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Counting lines on surfaces

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Abstract. This paper deals with surfaces with many lines. It is well-known that a cubic contains 27 of them and that the maximal number for a quartic is 64. In higher degree the question remains open. Here we study classical and new constructions of surfaces with high number of lines. We obtain a symmetric octic with 352 lines, and give examples of surfaces of degree d containing a sequence of $d(d-2)+4$ skew lines.

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Cubic surface with 27 lines[1]

1. Introduction

Motivation for this paper is the article of 1943 by Segre [12] which studies the following classical problem: What is the maximum number of lines that a surface of degree d in $\mathbb{P}_3$ can have? Segre answers this question for $d = 4$ by using some nice geometry, showing that it is exactly 64. For the degree three it is a classical result that each smooth cubic in $\mathbb{P}_3$ contains 27 lines, but for $d \geq 5$ this number is still not known. In this case, Segre shows in [12] that the maximal number is less

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[1] `http://enriques.mathematik.uni-mainz.de/surf/logo.jpg`

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than or equal to $(d-2)(11d-6)$ but this bound is far from being sharp. Indeed, already in degree four it gives 76 lines which is not optimal. So on one hand one can try to improve the upper bound for the number of lines $\ell(d)$ that a surface of degree d in $\mathbb{P}_3$ can have, on the other hand it is interesting to construct surfaces with as many lines as possible to give a lower bound for $\ell(d)$.

It is notoriously difficult to construct examples of surfaces with many lines. There are two classical constructions:

(1) Surfaces of the kind $\phi(x, y) = \psi(z, t)$ where ϕ and ψ are homogeneous polynomials of degree d.
(2) d-coverings of the plane $\mathbb{P}_2$ branched over a curve of degree d.

Segre in [13] studies surfaces of the kind (1) in the case $d = 4$, showing that the possible numbers of lines are 16, 32, 48, 64. Caporaso-Harris-Mazur in [3], by using similar methods as Segre, study the maximal number of lines N_d on such surfaces in any degree d, showing that $N_d \geq 3d^2$ for each d and $N_4 \geq 64$, $N_6 \geq 180$, $N_8 \geq 256$, $N_{12} \geq 864$, $N_{20} \geq 1600$. In Section 4 we establish the exactness of these results:

Proposition 4.1. *The maximal numbers of lines on $\phi(x, y) = \psi(z, t)$ are:*

- $N_d = 3d^2$ *for* $d \geq 3$, $d \neq 4, 6, 8, 12, 20$;
- $N_4 = 64$, $N_6 = 180$, $N_8 = 256$, $N_{12} = 864$, $N_{20} = 1600$.

In particular, we show that it is not possible, with these surfaces, to obtain better examples and a better lower bound for $\ell(d)$ (Proposition 4.2). Turning to surfaces of the kind (2), we prove in Section 6:

Proposition 6.2. *Let $\mathcal{C}$ be a smooth plane curve with β total inflection points. Then the surface $\mathcal{S}$ obtained as the d-covering of $\mathbb{P}_2$ branched over $\mathcal{C}$ contains exactly $\beta \cdot d$ lines. In particular, it contains no more than $3d^2$ lines.*

As one sees, one can not find more lines by using these two classical methods. To find better examples, one has to use new ideas: We study symmetric surfaces in $\mathbb{P}_3$ (Section 3). The method is based on the following idea: If a surface has many automorphisms (many symmetries) then possibly it contains many orbits of lines. This approach was used successfully in the study of surfaces with many nodes. The first main result of this paper is:

Theorem 3.1. *There exists a smooth octic in $\mathbb{P}_3$ with bioctahedral symmetries containing* 352 *lines.*

This shows $\ell(8) \geq 352$, improving the previous bound 256 of [3].

In another direction, one can try to improve the upper bound for $\ell(d)$. Following the idea of Segre [12] and imposing some extra conditions on the lines on a surface, we find the interpolation $d(7d-12)$ which surprisingly agrees with the maximal known examples in degrees 4, 6, 8, 12 (Section 7.1).

