Periodic solutions of forced Kirchhoff equations

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Abstract. We consider the Kirchhoff equation for a vibrating body, in any dimension, in the presence of a time-periodic external forcing with period $2\pi/\omega$ and amplitude ε. We treat both Dirichlet and space-periodic boundary conditions, and both analytic and Sobolev regularity.

We prove the existence, regularity and local uniqueness of time-periodic solutions, using a Nash-Moser iteration scheme. The results hold for parameters (ω, ε) in a Cantor set with asymptotically full measure as $\varepsilon \to 0$.

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1. Introduction

We consider the Kirchhoff equation

$$u_{tt} - \Delta u \left(1 + \int_\Omega |\nabla u|^2 \, dx \right) = \varepsilon g(x, t), \qquad x \in \Omega, \ t \in \mathbb{R}, \qquad (1.1)$$

where g is a time-periodic external forcing with period $2\pi/\omega$, ε is an amplitude parameter, and the displacement $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is the unknown. We consider both Dirichlet boundary conditions

$$u(x, t) = 0 \quad \forall x \in \partial\Omega, \ t \in \mathbb{R}, \qquad (1.2)$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded, connected open set with smooth boundary, and periodic boundary conditions

$$u(x, t) = u(x + 2\pi m, t) \quad \forall m \in \mathbb{Z}^d, \ x \in \mathbb{R}^d, \ t \in \mathbb{R}, \qquad (1.3)$$

where $\Omega = (0, 2\pi)^d$.

Equation (1.1) is a quasi-linear integro-partial differential equation having the structure of an infinite-dimensional Hamiltonian system, with time-dependent Hamiltonian

$$H(u, v) = \int_\Omega \frac{v^2}{2} \, dx + \int_\Omega \frac{|\nabla u|^2}{2} \, dx + \left(\int_\Omega \frac{|\nabla u|^2}{2} \, dx \right)^2 - \int_\Omega \varepsilon g u \, dx.$$

It describes the nonlinear forced vibrations of a d-dimensional body (in particular, a string for $d = 1$ and a membrane for $d = 2$).

This model was first proposed in 1876 by Kirchhoff [23], in dimension one, without the forcing term and with Dirichlet boundary conditions,

$$u_{tt} - u_{xx} \left(1 + \int_0^\pi u_x^2 \, dx \right) = 0, \quad u(0, t) = u(\pi, t) = 0, \tag{1.4}$$

to describe transversal free vibrations of a clamped string in which the dependence of the tension on the deformation cannot be neglected. Independently, Carrier [15, 16] and Narasimha [31] rediscovered the same equation as a nonlinear approximation of the exact model for the vibrations of a stretched string.

The Cauchy problem, $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$, for Kirchhoff equations has been studied by many authors. Starting from the pioneering paper of Bernstein [6], both local and global existence theories have been developed, for initial data having Sobolev or analytic regularity; see, for example, [3,4,20,25,28,33] and the surveys [2, 37]. We remark that the global existence in Sobolev (or even C^∞) spaces is still an open problem, except for special cases (for example, for $\Omega = \mathbb{R}^d$ [19, 22]).

On the other hand, to the best of our knowledge, nothing is known about the existence of periodic solutions of Kirchhoff equations, except for the normal modes which Kirchhoff found when $\varepsilon = 0$. Thanks to its special structure, the unforced Kirchhoff equation possesses a sequence of normal modes, which are solutions of the form $u(x, t) = u_j(t)\varphi_j(x)$, where $\varphi_j(x)$ is an eigenfunction of the Laplacian on Ω, and $u_j(t)$ is periodic. However, in presence of a forcing term $g(x, t)$ the normal modes do not survive, except in the one-mode case $g(x, t) = g_j(t)\varphi_j(x)$. Indeed, decomposing $u(x, t) = \sum_j u_j(t)\varphi_j(x)$ shows that all components $u_j(t)$ are coupled in the integral term $\int_\Omega |\nabla u|^2 dx$, and problem (1.1) is equivalent to an infinite system of coupled nonlinear ODEs of the form

$$u_j''(t) + \lambda_j^2 u_j(t) \left(1 + \sum_k \lambda_k^2 u_k^2(t) \right) = \varepsilon g_j(t), \quad j = 1, 2, \ldots, \tag{1.5}$$

where $g(x, t) = \sum_j g_j(t)\varphi_j(x)$ and λ_j^2 are the eigenvalues of the Laplacian on Ω.

In this paper we prove the existence of periodic solutions of (1.1). We consider the amplitude ε and the frequency ω of the forcing term g as parameters of the problem. We prove that there exist periodic solutions with amplitude of order ε and

period $2\pi/\omega$, when ε is small and (ε, ω) belong to a Cantor set which is large in a Lebesgue measure sense, namely it has positive, asymptotically full measure as $\varepsilon \to 0$. We prove regularity estimates for the solutions, both in Sobolev spaces and in spaces of analytic functions. We also prove the local uniqueness.

The solutions found here are possibly the first examples of global solutions of the forced Kirchhoff equation (1.1) with boundary conditions (1.2) or (1.3), when the forcing term has only Sobolev regularity.

We proceed as follows. After normalising the time $t \to \omega t$ and rescaling the amplitude $u \to \varepsilon^{1/3} u$, equation (1.1) becomes

$$\omega^2 u_{tt} - \Delta u = \mu \left(\Delta u \int_\Omega |\nabla u|^2 \, dx + g(x, t) \right), \tag{1.6}$$

where $\mu := \varepsilon^{2/3}$ and g, u are 2π-periodic.

The main difficulty in finding periodic solutions of (1.6) is the so-called "small divisors problem", caused by resonances between the forcing frequency ω with its superharmonics and the eigenvalues λ_j^2 of the Laplacian on Ω. The spectrum of the d'Alembert operator $\omega^2 \partial_{tt} - \Delta$ is

$$\{-\omega^2 l^2 + \lambda_j^2 \; : \; l \in \mathbb{N}, \; j = 1, 2, \ldots\}.$$

If $\omega l \neq \lambda_j$ for all l, j, that is for almost every ω, the spectrum does not contain zero and the d'Alembert operator is invertible. However, for almost every ω the quantities $|-\omega^2 l^2 + \lambda_j^2|$ accumulate to zero. As a consequence, the inverse operator $(\omega^2 \partial_{tt} - \Delta)^{-1}$, whose spectrum is

$$\left\{ \frac{1}{-\omega^2 l^2 + \lambda_j^2} \; : \; l \in \mathbb{N}, \; j = 1, 2, \ldots \right\},$$

is in general an unbounded operator, which does not map a function space into itself, but only into a larger space of less regular functions. This makes it impossible to apply the standard implicit function theorem.

Our proof overcomes this difficulty by using a Nash-Moser method, which is a modified Newton iteration method. At each step of the iteration we impose a "non-resonance condition" on the parameters to control the small divisors. For non-resonant frequencies we invert the linearised operator, which is a perturbation of the d'Alembertian, losing some regularity in the process. In this way we construct inductively a sequence of approximate solutions. The loss of regularity, which occurs at each step of the iteration, is compensated for by smoothing operators and by the high speed of convergence of the scheme. The process converges to a solution of the problem for those values of parameters that remain after infinitely many exclusions. The remaining set, by construction, has the structure of a Cantor set.

The application of Nash-Moser methods to infinite-dimensional dynamical systems having small divisors problems was introduced in the nineties by Craig,

Wayne and Bourgain, in analytic and Gevray classes [13, 14, 17, 18]. Further developments are to be found, for example, in [5, 7, 8, 32]. This technique, combined with the Lyapunov-Schmidt reduction, is a flexible alternative to KAM procedures [24, 34, 38]. In particular, KAM methods currently available seem not to apply to the quasi-linear problem (1.1).

Because of the presence in (1.1) of the integral on the space domain, the Kirchhoff nonlinearity $\Delta u \int_\Omega |\nabla u|^2 \, dx$ is diagonal with respect to the spatial basis $\{\varphi_j(x)\}$. This very special structure plays a fundamental role in the inversion of the linearised operator, and makes it possible to solve a d-dimensional quasi-linear problem such as (1.1), whereas, in general, the presence of derivatives, especially those of order ≥ 1, in the nonlinearity causes significant difficulties when controlling the small divisors in the study of periodic solutions of wave equations. The problem of periodic solutions for quasi-linear wave equations was studied in dimension one by Rabinowitz [35] in presence of a dissipative term,

$$u_{tt} - u_{xx} + \alpha u_t = \varepsilon f(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}),$$

for the frequency $\omega = 1$; by Craig [17] for pseudo-differential operators

$$u_{tt} - u_{xx} = a(x)u + b(x, |\partial_x|^\beta u) = 0, \quad \beta < 1;$$

by Bourgain [14] for

$$u_{tt} - u_{xx} + \rho u + u_t^2 = 0,$$

and, for quasiperiodic solutions, [11] $u_{tt} - u_{xx} = a(x)u + \varepsilon \partial_x^{1/2}(h(x, u))$.

