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A priori estimates in L^∞ for non-diagonal perturbed quasilinear systems

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Abstract. We present a way to derive a priori estimates in L^∞ for a class of quasilinear systems containing examples with a leading part which is neither diagonal nor of Uhlenbeck type. Moreover, a perturbation term with natural growth in first order derivatives is allowed.

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1. Introduction

We obtain a priori estimates in L^∞ for solutions of the Dirichlet problem for a class of quasilinear systems of second order, a prototype of which is

$$\begin{aligned} -\operatorname{div} Q^j(\nabla u) + |\nabla u|^p \,(Ru)^j + f^j(x) = 0 \;\text{ in } \Omega \quad \text{for } j = 1, \dots, M, \\ u = 0 \;\text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $f = (f^1, \dots, f^M) \in L^q(\Omega; \mathbb{R}^M)$ with $q > N/p$ and $R \in \mathbb{R}^{M \times M}$ non-negative definite are fixed, $\Omega \subset \mathbb{R}^N$ is a bounded domain, and $u = (u^1, \dots, u^M) : \Omega \to \mathbb{R}^M$. A simple example for admissible Q is the vector p-Laplacian in the form $Q^j(\nabla u) = |\nabla u|^{p-2} \nabla u^j$ for some $1 < p < \infty$, but the results presented here are also applicable for a (slightly) more general class containing leading parts which are not of Uhlenbeck type.

Concerning the existence of (weak) solutions of the Dirichlet problem for this system, a key feature of the second term (henceforth called "the perturbation") is its so–called natural growth with respect to ∇u, which should be seen in comparison with the growth of the leading part. There is a well established and fairly complete theory for the existence of weak solutions for scalar equations ($M = 1$) of this type, for instance see [1,2,9,12,16] and the monographs [4] and [17]. The main structural assumption employed in that context is a sign condition on the perturbation, $R \geq 0$ in our example. This condition also contributes to obtaining a priori estimates in $W^{1,p}$. While these estimates easily extend to systems, basically just using that the

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leading part is coercive in a suitable sense and that the matrix R is non-negative definite, so far it remains an open problem whether quasilinear elliptic systems admit the existence of a weak solution for any such R. Partial results are available, though. For instance, a simple argument shows existence whenever a sequence of approximating solutions can be bounded in L^∞ by a suitable small constant (*e.g.* [6]). In the special case where R is a positive multiple of the identity matrix (*i.e.*, the angle formed by the perturbation and its argument is zero at least in an asymptotic sense, "angle condition"), R. Landes proved existence without such a smallness assumption [11] (see also [3] for a generalization). For that purpose, additional structural properties of the leading part are needed. More recently, it turned out that an existence result which in some sense lies in between the two aforementioned approaches is also possible [13]. It exhibits a relationship of the maximal admissible constant for the a priori bound in L^∞ and the maximal angle formed by the perturbation and its argument (the angle between u and Ru in the example above), roughly speaking, one of those two quantities has to be sufficiently small with respect to the other.

Of course, quantitative bounds in L^∞ are somewhat hard to verify, in particular in view of the fact that for systems of general form, the possible approaches to obtaining such a bound are rather limited, even if one does not care about the numerical value of the bound. In the case of a scalar quasilinear elliptic equation, a powerful theory is provided in [10]. In particular, this approach requires that the terms of the equation satisfy suitable coercivity conditions, such as (Q:2) and (b:2) for (2.1) below, and if these are violated, unbounded solutions may exist [7]. Moreover, in case of quasilinear elliptic systems, imposing coercivity is still not enough; see [8] for instance, where several counterexamples concerning a priori bounds and regularity are collected. As already observed in [10], a notable exception is formed by systems with a leading part in diagonal form, *i.e.*, with $Q^j(\nabla u) = q(\nabla u^j)$ for a $q : \mathbb{R}^N \to \mathbb{R}^N$. The same method can also be employed to obtain an a priori bound in L^∞ if the leading part of the system is of Uhlenbeck type, *i.e.*, of the form $Q^j(\nabla u) = \tilde{q}(\nabla u)\nabla u^j$ with $\tilde{q} : \mathbb{R}^{M\times N} \to \mathbb{R}$, as shown in [13]. For the special case of homogeneous systems ($f = 0$ in (1.1)) with inhomogeneous boundary conditions we also refer to [14].