A related problem is to determine the maximal number $m(d)$ of sequences of skew lines a surface of degree d in $\mathbb{P}_3$ can have. It is well-known that $m(3) = 6$ and $m(4) = 16$, and Miyaoka in [7] gives the upper bound $m(d) \leq 2d(d-2)$ for $d \geq 5$. It seems to be difficult to construct examples of surfaces with many skew lines: The best examples so far are those of Rams [9, 10] giving examples of surfaces with $d(d-2)+2$ skew lines ($d \geq 5$) and with 19 skew lines for $d = 5$. The second main result of this paper improves these results:

Theorem 5.1. *For $d \geq 7$ and $\gcd(d, d-2) = 1$ there exists a smooth surface in $\mathbb{P}_3$ containing a sequence of $d(d-2)+4$ skew lines.*

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2. General results

It is a well-known fact that, in $\mathbb{P}_3$, each smooth quadric surface contains an infinite number of lines, each smooth cubic surface contains exactly 27 lines and a generic smooth surface of degree $d \geq 4$ contains no line ([1, 2, 5]). This leads to the problem of finding surfaces of degree $d \geq 4$ with an optimal number of lines. The best upper bound known so far is:

Theorem. (Segre [12])

- *The number of lines lying on a smooth surface of degree $d \geq 4$ does not exceed* $(d-2)(11d-6)$.
- *The maximum number of lines lying on a smooth quartic is exactly* 64.

This bound is effective for $d = 4$ but for $d \geq 5$ it is believed that it could be improved. For instance, already for $d = 4$ the uniform bound $(d-2)(11d-6)$ is too big.

A natural question related to the number of lines on a surface is the study of maximal sequences of pairwise disjoint lines on a smooth surface in $\mathbb{P}_3$. The best upper bound known so far is:

Theorem. (Miyaoka [7, Section 2.2]) *The maximal length of a sequence of skew lines on a smooth surface of degree $d \geq 4$ is $2d(d-2)$.*

For $d = 3$, each cubic surface contains a maximal sequence of 6 skew lines: This comes from the study of the configuration of the 27 lines ([5, Theorem V.4.9]). For $d = 4$, Kummer surfaces contain a maximal sequence of 16 skew lines ([8]) so the bound is optimal. But for $d \geq 5$, it is not known if it is sharp.

3. A smooth octic with 352 lines

The octic we construct is a surface with bioctahedral symmetries. The rough idea is as follows: If a surface has many symmetries, one can expect that it contains many lines, since if the surface contains a line then it contains the whole orbit, and if the symmetry group is big, hopefully this orbit has a big length.

Let $G \in \mathrm{SO}(3,\mathbb{R})$ be a polyhedral group. Consider the exact sequence:

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2) \xrightarrow{\phi} \mathrm{SO}(3,\mathbb{R}) \longrightarrow 0.$$

The inverse image $\widetilde{G} := \phi^{-1}G$ is a *binary polyhedral group*. Now consider the exact sequence:

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{SU}(2)\times\mathrm{SU}(2) \xrightarrow{\sigma} \mathrm{SO}(4,\mathbb{R}) \longrightarrow 0.$$

The direct image $\sigma(\widetilde{G}\times\widetilde{G}) \subset \mathrm{SO}(4,\mathbb{R})$ is a *bipolyhedral group*. Denote by $\mathcal{T}$, $\mathcal{O}$ and $\mathcal{I}$ respectively the tetrahedral, octahedral and icosahedral group. We shall make use of the following particular groups:

- $G_6 = \sigma(\widetilde{\mathcal{T}}\times\widetilde{\mathcal{T}})$ of order 288;
- $G_8 = \sigma(\widetilde{\mathcal{O}}\times\widetilde{\mathcal{O}})$ of order 1152;
- $G_{12} = \sigma(\widetilde{\mathcal{I}}\times\widetilde{\mathcal{I}})$ of order 7200.

The polynomial invariants of these groups were studied by Sarti in [11, Section 4]. First note that the quadratic form $Q := x^2+y^2+z^2+t^2$ is an invariant of the action of these groups.

Theorem (Sarti). *For $d = 6, 8, 12$ there is a one-dimensional family of G_d-invariant surfaces of degree d. The equation of the family is $S_d + \lambda Q^{d/2} = 0$. The base locus of the family consists in $2d$ lines, d in each ruling of Q. The general member of each family is smooth and there are exactly five singular surfaces in each family.*

From this theorem immediately follows that each member of the family contains at least $2d$ lines.

Consider the group G_8. Denote by $\mathcal{S}_8$ the surface $S_8 = 0$ where:

$$\begin{aligned} S_8 = {} & x^8+y^8+z^8+t^8+168x^2y^2z^2t^2 \\ & + 14(x^4y^4+x^4z^4+x^4t^4+y^4z^4+y^4t^4+z^4t^4). \end{aligned}$$

Theorem 3.1. *The surface $\mathcal{S}_8$ is smooth and contains exactly 352 lines.*

Proof. The proof goes as follows: first we introduce Plücker coordinates for the lines in $\mathbb{P}_3$, then we compute explicitly all the lines contained in the surface.

