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L^p Boundedness of the Riesz transform related to Schrödinger operators on a manifold

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Abstract. We establish various L^p estimates for the Schrödinger operator $-\Delta+V$ on Riemannian manifolds satisfying the doubling property and a Poincaré inequality, where Δ is the Laplace-Beltrami operator and V belongs to a reverse Hölder class. At the end of this paper we apply our result to Lie groups with polynomial growth.

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1. Introduction

The main goal of this paper is to establish the L^p boundedness for the Riesz transforms $\nabla(-\Delta + V)^{-\frac{1}{2}}$, $V^{\frac{1}{2}}(-\Delta + V)^{-\frac{1}{2}}$ and related inequalities on certain classes of Riemannian manifolds. Here, V is a non-negative, locally integrable function on M.

For the Euclidean case, this subject was studied by many authors under different conditions on V. We mention the works of Helffer-Nourrigat [35], Guibourg [31], Shen [51], Sikora [52], Ouhabaz [47] and others.

Recently, Auscher-Ben Ali [3] proved L^p maximal inequalities for these operators under less restrictive assumptions. They assumed that V belongs to some reverse Hölder class RH_q (for a definition, see Section 2). A natural further step is to extend the above results to the case of Riemannian manifolds.

For Riemannian manifolds, the L^p boundedness of the Riesz transform of $-\Delta + V$ was discussed by many authors. We mention Meyer [45], Bakry [9] and Yosida [59]. The most general answer was given by Sikora [52]. Let M satisfy the doubling property (D) and assume that the heat kernel verifies $\|p_t(x, .)\|_2 \leq \frac{C}{\mu(B(x,\sqrt{t}))}$ for all $x \in M$ and $t > 0$. Under these hypotheses, Sikora proved that if $V \in L^1_{\text{loc}}(M)$, $V \geq 0$, then the Riesz transforms of $-\Delta + V$ are L^p bounded for $1 < p \leq 2$ and of weak type (1,1).

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Li [41] obtained boundedness results on Nilpotent Lie groups under the restriction $V \in RH_q$ and $q \geq \frac{D}{2}$, D being the dimension at infinity of G (see [23]).

Following the method of [3], we obtain new results for $p > 2$ on complete Riemannian manifolds satisfying the doubling property (D), a Poincaré inequality (P_2) and taking V in some RH_q. For manifolds of polynomial type we obtain additional results. This includes Nilpotent Lie groups.

Let us summarize the content of this paper. Let M be a complete Riemannian manifold satisfying the doubling property (D) and admitting a Poincaré inequality (P_2). First we obtain the range of p for the following maximal inequality valid for $u \in C_0^\infty(M)$:

$$\|\Delta u\|_p + \|Vu\|_p \lesssim \|(-\Delta + V)u\|_p. \tag{1.1}$$

Here and after, we use $u \lesssim v$ to say that there exists a constant C such that $u \leq Cv$. The starting step is the following L^1 inequality for $u \in C_0^\infty(M)$,

$$\|\Delta u\|_1 + \|Vu\|_1 \leq 3\|(-\Delta + V)u\|_1 \tag{1.2}$$

which holds for any non-negative potential $V \in L^1_{\mathrm{loc}}(M)$. This allows us to define $-\Delta + V$ as an operator on $L^1(M)$ with domain $\mathcal{D}_1(\Delta) \cap \mathcal{D}_1(V)$.

For larger range of p, we assume that $V \in L^p_{\mathrm{loc}}(M)$ and $-\Delta + V$ is a priori defined on C_0^∞. The validity of (1.1) can be obtained if one imposes the potential V to be more regular:

Theorem 1.1. *Let M be a complete Riemannian manifold satisfying (D) and (P_2). Consider $V \in RH_q$ for some $1 < q \leq \infty$. Then there is $\epsilon > 0$ depending only on V such that* (1.1) *holds for* $1 < p < q + \epsilon$.

This new result for Riemannian manifolds is an extension of the one of Li [41] in the Nilpotent Lie groups setting obtained under the restriction $q \geq \frac{D}{2}$.

The second purpose of our work is to establish some L^p estimates for the square root of $-\Delta + V$. Notice that we always have the identity

$$\| \, |\nabla u| \, \|_2^2 + \|V^{\frac{1}{2}} u\|_2^2 = \|(-\Delta + V)^{\frac{1}{2}} u\|_2^2, \quad u \in C_0^\infty(M). \tag{1.3}$$

The weak type (1,1) inequality proved by Sikora [52] is satisfied under our hypotheses:

$$\| \, |\nabla u| \, \|_{1,\infty} + \|V^{\frac{1}{2}} u\|_{1,\infty} \lesssim \|(-\Delta + V)^{\frac{1}{2}} u\|_1. \tag{1.4}$$

Interpolating (1.3) and (1.4), we obtain

$$\| \, |\nabla u| \, \|_p + \|V^{\frac{1}{2}} u\|_p \lesssim \|(-\Delta + V)^{\frac{1}{2}} u\|_p \tag{1.5}$$

when $1 < p < 2$ and $u \in C_0^\infty(M)$. Here, $\| \; \|_{p,\infty}$ is the norm in the Lorentz space $L^{p,\infty}$.

It remains to find good assumptions on V and M to obtain (1.5) for some/all $2 < p < \infty$. First recall the following result:

Proposition 1.2 ([4]). *Let M be a complete Riemannian manifold satisfying (D) and (P_2). Then there exists $p_0 > 2$ such that the Riesz transform $T = \nabla(-\Delta)^{-\frac{1}{2}}$ is L^p bounded for $1 < p < p_0$.*

We now let $p_0 = \sup\left\{p \in]2, \infty[;\ \nabla(-\Delta)^{-\frac{1}{2}} \text{ is } L^p \text{ bounded}\right\}$. We obtain the following theorem:

Theorem 1.3. *Let M be a complete Riemannian manifold. Let $V \in RH_q$ for some $q > 1$ and $\epsilon > 0$ such that $V \in RH_{q+\epsilon}$.*

1. *Assume that M satisfies (D) and (P_2). Then for all $u \in C_0^\infty(M)$,*

$$\|\,|\nabla u|\,\|_p \lesssim \|(-\Delta + V)^{\frac{1}{2}} u\|_p \quad \textit{for } 1 < p < \inf(p_0, 2(q+\epsilon)); \tag{1.6}$$

$$\|V^{\frac{1}{2}} u\|_p \lesssim \|(-\Delta + V)^{\frac{1}{2}} u\|_p \quad \textit{for } 1 < p < 2(q+\epsilon). \tag{1.7}$$

2. *Assume that M is of polynomial type and admits (P_2). Suppose that $D < p_0$, where D is the dimension at infinity and that $\frac{D}{2} \le q < \frac{p_0}{2}$.*
   a. *If $q < D$, then (1.6) holds for $1 < p < \inf(q_D^* + \epsilon, p_0)$, ($q_D^* = \frac{Dq}{D-q}$).*
   b. *If $q \ge D$, then (1.6) holds for $1 < p < p_0$.*

Some remarks concerning this theorem are in order:

1. Note that item 1 is true without any additional assumption on the volume growth of balls other than (D). Our assumption that M is of polynomial type in item 2 –which is stronger than the doubling property (see Section 2)– is used only to improve the L^p boundedness of $\nabla(-\Delta + V)^{-\frac{1}{2}}$ when $\frac{D}{2} < q < \frac{p_0}{2}$. We do not need it to prove L^p estimates for $V^{\frac{1}{2}}(-\Delta + V)^{-\frac{1}{2}}$.
2. If $q > \frac{p_0}{2}$ then we can replace q in item 2 by any $q' < \frac{p_0}{2}$ since $V \in RH_{q'}$ (see Proposition 2.11 in Section 2).
3. If $p_0 \le D$ and $q \ge \frac{D}{2}$, then (1.6) holds for $1 < p < p_0$ and that is why we assumed $D < p_0$ in item 2.
4. Finally the parameter ϵ depends on the self-improvement of the reverse Hölder condition (see Theorem 2.11 in Section 2).

We establish also a converse theorem which is a crucial step in proving Theorem 1.3.

Theorem 1.4. *Let M be a complete Riemannian manifold satisfying (D) and (P_l) for some $1 \le l < 2$. Consider $V \in RH_q$ for some $q > 1$. Then*

$$\|(-\Delta + V)^{\frac{1}{2}} u\|_{l,\infty} \lesssim \|\,|\nabla u|\,\|_l + \|V^{\frac{1}{2}} u\|_l \quad \textit{for every } u \in C_0^\infty(M) \tag{1.8}$$

and

$$\|(-\Delta + V)^{\frac{1}{2}} u\|_p \lesssim \|\,|\nabla u|\,\|_p + \|V^{\frac{1}{2}} u\|_p \quad \textit{for every } u \in C_0^\infty(M) \tag{1.9}$$

and $l < p < 2$.

Using the interpolation result of [8], we remark that (1.9) follows directly from (1.8) and the the L^2 estimate (1.3).

Remark 1.5. The estimate (1.9) always holds in the range $p > 2$. This follows from the fact that (1.5) holds for $1 < p \leq 2$ and that (1.5) for p implies (1.9) for p', where p' is the conjugate exponent of p.

In the following corollaries we give examples of manifolds satisfying our hypotheses and to which we can apply the theorems above.

Corollary 1.6. *Let M be a complete n-Riemannian manifold with non-negative Ricci curvature. Then Theorem* 1.1*, item 1 of Theorem* 1.3 *and Theorem* 1.4 *hold with $p_0 = \infty$. Moreover, if M satisfies the maximal volume growth $\mu(B) \geq cr^n$ for all balls B of radius $r > 0$ then item 2 of Theorem* 1.3 *also holds.*

Proof. It suffices to note that in this case M satisfies (D) with $\log_2 C_d = n$, (P_1) – see Proposition 2.9 below–, that the Riesz transform is L^p bounded for $1 < p < \infty$ [9] and that M has at most an Euclidean volume growth, that is $\mu(B) \leq Cr^n$ for any ball B of radius $r > 0$ – in [15, Theorem 3.9]. □

Corollary 1.7. *Let $C(N) = \mathbb{R}^+ \times N$ be a conical manifold with compact basis N of dimension $n - 1 \geq 1$. Then Theorem* 1.1*, Theorem* 1.4 *and Theorem* 1.3 *hold with $d = D = n$, $p_0 = p_0(\lambda_1) > n$ where λ_1 is the first positive eigenvalue of the Laplacian on N.*

Proof. Note that such a manifold is of polynomial type n

$$C^{-1}r^n \leq \mu(B) \leq Cr^n$$

for all ball B of $C(N)$ of radius $r > 0$ [43, Proposition 1.3]. $C(N)$ admits (P_2) [21], and even (P_1) using the methods in [30]. For the L^p boundedness of the Riesz transform it was proved by Li [42] that $p_0 = \infty$ when $\lambda_1 \geq n - 1$ and $p_0 = \frac{n}{\frac{n}{2} - \sqrt{\lambda_1 + (\frac{n-1}{2})^2}} > n$ when $\lambda_1 < n - 1$. □

Remark 1.8. Related results for asymptotically conical manifolds are obtained in [32] (see the Introduction and Theorem 1.5). Under our geometric assumptions (*i.e.* conical manifolds), loc. cit. and our work are partially complementary. Indeed, the potentials in [32] are required to have some kind of fast decay at infinity, while the Reverse Hölder condition we imposed rules out this possibility.

Our main tools to prove these theorems are:

- the fact that V belongs to a Reverse Hölder class;
- an improved Fefferman-Phong inequality;
- a Calderón-Zygmund decomposition;
- reverse Hölder inequalities involving the weak solution of $-\Delta u + Vu = 0$;
- complex interpolation;

- the boundedness of the Riesz potential when M satisfies $\mu(B(x,r)) \geq Cr^{\lambda}$ for all $r > 0$.

Many arguments follow those of [3] –with additional technical problems due to the geometry of the Riemannian manifold– but those for the Fefferman-Phong inequality require some sophistication. This Fefferman-Phong inequality with respect to balls is new even in the Euclidean case. In [3], this inequality was proved with respect to cubes instead of balls which greatly simplifies the proof.

We end this introduction with a plan of the paper. In Section 2, we recall the definitions of the doubling property, Poincaré inequality, reverse Hölder classes and homogeneous Sobolev spaces associated to a potential V. Section 3 is devoted to define the Schrödinger operator. In Section 4 we give the principal tools to prove the theorems mentioned above. We establish an improved Fefferman-Phong inequality, make a Calderón-Zygmund decomposition, give estimates for positive subharmonic functions. We prove Theorem 1.1 in Section 5. We handle the proof of Theorem 1.3, item 1 in Section 6. Section 7 is concerned with the proof of Theorem 1.4. In Section 8, we give different estimates for the weak solution of $-\Delta u + Vu = 0$ and complete the proof of item 2 of Theorem 1.3. Finally, in Section 9, we apply our result on Lie groups with polynomial growth.

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2. Preliminaries

Let M denote a complete non-compact Riemannian manifold. We write ρ for the geodesic distance, μ for the Riemannian measure on M, ∇ for the Riemannian gradient, Δ for the Laplace-Beltrami operator, $|\cdot|$ for the length on the tangent space (forgetting the subscript x for simplicity) and $\|\cdot\|_p$ for the norm on $L^p(M,\mu)$, $1 \leq p \leq +\infty$.

2.1. The doubling property and Poincaré inequality

Definition 2.1 (Doubling property). Let M be a Riemannian manifold. Denote by $B(x,r)$ the open ball of center $x \in M$ and radius $r > 0$. One says that M satisfies the doubling property (D) if there exists a constant $C_d > 0$, such that for all $x \in M,\ r > 0$ we have

$$\mu(B(x,2r)) \leq C_d \mu(B(x,r)). \tag{D}$$

Lemma 2.2. *Let M be a Riemannian manifold satisfying (D) and let $s = \log_2 C_d$. Then for all $x,\ y \in M$ and $\theta \geq 1$*

$$\mu(B(x,\theta R)) \leq C\theta^{s}\mu(B(x,R)) \tag{2.1}$$

and

$$\mu(B(y,R)) \le C\left(1+\frac{d(x,y)}{R}\right)^s \mu(B(x,R)). \tag{2.2}$$

We have also the following lemma:

Lemma 2.3. *Let M be a Riemannian manifold satisfying (D). Then for $x_0 \in M$, $r_0 > 0$, we have*

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \ge 4^{-s}\left(\frac{r}{r_0}\right)^s$$

whenever $x \in B(x_0,r_0)$ and $r \le r_0$.

Theorem 2.4 (Maximal theorem [18]). *Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy-Littlewood maximal function over open balls of X defined by*

$$\mathcal{M}f(x) = \sup_{B: x\in B} |f|_B$$

where $f_E := \fint_E f\,d\mu := \frac{1}{\mu(E)}\int_E f\,d\mu$. Then

1. $\mu(\{x : \mathcal{M}f(x) > \lambda\}) \le \frac{C}{\lambda}\int_M |f|\,d\mu$ *for every* $\lambda > 0$*;*
2. $\|\mathcal{M}f\|_p \le C_p\|f\|_p$, *for* $1 < p \le \infty$.

