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A family of adapted complexifications for $SL_2(\mathbb{R})$

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Abstract. Let G be a non-compact, real semisimple Lie group. We consider maximal complexifications of G which are adapted to a distinguished one-parameter family of naturally reductive, left-invariant metrics. In the case of $G = SL_2(\mathbb{R})$ their realization as equivariant Riemann domains over $G^{\mathbb{C}} = SL_2(\mathbb{C})$ is carried out and their complex-geometric properties are investigated. One obtains new examples of non-univalent, non-Stein, maximal adapted complexifications.

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1. Introduction

Let ∇ be a linear connection on a real-analytic manifold M which is identified with the zero section in the tangent bundle TM. A complex structure defined on a domain Ω of TM containing M is adapted to the connection if for any ∇-geodesic γ its complexification γ_*, given by $(x + iy) \to y\,\gamma'(x)$, is holomorphic on $(\gamma_*)^{-1}(\Omega)$. In this situation we refer to Ω as an adapted complexification of (M, ∇). In the case where ∇ is real-analytic, R. Bielawski [3] and R. Szőke [26] recently showed that the adapted complex structure exists in a neighborhood of M. In [26] one also finds a uniqueness result. For the Levi Civita connection of a real-analytic Riemannian manifolds, such results were known since the pioneering works of Guillemin-Stenzel [13] and Lempert-Szőke [19].

In the presence of a "large enough" Lie group acting on M and preserving the geodesic flow induced by ∇, one can prove that there exists a maximal domain Ω for the adapted complex structure, *i.e.* every adapted complexification is necessarily contained in Ω (see Proposition 3.1, *cf.* [14]). If M is a non-compact, Riemannian symmetric space such a maximal complexification is well-known under the name of Akhiezer-Gindikin domain (see [2], *cf.* [10]).

Recall that M is the fixed point set of the anti-holomorphic involution on $\Omega \subset TM$ given by $v \to -v$ and in the case of the Levi Civita connection associated to

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a pseudo-Riemannian manifold the metric ν appears as the restriction of a pseudo-Kähler metric κ with the same index as ν. Moreover κ admits a global potential whose properties give important geometric insights of Ω ([26], *cf.* [5, 19, 21, 23, 24]).

For a connected, non-compact, real semisimple Lie group G, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of its Lie algebra $\mathfrak{g}$ with respect to a maximal compact subalgebra $\mathfrak{k}$. Denote by B the Killing form of G and consider the distinguished one-parameter family of left-invariant metrics ν_m (degenerate for $m = 0$) uniquely defined by

$$\nu_m|_{\mathfrak{g}}(X, Y) = -mB(X_{\mathfrak{k}}, Y_{\mathfrak{k}}) + B(X_{\mathfrak{p}}, Y_{\mathfrak{p}}) ,$$

for any $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ and $Y = Y_{\mathfrak{k}} + Y_{\mathfrak{p}}$ in $\mathfrak{k} \oplus \mathfrak{p}$. Note that for all real m, the action of the group $L := G \times K$, given by left and right multiplication on G, is isometric. Here K is the connected subgroup of G generated by $\mathfrak{k}$. Recall that for $m > 0$ these metrics appear in the classification of all naturally reductive, left-invariant Riemannian metrics on G given by C. Gordon in [12].

Our main goal is to present new examples of maximal complexifications adapted, in the non-degenerate cases, to the Levi Civita connection associated to a metric of the above family. In the degenerate case one finds a maximal complexification adapted to the unique real-analytic linear connection which is obtained as the limit of such Levi Civita connections. For $G = SL_2(\mathbb{R})$ we give a precise description of these complexifications and we determine their basic complex-geometric properties. For positive m this gives, along with previous results (see [6, 15, 25]), examples among all classes of 3-dimensional, naturally reductive, Riemannian homogeneous spaces (*cf.* [7]). For $m = -1$ one obtains the symmetric pseudo-Riemannian case which has been investigated, among others, by G. Fels [9] and R. Bremigan [4].

The paper is organized as follows. Basic results and properties of the above metrics ν_m are recalled in Section 2. There we also point out with an example that in order to perform (pseudo) Kählerian reduction in this pseudo-Riemannian context, one may need more conditions than those necessary in the Riemannian case (Remark 2.3, *cf.* [1]).

In Section 3 we give a version of the characterization of maximal adapted complexifications given in [14] which is suitable in our situation (Proposition 3.1). In particular we show the existence of the maximal complexification Ω_m adapted to ν_m. This is realized as an L-equivariant Riemann domain over the universal complexification $G^{\mathbb{C}}$ of G, with *polar map* $P_m : \Omega_m \to G^{\mathbb{C}}$. By considering the usual identification $TG \cong G \times \mathfrak{g}$, such complexification can be described via a slice for the induced L-action by

$$\Omega_m = L \cdot \Sigma_m ,$$

where $\Sigma_m \subset \{e\} \times \mathfrak{g}$ is a semi-analytic subset of the product of $\mathfrak{k}$ and the closure of a Weyl chamber in a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. That is, $(e, X) \in \Sigma_m$

if and only if X belongs to the intersection of sublevel sets of certain real-analytic functions of $\mathfrak{k} \oplus \mathfrak{a}$ (Proposition 3.2 and 4.2).

The case $G = SL_2(\mathbb{R})$ is carried out in detail and the defining functions for Σ_m as well as the polar map P_m are explicitly determined in terms of a fixed basis of $\mathfrak{k} \oplus \mathfrak{a}$ in Sections 4 and 6. For this it is useful to have concrete realizations of slices and quotients of $SL_2(\mathbb{C})$ with respect to the involved actions, *i.e.* those of $SL_2(\mathbb{R})$, $SL_2(\mathbb{R}) \times SO_2(\mathbb{R})$ and $SL_2(\mathbb{R}) \times SO_2(\mathbb{C})$. This is separately discussed in Section 5.

Finally, in Sections 7 and 8 we single out the following different situations, whose boundary cases are given by the symmetric pseudo-Riemannian $m = -1$ and the degenerate $m = 0$. For $m < -1$ all maximal adapted complexifications are biholomorphic via P_m to a non-Stein L-invariant domain. Namely $SL_2(\mathbb{C})$ without a single $SL_2(\mathbb{R}) \times SO_2(\mathbb{C})$-orbit (Theorem 7.2). For $-1 \leq m \leq 0$ the polar map P_m remains injective but its non-Stein image misses more and more $SL_2(\mathbb{R}) \times SO_2(\mathbb{C})$-orbits. If $m > 0$, the maximal adapted complexifications turn out to be neither holomorphically convex, nor holomorphically separable (Theorem 8.4). In all cases the envelope of holomorphy of Ω_m is shown to be biholomorphic to $SL_2(\mathbb{C})$ (*cf.* [26, Section 9]).

Note that in the Riemannian context $m > 0$ all metrics ν_m have mixed sign sectional curvature. A similar situation can be noticed in the examples discussed in [14]. In these examples certain left-invariant Riemannian metrics on the generalized Heisenberg group were considered. Their sectional curvature has mixed sign and the associated maximal adapted complexifications have similar complex-geometric properties. It would be interesting to know if this is only a coincidence.

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2. Preliminaries

Let G be a non-compact, semisimple Lie group. Here we recall basic properties of the one-parameter family of left-invariant metrics on G which will be considered in the sequel. Such a family contains degenerate, Riemannian and pseudo-Riemannian, naturally reductive metrics. The Riemannian ones appear in the classification given by C. Gordon in [12]. More details and curvature computations can be found in [15].

We also recall those facts on adapted complex structures which are needed in the present paper. Finally we point out an example showing that the reduction procedure indicated by R. Aguilar [1] in the Riemannian context does not apply automatically when dealing with pseudo-Riemannian geometry.

Definition 2.1 (*cf.* **[20]).** A pseudo-Riemannian metric ν on a homogeneous manifold M is naturally reductive if there exist a connected Lie subgroup L of $\mathrm{Iso}(M)$

acting transitively on M and a decomposition $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{l}$, where $\mathfrak{h}$ is the Lie algebra of the isotropy group H at some point of M, such that $\mathrm{Ad}(H)\,\mathfrak{m} \subset \mathfrak{m}$ and

$$\tilde{\nu}([X,\, Y]_{\mathfrak{m}},\, Z) = \tilde{\nu}(X,\, [Y,\, Z]_{\mathfrak{m}})$$

for all $X,\, Y,\, Z \in \mathfrak{m}$. Here $[\ ,\ ]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of $[\ ,\ ]$ and $\tilde{\nu}$ is the pull-back of ν to $\mathfrak{m}$ via the natural projection $L \to L/H \cong M$. In this setting we refer to $\mathfrak{h} \oplus \mathfrak{m}$ as a naturally reductive decomposition and to L/H as a naturally reductive realization of M.

For a naturally reductive realization L/H every geodesic through the base point eH is the orbit of a one-parameter subgroup of L generated by some $X \in \mathfrak{m}$ (see [20], page 313). In fact for a Riemannian homogeneous manifold L/H with an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{h} \oplus \mathfrak{m}$ this property implies that L/H is a naturally reductive realization (see, *e.g.* [7]).

Let G be a connected, non-compact, semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of its Lie algebra with respect to a maximal compact Lie subalgebra $\mathfrak{k}$. Let B denote the Killing form on $\mathfrak{g}$ and, for every real m, assign a left-invariant metric ν_m on G by defining its restriction on $\mathfrak{g} \cong T_eG$ as follows:

$$\nu_m\big|_{\mathfrak{g}}(X, Y) = -mB(X_{\mathfrak{k}}, Y_{\mathfrak{k}}) + B(X_{\mathfrak{p}}, Y_{\mathfrak{p}}), \tag{2.1}$$

for any $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ and $Y = Y_{\mathfrak{k}} + Y_{\mathfrak{p}}$ in $\mathfrak{k} \oplus \mathfrak{p}$. Since B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, these metrics are Riemannian, degenerate or pseudo-Riemannian when $m > 0$, $m = 0$ or $m < 0$, respectively.

Let K be the connected subgroup of G generated by $\mathfrak{k}$ (which is compact if G is a finite covering of a real form of a complex semisimple Lie group). Since $\mathfrak{k}$, $\mathfrak{p}$ and B are $\mathrm{Ad}(K)$-invariant, ν_m is right K-invariant, *i.e.* the action of $G \times K$ on G defined by $(g,\, k) \cdot l := glk^{-1}$ is by isometries. Here we allow discrete ineffectivity given by the diagonal in $Z(G) \times Z(G)$, where $Z(G) \subset K$ is the center of G. One has $G = (G \times K)/H$ with H the diagonal in $K \times K$.

Note that a different choice of a maximal compact connected subalgebra $\mathfrak{k}'$ induces an *equivalent* left-invariant Riemannian structure, *i.e.* there exists an isometric isomorphism

$$(G, \nu_m) \to (G, \nu'_m)\,.$$

This is given by the internal conjugation transforming $\mathfrak{k}$ in $\mathfrak{k}'$.

We summarize the main properties of the above metrics in the following proposition where the degenerate metric ν_0 can be regarded as a limit case of non-degenerate ones.

Proposition 2.2 ([15, Section 3], ***cf.*** **[12, proof of Theorem 5.2]).** *Let G be a non-compact, semisimple Lie group and, for* $m \in \mathbb{R}$, *let* ν_m *be the above defined left-invariant metric. Then*

i) *the action of* $G \times K$ *by left and right multiplication is by isometries and*

ii) *the direct sum* $\mathfrak{h} \oplus \mathfrak{m}$*, with* $\mathfrak{h}$ *the isotropy Lie algebra and*

$$\mathfrak{m} := \{ (-mX_{\mathfrak{k}} + X_{\mathfrak{p}},\ -(1+m)\, X_{\mathfrak{k}}) \in \mathfrak{g} \times \mathfrak{k}\ :\ X_{\mathfrak{k}} + X_{\mathfrak{p}} \in \mathfrak{k} \oplus \mathfrak{p} \},$$

is a naturally reductive decomposition of $\mathfrak{g} \times \mathfrak{k} = Lie(G \times K)$*. In particular for every* $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ *in* $\mathfrak{k} \oplus \mathfrak{p} \cong T_e G$ *the unique geodesic through* e *and tangent to* X *is given by* $\gamma_X : \mathbb{R} \to G$*,*

$$t \longrightarrow \exp_G\, t\,(-mX_{\mathfrak{k}} + X_{\mathfrak{p}})\, \exp_K\, t\,(1+m)X_{\mathfrak{k}}\,.$$

From [12, Section 5], it follows that in the Riemannian cases $m > 0$ the connected component of the isometry group is essentially given by $G \times K$ (here discrete ineffectivity is allowed) while in the pseudo-Riemannian symmetric case $m = -1$ it coincides with $G \times G$. Further information regarding the Levi Civita connections (or their limit when m vanishes), the curvature tensor, the scalar and Ricci curvature can be found in [15, Section 3].

Let M be a complete real-analytic Riemannian manifold. Following the results of Guillemin-Stenzel [13] and Lempert-Szőke [19] one can introduce a complex structure on a subdomain of the tangent bundle TM which is *canonically* adapted to the given Riemannian structure. Recently R. Bielawski [3] and R. Szőke [26] have pointed out that for the existence of such a complex structure it is enough to have a real-analytic linear connection ∇. A real-analytic complex structure on a domain Ω of TM is adapted to ∇ if all leaves of the induced foliation are complex submanifolds with their natural complex structure. That is, for any ∇-geodesic $\gamma : I \to M$ the induced map $\gamma_* : TI \subset \mathbb{C} \to TM$ defined by $(x+iy) \mapsto y\,\gamma'(x)$ is holomorphic on $(\gamma_*)^{-1}(\Omega)$ with respect to the adapted complex structure. Here $y\,\gamma'(x) \in T_{\gamma(x)}M$ is the scalar multiplication in the vector space $T_{\gamma(x)}M$.