• *Plücker coordinates.* Let $\mathbb{G}(1,3)$ be the Grassmannian of lines in $\mathbb{P}_3$, or equivalently of 2-planes in $\mathbb{C}^4$. Such a line L is given by a rank-two matrix:

$$\begin{pmatrix} a & e \\ b & f \\ c & g \\ d & h \end{pmatrix}.$$

The 2-minors (*Plücker coordinates*):

$$\begin{array}{lll} p_{12} := af - be & p_{13} := ag - ce & p_{14} := ah - de \\ p_{23} := bg - cf & p_{24} := bh - df & p_{34} := ch - dg \end{array}$$

are not simultaneously zero, and induce a regular map $\mathbb{G}(1,3) \longrightarrow \mathbb{P}_5$. This map is injective, and its image is the hypersurface $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. In order to list once all lines with these coordinates, we inverse the Plücker embedding in the *Plücker stratification*:

(1)	(2)	(3)
$p_{12} = 1$	$p_{12} = 0,\ p_{13} = 1$	$p_{12} = 0,\ p_{13} = 0,\ p_{14} = 1$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -p_{23} & p_{13} \\ -p_{24} & p_{14} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ p_{23} & 0 \\ 0 & 1 \\ -p_{34} & p_{14} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ p_{24} & 0 \\ p_{34} & 0 \\ 0 & 1 \end{pmatrix}$

(4)	(5)	(6)
$p_{12} = 0,\ p_{13} = 0$ $p_{14} = 0,\ p_{23} = 1$	$p_{12} = 0,\ p_{13} = 0,\ p_{14} = 0$ $p_{23} = 0,\ p_{24} = 1$	$p_{12} = 0,\ p_{13} = 0,\ p_{14} = 0$ $p_{23} = 0,\ p_{24} = 0,\ p_{34} = 1$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -p_{34} & p_{24} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ p_{34} & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

• *Counting the lines.* The line L is contained in the surface $\mathcal{S}_8$ if and only if the function $(u, v) \mapsto S_8(ua + ve, ub + vf, uc + vg, ud + vh)$ is identically zero, or equivalently if all coefficients of this polynomial in u, v are zero. The conditions for the line to be contained in the surface is then given by a set of polynomial equations in a, b, c, d, e, f, g, h. In order to count the lines, we restrict the equations to each Plücker stratum and compute the solutions (this computation is not difficult if left to SINGULAR [4]).

1. *The stratum* $p_{12} = 1$. Set $p_{23} = c$, $p_{24} = d$, $p_{13} = g$, $p_{14} = h$. The equations for such a line to be contained in the surface are:

$$\begin{aligned}
c^7g + d^7h + 7c^3g + 7d^3h + 7c^4d^3h + 7c^3gd^4 &= 0 \\
c^6g^2 + d^6h^2 + 3c^4d^2h^2 + 8c^3gd^3h + 3c^2g^2d^4 & \\
+6c^2d^2 + 3c^2g^2 + 3d^2h^2 &= 0 \\
c^5g^3 + d^5h^3 + c^4dh^3 + cg^3d^4 + 6c^3gd^2h^2 & \\
+6c^2g^2d^3h + cg^3 + dh^3 + 6c^2dh + 6cgd^2 &= 0
\end{aligned}$$

$$\begin{aligned}
1 + g^4 + 5c^4g^4 + 5d^4h^4 + c^4 + d^4 + c^4h^4 & \\
+g^4d^4 + 16c^3gdh^3 + 36c^2g^2d^2h^2 + 16cg^3d^3h & \\
+h^4 + 12c^2h^2 + 12g^2d^2 + 48cgdh &= 0 \\
c^3g + d^3h + c^3gh^4 + c^3g^5 + d^3h^5 + 6c^2g^2dh^3 & \\
+g^4d^3h + 6cg^3d^2h^2 + 6cgh^2 + 6g^2dh &= 0 \\
3c^2g^2 + 3d^2h^2 + 3c^2g^2h^4 + 3g^4d^2h^2 + c^2g^6 & \\
+d^2h^6 + 8cg^3dh^3 + 6g^2h^2 &= 0 \\
cg^7 + dh^7 + 7cg^3 + 7dh^3 + 7cg^3h^4 + 7g^4dh^3 &= 0 \\
1 + g^8 + h^8 + 14g^4 + 14h^4 + 14g^4h^4 &= 0\,.
\end{aligned}$$

After simplification of the ideal with SINGULAR (that we do not reproduce here), the solutions give 320 lines of the kind $z = cx + gy$, $t = dx + hy$.