Definition 2.5. A Riemannian manifold M is of polynomial type if there is $c, C > 0$ such that

$$c^{-1}r^d \le \mu(B(x,r)) \le cr^d \tag{LU_l}$$

for all $x \in M$ and $r \le 1$ and

$$C^{-1}r^D \le \mu(B(x,r)) \le Cr^D \tag{LU_∞}$$

for all $x \in M$ and $r \ge 1$.

We call d the local dimension and D the dimension at infinity. Note that if M is of polynomial type then it satisfies (D) with $s = \max(d,D)$. Moreover, for every $\lambda \in [\min(d,D), \max(d,D)]$,

$$\mu(B(x,r)) \ge cr^\lambda \tag{L_λ}$$

for all $x \in M$ and $r > 0$.

Definition 2.6 (Poincaré inequality). Let M be a complete Riemannian manifold, $1 \le l < \infty$. We say that M admits a Poincaré inequality (P_l) if there exists a constant $C > 0$ such that, for every function $f \in C_0^\infty(M)$, and every ball B of M of radius $r > 0$, we have

$$\left(\fint_B |f - f_B|^l\,d\mu\right)^{\frac{1}{l}} \le Cr\left(\fint_B |\nabla f|^l\,d\mu\right)^{\frac{1}{l}}. \tag{P_l}$$

Remark 2.7. Note that if (P_l) holds for all $f \in C_0^\infty$, then it holds for all $f \in W^1_{p,\mathrm{loc}}$ for $p \geq l$ (see [34], [38]).

The following result from Keith-Zhong [38] improves the exponent in the Poincaré inequality.

Lemma 2.8. *Let (X, d, μ) be a complete metric-measure space satisfying (D) and admitting a Poincaré inequality (P_l), for some $1 < l < \infty$. Then there exists $\epsilon > 0$ such that (X, d, μ) admits (P_p) for every $p > l - \epsilon$.*

Proposition 2.9. *Let M be a complete Riemannian manifold M with non-negative Ricci curvature. Then M satisfies (D) (with $C_d = 2^n$) and admits a Poincaré inequality (P_1).*

Proof. Indeed if the Ricci curvature is non-negative that is there exists $a > 0$ such that $R_{ic} \geq -a^2 g$, a result by Gromov [16] shows that

$$\mu(B(x, 2r)) \leq 2^n \mu(B(x, r)) \text{ for all } x \in M,\ r > 0.$$

Here n means the topologic dimension.

On the other hand, Buser's inequality [13] gives us

$$\int_B |u - u_B|\, d\mu \leq c(n) r \int_B |\nabla u|\, d\mu.$$

Thus we get (D) and (P_1) (see also [48]). □

2.2. Reverse Hölder classes

Definition 2.10. Let M be a Riemannian manifold. A weight w is a non-negative locally integrable function on M. The reverse Hölder classes are defined in the following way: $w \in RH_q,\ 1 < q < \infty$, if

1. $w d\mu$ is a doubling measure;
2. there exists a constant C such that for every ball $B \subset M$

$$\left(\fint_B w^q d\mu \right)^{\frac{1}{q}} \leq C \fint_B w d\mu. \tag{2.3}$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, $w d\mu$ is doubling and for any ball B,

$$w(x) \leq C \fint_B w \quad \text{for } \mu - a.e.\ x \in B. \tag{2.4}$$

On $\mathbb{R}^n$, the condition $w d\mu$ doubling is superfluous. It could be the same on a Riemannian manifold.

Proposition 2.11 ([28, 57]).

1. $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \leq q \leq \infty$.
2. *If* $w \in RH_q$, $1 < q < \infty$, *then there exists* $q < p < \infty$ *such that* $w \in RH_p$.
3. *We say that* $w \in A_p$ *for* $1 < p < \infty$ *if there is a constant* C *such that for every ball* $B \subset M$

$$\left(\fint_B w d\mu\right)\left(\fint_B w^{\frac{1}{1-p}} d\mu\right)^{p-1} \leq C.$$

For $p = 1$, $w \in A_1$ *if there is a constant* C *such that for every ball* $B \subset M$

$$\fint_B w d\mu \leq C w(y) \quad \textit{for } \ \mu - a.e. y \in B.$$

We let $A_\infty = \bigcup_{1 \leq p < \infty} A_p$. *Then* $A_\infty = \bigcup_{1 < q \leq \infty} RH_q$.

Proposition 2.12 (see [3, Section 11], [36]). *Let* V *be a non-negative measurable function. Then the following properties are equivalent:*

1. $V \in A_\infty$.
2. *For all* $r \in]0, 1[$, $V^r \in RH_{\frac{1}{r}}$.
3. *There exists* $r \in]0, 1[$, $V^r \in RH_{\frac{1}{r}}$.

We end this subsection with the following lemma:

Lemma 2.13 ([12, Lemma 1.4]). *Let* G *be an open subset of an homogeneous space* (X, d, μ) *and let* $\mathcal{F}(G)$ *be the set of metric balls contained in* G. *Suppose that for some* $0 < q < p$ *and non-negative* $f \in L^p_{\text{loc}}$, *there is a constant* $A > 1$ *and* $1 \leq \sigma_0 \leq \sigma_0'$ *such that*

$$\left(\fint_B f^p d\mu\right)^{\frac{1}{p}} \leq A \left(\fint_{\sigma_0 B} f^q d\mu\right)^{\frac{1}{q}} \quad \forall B : \sigma_0' B \in \mathcal{F}(G).$$

Then for any $0 < r < q$ *and* $1 < \sigma \leq \sigma' < \sigma_0'$, *there exists a constant* $A' > 1$ *such that*

$$\left(\fint_B f^p d\mu\right)^{\frac{1}{p}} \leq A' \left(\fint_{\sigma B} f^r d\mu\right)^{\frac{1}{r}} \quad \forall B : \sigma' B \in \mathcal{F}(G).$$

2.3. Homogeneous Sobolev spaces associated to a weight V

Definition 2.14 ([8]). Let M be a Riemannian manifold, $V \in A_\infty$. Consider for $1 \leq p < \infty$, the vector space $\dot{W}^1_{p,V}$ of distributions f such that $|\nabla f|$ and $Vf \in L^p$. It is well known that the elements of $\dot{W}^1_{p,V}$ are in L^p_{loc}. We equip $\dot{W}^1_{p,V}$ with the semi norm

$$\|f\|_{\dot{W}^1_{p,V}} = \| \, |\nabla f| \, \|_p + \|Vf\|_p.$$

In fact, this expression is a norm since $V \in A_\infty$ yields $V > 0$ $\mu - a.e.$

For $p = \infty$, we denote $\dot{W}^1_{\infty,V}$ the space of all Lipschitz functions f on M with $\|Vf\|_\infty < \infty$.

Proposition 2.15 ([8]). *Assume that M satisfies (D) and admits a Poincaré inequality (P_s) for some $1 \leq s < \infty$ and that $V \in A_\infty$. Then, for $s \leq p \leq \infty$, $\dot{W}^1_{p,V}$ is a Banach space.*

Proposition 2.16. *Under the same hypotheses as in Proposition* 2.15*, the Sobolev space $\dot{W}^1_{p,V}$ is reflexive for $s \leq p < \infty$.*

Proof. The Banach space $\dot{W}^1_{p,V}$ is isometric to a closed subspace of $L^p(M, \mathbb{R} \times T^*M)$ which is reflexive. The isometry is given by the linear operator $T : \dot{W}^1_{p,V} \to L^p(M, \mathbb{R} \times T^*M)$ such that $Tf = (Vf, \nabla f)$ by definition of the norm of $\dot{W}^1_{p,V}$ and Proposition 2.15. □

Theorem 2.17 ([8]). *Let M be a complete Riemannian manifold satisfying (D). Let $V \in RH_q$ for some $1 < q \leq \infty$ and assume that M admits a Poincaré inequality (P_l) for some $1 \leq l < q$. Then, for $1 \leq p_1 < p < p_2 \leq q$, with $p > l$, $\dot{W}^1_{p,V}$ is a real interpolation space between $\dot{W}^1_{p_1,V}$ and $\dot{W}^1_{p_2,V}$.*

3. Definition of Schrödinger operator

Let V be a non-negative, locally integrable function on M. Consider the sesquilinear form

$$\mathcal{Q}(u, v) = \int_M (\nabla u \cdot \overline{\nabla v} + V u \,\overline{v}) d\mu$$

with domain

$$\mathcal{V} = \mathcal{D}(\mathcal{Q}) = W^1_{2,V^{\frac{1}{2}}} = \{f \in L^2(M) ;\ |\nabla f| \ \&\ V^{\frac{1}{2}} f \in L^2(M)\}$$

equipped with the norm

$$\|f\|_{\mathcal{V}} = \left(\|f\|_2^2 + \|\nabla f\|_2^2 + \|V^{\frac{1}{2}} f\|_2^2\right)^{\frac{1}{2}}.$$

Clearly $\mathcal{Q}(.,.)$ is a positive, symmetric closed form. It follows that there exists a unique positive self-adjoint operator, which we call $H = -\Delta + V$, such that

$$\langle Hu, v\rangle = \mathcal{Q}(u, v) \quad \forall\, u \in \mathcal{D}(H),\ \forall\, v \in \mathcal{V}.$$

When $V = 0$, $H = -\Delta$ is the Laplace-Beltrami operator. Note that $C_0^\infty(M)$ is dense in $\mathcal{V}$ (see [8, in appendix]).

The Beurling-Deny theory holds on M, which means that $\epsilon(H + \epsilon)^{-1}$ is a positivity-preserving contraction on $L^p(M)$ for all $1 \leq p \leq \infty$ and $\epsilon > 0$. Moreover, if $V' \in L^1_{\mathrm{loc}}(M)$ such that $0 \leq V' \leq V$ and H' is the corresponding operator then one has for any $\epsilon > 0$ and for any $f \in L^p$, $1 \leq p \leq \infty$, $f \geq 0$

$$0 \leq (H + \epsilon)^{-1} f \leq (H' + \epsilon)^{-1} f.$$

It is equivalent to a pointwise comparison of the kernels of resolvents. In particular, if V is bounded from below by some positive constant $\epsilon > 0$, then H^{-1} is bounded on L^p for $1 \leq p \leq \infty$ and is dominated by $(-\Delta + \epsilon)^{-1}$ (see Ouhabaz [47]).

Let $\dot{\mathcal{V}}$ be the closure of $C_0^\infty(M)$ under the semi-norm

$$\|f\|_{\dot{\mathcal{V}}} = \big(\| |\nabla f| \|_2^2 + \|V^{\frac{1}{2}} f\|_2^2\big)^{\frac{1}{2}}.$$

Assume that M satisfies (D) and (P_2). By Fefferman-Phong inequality –Lemma 4.1 in Section 4 below–, there is a continuous inclusion $\dot{\mathcal{V}} \subset L^2_{\mathrm{loc}}$ if V is not identically 0, which is assumed from now on, hence, this is a norm. Let $f \in \dot{\mathcal{V}}'$. Then, there exists a unique $u \in \dot{\mathcal{V}}$ such that

$$\int_M \nabla u \cdot \nabla \overline{v} + V u\, \overline{v} = \langle f, v\rangle \quad \forall\, v \in C_0^\infty(M). \tag{3.1}$$

In particular, $-\Delta u + V u = f$ holds in the distributional sense. We can obtain u for a nice f by the next lemma.

Lemma 3.1. *Assume that M satisfies (D) and (P_2). Consider $f \in C_0^\infty(M) \cap L^2(M)$. For $\epsilon > 0$, let $u_\epsilon = (H + \epsilon)^{-1} f \in \mathcal{D}(H)$. Then (u_ϵ) is a bounded sequence in $\dot{\mathcal{V}}$ which converges strongly to $H^{-1} f$.*

Proof. The proof is analogous to the proof of Lemma 3.1 in [3]. □

Remark 3.2. Assume that M satisfies (D) and (P_2). The continuity of the inclusion $\dot{\mathcal{V}} \subset L^2_{\mathrm{loc}}(M)$ has two further consequences. First, we have that $L^2_{\mathrm{comp}}(M)$, the space of compactly supported L^2 functions on M, is continuously contained in $\dot{\mathcal{V}}' \cap L^2(M)$. Second, (u_ϵ) has a subsequence converging to u almost everywhere.

Finally as H is self-adjoint, it has a unique square root which we denote $H^{\frac{1}{2}}$. $H^{\frac{1}{2}}$ is defined as the unique maximal-accretive operator such that $H^{\frac{1}{2}} H^{\frac{1}{2}} = H$. We have that $H^{\frac{1}{2}}$ is self-adjoint with domain $\mathcal{V}$ and for all $u \in C_0^\infty(M)$, $\|H^{\frac{1}{2}} u\|_2^2 = \| |\nabla u| \|_2^2 + \|V^{\frac{1}{2}} u\|_2^2$. This allows us to extend $H^{\frac{1}{2}}$ from $\dot{\mathcal{V}}$ into $L^2(M)$. If S denotes this extension, then we have $S^\star S = H$ where $S^\star \colon L^2(M) \to \dot{\mathcal{V}}'$ is the adjoint of S.

4. Principal tools

We gather in these section the main tools that we need to prove our results. Some of them are of independent interest.

4.1. An improved Fefferman-Phong inequality

Lemma 4.1. *Let M be a complete Riemannian manifold satisfying (D). Let $w \in A_\infty$ and $1 \leq p < \infty$. Assume that M admits also a Poincaré inequality (P_p). Then there are constants $C > 0$ and $\beta \leq 1$ depending only on the A_∞ constant of w, p and the constants in (D), (P_p), such that for every ball B of radius $R > 0$ and $u \in W^1_{p,\mathrm{loc}}$*

$$\int_B (|\nabla u|^p + w|u|^p)d\mu \geq \frac{Cm_\beta(R^p w_B)}{R^p} \int_B |u|^p d\mu \tag{4.1}$$

where $m_\beta(x) = x$ for $x \leq 1$ and $m_\beta(x) = x^\beta$ for $x \geq 1$.

Proof. Since M admits a (P_p) Poincaré inequality, we have

$$\int_B |\nabla u|^p d\mu \geq \frac{C}{R^p \mu(B)} \int_B \int_B |u(x) - u(y)|^p d\mu(x) d\mu(y).$$

This and

$$\int_B w|u|^p d\mu = \frac{1}{\mu(B)} \int_B \int_B w(x)|u(x)|^p d\mu(x) d\mu(y)$$

lead easily to

$$\int_B (|\nabla u|^p + w|u|^p)d\mu \geq [\min(CR^{-p}, w)]_B \int_B |u|^p d\mu.$$

Now we use that $w \in A_\infty$. There exists $\varepsilon > 0$, independent of B, such that $E = \{x \in B : w(x) > \varepsilon w_B\}$ satisfies $\mu(E) > \frac{1}{2}\mu(B)$. Hence

$$[\min(CR^{-p}, w)]_B \geq \frac{1}{2} \min(CR^{-p}, \varepsilon w_B) \geq C \min(R^{-p}, w_B).$$

This proves the desired inequality when $R^p w_B \leq 1$.