The adapted complex structure exists and it is unique on a sufficiently small neighborhood of M, which is identified with the zero section in its tangent bundle TM. If Ω is a domain of TM containing M on which this structure is defined, then we refer to it as an *adapted complexification*.

Associated to every non-degenerate metric ν_m of the above introduced one-parameter family one has the Levi Civita connection. For X and Y in $\mathfrak{g}$ this is given by the formula (*cf.* [15])

$$\nabla_{m\,X} Y = \frac{1}{2}\left([X,Y] + (1+m)\left([X_{\mathfrak{k}}, Y_{\mathfrak{p}}] + [Y_{\mathfrak{k}}, X_{\mathfrak{p}}]\right)\right).$$

Note that this uniquely defines a left-invariant, real-analytic, linear connection also in the degenerate case $m = 0$. Therefore for all real m one has an adapted complex structure at least in a neighborhood of M in TM. In the next section we will see that there exists an adapted complexification which is maximal in the sense of containing any other adapted complexification.

Remark 2.3. Let M be a real-analytic, Riemannian manifold with a free action by isometries of a compact Lie group K and endow M/K with the unique Riemannian metric such that the natural projection $M \to M/K$ becomes a Riemannian submersion. Then, as a consequence of results in [1], if the adapted complex structure exists on all of TM, so it does on $T(M/K)$. In the pseudo-Riemannian context the analogous result does not hold in this generality.

For instance, let U be a compact, semisimple Lie group and denote by $U^{\mathbb{C}}$ its universal complexification. Let G be a non-compact, real form of $U^{\mathbb{C}}$ and $K = G \cap U$. Denote by $\mathfrak{k}$ and $\mathfrak{u}$ the Lie algebras of K and U, respectively, and by B the Killing form on U. Consider the unique left-invariant Riemannian metric ν on U defined for all $X, Y \in \mathfrak{u}$ by

$$\nu|_{\mathfrak{u}}(X, Y) = -2B(X_{\mathfrak{k}}, Y_{\mathfrak{k}}) - B(X_{\mathfrak{p}}, Y_{\mathfrak{p}}),$$

where $\mathfrak{p} := \mathfrak{k}^{\perp_B}$. Endow $U \times K$ with the unique bi-invariant pseudo-metric $\tilde{\nu}$ such that

$$\tilde{\nu}|_{\mathfrak{u}\times\mathfrak{k}}((X, Z), (Y, W)) = -B(X, Y) + 2B(Z, W)$$

for all (X, Z), (Y, W) in $\mathfrak{u}\times\mathfrak{k}$. Then the projection $U\times K \to U$, $(u, k) \to uk^{-1}$ turns out to be a pseudo-Riemannian submersion. Moreover the adapted complex structure is defined on all of $T(U \times K)$. Indeed $(U \times K, \tilde{\nu})$ is essentially (up to the sign of the metric in the second component) the product of two symmetric Riemannian spaces of the compact type, thus this is a consequence of results in [24] and [26].

However (U, ν) has some negative sectional curvatures (see [8], *cf.* [15, Section 3]), thus by [19, Theorem 2.4] the adapted complex structure is not defined on all of TU. □

3. A family of maximal adapted complexifications

Let G be a connected, non-compact, semisimple Lie group and consider the one-parameter family of left-invariant metrics (pseudo-Riemannian for $m < 0$, degenerate for $m = 0$) introduced in Section 2 and defined by

$$\nu_m|_{\mathfrak{g}} := -mB(X_{\mathfrak{k}}, Y_{\mathfrak{k}}) + B(X_{\mathfrak{p}}, Y_{\mathfrak{p}}).$$

Then (*cf.* Proposition 2.2) the action of $L = G \times K$ by left and right multiplication is by isometries, the isotropy in e is $H = \{(k, k) \in G \times K : k \in K\}$ and the quotient L/H is a natural reductive realization of (G, ν_m). The Riemannian exponential map Exp_e in e is given by

$$\begin{aligned}\mathrm{Exp}_e(X) &= \exp_L(-mX_{\mathfrak{k}} + X_{\mathfrak{p}},\ -(1+m)X_{\mathfrak{k}}) \cdot e \\ &= \exp_G(-mX_{\mathfrak{k}} + X_{\mathfrak{p}}) \exp_K(1+m)X_{\mathfrak{k}}\,,\end{aligned}$$

for every $X \in \mathfrak{g} \cong T_eG$. Note that the L-action on G induces an action on the tangent space TG just by differentiation. If one identifies TG with $G \times \mathfrak{g}$ as usual, this action reads as

$$(g, k) \cdot (g', X) = (gg'k^{-1},\ \mathrm{Ad}_k(X)).$$

The group L also acts on the universal complexification $G^{\mathbb{C}}$ of G by left and right multiplication.

Next we show the existence of a maximal complexification in TG adapted to the connection associated to ν_m for all real m. It can be characterized as follows:

Proposition 3.1. *Let G be a connected, non-compact, semisimple Lie group endowed with a metric ν_m of the above family. Then there exists a maximal complexification Ω_m adapted to the connection associated to ν_m. Let*

$$P_m : G \times \mathfrak{g} \to G^{\mathbb{C}}$$

be the L-equivariant map defined by

$$(g, X) \to g \exp_{G^{\mathbb{C}}} i(-mX_{\mathfrak{k}} + X_{\mathfrak{p}}) \exp_{K^{\mathbb{C}}} i(1+m)X_{\mathfrak{k}} \, .$$

Then Ω_m is given by the connected component of $\{\,|DP_m| \neq 0\,\}$ containing $G \times \{0\}$. The polar map $P_m|_{\Omega_m}$ is locally biholomorphic.

Proof. Note that the universal complexifications of L and H are $L^{\mathbb{C}} = G^{\mathbb{C}} \times K^{\mathbb{C}}$ and $H^{\mathbb{C}} = \{\,(k, k) \in L^{\mathbb{C}} \ : \ k \in K^{\mathbb{C}}\}$, respectively. Let L act on $L^{\mathbb{C}}/H^{\mathbb{C}}$ by left multiplication and consider the L-equivariant map $\tilde{P}_m : TG \to L^{\mathbb{C}}/H^{\mathbb{C}}$ defined, for $l \in L$ and $X \in T_eG$, by

$$l_*(X) \to l \exp_{L^{\mathbb{C}}}(i(-mX_{\mathfrak{k}} + X_{\mathfrak{p}}, \, -(1+m)X_{\mathfrak{k}}))H^{\mathbb{C}} \, .$$

Identify $G = L/H$ with the zero section in TG. Then an analogous argument as in [14, Corollary 3.3], applies to show that the connected component Ω_m of $\{\,|D\tilde{P}_m| \neq 0\,\}$ which contains G is the maximal adapted complexification and the restriction $\tilde{P}_m|_{\Omega_m}$ is locally biholomorphic.

Since $\exp_{L^{\mathbb{C}}} = \exp_{G^{\mathbb{C}}} \times \exp_{K^{\mathbb{C}}}$, the statement is a consequence of the following real-analytic, L-equivariant identification

$$L^{\mathbb{C}}/H^{\mathbb{C}} \to G^{\mathbb{C}}, \quad (g, k)H^{\mathbb{C}} \to gk^{-1} \, . \qquad \square$$

In order to describe Ω_m it is convenient to determine a slice for the L-action. Let $\mathfrak{a}^+$ be the closure of a Weyl chamber in a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and define

$$\Sigma := \{\,(e, X) \in G \times \mathfrak{g} \ : \ X_{\mathfrak{p}} \in \mathfrak{a}^+\} \, .$$

Since $\mathfrak{a}^+$ is a fundamental domain for the Ad_K action on $\mathfrak{p}$, every L-orbit of $G \times \mathfrak{g}$ meets Σ. Then one has:

Proposition 3.2. *Denote by Σ_m the connected component of $(e, 0)$ in the subset $\{\,(e, X) \in \Sigma \ : \ |(DP_m)_{(e,X)}| \neq 0\,\}$ of Σ. Then $\Omega_m = L \cdot \Sigma_m$.*

Proof. Note that the L-equivariance of P_m induces that of DP_m, *i.e.*

$$(DP_m)_{l\cdot(e,X)} \circ Dl_{(e,X)} = Dl_{P_m(e,X)} \circ (DP_m)_{(e,X)} \, ,$$

for all $l \in L$ and $(e, X) \in \Sigma$. In particular $(DP_m)_{l\cdot(e,X)}$ has maximal rank if and only if so does $(DP_m)_{(e,X)}$, implying the statement. $\square$

4. The case of $SL_2(\mathbb{R})$

Let $G = SL_2(\mathbb{R})$. Here we choose $K = SO_2(\mathbb{R})$ and give an explicit description of Σ_m in terms of fixed basis of $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{p}$. By Proposition 3.2 this determines the maximal adapted complexification Ω_m associated to the connection of ν_m. Identify $\mathfrak{sl}_2(\mathbb{R})$ with the set of zero trace matrices and let

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\{U\}$ is a basis of $\mathfrak{k}$, while $\{H, W\}$ gives a basis of $\mathfrak{p}$. Consider the polar map $P_m : G \times \mathfrak{g} \to G^{\mathbb{C}}$ introduced in Section 3 and choose Σ by fixing $\mathfrak{a}^+ := \{aH : a \geq 0\}$. Given $(e, X) \in \Sigma$ consider the natural identification $T_{(e,X)}(G \times \mathfrak{g}) \cong \mathfrak{g} \times \mathfrak{g}$ and note that for $Y \in \mathfrak{g}$ one has

$$\begin{aligned}(DP_m)_{(e,X)}(Y, 0) &= \frac{d}{ds}\Big|_0 P_m(\exp sY, X) \\ &= \frac{d}{ds}\Big|_0 \exp sY \exp i(-mX_{\mathfrak{k}} + X_{\mathfrak{p}}) \exp i(1+m)X_{\mathfrak{k}} \\ &= DR_{\exp i(1+m)X_{\mathfrak{k}}} \circ DR_{\exp i(-mX_{\mathfrak{k}}+X_{\mathfrak{p}})} Y,\end{aligned}$$

$$\begin{aligned}(DP_m)_{(e,X)}(0, Y_{\mathfrak{k}}) &= \frac{d}{ds}\Big|_0 P_m(0, X + sY_{\mathfrak{k}}) \\ &= \frac{d}{ds}\Big|_0 \exp i(-m(X_{\mathfrak{k}} + sY_{\mathfrak{k}}) + X_{\mathfrak{p}}) \exp i(1+m)(X_{\mathfrak{k}} + sY_{\mathfrak{k}}) \\ &= DR_{\exp i(1+m)X_{\mathfrak{k}}} \circ (D\exp)_{i(-mX_{\mathfrak{k}}+X_{\mathfrak{p}})}(-imY_{\mathfrak{k}}) \\ &\quad + DL_{\exp i(-mX_{\mathfrak{k}}+X_{\mathfrak{p}})} \circ (D\exp)_{i(1+m)X_{\mathfrak{k}}}(i(1+m)Y_{\mathfrak{k}}),\end{aligned}$$

$$\begin{aligned}(DP_m)_{(e,X)}(0, Y_{\mathfrak{p}}) &= \frac{d}{ds}\Big|_0 P_m(0, X + sY_{\mathfrak{p}}) \\ &= \frac{d}{ds}\Big|_0 \exp i(-mX_{\mathfrak{k}} + X_{\mathfrak{p}} + sY_{\mathfrak{p}}) \exp i(1+m)X_{\mathfrak{k}} \\ &= DR_{\exp i(1+m)X_{\mathfrak{k}}} \circ (D\exp)_{i(-mX_{\mathfrak{k}}+X_{\mathfrak{p}})}(iY_{\mathfrak{p}}),\end{aligned}$$

where L_g and R_g denote left and right multiplication by $g \in G^{\mathbb{C}}$, respectively, and $\exp$ is the exponential map of $G^{\mathbb{C}}$. Recall that by identifying $T_{\exp X}G^{\mathbb{C}}$ with

$\mathfrak{g}^{\mathbb{C}}$ via left multiplication one has

$$(D\exp)_X(Y) = \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+1)!}\mathrm{ad}^l(X)Y$$

for all $X, Y \in \mathfrak{g}^{\mathbb{C}}$ (see, *e.g.* [27]). Fix $X = uU + aH$ in Σ. By using the above formulae, one shows that the basis $\{(U,0),\ (H,0),\ (W,0),\ (0,U),\ (0,H),\ (0,W)\}$ of $\mathfrak{g}\times\mathfrak{g} \cong T_{(e,X)}(G\times\mathfrak{g})$ is mapped by the differential $DP_m|_{(e,X)}$ into the image via $\mathrm{Ad}_{\exp -i(1+m)X_{\mathfrak{k}}}$ of the following six vectors

$$\mathrm{Ad}_{\exp i(umU-aH)}U,\quad \mathrm{Ad}_{\exp i(umU-aH)}H,\quad \mathrm{Ad}_{\exp i(umU-aH)}W,$$

$$\sum_{l=0}^{\infty} \frac{(-1)^l}{(l+1)!}\mathrm{ad}^l(i(-umU+aH))(-imU)\ + i(1+m)U,$$

$$\sum_{l=0}^{\infty} \frac{(-1)^l}{(l+1)!}\mathrm{ad}^l(i(-umU+aH))(iH),\quad \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+1)!}\mathrm{ad}^l(i(-umU+aH))(iW).$$

Now note that $w \to \cosh w$ and $w \to \frac{\sinh w}{w}$ are even holomorphic, thus the functions $z \to \cosh\sqrt{z}$ and $z \to \frac{\sinh\sqrt{z}}{\sqrt{z}}$ are well defined and holomorphic. Moreover the cofficients of their Taylor series around the origin are real. Then, by restriction one obtains two real-analytic functions which will be denoted in the sequel by C and S, respectively.