The analysis of the small divisors and the inversion of the linearised operator becomes more difficult in higher dimension, because of the sublinear growth of the eigenvalues λ_j of the Laplacian; see, for example, Bourgain [12, 13] and, recently, Berti-Bolle [10]. Nonetheless, since problem (1.1) is diagonal in space, nonlinear interactions occur in time, as it is specified by system (1.5), and the inversion of the linearised operator is possible in any dimension.

The Nash-Moser scheme that we use does not rely on analyticity, and in this respect differs from that of Craig, Wayne, Bourgain and [5, 8]. Rather, it goes back directly to ideas of the original methods of [29, 30, 39] as developed in [7]. Recently [9] this technique made it possible to prove the existence of periodic solutions of nonlinear wave equations for nonlinearities having only C^k differentiability. Some of the difficulties of [9] are not present here, thanks to the structure of the Kirchhoff nonlinearity. Note that the roles played here by space and time are inverted with respect to [5, 8, 9].

The paper is organised as follows. In Section 2 we introduce the functional setting and state the results, Theorem 2.1 for Dirichlet, and Theorem 2.5 for periodic, boundary conditions.

Section 3 is devoted to the inversion of the linearised operator (Theorem 3.1), which is the crucial ingredient in the Nash-Moser method. The proof, which is based on a time-periodic Hill spectral analysis, seems to be the main novelty of the present work. The linearised operator, which is diagonal in space, naturally

splits into a main part D and a remainder S. With respect to the eigenfunctions of the corresponding Hill problem (3.2), D turns out to be essentially diagonal also in time. Then we invert D for parameters (ω, μ) sufficiently far from the resonances (3.3) that are created by the periodic spectrum in time and the Laplacian spectrum in space (Lemma 3.3). That S is relatively bounded with respect to D is a straightforward consequence of its space-diagonal, regularising nature (Lemma 3.4).

In Section 4 we set up the Newton-Nash-Moser iteration (4.4), and calculate an estimate of the measure for the infinitely many excluded subsets of parameters. In this way we prove the large relative measure of the remaining Cantor set $\mathcal{A}_\gamma$ of parameters for which the iteration scheme produces a sequence converging to a solution of (1.6) (Lemma 4.4).

In Section 5 we collect technical proofs of the Nash-Moser method, concerning the inductive construction of the approximating sequence, its dependence on the parameter ω, its convergence, and the local uniqueness of the solution, completing the proof of Theorems 2.1 and 2.5.

Basic properties of Hill's eigenvalue problems are proved in Section 6, via the classical spectral theory and variational characterisation of eigenvalues.

2. Functional setting and results

2.1. Case of Dirichlet boundary conditions

Assume that $\partial\Omega$ is C^∞. Let $\lambda_j^2, \varphi_j(x)$, $j = 1, 2, \ldots$ be the eigenvalues and eigenfunctions of the boundary-value problem

$$\begin{cases} -\Delta\varphi_j = \lambda_j^2 \varphi_j & \text{in } \Omega, \\ \varphi_j = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\int_\Omega \varphi_j^2 \, dx = 1$ and $\lambda_1 < \lambda_2 \leq \ldots$. Weyl's formula for the asymptotic distribution of the eigenvalues gives $\lambda_j = O(j^{1/d})$ as $j \to \infty$, hence

$$C j^{1/d} \leq \lambda_j \leq C' j^{1/d} \quad \forall j = 1, 2, \ldots, \tag{2.1}$$

for some positive C, C' depending on the dimension d and on the domain Ω; see, for example, [36, Vol. IV, XIII.15].

By expansion in the basis $\{\varphi_j(x)\}$, we define the spaces

$$V_{\sigma,s}(\Omega) := \left\{ v(x) = \sum_j v_j \varphi_j(x) : \sum_j |v_j|^2 \lambda_j^{2s} \, e^{2\sigma\lambda_j} < \infty \right\}$$

for $s \geq 0$, $\sigma \geq 0$. Spaces $V_{0,s}$ with $\sigma = 0$ are used in [4]. They are the domains of the fractional powers $\Delta^{s/2}$ of the Laplace operator. See [4,21] for a characterisation; for instance, $V_{0,2} = H^2(\Omega) \cap H_0^1(\Omega)$. Note that, if $u \in V_{0,s}(\Omega)$, then $\Delta^k u \in H_0^1(\Omega)$ for all $0 \leq k \leq (s-1)/2$.

Spaces $V_{\sigma,0}$ with $s = 0$ are used in [3], where it is proved that $\cup_{\sigma>0} V_{\sigma,0}$ is the class of the $(-\Delta)$-analytic functions, which is, by definition, the set of functions $v(x) \in H_0^1(\Omega)$ such that

$$\Delta^k v \in H_0^1(\Omega) \quad \text{and} \quad \left| \int_\Omega v \Delta^k v \, dx \right|^{1/2} \leq C A^k k! \qquad \forall k = 0, 1, \ldots,$$

for some constants C, A. In [3] it is observed that an important subset of $\cup_{\sigma>0} V_{\sigma,0}$ consists of the functions $v(x)$, analytic on some neighbourhood of $\overline{\Omega}$, such that

$$\Delta^k v = 0 \quad \text{on } \partial\Omega \qquad \forall k = 0, 1, \ldots.$$

This subset coincides with the whole class of $(-\Delta)$-analytic functions when $\partial\Omega$ is a real analytic manifold of dimension $(d-1)$, leaving Ω on one side [26,27], or when Ω is a parallelepiped [1].

Clearly $V_{\sigma,s} = \{u \in V_{\sigma,0} : \Delta^{s/2} u \in V_{\sigma,0}\}$ and $V_{\sigma,0} \subset V_{\sigma',s} \subset V_{\sigma',0}$ for all $s > 0$, $\sigma > \sigma' > 0$. Moreover, all finite sums $\sum_{j \leq N} v_j \varphi_j(x)$ belong to $V_{\sigma,s}$ for all σ, s.

We set the problem in the spaces $X_{\sigma,s} = H^1(\mathbb{T}, V_{\sigma,s})$ of 2π-periodic functions $u : \mathbb{T} \to V_{\sigma,s}$, $t \mapsto u(\cdot, t)$ with H^1 regularity, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, namely

$$X_{\sigma,s} := \left\{ u(x,t) = \sum_{j \geq 1} u_j(t) \varphi_j(x) : u_j \in H^1(\mathbb{T}, \mathbb{R}), \right.$$
$$\left. \|u\|_{\sigma,s}^2 := \sum_{j \geq 1} \|u_j\|_{H^1}^2 \lambda_j^{2s} \, e^{2\sigma\lambda_j} < \infty \right\}.$$

Theorem 2.1 (Case of Dirichlet boundary conditions). *Suppose that $g \in X_{\sigma,s_0}$ for some $\sigma \geq 0$, $s_0 > 2d$. Let $s_1 \in (1 + d, 1 + s_0/2)$. There exist positive constants δ, C with the following properties.*

For every $\gamma \in (0, \lambda_1^{s_1})$ there exists a Cantor set $\mathcal{A}_\gamma \subset (0, +\infty) \times (0, \delta\gamma)$ of parameters such that for every $(\omega, \mu) \in \mathcal{A}_\gamma$ there exists a classical solution $u(\omega, \mu) \in X_{\sigma,s_1}$ of (1.6),(1.2). Such a solution satisfies

$$\|u(\omega, \mu)\|_{\sigma,s_1} \leq \frac{\mu}{\gamma} C, \qquad \|u(\omega, \mu)_{tt}\|_{\sigma,s_1-2} \leq \frac{\mu}{\gamma\omega^2} C,$$

and it is unique in the ball $\{\|u\|_{\sigma,s_1} < 1\}$.

The Lebesgue measures of the set $\mathcal{A}_\gamma$ and its sections $\mathcal{A}_\gamma(\mu) := \{\omega : (\omega, \mu) \in \mathcal{A}_\gamma\}$ have the following properties. For every interval $I = (\bar{\omega}_1, \bar{\omega}_2)$, with $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$, there exists a constant $\bar{C} = \bar{C}(I)$, independent on γ and μ, such that

$$\frac{|I \cap \mathcal{A}_\gamma(\mu)|}{|I|} > 1 - \bar{C}\gamma \quad \forall \mu < \delta\gamma, \qquad \frac{|\mathcal{R}_\gamma \cap \mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1 - \bar{C}\gamma,$$

where $\mathcal{R}_\gamma$ is the rectangular region $\mathcal{R}_\gamma = I \times (0, \delta\gamma)$.

We recall that (1.6) is obtained from (1.1) by the normalisation $t \to \omega t$ and the rescaling $u \to \varepsilon^{1/3} u$. Hence the solution $u(\omega, \mu)$ of (1.6) found in Theorem 2.1 gives a solution of (1.1) of order ε and period $2\pi/\omega$.