Assume now $R^p w_B > 1$. We say that a ball B of radius R is of type 1 if $R^p w_B < 1$ and of type 2 if not. Take $\delta, \epsilon > 0$ such that $2\delta < \epsilon < 1$. We consider a maximal covering of $(1-\epsilon)B$ by balls $(B_i^1)_i := (B(x_i^1, \delta R))_i$ such that the balls $\frac{1}{2}B_i^1$ are pairwise disjoint. By (D) there exists N independent of δ and R such that $\sum_{i \in I} 1\!\!1_{B_i^1} \leq N$. Since $2\delta < \epsilon$, we have $B_i^1 \subset B$ for all $i \in I$. Denote G_1 the union of all balls B_i^1 of type 1 and $\widetilde{G}_1 = \{x \in M : d(x, G_1) \leq \epsilon\delta R\}$. Set $\widetilde{E}_1 = (1-\epsilon\delta)B \setminus \widetilde{G}_1$. This time we consider a maximal covering of $\widetilde{E}_1$ by balls $(B_i^2)_i := (B(x_i^2, \delta^2 R))_i$ such that the balls $\frac{1}{2}B_i^2$ are pairwise disjoint. Therefore with the same N one has $\sum_{i \in I} 1\!\!1_{B_i^2} \leq N$. Let G_2 be the union of all balls B_i^2 of type 1 and $\widetilde{G}_2 = \{x \in M : d(x, G_1 \cup G_2) \leq \epsilon\delta^2 R\}$, $\widetilde{E}_2 = (1-\epsilon\delta^2)B \setminus \widetilde{G}_1$. We iterate this process. Note that the G_j's are pairwise disjoint (from $2\delta < \epsilon$). We claim then that $\mu(\setminus \bigcup_j G_j) = 0$. Indeed, for almost $x \in B$, $w_{B'}$ converges to $w(x)$ whenever $r(B') \to 0$ and $x \in B'$. Take such an x and assume that $x \notin \bigcup_j G_j$. Then, for every j there exists x_k^j such that $x \in B(x_k^j, \delta^j R)$ and $(\delta^j R)^p w_{B(x_k^j, \delta^j R)} \geq 1$. This is a contradiction since $(\delta^j R)^p w_{B(x_k^j, \delta^j R)} \to 0$ when $j \to \infty$. Note also that there exists $0 < A < 1$ such that for all j, k and ball B_k^j of type 1,

$$(\delta^j R)^p w_{B_k^j} > A. \tag{4.2}$$

Indeed, let B_k^j be of type 1. There exists B_l^{j-1} such that $x_k^j \in B_l^{j-1}$ and B_l^{j-1} must be of type 2 because $x_k^j \notin G_{j-1}$. Hence $B_k^j \subset B(x_l^{j-1}, \delta^j(1+\delta^{-1})R)$. Since $w d\mu$ is doubling, we get

$$\begin{aligned}
w(B_l^{j-1}) &\leq w\left(B(x_l^{j-1}, \delta^j(1+\delta^{-1})R)\right) \\
&\leq C'(1+\delta^{-1})^{s'} w\left(B(x_l^{j-1}, \delta^j R)\right) \\
&\leq C'^2(1+\delta^{-1})^{s'} \left(1 + \frac{d(x_l^{j-1}, x_k^j)}{\delta^j R}\right)^{s'} w(B_k^j) \\
&\leq C'^2(1+\delta^{-1})^{2s'} w(B_k^j)
\end{aligned}$$

where $s' = \log_2 C'$ and C' is the doubling constant of $w d\mu$. On the other hand, since $d\mu$ is doubling

$$\begin{aligned}
\mu(B_l^{j-1}) &\geq C^{-1}(1+\delta)^{-s}\mu(B(x_l^{j-1}, \delta^{j-1}(1+\delta)R)) \\
&\geq C^{-1}(1+\delta)^{-s}\mu(B_k^j).
\end{aligned}$$

Since B_l^{j-1} is of type 2, we obtain

$$\begin{aligned}
(\delta^j R)^p w_{B_k^j} &\geq C'^{-2}C^{-1}(1+\delta^{-1})^{-2s'}(1+\delta)^{-s}\delta^p(\delta^{j-1}R)^p w(B_l^{j-1}) \\
&> C'^{-2}C^{-1}(1+\delta^{-1})^{-2s'}(1+\delta)^{-s}\delta^p.
\end{aligned}$$

Thus we get (4.2) with $A = C'^{-2}C^{-1}(1+\delta^{-1})^{-2s'}(1+\delta)^{-s}\delta^p$. From all these facts we deduce that

$$\begin{aligned}\int_B (|\nabla u|^p + w|u|^p)d\mu &\geq \frac{1}{N} \sum_{j,\,k: B_k^j \text{ of type } 1} \int_{B_k^j} (|\nabla u|^p + w|u|^p)d\mu \\ &\geq C\frac{1}{N} \sum_{j,\,k: B_k^j \text{ of type } 1} \min((\delta^j R)^{-p}, w_{B_k^j}) \int_{B_k^j} |u|^p d\mu \\ &\geq \frac{C}{N} A \sum_{j,\,k: B_k^j \text{ of type } 1} (\delta^j R)^{-p} \int_{B_k^j} |u|^p d\mu \\ &\geq \frac{C}{N} A \min_j \left(\frac{R}{\delta^j R}\right)^p R^{-p} \int_B |u|^p d\mu.\end{aligned}$$

We used Fefferman-Phong inequality in the second estimate, (4.2) in the penultimate one, and that the B_k^j of type 1 cover B up to a μ-null set in the last one. It remains to estimate $\min_j \left(\frac{R}{R_j}\right)^p$ from below with $R_j = \delta^j R$. Let $1 \leq \alpha < \infty$ be such that $w \in A_\alpha$ (the Muckenhoupt class). Then for any ball B and measurable subset E of B we have

$$\left(\frac{w_E}{w_B}\right) \geq C \left(\frac{\mu(E)}{\mu(B)}\right)^{\alpha-1}.$$

Applying this to $E = B_k^j$ and B we obtain

$$\begin{aligned}\left(\frac{R}{R_j}\right)^p &= \frac{R^p w_B}{R_j^p w_{B_k^j}} \frac{w_{B_k^j}}{w_B} \geq R^p w_B \frac{w_{B_k^j}}{w_B} \\ &\geq C R^p w_B \left(\frac{\mu(B_k^j)}{\mu(B)}\right)^{\alpha-1} \geq C R^p w_B \left(\frac{R_j}{R}\right)^{s(\alpha-1)}\end{aligned}$$

where we used Lemma 2.3. This yields $\min_j (\frac{R}{R_j})^p \geq C(R^p w_B)^\beta$ with $\beta = \frac{p}{p+s(\alpha-1)}$. The lemma is proved. □

4.2. Calderón-Zygmund decomposition

We now proceed to establish the following Calderón-Zygmund decomposition:

Proposition 4.2. *Let M be a complete Riemannian manifold satisfying (D) and (P_l) for some $1 \leq l < 2$. Let $l \leq p < 2$, $V \in A_\infty$, $f \in \dot{W}^1_{p,V^{\frac{1}{2}}}$ and $\alpha > 0$. Then, one can find a collection of balls (B_i), functions g and b_i satisfying the following* *properties:*

$$f = g + \sum_i b_i \tag{4.3}$$

$$\| |\nabla g| \|_2 + \|V^{\frac{1}{2}} g\|_2 \leq C\alpha^{1-\frac{p}{2}} \big(\| |\nabla f| \|_p + \|V^{\frac{1}{2}} f\|_p\big)^{\frac{1}{2}}, \tag{4.4}$$

$$\operatorname{supp} b_i \subset B_i \text{ and } \int_{B_i} \big(|\nabla b_i|^l + |V^{\frac{1}{2}} b_i|^l + R_i^{-l} |b_i|^l\big) d\mu \leq C\alpha^l \mu(B_i), \tag{4.5}$$

$$\sum_i \mu(B_i) \leq C\alpha^{-p} \int_M (|\nabla f|^p + |V^{\frac{1}{2}} f|^p) d\mu, \tag{4.6}$$

$$\sum_i \mathbb{1}_{B_i} \leq N, \tag{4.7}$$

where N depends only on the doubling constant, and C on the doubling constant, p, l and the A_∞ constant of V. Here, R_i denotes the radius of B_i and gradients are taken in the distributional sense on M.

Remark 4.3. It follows from the proof of Proposition 4.2 that the function g is Lipschitz with the Lipschitz constant controlled by $C\alpha$ (see page 741 below).

Proof of Proposition 4.2. Let Ω be the open set $\{x \in M; \mathcal{M}(|\nabla f|^l + |V^{\frac{1}{2}} f|^l)(x) > \alpha^l\}$. If Ω is empty, then set $g = f$ and $b_i = 0$. Otherwise, the maximal theorem (Theorem 2.4) yields

$$\mu(\Omega) \leq C\alpha^{-p} \int_M (|\nabla f|^p + |V^{\frac{1}{2}} f|^p) d\mu < \infty. \tag{4.8}$$

In particular $\Omega \neq M$ as $\mu(M) = \infty$. Let F be the complement of Ω. Since Ω is an open set distinct of M, let $(\underline{B_i})$ be a Whitney decomposition of Ω [19]. That is, the balls $\underline{B_i}$ are pairwise disjoint and there is two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \bigcup_i B_i$ with $B_i = C_1 \underline{B_i}$ are contained in Ω and the balls $(B_i)_i$ have the bounded overlap property;
2. $r_i = r(B_i) = \frac{1}{2} d(x_i, F)$ and x_i is the center of B_i;
3. each ball $\overline{B_i} = C_2 \underline{B_i}$ intersects F ($C_2 = 4C_1$ works).

For $x \in \Omega$, denote $I_x = \{i : x \in B_i\}$. By the bounded overlap property of the balls B_i, we have that $\sharp I_x \leq N$. Fixing $j \in I_x$ and using the properties of the B_i's, we easily see that $\frac{1}{3} r_i \leq r_j \leq 3r_i$ for all $i \in I_x$. In particular, $B_i \subset 7B_j$ for all $i \in I_x$.

Condition (4.7) is nothing but the bounded overlap property of the B_i's and (4.6) follows from (4.7) and (4.8). We remark that since $V \in A_\infty$, Proposition 2.12 and Proposition 2.11, item 3 yield $V^{\frac{l}{2}} \in A_\infty$. Applying Lemma 4.1, we obtain

$$\int_{B_i} (|\nabla f|^l + |V^{\frac{1}{2}} f|^l) d\mu \geq C \min((V^{\frac{l}{2}})_{B_i}, R_i^{-l}) \int_{B_i} |f|^l d\mu. \tag{4.9}$$

We declare B_i of type 1 if $(V^{\frac{l}{2}})_{B_i} \geq R_i^{-l}$ and of type 2 if $(V^{\frac{l}{2}})_{B_i} < R_i^{-l}$.

Let us now define the functions b_i. Let (χ_i) be a partition of unity on Ω subordinated to the covering (B_i). Each χ_i is a C^1 function supported in B_i with $\|\chi_i\|_\infty + R_i \| \, |\nabla \chi_i| \, \|_\infty \leq C$. Set

$$b_i = \begin{cases} f\chi_i, & \text{if } B_i \text{ is of type 1,} \\ (f - f_{B_i})\chi_i, & \text{if } B_i \text{ is of type 2.} \end{cases}$$

If B_i is of type 2, then it is a direct consequence of the Poincaré inequality (P_l) that

$$\int_{B_i} (|\nabla b_i|^l + R_i^{-l} |b_i|^l) d\mu \leq C \int_{B_i} |\nabla f|^l d\mu.$$

As $\int_{\overline{B_i}} |\nabla f|^l d\mu \leq \alpha^l \mu(\overline{B_i})$ we get the desired inequality in (4.5). For $V^{\frac{1}{2}} b_i$ we have

$$\begin{aligned}
\int_{B_i} |V^{\frac{1}{2}} b_i|^l d\mu &= \int_{B_i} |V^{\frac{1}{2}} (f - f_{B_i})\chi_i|^l d\mu \\
&\leq C \left(\int_{B_i} |V^{\frac{1}{2}} f|^l d\mu + \int_{B_i} |V^{\frac{1}{2}} f_{B_i}|^l d\mu \right) \\
&\leq C \left((|V^{\frac{1}{2}} f|^l)_{B_i} \mu(B_i) + C (V^{\frac{l}{2}})_{B_i} (|f|^l)_{B_i} \mu(B_i) \right) \\
&\leq C \left(\alpha^l \mu(B_i) + \left(|\nabla f|^l + |V^{\frac{1}{2}} f|^l \right)_{B_i} \mu(B_i) \right) \\
&\leq C \alpha^l \mu(B_i).
\end{aligned}$$

We used that $\overline{B_i} \cap F \neq \emptyset$, Jensen's inequality and (4.9), noting that B_i is of type 2. If $\underline{B_i}$ is of type 1,

$$\int_{B_i} R_i^{-l} |b_i|^l d\mu \leq \int_{B_i} R_i^{-l} |f|^l \leq C \int_{B_i} (|\nabla f|^l + |V^{\frac{1}{2}} f|^l) d\mu.$$

As the same integral but on $\overline{B_i}$ is controlled by $\alpha^l \mu(\overline{B_i})$ we get $\int_{B_i} R_i^{-l} |b_i|^l d\mu \leq C\alpha^l \mu(B_i)$. Since $\nabla b_i = \chi_i \nabla f + f \nabla \chi_i$ we obtain the same bound for $\int_{B_i} |\nabla b_i|^l d\mu$.

Noting that $\overline{B_i} \cap F \neq \emptyset$ and B_i is of type 1, we easily deduce that $\int_{B_i} |V^{\frac{1}{2}} b_i|^l \leq C\alpha^l \mu(B_i)$.

Set $g = f - \sum b_i$ where the sum is over both types of balls and is locally finite by (4.7). It is clear that $g = f$ on $F = M \setminus \Omega$ and $g = \sum^2 f_{B_i} \chi_i$ on Ω, where $\sum^j$ means that we are summing over cubes of type j. Let us prove (4.3).