Letting $x = 4u^2m^2 - 4a^2$, a further computation (see Appendix) shows that the above six vectors can be written as

$$\left(1 - 4a^2\frac{C(x)-1}{x}\right)U\ +\ 4aum\frac{C(x)-1}{x}H\ +\ 2aS(x)\,iW\,,$$

$$-4aum\frac{C(x)-1}{x}U\ +\ \left(1 + 4u^2m^2\frac{C(x)-1}{x}\right)H\ +\ 2umS(x)\,iW\,,$$

$$2aS(x)\,iU\ -\ 2umS(x)\,iH\ +\ C(x)\,W\,,$$

$$\left(1 + 4a^2m\frac{S(x)-1}{x}\right)iU\ -\ 4aum^2\frac{S(x)-1}{x}iH\ +\ 2am\frac{C(x)-1}{x}W\,,$$

$$-4aum\frac{S(x)-1}{x}iU\ +\ \left(1 + 4u^2m^2\frac{S(x)-1}{x}\right)iH\ -\ 2um\frac{C(x)-1}{x}W\,,$$

$$-2a\frac{C(x)-1}{x}U\ +\ 2um\frac{C(x)-1}{x}H\ +\ S(x)\,iW\,.$$

Let us point out without proof some properties of the functions $C : \mathbb{R} \to \mathbb{R}$, $x \to \cosh\sqrt{x}$, and $S : \mathbb{R} \to \mathbb{R}$, $x \to \sinh\sqrt{x}/\sqrt{x}$, which are used in the sequel.

Lemma 4.1. *For $x > -\pi^2$ the real-analytic functions S, S' are strictly positive and S, C, S', C/S, S/S' are strictly increasing. Moreover*

$$C'(x) = \frac{1}{2} S(x)\,, \qquad S'(x) = \frac{C(x) - S(x)}{2x} \qquad \textit{and} \qquad x < \frac{C(x)}{2S'(x)}.$$

We can now determine the maximal adapted complexification Ω_m by computing the slice Σ_m introduced in Proposition 3.2.

Proposition 4.2. *The slice Σ_m for the maximal adapted complexification consists of the elements $(e, uU + aH)$ in Σ such that*

$*)$ $\quad 4u^2m^2 - 4a^2 > -\pi^2$

$**)$ $\quad 4u^2m^2 + m4a^2 < f(4u^2m^2 - 4a^2)\,,$

where the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) := \frac{x\,C(x)}{C(x)-S(x)} = \frac{C(x)}{2S'(x)}$.

Proof. Let $X = uU + aH \in \Sigma$ and $x = 4u^2m^2 - 4a^2$. By the above computations $(DP_m)_{(e,\,X)}$ is non-singular if and only if the two determinants

$$\begin{vmatrix} 1 - 4a^2\frac{C(x)-1}{x} & 4aum\frac{C(x)-1}{x} & 2aS(x) \\ -4aum\frac{C(x)-1}{x} & 1 + 4u^2m^2\frac{C(x)-1}{x} & 2umS(x) \\ -2a\frac{C(x)-1}{x} & 2um\frac{C(x)-1}{x} & S(x) \end{vmatrix}$$

and

$$\begin{vmatrix} 2aS(x) & -2umS(x) & C(x) \\ 1 + 4a^2m\frac{S(x)-1}{x} & -4aum^2\frac{S(x)-1}{x} & 2am\frac{C(x)-1}{x} \\ -4aum\frac{S(x)-1}{x} & 1 + 4u^2m^2\frac{S(x)-1}{x} & -2um\frac{C(x)-1}{x} \end{vmatrix},$$

do not vanish. A straightforward computation yields the two conditions above. □

Note that in the pseudo-Riemannian symmetric case $m = -1$, conditions $*)$ and $**)$ coincide and yield the set $\{\,(e, X) \in \Sigma \;:\; D\exp_X \text{ not singular}\,\}$.

Definition 4.3. For $m \in \mathbb{R}$ let Σ_m^* be the subdomain of Σ defined by condition $*)$ in Proposition 4.2, *i.e.*

$$\Sigma_m^* := \{\,(e, uU + aH) \in \Sigma \;:\; 4m^2u^2 - 4a^2 > -\pi^2\,\}.$$

Remark 4.4 (see Figure 4.1). For $m \leq -1$ one has $\Sigma_m = \Sigma_m^*$. Indeed by Lemma 4.1 one has $x < f(x)$, for $x > -\pi^2$. Consequently for any $(e, uU + aH) \in \Sigma_m^*$

$$4u^2m^2 + 4a^2m \leq 4u^2m^2 - 4a^2 < f(4u^2m^2 - 4a^2),$$

thus condition $**)$ is automatically fulfilled.

For $m > -1$ the closure of Σ_m in Σ is contained in Σ_m^*. Indeed, for $4u^2m^2 - 4a^2 = -\pi^2$, one has

$$4u^2m^2 + 4a^2m > 4u^2m^2 - 4a^2 = -\pi^2 = f(-\pi^2) = f(4u^2m^2 - 4a^2)\,,$$

therefore condition $**)$ is not fulfilled. In particular the boundary of Σ_m in Σ is defined by

$$\Big\{ (e, uU + aH) \in \Sigma_m^* \ : \ 4u^2m^2 + 4a^2m = f(4u^2m^2 - 4a^2) \Big\}.$$

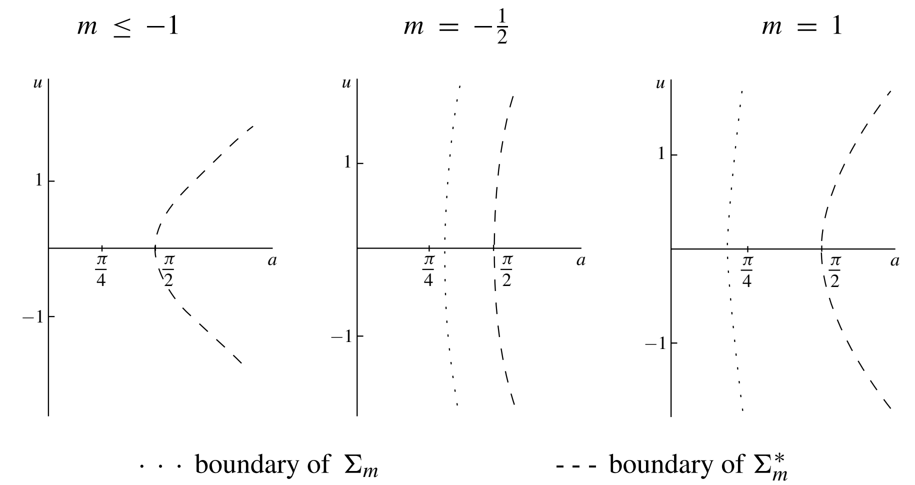


Figure 4.1.

Remark 4.5. For $m > -1$ one checks that the vector fields tangential to $P_m(\Sigma)$, *i.e.* $(DP_m)_{(e,X)}(0, U)$ and $(DP_m)_{(e,X)}(0, H)$, remain linearly independent on the boundary of $P(\Sigma_m)$. Thus the boundary of the maximal adapted complexification can be characterized by saying that L-orbits (in fact $\exp(W)$-orbits) become tangential to $P_m(\Sigma)$.

On the other hand for $m \leq -1$ it is the dimension of L-orbits at the boundary of $P(\Sigma_m)$ to drop from 4 to 3 (see i) of Proposition 5.4 below), giving again a characterization of the maximal adapted complexification. □

Before studying the restriction of P_m to Σ_m it will be useful to obtain a concrete realization of quotients of $G^{\mathbb{C}}$ with respect to those actions which are involved.

5. Slices and quotients of $SL_2(\mathbb{C})$

Here G and K denote $SL_2(\mathbb{R})$ and $SO_2(\mathbb{R})$, respectively. Let $L = G \times K$ act on $G^{\mathbb{C}}$ by left and right multiplication. Note that since K is compact, this action is proper. The main goal of this section is to present models for the quotients $G\backslash G^{\mathbb{C}}$, $G^{\mathbb{C}}/L$, $G^{\mathbb{C}}/(G \times K^{\mathbb{C}})$ and for the relative quotient maps. First consider the map

$$\Pi_1 : G^{\mathbb{C}} \to G^{\mathbb{C}}, \quad g \to \sigma_G(g)^{-1} g,$$

where $\sigma_G : G^{\mathbb{C}} \to G^{\mathbb{C}}$, $g \to \overline{g}$, is the conjugation in $G^{\mathbb{C}}$, *i.e.* the unique anti-holomorphic involutive automorphism of $G^{\mathbb{C}}$ whose fixed point set is G. Let G act on $G^{\mathbb{C}}$ by left multiplication and note that every fiber of this map consist of a single G-orbit. Thus $\Pi_1(G^{\mathbb{C}})$ is set theoretically equivalent to $G\backslash G^{\mathbb{C}}$ and

$$\Pi_1 : G^{\mathbb{C}} \to \Pi_1(G^{\mathbb{C}})$$

is a realization of the quotient map. Moreover, a simple computation shows that (*cf.*, *e.g.* [28])

$$\Pi_1(G^{\mathbb{C}}) = \{ g \in G^{\mathbb{C}} \ : \ \sigma_G(g) = g^{-1} \}.$$

It is convenient to consider the automorphism $A : G^{\mathbb{C}} \to G^{\mathbb{C}}$ transforming $SL(2, \mathbb{R})$ onto $SU(1, 1)$. This is induced by the unique complex Lie algebra morphism of $\mathfrak{g}^{\mathbb{C}}$ mapping the basis $\{U, H, W\}$ (*cf.* beginning of Section 4) into $\{iH, iU, W\}$. Recall that the involution of $G^{\mathbb{C}}$ defining $SU(1, 1)$ is given by $\sigma_{SU(1,1)}(g) = J^t\overline{g}^{-1} J$, where

$$J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the elements of the above basis are fixed by the Lie algebra automorphisms induced by σ_G and $\sigma_{SU(1,1)}$ respectively, it follows that

$$A \circ \sigma_G = \sigma_{SU(1,1)} \circ A \,. \tag{5.1}$$

Note that $\mathcal{Q} := A \circ \Pi_1(G^{\mathbb{C}})$ can be identified with $\Pi_1(G^{\mathbb{C}}) \cong G\backslash G^{\mathbb{C}}$. Then

$$\begin{aligned} \mathcal{Q} &= \{ A(g) \ : \ g \in G^{\mathbb{C}} \text{ and } \sigma_G(g) = g^{-1} \} \\ &= \{ g \in G^{\mathbb{C}} \ : \ \sigma_{SU(1,1)}(g) = g^{-1} \} \\ &= \left\{ \begin{pmatrix} s & b \\ -\overline{b} & t \end{pmatrix} \ : \ s, t \in \mathbb{R}, \ b \in \mathbb{C} \text{ and } st + |b|^2 = 1 \right\} \end{aligned}$$

gives a model of the quotient $G\backslash G^{\mathbb{C}}$. Let us describe how the right $K^{\mathbb{C}}$-action on $G^{\mathbb{C}}$ is transformed after applying $A \circ \Pi_1$. For $\lambda \in \mathbb{C}$ and $g \in G^{\mathbb{C}}$ one has

$$\begin{aligned} A \circ \Pi_1(g \exp(-\lambda U)) &= A(\sigma_G(g \exp(-\lambda U))^{-1} g \exp(-\lambda U)) \\ &= A(\sigma_G(\exp(-\lambda U)))^{-1} A(\sigma_G(g))^{-1} A(g) A(\exp(-\lambda U))) \end{aligned}$$

and by (5.1) this gives

$$\sigma_{SU(1,1)}(A(\exp(-\lambda U)))^{-1}A(\sigma_G(g)^{-1}g)A(\exp(-\lambda U)))$$

$$= (J\overline{\exp(-i\lambda H)}J)^{-1}A(\Pi_1(g))\exp(-i\lambda H)$$

$$= \exp(i\overline{\lambda}H)A(\Pi_1(g))\exp(-i\lambda H).$$

Thus $A \circ \Pi_1 : G^{\mathbb{C}} \to \mathcal{Q}$ is $K^{\mathbb{C}}$-equivariant, if one let $K^{\mathbb{C}}$ act on $G^{\mathbb{C}}$ by right multiplication and on $\mathcal{Q}$ by

$$\exp(\lambda U) \cdot \begin{pmatrix} s & b \\ -\overline{b} & t \end{pmatrix} := \exp(i\overline{\lambda}H) \begin{pmatrix} s & b \\ -\overline{b} & t \end{pmatrix} \exp(-i\lambda H)$$

$$= \begin{pmatrix} e^{2y}s & e^{2ix}b \\ -e^{-2ix}\overline{b} & e^{-2y}t \end{pmatrix},$$

for every $x + iy = \lambda \in \mathbb{C}$. In particular, after applying $A \circ \Pi_1$, the right K-action on $G^{\mathbb{C}}$ reads as rotations on b. Let

$$\mathcal{P} := \{ (s, t) \in \mathbb{R}^2 \ : \ st \leq 1\}$$

and define $\Pi_2 : \mathcal{Q} \to \mathcal{P}$ by

$$\begin{pmatrix} s & b \\ -\overline{b} & t \end{pmatrix} \to (s, t) \, .$$

For every $(s, t) \in \mathcal{P}$ the inverse image $\Pi_2^{-1}(s,\ t)$ consists of a single K-orbit given by

$$\left\{ \begin{pmatrix} s & b \\ -\overline{b} & t \end{pmatrix} \in \mathcal{Q} \ : \ |b|^2 = 1 - st \right\}$$