Remark 2.2. Theorem 2.1 covers both Sobolev and analytic cases:

- *Sobolev regularity.* If g belongs to the Sobolev space X_{0,s_0}, then the solution u found in the theorem belongs to the Sobolev space X_{0,s_1}.
- *Analytic regularity.* If g belongs to the analytic space $X_{\sigma_0,0}$, then $g \in X_{\sigma_1,s_0}$ for all $\sigma_1 \in (0, \sigma_0)$. Indeed,

$$\frac{\xi^{s_0}}{\exp[(\sigma_0 - \sigma_1)\xi]} \leq \left(\frac{s_0}{(\sigma_0 - \sigma_1)e}\right)^{s_0} =: C' \quad \forall \xi \geq 0,$$

therefore

$$\|g\|_{\sigma_1,s_0}^2 = \sum_j \|g_j\|_{H^1}^2 \lambda_j^{2s_0} e^{2\sigma_1\lambda_j} \frac{e^{2\sigma_0\lambda_j}}{e^{2\sigma_0\lambda_j}} \leq C'^2 \|g\|_{\sigma_0,0}^2.$$

Since $g \in X_{\sigma_1,s_0}$, the solution u found in the theorem belongs to the analytic space $X_{\sigma_1,s_1} \subset X_{\sigma_1,0}$.

Remark 2.3. If $g(x, \cdot) \in H^r(\mathbb{T})$, $r \geq 1$, then the solution u of (1.1) found in the theorem satisfies $u(x, \cdot) \in H^{r+2}(\mathbb{T})$ by bootstrap.

Remark 2.4 (Nonplanar vibrations). Consider the Kirchhoff equation for a string in the 3-dimensional space

$$u_{tt} - u_{xx} \left(1 + \int_0^\pi |u_x|^2 \, dx\right) = \varepsilon g(x, t), \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (2.2)$$

where the forcing g and the displacement u are $\mathbb{R}^2$-vectors belonging to the plane which is orthogonal to the rest position of the string, see [15, 31]. In this case nonplanar vibrations of the string are permitted.

Setting $\|u_j\|_{H^1}^2 := \|u_{1,j}\|_{H^1}^2 + \|u_{2,j}\|_{H^1}^2$ in the definition of the spaces $X_{\sigma,s}$, Theorem 2.1 holds also for problem (2.2).

2.2. Case of periodic boundary conditions

The eigenvalues and eigenfunctions of the Laplacian on $\mathbb{T}^d$ are $|m|^2$, $e^{im\cdot x}$, with $m \in \mathbb{Z}^d$. We consider a bijective numbering $\{m_j : j \in \mathbb{N}\}$ of $\mathbb{Z}^d$ such that $|m_j| \leq |m_{j+1}|$ for all $j \in \mathbb{N} = \{0, 1, \ldots\}$, and denote

$$\tilde{\lambda}_j^2 := |m_j|^2, \quad \tilde{\varphi}_j(x) := e^{im_j\cdot x} \quad \forall j \in \mathbb{N}.$$

We note that $\tilde{\lambda}_0 = 0$, $\tilde{\varphi}_0(x) \equiv 1$, and $\tilde{\lambda}_j \geq 1$ for all $j \geq 1$.

Weyl's estimate (2.1) holds also for $\tilde{\lambda}_j$, because the number of integer vectors $m \in \mathbb{Z}^d$ such that $|m| \leq \lambda$ is $O(\lambda^d)$ for $\lambda \to +\infty$; see [36, Vol. IV, XIII.15]. We define

$$\tilde{X}_{\sigma,s} := \left\{ u(x, t) = \sum_{j\geq 0} u_j(t)\tilde{\varphi}_j(x) : u_j \in H^1(\mathbb{T}, \mathbb{R}), \right.$$

$$\left. \|u\|_{\sigma,s}^2 := \|u_0\|_{H^1}^2 + \sum_{j\geq 1} \|u_j\|_{H^1}^2 \tilde{\lambda}_j^{2s} e^{2\sigma\tilde{\lambda}_j} < \infty \right\}.$$

Theorem 2.5 (Case of periodic boundary conditions). *Suppose that $g \in \tilde{X}_{\sigma,s_0}$ for some $\sigma \geq 0$, $s_0 > 2d$, and*

$$\int_{(0,2\pi)^{d+1}} g(x, t)\, dx\, dt = 0. \tag{2.3}$$

Let $s_1 \in (1 + d, 1 + s_0/2)$. There exist positive constants δ, C with the following properties.

For every $\gamma \in (0, 1)$ there exists a Cantor set $\mathcal{A}_\gamma \subset (0, +\infty) \times (0, \delta\gamma)$ of parameters such that for every $(\omega, \mu) \in \mathcal{A}_\gamma$ there exists a classical solution $u(\omega, \mu) \in \tilde{X}_{\sigma,s_1}$ of (1.6),(1.3), with

$$\int_{(0,2\pi)^{d+1}} u(\omega, \mu)(x, t)\, dx\, dt = 0.$$

Such a solution satisfies

$$\|u(\omega, \mu)\|_{\sigma,s_1} \leq \frac{\mu}{\gamma}\left(1 + \frac{1}{\omega^2}\right) C, \qquad \|u(\omega, \mu)_{tt}\|_{\sigma,s_1-2} \leq \frac{\mu}{\gamma\omega^2} C, \tag{2.4}$$

and it is unique in the ball $\{\int_{(0,2\pi)^{d+1}} u(x, t)\, dxdt = 0, \ \|u\|_{\sigma,s_1} < 1\}$.

The Lebesgue measures of the sets $\mathcal{A}_\gamma$, $\mathcal{A}_\gamma(\mu)$ are as in Theorem 2.1.

Remark 2.6. If $u(\omega, \mu)$ is a solution of (1.6), (1.3), then also $u(\omega, \mu) + c$, with $c \in \mathbb{R}$, solves (1.6), (1.3).

3. Inversion of the linearised operator

We denote
$$f(u) := \Delta u \int_\Omega |\nabla u|^2 \, dx$$
the Kirchhoff nonlinearity. Note that, because of the presence of the integral, f is not a composition operator. It is cubic, because $f(u) = A[u, u, u]$, where A is the three-linear map $A[u, v, w] = \Delta u \int_\Omega \nabla v \cdot \nabla w \, dx$. Its quadratic remainder at u is

$$
\begin{aligned}
Q(u, h) &:= f(u + h) - f(u) - f'(u)[h] \\
&= \Delta u \int_\Omega |\nabla h|^2 dx + \Delta h \int_\Omega (2\nabla u \cdot \nabla h + |\nabla h|^2) \, dx.
\end{aligned}
\tag{3.1}
$$

Let
$$L_\omega := \omega^2 \partial_{tt} - \Delta, \quad F(u) := L_\omega u - \mu f(u) - \mu g,$$
so that (1.6) reads $F(u) = 0$. The linearised operator at u is

$$
\begin{aligned}
F'(u)h &= L_\omega h - \mu f'(u)[h] \\
&= \omega^2 h_{tt} - \Delta h \left(1 + \mu \int_\Omega |\nabla u|^2 \, dx\right) - \mu \Delta u \int_\Omega 2\nabla u \cdot \nabla h \, dx.
\end{aligned}
$$

For $N > 0$, let
$$X^{(N)} := \left\{ u(x, t) = \sum_{\lambda_j \leq N} u_j(t) \varphi_j(x) \right\}.$$

$X^{(N)}$ is a finite-dimensional space with respect to the basis $\varphi_j(x)$, hence it is a subspace of $X_{\sigma,s}$, for all σ, s. Since the integral term $\int_\Omega |\nabla u|^2 dx$ depends only on time,
$$f(u) \in X^{(N)} \quad \forall u \in X^{(N)}.$$

Consider the Hill eigenvalue problem

$$
\begin{cases}
y'' + p^2 (1 + \mu a(t)) \, y = 0, \\
y(t) = y(t + 2\pi),
\end{cases}
\tag{3.2}
$$

where $a(t) := \int_\Omega |\nabla u|^2 \, dx$, and denote $p_l^2, l \in \mathbb{N}$, its eigenvalues.

Theorem 3.1 (Inversion of the linearised operator). *There exist universal constants R, K, K' with the following property. Let $u \in X_{0,1}$, with $\|u\|_{0,1} < R$, $\mu \in (0, 1)$, $\tau > d$, and $\gamma \in (0, \lambda_1^{\tau+1})$. Let p_l^2 be the eigenvalues of (3.2). Suppose that $\omega > 0$ satisfies*

$$|\omega p_l - \lambda_j| > \frac{\gamma}{\lambda_j^\tau} \quad \forall \lambda_j \leq N, \, l \in \mathbb{N}, \tag{3.3}$$

for some $N > 0$. If

$$\frac{\mu}{\gamma} \|u\|_{\sigma,\tau+1}^2 < K', \tag{3.4}$$

with $\sigma \geq 0$, then $F'(u)$ is invertible on $X^{(N)}$, $F'(u)^{-1} : X^{(N)} \to X^{(N)}$, and

$$\|F'(u)^{-1}h\|_{\sigma,0} \leq \frac{K}{\gamma} \|h\|_{\sigma,\tau-1} \quad \forall h \in X^{(N)}. \tag{3.5}$$

The proof of Theorem 3.1 is divided in some lemmas. First, we write $F'(u) = D + S$, where

$$Dh := \omega^2 h_{tt} - \Delta h (1 + \mu a(t)), \quad Sh := -\mu \Delta u \int_\Omega 2\nabla u \cdot \nabla h \, dx.$$

We recall some results on Hill's eigenvalue problems.