One now may ask if it is possible to extend the method of [10] to systems with a leading part of more general form, at least if it satisfies the structure condition used in [13] to prove existence of weak solutions *assuming* a suitable quantitative a priori estimate in L^∞ is valid. However, such a generalization is by no means obvious. Here, the technical problem is due to the typical way of truncating the vector valued solution (which is employed as a test function in this context), because it can interfere with the action of the leading part in a bad way. Nevertheless, as we show in the proof of Proposition 3.3, if the leading part satisfies the structure condition (Q:3) below, a suitable modification of the method of [10] still yields the desired estimate. The main novelty is the use of a modified truncation, which at first glance has the rather bad property of not preserving continuity and thus leaving the class of admissible functions. Of course, this can be circumvented by using a sequence of mollified versions instead, at the cost of a perturbation term in the estimates arising from the discontinuity created in the limit. Somewhat surprisingly, this term turns out to be of a form which still can be handled.

Notation

As usual, $W^{k,p}(\Omega; V)$ is the Sobolev space of functions $u : \Omega \to V$ with distributional derivatives up to order k in $L^p(\Omega; V)$, where $\Omega \subset \mathbb{R}^N$ is a domain (*i.e.*, open and connected) and V is some finite dimensional euclidean vector space. The closed subspace $W_0^{k,p}(\Omega; V)$ consists of the closure of $C_0^\infty(\Omega)$, the smooth functions with compact support, in $W^{k,p}$. Norms of infinite dimensional spaces are denoted by $||\cdot||_X$, where the corresponding space X is given in the index. In a finite dimensional euclidean vector space, the norm generated by the (standard) scalar product is denoted by $|\cdot|$, as is the real modulus. Moreover, if $A \subset \mathbb{R}^N$ is a measurable set, $\mathcal{L}^N(A)$ is its Lebesgue measure. The tensor product $\mu \otimes \nu \in \mathbb{R}^{k\times l}$ of two vectors $\mu \in \mathbb{R}^k$, $\nu \in \mathbb{R}^l$ is the matrix with entries $(\mu \otimes \nu)_{ij} = \mu_i \nu_j$, and the positive part of a real or a real-valued function is denoted by $(f)^+ := \max\{0, f\}$. For subsets of Ω defined by a logical expression $A(x)$, we frequently use the abbreviation $\{A\} = \{A(x)\} := \{x \in \Omega \mid A(x) \text{ is true}\}$.

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2. Assumptions and main result

Throughout this paper, $\Omega \subset \mathbb{R}^N$ is a bounded domain and $1 < p < N$. We study solutions of the system

$$\begin{gathered} -\operatorname{div} Q^i(x, u, \nabla u) + b^i(x, u, \nabla u) = 0 \ \text{ in } \Omega \text{ for } i = 1, \dots, M, \\ u = 0 \ \text{ on } \partial\Omega \end{gathered} \tag{2.1}$$

in the set

$$A := \left\{ u \in W_0^{1,p}(\Omega; \mathbb{R}^M) \;\middle|\; \begin{array}{l} Q(\cdot, u, \nabla u) \in L^{\frac{p}{p-1}}(\Omega; \mathbb{R}^{M\times N}), \\ b(\cdot, u, \nabla u) \in L^1(\Omega; \mathbb{R}^M) \text{ and} \\ b(\cdot, u, \nabla u) \cdot u \in L^1(\Omega) \end{array} \right\}.$$

Here, (2.1) is understood in a suitable weak sense, that is, $u \in A$ satisfies

$$F(u)[\varphi] = 0 \ \text{ for every } \varphi \in X := W_0^{1,p}(\Omega; \mathbb{R}^M) \cap L^\infty(\Omega; \mathbb{R}^M), \tag{2.2}$$

with $F : A \to X'$ defined by

$$F(u)[\varphi] := \int_\Omega \big[Q(x, u, \nabla u) : \nabla\varphi + b(x, u, \nabla u) \cdot \varphi\big]\, dx, \tag{2.3}$$

where X' denotes the dual of X. In the integrand in (2.3), the symbols $:$ and $\cdot$ denote the standard scalar products in $\mathbb{R}^{M\times N}$ and $\mathbb{R}^M$, respectively. Moreover,

$$Q = (Q^i_j) : \ \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to \mathbb{R}^{M\times N} \ \text{ is a Carathéodory function and} \tag{Q:0}$$

$$b = (b^i) : \ \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to \mathbb{R}^M \ \text{ is a Carathéodory function,} \tag{b:0}$$

i.e. the functions are measurable in their first variable x and continuous in the other variables for a.e. $x \in \Omega$. The remaining assumptions on Q and b stated below are understood to be valid for a.e. $x \in \Omega$, every $\mu \in \mathbb{R}^M$ and every $\xi \in \mathbb{R}^{M\times N}$. In the following,

$$\begin{aligned} &h_s \text{ denotes a fixed non-negative function in } L^s(\Omega) \text{ (for } s \in [1, \infty]\text{)},\\ &\text{and } g : [0, \infty) \to [0, \infty) \text{ is a fixed continuous function with } g(0) = 0. \end{aligned} \tag{2.4}$$