First, by the differentiation theorem, $V^{\frac{1}{2}}|f| \leq \alpha$ almost everywhere on F. Next, as we explained before, $V \in A_\infty$ implies $V^{\frac{l}{2}} \in RH_{\frac{2}{l}}$ and therefore $V_{B_i} \leq C((V^{\frac{l}{2}})_{B_i})^{\frac{2}{l}}$. Therefore

$$\int_\Omega V|g|^2 d\mu \leq \sum{}^2 \int_{B_i} V|f_{B_i}|^2 \leq C \sum{}^2 \left((V^{\frac{l}{2}})_{B_i})|f_{B_i}|^l\right)^{\frac{2}{l}} \mu(B_i).$$

Now, by construction of the type 2 balls and the L^l version of Fefferman-Phong inequality,

$$(V^{\frac{l}{2}})_{B_i}|f_{B_i}|^l \leq C(|\nabla f|^l + |V^{\frac{1}{2}} f|^l)_{B_i} \leq C\alpha^l.$$

It comes that

$$\int_\Omega V|g|^2 d\mu \leq C \sum{}^2 \alpha^{2-l}\mu(B_i) \leq C'\alpha^{2-l} \int_M (|\nabla f|^l + |V^{\frac{1}{2}} f|^l) d\mu.$$

Combining the estimates on F and Ω, we obtain the desired bound for $\int_M V|g|^2 d\mu$. We finish the proof by estimating $\| \, |\nabla g| \, \|_\infty$ and $\| \, |\nabla g| \, \|_l$. Observe that g is a locally integrable function on M. Indeed, let $\varphi \in L_\infty$ with compact support. Since $d(x, F) \geq R_i$ for $x \in \operatorname{supp} \ b_i$, we obtain

$$\int \sum_i |b_i| \, |\varphi| \, d\mu \leq \left(\int \sum_i \frac{|b_i|}{R_i} \, d\mu\right) \sup_{x \in M} \Big(d(x, F)|\varphi(x)|\Big).$$

If B_i is of type 2

$$\begin{aligned} \int \frac{|b_i|}{R_i} d\mu \leq \mu(B_i)^{1-\frac{1}{l}} \int \frac{|b_i|^l}{R_i^l} d\mu \leq C\mu(B_i)^{1-\frac{1}{l}} \int_{B_i} |\nabla f|^l d\mu \\ \leq C\alpha\mu(B_i). \end{aligned}$$

We used the Hölder inequality, (P_l) and that $\overline{B_i} \cap F \neq \emptyset$, q' being the conjugate of q.

If B_i is of type 1,

$$\int \frac{|b_i|}{R_i} d\mu \leq \mu(B_i)^{1-\frac{1}{l}} \int \frac{|b_i|^l}{R_i^l} d\mu \leq C\alpha\mu(B_i).$$

Hence $\int \sum_i |b_i||\varphi| d\mu \leq C\alpha\mu(\Omega)^{\frac{1}{l}} \sup_{x\in M}\Big(d(x,F)|\varphi(x)|\Big)$. Since $f \in L^1_{\mathrm{loc}}$, we conclude that $g \in L^1_{\mathrm{loc}}$. Thus $\nabla g = \nabla f - \sum \nabla b_i$. It follows from the L^l estimates on ∇b_i and the bounded overlap property that

$$\Big\| \sum |\nabla b_i| \Big\|_l \leq C'(\| \, |\nabla f| \, \|_l + \|V^{\frac{1}{2}} f\|_l).$$

As $g = f - \sum b_i$, the same estimate holds for $\| \, |\nabla g| \, \|_l$. Next, a computation of the sum $\sum \nabla b_i$ leads us to

$$\nabla g = 1\!\!1_F(\nabla f) - \sum{}^1 f \nabla \chi_i - \sum{}^2 (f - f_{B_i}) \nabla \chi_i.$$

Set $h_i = \sum^i (f - f_{B_i}) \nabla \chi_i$ and $h = h_1 + h_2$. Then

$$\nabla g = (\nabla f) 1\!\!1_F - \sum{}^1 f \nabla \chi_i - (h - h_1) = (\nabla f) 1\!\!1_F + \sum{}^1 f_{B_i} \nabla \chi_i - h.$$

By definition of F and the differentiation theorem, $|\nabla g|$ is bounded by α almost everywhere on F. By already seen arguments for type 1 balls, $|f_{B_i}| \leq C\alpha R_i$. Therefore, $|\sum^1 f_{B_i} \nabla \chi_i| \leq C \sum^1 1\!\!1_{B_i} \alpha \leq CN\alpha$. It remains to control $\|h\|_\infty$. For this, note first that h vanishes on F and the sum defining h is locally finite on Ω. Then fix $x \in \Omega$. Observe that $\sum_i \nabla \chi_i(x) = 0$ and by definition of I_x, the sum reduces $i \in I_x$. For all $i \in I_x$, we have $|f(x) - f_{B_i}| \leq C r_i \alpha$. Hence, we have for all $j \in I_x$,

$$\sum_i (f(x) - f_{B_i}) \nabla \chi_i(x) = \sum_{i \in I_x} (f(x) - f_{B_i}) \nabla \chi_i(x) = \sum_{i \in I_x} (f_{B_j} - f_{B_i}) \nabla \chi_i(x).$$

We claim that $|f_{B_j} - f_{B_i}| \leq C r_j \alpha$ with C independent of $i, j \in I_x$ and $x \in \Omega$. Indeed, we use that B_i and B_j are contained in $7B_j$, Poincaré inequality (P_l), the comparability of r_i and r_j, and that $\overline{B_i} \cap F \neq \emptyset$. Since I_x has cardinal bounded by N, we are done.

We conclude that $\|h\|_\infty \leq C\alpha$ and interpolating $\| \, |\nabla g| \, \|_l$ and $\| \, |\nabla g| \, \|_\infty$, we therefore finish the proof. □

Proposition 4.4. *Let M be a complete Riemannian manifold satisfying (D). Let $V \in A_\infty$. Moreover assume that M admits a Poincaré inequality (P_p) for some $1 < p < 2$. Then,* $\mathrm{Lip}(M) \cap \dot{W}^1_{2,V^{\frac{1}{2}}} \cap \dot{W}^1_{p,V^{\frac{1}{2}}}$ [1] *is dense in* $\dot{W}^1_{p,V^{\frac{1}{2}}}$.

Proof. Theorem 2.8 proves that M admits a Poincaré inequality (P_l) for some $1 \leq l < p$. Let $f \in \dot{W}^1_{p,V^{\frac{1}{2}}}$. For every $n \in \mathbb{N}^*$, consider the Calderón-Zygmund

[1] $\mathrm{Lip}(M)$ is the set of all Lipschitz functions on M.

decomposition of Proposition 4.2 with $\alpha = n$. Take a compact K of M. We have

$$\begin{aligned}\int_K |f-g_n|^l d\mu &= \int_{K\cap(\bigcup_i B_i)} |\sum_i b_i|^l d\mu \\ &= \int_{\bigcup_i K\cap B_i} |\sum_i b_i|^l d\mu \\ &\leq C\sum{}^2 \int_{K\cap B_i} \frac{|f-f_{B_i}|^l}{R_i^l} d(x,F_n)^l d\mu \\ &\quad + C\sum{}^1 \int_{K\cap B_i} \frac{|f|^l}{R_i^l} d(x,F_n)^l d\mu \\ &\leq C \sup_{x\in K}(d(x,F_n))^l \sum_i \int_{B_i} (|\nabla f|^l + |V^{\frac{1}{2}} f|^l) d\mu \\ &\leq C \sup_{x\in K}(d(x,F_1))^l \sum_i n^l \mu(B_i) \\ &\leq Cn^{l-p}(\| \, |\nabla f| \, \|_p^p + \| \, |V^{\frac{1}{2}} f| \, \|_p^p).\end{aligned}$$

Letting $n\to\infty$, we get that $\int_K |f-g_n|^l d\mu \to 0$. Hence $(f-g_n)$ converges to 0 when $n\to\infty$ in the distributional sense.
Let us check that $(V^{\frac{1}{2}}(f-g_n))_n$ is bounded in L^p. Indeed,

$$\begin{aligned}\int_M |V^{\frac{1}{2}}(f-g_n)|^p d\mu &\leq \int_{\Omega_n} |V^{\frac{1}{2}} f|^p d\mu + \sum{}^2 \int_{\Omega_n} V^{\frac{p}{2}} |f_{B_i}|^p d\mu \\ &\leq \int_{\Omega_n} |V^{\frac{1}{2}} f|^p d\mu + \sum{}^2 \left((V^{\frac{l}{2}})_{B_i} |f_{B_i}|^l\right)^{\frac{p}{l}} \mu(B_i) \\ &\leq \int_{\Omega_n} |V^{\frac{1}{2}} f|^p d\mu + Cn^p \mu(\Omega_n) \\ &\leq C(\| \, |\nabla f| \, \|_p^p + \|V^{\frac{1}{2}} f\|_p^p).\end{aligned}$$

Similarly,

$$\int_M |\nabla f - \nabla g_n|^p d\mu = \int_{\Omega_n} |\nabla f - \nabla g_n|^p d\mu \leq C\int_{\Omega_n} |\nabla f|^p d\mu + Cn^p \mu(\Omega_n) \leq C.$$

Thus, $(\nabla f - \nabla g_n)_n$ is bounded in L^p. So $(f-g_n)_n$ is bounded in $\dot{W}^1_{p,V^{\frac{1}{2}}}$. Since $\dot{W}^1_{p,V^{\frac{1}{2}}}$ is reflexive –Proposition 2.16–, there exists a subsequence, which we denote also by $(f-g_n)_n$, converging weakly in $\dot{W}^1_{p,V^{\frac{1}{2}}}$ to a function h. The uniqueness of the limit in the distributional sense yields $h=0$. By Mazur's Lemma, we find a sequence (h_n) of convex combinations of $(f-g_n)$ such that $h_n = \sum_{k=1}^n a_{n,k}(f-$

$g_k)$, $a_{n,k} \geq 0$, $\sum_{k=1}^{n} a_{n,k} = 1$, that converges to 0 in $\dot{W}^1_{p,V^{\frac{1}{2}}}$. Since $\nabla h_n = \nabla f - \nabla l_n$ and $V^{\frac{1}{2}} h_n = V^{\frac{1}{2}}(f - l_n)$ with $l_n = \sum_{k=1}^{n} a_{n,k} g_k$, we obtain $l_n \underset{n\to\infty}{\longrightarrow} f$ in $\dot{W}^1_{p,V^{\frac{1}{2}}}$ and the proposition follows on noting that g_n, hence l_n, also belongs to $\mathrm{Lip}(M) \cap \dot{W}^1_{2,V^{\frac{1}{2}}}$. □

4.3. Estimates for subharmonic functions

Fix an open set $\Omega \subset M$. A subharmonic function on Ω is a function $v \in L^1_{\mathrm{loc}}(\Omega)$ such that $\Delta v \geq 0$ in $D'(\Omega)$.

Lemma 4.5. *Let M be a Riemannian manifold satisfying (D) and (P_2). Let $R > 0$ and x_0 be a point such that a neighborhood of $\overline{B(x_0, 4R)}$ is contained in M. Suppose that f is a non-negative subharmonic function defined on this neighborhood. Then, there is a constant $C > 0$ independent of f, x_0, R such that*

$$\sup_{x\in B(x_0,R)} f(x) \leq C \left(\fint_{B(x_0,4R)} f^2(y)\,d\mu(y) \right)^{\frac{1}{2}}. \tag{4.10}$$

It readily follows from Lemma 2.13 *that for all $r > 0$, $1 < \eta < 4$, there is $C > 0$ such that*

$$\sup_{x\in B(x_0,R)} f(x) \leq C \left(\fint_{B(x_0,\eta R)} f^r(y)\,d\mu(y) \right)^{\frac{1}{r}}. \tag{4.11}$$

Proof. In [44, Theorem 7.1], this lemma is stated for Riemannian manifolds with non-negative Ricci curvature. The proof relies on the following properties of the manifold. First, the Harnack inequality for non-negative harmonic functions which holds for complete Riemannian manifolds satisfying (D) and (P_2) (see [29]). Secondly, the Poincaré inequality (P_2). Finally, the Caccioppoli inequality for non-negative subharmonic functions in [44, Lemma 7.1] which is valid on any complete Riemannian manifold. We then get this lemma under the hypotheses (D) and (P_2). □

Other forms of the mean value inequality for subharmonic functions still hold if the volume form is replaced by a weighted measure of Muckenhoupt type. More precisely:

Lemma 4.6. *Consider a complete Riemannian manifold M satisfying (D) and (P_2). Let $V \in A_\infty$ and f a non-negative subharmonic function defined on a neighborhood of $\overline{B(x_0, 4R)}$, $0 < s < \infty$ and $1 < \eta < 4$. Then for some C depending on the A_∞ constant of V, s (and independent of f and x_0, R), we have*

$$\sup_{x\in B(x_0,R)} f(x) \leq \left(\frac{C}{V(B(x_0,\eta R))} \int_{B(x_0,\eta R)} V f^s\,d\mu \right)^{\frac{1}{s}}.$$

Here $V(E) = \int_E V d\mu$. As A_∞ weights have the doubling property we have $V_{B(x_0,\eta R)} \sim V_{B(x_0,R)}$ and the inequality above is the same as

$$V_{B(x_0,R)}(\sup_{B(x_0,R)} f^s) \le C(Vf^s)_{B(x_0,\eta R)}. \tag{4.12}$$

Proof. Since $V \in A_\infty$, there is $t < \infty$ such that $V \in A_t$. Hence for any non-negative measurable function g we have

$$\begin{aligned} g_{B(x_0,\eta R)} &\le C\left(\frac{1}{V(B(x_0,\eta R))}\int_{B(x_0,\eta R)} V g^t d\mu\right)^{\frac{1}{t}} \\ &= C\left((Vg^t)_{B(x_0,\eta R)}\right)^{\frac{1}{t}}\left(V_{B(x_0,\eta R)}\right)^{-\frac{1}{t}}. \end{aligned}$$

Applying (4.11) with $r = \frac{s}{t}$ yields

$$f(x) \le C\left((f^{\frac{s}{t}})_{B(x_0,\eta R)}\right)^{\frac{t}{s}} \le C\left((Vf^s)_{B(x_0,\eta R)}\right)^{\frac{1}{s}}\left(V_{B(x_0,\eta R)}\right)^{-\frac{1}{s}}.$$

□

Corollary 4.7. *Let M be a complete Riemannian manifold satisfying (D) and (P_2). Let $V \in RH_r$ for some $1 < r \le \infty$, $0 < s < \infty$ and $1 < \eta \le 4$. Then there is $C \ge 0$ depending only on the RH_r constant of V, s such that for any ball $B(x_0,R)$ and any non-negative subharmonic function defined on a neighborhood of $\overline{B(x_0,4R)}$ we have*

$$\left(((Vf^s)^r)_{B(x_0,R)}\right)^{\frac{1}{r}} \le C(Vf^s)_{B(x_0,\eta R)}.$$

Proof. We have

$$\begin{aligned} \left(((Vf^s)^r)_{B(x_0,R)}\right)^{\frac{1}{r}} &\le C\left((V^r)_{B(x_0,R)}\right)^{\frac{1}{r}} \sup_{B(x_0,R)} f^s \\ &\le C V_{B(x_0,R)} \sup_{B(x_0,R)} f^s \le C(Vf^s)_{B(x_0,\eta R)}. \end{aligned}$$

The second inequality uses the RH_r condition on V and the last inequality is (4.12). □

5. Maximal inequalities

This section is devoted to the proof of Theorem 1.1. Let $1 < q \le \infty$ and $V \in RH_q$. The following lemma is classical in an Euclidean setting [27,37] (see also [3]).