2. *The stratum* $p_{12} = 0$, $p_{13} = 1$. Set $p_{23} = b$, $-p_{34} = d$, $p_{14} = h$. The equations for such a line to be contained in the surface are (after simplification):

$$\begin{aligned}
d &= 0 \\
b^4h^2 - b^2h^4 - b^2 + h^2 &= 0 \\
b^6 - h^6 + 13b^2 - 13h^2 &= 0 \\
h^8 + 14h^4 + 1 &= 0 \\
b^2h^6 + b^4 + 13b^2h^2 + 1 &= 0\,.
\end{aligned}$$

The solutions give 32 lines of the kind $y = bz$, $t = hx$, since there are eight possible values for h, and for each of them there are four values of b.

An easy computation shows that the other strata contain no line, so there are exactly 352 lines on the surface. $\square$

Remark 3.2. To our knowledge, this is the best example so far of an octic surface with many lines. This improves widely the bound 256 of Caporaso-Harris-Mazur [3].

Consider now the group G_6. We take:

$$S_6 = x^6 + y^6 + z^6 + t^6 + 15(x^2y^2z^2 + x^2y^2t^2 + x^2z^2t^2 + y^2z^2t^2).$$

Proposition 3.3. *The surface* $8S_6 - 5Q^3 = 0$ *contains exactly* 132 *lines.*

This result can be shown in a similar way as in the G_8 case. We do not reproduce the computation since there are sextics with more lines (see Section 4.4), but this shows the existence of a sextic with 132 lines. In Section 4.4 we shall give more examples of constructions of symmetric surfaces.

4. Surfaces of the kind $\phi(x, y) = \psi(z, t)$

Consider a smooth surface $\mathcal{S}$ given by an equation $\phi(x, y) = \psi(z, t)$ for two homogeneous polynomials ϕ, ψ of degree d. Segre gives a complete description of the possible and maximal numbers of lines in the case $d = 4$ ([13, Section VIII]). Caporaso-Harris-Mazur in [3, Lemma 5.1] generalized the argument to all degrees and gave a lower bound for the number of lines on such surfaces. We recall briefly the argument in order to establish the exactness of their result and conclude with the maximal numbers of lines for such surfaces.

4.1. Configuration of the lines

Let $Z(\phi)$, respectively $Z(\psi)$ denote the set of distinct zeros of $\phi(x, y)$, respectively $\psi(z, t)$ in $\mathbb{P}_1$.

Proposition 4.1. *The number N_d of lines on $\mathcal{S}$ is exactly $N_d = d(d + \alpha_d)$ where α_d is the order of the group of isomorphisms of $\mathbb{P}_1$ mapping $Z(\phi)$ to $Z(\psi)$.*

Sketch of proof. Let L be the line $z = t = 0$ and L' be the line $x = y = 0$. Set $Z(\phi) = \mathcal{S} \cap L = \{P_1, \dots, P_d\}$ and $Z(\psi) = \mathcal{S} \cap L' = \{P'_1, \dots, P'_d\}$. One shows easily (see [3, 13]) that:

- Each line $L_{i,j}$ joining a P_i to a P'_j is contained in $\mathcal{S}$: this gives d^2 lines.
- Each line contained in $\mathcal{S}$ and intersecting L and L' is one of the previous lines.
- Let D be a line contained in $\mathcal{S}$ and not intersecting L. Then D does not intersect L' (and *vice-versa*) and equations for such a line D are:

$$\begin{cases} x = \alpha z + \beta t \\ y = \gamma z + \delta t \,. \end{cases}$$

These equations define a linear isomorphism between L' and L inducing a bijection between $Z(\psi)$ and $Z(\phi)$.