We assume the growth conditions

$$|Q(x, \mu, \xi)| \le C\big(g(|\mu|) + 1\big)\big(h_p(x) + |\xi|\big)^{p-1}, \tag{Q:1}$$

$$\begin{aligned} &|b(x, \mu, \xi)| \le C\big(g(|\mu|) + 1\big)\big(h_p(x) + |\xi|\big)^{p},\\ &\text{with a constant } C \ge 0. \end{aligned} \tag{b:1}$$

Moreover, we assumed coercivity in the sense that

$$Q(x, \mu, \xi) : \xi \ \ge\ \zeta_2 |\xi|^p - h_q(x)(1 + |\mu|^r), \tag{Q:2}$$

$$\begin{aligned} &b(x, \mu, \xi) \cdot \mu \ \ge\ -\zeta_3 |\xi|^p - h_q(x)(1 + |\mu|^r),\\ &\text{with constants } \zeta_3 < \zeta_2,\ q > N/p \text{ and } 0 \le r < p. \end{aligned} \tag{b:2}$$

Last but not least, we need the following structure condition on the leading part:

$$\begin{aligned} &Q(x, \mu, \xi) : \left[\left(\frac{\mu \otimes \mu}{|\mu|^2}\right)\xi\right] \ \ge\ -C \ \text{ whenever } |\mu| \ge U\\ &\text{for a suitable constant } U \ge 0. \end{aligned} \tag{Q:3}$$

Observe that as a consequence of these assumptions, F is well defined.

Our main result is the following estimate:

Theorem 2.1. *Assume* (Q:0)–(Q:3) *and* (b:0)–(b:2). *Moreover, let $u \in A$ be a solution of* (2.2). *Then $u \in L^\infty(\Omega; \mathbb{R}^M)$, and its norm in L^∞ is bounded by a constant which only depends on N, M, $\zeta_2 - \zeta_3$, p, C, U, r, q, $\mathcal{L}^N(\Omega)$, $\|h_q\|_{L^q(\Omega)}$ and $\|h_p\|_{L^p(\Omega)}$.*

The theorem is a simple consequence of Proposition 3.2 and Proposition 3.3 in the next section.

Remark 2.2.

(i) The upper bound on p is actually superfluous since our arguments are valid also for $p \geq N$. Of course, the case $p > N$ can be dealt with in a much simpler way even under more general assumptions, using the embedding of $W_0^{1,p}$ into L^∞.

(ii) The conditions listed here are **not** sufficient to ensure the existence of weak solutions. In particular, a monotonicity (ellipticity) condition is missing, which we do not need for the a priori estimate.

(iii) The proof of Proposition 3.3 actually allows us to explicitly calculate the bound on $||u||_{L^\infty}$, just as the bound on $||u||_{W^{1,p}}$ in Proposition 3.2 can be made explicit. Moreover, these bounds only depend on the quantities listed and not on the explicit form of Q and b which in particular means that it holds uniformly for sequences of solutions of regularized problems obtained by truncating b, as used in [13] and [3]. As a consequence, if the additional conditions on Q and b of [13] or [3] are satisfied, the explicit form of our a priori estimate can be used to obtain sufficient conditions for the existence of weak solutions. However, our approach suffers from the same problem as its more restricted counterpart derived in [13]: the dependence of the bound on the constants is complicated (in particular, it involves determining the largest positive zero of two nonlinear functions to get the value of the constants of Proposition 3.2 and Lemma 3.5) and most likely not optimal.

(iv) Structure conditions similar to (Q:3) are a common assumption for quasi-linear systems in divergence form with perturbations having natural growth with respect to the gradient. Together with the angle condition on the perturbation described in the introduction, a predecessor being slightly weaker than (Q:3) was introduced in [11]. More recently, similar assumptions implying (Q:3) have been employed successfully in [13, 14] and [15]. So far, the role of these conditions remains rather unclear, apart from their technical merits with respect to the use of truncated test functions. In particular, systems with a leading part of the form

$$Q(x,\mu,\xi) = \xi q(x,\mu,\xi) \text{ with } q : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to (\mathbb{R}^{N\times N})_{\geq 0}, \quad (2.5)$$

