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Kähler manifolds and their relatives

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Abstract. Let M_1 and M_2 be two Kähler manifolds. We call M_1 and M_2 *relatives* if they share a non-trivial Kähler submanifold S, namely, if there exist two holomorphic and isometric immersions (Kähler immersions) $h_1 : S \to M_1$ and $h_2 : S \to M_2$. Moreover, two Kähler manifolds M_1 and M_2 are said to be *weakly relatives* if there exist two locally isometric (not necessarily holomorphic) Kähler manifolds S_1 and S_2 which admit two Kähler immersions into M_1 and M_2 respectively. The notions introduced are not equivalent (*cf.* Example 2.3). Our main results in this paper are Theorem 1.2 and Theorem 1.4. In the first theorem we show that a complex bounded domain $D \subset \mathbb{C}^n$ with its Bergman metric and a projective Kähler manifold (*i.e.* a projective manifold endowed with the restriction of the Fubini–Study metric) are not relatives. In the second theorem we prove that a Hermitian symmetric space of noncompact type and a projective Kähler manifold are not weakly relatives. Notice that the proof of the second result does not follows trivially from the first one. We also remark that the above results are of local nature, *i.e.* no assumptions are used about the compactness or completeness of the manifolds involved.

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1. Introduction

The study of holomorphic and isometric immersions between Kähler manifolds (called Kähler immersions in the sequel) was started by Eugenio Calabi who, in his pioneering work [6] of 1953 (see also [9]), solved the problem of deciding about the existence of Kähler immersions between complex space forms. More specifically, he proved that two complex space forms with curvature of different sign cannot be Kähler immersed one into another and, in particular that, for complex space forms of the same type, just projective spaces can be embedded between themselves in a non trivial way by using Veronese mappings. Unfortunately, this subject has not been further explored by other authors as pointed out by Marcel Berger in [3] who referring to Calabi's paper wrote: *"this wonderful text remains quite unknown and almost unused..."*.

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The authors believe that the results of the present paper are in accordance with Berger's opinion (see also in [4, page 528]) and could stimulate future research in this field.

In order to state our first result we give the following:

Definition 1.1. Let r be a positive integer. Two Kähler manifolds M_1 and M_2 are said to be *r-relatives* if they have in common a complex r-dimensional Kähler submanifold S, *i.e.* there exist two Kähler immersions $h_1 : S \to M_1$ and $h_2 : S \to M_2$. Otherwise, we say that M_1 and M_2 are not relatives.

Our first result is the following:

Theorem 1.2. *A bounded domain $D \subset \mathbb{C}^n$ endowed with its Bergman metric and a projective Kähler manifold endowed with the restriction of the Fubini–Study metric are not relatives.*

It is worth pointing out that the previous definition and the previous theorem are inspired by the work of Masaaki Umehara [14] on Kähler embeddings between complex space forms. Indeed, his main result in [14] can be stated in our terminology by saying that two finite dimensional complex space forms with holomorphic sectional curvatures of different signs cannot be relatives (*cf.* also Remark 2.6).

The following definition generalizes the previous one:

Definition 1.3. Let r be a positive integer. Two Kähler manifolds M_1 and M_2 are said to be *weakly r-relatives* if there exist two locally isometric Kähler manifolds S_1 and S_2 of complex dimension r which admit two Kähler immersions $h_1 : S_1 \to M_1$ and $h_2 : S_2 \to M_2$.

We remark that here the local isometry between S_1 and S_2 is not assumed to be holomorphic (*cf.* Example 2.3 below). Observe also that it is immediate to see that two Kähler manifolds of complex dimension one which are weakly relatives are also relatives.

Let us now state our second result.

Theorem 1.4. *An irreducible Hermitian symmetric space of noncompact type and a projective Kähler manifold are not weakly relatives.*

Notice that even if the word *weakly relatives* in the above theorem is replaced by the word *relatives* the proof of Theorem 1.4 does not directly follow from Theorem 1.2 (*cf.* discussion before Example 2.3).