• Conversely, let $\sigma : L' \to L$ be an isomorphism mapping the points P'_j to the points P_i, and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ a matrix defining σ. Consider the smooth quadric Q_σ : $x(\gamma z + \delta t) - y(\alpha z + \beta t) = 0$. Its first ruling is the family of lines $(p, \sigma(p))$ for $p \in L'$. Its second ruling consists in the family of lines of equations

$$\mathbb{I}_{[a:b]} : \begin{cases} ax - b(\alpha z + \beta t) = 0 \\ ay - b(\gamma z + \delta t) = 0 \end{cases}$$

for $[a : b] \in \mathbb{P}_1$. In each ruling, the lines are disjoint to each other, and each line of one ruling intersects each line of the other ruling. Since the intersection $\mathcal{S} \cap Q$ contains exactly the d different lines $(P'_j, \sigma(P'_j))$ of the first ruling, it contains also d lines of the second ruling. To see that these lines are really distinct we proceed as follows. Denoting by $\mathcal{U}_d$ the group of d-th roots of the unit, the group $\mathcal{U}_d \times \mathcal{U}_d$ acts on $\mathbb{P}_3$ by $(\xi, \eta) \cdot [x : y : z : t] = [\xi x : \xi y : \eta z : \eta t]$, leaving the surface $\mathcal{S}$ globally invariant. Since $(\xi, \eta) \cdot \mathbb{I}_{[a:b]} = \mathbb{I}_{[\xi^{-1}a:\eta^{-1}b]}$, each line of the second ruling produces a length d orbit through the action.
• Therefore, each isomorphism $\sigma : L' \to L$ mapping $Z(\psi)$ to $Z(\phi)$ gives exactly d lines. Denote by α_d the number of isomorphims $\sigma : L' \to L$ mapping $Z(\psi)$ to $Z(\phi)$. The exact number of lines contained in the surface $\mathcal{S}$ is: $N_d = d^2 + \alpha_d d$. □

4.2. Possible numbers of lines

We describe now the possible values of α_d. One has $\alpha_3 = 6$. For $d \geq 4$, assuming that there is at least one isomorphism σ, we are lead to the problem of determining the possible groups Γ_d of automorphisms of $\mathbb{P}_1$ acting on a given set of $d \geq 4$ points on $\mathbb{P}_1$. The following classification is easy to obtain:

(1) $\Gamma_d = \{\mathrm{id}\}$. This forces $d \neq 4$.
(2) Γ_d *is a cyclic group:* $\Gamma_d \cong \mathbb{Z}/k\mathbb{Z}$ $(k \geq 2)$. The action on $\mathbb{P}_1$ has two fix points so $d = \alpha + \beta k$ with $\alpha \in \{0, 1, 2\}$ and $\beta \geq 1$, with the restrictions:
   - If $\alpha = 0$ then $\beta \geq 3$.
   - If $\alpha = 1$ then $d = 1 + \beta k \geq 5$, with $\beta \geq 2$ if $k = 3$.
   - If $\alpha = 2$ then $\beta \geq 3$.
(3) Γ_d *is a dihedral group:* $\Gamma_d \cong \mathbb{Z}/k\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ $(k \geq 2)$. There is one length 2 orbit and one length k orbit so $d = 2\alpha + \beta k + \gamma 2k$ with $\alpha, \beta \in \{0, 1\}$, $\gamma \geq 0$ and if $\gamma = 0$, $\alpha = 1$ and $\beta = 1$ then $k \neq 2, 4$.
(4) Γ_d *is the tetrahedral group* $\mathcal{T}$. There are two length four orbits and one length six orbit so $d = 4\alpha + 6\beta + 12\gamma$ with $\alpha \in \{0, 1, 2\}$, $\beta \in \{0, 1\}$, $\gamma \geq 0$. The only possibilities are:

$$\gamma = 0, \beta = 0, \alpha = 1; \;\; \gamma = 0, \beta = 1, \alpha = 1;$$
$$\gamma = 0, \beta = 1, \alpha = 2; \;\; \gamma \neq 0.$$

(5) Γ_d *is the octahedral group* $\mathcal{O}$. There is one length six orbit, one length eight orbit and one length twelve orbit, so $d = 6\alpha + 8\beta + 12\gamma + 24\delta$ with $\alpha, \beta, \gamma \in \{0, 1\}$ and $\delta \geq 0$.

(6) Γ_d *is the icosahedral group* $\mathcal{I}$. There is one length twelve orbit, one length 20 orbit and one length 30 orbit, so $d = 12\alpha+20\beta+30\gamma+60\delta$ with $\alpha, \beta, \gamma \in \{0, 1\}$ and $\delta \geq 0$.