Lemma 3.2 (Hill's problems). *There exist universal constants δ_0, $K_0 > 0$ with the following properties. For $\alpha(t) \in H^1(\mathbb{T})$, with $\|\alpha\|_{H^1} < \delta_0$, the eigenvalues p^2 of the periodic problem*

$$\begin{cases} y'' + p^2(1 + \alpha(t)) \, y = 0 \\ y(t) = y(t + 2\pi) \end{cases} \tag{3.6}$$

form a sequence $\{p_l^2(\alpha)\}_{l \in \mathbb{N}}$ such that

$$\frac{1}{3} l \leq p_l(\alpha) \leq 2l \quad \forall l \in \mathbb{N}. \tag{3.7}$$

For $\|\alpha\|_{H^1}, \|\beta\|_{H^1} < \delta_0$,

$$\left| p_l(\alpha) - p_l(\beta) \right| \leq K_0 \, l \, \|\alpha - \beta\|_{H^1} \quad \forall l \in \mathbb{N}. \tag{3.8}$$

The eigenfunctions $\psi_l(\alpha)(t)$ of (3.6) form an orthonormal basis of $L^2(\mathbb{T})$ with respect to the scalar product

$$(u, v)_{L_\alpha^2} = \int_0^{2\pi} uv(1 + \alpha) \, dt,$$

and also an orthogonal basis of $H^1(\mathbb{T})$ with respect to the scalar product

$$(u, v)_{H_\alpha^1} = \int_0^{2\pi} u'v' \, dt + (u, v)_{L_\alpha^2}.$$

The corresponding norms are equivalent to the standard Sobolev norms,

$$\frac{1}{2} \|y\|_{L^2} \leq \|y\|_{L_\alpha^2} \leq 2 \|y\|_{L^2}, \quad \frac{1}{2} \|y\|_{H^1} \leq \|y\|_{H_\alpha^1} \leq 2 \|y\|_{H^1}. \tag{3.9}$$

Proof. The proof is in Section 6. □

Let R be a universal constant such that $\|a(t)\|_{H^1(\mathbb{T})} < \delta_0$, where $a(t) := \int_\Omega |\nabla u|^2\, dx$, for all $u \in X_{0,1}$ with $\|u\|_{0,1} < R$.

Lemma 3.3 (Inversion of D). *There exists a universal constant $\bar{K}$ with the following property. Let $\|a(t)\|_{H^1} < \delta_0$, $\mu \in (0,1)$, $\tau > d$, and $\gamma \in (0, \lambda_1^{\tau+1})$. Let p_l^2 be the eigenvalues of (3.2). Suppose that $\omega > 0$ satisfies (3.3) for some $N > 0$. Then D is invertible on $X^{(N)}$, $D^{-1} : X^{(N)} \to X^{(N)}$, and*

$$\|D^{-1}h\|_{\sigma,0} \leq \frac{\bar{K}}{\gamma}\, \|h\|_{\sigma,\tau-1} \quad \forall h \in X^{(N)}, \ \ \forall \sigma \geq 0.$$

Proof. If $h = \sum h_j(t)\, \varphi_j(x)$, then $Dh = \sum D_j h_j(t)\, \varphi_j(x)$, where

$$D_j z(t) = \omega^2 z''(t) + \lambda_j^2\, z(t) \rho(t), \quad \rho(t) := 1 + \mu a(t).$$

Using the eigenfunctions $\psi_l(t)$ of (3.2) as a basis of $H^1(\mathbb{T})$,

$$z = \sum_{l \in \mathbb{N}} \hat{z}_l \, \psi_l(t) \quad \Rightarrow \quad D_j z(t) = \sum_{l \in \mathbb{N}} (\lambda_j^2 - \omega^2 p_l^2)\, \hat{z}_l \, \psi_l(t)\, \rho(t),$$

and $G_j := (1/\rho)D_j$ is the diagonal operator $\{\lambda_j^2 - \omega^2 p_l^2\}_{l \in \mathbb{N}}$. Since

$$|\lambda_j^2 - \omega^2 p_l^2| \geq |\lambda_j - \omega p_l|\lambda_j > \frac{\gamma}{\lambda_j^{\tau-1}} \quad \forall \lambda_j \leq N, \ l \in \mathbb{N},$$

G_j is invertible for all j such that $\lambda_j \leq N$, and

$$\|G_j^{-1} z\|_{H_{\mu a}^1}^2 = \sum_{l \in \mathbb{N}} \left| \frac{\hat{z}_l}{\lambda_j^2 - \omega^2 p_l^2} \right|^2 \|\psi_l\|_{H_{\mu a}^1}^2 \leq \frac{\lambda_j^{2(\tau-1)}}{\gamma^2}\, \|z\|_{H_{\mu a}^1}^2.$$

Hence, by (3.9), $\|G_j^{-1} z\|_{H^1} \leq 4(\lambda_j^{\tau-1}/\gamma)\, \|z\|_{H^1}$. Since $D_j^{-1} z = G_j^{-1}(z/\rho)$ and $\|1/\rho\|_{H^1}$ is smaller than a universal constant,

$$\|D_j^{-1} z\|_{H^1} \leq \frac{C\lambda_j^{\tau-1}}{\gamma}\, \|z\|_{H^1}.$$

For $h \in X^{(N)}$, $D^{-1}h = \sum_{\lambda_j \leq N} D_j^{-1} h_j(t)\, \varphi_j(x)$, and the lemma follows. □

Lemma 3.4 (Control of S). *There exists a universal constant $\widetilde{K}$ such that, if σ, $s \geq 0$ and $u \in X_{\sigma,s+2}$, then $S : X_{0,0} \to X_{\sigma,s}$ is bounded, and*

$$\|Sh\|_{\sigma,s} \leq \widetilde{K}\mu\|u\|_{\sigma,s+2}\|u\|_{0,2}\|h\|_{0,0} \quad \forall h \in X_{0,0}.$$

Proof. Since $\int_\Omega \nabla u \cdot \nabla h \, dx$ does not depend on x,

$$\left\| \Delta u \int_\Omega \nabla u \cdot \nabla h \, dx \right\|_{\sigma,s} \leq \|\Delta u\|_{\sigma,s} \left\| \int_\Omega \nabla u \cdot \nabla h \, dx \right\|_{H^1(\mathbb{T})}.$$

$\int_\Omega \nabla u \cdot \nabla h \, dx = \sum_j \lambda_j^2 u_j(t) h_j(t)$, therefore, by Hölder inequality,

$$\left\| \int_\Omega \nabla u \cdot \nabla h \, dx \right\|_{H^1(\mathbb{T})} \leq C_{H^1} \|u\|_{0,2} \|h\|_{0,0},$$

where C_{H^1} is the algebra constant of $H^1(\mathbb{T})$.   □

Proof of Theorem 3.1. $F'(u) = D + S = (I + SD^{-1})D$, where I is the identity map on $X^{(N)}$. By Lemma 3.3, it remains to prove the invertibility of $I + SD^{-1}$ in norm $\| \, \|_{\sigma,\tau-1}$. By Neumann series, it is sufficient to show that

$$\|SD^{-1}h\|_{\sigma,\tau-1} \leq \frac{1}{2} \|h\|_{\sigma,\tau-1} \quad \forall h \in X^{(N)}. \tag{3.10}$$

By Lemmas 3.3 and 3.4,

$$\|SD^{-1}h\|_{\sigma,\tau-1} \leq \widetilde{K}\mu\|u\|_{\sigma,\tau-1+2}\|u\|_{0,2}\|D^{-1}h\|_{0,0} \leq \frac{\bar{K}\widetilde{K}\mu}{\gamma} \|u\|_{\sigma,\tau+1}^2 \|h\|_{\sigma,\tau-1},$$

because $\sigma \geq 0$ and $\tau > d \geq 1$. Thus the condition

$$\frac{\mu}{\gamma} \|u\|_{\sigma,\tau+1}^2 \leq \frac{1}{2\bar{K}\widetilde{K}} =: K'$$

implies (3.10), and, by Neumann series, $\|(I + SD^{-1})^{-1}h\|_{\sigma,\tau-1} \leq 2\|h\|_{\sigma,\tau-1}$.   □

4. Iteration scheme and Cantor set of parameters

Fix $\sigma \geq 0$ once and for all. For convenience, we write $X_s := X_{\sigma,s}$, $\|u\|_s := \|u\|_{\sigma,s}$. All the following calculations hold both in the Sobolev case $\sigma = 0$ and in the analytic case $\sigma > 0$; indeed, the only index that is used in the present Nash-Moser scheme is s.