satisfy (Q:3). Here, $(\mathbb{R}^{N\times N})_{\geq 0}$ denotes the set of non-negative definite matrices in $\mathbb{R}^{N\times N}$. Nonlinear examples more general than (2.5) can be obtained as follows: One can check that for any matrix $B \in \mathbb{R}^{M\times M}$, $N = M$, $p = M+2$ and $\varepsilon > 0$ small enough (depending on B),

$$Q(\xi) := |\xi|^M \xi + \varepsilon\big(\det \xi\big) B\xi \quad (2.6)$$

satisfies (Q:0)–(Q:3). Moreover, it is also strictly monotone in the sense that $\big(Q(\xi_1) - Q(\xi_2)\big) : (\xi_1 - \xi_2) \geq c(|\xi_1|^M + |\xi_2|^M)\,|\xi_1 - \xi_2|^2$ for some $c > 0$, provided ε is small enough.

3. Proof of the a priori estimate

For technical reasons, it is convenient to be able to use the solution as a test function in (2.2) even if we do not know that it belongs to L^∞. As a matter of fact, this is possible due to the definition of A.

Proposition 3.1. *Assume* (Q:0), (Q:1), (b:0) *and* (b:1). *If $u \in A$ is a solution of* (2.2), *and if $a \in L^\infty(\Omega)$ is a function such that $v(x) := a(x)u(x)$ belongs to $W_0^{1,p}(\Omega; \mathbb{R}^M)$, then $F(u)[v]$ given by* (2.3) *with $\varphi = v$ is well defined and we have $F(u)[v] = 0$.*

Proof. Using $\varphi = v_h := \frac{v}{|v|} \min\{h, |v|\} \in X$ in (2.2), we have $F(u)[v_h] = 0$ for every $h > 0$. Since $\nabla v_h \to \nabla v$ strongly in L^p and $v_h \to v$ pointwise a.e., we can pass to the limit as $h \to \infty$ to get the assertion. Here, note that $||a||_{L^\infty} |b(x, u, \nabla u) \cdot u|$ is integrable and a majorant of $b(x, u, \nabla u) \cdot v_h$. □

As a consequence, we have the usual a priori estimate in $W_0^{1,p}$.

Proposition 3.2. *Assume* (Q:0)–(Q:2) *and* (b:0)–(b:2). *Moreover, let $u \in A$ be a solution of* (2.2). *Then*

$$||\nabla u||_{L^p(\Omega;\mathbb{R}^M)} \le C_0,$$

where $C_0 > 0$ is a constant which only depends on p, N, M, $\mathcal{L}^N(\Omega)$, $\zeta_2 - \zeta_3$, r and $||h_q||_{L^q}$.

Proof. Testing (2.2) with $\varphi = u$, a standard calculation using Hölder's inequality, the Sobolev inequality $||u||_{L^{p^*}(\Omega;\mathbb{R}^M)} \le S\,||\nabla u||_{L^p(\Omega;\mathbb{R}^M)}$ with the optimal constant $S = S(p, N, M)$ and (Q:2), (b:2) yields

$$g(U) := (\xi_2 - \xi_3)U^p - 2\,||h_q||_{L^{\frac{p^*}{p^*-r}}} S^r U^r - 2\,||h_q||_{L^{\frac{p^*}{p^*-1}}} SU \le 0,$$

where $U := ||\nabla u||_{L^p}$. Observe that $g(t) \to +\infty$ as $t \to \infty$ since $p > r$ and $\xi_2 > \xi_3$. In particular, $||\nabla u||_{L^p} \le C_0$ with C_0 being the largest positive zero of g. □

By the Sobolev embedding, we also have $||u||_{L^{p^*}(\Omega;\mathbb{R}^M)} \le SC_0$. Next, we improve this bound to an estimate in L^∞.

Proposition 3.3. *Assume* (Q:0)–(Q:3) *and* (b:0)–(b:2). *Moreover, let $u \in A$ be a solution of* (2.2) *and assume that $||u||_{L^{p^*}(\Omega;\mathbb{R}^M)} \le \tilde{C}_0$ for some constant $\tilde{C}_0 \ge 0$. Then $u \in L^\infty$, and*

$$||u||_{L^\infty(\Omega;\mathbb{R}^M)} \le \bar{C},$$

where $\bar{C} > 0$ is a constant which only depends on N, M, $\zeta_2 - \zeta_3$, p, C, U, r, q, $\mathcal{L}^N(\Omega)$, $||h_q||_{L^q(\Omega)}$ and $\tilde{C}_0$.