4.3. Maximal number of lines

As a corollary of Proposition 4.1 and the preceding discussion of cases, we get the following maximality result:

Proposition 4.2. *The maximal numbers of lines on $\mathcal{S}$ are:*

- $N_d = 3d^2$ *for* $d \geq 3$, $d \neq 4, 6, 8, 12, 20$*;*
- $N_4 = 64$*,* $N_6 = 180$*,* $N_8 = 256$*,* $N_{12} = 864$*,* $N_{20} = 1600$.

Proof. Looking up at the discussion above, it appears that $\alpha_d = 2d$ is maximal when the group of automorphisms can not be a group $\mathcal{T}$, $\mathcal{O}$ or $\mathcal{I}$ and that $\alpha_4 = 12$, $\alpha_6 = \alpha_8 = 24$ and $\alpha_{12} = \alpha_{20} = 60$ are maximal. For other values of d, if the automorphism group is $\mathcal{T}$, respectively $\mathcal{O}$, respectively $\mathcal{I}$ then the number of lines is:

$$d^2 + 12d, \quad \text{respectively } d^2 + 24d, \quad \text{respectively } d^2 + 60d$$

and these numbers are bigger than $3d^2$ only if

$$d < 6, \quad \text{respectively } d < 12, \quad \text{respectively } d < 30.$$

So it just remains to check that the degree $d = 10$ is not possible for $\mathcal{O}$ and $\mathcal{I}$ and that the degrees $d = 14, 16, 18, 22, 24, 26, 28$ are not possible for $\mathcal{I}$, that is we cannot decompose such a d as a sum of lengths of orbits for the groups $\mathcal{O}$ or $\mathcal{I}$. This is clear with the restrictions on the numbers of orbits of each type. □

4.4. Examples

1. For d generic, the Fermat surface $x^d - y^d = z^d - t^d$ gives the best example for surfaces of the kind $\phi(x, y) = \psi(z, t)$.
2. For $d=4$, $\Gamma_4 \in \{\emptyset, D_2, D_4, \mathcal{T}\}$: the possible numbers of lines are 16, 32, 48, 64. This agrees with Segre's result (Section 2).
3. For $d = 5$, $\Gamma_5 \in \{\emptyset, \{\mathrm{id}\}, C_4, D_3, D_5\}$: the possible numbers of lines are 25, 30, 45, 55, 75. The general bound of Segre gives 147.
4. For $d = 6$, $\Gamma_6 \in \{\emptyset, \{\mathrm{id}\}, C_2, D_2, D_3, D_6, \mathcal{O}\}$: the possible numbers of lines are 36, 42, 48, 60, 72, 108, 180. The general bound of Segre gives 240.

Remark 4.3. It is an interesting related problem to find surfaces of any degree d with as many real lines as possible. For surfaces of the kind $\phi(x, y) = \phi(z, t)$, if the zeros of ϕ are all real, one gets already d^2 real lines. Then, for each isomophism in the group Γ_d represented by a real matrix, one gets one more real line if d is odd and two more real lines if d is even.

In Section 3 we constructed an octic with bipolyhedral symmetries and 352 lines. The surfaces of the kind $\phi(x, y) = \phi(z, t)$ produce good examples of surfaces with polyhedral symmetries. Set Γ the group of isomorphisms of $\mathbb{P}_1$ permuting the zeros of ϕ in $\mathbb{P}_1$: ϕ is a projective invariant for the action of Γ on $\mathbb{C}^2$. This implies that the surface $\phi(x, y) = \phi(z, t)$ is invariant for the diagonal action of Γ given by $g(x, y, z, t) = (g(x, y), g(z, t))$ for $g \in \Gamma$. Its number of lines is given by Proposition 4.1. Looking at the projective invariant polynomials of the groups $\mathcal{O}, \mathcal{I}$ in Klein [6, I.2, Section 11-12-13], one gets:

- A surface of degree six with octahedral symmetries and 180 lines:

$$\phi(x, y) = xy(x^4 - y^4).$$

- A surface of degree eight with octahedral symmetries and 256 lines:

$$\phi(x, y) = x^8 + 14x^4y^4 + y^8.$$

- A surface of degree twelve with octahedral symmetries and 432 lines:

$$\phi(x, y) = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}.$$

- A surface of degree twelve with icosahedral symmetries and 864 lines:

$$\phi(x, y) = xy(x^{10} + 11x^5y^5 - y^{10}).$$

- A surface of degree 20 with icosahedral symmetries and 1600 lines:

$$\phi(x, y) = -(x^{20} + y^{20}) + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10}.$$