In fact $\mathcal{P}$ is a realization of the quotient $\mathcal{Q}/K \cong G^{\mathbb{C}}/L$. Recall that $L^{\mathbb{C}} = G^{\mathbb{C}} \times K^{\mathbb{C}}$ act on $G^{\mathbb{C}}$ by left and right multiplication. Let the one-parameter subgroup of $L^{\mathbb{C}}$ defined by $R := \{e\} \times \exp i\mathfrak{k}$ act on $\mathcal{P}$ by $(\{e\} \times \exp(iyU)) \cdot (s,\ t) := (e^{2y}s, e^{-2y}t)$, for all $y \in \mathbb{R}$ and $(s, t) \in \mathcal{P}$. One has:

Lemma 5.1. *Let $K^{\mathbb{C}}$ act on $\mathcal{Q}$ by*

$$\exp(x + iy)U \cdot \begin{pmatrix} s & b \\ -\overline{b} & t \end{pmatrix} = \begin{pmatrix} e^{2y}s & e^{2ix}b \\ -e^{-2ix}\overline{b} & e^{-2y}t \end{pmatrix},$$

for $x + iy \in \mathbb{C}$. Then

i) *a model for the quotient $G\backslash G^{\mathbb{C}}$ is $\mathcal{Q}$ with $K^{\mathbb{C}}$-equivariant quotient map $A \circ \Pi_1 : G^{\mathbb{C}} \to \mathcal{Q}$. The fixed point set of the K-action on $\mathcal{Q}$ is given by $\left\{ \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \ : \ st = 1 \right\}$,*

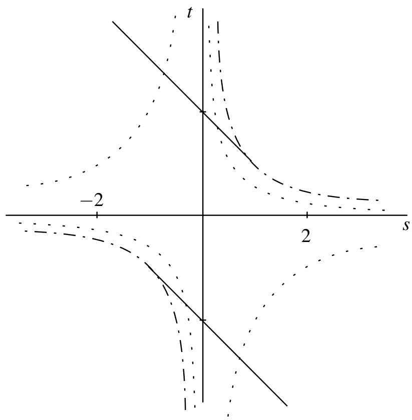
ii) *a model for* $\mathcal{Q}/K$ *is given by* $\mathcal{P}$ *with quotient map* $\Pi_2 : \mathcal{Q} \to \mathcal{P}$ *defined by* $\begin{pmatrix} s & b \\ -\overline{b} & t \end{pmatrix} \to (s, t)$,

iii) *a model for the quotient* $G^{\mathbb{C}}/L$ *is* $\mathcal{P}$ *with* R*-equivariant quotient map* $F := \Pi_2 \circ A \circ \Pi_1 : G^{\mathbb{C}} \to \mathcal{P}$. *In particular* $G\backslash G^{\mathbb{C}}/K^{\mathbb{C}} \cong \mathcal{P}/R$.

Remark 5.2. Let $N_{G^{\mathbb{C}}}(K^{\mathbb{C}})$ denote the normalizer of $K^{\mathbb{C}}$ in $G^{\mathbb{C}}$. Then e and $\tilde{e} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ represent the two elements of the quotient $N_{G^{\mathbb{C}}}(K^{\mathbb{C}})/K^{\mathbb{C}}$. A simple computation shows that F is equivariant with respect to right (or left) multiplication by $\tilde{e}$ in $G^{\mathbb{C}}$ and reflection with respect to the origin in $\mathcal{P}$. Furthermore F is equivariant with respect to conjugation by $\tilde{e}$ in $G^{\mathbb{C}}$ and reflection with respect to the line of equation $s = t$ in $\mathcal{P}$.

Remark 5.3. The action of R on $\mathcal{P}$ is not proper. Set theoretically $\mathcal{P}/R$ can be identified with the slice for the R-action on $\mathcal{P}$ defined by the union $\{ s = t,\ st \leq 1 \} \cup \{ s = -t \} \cup \{ p_1,\ p_2,\ p_3,\ p_4 \}$, where $p_1 := (2, 0)$, $p_2 := (0, 2)$, $p_3 := (-2, 0)$ and $p_4 := (0, -2)$ correspond to the non-closed R-orbits in $\mathcal{P}$. The closure of these orbits is obtained by adding the unique fixed point $p_0 = (0, 0)$. However for our purposes it is more convenient to consider the following slice (*cf.* Figure 5.1)

$$\mathcal{S} := \{ s + t = 2,\ t \geq s \} \cup \{ s + t = -2,\ s \geq t \} \cup \{ p_0,\ p_1,\ p_3 \}.$$



— the slice $\mathcal{S}$ - - - boundary of $\mathcal{P}$ · · · R-orbits

Figure 5.1.

Here and in the sequel a slice is assumed to intersect every orbit in a single point. Let $\Sigma = \{ (e, X) \in G \times \mathfrak{g} \ :\ X_{\mathfrak{p}} \in \mathfrak{a}^+ \}$ be the slice in $TG \cong G \times \mathfrak{g}$ introduced in Section 3. Consider the subdomain

$$\Sigma_{AG} = \left\{ (e, uU + aH) \in \Sigma \ : \ 4u^2 - 4a^2 > -\frac{\pi^2}{4} \right\}$$

and denote by $\overline{\Sigma}_{AG}$ its closure in Σ. One has:

Proposition 5.4. *Let $G \times K^{\mathbb{C}} \subset L^{\mathbb{C}}$ act on $G^{\mathbb{C}}$ by left and right multiplication.*

i) *A slice for the L-action on $G^{\mathbb{C}}$ is given by*

$$S_1 := \exp i\overline{\Sigma}_{AG} \cup \tilde{e} \exp i\Sigma_{AG}.$$

All L-orbits are closed (the action is proper), the union of all 3-dimensional L-orbits is given by $F^{-1}(\{ st = 1 \}) = L \cdot (\exp i\mathfrak{k} \cup \tilde{e} \exp i\mathfrak{k})$. All other orbits are 4-dimensional with discrete isotropy given by the ineffectivity $\pm(e, e)$ of L.

ii) *A slice for the $G \times K^{\mathbb{C}}$-action is given by*

$$S_2 := \{ \exp \rho i(U+H) : \rho \geq 0 \} \cup \tilde{e} \{ \exp \rho i(U+H) : \rho \geq 0 \} \cup \{ g_0,\ g_1,\ g_3 \},$$

where $g_0 = \exp i\frac{\pi}{4}H$, $g_1 = \exp -\frac{i}{2}(U + H)$ and $g_3 = \tilde{e}g_1$. The only non-closed $G \times K^{\mathbb{C}}$-orbits are those through $g_1,\ g_2\ g_3,\ g_4$, with $g_2 = \exp \frac{i}{2}(U + H)$ and $g_4 = \tilde{e}g_2$. Their closure is obtained by adding the orbit through g_0. The only 4-dimensional orbits are those through $e,\ \tilde{e}$ and g_0, all other orbits have maximal dimension.

Proof. For all $(e, uU + aH)$ in $\overline{\Sigma}_{AG}$ one has

$$\begin{aligned} F(\exp i(uU + aH)) &= \Pi_2 \circ A(\exp 2i(uU + aH)) = \Pi_2(\exp -2(uH + aU)) \\ &= \Pi_2 \begin{pmatrix} C(x) - 2uS(x) & 2aS(x) \\ -2aS(x) & C(x) + 2umS(x) \end{pmatrix} \\ &= (C(x) - 2uS(x),\ C(x) + 2uS(x)), \end{aligned}$$

where $x = 4u^2 - 4a^2$. Fix $x \geq -\pi^2/4$ and note that the set $Q_x := \{ (e, uU + aH) \in \overline{\Sigma}_{AG} \ : \ 4u^2 - 4a^2 = x \}$ is given by $\{ (e, uU + aH) \ : \ 4u^2 \geq x,\ 0 \leq a = \sqrt{4u^2 - x}/2 \}$. Then, from the above formula it follows that $F(\exp i Q_x)$ consists of the intersection of $\mathcal{P}$ with the line of equation $s + t = 2C(x)$. As a consequence $\exp i\overline{\Sigma}_{AG}$ is mapped bijectively onto $\mathcal{P} \cap \{s \geq -t\}$. This and Remark 5.2 imply that F maps S_1 bijectively onto $\mathcal{P}$. Thus S_1 is a slice for the L-action on $G^{\mathbb{C}}$.

Since $F = \Pi_2 \circ A \circ \Pi_1$, by i) of Lemma 5.1 the only 3-dimensional L-orbits are those through $F^{-1}(\{st = 1\})$ and all others are 4-dimensional. Moreover the isotropy of the K-action on $\mathcal{Q} \setminus \Pi_2^{-1}(\{ st = 1 \})$ is given by the ineffectivity $\pm(e, e)$, implying that last claim in i).

For ii) note that $F(\exp i\rho(U+H)) = (1-2\rho, 1+2\rho)$ (*cf.* the above formula). This and Remark 5.2 apply to show that the restriction $F|_{S_2} : S_2 \to \mathcal{S}$ is bijective, where $\mathcal{S}$ is the R-slice in $\mathcal{P}$ introduced in Remark 5.3. One has a commutative diagram of canonical quotients

$$\begin{array}{ccc} G^{\mathbb{C}} & \xrightarrow{F} & G^{\mathbb{C}}/L \cong \mathcal{P} \\ & \searrow & \downarrow \\ & & G^{\mathbb{C}}/(G \times K^{\mathbb{C}}) \cong \mathcal{P}/R \end{array}$$

where, by iii) of Lemma 5.1, the map F is R-equivariant. This gives a one to one correspondence between the $G \times K^{\mathbb{C}}$-orbit space of $G^{\mathbb{C}}$ and the R-orbit space of $\mathcal{P}$. As a consequence S_2 is a slice for the $G \times K^{\mathbb{C}}$-action on $G^{\mathbb{C}}$. Moreover the $G \times K^{\mathbb{C}}$-orbits through $g_0, \ldots, g_4$ correspond to the five R-orbits in $\{st = 0\} \subset \mathcal{P}$, implying the topological claim. Finally, the dimension of every $G \times K^{\mathbb{C}}$ orbit can be obtained by adding to the dimension of the corresponding R-orbit the dimension of the L-orbit given in i). Since every R-orbit different from the fixed point $(0,0)$ is one-dimensional, this concludes the proof. □

Regard G/K as a Riemannian symmetric space of rank one. Its maximal adapted complexification can be realized in $G^{\mathbb{C}}/K^{\mathbb{C}}$ and it is usually described as $\Omega_{AG} = G\exp(i[0, \frac{\pi}{4})H)K^{\mathbb{C}}$ (see [2, 6]). Its boundary is given by $\cup_{j=1}^{3} Gg_jK^{\mathbb{C}}$ and we refer to $Gg_0K^{\mathbb{C}}$ as its singular boundary. For later use we note the following direct consequence of [11, Theorem 6.1 and Example 6.3].

Lemma 5.5. *For $G = SL_2(\mathbb{R})$, let Ω be a G-invariant domain of $G^{\mathbb{C}}/K^{\mathbb{C}}$ which contains Ω_{AG} and its non-singular boundary, i.e. Ω contains the invariant subset $\overline{\Omega}_{AG} \setminus G\exp(i\frac{\pi}{4}H)K^{\mathbb{C}}$. Then Ω is holomorphically convex if and only if it coincides with $G^{\mathbb{C}}/K^{\mathbb{C}}$.*

6. The reduced polar map

Let $G = SL_2(\mathbb{R})$ be endowed with one of the metrics ν_m. By Proposition 3.2 the associated maximal adapted complexification is given by $\Omega_m = L \cdot \Sigma_m$, where Σ_m consists of the elements of Ω_m which are in the slice Σ for the L-action on $G \times \mathfrak{g} \cong TG$. Note that every L-orbit in $G \times \mathfrak{g}$ intersects Σ in a single element, thus $\Sigma \cong TG/L$. One has a commutative diagram

$$\begin{array}{ccc} TG \cong G \times \mathfrak{g} & \overset{P_m}{\to} & G^{\mathbb{C}} \\ \downarrow & & \downarrow F \\ TG/L \cong \Sigma & \overset{\hat{P}_m}{\to} & G^{\mathbb{C}}/L \cong \mathcal{P}\,, \end{array}$$

where $F : G^{\mathbb{C}} \to \mathcal{P}$ is the realization of the quotient map given in Lemma 5.1 and $\hat{P}_m := F \circ P_m|_\Sigma$. Here we show that the polar map $P_m|_{\Omega_m}$ is injective if and only if so is the restriction $\hat{P}_m|_{\Sigma_m}$. We also point out certain properties of the map $\hat{P}_m$ which are used in the remaining sections in order to discuss injectivity of $P_m|_{\Omega_m}$. It will turn out that Ω_m is biholomorphic to an L-invariant domain of $G^{\mathbb{C}}$ if $m \leq 0$, while it is a non-holomorphically separable Riemann domain over $G^{\mathbb{C}}$ if $m > 0$. In both cases Ω_m is not holomorphically convex and its envelope of holomorphy is biholomorphic to $G^{\mathbb{C}}$.