Lemma 5.1. *Let M be a complete Riemannian manifold. We assume that $V \in L^1_{\mathrm{loc}}(M)$ is not identically 0.*

Let $u \in C_0^\infty(M)$. Then

$$\int_M V|u|d\mu \leq \int_M |(-\Delta + V)u|d\mu,$$

$$\int_M |\Delta u|d\mu \leq 2\int_M |(-\Delta + V)u|d\mu.$$

Proof. Let us prove the estimate for $V|u|$. Take $p_n : \mathbb{R} \to \mathbb{R}$ a sequence of C^1 functions such that $|p_n| \leq C$, $p_n'(t) \geq 0$ and $p_n(t) \to \operatorname{sign}(t)$ for every $t \in \mathbb{R}$. Using the Lebesgue convergence theorem we see that

$$-\int_M \operatorname{sign}(u)\Delta u d\mu = -\lim_n \int_M p_n(u)\Delta u d\mu = \lim_n \int_M |\nabla u|^2 p_n'(u) d\mu \geq 0.$$

If $-\Delta u + Vu = f$, we get

$$\int_M V|u|d\mu \leq \int_M \operatorname{sign}(u)(-\Delta + V)u d\mu = \int_M f \operatorname{sign}(u) d\mu \leq \int_M |f|d\mu.$$

This gives the desired estimation for $V|u|$.

The estimate for Δu follows from that of Vu since $-\Delta u + Vu = f$. □

Let $\mathcal{D}_1(H) = \{u \in L^1_{\mathrm{loc}}\,;\ Vu \in L^1_{\mathrm{loc}},\ (-\Delta + V)u \in L^1\}$. One can easily check that C_0^∞ is dense in $\mathcal{D}_1(H)$ [14, for a proof in the Euclidean parabolic case] thanks to the Kato inequality on manifolds [11, Theorem 5.6]. Thus the above estimates for $\int V|u|$ and $\int |\Delta u|$ still holds for any $u \in \mathcal{D}_1(H)$. Lemma 5.1 shows that $\mathcal{D}_1(H) = \{u \in L^1_{\mathrm{loc}}\,;\ \Delta u \in L^1, Vu \in L^1\}$ equipped with the topology defined by the semi-norms for L^1_{loc}, $\|\Delta u\|_1$ and $\|Vu\|_1$. We have therefore obtained:

Theorem 5.2. *The operator H^{-1} a priori defined on $L_0^\infty(M)$ –the set of compactly supported bounded functions defined on M– extends to a bounded operator from $L^1(M)$ into $\mathcal{D}_1(H)$. Denoting again H^{-1} this extension, VH^{-1} is a positivity-preserving contraction on $L^1(M)$ and $\frac{1}{2}\Delta H^{-1}$ is a contraction on $L^1(M)$.*

Proposition 5.3. *Assume that M satisfies (D) and (P_2). Let $f \in L^1(M)$. Then there is unique of solution of the equation $-\Delta u + Vu = f$ in the class $L^1(M) \cap \mathcal{D}_1(H)$. In particular, if $u \in C_0^\infty(M)$ and $f = -\Delta u + Vu$, then $u = H^{-1}f$.*

Proof. Assume $-\Delta u + Vu = 0$, then for $\epsilon > 0$ we have $-\Delta u + Vu + \epsilon u = \epsilon u$. As $u \in L^1(M)$, we can write $|u| \leq (-\Delta + \epsilon)^{-1}(\epsilon|u|) = (-\epsilon^{-1}\Delta + 1)^{-1}|u|$. Using the upper bound of the kernel of $(-\epsilon^{-1}\Delta + 1)^{-1}$ which follows from (D) and (P_2), and taking limits when $\epsilon \to 0$ we get $u = 0$. □

Corollary 5.4. *Assume (D) and (P_2). Then equation* (1.2) holds.

Proof. If $u \in C_0^\infty(M)$ and $f = -\Delta u + Vu$, then $Vu = VH^{-1}f$ and $\Delta u = \Delta H^{-1}f$ by the proposition above. Applying Theorem 5.2 we get $\|Vu\|_1 \leq \| -\Delta u + Vu\|_1$ and $\|\Delta u\|_1 \leq 2\| -\Delta u + Vu\|_1$. $\square$

We now give the following criterion for L^p boundedness:

Theorem 5.5 ([7]). *Let M be a complete Riemannian manifold satisfying (D). Let $1 \leq r_0 < q_0 \leq \infty$. Suppose that T is a bounded sublinear operator on $L^{r_0}(M)$. Assume that there exist constants $\alpha_2 > \alpha_1 > 1$, $C > 0$ such that*

$$\Big(\fint_B |Tf|^{q_0}\Big)^{\frac{1}{q_0}} \leq C \left\{ \Big(\fint_{\alpha_1 B} |Tf|^{r_0}\Big)^{\frac{1}{r_0}} + (S|f|)(x) \right\}, \tag{5.1}$$

for any ball B, $x \in B$ and all $f \in L_0^\infty(M)$ with support in $M \setminus \alpha_2 B$, where S is a positive operator. Let $r_0 < p < q_0$. If S is bounded on $L^p(M)$, then, there is a constant C such that

$$\|Tf\|_p \leq C\,\|f\|_p$$

for all $f \in L_0^\infty(M)$.

Note that the space $L_0^\infty(M)$ can be replaced by $C_0^\infty(M)$.
Now we use the L^1 estimate (1.2) and Theorem 5.5 to get

Theorem 5.6. *Let M be a complete Riemannian manifold satisfying (D) and (P_2). Consider $V \in RH_q$, with $q > 1$. Then, there exists $r > q$, such that VH^{-1} and ΔH^{-1} defined on $L^1(M)$ by Theorem 5.2 extend to $L^p(M)$ bounded operators for all $1 < p < r$.*

Proof. By difference, it suffices to prove the theorem for VH^{-1}. We know that this is a bounded operator on $L^1(M)$. Let r be given by the self-improvement of the reverse Hölder condition of V. Fix a ball B and let $f \in L^\infty(M)$ with compact support contained in $M \setminus 4B$. Then $u = H^{-1}f$ is well-defined in $\dot{\mathcal{V}}$ and is a weak solution of $-\Delta u + Vu = 0$ in $4B$. Since $|u|^2$ is subharmonic (*cf.* Section 8.1), we can apply Corollary 4.7 with V, $f = |u|^2$ and $s = \frac{1}{2}$. Thus (5.1) holds with $T = VH^{-1}$, $r_0 = 1$, $q_0 = r$, $S = 0$, $\alpha_1 = 2$ and $\alpha_2 = 4$. Hence, Theorem 5.5 asserts that $T = VH^{-1}$ is bounded on $L^p(M)$ for $1 < p < r$. $\square$

Proof of Theorem 1.1 . Let $u \in C_0^\infty(M)$ and $f = -\Delta u + Vu$. Proposition 5.3 shows that $u = H^{-1}f$. Since $V \in RH_q$, Theorem 5.6 shows that VH^{-1} and ΔH^{-1} have bounded extensions on $L^p(M)$ for $1 < p < q + \epsilon$ for some $\epsilon > 0$ depending on V. This means that $\|Vu\|_p + \|\Delta u\|_p \lesssim \|f\|_p$ which is the desired result. $\square$

6. Complex interpolation

We shall use complex interpolation to obtain item 1 of Theorem 1.3. This method is based on the boundedness of imaginary powers of H and of the Laplace-Beltrami operator. Then we use Stein's interpolation theorem to prove the boundedness of $\nabla H^{-\frac{1}{2}}$ on $1 < p < \inf(p_0,\, 2(q+\epsilon))$ and $V^{\frac{1}{2}} H^{-\frac{1}{2}}$ on $L^p(M)$ for $1 < p < 2(q+\epsilon)$ and therefore obtain item 1 of Theorem 1.3.

Let $y \in \mathbb{R}$, the operator H^{iy} is defined via spectral theory. One has

$$\|H^{iy}\|_{2\to 2} = 1.$$

Theorem 6.1. *Let M be a complete Riemannian manifold satisfying (D) and assume that the heat kernel verifies the following upper bound: for all $x \in M$ and $t > 0$*

$$p_t(x,x) \le \frac{C}{\mu(B(x,\sqrt{t}))}. \tag{6.1}$$

Let V be a non-negative locally integrable function on M. Then for all $\gamma \in \mathbb{R}$, $H^{i\gamma}$ has a bounded extension on $L^p(M)$, $1 < p < \infty$, and for fixed p its operator norm does not exceed $C(\delta, p)e^{\delta|\gamma|}$ for some $\delta > 0$.

Remark 6.2. The operator norm is far from optimal but sufficient for us. Nevertheless, as pointed out by the referee, the operator norm can be improved to $C(1+|\gamma|)^{s/2}$ where $s = \log_2 C_d$. This can be checked by a careful proof reading of [53, Theorem 1]. Also a stronger result can be found in [24].

Proof of Theorem 6.1. For $V = 0$, this follows from the universal multiplier theorem for Markovien semi groups [56, Corollary 4, page 121]. However, the following proof works for all V. Indeed, the remark after [26, Theorem 3.1] applies to H: H has a bounded holomorphic functional calculus on $L^2(M)$ in any sector $|\arg z| < \theta$, $0 < \theta < \pi$ and the kernel $h_t(x,y)$ of e^{-tH} has a Gaussian upper bound. This follows from the domination of e^{-tH} by $e^{-t\Delta}$, (D) and (6.1). We have

$$|h_t(x,y)| \le \frac{C}{\mu(B(x,\sqrt{t}))} e^{-c\frac{d^2(x,y)}{t}}$$

for every $t > 0$, $x,\ y \in M$.

Thus a variant of [26, Theorem 3.1, see page 104] shows that H has a bounded holomorphic functional calculus on $L^p(M)$ in any sector $|\arg z| < \mu$, $\frac{\pi}{2} < \mu \le \pi$ for $1 < p < \infty$. This implies

$$\|H^{i\gamma}\|_{p\to p} \le C(p,\mu) \sup_{|\arg z|<\mu} |z^{i\gamma}| \le C_{p,\mu} e^{|\gamma|\mu}.$$

□

Lemma 6.3. *The space $\mathcal{D} = \mathcal{R}(H) \cap L^1(M) \cap L^\infty(M)$ is dense in $L^p(M)$ for $1 < p < \infty$.*

Proof. Same proof as [3, Lemma 6.2]. □

Proposition 6.4. *Assume that M satisfies (D) and (P_2). Let $V \in RH_q$ for some $1 \leq q < \infty$. Then, for $0 < \alpha < 1$, there exists $\epsilon > 0$ such that the operators $\Delta^\alpha H^{-\alpha}$, $V^\alpha H^{-\alpha}$ are bounded on $L^p(M)$ for $1 < p < \frac{1}{\alpha}q + \epsilon$.*

Proof. From Theorem 6.1, we have that $\Delta^{i\gamma}$ and $H^{i\gamma}$ are $L^p(M)$ bounded for $1 < p < \infty$ and $\gamma \in \mathbb{R}$. Moreover, Theorem 1.1 asserts that ΔH^{-1} and VH^{-1} are $L^p(M)$ bounded for $1 < p < q + \epsilon$ for some $\epsilon > 0$. It follows from Stein's interpolation theorem [54] that $\Delta^\alpha H^{-\alpha}$, $V^\alpha H^{-\alpha}$ are bounded on $L^p(M)$ for $1 < p < \frac{1}{\alpha}(q + \epsilon)$ (see [3] for details). □

We can now prove item 1 of Theorem 1.3. Fix $1 < p < 2(q + \epsilon)$. Let $u \in C_0^\infty(M)$. Since $u \in \mathcal{V}$, $f = H^{\frac{1}{2}}u$ is well-defined. We assume that $f \in L^p(M)$, otherwise there is nothing to prove. Applying Proposition 6.4 to $V^{\frac{1}{2}}$, it comes that $\|V^{\frac{1}{2}}u\|_p \leq C_p\|f\|_p$. The $L^p(M)$ boundedness of the Riesz transform which holds for all $1 < p < p_0$ with $p_0 > 2$ on a complete Riemannian manifold satisfying (D) and (P_2) and again Proposition 6.4 yield

$$\| \, |\nabla u| \, \|_p \leq C(p)\|\Delta^{\frac{1}{2}}H^{-\frac{1}{2}}f\|_p \leq C'(p)\|f\|_p$$

for $1 < p < \inf(p_0, 2(q + \epsilon))$ and finishes the proof.

Remark 6.5. This interpolation argument also gives us a proof of the $L^p(M)$ boundedness of ∇H^{-1} and $V^{\frac{1}{2}}H^{-\frac{1}{2}}$ for $1 < p < 2$ for all non zero $V \in L^1_{\mathrm{loc}}(M)$.

7. Proof of Theorem 1.4

The proof is similar to that of item 2 of [3, Theorem 1.2] with some modifications. We write it for the sake of completeness. Assume that $1 < l < 2$. Let $f \in \mathrm{Lip}(M) \cap \dot{W}^1_{l,V^{\frac{1}{2}}} \cap \dot{W}^1_{2,V^{\frac{1}{2}}}$. By the spectral theory we have

$$H^{\frac{1}{2}}f = c\int_0^\infty He^{-t^2H}f\,dt$$

where $c = \sqrt{\pi}/2$. It suffices to obtain the result for the truncated integrals $\int_\epsilon^R \ldots$ with bounds independent of ϵ, R, and then to let $\epsilon \searrow 0$ and $R \nearrow \infty$. For the truncated integrals, all the calculations are justified. We thus consider that $H^{\frac{1}{2}}$ is one of the truncated integrals but we still write the limits as 0 and $+\infty$ to simplify the exposition. As f does not belong to $C_0^\infty(M)$, we have to give a meaning to $He^{-t^2H}f$ for $t > 0$. Take η_r a smooth function on M, $0 \leq \eta_r \leq 1$, $\eta_r = 1$ on a ball B of radius $r > 0$, $\eta_r = 0$ outside $2B$ and $\| |\nabla \eta_r| \|_\infty \leq \frac{C}{r}$. For $\varphi \in C_0^\infty(M)$,

$$\begin{aligned}\int_M f\, He^{-t^2H}\varphi d\mu &= = \lim_{r\to\infty}\int_M \eta_r f He^{-t^2H}\varphi d\mu\\ &= \int_M \eta_r \nabla f.\nabla e^{-t^2H}\varphi d\mu + \int_M f\nabla\eta_r.\nabla e^{-t^2H}\varphi d\mu\\ &\quad + \int_M \eta_r f\, Ve^{-t^2H}\varphi d\mu\\ &= I_r + II_r + III_r.\end{aligned}$$

We used Fubini and Stokes theorems. Note that $\int_M |\nabla_x h_t(x,y)|^2 e^{\gamma\frac{d^2(x,y)}{t}}d\mu(x) \leq \frac{C}{t\mu(B(y,\sqrt{t}))}$. This is due to the Gaussian upper estimate of the kernel h_t of e^{-tH} and that of $\partial_t h_t$ under (D) and (P_2) (see [20, Lemma 2.3], for the heat kernel p_t of $e^{-t\Delta}$). Since $|\nabla f| \in L^2(M)$ then $I_r \to \int_M \nabla f.\nabla e^{-t^2H}\varphi d\mu$. Since f is Lipschitz, $II_r \to 0$. We also have $\int_M |V^{\frac{1}{2}}(x)h_t(x,y)|^2 e^{\gamma\frac{d^2(x,y)}{t}}d\mu(x) \leq \frac{C}{\mu(B(y,\sqrt{t}))}$ and $V^{\frac{1}{2}}f \in L^2(M)$. Thus $III_r \to \int_M fVe^{-t^2H}\varphi d\mu$. This proves that $He^{-t^2H}f$ is defined as a distribution by

$$\langle He^{-t^2H}f,\varphi\rangle = \int_M \nabla f.\nabla e^{-t^2H}\varphi d\mu + \int_M V^{\frac{1}{2}}fV^{\frac{1}{2}}e^{-t^2H}\varphi d\mu.$$

Therefore, integrating in t yields

$$\langle H^{\frac{1}{2}}f,\varphi\rangle = \langle \nabla f, \nabla H^{-\frac{1}{2}}\varphi\rangle + \langle V^{\frac{1}{2}}f, V^{\frac{1}{2}}H^{-\frac{1}{2}}\varphi\rangle.$$

We return to the proof of Theorem 1.4. Apply the Calderón-Zygmund decomposition of Lemma 4.2 to f at height α and write $f = g + \sum_i b_i$.