- A surface of degree 30 with icosahedral symmetries and 2700 lines:

$$\phi(x, y) = (x^{30} + y^{30}) + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20}).$$

5. Sequences of skew lines

In [10], Rams considers the surfaces $x^{d-1}y+y^{d-1}z+z^{d-1}t+t^{d-1}x = 0$ and proves that they contain a family of $d(d-2)+2$ skew lines for any d. In [9, Example 2.3], he also gives an example of a surface of degree five containing a sequence of 19 skew lines. We generalize his result, improving the number of skew lines to $d(d-2)+4$ in the case $d \geq 7$ and $\gcd(d, d-2) = 1$.

Consider the surface $\mathcal{R}_d : x^{d-1}y + xy^{d-1} + z^{d-1}t + zt^{d-1} = 0$. By our study in Section 4, this surface contains exactly $3d^2 - 4d$ lines if $d \neq 6$ and 180 lines for $d = 6$. We prove:

Theorem 5.1. *The surface* $\mathcal{R}_d$ *with* $\gcd(d, d-2) = 1$ *contains a sequence of* $d(d-2)+4$ *disjoint lines.*

Proof. Denote by ϵ, γ the primitive roots of the unit of degrees $d-2$ and d, and let $\eta := \epsilon^l \gamma^s$, with $0 \leq l \leq d-3$, $0 \leq s \leq d-1$. Since $\gcd(d, d-2) = 1$ we have $d(d-2)$ such η. Now consider the points

$$(0 : 1 : 0 : -\eta^{d-1}), (-\eta : 0 : 1 : 0)$$

then the line through the two points is

$$C_{l,s} : (-\eta\lambda : \mu : \lambda : -\eta^{d-1}\mu)$$

An easy computation shows that these lines are contained in $\mathcal{R}_d$ and are $d(d-2)$. This form a set of $d(d-2)+4$ skew lines together with the lines

$$\{x = 0, z + \epsilon t = 0\},\ \{y = 0, z + t = 0\},$$
$$\{z = 0, x + \epsilon y = 0\},\ \{t = 0, x + y = 0\}. \qquad \square$$

6. Surfaces of the kind $t^d = f(x, y, z)$

Let $\mathcal{C} : f(x, y, z) = 0$ be a smooth plane curve defined by a homogeneous polynomial f of degree d and consider the smooth surface $\mathcal{S}$ in $\mathbb{P}_3$ given by the equation $t^d = f(x, y, z)$. Set $p = [0 : 0 : 0 : 1] \in \mathbb{P}_3$. The projection $(\mathbb{P}_3 - \{p\}) \to \mathbb{P}_2$, $[x : y : z : t] \mapsto [x : y : z]$ induces a d-covering $\pi : \mathcal{S} \to \mathbb{P}_2$ ramified along the curve $\mathcal{C}$. Recall that a point $x \in \mathcal{C}$ is a *d-point* (or *total inflection point*) if the intersection multiplicity of $\mathcal{C}$ and its tangent line at x is equal to d. The following lemma is easy to prove:

Lemma 6.1.

1. *Assume L is a line contained in $\mathcal{S}$. Then $\pi(L)$ is a line.*
2. *Let $x \in \mathcal{C}$ and L be the tangent at $\mathcal{C}$ in x, then the preimage $\pi^{-1}(L)$ consists in d different lines contained in $\mathcal{S}$ if and only if x is a d-point.*
3. *Let L be a line in $\mathbb{P}_2$. Then $\pi^{-1}(L)$ contains a line if and only if L is tangent to $\mathcal{C}$ at a d-point.*

Sketch of proof.

(2) Assume that x is a d-point. Let Δ be a line of equation $\delta = 0$ intersecting L at x. Then $d \cdot (\Delta \cdot L) = (\mathcal{C} \cdot L)$ so up to a scalar factor $f_{|_L} = \delta^d_{|_L}$: the covering restricted to L is trivial and $\pi^{-1}(L)$ consists in d lines. Conversely, if the covering splits, there exists a section $\gamma \in H^0(L, \mathcal{O}_L(1))$ such that $\gamma^d = f_{|_L} \in H^0(L, \mathcal{O}_L(d))$ so L intersects $\mathcal{C}$ at x with multiplicity d.