We first compute the two components of $\hat{P}_m$ with respect to the basis $\{U, H\}$. One has

$$\begin{aligned}
&\hat{P}_m(e, uU + aH) = F \circ P_m(e, uU + aH) \\
&= F\big(\exp i(-umU + aH) \exp iu(1+m)U\big) \\
&= \Pi_2 \circ A\big(\exp iu(1+m)U \exp 2i\,(-umU + aH) \exp iu(1+m)U\big) \\
&= \Pi_2\big(\exp -u(1+m)H \exp 2(umH - aU) \exp -u(1+m)H\big) \\
&= \Pi_2 \begin{pmatrix} e^{-2u(1+m)}\,(C(x) + 2umS(x)) & 2aS(x) \\ -2aS(x) & e^{2u(1+m)}\,(C(x) - 2umS(x)) \end{pmatrix},
\end{aligned}$$

where $x = 4u^2m^2 - 4a^2$. This gives

$$\hat{P}_m(e, uU + aH) = \Big(e^{-2u(1+m)}(C(x) + 2umS(x)),\, e^{2u(1+m)}(C(x) - 2umS(x))\Big). \quad (6.1)$$

Remark 6.1. The map $\hat{P}_m$ has maximal rank on $\{(e, X) \in \Sigma_m : X_{\mathfrak{a}^+} \neq 0\}$. Indeed it is easy to check that F has maximal rank on $F^{-1}(\{st \neq 1\})$, therefore so does $\hat{P}_m$ on the set $\Sigma_m \cap \hat{P}_m^{-1}(\{st \neq 1\})$. From formula (6.1) it follows that this set coincides with $\Sigma_m \cap \{(e, uU + aH) : a > 0\}$. □

Recall that for all real m the slice Σ_m is contained in the domain $\Sigma_m^* = \{(e, uU + aH) \in \Sigma : 4m^2u^2 - 4a^2 > -\pi^2\}$ (*cf.* Remark 4.4). For such bigger domain one has:

Lemma 6.2. *The restriction of P_m to any L-orbit of $L \cdot \Sigma_m^*$ is a diffeomorphism onto an L-orbit of $G^{\mathbb{C}}$.*

Proof. Note that the isotropy of L at a point (e, X) of Σ_m^* is given by $\{(k, k) : k \in K\}$ if $X \in \mathfrak{k}$, it consists of the ineffectivity $\pm(e, e)$ otherwise. Moreover since K and $\exp i\mathfrak{k}$ commute, the identity

$$(g, k) \cdot P_m(e, X) = P_m(e, X)$$

holds true if and only if

$$g \exp i(-muU + aH)k = \exp i(-muU + aH) .$$

Now, a similar computation as in formula (6.1) yields

$$F(\exp i(X)) = (\, C(x) + 2umS(x)\, ,\; C(x) - 2umS(x)\,)\, ,$$

which belongs to $\{ st = 1 \}$ if and only if $a = 0$. Then, as a consequence of i) of Proposition 5.4, the isotropy at $P_m(e, uU + aH)$ is given by $\{ (k, k) \ : \ k \in K \}$ if $a = 0$ or $\pm(e, e)$ otherwise, which proves the statement. □

Remark 6.3. For $(e,\ uU + aH) \in \partial\Sigma_m^*$ one has $4u^2m^2 - 4a^2 = -\pi^2$. Thus, by formula (6.1)

$$F \circ P_m(e,\ uU + aH) = (-e^{-u(1+m)}, -e^{u(1+m)}) \in \{ (s,\ t) \in \mathcal{P} \ : \ st = 1\} .$$

Then i) of Proposition 5.4 implies that the dimension of the L-orbit through $P_m(e, uU + aH)$ is only three. In particular an analogous statement as in the above lemma does not hold on domains larger than Σ_m^*. □

Since $\Omega_m = L \cdot \Sigma_m$ and Σ_m is contained in Σ_m^*, from the above lemma it follows that $P_m|_{\Omega_m}$ is injective if and only if different L-orbits in Ω_m are mapped by P_m to different L-orbits in $G^{\mathbb{C}}$. Recalling that for $G = SL_2(\mathbb{R})$ every orbit intersects Σ_m in a single point and that $F : G^{\mathbb{C}} \to \mathcal{P}$ is a realization of the quotient map with respect to the L-action on $G^{\mathbb{C}}$, one has:

Proposition 6.4. *The polar map* $P_m|_{\Omega_m}$ *is injective if and only if* $\hat{P}_m|_{\Sigma_m}$ *is injective.*

We conclude this section with a technical result which will be repeatedly used in the sequel. Consider the two involutions

$$\alpha : \Sigma_m^* \to \Sigma_m^* \qquad X_{\mathfrak{k}} + X_{\mathfrak{p}} \to -X_{\mathfrak{k}} + X_p$$

and

$$\beta : \mathcal{P} \to \mathcal{P} \qquad (s, t) \to (t, s)$$

and denote by $\mathrm{fix}(\alpha) = \{ X \in \Sigma_m^* \ : \ X_{\mathfrak{k}} = 0 \}$ and $\mathrm{fix}(\beta) = \{ (s, t) \in \mathcal{P} \ : s = t \}$ the associated fixed point sets.

Note that $\hat{P}_m|_{\Sigma_m^*} : \Sigma_m^* \to \mathcal{P}$ is equivariant with respect to these involutions. As a consequence $\hat{P}_m(\mathrm{fix}(\alpha))$ is contained in $\mathrm{fix}(\beta)$. Also consider the α-invariant map $\Gamma : \Sigma_m^* \to \mathbb{R}$ defined by

$$\Gamma(uU + aH) := \frac{m}{1+m}\,\frac{C(4u^2(1+m)^2)}{S(4u^2(1+m)^2)} - \frac{C(x)}{S(x)}\, ,$$

with $x = 4u^2m^2 - 4a^2$.

Lemma 6.5. *Let $\alpha : \Sigma_m^* \to \Sigma_m^*$ and $\beta : \mathcal{P} \to \mathcal{P}$ be the two involutions defined above.*

i) *For $m \le -1$ one has*

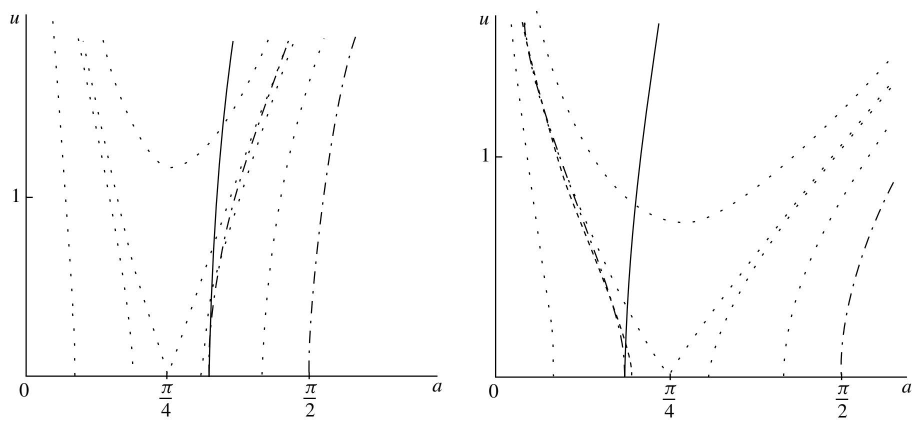
$$\left(\hat{P}_m|_{\Sigma_m^*}\right)^{-1}(\mathrm{fix}(\beta)) = \mathrm{fix}(\alpha)\,.$$

ii) *For $m > -1$ one has* (*cf.* Figure 6.1)

$$\left(\hat{P}_m|_{\Sigma_m^*}\right)^{-1}(\mathrm{fix}(\beta)) = \mathrm{fix}(\alpha) \cup \mathrm{graph}(\gamma)$$

where $\gamma : \mathfrak{k} \to \mathfrak{a}$ is the real-analytic map implicitly defined by $\{\,\Gamma = 0\,\}$.

$m = -\frac{1}{2}$ $m = 1$



— $\partial\Sigma_m$ - · - $\partial\Sigma_m^*$ - - - $\{\,\Gamma = 0\,\}$ · · · ℓ_c for $c \approx 0.7,\ 0.2,\ 0,\ -0.7$

Figure 6.1.

Proof. Formula (6.1) implies that the set $\left(\hat{P}_m|_{\Sigma_m^*}\right)^{-1}(\mathrm{fix}(\beta))$ is given by

$$\begin{aligned}\{uU + aH \in \Sigma_m^* : e^{-2u(1+m)}(C(x) + 2umS(x)) &= e^{2u(1+m)}(C(x) - 2umS(x))\}\\ &= \{\,\cosh(2u(1+m))\,2um\,S(x)\\ &= \sinh(2u(1+m))\,C(x)\,\}\,.\end{aligned}$$

Then the cases $m = -1$ and $m = 0$ are straightforward. For $m \neq -1$ this set can be written as $\{\,u = 0\,\} \cup \{\,\Gamma = 0\,\}$, with Γ as in the statement.

For $m < -1$ let $u > 0$ and note that

$$\frac{\cosh(2u(1+m))}{\sinh(2u(1+m))} < \frac{\cosh(2um)}{\sinh(2um)}.$$

Recalling that $x \to C(x)/S(x)$ is strictly increasing this yields

$$\begin{aligned}\frac{m}{1+m}\frac{C(4u^2(1+m)^2)}{S(4u^2(1+m)^2)} &= \frac{\cosh(2u(1+m))\,2um}{\sinh(2u(1+m))}\\ &> \frac{\cosh(2um)\,2um}{\sinh(2um)}\\ &= \frac{C(4u^2m^2)}{S(4u^2m^2)}\\ &\geq \frac{C(4u^2m^2-4a^2)}{S(4u^2m^2-4a^2)},\end{aligned}$$

for any $a \geq 0$, implying that $\{\Gamma = 0\}$ has no solution for $u > 0$. Along with the α-invariance of Γ, this implies i).

For $m > -1$, $m \neq 0$ and $u > 0$ fixed, an analogous argument shows that

$$\frac{m}{1+m}\frac{C(4u^2(1+m)^2)}{S(4u^2(1+m)^2)} < \frac{C(4u^2m^2)}{S(4u^2m^2)}.$$

Since $C(x)/S(x)$ is strictly increasing for $x > -\pi^2$ and $C(x)/S(x) \to -\infty$ for $x \to -\pi^2$, it follows that there exists a unique $a \in \mathbb{R}$ with $4u^2m^2 - 4a^2 > -\pi^2$ such that $\Gamma(uU + aH) = 0$. This and α-invariance of Γ yield ii). □

7. The case of $m \leq -1$

Let $G = SL_2(\mathbb{R})$ be endowed with one of the metrics ν_m for some $m \leq -1$. In this case we show that the polar map P_m is a biholomorphism from the maximal adapted complexification Ω_m onto an L-invariant domain of $G^{\mathbb{C}}$. This domain is given by removing from $G^{\mathbb{C}}$ a single 4-dimensional $G \times K^{\mathbb{C}}$-orbit. Its envelope of holomorphy turns out to be biholomorphic to $G^{\mathbb{C}}$.

Consider the one parameter subgroup $R := \{e\} \times \exp(i\mathfrak{k})$ of $L^{\mathbb{C}}$. The $\{e\} \times K$-action on Ω_m induces a local R-action whose infinitesimal generator is given, for all (g, X) in Ω_m, by

$$iU \to J_m\left(\frac{d}{dy}\Big|_0 (\{e\} \times \exp(yU)) \cdot (g, X)\right).$$

Here J_m denotes the adapted complex structure of Ω_m.

Since P_m is holomorphic and $(\{e\} \times K)$-equivariant, it is also locally equivariant with respect to such local R-action on Ω_m and the global R-action on $G^{\mathbb{C}}$. Furthermore both (local) actions commute with the L-actions on Ω_m and on $G^{\mathbb{C}}$, thus they push down to (local) R-actions on $\Omega_m/L \cong \Sigma_m$ and on $G^{\mathbb{C}}/L \cong \mathcal{P}$, respectively. Since one has the commutative diagram

$$\begin{array}{ccc} \Omega_m & \overset{P_m}{\to} & G^{\mathbb{C}} \\ \downarrow & & \downarrow F \\ \Omega_m/L \cong \Sigma_m & \overset{\hat{P}_m}{\to} & \mathcal{P} \cong G^{\mathbb{C}}/L\,, \end{array}$$

the restriction $\hat{P}_m|_{\Sigma_m}$ is locally R-equivariant. Recall that by iii) of Lemma 5.1, the R-action on $\mathcal{P}$ is explicitly given by $(\{e\}\times\exp iyU)\cdot(s,t) = (e^{2y}s,\ e^{-2y}t)$. In particular the function $\mathcal{P} \to \mathbb{R}$, defined by $(s,t) \to st$, is R-invariant. This is used to determine local R-orbits in Σ_m as follows. For $c \leq 1$ let $\ell_c := (\hat{P}_m|_{\Sigma_m})^{-1}(\{\, st = c\,\})$ and denote by Σ_m^+ the set $\{\,(e, uU + aH) \in \Sigma_m \ :\ a > 0,\ u > 0\,\}$. One has:

Lemma 7.1. *Let $m \leq -1$. Then*

- i) *a local R-orbit of Σ_m^+ coincides with a connected component of $\ell_c \cap \Sigma_m^+$,*
- ii) *the intersection $\ell_c \cap \Sigma_m^+$ has two connected components if $0 \leq c < 1$, it is connected if $c < 0$,*
- iii) *$\hat{P}_m$ maps different local R-orbits of Σ_m^+ to different R-orbits of $\mathcal{P}$,*
- iv) *$\hat{P}_m$ is injective on Σ_m,*
- v) *$\hat{P}_m$ maps different local R-orbits of Σ_m to different R-orbits of $\mathcal{P}$,*
- vi) *local R-orbits closed to $\{e\} \times \mathfrak{k}$ are mapped bijectively by $\hat{P}_m$ to R-orbits of $\mathcal{P}$, i.e. R acts globally in a R-invariant neighborhood of $\{e\} \times \mathfrak{k}$.*

Proof. Note that $(0,0)$ is the unique fixed point for the R-action on $\mathcal{P}$ and that $(\hat{P}_m)^{-1}(0,0) = (e, \frac{\pi}{4})$ does not belong to Σ_m^+. Furthermore, the restriction $\hat{P}_m|_{\Sigma_m^+}$ is locally diffeomorphic by Remark 6.1 and locally R-equivariant by construction. This implies that every local R-orbit of Σ_m^+ is one-dimensional.

