Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XX (2020), 1621-1663

Extremal conformal structures on projective surfaces

THOMAS METTLER

Abstract. We introduce a new functional $\mathcal{E}_{\mathfrak{p}}$ on the space of conformal structures on an oriented projective manifold $(M, \mathfrak{p})$. The nonnegative quantity $\mathcal{E}_{\mathfrak{p}}([g])$ measures how much $\mathfrak{p}$ deviates from being defined by a $[g]$-conformal connection. In the case of a projective surface $(\Sigma, \mathfrak{p})$, we canonically construct an indefinite Kähler-Einstein structure $(h_{\mathfrak{p}}, \Omega_{\mathfrak{p}})$ on the total space Y of a fibre bundle over Σ and show that a conformal structure $[g]$ is a critical point for $\mathcal{E}_{\mathfrak{p}}$ if and only if a certain lift $\widetilde{[g]} : (\Sigma, [g]) \to (Y, h_{\mathfrak{p}})$ is weakly conformal. In fact, in the compact case $\mathcal{E}_{\mathfrak{p}}([g])$ is – up to a topological constant – just the Dirichlet energy of $\widetilde{[g]}$. As an application, we prove a novel characterisation of properly convex projective structures among all flat projective structures. As a by-product, we obtain a Gauss-Bonnet type identity for oriented projective surfaces.

Mathematics Subject Classification (2010): 53A20 (primary); 53C28, 58E20 (secondary).