Since an irreducible Hermitian symmetric space of compact type admits a Kähler embedding into a complex projective space (see *e.g.* [10] and [11]), Theorem 1.4 yield to the following appealing:

Corollary 1.5. *An irreducible Hermitian symmetric space of noncompact type and an irreducible Hermitian symmetric space of compact type, are not weakly relatives.*

2. Proof of the main results

Let $\Phi_B(z,z)$ be a (globally defined) Kähler potential for the Bergman metric g_B on a bounded domain $D \subset \mathbb{C}^n$. Then $\Phi_B(z,z) = \log K_B(z,z)$ where $K_B(z,z)$ is the Bergman kernel function on D, namely $K_B(z,z) = \sum_{j=0}^{+\infty} |F_j(z)|^2$ where F_j, $j = 0, 1, \dots$ is an orthonormal basis for the Hilbert space $\mathcal{H}$ consisting of square integrable holomorphic functions on D. Observe that, from the boundedness of D, $\mathcal{H}$ contains all polynomials. In particular each element of the sequence z_1^k, $k = 0, 1, \dots$ belongs to $\mathcal{H}$, where z_1 is the first variable of $z = (z_1, \dots, z_n) \in D \subset \mathbb{C}^n$. By applying the Gram–Schmidt orthonormalization procedure to the sequence z_1^k we can assume that there exists a sequence of linearly independent polynomials $P_k(z_1)$, $k = 0, 1 \dots$ in the variable z_1 such that $P_0(z_1) = F_0(z_1, \dots, z_n) = \lambda_0 \in \mathbb{C}^*$ and $P_k(z_1) = F_k(z_1, \dots, z_n)$, $\forall k = 1, \dots$. Consider now the holomorphic map of D into the standard complex Hilbert space $l^2(\mathbb{C})$ given by:

$$\phi : D \to l^2(\mathbb{C}), z = (z_1, \dots, z_n) \mapsto (P(z_1), F(z)), \tag{2.1}$$

where $P(z_1) = (P_0(z_1), P_1(z_1), \dots)$ and $F(z)$ is the sequence obtained by deleting the sequence z_1^k from the sequence $F_j(z)$.

Observe that $l^2(\mathbb{C})$ can be seen as the affine chart $Z_0 \neq 0$ of the infinite dimensional complex projective space $\mathbb{C}P^\infty$, endowed with homogeneous coordinates $[Z_0, \dots, Z_j, \dots]$. Moreover, the Fubini-Study metric g_{FS}^∞ of $\mathbb{C}P^\infty$ restricts to the Kähler metric

$$\frac{i}{2}\partial\bar{\partial}\log\left(1 + \sum_{j=1}^{+\infty} |w_j|^2\right) \left(w_j = \frac{Z_j}{Z_0}\right)$$

on $l^2(\mathbb{C})$ and it follows by the very definition of the Bergman metric that the map (2.1) induces a Kähler immersion

$$\Phi(z) = [P(z_1), F(z)] : (D, g_B) \to (\mathbb{C}P^\infty, g_{FS}^\infty). \tag{2.2}$$

Remark 2.1. The fact that a bounded domain endowed with its Bergman metric admits a Kähler immersion Φ into the infinite dimensional complex projective space is well-known and was first pointed out by Kobayashi [11]. For the proof of our main result it is crucial that the map Φ can be put in the special form (2.2).

For later use we give the following definition. Let S be a complex manifold. We say that a holomorphic map $\Psi : S \to \mathbb{C}P^\infty$ is *non-degenerate* iff $\Psi(S)$ is not contained in any *finite* dimensional complex projective space $\mathbb{C}P^N \subset \mathbb{C}P^\infty$. The following lemma summarizes what we need about non-degenerate maps.

Lemma 2.2. *Let $S \subset \mathbb{C}^n$ be an open subset of $\mathbb{C}^n$ and let*

$$\Psi : S \to \mathbb{C}P^\infty : z \mapsto [\psi_0(z), \psi_1(z), \dots]$$

be a holomorphic map induced by the holomorphic map

$$\psi : S \to l^2(\mathbb{C}) : z \mapsto (\psi_0(z), \psi_1(z), \dots)$$

where ψ_j, $j = 0, 1 \dots$ is an infinite sequence of holomorphic functions on S. Assume that there exists an infinite subsequence ψ_{j_α} of ψ_j, consisting of linearly independent functions such that for all $s \in S$ there exists a function of this subsequence non-vanishing at s. Then Ψ is non-degenerate. Furthermore, if Ψ is non-degenerate and $\tilde{\Psi} : S \to \mathbb{C}P^\infty$ is another holomorphic immersion which induces on S the same Kähler metric induced by Ψ, i.e. $\Psi^*(g_{FS}) = \tilde{\Psi}^*(g_{FS})$, then also $\tilde{\Psi}$ is non-degenerate.