One checks that $\ell_c = \{\,(e,\ uU + aH) \in \Sigma_m^* \ :\ \phi_m(a,u) = \sqrt{1-c}\,\}$, where $\phi_m(a,u) := 2aS(4m^2u^2 - 4a^2)$. Moreover $\partial\phi_m/\partial a \neq 0$ on Σ_m^+, therefore $\ell_c \cap \Sigma_m^+$ is a one-dimensional manifold. By construction, ℓ_c is locally R-invariant, hence local R-orbits of Σ_m^+ are open and closed in $\ell_c \cap \Sigma_m^+$, implying i).

For ii), recall that $\Sigma_m = \Sigma_m^*$ by Remark 4.4. As a consequence ϕ_m vanishes on the boundary of Σ_m. Also note that $\phi_m(a,0) = \sin(2a)$ and that for u fixed and a such that $(e,\ uU + aU) \in \Sigma_m^+$, the map $\mathbb{R}^{\geq 0} \to \mathbb{R}$, $u \to \phi_m(a,u)$ is strictly increasing (*cf.* Lemma 4.1). This implies that $\ell_c \cap \Sigma_m^+$ is the (connected) graph of a function defined on $\mathfrak{a}^+ \setminus \{0\}$ for $c < 0$, while, for $0 \leq c < 1$, it consists of two connected components (which are contained in $\Sigma_m^+ \setminus \{\,(e, uU + aH)\ : 0 < a < \frac{\pi}{2}$ and $\sin(2a) \geq \sqrt{1-c}\,\}$).

For iii) first consider the case of $0 < c < 1$. One needs to show that the two components of $\ell_c \cap \Sigma_m^+$ are mapped to different components of the hyperbola in $\mathcal{P}$ defined by $\{ st = c \}$. For this it is enough to note that this holds for the two limit points given by $\{ (e, aH) \ : \ 0 < a < \frac{\pi}{2} \text{ and } \sin(2a) = \sqrt{1-c} \}$.

The only other non-connected case is $c = 0$, when the two components of $\ell_0 \cap \Sigma_m^+$ have the same limit point $(e, \frac{\pi}{4}H)$. By Remark 6.1 the map $\hat{P}_m$ is a diffeomorphism in a neighborhood (in Σ_m) of this point, thus these two components are mapped to different components of $\{ (s,t) \in \mathcal{P} \ : \ st = 0, \ (s,t) \neq (0,0) \}$, as claimed. The case $c < 0$ is straightforward.

Since all R-orbits in $\mathcal{P}$ have connected isotropy and $\hat{P}_m|_{\Sigma_m^+}$ is a local diffeomorphism, $\hat{P}_m$ is necessarily injective on every local R-orbit of Σ_m. Therefore iii) implies that $\hat{P}_m|_{\Sigma_m^+}$ is injective. Moreover, from i) of Lemma 6.5 it follows that Σ_m^+ is mapped to one of the two connected components of $\mathcal{P} \setminus \mathrm{fix}(\beta)$. By α-β-equivariance of $\hat{P}_m$ this implies that $\Sigma_m^- := \alpha(\Sigma_m^+)$ is injectively mapped to the other connected component. Finally it is easy to check that $\mathfrak{k} \cup \mathrm{fix}(\beta)$ is injectively mapped into $\{ st = 1 \} \cup \mathrm{fix}(\beta)$, implying iv).

Now note that a local R-orbit of Σ_m either meets $\mathrm{fix}(\alpha)$ in a unique point or is contained in $\Sigma_m^+ \cup \Sigma_m^-$. As noticed, $\hat{P}_m$ maps different elements of $\mathrm{fix}(\alpha)$ into different R-orbits of $\mathcal{P}$. Then iii) and α-β-equivariance of $\hat{P}_m$, imply v).

For vi), one can move towards infinity (topologically) along local R-orbits of Σ_m which are closed to $\{e\} \times \mathfrak{k}$, apply $\hat{P}_m$ and check that one is moving towards infinity along R-orbits in $\mathcal{P}$. Recalling that $e^{2u(1+m)}\,(C(x) - 2umS(x))$ is the second component of $\hat{P}_m$, this follows by showing that for $\varepsilon > 0$ small enough and $uU + aH \in \Sigma_m$, with $a < \frac{\pi}{4}$, such that $2aS(x) = \varepsilon$, one has

$$\lim_{u \to \infty} e^{2u(1+m)}\,(C(x) - 2umS(x)) = \infty \,.$$

The details of this computation are omitted. □

Theorem 7.2. *Let $G = SL_2(\mathbb{R})$ endowed with a metric ν_m, with $m \leq -1$. Then the polar map $P_m|_{\Omega_m} : \Omega_m \to G^{\mathbb{C}}$ is injective and consequently Ω_m is L-equivariantly biholomorphic to $P_m(\Omega_m)$. This domain is not holomorphically convex and its envelope of holomorphy is biholomorphic to $G^{\mathbb{C}}$.*

Proof. Injectivity follows from iv) of Lemma 7.1 and Proposition 6.4. Assume by contradiction that Ω_m is holomorphically convex, *i.e.* that the domain $P_m(\Omega_m)$ is Stein. Then the categorical quotient $P_m(\Omega_m)//K$ with respect to the K-action is Stein (see [16, Section 6.5]). Note that all local $K^{\mathbb{C}}$-orbits are closed in $P_m(\Omega_m)$ and by v) of Lemma 7.1 the domain $P_m(\Omega_m)$ is K-orbit-convex in $G^{\mathbb{C}}$. It follows that $P_m(\Omega_m)//K$ is biholomorphic to $\Pi(P_m(\Omega_m))$, where $\Pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$ is the canonical projection. In particular $\Pi(P_m(\Omega_m))$ is Stein.

Denote by "$\hat{P}_m(\Sigma_m)/L$" the image of $\hat{P}_m(\Sigma_m)$ in $\mathcal{P}/L$ via the canonical projection. Consider the commutative diagram

$$\begin{array}{ccc} P_m(\Omega_m) \subset G^{\mathbb{C}} & \stackrel{F}{\longrightarrow} & \hat{P}_m(\Sigma_m) \subset \mathcal{P} \\ \Pi \downarrow & & \downarrow \\ \Pi(P_m(\Omega_m)) \subset G^{\mathbb{C}}/K^{\mathbb{C}} & \longrightarrow & G\backslash\Pi(P_m(\Omega_m)) \cong \text{``}\hat{P}_m(\Sigma_m)/L\text{''} \subset \mathcal{P}/L, \end{array}$$

where $F : G^{\mathbb{C}} \to \mathcal{P}$ is our usual quotient map. One checks (*cf.* the proof of Lemma 7.1) that $\hat{P}_m(\Sigma_m)$ intersects all local R-orbits of $\mathcal{P}$ but one, namely $R \cdot (-1, -1)$. Moreover, for $\tilde{e} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, one has $F(\tilde{e}) = (-1, -1)$. As a consequence $\Pi(P_m(\Omega_m)) = G^{\mathbb{C}}/K^{\mathbb{C}} \setminus G\tilde{e}K^{\mathbb{C}}$ which, by Lemma 5.5, is not holomorphically convex. This gives a contradiction, showing that Ω_m is not holomorphically convex.

For the last statement, identify $P_m(\Omega_m)$ with Ω_m and note that its envelope of holomorphy $\hat{\Omega}_m$ is a Stein, L-equivariant, Riemann domain over $G^{\mathbb{C}}$. One has an induced Stein, G-equivariant, Riemann domain $q : \hat{\Omega}_m//K \to G^{\mathbb{C}}/K^{\mathbb{C}}$, where $\hat{\Omega}_m//K$ denotes the K-categorical quotient of $\hat{\Omega}_m$ (see [11, Section 3]). In fact, this can be regarded as a $G/\{\pm e\}$-equivariant, Riemann domain over $G^{\mathbb{C}}/K^{\mathbb{C}}$, since the subgroup $\{\pm(e, e)\}$ of L acts trivially on Ω_m. Then, by [11, Theorem 7.6] the map q is injective and [11, Corollary 3.3] implies that $\hat{\Omega}_m$ is univalent. That is, $\hat{\Omega}_m$ is a Stein, L-invariant domain of $G^{\mathbb{C}}$. As a consequence $\hat{\Omega}_m//K$ is biholomorphic to $\Pi(\hat{\Omega}_m)$ and, by Lemma 5.5, it necessarily coincides with $G^{\mathbb{C}}/K^{\mathbb{C}}$.

Choose, on a neighborhood U of $eK^{\mathbb{C}}$ in $G^{\mathbb{C}}/K^{\mathbb{C}}$, a Stein local trivialization $U \times \mathbb{C}^* \subset G^{\mathbb{C}}$ of the principal $\mathbb{C}^*$-bundle $\Pi : G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$. Then one has $\hat{\Omega}_m \cap (U \times \mathbb{C}^*) = \{(z, \lambda) \in U \times \mathbb{C}^* \ : \ a(z) < |\lambda| < b(z)\}$, with $a < b$ functions on U with values in the extended real line. Since $\hat{\Omega}_m \cap (U \times \mathbb{C}^*)$ is Stein, the functions $\log a$ and $-\log b$ are plurisubharmonic. Moreover, vi) of Lemma 7.1 implies that $\log a(z) = -\log b(z) = -\infty$ for z close to $eK^{\mathbb{C}}$. Thus $a \equiv 0$ and $b \equiv \infty$ on U. Finally a connectedness argument using local trivializations covering all $G^{\mathbb{C}}$ shows that this holds for any trivialization. Hence $\hat{\Omega}_m = G^{\mathbb{C}}$ as claimed. □

8. The case $m > -1$

Let $G = SL_2(\mathbb{R})$ be endowed with one of the metrics ν_m. For $-1 < m \leq 0$, similarly to the cases considered in Section 7, we show that the polar map P_m is a biholomorphism from the maximal adapted complexification Ω_m onto an L-invariant domain of $G^{\mathbb{C}}$ which is not holomorphically convex. However, note that here the projection $\Pi(P_m(\Omega_m))$ of Ω_m to $G^{\mathbb{C}}/K^{\mathbb{C}}$ misses more than one G-orbit.

Finally, in the Riemannian cases $m > 0$ the polar map P_m is not injective and its fibers consists of at most two elements. The maximal complexification Ω_m

is neither holomorphically separable, nor holomorphically convex. For all m, its envelope of holomorphy is shown to be biholomorphic to $G^{\mathbb{C}}$.

Lemma 8.1. (*cf.* Figure 6.1) *Let* $m > -1$. *Then*

i) *different local* R*-orbits of* $\Sigma_m \cap \{u > 0\}$ *are mapped by* $\hat{P}_m$ *to different* R*-orbits of* $\mathcal{P}$,
ii) *local* R*-orbits closed to* $\{e\} \times \mathfrak{k}$ *are mapped bijectively by* $\hat{P}_m$ *to* R*-orbits of* $\mathcal{P}$, *i.e.* R *acts globally in an* R*-invariant neighborhood of* $\{e\} \times \mathfrak{k}$,
iii) *for* $-1 < m \leq 0$ *one has* $(\hat{P}_m|_{\Sigma_m})^{-1}(\mathrm{fix}\beta) \subset \mathrm{fix}(\alpha)$ *and different local* R*-orbits of* Σ_m *are mapped by* $\hat{P}_m$ *to different* R*-orbits of* $\mathcal{P}$,
iv) *for* $-1 < m \leq 0$ *the polar map* $\hat{P}_m$ *is injective on* Σ_m,
v) *for* $m > 0$ *one has* $(\hat{P}_m|_{\Sigma_m})^{-1}(\mathrm{fix}(\beta)) \not\subset \mathrm{fix}(\alpha)$. *More precisely* Σ_m *contains the graph of* $\gamma|_{\mathfrak{k}\setminus\{0\}}$, *with* γ *defined as in Lemma* 6.5.

Proof. Recall that by Lemma 4.4 the slice Σ_m is a proper subdomain of Σ_m^*. For $c \leq 1$ let $\ell_c := (\hat{P}_m|_{\Sigma_m^*})^{-1}(\{st = c\})$. A similar argument as in Lemma 7.1 shows that different components of $\ell_c \cap \{(e, uU + aH) \in \Sigma_m^* \ : \ a > 0, \ u > 0\}$ are mapped by $\hat{P}_m$ to different R-orbits of $\mathcal{P}$. One also checks that local R-orbits of $\Sigma_m^+ := \{(e, uU + aH) \in \Sigma_m \ : \ a > 0, \ u > 0\}$ are connected components of $\ell_c \cap \Sigma_m^+$. Then in order to prove i) it is enough to show that for every $c < 1$ and every connected component O of $\ell_c \cap \{(e, uU + aH) \in \Sigma_m^* \ : \ a > 0, \ u > 0\}$ the locally R-invariant set $O \cap \Sigma_m^+$ is connected.

For this recall that (*cf.* Proposition 4.2 and Remark 4.4) the boundary of Σ_m^+ in the set $\{(e, uU + aH) \in \Sigma_m^* \ : \ u > 0, \ a > 0\}$ is given by $y = f(x)$ and that O is a connected component of $4a^2S^2(x) = 1 - c$, where

$$x = 4u^2m^2 - 4a^2, \quad y = 4u^2m^2 + 4a^2m \quad \text{and} \quad f(x) = \frac{C(x)}{2S'(x)}.$$

Since $4a^2 = \frac{y-x}{1+m}$, in the coordinates x, y such equations read as $y - f(x) = 0$ and $y = \frac{(1+m)(1-c)}{S^2(x)} + x$. Thus it is enough to note that the function

$$\frac{(1+m)(1-c)}{S^2(x)} + x - f(x)$$

can be rewritten as

$$\frac{(1+m)(1-c)}{S^2(x)} - \frac{xS(x)}{C(x) - S(x)},$$

hence it is strictly decreasing by Lemma 4.1. This proves i).