For $\chi \in (1, 2)$, we define

$$N_n := \exp(\chi^n), \tag{4.1}$$

with $n \in \mathbb{N}$, we consider the increasing sequence of finite-dimensional subspaces

$$X^{(n)} := X^{(N_n)} = \left\{ u(x,t) = \sum_{\lambda_j \leq N_n} u_j(t)\varphi_j(x) \right\},$$

and denote P_n the projector onto $X^{(n)}$ (truncation operator). For all $s, \alpha \geq 0$, P_n enjoys the smoothing properties

$$\|P_n u\|_{s+\alpha} \leq N_n^\alpha \|u\|_s \quad \forall u \in X_s, \tag{4.2}$$

$$\|(I - P_n)u\|_s \leq N_n^{-\alpha} \|u\|_{s+\alpha} \quad \forall u \in X_{s+\alpha}, \tag{4.3}$$

where I is the identity map.

We adapt the original Newton-Nash-Moser iteration to the special structure of problem (1.6), and define

$$u_0 := 0, \quad u_{n+1} := u_n - F'(u_n)^{-1}[L_\omega u_n - \mu f(u_n) - \mu P_{n+1} g], \tag{4.4}$$

provided the inverse operator $F'(u_n)^{-1}$ is defined and bounded on $X^{(n+1)}$. By Theorem 3.1, $F'(u)$ is invertible if the parameters (ω, μ) satisfy the nonresonance condition (3.3). Thus, let

$$A_0 := (0, +\infty) \times (0, 1).$$

By induction, suppose that we have constructed the set A_n and u_n. We consider the eigenvalues $(p_l^{(n)})^2$, $l \in \mathbb{N}$, of Hill's problem (3.2), with

$$a(t) = a_n(t) := \int_\Omega |\nabla u_n|^2 \, dx. \tag{4.5}$$

For $\tau > d$ and $\gamma \in (0, \lambda_1^{\tau+1})$, we define

$$A_{n+1} := \left\{ (\omega, \mu) \in A_n : |\omega p_l^{(n)} - \lambda_j| > \frac{\gamma}{\lambda_j^\tau} \quad \forall \lambda_j \leq N_{n+1}, \ l \in \mathbb{N} \right\}. \tag{4.6}$$

Remark 4.1. For all $\mu \in (0, 1)$, $n \in \mathbb{N}$, the set $A_n(\mu) := \{\omega : (\omega, \mu) \in A_n\}$ is open. Indeed, by (2.1) and (3.7), for every $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$ the intersection $(\bar{\omega}_1, \bar{\omega}_2) \cap A_n(\mu)$ is defined by a finite number of strict inequalities.

(A_n, u_n), defined by induction, form a sequence only if the parameters (ω, μ) belong to A_n for all n. In Lemma 4.4 we prove that $\{(\omega, \mu) \in A_n \ \forall n \in \mathbb{N}\}$ is a nonempty set, and we estimate its large Lebesgue measure. For this purpose we need to control the difference between two successive terms u_n, u_{n+1}, and the dependence of u_n on the parameter ω.

Lemma 4.2 (Construction of the approximating sequence). *Let* $g \in X_{s_0}$, *with* $s_0 > 2d$, *and let* $\tau \in (d, s_0/2)$. *There exist a choice for* χ *in the definition* (4.1) *and positive constants* b, δ_1, K_1, *with* $b(2 - \chi) > \tau + 1$, *satisfying the following properties.*

First step. If $(\omega, \mu) \in A_1$ *and* $\mu/\gamma < \delta_1$, *then there exists* $u_1 \in X^{(1)}$ *defined by* (4.4), *with*

$$\|u_1\|_0 < K_1 \frac{\mu}{\gamma} \exp(-b\chi). \tag{4.7}$$

Induction step. Suppose that $u_1, \ldots, u_n$ are constructed by (4.4), $(\omega, \mu) \in A_n$, $n \geq 1$, where each A_{k+1} is defined by means of u_k by (4.6), and $u_k \in X^{(k)}$, $k = 1, \ldots, n$. Suppose that $\mu/\gamma < \delta_1$. Let

$$h_k := u_k - u_{k-1}.$$

Suppose that, for all $k = 1, \ldots, n$,

$$\|h_k\|_0 < K_1 \frac{\mu}{\gamma} \exp(-b\chi^k). \tag{4.8}$$

If $(\omega, \mu) \in A_{n+1}$, then there exists $u_{n+1} = u_n + h_{n+1} \in X^{(n+1)}$ defined by (4.4), with

$$\|h_{n+1}\|_0 < K_1 \frac{\mu}{\gamma} \exp(-b\chi^{n+1}). \tag{4.9}$$

Proof. The proof is in Section 5.   □

By (4.2) and (4.1), (4.8) implies that

$$\|h_k\|_\alpha \leq N_k^\alpha \|h_k\|_0 < K_1 \frac{\mu}{\gamma} \exp[(-b+\alpha)\chi^k], \tag{4.10}$$

for all $\alpha \geq 0$. Since $u_n = h_1 + \ldots + h_n$, if $\alpha < b$, then

$$\|u_n\|_\alpha \leq \sum_{k=1}^n \|h_k\|_\alpha \leq K_1 \frac{\mu}{\gamma} \sum_{k=1}^\infty \exp[(-b+\alpha)\chi^k] = C\frac{\mu}{\gamma}, \tag{4.11}$$

for some C independent on n. Note that $2 \leq 1 + d < 1 + \tau < \frac{1+\tau}{2-\chi} < b$.

Lemma 4.3 (Dependence on the parameter ω). *Assume the hypotheses of Lemma 4.2. There exist $\delta_2 \in (0, \delta_1]$ and $K_2 > 0$ such that all the maps*

$$h_k : A_k \cap \{(\omega, \mu) : \mu/\gamma < \delta_2\} \to X^{(k)}, \quad (\omega, \mu) \mapsto h_k(\omega, \mu)$$

are differentiable with respect to ω, and

$$\|\partial_\omega u_n\|_0 \leq K_2 \frac{\mu}{\gamma^2 \omega}. \tag{4.12}$$

Proof. The proof is in Section 5.   □

Lemma 4.4 (The Cantor set of parameters). *There exists $\delta_3 \in (0, \delta_2]$ such that, for all $\gamma \in (0, \lambda_1^{\tau+1})$, the Lebesgue measures of the Cantor set*

$$\mathcal{A}_\gamma := \{(\omega, \mu) \in A_n \ \forall n \in \mathbb{N}, \ \mu < \delta_3 \gamma\}$$

and its sections $\mathcal{A}_\gamma(\mu) := \{\omega : (\omega, \mu) \in \mathcal{A}_\gamma\}$ *have the following property. For every interval* $I = (\bar{\omega}_1, \bar{\omega}_2)$, *with* $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$, *there exists a constant* $\bar{C} = \bar{C}(I)$, *independent on* γ *and* μ, *such that*

$$\frac{|I \cap \mathcal{A}_\gamma(\mu)|}{|I|} > 1 - \bar{C}\gamma \quad \forall \mu < \delta_3\gamma, \qquad \frac{|\mathcal{R}_\gamma \cap \mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1 - \bar{C}\gamma, \qquad (4.13)$$

where $\mathcal{R}_\gamma$ *is the rectangular region* $\mathcal{R}_\gamma = I \times (0, \delta_3\gamma)$.