For g, we have

$$\begin{aligned}\mu\left(\left\{x\in M; |H^{\frac{1}{2}}g(x)| > \frac{\alpha}{3}\right\}\right) &\leq \frac{9}{\alpha^2}\int |H^{\frac{1}{2}}g|^2 d\mu \leq \frac{9}{\alpha^2}\int (|\nabla g|^2 + V|g|^2)d\mu\\ &\leq \frac{C}{\alpha^l}\int (|\nabla f|^l + |V^{\frac{1}{2}}f|^l)d\mu.\end{aligned}$$

We used a similar argument as above to compute $H^{\frac{1}{2}}g$ (see [4]) and the L^2 estimate follows. For the last inequality we used (4.4) of the Calderón-Zygmund decomposition and that $l < 2$.

The argument to estimate $H^{\frac{1}{2}}b_i$ will use the Gaussian upper bound of h_t. As we mentioned above, under our assumptions we have the Gaussian upper bound for the kernel of e^{-t^2H} and by analyticity for He^{-t^2H}. As b_i is supported in a ball and integrable $He^{-t^2H}b_i$ is defined by the convergent integral

$\int_M \frac{-1}{2t}\partial_t h_{t^2}(x,y)b_i(y)d\mu(y)$. Let $r_i = 2^k$ if $2^k \le R_i < 2^{k+1}$ (R_i is the radius of B_i) and set $T_i = \int_0^{r_i} He^{-t^2H}\,dt$ and $U_i = \int_{r_i}^{\infty} He^{-t^2H}\,dt$. It is enough to estimate

$$A = \mu\left(\left\{x \in M; |\sum_i T_i b_i(x)| > \frac{\alpha}{3}\right\}\right)$$

and

$$B = \mu\left(\left\{x \in M; \left|\sum_i U_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right).$$

First,

$$A \le \mu\left(\bigcup_i \overline{B_i}\right) + \mu\left(\left\{x \in M \setminus \bigcup_i \overline{B_i}; \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right),$$

and by (4.6), $\mu(\bigcup_i \overline{B_i}) \le \frac{C}{\alpha^l}\int(|\nabla f|^l + |V^{\frac{1}{2}} f|^l)d\mu$.

For the other term, we have

$$\mu\left(\left\{x \in M \setminus \bigcup_i \overline{B_i}; \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right) \le \frac{C}{\alpha^2}\int\left|\sum_i h_i\right|^2$$

with $h_i = \mathbb{1}_{(\overline{B_i})^c}|T_i b_i|$. To estimate the L^2 norm, we dualize against $u \in L^2(M)$ with $\|u\|_2 = 1$:

$$\int |u| \sum_i h_i = \sum_i \sum_{j=2}^{\infty} A_{ij}$$

where

$$A_{ij} = \int_{C_j(B_i)} |T_i b_i||u|d\mu, \quad C_j(B_i) = 2^{j+1}B_i \setminus 2^j B_i.$$

By the Minkowski integral inequality, for some appropriate positive constants C, c,

$$\|T_i b_i\|_{L^2(C_j(B_i))} \le \int_0^{r_i} \|He^{-t^2H} b_i\|_{L^2(C_j(B_i))}\,dt.$$

By the well-known Gaussian upper bounds for the kernels of tHe^{-tH}, $t > 0$, valid since we have (D) and (P_2)

$$|He^{-t^2H} b_i(x)| \le \int_M \frac{C}{t^2\mu(B(y,t))} e^{-\frac{cd^2(x,y)}{t^2}} |b_i(y)|d\mu(y).$$

Now $y \in$ supp b_i, that is B_i, and $x \in C_j(B_i)$, hence one may replace $d(x,y)$ by $2^j r_i$ in the Gaussian term since $r_i \sim R_i$. Also if x_i denotes the center of B_i, we have

$$\frac{\mu(B(x_i,t))}{\mu(B(y,t))} = \frac{\mu(B(x_i,t))}{\mu(B(x_i,r_i))}\frac{\mu(B(x_i,r_i))}{\mu(B(y,r_i))}\frac{\mu(B(y,r_i))}{\mu(B(y,t))}.$$

By (D) and Lemma 2.3 as $t \leq r_i$, we have

$$\frac{\mu(B(x_i,t))}{\mu(B(y,t))} \leq C\left(2\frac{r_i}{t}\right)^s.$$

Using the estimate (4.5), $\|b_i\|_1 \leq c\alpha R_i\mu(B_i)$, and $\mu(B_i) \sim \mu(B(x_i,r_i))$, it comes that

$$\begin{aligned}|He^{-t^2H}b_i(x)| &\leq \frac{C}{t^2\mu(B(x_i,t))}\left(\frac{r_i}{t}\right)^s e^{-\frac{c4^j r_i^2}{t^2}}\int_{B_i}|b_i|d\mu\\ &\leq \frac{Cr_i}{t^2}\left(\frac{r_i}{t}\right)^{2s} e^{-\frac{c4^j r_i^2}{t^2}}\alpha.\end{aligned}$$

Thus

$$\|He^{-t^2H}b_i\|_{L^2(C_j(B_i))} \leq \frac{Cr_i}{t^2}\left(\frac{r_i}{t}\right)^{2s} e^{-\frac{c4^j r_i^2}{t^2}}\mu(2^{j+1}B_i)^{\frac{1}{2}}\alpha.$$

Plugging this estimate inside the integral, we get

$$\|T_ib_i\|_{L^2(C_j(B_i))} \leq C\alpha e^{-c4^j}\mu(2^{j+1}B_i)^{\frac{1}{2}}.$$

Now remark that for any $y \in B_i$ and any $j \geq 2$,

$$\left(\int_{C_j(B_i)}|u|^2\right)^{\frac{1}{2}} \leq \left(\int_{2^{j+1}B_i}|u|^2\right)^{\frac{1}{2}} \leq (2^{s(j+1)}\mu(B_i))^{\frac{1}{2}}\big(\mathcal{M}(|u|^2)(y)\big)^{\frac{1}{2}}.$$

Applying the Hölder inequality, one obtains

$$A_{ij} \leq C\alpha 2^{sj}e^{-c4^j}\mu(B_i)\big(\mathcal{M}(|u|^2)(y)\big)^{\frac{1}{2}}.$$

Averaging over B_i yields

$$A_{ij} \leq C\alpha 2^{sj}e^{-c4^j}\int_{B_i}\big(\mathcal{M}(|u|^2)(y)\big)^{\frac{1}{2}}\,d\mu(y).$$

Summing over $j \geq 2$ and i, it follows that

$$\int |u|\sum_i h_i d\mu \leq C\alpha\int\sum_i \mathbb{1}_{B_i}(y)\big(\mathcal{M}(|u|^2)(y)\big)^{\frac{1}{2}}\,d\mu(y).$$

Using finite overlap (4.7) of the balls B_i and Kolmogorov's inequality, one obtains

$$\int |u|\sum_i h_i d\mu \leq C'N\alpha\mu\Big(\bigcup_i B_i\Big)^{\frac{1}{2}}\||u|^2\|_1^{\frac{1}{2}}.$$

Hence

$$\mu\left(\left\{x \in M \setminus \bigcup_i \overline{B_i};\ \left|\sum_i T_i b_i(x)\right| > \frac{\alpha}{3}\right\}\right) \le C\mu(\bigcup_i B_i)$$
$$\le \frac{C}{\alpha^l}\int (|\nabla f|^l + |V^{\frac{1}{2}} f|^l)d\mu$$

by (4.7) and (4.6).

It remains to handle the term B. Using functional calculus for H one can compute U_i as $r_i^{-1}\psi(r_i^2 H)$ with ψ the holomorphic function on the sector $|\arg z| < \frac{\pi}{2}$ given by

$$\psi(z) = \int_1^\infty e^{-t^2 z} z\, dt.$$

It is easy to show that $|\psi(z)| \le C|z|^{\frac{1}{2}} e^{-c|z|}$, uniformly on subsectors $|\arg z| \le \mu < \frac{\pi}{2}$.

The (P_l) Poincaré inequality gives us if B_i is of type 2

$$\|b_i\|_l^l \le C R_i^l \int_{B_i} |\nabla f|^l d\mu \le C R_i^l \alpha^l \mu(B_i).$$

If B_i is of type 1

$$b_i = (b_i - (b_i)_{B_i})\mathbb{1}_{B_i} + (b_i)_{B_i}\,\mathbb{1}_{B_i}. \tag{7.1}$$

Therefore using the type 1 property of B_i and also (7.1) yield

$$\begin{aligned}
\int_{B_i} |b_i|^l d\mu &\le 2^{l-1}\left(\int_{B_i} |b_i - (b_i)_{B_i}|^l + \mu(B_i)\left|\fint_{B_i} b_i d\mu\right|^l\right)\\
&\le C R_i^l \mu(B_i)^{1-l}\int_{B_i} |\nabla b_i|^l d\mu + C\mu(B_i) R_i^l \fint_{B_i} (|\nabla f|^l + |V^{\frac{1}{2}} f|^l)d\mu\\
&\le C R_i^l \mu(B_i)^{1-l}\int_{B_i} |\nabla f|^l d\mu + C\mu(B_i) R_i^l \int_{B_i} (|\nabla f|^l + |V^{\frac{1}{2}} f|^l)d\mu\\
&\le C\alpha^l R_i^l \mu(B_i).
\end{aligned}$$

Hence $\|b_i\|_l^l \le C\alpha^l R_i^l \mu(B_i)$. We invoke the estimate

$$\left\|\sum_{k\in\mathbb{Z}} \psi(4^k H)\beta_k\right\|_l \lesssim \left\|\left(\sum_{k\in\mathbb{Z}} |\beta_k|^2\right)^{\frac{1}{2}}\right\|_l. \tag{7.2}$$

Indeed, by duality, this is equivalent to the Littlewood-Paley inequality

$$\left\|\left(\sum_{k\in\mathbb{Z}} |\psi(4^k H)\beta|^2\right)^{\frac{1}{2}}\right\|_{l'} \lesssim \|\beta\|_{l'}.$$

This is a consequence of the Gaussian estimates for the kernels of e^{-tH}, $t > 0$ (this was first proved in [5] using the vector-valued version of the work in [25]. See [2] or [6] for a more general argument in this spirit or [39] for an abstract proof relying on functional calculus). To apply (7.2), observe that the definitions of r_i and U_i yield

$$\sum_i U_i b_i = \sum_{k\in\mathbb{Z}} \psi(4^k H)\beta_k$$

with

$$\beta_k = \sum_{i, r_i = 2^k} \frac{b_i}{r_i}.$$

Using the bounded overlap property (4.7), one has that

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\beta_k|^2 \right)^{\frac{1}{2}} \right\|_l^l \le C \int \left(\sum_i \frac{|b_i|^l}{r_i^l} \right) d\mu.$$

Using $R_i \sim r_i$,

$$\int \left(\sum_i \frac{|b_i|^l}{r_i^l} \right) d\mu \le C\alpha^l \sum_i \mu(B_i).$$

Hence, by (4.6)

$$\mu\left(\left\{ x \in M;\, \left| \sum_i U_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right) \le C \sum_i \mu(B_i) \le \frac{C}{\alpha^l} \int_M (|\nabla f|^l + |V^{\frac{1}{2}} f|^l) d\mu.$$

Thus, we have obtained

$$\mu\left(\{x \in M;\, |H^{\frac{1}{2}} f(x)| > \alpha\} \right) \le \frac{C}{\alpha^l} \int_M (|\nabla f|^l + V^{\frac{1}{2}} f|^l) d\mu$$

for all $f \in \mathrm{Lip}(M) \cap \dot{W}^1_{l,V^{\frac{1}{2}}} \cap \dot{W}^1_{l,V^{\frac{1}{2}}}$.

Moreover, using the density argument of Proposition 4.4 we extend $H^{\frac{1}{2}}$ to a bounded operator acting from $\dot{W}^1_{l,V^{\frac{1}{2}}}$ to $L^{l,\infty}$. We already have

$$\|H^{\frac{1}{2}} f\|_2 \le \|\,|\nabla f|\,\|_2 + \|V^{\frac{1}{2}} f\|_2.$$

Since $V \in A_\infty$ implies $V^{\frac{1}{2}} \in RH_2$ (Proposition 2.12), we see from Corollary 2.17 that

$$\|H^{\frac{1}{2}} f\|_p \le C_p \left(\|\,|\nabla f|\,\|_p + \|V^{\frac{1}{2}} f\|_p \right) \tag{7.3}$$

for all $l < p \le 2$ and $f \in \dot{W}^1_{p,V}$.

If $l = 1$, we take $1 < p < 2$. There exists $\epsilon > 0$ such that $1 < 1 + \epsilon < p$. The same argument works replacing $l = 1$ by $1 + \epsilon$.

8. Proof of item 2 of Theorem 1.3

We first give some estimates for the weak solutions of $-\Delta u + Vu = 0$. Then we proceed to a reduction and then give the proof of item 2 of Theorem 1.3.