The analogous proof as in vi) of 7.1 implies ii).

For $m = 0$ one checks directly that $(\hat{P}_m|_{\Sigma_m^*})^{-1}(\mathrm{fix}(\beta)) = \mathrm{fix}(\alpha) \cup \partial\Sigma_m = \{u = 0\} \cup \{a = \frac{\pi}{4}\}$ (*cf.* Lemma 6.5). Thus iii) holds for $m = 0$. Now let $-1 < m < 0$

and let (e, X) be an element of $(\hat{P}_m|_{\Sigma_m^*})^{-1}(\text{fix}(\beta)) \setminus \text{fix}(\alpha)$. That is, $X = uU + aH$ with $u \neq 0$ and

$$\frac{m}{1+m}\frac{C(4u^2(1+m)^2)}{S(4u^2(1+m)^2)} = \frac{C(x)}{S(x)}, \tag{8.1}$$

where $x = 4u^2m^2 - 4a^2 > -\pi^2$. In order to show that (e, X) does not belong to Σ_m we need to see that $y > f(x)$, i.e that (*cf.* Lemma 4.1)

$$4u^2m^2 + 4a^2m > \frac{xC(x)}{C(x) - S(x)}.$$

Since $-1 < m < 0$, equation (8.1) above implies that $\frac{C(x)}{S(x)} < 0$. Then this inequality can be written as

$$\left(1 - \frac{S(x)}{C(x)}\right)(4u^2m^2 + 4a^2m) > (4u^2m^2 - 4a^2)$$

which becomes

$$4a^2m + 4a^2 > \frac{S(x)}{C(x)}(4u^2m^2 + 4a^2m).$$

Note that $\frac{C(x)}{S(x)} < 0$ also implies $a \neq 0$. Then, by using equation (8.1), one obtains

$$1 > \frac{S(x)}{C(x)}\frac{(4u^2m^2 + 4a^2m)}{4a^2(1+m)} = \frac{S(4u^2(1+m)^2)}{C(4u^2(1+m)^2)}\left(\frac{u^2}{a^2}m + 1\right)$$

which is easily checked to hold, since $0 < \frac{S(4u^2(1+m)^2)}{C(4u^2(1+m)^2)} < 1$ and $(\frac{u^2}{a^2}m + 1) < 1$. Finally, a similar argument as in Lemma 7.1 implies the last statement in iii). Part iv) follows from iii) by using the same argument as in Lemma 7.1

For v) first note that $\Sigma_m \cap \{u = 0\} = \{(e, aH) \ : \ 0 \leq a < \tilde{a}\}$, with $\tilde{a}$ uniquely defined by $0 < \tilde{a} < \pi/2$ and $\tan(2\tilde{a}) = 2\tilde{a}\frac{1+m}{m}$. Rewrite condition $**$) of Proposition 4.2 as $x + (1+m)4a^2 < \frac{xC(x)}{C(x)-S(x)}$. That is

$$(1+m)4a^2 < \frac{S(x)}{2S'(x)}.$$

Recall that $\frac{S(x)}{S'(x)}$ is strictly increasing for $x > -\pi^2$ (Lemma 4.1). Then for $(e, aH) \in \Sigma_m$ and $u \in \mathbb{R}$ one has

$$(1+m)4a^2 < \frac{S(-4a^2)}{2S'(-4a^2)} \leq \frac{S(x)}{2S'(x)}$$

implying that $\{(e, uU + aH) \ : \ u \in \mathbb{R},\ 0 \leq a < \tilde{a}\} \subset \Sigma_m$. Since one sees that $(e, \tilde{a}H) = \gamma(0)$, in order to prove that the graph of $\gamma|_{\mathfrak{k}\setminus 0}$ is contained in Σ_m it is enough to check that γ is strictly decreasing (increasing) for $u > 0$ ($u < 0$). This can be done by a direct computation showing that

$$\left.\frac{\partial \Gamma}{\partial a}\right|_{\{\Gamma=0\}} < 0 \text{ and } \left.\frac{\partial \Gamma}{\partial u}\right|_{\{\Gamma=0\}} < 0 \qquad \left(\left.\frac{\partial \Gamma}{\partial a}\right|_{\{\Gamma=0\}} < 0 \text{ and } \left.\frac{\partial \Gamma}{\partial u}\right|_{\{\Gamma=0\}} > 0\right). \ \square$$

Corollary 8.2. *Let $m > -1$. Then the restrictions $P_m|_{L\cdot\{\Sigma_m \cap \{u \geq 0\}\}}$ and $P_m|_{L\cdot\{\Sigma_m \cap \{u \leq 0\}\}}$ are injective.*

Proof. First consider the case $u \geq 0$. Since $\hat{P}_m$ is necessarily injective on local R-orbits, i) of Lemma 8.1 implies that $\hat{P}_m$ is injective on $\Sigma_m^+ = \Sigma_m \cap \{u > 0\}$. Moreover, a straightforward computation shows that $\hat{P}_m$ is also injective on $\Sigma_m \cap \{u = 0\}$.

Then, for $-1 < m \leq 0$, the map $\hat{P}_m$ is injective on $\Sigma_m \cap \{u \geq 0\}$ by iii) of Lemma 8.1. Now recall that every L-orbit of Ω_m meets Σ_m in a single element and P_m is injective on such an orbit by Lemma 6.2. Then the statement follows in the case $-1 < m \leq 0$.

For $m > 0$ note that by v) of Lemma 8.1 the set $(\hat{P}_m|_{\Sigma_m \cap \{u \geq 0\}})^{-1}(\text{fix}(\beta))$ consists of a one-dimensional manifold with two connected components. Then in these cases one can essentially argue as follows.

Assume by contradiction that $\hat{P}_m(e, X) = \hat{P}_m(e, Y)$ for some (e, X) in $\{u = 0\}$ and (e, Y) in graph$(\gamma|_{\mathfrak{k}\setminus 0})$. Since $(e, X) \in \partial\Sigma_m^+$, $(e, Y) \in \Sigma_m^+$ are both in Σ_m, where $\hat{P}_m$ is a local diffeomorphism, this would imply that $\hat{P}_m|_{\Sigma_m^+}$ is not injective, which gives a contradiction.

Finally the case $u \leq 0$ follows from α-β-equivariance of $\hat{P}_m$. □

Let Ω be a complex S^1-manifold. We need the following remark on limits of local orbits for the induced local $\mathbb{C}^*$-action. For details on induced local actions in this particular situation we refer to [17].

Proposition 8.3. *Let Ω be a complex S^1-manifold and consider the induced local $\mathbb{C}^*$-action. Assume that there exist a sequence $\{x_n\}$ of Ω and an element $X \in Lie(S^1)$ such that $\exp(iX)$ acts on every x_n and the sequences $\{x_n\}$, $\{\exp(iX) \cdot x_n\}$ converge to different local $\mathbb{C}^*$-orbits. Then Ω admits no continuous plurisubharmonic exhaustion. In particular it is not holomorphically convex.*

Proof. Assume by contradiction that a continuous, plurisubharmonic exhaustion φ of Ω exists. After integration over S^1, this function can be assumed to be S^1-invariant. Let x and y be the limit points of x_n and of $\exp(iX) \cdot x_n$. Denote by O_x and O_y the local $\mathbb{C}^*$-orbits through x and y, respectively, and choose $M \in \mathbb{R}$ such that $\varphi(x) < M$ and $\varphi(y) < M$. By assumption $O_x \cap O_y = \emptyset$, therefore O_x is given by $\{\exp(\lambda X) \cdot x \ : \ a < \text{Im}\lambda < b\}$, with $-\infty \leq a < 0$ and $0 < b < 1$. Then $\exp(itX) \cdot x \to \infty$ as $t \to b$, in the sense of leaving all compact subsets of X. Thus there exists a real $\tilde{b}$ with $0 < \tilde{b} < b$ such that $\varphi(\exp(i\tilde{b}X) \cdot x) > M$. Furthermore $\exp(i\tilde{b}X) \cdot x_n \to \exp(i\tilde{b}X) \cdot x$, thus for n large enough $\varphi(\exp(i\tilde{b}X) \cdot x_n) > M$, while $\varphi(x_n) < M$ and $\varphi(\exp(iX) \cdot x_n) < M$.

However S^1-invariance and plurisubharmonicity of φ imply that the function $t \to \varphi(\exp(itX) \cdot x_n)$ is convex. This gives a contradiction and concludes the proof. □

In fact a similar argument yields the analogous result for actions of compact Lie groups on holomorphically separable complex manifolds. The remaining cases with $m > -1$ are discussed in the following theorem.

Theorem 8.4. *Let $G = SL_2(\mathbb{R})$ endowed with a left invariant metric ν_m, with $m > -1$. The polar map $P_m|_{\Omega_m} : \Omega_m \to G^{\mathbb{C}}$ is*

i) *injective for $-1 < m \leq 0$ and consequently Ω_m is L-equivariantly biholomorphic to $P_m(\Omega_m)$. Such domain is not holomorphically convex,*

ii) *not injective for $m > 0$ and its fibers have at most two elements. The maximal complexification Ω_m is a Riemann domain over $G^{\mathbb{C}}$ which is neither holomorphically separable, nor holomorphically convex.*

In both cases the envelope of holomorphy of Ω_m is biholomorphic to $G^{\mathbb{C}}$.

Proof. From iv) of Lemma 8.1 and Proposition 6.4 it follows that $P_m|_{\Omega_m}$ is injective for $-1 < m \leq 0$. Assume by contradiction that $P_m(\Omega_m)$ is Stein. Then the analogous argument as in Theorem 7.2 shows that $\Pi(P_m(\Omega_m))$ is Stein. However one checks that $[0, \frac{\pi}{4}]H \in \Sigma_m$ (*cf.* proof of Lemma 8.1). Then, by G-invariance of Ω_m and G-equivariance of P_m, one has $G\exp(i[0, \frac{\pi}{4}]H)K^{\mathbb{C}} \subset \Pi(P_m(\Omega_m))$. As a consequence the closure of the Akhiezer-Gindikin domain Ω_{AG} is contained in $\Pi(P_m(\Omega_m))$. Since one sees that $\Pi(P_m(\Omega_m)) \neq G^{\mathbb{C}}/K^{\mathbb{C}}$ (*cf.* proof of Lemma 8.1), Lemma 5.5 implies that $\Pi(P_m(\Omega_m))$ is not Stein. This gives a contradiction, implying i).

For ii) note that by v) of Lemma 8.1 there exists an element (e, X) in $\Sigma_m \cap (\hat{P}_m)^{-1}(\text{fix}\beta)$ with $X_{\mathfrak{k}} \neq 0$. Then by α-β-equivariance of $\hat{P}_m$ one has $\hat{P}_m(e, X_{\mathfrak{k}} + X_{\mathfrak{a}}) = \hat{P}_m(e, -X_{\mathfrak{k}} + X_{\mathfrak{a}})$, showing that $\hat{P}_m|_{\Sigma_m}$ is not injective. Then, by Proposition 6.4 the polar map $P_m|_{\Omega_m}$ is not injective as well.

Since $\Sigma_m = (\Sigma_m \cap \{u \geq 0\}) \cup (\Sigma_m \cap \{u \leq 0\})$, from Corollary 8.2 and Proposition 6.4 it follows that the fibers of P_m consist either of one point or two points p^+ and p^-. In the second case, necessarily $p^+ \in L \cdot \Sigma_m^+$ and $p^- \in L \cdot \Sigma_m^-$, where $\Sigma_m^- = \alpha(\Sigma_m^+) = \{(e, uU + aH) : u < 0,\ a > 0\}$.

Now let $(e, [0, \tilde{a})H) = \Sigma_m \cap \mathfrak{a}^+$ and note that local $\mathbb{C}^*$-orbits through (e, tH) accumulate, for $t \to \tilde{a}$, to different local $\mathbb{C}^*$-orbits. Indeed this holds true for their images in the quotient $\Sigma_m \cong \Omega/L$. Such images are given by the R-orbits through (e, tH) for which, chosen ε small enough, $(\{e\} \times \exp(i\varepsilon U)) \cdot (e, tH)$ and $(\{e\} \times \exp(-i\varepsilon U)) \cdot (e, tH)$ accumulate to different local R-orbits. Namely the two connected components of $\ell_{\tilde{c}} \cap \Sigma_m$, with $\tilde{c}$ such that $\hat{P}_m(e, \tilde{a}H) \in \{st = \tilde{c}\}$ (*cf.* the proof of Lemma 8.1). Then Proposition 8.3 implies that Ω_m is not holomorphically convex.

Assume by contradiction that Ω_m is holomorphically separable. Then it embeds in its envelope of holomorphy $\hat{\Omega}_m$, which is a non-univalent, Stein, L-equivariant, Riemann domain over $G^{\mathbb{C}}$. By Corollary 3.3 in [11], the induced Stein, G-equivariant, Riemann domain $q : \hat{\Omega}_m//K \to G^{\mathbb{C}}/K^C$ is also non-univalent.

However the subgroup $\{\pm(e, e)\}$ of L acts trivially on $\hat{\Omega}_m$, thus $\hat{\Omega}_m//K$ can be regarded as a $G/\{\pm e\}$-equivariant, Riemann domain. Then [11, Theorem 7.6] implies that q is injective, giving a contradiction.