Remark 3.4. Proposition 3.3 stays valid even if the assumption $r < p$ in (Q:2) and (b:2) is replaced by $r < p^*\bigl(1 - \frac{1}{q}\bigr) = p + p^*\bigl(\frac{p}{N} - \frac{1}{q}\bigr)$. This coincides with the condition commonly used in the scalar case to bound the norm in L^∞ in terms of the norm in L^{p^*}, *cf.* [10, Theorem 7.1 in Chapter 4].

Proof of Proposition 3.3. We use test functions of the form $\varphi = g_{\delta,\theta}(|u|)u$. Here, $\delta > 0$ and $\theta > 0$ are parameters, and $g_{\delta,\theta} : [0,\infty) \to [0,\infty)$ is defined by

$$g_{\delta,\theta}(t) := \begin{cases} 0 & \text{if } 0 \le t \le \theta, \\ 1 - \dfrac{\theta}{2t} & \text{if } \theta + \delta < t, \\ 1 - \dfrac{\theta}{2t}\left(1 + \dfrac{1}{\delta}(\theta + \delta - t)\right) & \text{if } \theta < t \le \theta + \delta, \end{cases}$$

which is a bounded, continuous and piecewise continuously differentiable function if $\delta > 0$ (whereas $g_{0,\theta}$ jumps at $t = \theta$). For $\theta < t < \theta + \delta$, its first derivative is $g'_{\delta,\theta}(t) = \frac{\theta}{2t^2}\left(1 + \frac{\theta+\delta}{\delta}\right)$. Also note that

$$0 \le g_{\delta,\theta}(t) \le 1 \quad\text{and}\quad 0 \le g'_{\delta,\theta}(t) \le \frac{1}{2\theta}\left(1 + \frac{\theta+\delta}{\delta}\right) \quad \text{for all } t \ge 0.$$

For $u \in W_0^{1,p}$, we have $g_{\delta,\theta}(|u|)u \in W_0^{1,p}$ by chain and product rule, and

$$\nabla\bigl(g_{\delta,\theta}(|u|)u\bigr) = \begin{cases} 0 & \text{on } \{|u| \le \theta\}, \\ \left(1 - \dfrac{\theta}{2|u|}\right)\nabla u + \dfrac{\theta}{2|u|^3}(u \otimes u)\nabla u & \text{on } \{\theta + \delta < |u|\}, \\ g_{\delta,\theta}(|u|)\nabla u + \dfrac{g'_{\delta,\theta}(|u|)}{|u|}(u \otimes u)\nabla u & \text{on } \{\theta < |u| \le \theta + \delta\}. \end{cases}$$

Hence, from (2.2) with $\varphi = g_{\delta,\theta}(|u|)u$, which is admissible due to Proposition 3.1, we obtain that

$$\begin{aligned} 0 \ge &\int_{\{|u|>\theta+\delta\}} \left(1 - \frac{\theta}{2|u|}\right)(\zeta_2 - \zeta_3)\,|\nabla u|^p \,dx \\ &- \int_{\{|u|>\theta+\delta\}} \left(C\frac{\theta}{2|u|} + 2\left(1 - \frac{\theta}{2|u|}\right)(1 + |u|^r)h_q(x)\right) dx \\ &+ \int_{\{\theta<|u|\le\theta+\delta\}} g_{\delta,\theta}(|u|)(\zeta_2 - \zeta_3)\,|\nabla u|^p \,dx \\ &- \int_{\{\theta<|u|\le\theta+\delta\}} \Bigl(C g'_{\delta,\theta}(|u|)\,|u| + 2g_{\delta,\theta}(|u|)(1 + |u|^r)h_q(x)\Bigr)\,dx \end{aligned} \tag{3.1}$$

for every $\theta \geq U$, where we used (Q:2), (Q:3) and (b:2). Accordingly, all statements below are understood to be valid for $\theta \geq U$. Passing to the limit as $\delta \searrow 0$ yields

$$
\begin{aligned}
0 \geq & \int_{\{|u|>\theta\}} \left(1-\frac{\theta}{2\,|u|}\right)(\zeta_2-\zeta_3)\,|\nabla u|^p\;dx \\
& - \int_{\{|u|>\theta\}} \left(C\frac{\theta}{2\,|u|}+2\left(1-\frac{\theta}{2\,|u|}\right)(1+|u|^r)h_q(x)\right)dx \\
& - C\liminf_{\delta\searrow 0}\int_{\{\theta<|u|\leq\theta+\delta\}} g'_{\delta,\theta}(|u|)\,|u|\;dx
\end{aligned}
\tag{3.2}
$$

due to dominated convergence, using that $\mathcal{L}^N(\{\theta < |u| \leq \theta+\delta\}) \to 0$ as $\delta \searrow 0$ for a.e. θ. In the following, we employ the abbreviation