Proof. Fix μ, and let

$$E_n := A_n(\mu) \setminus A_{n+1}(\mu), \quad n \in \mathbb{N}.$$

We prove that $\cup_{n \in \mathbb{N}} E_n$ has small measure; as a consequence, its complementary set $\cap_{n \in \mathbb{N}} A_n(\mu)$ is a large set. Let

$$\Omega_{j,l}^n := \left\{ \omega : |\omega p_l^{(n)}(\omega, \mu) - \lambda_j| \leq \frac{\gamma}{\lambda_j^\tau} \right\}.$$

Note that $\Omega_{j,0}^n = \emptyset$ for all j, n, because $\gamma < \lambda_1^{\tau+1}$ and $p_0^{(n)} = 0$. If $\omega \in \Omega_{j,l}^n$, then $C\lambda_j < \omega l < C'\lambda_j$ for some C, C' by (3.7). For $\omega_1, \omega_2 \in A_n(\mu)$, with $\omega_1 < \omega_2$,

$$\left| p_l^{(n)}(\omega_2, \mu) - p_l^{(n)}(\omega_1, \mu) \right| \leq K_0 l \mu \|a_n(\omega_2, \mu) - a_n(\omega_1, \mu)\|_{H^1},$$

by (3.8). Recalling the definition (4.5) of a_n, and using (4.11) with $\alpha = 2$ and (4.12), for all $\omega \in A_n(\mu)$

$$\|\partial_\omega a_n(\omega, \mu)\|_{H^1} = 2 \left\| \int_\Omega \nabla u_n \cdot \nabla(\partial_\omega u_n) \, dx \right\|_{H^1} \leq 2\|u_n\|_2 \|\partial_\omega u_n\|_0 \leq C \frac{\mu^2}{\gamma^3 \omega},$$

whence

$$\left| p_l^{(n)}(\omega_2, \mu) - p_l^{(n)}(\omega_1, \mu) \right| \leq Cl \frac{\mu^3}{\gamma^3} \frac{|\omega_2 - \omega_1|}{\omega_1}.$$

Then, by (3.7),

$$\left| \omega_2 \, p_l^{(n)}(\omega_2, \mu) - \omega_1 \, p_l^{(n)}(\omega_1, \mu) \right| \geq |\omega_2 - \omega_1| \, p_l^{(n)}(\omega_2, \mu) - \omega_1 \, Cl \frac{\mu^3}{\gamma^3} \frac{|\omega_2 - \omega_1|}{\omega_1}$$

$$> |\omega_2 - \omega_1| \frac{l}{4}, \qquad (4.14)$$

provided μ/γ is small enough, say $\mu/\gamma < \delta_3$.

Fix $0 < \bar{\omega}_1 < \bar{\omega}_2$. If $\Omega_{j,l}^n \cap A_n(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2)$ is nonempty, then, by (4.14),

$$|\Omega_{j,l}^n| < \frac{8\gamma}{l\lambda_j^\tau} < C\bar{\omega}_2 \frac{\gamma}{\lambda_j^{\tau+1}}, \quad l \in \left(\frac{C'}{\bar{\omega}_2} \lambda_j, \frac{C''}{\bar{\omega}_1} \lambda_j \right) =: \Lambda(j), \qquad (4.15)$$

for some C, C', C''. Since $E_0 = \cup_{\lambda_j \leq N_1, l \geq 1} \Omega^0_{j,l}$,

$$|E_0 \cap (\bar{\omega}_1, \bar{\omega}_2)| \leq \sum_{\lambda_j \leq N_1} \sum_{l \in \Lambda(j)} |\Omega^0_{j,l}| < \widetilde{C}\gamma \sum_{\lambda_j \leq N_1} \frac{1}{\lambda_j^\tau}, \qquad (4.16)$$

for some $\widetilde{C}$ depending on $(\bar{\omega}_1, \bar{\omega}_2)$.

To estimate $|E_n \cap (\bar{\omega}_1, \bar{\omega}_2)|$, $n \geq 1$, we note that

$$E_n = \bigcup_{\lambda_j \leq N_{n+1}, l \geq 1} \Omega^n_{j,l} \cap A_n(\mu).$$

For the sets $\Omega^n_{j,l}$ with $N_n < \lambda_j \leq N_{n+1}$ we use (4.15), whence

$$\left| \bigcup_{\substack{N_n < \lambda_j \leq N_{n+1} \\ l \in \Lambda(j)}} \Omega^n_{j,l} \cap A_n(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2) \right| < \widetilde{C}\gamma \sum_{N_n < \lambda_j \leq N_{n+1}} \frac{1}{\lambda_j^\tau},$$

where $\widetilde{C}$ is the constant of (4.16). To estimate the remaining sets, suppose that $\omega \in \Omega^n_{j,l} \cap A_n(\mu)$ for some $\lambda_j \leq N_n$, $l \geq 1$. Then, by (3.8),

$$\begin{aligned} |\lambda_j - \omega p_l^{(n-1)}| &\leq |\lambda_j - \omega p_l^{(n)}| + \omega |p_l^{(n)} - p_l^{(n-1)}| \\ &\leq \frac{\gamma}{\lambda_j^\tau} + C\omega l\mu \|a_n - a_{n-1}\|_{H^1}. \end{aligned}$$

Now, $a_n - a_{n-1} = \int_\Omega \nabla h_n \cdot (2\nabla u_{n-1} + \nabla h_n)\, dx$, hence, by (4.8),(4.10) and (4.11),

$$\|a_n - a_{n-1}\|_{H^1} \leq \|h_n\|_0 \|2u_{n-1} + h_n\|_2 \leq C\frac{\mu^2}{\gamma^2} \exp(-b\chi^n),$$

for some C. Since $\omega l \leq C'\lambda_j$,

$$|\lambda_j - \omega p_l^{(n-1)}| \leq \frac{\gamma}{\lambda_j^\tau} + C\lambda_j \frac{\mu^3}{\gamma^2} \exp(-b\chi^n),$$

for some C. Then

$$\Omega^n_{j,l} \cap A_n(\mu) \subseteq \left\{ \omega : \frac{\gamma}{\lambda_j^\tau} < |\lambda_j - \omega p_l^{(n-1)}| \leq \frac{\gamma}{\lambda_j^\tau} + C\lambda_j \frac{\mu^3}{\gamma^2} \exp(-b\chi^n) \right\},$$

therefore, by (4.14),

$$|\Omega^n_{j,l} \cap A_n(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2)| \leq C\bar{\omega}_2 \frac{\mu^3}{\gamma^2} \exp(-b\chi^n),$$

because $l \in \Lambda(j)$. It follows that

$$\left| \bigcup_{\substack{\lambda_j \leq N_n \\ l \in \Lambda(j)}} \Omega^n_{j,l} \cap A_n(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2) \right| \leq C \frac{\mu^3}{\gamma^2} \exp(-b\chi^n) \sum_{\lambda_j \leq N_n} \lambda_j,$$

for some C depending on $(\bar{\omega}_1, \bar{\omega}_2)$. Now, by (2.1), $\lambda_j \leq N_n$ implies that $j \leq C N_n^d$ for some C, hence

$$\sum_{\lambda_j \leq N_n} \lambda_j \leq \sum_{1 \leq j \leq C N_n^d} C' j^{1/d} \leq C' \int_0^{C N_n^d} \xi^{1/d} d\xi = C'' N_n^{d+1},$$

for some C''. As a consequence,

$$\exp(-b\chi^n) \sum_{\lambda_j \leq N_n} \lambda_j \leq C'' \exp[(-b+d+1)\chi^n].$$

Then

$$|E_n \cap (\bar{\omega}_1, \bar{\omega}_2)| \leq C\gamma \left(\exp[(-b+d+1)\chi^n] + \sum_{N_n < \lambda_j \leq N_{n+1}} \frac{1}{\lambda_j^\tau} \right)$$

for some C, and

$$\left| \bigcup_{n \in \mathbb{N}} E_n \cap (\bar{\omega}_1, \bar{\omega}_2) \right| \leq C\gamma \left(\sum_{n \in \mathbb{N}} \exp[(-b+d+1)\chi^n] + \sum_{j \geq 1} \frac{1}{\lambda_j^\tau} \right).$$

The first series converges because $b > \tau + 1 > d + 1$. The second series converges because $\tau > d$ and, by (2.1),

$$\sum_{j \geq 1} \frac{1}{\lambda_j^\tau} \leq C \sum_{j \geq 1} \frac{1}{j^{\tau/d}} < \infty.$$

It follows that

$$\left| \bigcup_{n \in \mathbb{N}} E_n \cap (\bar{\omega}_1, \bar{\omega}_2) \right| \leq C\gamma,$$

for some C depending on $(\bar{\omega}_1, \bar{\omega}_2)$. Since $\mathcal{A}_\gamma(\mu)$ is the complementary set of $\cup_{n \in \mathbb{N}} E_n$, the first estimate in (4.13) holds for all $\mu/\gamma < \delta_3$.