Finally, in view of ii) of Lemma 8.1, the last statement follows from the same argument as in Theorem 7.2. $\square$

9. Appendix

Here we carry out the computation used in Section 4. For this note that $[U, H] = 2W$, $[U, W] = -2H$ and $[H, W] = -2U$. Thus

$$\begin{aligned}
\operatorname{ad}(i(umU-aH))U &= [i(umU - aH), U] = 2aiW,\\
\operatorname{ad}^2(i(umU-aH))U &= [i(umU - aH), 2aiW] = -4a(aU-umH),\\
\operatorname{ad}^3(i(umU-aH))U &= [i(umU-aH), -4a(aU-umH)] = 8a(u^2m^2 - a^2)iW,\\
\operatorname{ad}^4(i(umU-aH))U &= [i(umU - aH), 8a(u^2m^2 - a^2)iW]\\
&= -16a(u^2m^2 - a^2)(aU - umH).
\end{aligned}$$

Then, by recalling that $\operatorname{Ad}_{\exp} = e^{\operatorname{ad}}$, one has

$$\begin{aligned}
\operatorname{Ad}_{\exp i(umU-aH)}U &= e^{\operatorname{ad}(i(umU-aH))}U = U + 2aiW - \frac{4a}{2!}(aU - umH)\\
&\quad + \frac{8a(u^2m^2-a^2)}{3!}iW - \frac{16a(u^2m^2-a^2)}{4!}(aU-umH) + \dots\\
&= U - 4a\frac{\cosh\sqrt{x} - 1}{x}(aU - umH) + 2a\frac{\sinh\sqrt{x}}{\sqrt{x}}iW\\
&= \left(1 - 4a^2\frac{C(x) - 1}{x}\right)U + 4aum\frac{C(x) - 1}{x}H + 2aS(x)iW,
\end{aligned}$$

where $x = 4u^2m^2 - 4a^2$. Similarly, for the second vector one has

$$\begin{aligned}
\operatorname{ad}(i(umU - aH))H &= [i(umU - aH), H] = 2umiW,\\
\operatorname{ad}^2(i(umU - aH))H &= [i(umU - aH), 2umiW] = -4um(aU - umH),\\
\operatorname{ad}^3(i(umU - aH))H &= [i(umU - aH), -4um(aU - umH)]\\
&= 8um(u^2m^2 - a^2)iW,\\
\operatorname{ad}^4(i(umU - aH))H &= [i(umU - aH), 8um(u^2m^2 - a^2)iW]\\
&= -16um(u^2m^2 - a^2)(aU - umH).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Ad}_{\exp i(umU-aH)} H &= e^{\mathrm{ad}(i(umU-aH))} H \\
&= H + 2umiW - \frac{4um}{2!}(aU - umH) + \frac{8um(u^2m^2 - a^2)}{3!} iW \\
&\quad - \frac{16um(u^2m^2 - a^2)}{4!}(aU - umH) + \dots \\
&= H - 4um\frac{\cosh\sqrt{x} - 1}{x}(aU - umH) + 2um\frac{\sinh\sqrt{x}}{\sqrt{x}} iW \\
&= -4aum\frac{C(x) - 1}{x}U + \left(1 + 4u^2m^2\frac{C(x) - 1}{x}\right)H \\
&\quad + 2umS(x)iW.
\end{aligned}$$

For the third vector one has

$$\mathrm{ad}(i(umU - aH))W = [i(umU - aH), W] = 2i(aU - umH),$$

$$\mathrm{ad}^2(i(umU - aH))W = [i(umU - aH), 2i(aU - umH)] = 4(u^2m^2 - a^2)W,$$

$$\begin{aligned}
\mathrm{ad}^3(i(umU - aH))W &= [i(umU - aH), 4(u^2m^2 - a^2)W] \\
&= 8(u^2m^2 - a^2)i(aU - umH),
\end{aligned}$$

$$\begin{aligned}
\mathrm{ad}^4(i(umU-aH))W &= [i(umU-aH), 8(u^2m^2-a^2)i(aU - umH)] \\
&= 16(u^2m^2 - a^2)^2 W.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Ad}_{\exp i(umU-aH)} W &= e^{\mathrm{ad}(i(umU-aH))} W \\
&= W + 2i(aU - umH) + \frac{4(u^2m^2 - a^2)}{2!} W \\
&\quad + \frac{8(u^2m^2 - a^2)}{3!} i(aU - umH) + \frac{16(u^2m^2 - a^2)^2}{4!} W + \dots \\
&= 2\frac{\sinh\sqrt{x}}{\sqrt{x}} i(aU - 2umH) + \cosh\sqrt{x}\, W \\
&= 2aS(x)iU \; - \; 2umS(x)iH \; + \; C(x)W.
\end{aligned}$$

For the fourth vector one has

$$\begin{aligned}
\mathrm{ad}(-i(umU-aH))(-imU) &= [-i(umU-aH),-imU]\\
&= -2amW,\\
\mathrm{ad}^2(-i(umU-aH))(-imU) &= [-i(umU-aH),-2amW]\\
&= 4ami(aU-umH),\\
\mathrm{ad}^3(-i(umU-aH))(-imU) &= [-i(umU-aH),4ami(aU-umH)]\\
&= -8am(u^2m^2-a^2)W,\\
\mathrm{ad}^4(-i(umU-aH))(-imU) &= [-i(umU-aH),-8am(u^2m^2-a^2)W]\\
&= 16am(u^2m^2-a^2)i(aU-umH).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{l=0}^{\infty}\frac{(-1)^l}{(l+1)!}\mathrm{ad}^l(i(-umU+aH))(-imU)+i(1+m)U\\
&= iU+\frac{2am}{2!}W+\frac{4am}{3!}i(aU-umH)\\
&\quad+\frac{8am(u^2m^2-a^2)}{4!}W+\frac{16am(u^2m^2-a^2)}{5!}i(aU-umH)+\dots\\
&= iU+4am\frac{\sinh\sqrt{x}/\sqrt{x}-1}{x}i(aU-umH)+2am\frac{\cosh\sqrt{x}-1}{x}W\\
&= \left(1+4a^2m\frac{S(x)-1}{x}\right)iU \;-\; 4aum^2\frac{S(x)-1}{x}iH\\
&\quad+2am\frac{C(x)-1}{x}W.
\end{aligned}$$

For the fifth vector one has

$$\begin{aligned}
\mathrm{ad}(-i(umU-aH))iH &= [-i(umU-aH),iH]=2umW,\\
\mathrm{ad}^2(-i(umU-aH))iH &= [-i(umU-aH),2umW]\\
&= -4umi(aU-umH),\\
\mathrm{ad}^3(-i(umU-aH))iH &= [-i(umU-aH),-4umi(aU-umH)]\\
&= 8um(u^2m^2-a^2)W,\\
\mathrm{ad}^4(-i(umU-aH))iH &= [-i(umU-aH),8um(u^2m^2-a^2)W]\\
&= -16um(u^2m^2-a^2)i(aU-umH).
\end{aligned}$$

Therefore

$$\sum_{l=0}^{\infty} \frac{(-1)^l}{(l+1)!} \mathrm{ad}^l(i(-umU + aH))(iH)$$
$$= iH - \frac{2um}{2!}W - \frac{4um}{3!}i(aU - umH)$$
$$- \frac{8um(u^2m^2 - a^2)}{4!}W - \frac{16um(u^2m^2 - a^2)}{5!}i(aU - umH) - \dots$$
$$= iH - 4um\frac{\sinh\sqrt{x}/\sqrt{x} - 1}{x}i(umH - aU) - 2um\frac{\cosh\sqrt{x} - 1}{x}W$$
$$= -4aum\frac{S(x) - 1}{x}iU + \left(1 + 4u^2m^2\frac{S(x) - 1}{x}\right)iH - 2um\frac{C(x) - 1}{x}W,$$

For the sixth vector one has

$$\mathrm{ad}(-i(umU - aH))iW = [-i(umU - aH), iW] = 2(aU - umH),$$
$$\mathrm{ad}^2(-i(umU - aH))iW = [-i(umU - aH), 2(aU - umH)]$$
$$= 4(u^2m^2 - a^2)iW,$$
$$\mathrm{ad}^3(-i(umU - aH))iW = [-i(umU - aH), 4(u^2m^2 - a^2)iW]$$
$$= 8(u^2m^2 - a^2)(aU - umH),$$
$$\mathrm{ad}^4(-i(umU - aH))iW = [-i(umU - aH), 8(u^2m^2 - a^2)(aU - umH)]$$
$$= 16(u^2m^2 - a^2)^2 iW.$$

Therefore

$$\sum_{l=0}^{\infty} \frac{(-1)^l}{(l+1)!} \mathrm{ad}^l(i(-umU + aH))(iW)$$
$$= iW - \frac{2}{2!}(aU - umH) + \frac{4(u^2m^2 - a^2)}{3!}iW$$
$$- \frac{8(u^2m^2 - a^2)}{4!}(aU - umH) + \frac{16(u^2m^2 - a^2)^2}{5!}iW - \dots$$
$$= -2\frac{\cosh\sqrt{x} - 1}{x}(aU - umH) + \frac{\sinh\sqrt{x}}{\sqrt{x}}iW$$
$$= -2a\frac{C(x) - 1}{x}U \;+\; 2um\frac{C(x) - 1}{x}H \;+\; S(x)iW.$$

References

[1] R. AGUILAR, *Symplectic reduction and the homogeneous complex Monge-Ampère equation*, Ann. Global Anal. Geom. **19** (2001), 327–353.

[2] D. N. AKHIEZER and S. G. GINDIKIN, *On Stein extensions of real symmetric spaces*, Math. Ann. **286** (1990), 1–12.

[3] R. BIELAWSKI, *Complexification and hypercomplexification of manifolds with a linear connection*, Internat. J. Math. **14** (2003), 813–824.

[4] R. BREMIGAN, *Pseudokähler forms on complex Lie groups*, Doc. Math. **5** (2000), 595–611.

[5] D. BURNS, *On the uniqueness and characterization of Grauert tubes*, In: "Complex Analysis and Geometry", V. Ancona and A. Silva (eds.), Lecture Notes in Pure and Applied Math., Vol. 173, 1995, 119–133.

[6] D. BURNS, S. HALVERSCHEID and R. HIND, *The geometry of grauert tubes and complexification of symmetric spaces*, Duke Math. J. **118** (2003), 465–491.

[7] J. BERNDT, F. TRICERRI and L. VANHECKE, "Generalized Heisenberg Groups and Damek-Ricci Harmonic Spaces", Lecture Notes Math., Vol. 1598, Springer-Verlag, Berlin 1995.

[8] J. E. D'ATRI and W. ZILLER, "Naturally Reductive Metrics and Einstein Metrics on Compact Lie Groups", Mem. Amer. Math. Soc., Vol. 215, 1979.

[9] O. FELS, *Pseudo-Kählerian structure on domains over a complex semisimple Lie group*, Math. Ann. **232** (2002), 1–29.

[10] G. FELS, A. T. HUCKLEBERRY and J. A. WOLF, "Cycle Spaces of Flag Domains: A Complex Geometric Viewpoint", Progress in Mathematics, Vol. 245, Birkhäuser, Boston, 2005.

[11] L. GEATTI and A. IANNUZZI, *On univalence of equivariant Riemann domains over the complexification of a non-compact, Riemannian symmetric space*, Pacific J. Math. **238** (2008), 275–330.

[12] C. S. GORDON, *Naturally reductive homogeneous Riemannian manifolds*, Canad. J. Math. **37** (1985), 467–487.

[13] V. GUILLEMIN and M. STENZEL, *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Differential Geom. **34** (1991), 561–570 (first part); J. Differential Geom. **35** (1992), 627–641 (second part).

[14] S. HALVERSCHEID and A. IANNUZZI, *Maximal complexifications of certain Riemannian homogeneous spaces*, Trans. Amer. Math. Soc. **355** (2003), 4581–4594.

[15] S. HALVERSCHEID and A. IANNUZZI, *On naturally reductive left-invariant metrics of* $SL_2(\mathbb{R})$. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **5** (2006), 171–187.

[16] P. HEINZNER, *Geometric invariant theory on Stein spaces.* Math. Ann. **289** (1991), 631–662.

[17] P. HEINZNER and A. IANNUZZI, *Integration of local actions on holomorphic fiber spaces*, Nagoya Math. J. **146** (1997), 31–53

[18] S. HELGASON, "Differential Geometry, Lie Groups and Symmetric Spaces", Graduate Studies in Mathematics, Vol. 34, Amer. Math. Soc., Providence R.I., 2001.

[19] L. LEMPERT and R. SZŐKE, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundles of Riemannian manifolds*, Math. Ann. **290** (1991), 689–712.

[20] B. O'NEILL, "Semi-Riemannian Geometry", Academic Press, New York, 1983.

[21] G. PATRIZIO and P.-M. WONG, *Stein manifolds with compact symmetric center*, Math. Ann. **289** (1991), 355–382.

[22] H. ROSSI, *On envelopes of holomorphy*, Commun. Pure Appl. Math. **16** (1963), 9–17.

[23] W. STOLL, *The characterization of strictly parabolic manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), 87–154.

[24] R. SZŐKE, *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. **291** (1991), 409–428.

[25] R. SZŐKE, *Adapted complex structures and Riemannian homogeneous spaces*, In: "Complex Analysis and Applications" (Warsaw, 1997) Ann. Polon. Math. 70 (1998), 215–220.
[26] R. SZŐKE, *Canonical complex structures associated to connections*, Math. Ann. **329** (2004), 553-591.
[27] V. S. VARADARAJAN, "Lie Groups, Lie Algebras, and their Representations", Springer-Verlag, New York, 1984.
[28] P. ZHAO, "Invariant Stein Domains: A Contribution to the Program of Gelfand and Gindikin", PhD Thesis, Berichte aus der Mathematik, Shaker Verlag, Aachen 1996.

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