8.1. Estimates for weak solutions

Let M be a complete Riemannian manifold satisfying (D) and (P_2). Let $B = B(x_0, R)$ denotes a ball of radius $R > 0$ and u a weak solution of $-\Delta u + Vu = 0$ in a neighborhood of $\overline{B(x_0, 4R)}$. By a weak solution of $-\Delta u + Vu = 0$ in an open set Ω, we mean $u \in L^1_{\text{loc}}(\Omega)$ with $V^{\frac{1}{2}}u, \nabla u \in L^2_{\text{loc}}(\Omega)$ and the equation holds in the distribution sense on Ω. Remark that under the Poincaré inequality (P_2) if u is a weak solution, then $u \in L^2_{\text{loc}}(\Omega)$. It should be observed that if u is a weak solution in Ω of $-\Delta u + Vu = 0$ then

$$\Delta |u|^2 = 2V|u|^2 + 2|\nabla u|^2 \tag{8.1}$$

since $\Delta|u|^2 = 2\langle \Delta u, u\rangle + 2|\nabla u|^2$ (see [10]). In particular, $|u|^2$ is a non-negative subharmonic function in Ω. Hence the lemmas in Subsection 3 of Section 4 apply to $|u|^2$. In particular

$$\sup_{B(x_0,R)} |u| \leq C(r)\left((|u|^r)_{B(x_0,\mu R)}\right)^{\frac{1}{r}} \tag{8.2}$$

holds for any $0 < r < \infty$ and $1 < \lambda \leq 4$. We have also shown a mean value inequality against arbitrary A_∞ weights.

We state some further estimates that are interesting in their own right assuming $V \in A_\infty$. By splitting real and imaginary parts, we may suppose u real-valued. All constants are independent of B and u but they may depend on the constants in the A_∞ condition or the RH_q condition of V when assumed, on the doubling constant C_d and the Poincaré inequality (P_2). Let s be any real number such that $\frac{\mu(B)}{\mu(B_0)} \geq C(\frac{r}{r_0})^s$ whenever $B = B(x, r)$, $x \in B_0$, $r \leq r_0$ ($s = \log_2 C_d$ works).

The proofs of the next 3 lemmas are as in [3], we skip them.

Lemma 8.1. *For all $1 \leq \lambda < \lambda' \leq 4$ and $k > 0$, there is a constant C such that*

$$(|u|^2)_{\lambda B} \leq \frac{C}{(1+R^2V_B)^k}(|u|^2)_{\lambda' B}\,.$$

and

$$(|\nabla u|^2 + V|u|^2)_{\lambda B} \leq \frac{C}{(1+R^2V_B)^k}(|\nabla u|^2 + V|u|^2)_{\lambda' B}.$$

Lemma 8.2. *For all $1 \leq \lambda \leq 4$, $k > 0$, there is a constant C such that*

$$(RV_B)^2(|u|^2)_B \leq \frac{C}{(1+R^2V_B)^k}(V|u|^2)_{\lambda B}.$$

Lemma 8.3. *For all* $1 < \lambda \leq 4$, $k > 0$ *and* $\max(s, 2) < r < \infty$, *there is a constant* C *such that*

$$(RV_B)^2(|u|^2)_B \leq \frac{C}{(1+R^2V_B)^k}(|\nabla u|^r)_{\lambda B}^{\frac{2}{r}}.$$

The main tools to prove these lemmas are the improved Fefferman-Phong inequality of Lemma 4.1, the Caccioppoli type inequality which holds on complete Riemannian manifolds, Poincaré inequality, subharmonicity of $|u|^2$, Lemma 4.6 and the Morrey embedding theorem with exponent $\alpha = 1 - \frac{s}{r}$ ($s = \mathrm{Log}_2 C$, with C the doubling constant) [34, Theorem 5.1, page 23] to prove Lemma 8.3.

For the remaining lemmas, we moreover assume that M is of polynomial type: every ball B of radius $r > 0$ satisfies

$$\mu(B) \geq cr^\sigma, \tag{L_σ}$$

and

$$\mu(B) \leq Cr^\sigma \tag{U_σ}$$

with $\sigma = d$ if $r \leq 1$ and $\sigma = D$ for $r \geq 1$ and $d \leq D$. Note that if (L_σ) holds then $\sigma \geq n$ where n is the topological dimension of M (see [50]). Recall that under (L_σ) and (U_σ), $s = D$ works and that $\mu(B(x, r)) \geq cr^\lambda$ for all $r > 0$ with any $\lambda \in [d, D]$. We also recall that the exponent p_0 is that appearing in Proposition 1.2.

Lemma 8.4. *Assume* $V \in RH_q$. *Let* B *be a ball of radius* $R > 0$. *Set* $\tilde{q} = q^*_\sigma$ *if* $q^*_\sigma < p_0$ ($\frac{1}{q^*_\sigma} = \frac{1}{q} - \frac{1}{\sigma}$) *and* $\tilde{q}$ *arbitrary in* $]2, p_0[$ *if not. Then for all* $k > 0$ *there is a constant* $C = C(\sigma)$ *independent of* B *such that*

$$\left((|\nabla u|^{\tilde{q}})_B\right)^{\frac{1}{\tilde{q}}} \leq \frac{C}{(1+R^2V_B)^k}\left((|\nabla u|^2 + V|u|^2)_{4B}\right)^{\frac{1}{2}}.$$

Lemma 8.5. *Assume* $V \in RH_q$ *with* $\frac{D}{2} \leq q < \frac{p_0}{2}$. *Let* B *be a ball of radius* $R > 0$. *Set* $\tilde{q} = q^*_\sigma$ *if* $q^*_\sigma < p_0$ *and* $\tilde{q}$ *arbitrary in* $]2q, p_0[$ *if not. Then, there is a constant* $C = C(\sigma)$ *such that*

$$\left((|\nabla u|^{\tilde{q}})_B\right)^{\frac{1}{\tilde{q}}} \leq C\left((|\nabla u|^2)_{4B}\right)^{\frac{1}{2}},$$

We give the proofs of Lemma 8.4 and 8.5 since they are not exactly the same as the one in the Euclidean case. Before the proof of Lemma 8.4, we need the following theorem for the boundedness of the Riesz potential.

Theorem 8.6 ([17]). *Let* M *be a complete Riemannian manifold satisfying* (D) *and* (P_2). *Moreover, assume that* M *satisfies*

$$\mu(B) \geq cr^\lambda \tag{L_λ}$$

for every $x \in M$ *and* $r > 0$.

Then $(-\Delta)^{-\frac{1}{2}}$ *is* $L^p - L^{p*}$ *bounded with* $1 < p,\ p^* < \infty$ *and* $p^* = \frac{\lambda p}{\lambda - p}$, *that is,*

$$\|(-\Delta)^{-\frac{1}{2}} f\|_{p^*} \leq C(p, \lambda)\|f\|_p.$$

Proof. In [17], Chen proves this theorem for Riemannian manifolds with non-negative Ricci curvature. His proof still works under our hypotheses. The properties that he used for these manifolds are first the lower and upper gaussian estimates for the heat kernel which holds on Riemmanian manifolds satisfying (D) and (P_2). Secondly, he applied an argument from the proof of the $L^p - L^{p^*}$ boundedness of the Riesz potential in the Euclidean case [55, Chapter V, Theorem 1] which remains true since we have (D), (P_2) and (L_λ) with $\lambda \geq n = \dim M$. □

Proof of Lemma 8.4. First note that if $q \leq \frac{2\sigma}{\sigma+2}$ then $\tilde{q} \leq 2$ and the conclusion (useless for us) follows by a mere Hölder inequality. Henceforth, we assume $q > \frac{2\sigma}{\sigma+2}$. Also, by Lemma 8.1, it suffices to obtain the estimate with $k = 0$. Let us assume $\mu = 4$ for simplicity of the argument. Let v be the harmonic function on $4B$ with $v = u$ on $\partial(4B)$ and set $w = u - v$ on $4B$. Since $w = 0$ on $\partial(4B)$, the fact that an harmonic function minimises Dirichlet integral among functions with the same boundary implies

$$\left(\fint_{4B} |\nabla w|^2\right)^{\frac{1}{2}} \leq 2\left(\fint_{4B} |\nabla u|^2\right)^{\frac{1}{2}}.$$

By the elliptic estimate for the harmonic function v [4, Theorem 2.1], we have for $p < p_0$

$$\left(\fint_{B} |\nabla v|^p\right)^{\frac{1}{p}} \leq C\left(\fint_{4B} |\nabla v|^2\right)^{\frac{1}{2}} \leq 2C\left(\fint_{4B} |\nabla u|^2\right)^{\frac{1}{2}}. \tag{8.3}$$

Let $1 < \nu < \lambda < 4$ and η be a smooth non-negative function, bounded by 1, equal to 1 on νB with support contained in λB and whose gradient is bounded by $\frac{C}{R}$. As $\Delta w = \Delta u = Vu$ on $4B$, we have

$$\Delta(w\eta) = Vu\eta + \nabla w \cdot \nabla\eta + \operatorname{div}(w\nabla\eta) \quad \text{on } M.$$

It comes that

$$\begin{aligned}
\nabla(w\eta)(x) &= \nabla(-\Delta)^{-1}(-\Delta)(w\eta)(x) \\
&= \nabla(-\Delta)^{-\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}(-Vu\eta)(x) + \nabla(-\Delta)^{-\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}(-\nabla w.\nabla\eta)(x) \\
&\quad + \nabla(-\Delta)^{-1}(-\operatorname{div}(w\nabla\eta))(x) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Let us begin with

$$I_3 = \nabla(-\Delta)^{-\frac{1}{2}}(-\Delta)^{-\frac{1}{2}}\operatorname{div}(-w\nabla\eta)(x) = (\nabla(-\Delta)^{-\frac{1}{2}})(\nabla(-\Delta)^{-\frac{1}{2}})^*(-w\nabla\eta)(x).$$

Let η' be a smooth function, bounded by 1, equal to 1 on λB with support contained in $\lambda' B$ with $\lambda' < 4$ and whose gradient is bounded by $\frac{C}{R}$. The Riesz transform

$\nabla(-\Delta)^{-\frac{1}{2}}$ is $L^p(M)$ bounded for $1 < p < p_0$. By duality, $(\nabla(-\Delta)^{-\frac{1}{2}})^*$ is $L^p(M)$ bounded for $p_0' < p < \infty$. Hence for $2 < p < p_0$

$$\left(\int_M |I_3|^p d\mu\right)^{\frac{1}{p}} \leq C\left(\int_M |w\eta'|^p|\nabla\eta|^p d\mu\right)^{\frac{1}{p}}$$
$$\leq \frac{C}{R}\left(\int_M |\nabla(w\eta')|^{p_*} d\mu\right)^{\frac{1}{p_*}}.$$

We used the Sobolev inequality which holds under (D), (P_2) and $\mu(B(x,r)) \geq cr^\sigma$ for all $r > 0$ with $p_{*\sigma} < p$ defined by $p_{*\sigma} = \frac{\sigma p}{\sigma+p}$ that is $(p_*)^* = p$ (see [50]).

We use the $L^q - L^{q^*_\sigma}$ boundedness of the Riesz potential $(-\Delta)^{-\frac{1}{2}}$ and the L^p boundedness of the Riesz transform $\nabla(-\Delta)^{-\frac{1}{2}}$ for $1 < p < p_0$ to get the estimates for I_2 and I_1. First for I_2, we have for all $2 \leq p < p_0$

$$\left(\int_M |I_2|^p d\mu\right)^{\frac{1}{p}} \leq C\left(\int_M |(-\Delta)^{-\frac{1}{2}}(\nabla(w\eta').\nabla\eta)|^p d\mu\right)^{\frac{1}{p}}$$
$$\leq \frac{C}{R}\left(\int_M |\nabla(w\eta')|^{p_{*\sigma}} d\mu\right)^{\frac{1}{p_{*\sigma}}}$$
$$\leq \frac{C}{R}\left(\int_M |\nabla(w\eta')|^{p_{*\sigma}} d\mu\right)^{\frac{1}{p_{*\sigma}}}$$
$$= \frac{C}{R}\left(\int_M |\nabla(w\eta')|^{p_{*\sigma}} d\mu\right)^{\frac{1}{p_{*\sigma}}}.$$

Now, it remains to look at I_1. Take $p = q^*_\sigma$ if $q^*_\sigma < p_0$ and if not any $2 < p < p_0$. It follows that

$$\left(\int_M |I_1|^p d\mu\right)^{\frac{1}{p}} \leq C\left(\int_M |Vu\eta|^{p_{*\sigma}} d\mu\right)^{\frac{1}{p_{*\sigma}}}$$
$$\leq C\mu(B)^{\frac{1}{p_{*\sigma}}}\left(\fint_{\lambda B} |V|^q d\mu\right)^{\frac{1}{q}} \sup_{\mu B}|u|$$

since $p_{*\sigma} \leq q$ in the two cases. Using the RH_q condition on V, we obtain

$$\left(\int_M |I_1|^p d\mu\right)^{\frac{1}{p}} \leq C\mu(B)^{\frac{1}{p_{*\sigma}}} \fint_{\lambda B} V d\mu \sup_{\mu B}|u|. \tag{8.4}$$

Now, if $\lambda < \gamma < 4$, the subharmonicity of $|u|^2$ and Lemma 4.6 yield

$$\fint_{\lambda B} V d\mu \sup_{\lambda B}|u| \leq C \fint_{\gamma B} V d\mu \left(\fint_{\gamma B} |u|^2 d\mu\right)^{\frac{1}{2}}.$$

It follows from Lemma 8.2 and (U_σ) that $\left(\int_M I_1^p d\mu\right)^{\frac{1}{p}} \leq C\mu(B)^{\frac{1}{p}}\left(\fint_{4B} V|u|^2 d\mu\right)^{\frac{1}{2}}$. Therefore, we showed that

$$\left(\int_M |\nabla(w\eta)|^p d\mu\right)^{\frac{1}{p}} \leq \frac{C}{R}\left(\int_M |\nabla(w\eta')|^{p_*} d\mu\right)^{\frac{1}{p_*}} + C\mu(B)^{\frac{1}{p}}\left(\fint_{4B} V|u|^2 d\mu\right)^{\frac{1}{2}}.$$

We repeat the same process and after a finite iteration ($K = (\sigma[\frac{1}{2} - \frac{1}{p}] + 1)$ times), using (U_σ) we get

$$\left(\fint_B |\nabla w|^{\tilde{q}} d\mu\right)^{\frac{1}{\tilde{q}}} \leq C\left(\fint_{4B} |\nabla w|^2 d\mu\right)^2 + C\left(\fint_{4B} V|u|^2 d\mu\right)^{\frac{1}{2}}.$$

We derive therefore the desired inequality for ∇u from the estimates obtained for ∇v and ∇w. $\square$

Proof of Lemma 8.5. Since $V \in RH_q$ and $q \geq \frac{D}{2}$, we may assume $q > \frac{D}{2}$ by self-improvement. Let $\sigma = d$ if $R \leq 1$ and $\sigma = D$ if $R \geq 1$. We apply the same arguments as in the proof of the previous lemma. The only difference is that since $2q > s = D$, we use Lemma 8.3 with $k = 0$, $r = 2q$, and $s = D$ instead of Lemma 8.2 in the estimate for the term I. We then obtain

$$\left(\fint_B |\nabla u|^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}} \leq C\left(\fint_{4B} |\nabla u|^{2q}\right)^{\frac{1}{2q}} \tag{8.5}$$

where $p = q_\sigma^*$ if $q_\sigma^* < p_0$ and if not we take any $2 < p < p_0$. Since $2q < p_0$, if we take $p = \tilde{q} \in]2q, p_0[$ in (8.5) we can apply Lemma 2.13 and improve the exponent $2q$ to 2. Thus, we get

$$\left(\fint_B |\nabla u|^{\tilde{q}}\right)^{\frac{1}{\tilde{q}}} \leq C\left(\fint_{4B} |\nabla u|^2\right)^{\frac{1}{2}}.$$

Remark that when $q > D$, $q_\sigma^* = \infty$ and therefore we have our lemma for any $2q < p < p_0$. $\square$

8.2. A reduction

It is sufficient to prove the L^p boundedness of $\nabla H^{-\frac{1}{2}}$ and of $V^{\frac{1}{2}} H^{-\frac{1}{2}}$ for the appropriate range of p. As we have seen in the introduction, the case $1 < p \leq 2$ does not need any assumption on V. We henceforth assume $p > 2$ and $V \in A_\infty$.