Then integrating in μ on the interval $(0, \delta_3\gamma)$ yields

$$|\mathcal{A}_\gamma \cap R_\gamma| = \int_0^{\delta_3\gamma} |\mathcal{A}_\gamma(\mu) \cap (\bar{\omega}_1, \bar{\omega}_2)| \, d\mu. \qquad \square$$

5. Proofs

Proof of Lemma 4.2. *First step.* Since $u_0 = 0$ and $(\omega, \mu) \in A_1$, $F'(0) = L_\omega$ is invertible by Theorem 3.1. By (3.5), (4.7) holds if

$$K \|g\|_{s_0} < K_1 \exp(-b\chi), \quad \tau - 1 \le s_0. \tag{5.1}$$

Induction step. To define h_{n+1} via (4.4), we have to verify that the hypotheses of Theorem 3.1 hold. By (4.11), (3.4) holds if

$$b > \tau + 1, \quad (K_1 C_0)^2 \left(\frac{\mu}{\gamma}\right)^3 < K', \quad C_0 := \sum_{k \ge 1} \exp[(-b + \tau + 1)\chi^k], \tag{5.2}$$

and $\|u_n\|_{0,1} < R$ if

$$K_1 C_0 \frac{\mu}{\gamma} < R. \tag{5.3}$$

If (5.2) and (5.3) hold, by Theorem 3.1 we define h_{n+1} via (4.4). By Taylor's expansion (3.1) and the iteration scheme (4.4) at the previous step,

$$h_{n+1} = \mu F'(u_n)^{-1}[(P_{n+1} - P_n)g + Q(u_{n-1}, h_n)]. \tag{5.4}$$

Now, by (4.3),

$$\|(P_{n+1} - P_n)g\|_{\tau-1} \le N_n^{-\beta} \|P_{n+1}g\|_{\tau-1+\beta} \le N_n^{-\beta} \|g\|_{s_0},$$

where $\beta := s_0 - \tau + 1$. Note that $\beta > 0$ because, by assumption, $s_0 > 2\tau$. By (4.10), (4.11) and (4.2),

$$\|Q(u_{n-1}, h_n)\|_{\tau-1} < 3K_1 \frac{\mu}{\gamma} C_0 N_n^{\tau+1} \|h_n\|_0^2,$$

where C_0 is defined in (5.2). Then, by (5.4) and (3.5), (4.9) holds if

$$K N_n^{-\beta} \|g\|_{s_0} < \frac{1}{2} K_1 \exp(-b\chi^{n+1}), \tag{5.5}$$

$$3K \frac{\mu}{\gamma} C_0 N_n^{\tau+1} \|h_n\|_0^2 < \frac{1}{2} \exp(-b\chi^{n+1}). \tag{5.6}$$

(5.5) holds if

$$s_0 - \tau + 1 > b\chi, \quad K_1 > \frac{2K \|g\|_{s_0}}{\exp[(s_0 - \tau + 1 - b\chi)\chi]}, \tag{5.7}$$

and, by (4.8), (5.6) holds if

$$b(2 - \chi) > \tau + 1, \quad \frac{\mu}{\gamma} < \left\{\frac{\exp[(b(2 - \chi) - \tau - 1)\chi]}{6K C_0 K_1^2}\right\}^{1/3}. \tag{5.8}$$

Since $2\tau < s_0$, we fix $\chi \in (1, 2)$ such that

$$\tau - 1 + (\tau + 1)\frac{\chi}{2 - \chi} < s_0.$$

Next, we fix b in the interval

$$\frac{\tau + 1}{2 - \chi} < b < \frac{s_0 - \tau + 1}{\chi}.$$

Then we fix K_1 large enough to satisfy (5.7) and (5.1). Finally, we fix μ/γ small enough to satisfy (5.2), (5.3) and (5.8). □

Suppose that the hypotheses of Lemma 4.2 hold. For all $k = 1, \ldots, n$, by Taylor's expansion (3.1) and the iteration scheme (4.4) at two consecutive steps,

$$\omega^2(h_{k+1})_{tt} = \Delta h_{k+1} + \mu\{f'(u_k)[h_{k+1}] + (P_k - P_{k-1})g + Q(u_{k-1}, h_k)\}. \quad (5.9)$$

Hence, recalling that $\mu/\gamma < \delta_1$ and $\gamma < \lambda_1^{\tau+1}$, by (4.10), (5.5) and (5.6)

$$\|(h_{k+1})_{tt}\|_{\tau-1} \le C\frac{\mu}{\gamma\omega^2}\exp[(-b + \tau + 1)\chi^{k+1}], \quad (5.10)$$

for some C. Since $L_\omega h_1 = \mu P_1 g$, (5.10) holds also for $k = 0$.

Proof of Lemma 4.3. h_1 is defined for $(\omega, \mu) \in A_1$ and it solves $\omega^2(h_1)_{tt} = \Delta h_1 + \mu P_1 g$. By the implicit function theorem, recalling Remark 4.1 and Theorem 3.1, h_1 is differentiable with respect to ω, and

$$2\omega(h_1)_{tt} + L_\omega[\partial_\omega h_1] = 0.$$

Applying L_ω^{-1}, (4.12) for $n = 1$ follows by (3.5) and (5.10).

Assume that, for $n \ge 1$,

$$\|\partial_\omega h_k\|_0 \le C_1\frac{\mu}{\gamma^2\omega}\exp[(-b + \tau + 1)\chi^k] \quad \forall k = 1, \ldots, n, \quad (5.11)$$

for some C_1. h_{n+1} is defined for $(\omega, \mu) \in A_{n+1}$ and it solves (5.9). Hence, by the implicit function theorem and Theorem 3.1, it is differentiable with respect to ω, and

$$F'(u_n)[\partial_\omega h_{n+1}] = -2\omega(h_{n+1})_{tt} + \mu\{f''(u_n)[\partial_\omega u_n, h_{n+1}] + \partial_\omega(Q(u_{n-1}, h_n))\}.$$

Since $\|\partial_\omega u_n\|_0 \le \sum_{k=1}^n \|\partial_\omega h_k\|_0$, by (5.11),(4.8), (4.2) and (5.10) we estimate the $\|\ \|_{\tau-1}$ norm of the right-hand term. Then, applying $F'(u_n)^{-1}$,

$$\|\partial_\omega h_{n+1}\|_0 \le C\left(\frac{\mu}{\gamma}\right)^3\frac{\mu}{\gamma^2\omega}\exp[(-b + \tau + 1)\chi^{n+1}]$$

for some C. Hence (5.11) holds for $k = n+1$ if μ/γ is small enough, independently on n. (4.12) follows from (5.11) because $\|\partial_\omega u_n\|_0 \le \sum_{k=1}^n \|\partial_\omega h_k\|_0$. □

Lemma 5.1 (Existence of a solution). *Assume the hypotheses of Lemma* 4.2 *and suppose that* $(\omega, \varepsilon) \in \mathcal{A}_\gamma$. *Then the sequence* (u_n) *converges in* $X_{\tau+1}$ *to* $u_\infty :=$ $\sum_{k \geq 1} h_k$, *and* $(u_n)_{tt}$ *converges to* $(u_\infty)_{tt}$ *in* $X_{\tau-1}$, *with*

$$\|u_\infty\|_{\tau+1} \leq \frac{\mu}{\gamma} C, \quad \|(u_\infty)_{tt}\|_{\tau-1} \leq \frac{\mu}{\gamma \omega^2} C,$$

for some C. u_∞ *is a classical solution of* (1.6).

Proof. By (4.10) and (5.10), the series $\sum \|h_k\|_{\tau+1}$ and $\sum \|(h_k)_{tt}\|_{\tau-1}$ converge, therefore $\|u_\infty - u_n\|_{\tau+1}$ and $\|(u_\infty - u_n)_{tt}\|_{\tau-1} \to 0$. As a consequence, $\|F(u_n) - F(u_\infty)\|_{\tau-1} \to 0$.

On the other hand, by the iteration scheme (4.4), $F(u_n) = -\mu\{(I - P_n)g + Q(u_{n-1}, h_n)\}$, hence, by (5.5) and (5.6), $\|F(u_n)\|_{\tau-1} \to 0$. Then $F(u_\infty) = 0$. $\square$

Lemma 5.2 (Uniqueness of the solution). *Assume the hypotheses of Lemma* 5.1. *There exists* $\delta_4 \in (0, \delta_3]$ *such that, for* $\mu/\gamma < \delta_4$, u_∞ *is the unique solution of* (1.6) *in the ball* $\{u \in X_{\tau+1} : \|u\|_{\tau+1} < 1\}$.

Proof. Suppose that $F(v) = 0$, with $\|v\|_{\tau+1} < 1$. Let $v_n := P_n v$. Projecting the equation $F(v) = 0$ on $X^{(n)}$ gives

$$L_\omega v_n = \mu\big(f(v_n) + R_n(v) + P_n g\big), \quad R_n(v) := \Delta v_n \int_\Omega |\nabla(v - v_n)|^2 \, dx.$$

Since u_n solves (4.4), the difference $w_n := v_n - u_n$ satisfies

$$L_\omega w_n - \mu\big\{f(v_n) - f(u_n) + R_n(v) - (P_{n+1} - P_n)g\big\} = F'(u_n)h_{n+1}.$$

Since $f(v_n) - f(u_n) = f'(u_n)[w_n] + Q(u_n, w_n)$, applying $F'(u_n)^{-1}$,

$$w_n - \mu F'(u_n)^{-1} Q(u_n, w_n) = h_{n+1} + \mu F'(u_n)^{-1}[R_n(v) - (P_{n+1} - P_n)g]. \quad (5.12)$$

$\|w_n\|_{\tau+1}$ is bounded, because both v and u_∞ belong to $X_{\tau+1}$. Therefore $\|Q(u_n, w_n)\|_{\tau-1} \leq C\|w_n\|_0$, for some C. By (3.5),

$$\|\mu F'(u_n)^{-1} Q(u_n, w_n)\|_0 \leq K \frac{\mu}{\gamma} \|Q(u_n, w_n)\|_{\tau-1} \leq \frac{1}{2} \|w_n\|_0,$$

if μ/γ is sufficiently small. Then, by (5.12),

$$\frac{1}{2} \|w_n\|_0 \leq \|h_{n+1}\|_0 + K \frac{\mu}{\gamma} \|R_n(v) - (P_{n+1} - P_n)g\|_{\tau-1}.$$