(3) Assume that $\pi^{-1}(L)$ contains a line and that L is given by a linear function $z = l(x, y)$. The equation $t^d - f(x, y, l(x, y)) = 0$ of $\pi^{-1}(L)$ factorizes by $t - w(x, y)$ where $w(x, y)$ is a linear form, so $f(x, y, l(x, y)) = w(x, y)^d$ hence the preimage consists in d lines. This implies that the covering is trivial over L so by (6.1) x is a d-point. □

We deduce the number of lines contained in such surfaces:

Proposition 6.2. *Assume that the curve $\mathcal{C}$ has β total inflection points. Then the surface $\mathcal{S}$ contains exactly $\beta \cdot d$ lines. In particular, it contains no more than $3d^2$ lines.*

Proof. The first assertion follows directly from the Lemma 6.1. For the second one, the inflection points are the intersections of $\mathcal{C}$ with its Hessian curve $\mathcal{H}$ of degree $3(d-2)$ and at a total inflection point the intersection multiplicity of $\mathcal{C}$ and $\mathcal{H}$ is $d-2$, so by Bezout one gets $\beta \leq 3d$. □

Example 6.3.

- Each cubic curve has nine inflection points, so the induced cubic surface has 27 lines.
- The Fermat curves $x^d + y^d + z^d = 0$ have $3d$ total inflection points so the Fermat surfaces have $3d^2$ lines.

7. Final remarks

7.1. An interpolation

As we mentioned in Section 2, the uniform bound $(d-2)(11d-6)$ of Segre is too big already in degree four. We propose here a lower polynomial, which interpolates all maximal numbers of lines known so far in degrees 4, 6, 8, 12, including the octic of Section 3. Although there is no reason for this interpolation to be a maximal bound, it seems reasonable to expect that an effective construction of a surface with this number of lines is possible in all degrees.

Let $\mathcal{S}$ be a smooth surface of degree $d \geq 3$ and C a line contained in $\mathcal{S}$. Let $|H|$ be the linear system of planes H passing through C. Then $H \cap \mathcal{S} = C \cup \Gamma$ where Γ is a curve of degree $d-1$. The system $|\Gamma|$ is described by Segre in [12]: it is base-point free and any curve Γ does not contain C as a component. Then:

Proposition.(Segre) *Either each curve Γ intersects C in $d-1$ points which are inflections for Γ, or the points of C each of which is an inflection for a curve Γ are $8d-14$ in number. In particular, in this case C is met by no more than $8d-14$ lines lying on $\mathcal{S}$.*

Following Segre, C is called a line of the *second kind* if it intersects each Γ in $d-1$ inflections. A generalization of Segre's argument in [12, Section 9] gives the following result:

Proposition 7.1. *Assume that $\mathcal{S}$ contains d coplanar lines, none of them of the second kind. Then $\mathcal{S}$ contains at most $d(7d - 12)$ lines.*

Proof. Let P be the plane containing these d distinct lines. Then they are the complete intersection of P with $\mathcal{S}$. Hence each other line on $\mathcal{S}$ must intersect P in some of the lines. By the previous proposition, each of the d lines in the plane meets at most $8d - 14$ lines, so $8d - 14 - (d - 1)$ lines not on the plane. The total number of lines is at most $d + d(7d - 13) = d(7d - 12)$. □

7.2. Number of rational points on a plane curve

We give an application of our results to the *universal bound conjecture*, following Caporaso-Harris-Mazur [3]:

Universal bound conjecture. *Let $g \geq 2$ be an integer. There exists a number $N(g)$ such that for any number field K there are only finitely many smooth curves of genus g defined over K with more than $N(g)$ K-rational points.*

As mentioned in [3] an interesting way to find a lower bound of $N(g)$, or of the limit:

$$\overline{N} := \limsup_{g\to\infty} \frac{N(g)}{g}$$

is to consider plane sections of surfaces with many lines. Indeed, over the common field K of definition of the surface and its lines, a generic plane section is a curve containing at least as many K-rational points as the number of lines. In particular, they show that $N(21) \geq 256$. Since we obtain an octic surface with 352 lines and a generic plane section of this surface is a smooth curve of genus 21, we get:

Corollary 7.2. $N(21) \geq 352$.

As we remarked in Section 7.1, it seems to be possible to construct surfaces with $d(7d - 12)$ lines. This would improve the lower bound of $N(g)$ for many g. In particular, this would improve the known estimate $\overline{N} \geq 8$ to $\overline{N} \geq 14$.

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