$$
m_\theta := \mathcal{L}^N(\{|u| > \theta\})\,.
$$

Observe that $0 < \frac{\theta}{2|u|} < \frac{1}{2}$ on $\{|u| > \theta\}$ and that

$$
\liminf_{\delta\searrow 0}\int_{\{\theta<|u|\leq\theta+\delta\}} g'_{\delta,\theta}(|u|)\,|u|\;dx = \liminf_{\delta\searrow 0}\frac{\theta}{2}\frac{m_\theta-m_{\theta+\delta}}{\delta} = -\frac{\theta}{2}m'_\theta \quad \text{for a.e. } \theta.
$$

Hence (3.2) implies that

$$
\begin{aligned}
& \int_{\{|u|>\theta\}} \frac{1}{2}(\zeta_2-\zeta_3)\,|\nabla u|^p\;dx \\
& \quad \leq \frac{C}{2}\theta(-m'_\theta) + \int_{\{|u|>\theta\}}\left(\frac{C}{2}+2h_q(x)+2\,|u|^r\,h_q(x)\right)dx \\
& \quad \leq \frac{C}{2}\theta(-m'_\theta) + m_\theta^{1-\frac{1}{q}}\,\big|\big|2h_q+C/2\big|\big|_{L^q(\Omega)} \\
& \qquad\qquad + 2m_\theta^{1-\frac{1}{q}-\frac{r}{p^*}}\,\big|\big|h_q\big|\big|_{L^q(\Omega)}\,||u||^r_{L^{p^*}(\{|u|>\theta\};\mathbb{R}^M)}\,,
\end{aligned}
\tag{3.3}
$$

by Hölder's inequality. We proceed by estimating the term $||u||^r_{L^{p^*}(\{|u|>\theta\};\mathbb{R}^M)}$ as follows:

$$
\begin{aligned}
||u||^r_{L^{p^*}(\{|u|>\theta\};\mathbb{R}^M)} &= ||u||^{r-\tilde{r}}_{L^{p^*}(\{|u|>\theta\};\mathbb{R}^M)}\,||u||^{\tilde{r}}_{L^{p^*}(\{|u|>\theta\};\mathbb{R}^M)} \\
&\leq \tilde{C}\left(\big|\big|(|u|-\theta)^+\big|\big|^{\tilde{r}}_{L^{p^*}(\Omega)} + m_\theta^{\frac{\tilde{r}}{p^*}}\theta^{\tilde{r}}\right),
\end{aligned}
$$

where

$$
\tilde{r} := \min\{r, p\} \;\text{ and }\; \tilde{C} := 2^{\tilde{r}}(\tilde{C}_0)^{r-\tilde{r}},
$$

since $||u||_{L^{p*}} \leq \tilde{C}_0$. (As already mentioned in Remark 3.4, we prefer to employ the more general assumption $r < p^*\big(1-\frac{1}{q}\big)$ instead of $r < p$. This requires but little additional effort, anyway.) Thus,

$$\begin{aligned}
&\int_{\{|u|>\theta\}} \frac{1}{2}(\zeta_2-\zeta_3)\,|\nabla u|^p\,dx \\
&\quad\leq \frac{C}{2}\theta(-m_\theta') + \tilde{C} m_\theta^{1-\frac{1}{q}-\frac{r}{p^*}}\,\big|\big|h_q\big|\big|_{L^q(\Omega)}\,\big|\big|(|u|-\theta)^+\big|\big|_{L^{p^*}(\Omega)}^{\tilde{r}} \\
&\qquad + m_\theta^{1-\frac{1}{q}-\frac{r-\tilde{r}}{p^*}}\left[\mathcal{L}^N(\Omega)^{\frac{r-\tilde{r}}{p^*}}\left(\frac{C}{2}+2\,\big|\big|h_q\big|\big|_{L^q(\Omega)}\right)+\tilde{C}\theta^{\tilde{r}}\,\big|\big|h_q\big|\big|_{L^q(\Omega)}\right].
\end{aligned} \tag{3.4}$$