By (4.9) and (4.3) the right-hand side tends to 0 as $n \to \infty$, hence $\|v_n - u_n\|_0 \to 0$. Since $v_n \to v$ and $u_n \to u_\infty$ in X_0, it follows that $v = u_\infty$. $\square$

Proof of Theorem 2.1. Let $g \in X_{\sigma, s_0}$ and $2d < 2(s_1 - 1) < s_0$. Define

$$\tau := s_1 - 1.$$

Theorem 2.1 follows from Lemmas 4.4, 5.1 and 5.2.   □

Proof of Theorem 2.5. Let $g \in \tilde{X}_{\sigma, s_0}$ and $2d < 2(s_1 - 1) < s_0$. Define $\tau := s_1 - 1$. Decompose $\tilde{X}_{\sigma, s} = Y \oplus (W \cap \tilde{X}_{\sigma, s})$, where

$$Y := \{y(t) \in H^1(\mathbb{T}, \mathbb{R})\}, \quad W := \left\{ w \in \tilde{X}_{0,0} : w(x, t) = \sum_{j \geq 1} w_j(t) \tilde{\varphi}_j(x) \right\}.$$

Let Π_Y, Π_W be the projectors onto Y, W respectively, and let $g_Y(t) := \Pi_Y g$ and $g_W(x, t) := \Pi_W g$. Decompose $u(x, t) = y(t) + w(x, t)$, $y \in Y$, $w \in W$. Since ∇ and Δ involve only derivatives with respect to x,

$$f(u) = f(y + w) = f(w) \in W.$$

Projecting equation (1.6) on Y and W gives

$$\omega^2 y''(t) = \mu g_Y(t) \qquad (Y \text{ equation}), \qquad (5.13)$$
$$L_\omega w = \mu(f(w) + g_W) \qquad (W \text{ equation}). \qquad (5.14)$$

(5.13) is an ODE, and has 2π-periodic solutions if and only if

$$\int_0^{2\pi} g_Y(t) \, dt = 0,$$

that is (2.3). If $y(t)$ solves (5.13), then also $y(t) + c$ solves (5.13), for all $c \in \mathbb{R}$. Moreover, the unique solution $y(t)$ of (5.13) with zero mean satisfies

$$\|y\|_{H^1} \leq \|y''\|_{H^1} \leq \frac{\mu}{\omega^2} \|g_Y\|_{H^1}.$$

To solve (5.14), we consider all lemmas and calculations that lead to Theorem 2.1, replacing $X_{\sigma, s}$ with $\tilde{X}_{\sigma, s} \cap W$ and λ_j, $\varphi_j(x)$ with $\tilde{\lambda}_j$, $\tilde{\varphi}_j(x)$, $j \geq 1$. It follows the existence of a unique solution $w \in \tilde{X}_{\sigma, s_1} \cap W$ of (5.14), with

$$\|w\|_{\sigma, s_1} \leq \frac{\mu}{\gamma} C, \quad \|w_{tt}\|_{\sigma, s_1 - 2} \leq \frac{\mu}{\gamma \omega^2} C.$$

Then $u = y + w$ is a solution of (1.6),(1.3), and (2.4) follows from

$$\|u\|_{\sigma, s_1}^2 = \|y\|_{H^1}^2 + \|w\|_{\sigma, s_1}^2, \quad \|u_{tt}\|_{\sigma, s_1 - 2}^2 = \|y''\|_{H^1}^2 + \|w_{tt}\|_{\sigma, s_1 - 2}^2. \qquad □$$

6. Hill's eigenvalue problem

Proof of Lemma 3.2. Fix $\alpha \in H^1(\mathbb{T})$. Define the bilinear, symmetric form

$$B_\alpha(u, v) := \int_0^{2\pi} \left(\frac{u}{\sqrt{1+\alpha}}\right)' \left(\frac{v}{\sqrt{1+\alpha}}\right)' dt, \quad u, v \in H^1(\mathbb{T}). \qquad (6.1)$$

Let

$$H_\alpha := \left\{ u \in H^1(\mathbb{T}) : \int_0^{2\pi} u \sqrt{1+\alpha} \, dt = 0 \right\},$$

endowed with the usual $H^1(\mathbb{T})$ norm. For $\|\alpha\|_{H^1}$ smaller than a universal constant δ_0, B_α is continuous and coercive on $(H_\alpha, \| \, \|_{H^1})$. Hence, for every $f \in L^2(\mathbb{T})$, there exists a unique $u := T_\alpha f \in H_\alpha$ such that $B_\alpha(u, \phi) = (f, \phi)_{L^2}$ for all $\phi \in H_\alpha$. $T_\alpha : L^2 \to L^2$ is a compact, self-adjoint operator, with one-dimensional kernel spanned by $\sqrt{1+\alpha}$. Therefore, by the classical spectral theory, there exist an orthogonal basis $v_l(\alpha)$ of $L^2(\mathbb{T})$ and a non-decreasing sequence $p_l^2(\alpha)$ such that

$$B_\alpha(v_l(\alpha), \phi) = p_l^2(\alpha)(v_l(\alpha), \phi)_{L^2} \quad \forall \phi \in H_\alpha, \, l \in \mathbb{N},$$

with $p_0^2(\alpha) = 0$, $v_0(\alpha)$ a multiple of $\sqrt{1+\alpha}$, and $p_l^2(\alpha) > 0$ for all $l \geq 1$. Let

$$\psi_l(\alpha) := \frac{v_l(\alpha)}{\sqrt{1+\alpha}} \, .$$

$\psi_l(\alpha)$, $p_l^2(\alpha)$ are the eigenvectors and eigenvalues of (3.6).

Let Π_α denote the projector of H^1 onto H_α. Since $B_\alpha(u, u) = B_\alpha(\Pi_\alpha u, \Pi_\alpha u)$ for all $u \in H^1$, the variational characterisation of the eigenvalues is

$$p_l^2(\alpha) = \min_{\substack{E \subset H^1, \\ \dim E = l+1}} \left(\max_{\substack{u \in E, \\ \|u\|_{L^2}=1}} B_\alpha(u, u) \right), \quad l \in \mathbb{N}.$$

Then, for $\|\alpha\|_{H^1}, \|\beta\|_{H^1} < \delta_0$,

$$p_l^2(\alpha) - p_l^2(\beta) \geq B_\alpha(\bar{u}, \bar{u}) - B_\beta(\bar{u}, \bar{u}) = B_\alpha(\Pi_\alpha \bar{u}, \Pi_\alpha \bar{u}) - B_\beta(\Pi_\beta \bar{u}, \Pi_\beta \bar{u}),$$

where $\bar{u}$ is a maximiser for B_β on $\bar{E}$, with $\|\bar{u}\|_{L^2} = 1$, and $\bar{E}$ is a minimiser for $(\max_E B_\alpha)$ among the $(l+1)$-dimensional subspaces $E \subset H^1$. By triangular inequality and definition (6.1),

$$|B_\alpha(\Pi_\alpha \bar{u}, \Pi_\alpha \bar{u}) - B_\beta(\Pi_\beta \bar{u}, \Pi_\beta \bar{u})| \leq C \|\beta - \alpha\|_{H^1}(1 + \|\Pi_\alpha \bar{u}\|_{H^1}^2),$$

for some universal constant C. $\|\Pi_\alpha \bar{u}\|^2_{H^1} \leq C' B_\alpha(\bar{u}, \bar{u})$, because B_α is coercive on $(H_\alpha, \|\cdot\|_{H^1})$, and

$$B_\alpha(\bar{u}, \bar{u}) \leq \max_{\substack{u \in \bar{E}, \\ \|u\|_{L^2}=1}} B_\alpha(u, u) = p_l^2(\alpha).$$

After analogous calculations in the opposite direction,

$$-C\|\beta - \alpha\|_{H^1}(1 + p_l^2(\alpha)) \leq p_l^2(\alpha) - p_l^2(\beta) \leq C\|\beta - \alpha\|_{H^1}(1 + p_l^2(\beta)), \quad (6.2)$$

for all $l \in \mathbb{N}$, for some universal constant C. Taking a smaller δ_0 if necessary, (3.7) follows from (6.2) with $\beta = 0$, because $p_l(0)$ is the integer part of $(l+1)/2$.

Finally, from (6.2) and (3.7), for all $l \geq 1$,

$$|p_l(\beta) - p_l(\alpha)| = \frac{|p_l^2(\beta) - p_l^2(\alpha)|}{p_l(\beta) + p_l(\alpha)} \leq K_0 \, l \|\beta - \alpha\|_{H^1}. \qquad \square$$

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