By duality, we know that $H^{-\frac{1}{2}}\mathrm{div}$ and $H^{-\frac{1}{2}} V^{\frac{1}{2}}$ are bounded on L^p for $2 < p < \infty$. Thus, if $\nabla H^{-\frac{1}{2}}$ is also bounded on L^p, it follows that $\nabla H^{-1}\mathrm{div}$ and $\nabla H^{-1} V^{\frac{1}{2}}$ are bounded on L^p.

Reciprocally, if $\nabla H^{-1}\mathrm{div}$ and $\nabla H^{-1}V^{\frac{1}{2}}$ are bounded on L^p, then their adjoints are bounded on $L^{p'}$. Thus, if $F \in C_0^\infty(M, TM)$,

$$\begin{aligned}\|H^{-\frac{1}{2}}\mathrm{div}F\|_{p'} &= \|H^{\frac{1}{2}}H^{-1}\mathrm{div}F\|_{p'} \\ &\leq C(\| \, |\nabla H^{-1}\mathrm{div}F| \, \|_{p'} + \|V^{\frac{1}{2}}H^{-1}\mathrm{div}F\|_{p'}) \leq C\|F\|_{p'}\end{aligned}$$

where the first inequality follows from Theorem 1.4. By duality, we have that $\nabla H^{-\frac{1}{2}}$ is bounded on L^p.

The same treatment can be done on $V^{\frac{1}{2}}H^{-\frac{1}{2}}$. We have obtained

Lemma 8.7. *Let M be a complete Riemannian manifold. If $V \in A_\infty$ and $p > 2$, the L^p boundedness of $\nabla H^{-\frac{1}{2}}$ is equivalent to that of $\nabla H^{-1}\mathrm{div}$ and $\nabla H^{-1}V^{\frac{1}{2}}$, and the L^p boundedness of $V^{\frac{1}{2}}H^{-\frac{1}{2}}$ is equivalent to that of $V^{\frac{1}{2}}H^{-1}V^{\frac{1}{2}}$ and $V^{\frac{1}{2}}H^{-1}\mathrm{div}$.*

Hence, to prove Theorem 1.3, it suffices the L^p boundedness of the operators: $\nabla H^{-1}\mathrm{div}$, $\nabla H^{-1}V^{\frac{1}{2}}$, $V^{\frac{1}{2}}H^{-1}V^{\frac{1}{2}}$, $V^{\frac{1}{2}}H^{-1}\mathrm{div}$.

8.3. Proof of item 2. of Theorem 1.3

Proposition 8.8. *Let M be a complete Riemannian manifold satisfying (D) and (P_2). Assume that $V \in RH_q$ for some $q > 1$. Then for $2 < p < 2(q+\epsilon)$, for some $\epsilon > 0$ depending only on V, $f \in C_0^\infty(M, \mathbb{C})$ and $F \in C_0^\infty(M, TM)$,*

$$\|V^{\frac{1}{2}}H^{-1}V^{\frac{1}{2}}f\|_p \leq C_p\|f\|_p, \quad \|V^{\frac{1}{2}}H^{-1}\mathrm{div}F\|_p \leq C_p\|F\|_p.$$

Proposition 8.9. *Let M be a complete Riemannian manifold of polynomial type satisfying (P_2). Let $V \in RH_q$ for some $q > 1$. If $q_D^* < p_0$, let $p = q_D^*$. If $q_D^* \geq p_0$, we take any $2 < p < p_0$. Then for all $f \in C_0^\infty(M, C)$ and $F \in C_0^\infty(M, TM)$,*

$$\|\nabla H^{-1}V^{\frac{1}{2}}f\|_p \leq C_p\|f\|_p, \quad \| \, |\nabla H^{-1}\mathrm{div}F| \, \|_p \leq C_p\|F\|_p.$$

The interest of such a reduction is that this allows us to use properties of weak solutions of H. Note that Proposition 8.9 is void if $q \leq \frac{2D}{D+2}$ as $q_D^* \leq 2$. Note also that $q_D^* < 2q$ exactly when $q < \frac{D}{2}$. In this case, this statement yields a smaller range than the interpolation method in Section 6.

Proof of Proposition 8.8. Fix a ball $B = B(x_0, R)$ and let $f \in C_0^\infty(M)$ supported away from $\overline{4B}$. Then $u = H^{-1}V^{\frac{1}{2}}f$ is well defined on M with $\|V^{\frac{1}{2}}u\|_2 + \| \, |\nabla u| \, \|_2 \leq \|f\|_2$ by construction of H and

$$\int_M (Vu\varphi + \nabla u \cdot \nabla\varphi)d\mu = \int_M V^{\frac{1}{2}}f\varphi d\mu$$

for all $\varphi \in L^2(M)$ with $\|V^{\frac{1}{2}}\varphi\|_2 + \|\,|\nabla\varphi|\,\|_2 < \infty$. In particular, the support condition on f implies that u is a weak solution of $-\Delta u + Vu = 0$ in a neighborhood of $\overline{4B}$, hence $|u|^2$ is subharmonic there. Let r such that $V \in RH_r$. Note that by Proposition 2.12, $V^{\frac{1}{2}} \in RH_{2r}$. From Corollary 4.7 with $V^{\frac{1}{2}}$, $|u|^2$ and $s = \frac{1}{2}$, we get

$$\left(\fint_B (V^{\frac{1}{2}}|u|)^{2r} d\mu\right)^{\frac{1}{2r}} \leq C \fint_{4B} V^{\frac{1}{2}}|u| d\mu.$$

Thus, (5.1) holds with $T = V^{\frac{1}{2}}H^{-1}V^{\frac{1}{2}}$, $q_0 = 2r$, $r_0 = 2$ and $S = 0$. By Theorem 5.5, $V^{\frac{1}{2}}H^{-1}V^{\frac{1}{2}}$ is bounded on L^p for $2 < p < 2r$.

The argument is the same for $V^{\frac{1}{2}}H^{-1}\mathrm{div}$. This finishes the proof. □

Proof of Proposition 8.9. We assume $q > \frac{2D}{D+2}$, that is $q_D^* > 2$, otherwise there is nothing to prove. We consider first the operator $\nabla H^{-1}V^{\frac{1}{2}}$.

Assume $q < \frac{D}{2}$. Fix a ball B of radius R and let $f \in C_0^\infty(M)$ supported away from $\overline{4B}$. Let $u = H^{-1}V^{\frac{1}{2}}f$. As before, the support condition on f implies that u is a weak solution of $-\Delta u + Vu = 0$ in a neighborhood of $\overline{4B}$. Thanks to Lemma 8.4, (5.1) holds with $T = \nabla H^{-1}V^{\frac{1}{2}}$, $q_0 = q_D^* \leq q_d^*$ if $q_D^* < p_0$ and if not $q_0 = p_0 - \epsilon'$ for any $\epsilon' > 0$, and $S = \left(\mathcal{M}(|V^{\frac{1}{2}}H^{-1}V^{\frac{1}{2}}|^2)\right)^{\frac{1}{2}}$. The maximal theorem (Theorem 2.4) and Proposition 8.8 show that S is bounded on $L^p(M)$ for $1 < p < 2q$. Then Theorem 5.5 implies that $\nabla H^{-1}V^{\frac{1}{2}}$ is bounded on $L^p(M)$ for $2 < p < p_0$ if $q_D^* \geq p_0$. If $q_D^* < p_0$, by the self-improvement of reverse Hölder estimates we can replace q by a slightly larger value and, therefore we get the L^p boundedness of $\nabla H^{-1}V^{\frac{1}{2}}$ for $p \leq q_D^*$.

Assume next that $\frac{D}{2} \leq q < D$ and $2q < p_0$. Again, we may as well assume $q > \frac{D}{2}$. In this case $q_D^* > 2q$. Then, Lemma 8.5 yields, this time, (5.1) with $T = \nabla H^{-1}V^{\frac{1}{2}}$, $q_0 = q_D^*$ if $q_D^* < p_0$ and if not $q_0 = p_0 - \epsilon'$ for any $0 < \epsilon' < p_0 - 2q$, and $S = 0$. Theorem 5.5 asserts that $\nabla H^{-1}V^{\frac{1}{2}}$ is bounded on L^p for $2 < p < p_0$ if $q_D^* \geq p_0$ and, by the self-improvement of the RH_q condition, it holds for $p \leq q_D^*$ if $q_D^* < p_0$.

Finally, if $q \geq D$, then Lemma 8.5 yields (5.1) for any $2 < q_0 < p_0$ with $T = \nabla H^{-1}V^{\frac{1}{2}}$ and $S = 0$. Theorem 5.5 shows then that $\nabla H^{-1}V^{\frac{1}{2}}$ is bounded on L^p for $2 < p < p_0$.

The argument is the same for $\nabla H^{-1}\mathrm{div}$ and the proof is therefore complete. □

9. The case of Lie groups

Consider G a simply connected Lie group. Assume that G is unimodular and let $d\mu$ be a fixed Haar measure on G. Let $X_1, \ldots, X_k$ be a family of left invariant vector fields such that the X_i's satisfy a Hörmander condition. In this case the Carnot-Carathéodory metric ρ is a distance, and the metric space (G, ρ) is complete and has the same topology as G as a manifold (see [22, page 1148]). Denote $V(r) = \mu(B(x,r))$ for all $x \in G$. An important result of Guivarc'h [33] says that, either there exists an integer D such that $cr^D \leq V(r) \leq Cr^D$ for all $r > 1$, or $e^{cr} \leq V(r) \leq Ce^{Cr}$ for all $r > 1$ with $V(r) = \mu(B(x,r)) = \mu(B(y,r))$, for all $x,\ y \in G$ and $r > 0$. In the first case we say that G has polynomial growth, while in the second case G has exponential growth. For small r, a result of [46] implies that there exists an integer d such that $cr^d \leq V(r) \leq Cr^d$ for $0 < r < 1$. Suppose that G has polynomial growth. Then there exists $C_1 > 0$ such that

$$C_1^{-1} r^d \leq V(r) \leq C_1 r^d, \quad 0 \leq r \leq 1, \tag{9.1}$$

$$C_1^{-1} r^D \leq V(r) \leq C_1 r^D, \quad 1 \leq r < \infty. \tag{9.2}$$

We say that d is the local dimension of G and D is the dimension at infinity. We assume that $d \geq 3$ and $d \leq D$. If G is nilpotent and since G is simply connected, we have $d \leq D$ (see [23]). In particular (D) holds with $s = D$. Moreover G satisfies a Poincaré inequality (P_1): there exists $C > 0$ such that for all ball B of radius $r > 0$ we have for every smooth function u,

$$\int_B |u - u_B| d\mu \leq Cr \int_{2B} |Xu| d\mu \tag{P_1}$$

(see [49, 58]) where $|Xu| = \left(\sum_{i=1}^k |X_i u|^2\right)^{\frac{1}{2}}$.

For the rest of this section, we consider G a Lie group as above with polynomial growth and set $\Delta = \sum_{i=1}^k X_i^2$.

Let us check the validity of our approach to obtain Theorem 1.1, Theorem 1.3 and Theorem 1.4 for G. The main tools used to prove those theorems still hold:

- The Riesz transform $\nabla(-\Delta)^{-\frac{1}{2}}$ is L^p bounded for all $1 < p < \infty$. This result was proved by Alexopoulos [1].
- An improved Fefferman-Phong inequality of type (4.1) holds on G with $\beta = \frac{p}{p+D(\alpha-1)}$.
- We get a Calderón-Zygmund decomposition analogous to that of Proposition 4.2. Thanks to this decomposition, we get the analog of Theorem 1.4 as in Section 7.
- Theorem 6.1 proved in Section 6 remains true for Lie groups with polynomial growth (we use the same proof).
- The argument of complex interpolation (valid on G) allows us to obtain Theorem 1.3 part 1.
- Let u a weak solution of $-\Delta u + Vu = 0$ on G, then u satisfies some mean values inequalities as in Lemma 4.5, 4.6 and Corollary 4.7. We mention that the analogous of Lemma 4.5 was proved by Li [40, 41] for nilpotents groups using estimations for the heat kernel and its first and second derivatives.

- The lemmas in Section 8.1 still hold in our case: G is of polynomial type. The Sobolev inequality and the Morrey embedding (with $\alpha = 1 - \frac{n}{p}$ and $1 - \frac{n}{p} \notin \mathbb{N}$) hold for any $n \in [d, D]$ (see [23, Theorem VIII.2.10]). We also have that $\Delta^{-\frac{1}{2}}$ is bounded from L^p to $L^{\frac{np}{n-p}}$ for any $n \in [d, D]$ and $p < n$ [23, Theorem VIII.2.3]. Thus we get similar lemmas to that of Section 8.1 this time on a Lie group G of polynomial growth.

With all these ingredients, we establish the following theorem analogous to Theorem 1.3.

Theorem 9.1. *Let G be a simply connected Lie group with polynomial growth and assume $3 \leq d \leq D$. Let $V \in RH_q$ for some $q > 1$.*

1. *Then for any smooth function u,*

$$\| |\nabla u| \|_p + \|V^{\frac{1}{2}} u\|_p \lesssim \|(-\Delta + V)^{\frac{1}{2}} u\|_p \quad \textit{for } 1 < p < 2(q + \epsilon). \tag{9.3}$$

2. *Assume $q \geq \frac{D}{2}$. Consider*

$$\| |\nabla u| \|_p \lesssim \|(-\Delta + V)^{\frac{1}{2}} u\|_p \tag{9.4}$$

for all smooth function u.

a. *if $\frac{D}{2} < q < D$, (9.4) holds for $1 < p < q_D^* + \epsilon$,*

b. *if $q \geq D$, (9.4) holds for $1 < p < \infty$.*

Remark 9.2. Li [40,41] proved item 2 of Theorem 9.1 if G is in addition Nilpotent.

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