Since $|\nabla(|u|-\theta)| = |\nabla\,|u|| \leq |\nabla u|$ on $\{|u|>\theta\}$ and $(|u|-\theta)^+ \in W_0^{1,p}(\Omega)$, we also have the Poicaré–Sobolev inequality

$$\left(\big|\big|(|u|-\theta)^+\big|\big|_{L^{p^*}(\Omega)}\right)^p \leq S^p \int_{\{|u|>\theta\}} |\nabla u|^p\,dx,$$

with a constant $S = S(N,p) > 0$. Plugging this into (3.4) yields

$$\big|\big|(|u|-\theta)^+\big|\big|_{L^{p^*}(\Omega)}^{p-\tilde{r}} \leq \frac{C_1\theta(-m_\theta')+\big(C_2+C_3\theta^{\tilde{r}}\big)\,m_\theta^{1-\frac{1}{q}-\frac{r-\tilde{r}}{p^*}}}{\big|\big|(|u|-\theta)^+\big|\big|_{L^{p^*}(\Omega)}^{\tilde{r}}} + C_4 m_\theta^{1-\frac{1}{q}-\frac{r}{p^*}}. \tag{3.5}$$

Here, all terms independent of θ and u have been collected in suitable new constants $C_{1,2,3,4} \geq 0$. In the following, we abbreviate

$$f(\theta) := \big|\big|(|u|-\theta)^+\big|\big|_{L^1(\Omega)},\quad \varepsilon := 1-\frac{1}{q}-\frac{r-\tilde{r}+p}{p^*} \quad\text{and}\quad \delta := 1-\frac{1}{q}-\frac{r-\tilde{r}}{p^*}.$$

Note that $\varepsilon > 0$: If $p \leq r < p^*\big(1-\frac{1}{q}\big)$ this is due to the upper bound on r whereas if $r \leq p$, $\varepsilon = \frac{p}{N}-\frac{1}{q} > 0$ since $q > N/p$. By Hölder's inequality in the form $f(\theta) \leq m_\theta^{1-\frac{1}{p^*}}\,\big|\big|(|u|-\theta)^+\big|\big|_{L^{p^*}(\Omega)}$, (3.5) implies that

$$f(\theta)^p \;\leq\; C_1\theta(-m_\theta')m_\theta^{p+\varepsilon-\delta} + (C_2+C_3\theta^{\tilde{r}})m_\theta^{p+\varepsilon} + C_4 m_\theta^{p+\varepsilon-\tilde{r}} f(\theta)^{\tilde{r}}. \tag{3.6}$$

This can be further simplified using Young's inequality: For every $\beta > 0$, we have

$$m_\theta^{p+\varepsilon-\tilde{r}} f(\theta)^{\tilde{r}} \leq \beta^{-\frac{1}{p+\varepsilon-\tilde{r}}} m_\theta^{p+\varepsilon} + \beta f(\theta)^{p+\varepsilon} \leq \beta^{-\frac{1}{p+\varepsilon-\tilde{r}}} m_\theta^{p+\varepsilon} + \beta f(\theta)^p f(U)^\varepsilon,$$

since f is decreasing. If we choose $\beta := \frac{1}{2C_4} f(U)^{-\varepsilon}$, (3.6) thus turns into

$$\frac{1}{2} f(\theta)^p \;\leq\; C_1\theta(-m_\theta')m_\theta^{p+\varepsilon-\delta} + (C_2 + C_4\beta^{-\frac{1}{p-1+\varepsilon}} + C_3\theta^{\tilde{r}})m_\theta^{p+\varepsilon}.$$

Last but not least, observe that $f(\theta) = \int_\theta^\infty m_t\,dt$, whence $-f'(\theta) = m_\theta$ and $f''(\theta) = -m'_\theta$ for a.e. θ. Thus we have

$$f(\theta)^p \leq \tilde{C}_1\theta f''(\theta)[-f'(\theta)]^{p+\varepsilon-\delta} + (\tilde{C}_2 + \tilde{C}_3\theta^{\tilde{r}})[-f'(\theta)]^{p+\varepsilon} \text{ for every } \theta \geq U,$$

where $\tilde{C}_1 := 2C_1$, $\tilde{C}_2 := 2\big(C_2 + C_4\beta^{-\frac{1}{p-1+\varepsilon}}\big)$ and $\tilde{C}_3 := 2C_3$. Lemma 3.5 below with $T_0 := U$ now yields the assertion. Here, note that the initial values of f and f' are bounded:

$$0 \leq f(U) \leq ||u||_{L^1(\Omega;\mathbb{R}^M)} \leq \mathcal{L}^N(\Omega)^{1-\frac{1}{p^*}}\, ||u||_{L^{p^*}(\Omega;\mathbb{R}^M)} \text{ and } 0 \leq -f'(U) \leq \mathcal{L}^N(\Omega)\,.$$

Moreover, $f(U)f'(U) = 0$ if and only if $|u| \leq U$ almost everywhere. □

The final ingredient for the bound in L^∞ is a generalization of an observation of [10, Lemma 5.1 in Chapter 2].

