past the main question was the location of the critical points of a certain distance function, the main thrust of our approach is to look at the logarithm of this function as the sum of logarithmic potentials at r points. And the main goal is the topological analysis of the configuration of the singular solutions (of the associated differential equation). We look at the number and type of critical points, and we make a breakthrough in higher dimensions relating the question to modern methods of complex analysis in several variables, showing that any critical point has Hessian with positivity at least $(n-1)$; hence, for generic choice of the r points we get a Morse function whose only critical points are local minima and saddles of negativity 1.

Moreover we show that the critical points are isolated, so there are no curves of local minima. The existence or not of these curves, and the upper bound for the number of critical points are related to famous classical problems posed by Morse-Cairns [7] and by Maxwell [15] for the case of electrostatic potentials.

We provide examples where the number of extra local minima can be arbitrarily large.

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