Lemma 3.5. *Let $T_0 \geq 0$ and let $f : [T_0, \infty) \to [0, \infty)$ be convex and decreasing. Moreover, assume that $f(T_0) \in (0, C_0]$ and that $f'_+(T_0) \in [-C_0, 0)$, let $\bar{T} := \inf\{t > 0 : f$ is constant on $[t, \infty)\}$ (the first zero of f' if one exists, otherwise $\bar{T} = +\infty$) and assume that f satisfies*

$$\frac{f(t)^p}{(-f'(t))^{p+\varepsilon}} \leq C_1 t \frac{f''(t)}{(-f'(t))^\delta} + C_2 + C_3 t^r \text{ for a.e. } t \in (T_0, \bar{T}). \tag{3.7}$$

Here, $C_0, C_1, C_2, C_3 \geq 0$, $p \geq 1$, $\varepsilon > 0$, $\delta < 1$ and $0 \leq r < p + \varepsilon$ are constants. Then $\bar{T}$ is finite and bounded by a constant which only depends on C_0, C_1, C_2, C_4, p, ε, δ, r and T_0.

Remark 3.6. Note that since f is convex and decreasing, $f'_+(T_0)$, the derivative from the right at T_0, always exists in $[-\infty, 0]$. Since it is finite by assumption, f is globally Lipschitz continuous. Moreover, f is twice differentiable a.e. in $[T_0, \infty)$ by Aleksandrov's theorem (*e.g.* [5]).

Proof of Lemma 3.5. Fix an arbitrary $T \in (T_0, \bar{T})$ (in particular, $T < \infty$).

Integrating (3.7), we obtain

$$\begin{aligned}
&\int_{T_0}^T \frac{f(t)^p}{(-f'(t))^{p+\varepsilon}}\,dt \\
&\leq C_1 T \int_{T_0}^T \frac{f''(t)}{(-f'(t))^\delta}\,dt + C_2(T - T_0) + \frac{C_3}{r+1}(T^{r+1} - T_0^{r+1}) \\
&\leq \frac{C_1}{1-\delta} T \left[(-f'(T_0))^{1-\delta} - (-f'(T))^{1-\delta}\right] + C_2(T - T_0) + \frac{C_3}{r+1}(T^{r+1} - T_0^{r+1}) \\
&\leq \frac{C_1}{1-\delta} T(-f'(T_0))^{1-\delta} + C_2(T - T_0) + \frac{C_3}{r+1}(T^{r+1} - T_0^{r+1}).
\end{aligned}$$

(Observe that the version of the fundamental theorem of calculus used above holds with equality if and only if $f' \in BV$ actually belongs to $W^{1,1}$. In general, we only have an estimate because the function f'' neglects the part of the derivative of f' which is singular with respect to the Lebesgue measure. That part is a non-negative measure since f' is increasing.) The left hand side can also be estimated from below:

$$\begin{aligned}
&\int_{T_0}^{T} \frac{f(t)^p}{(-f'(t))^{p+\varepsilon}}\,dt \\
&\geq (T-T_0)\left(\frac{1}{T-T_0}\int_{T_0}^{T}\frac{-f'(t)}{f(t)^{\frac{p}{p+\varepsilon}}}\,dt\right)^{-(p+\varepsilon)} \\
&= (T-T_0)^{p+\varepsilon+1}\left(\frac{\varepsilon}{p+\varepsilon}\right)^{p+\varepsilon}\left[f(T_0)^{\frac{\varepsilon}{p+\varepsilon}} - f(T)^{\frac{\varepsilon}{p+\varepsilon}}\right]^{-(p+\varepsilon)} \\
&\geq (T-T_0)^{p+\varepsilon+1}\left(\frac{\varepsilon}{p+\varepsilon}\right)^{p+\varepsilon} f(T_0)^{-\varepsilon}.
\end{aligned}$$

In the calculation above, the first estimate is due to Jensen's inequality, using that $s \mapsto s^{-(p+\varepsilon)}$ is convex on $(0,\infty)$. Combining, we infer that

$$g(T) := (T-T_0)^{p+\varepsilon+1} - C_4 - C_5 T - C_6 T^{r+1} \leq 0 \quad \text{for every } T_0 < T < \bar{T},$$

where $C_3, C_4, C_5 \geq 0$ are suitable new constants which only depend on $C_0, C_1, C_2, C_3, T_0, r, p, \varepsilon$ and δ. Since $p+\varepsilon+1 > r+1$, $g(T) \to +\infty$ as $T \to \infty$, and thus $\bar{T}$ is less or equal than the largest positive zero of g. □

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