Proof. Let W be the infinite dimensional complex subspace of $l^2(\mathbb{C})$ spanned by the vectors e_{j_α}, where e_j is the canonical basis of $l^2(\mathbb{C})$. Denote by $\pi : l^2(\mathbb{C}) \to W$ the projection onto W. Therefore, the map $\pi \circ \psi : S \to W \subset l^2(\mathbb{C})$ induces a holomorphic and non-degenerate map $S \to \mathbb{C}P^\infty$. Hence, *a fortiori*, the map Ψ is non-degenerate. The proof of the second part of the lemma is an immediate consequence of Calabi's rigidity theorem (see [6]), which asserts that any two Kähler immersions Ψ_1, Ψ_2 of a Kähler manifold S into $\mathbb{C}P^\infty$ are related by a unitary transformation U of $\mathbb{C}P^\infty$, *i.e.* $U \circ \Psi_1 = \Psi_2$. □

Proof of Theorem 1.2. We can restrict ourself to prove that the domain $D \subset \mathbb{C}^n$ equipped with its Bergman metric is not relative to any complex projective space $\mathbb{C}P^m$. Assume by contradiction this is the case. Then, there exists an open subset $S \subset \mathbb{C}$ passing through the origin and two Kähler immersions $f : S \to D$ and $h : S \to \mathbb{C}P^m$. If $(f_1, \dots, f_n)$ denote the components of f we can assume that $\frac{\partial f_1}{\partial \xi}(0) \neq 0$, where ξ is the complex coordinate on S. Consider the Kähler immersion of S into $\mathbb{C}P^\infty$ given by the composition $\Phi \circ f : S \to \mathbb{C}P^\infty$, where Φ is given by (2.2). We claim that this map is indeed non-degenerate. In order to prove our claim observe that from (2.2) one gets:

$$(\Phi \circ f)(\xi) = [P(f_1(\xi)), F(f_1(\xi), \dots, f_n(\xi))].$$

Hence, by the first part of Lemma 2.2, it is enough to prove that $P_k(f_1(\cdot)), k = 0, 1, \dots$ is a sequence of linearly independent functions on S. So, let q be any positive integer and assume that there exist q complex numbers $a_0, \dots, a_q$ such that

$$a_0 P_0(f_1(\xi)) + \cdots a_q P_q(f_1(\xi)) = 0, \ \forall \xi \in S. \tag{2.3}$$

By the assumption on $f_1 : S \to \mathbb{C}$ it follows that $f_1(S)$ is on open subset of $\mathbb{C}$. Therefore, equality (2.3) is satisfied on all $\mathbb{C}$, and since $P_1, \dots P_q$ are linearly independent all the a_j's are forced to be zero, proving our claim. Next, consider the Kähler immersion of S into $\mathbb{C}P^\infty$ given by the composition $i \circ h$, where $i : \mathbb{C}P^m \hookrightarrow \mathbb{C}P^\infty$ is the natural inclusion. Since this map is obviously degenerate the second part of Lemma 2.2 yields the desired contradiction. □

Before proving Theorem 1.4 let us explain the two main problems one has to face in proving it.

The first one comes from the fact that a Hermitian symmetric space of non-compact type (D, g) may only be equivalent to a bounded symmetric domain with

its Bergman metric (D, g_B) up to homotheties, *i.e.* $g = cg_B, c > 0$. Hence, we cannot apply directly Theorem 1.2 even when *weakly relatives* is replaced by *relatives*. Indeed, one can easily exibit two Kähler manifolds which are not r-relatives (for all r) but become s-relatives (for some s) when one multiplies the metric of one of them by a suitable constant (for example $(\mathbb{C}P^1, g = \lambda g_{FS})$ endowed with an irrational multiple λ of the Fubini–Study metric is not r-relative to $(\mathbb{C}P^1, g_{FS})$ but the later is obviously 1-relative to itself).

The second problem is that weakly relatives Kähler manifolds may not be relatives as shown by the following:

Example 2.3. Let X be a $K3$ surface with its hyperkählerian structure (see *e.g.* [5, page 400]). It is well-known that its isometry group $\mathrm{Iso}(X)$ is finite (see [1] for a beautiful description of this group). Let J_1 and J_2 be two parallel complex structures which do not belong to the same $\mathrm{Iso}(X)$-orbit. Then, the two Kähler manifolds (X, J_1) and (X, J_2) are obviously weakly 2-relatives but not 2-relatives.

The following lemma allow us to avoid the previous difficulty.

Lemma 2.4. *Let (D, g) be a Hermitian symmetric space of noncompact type and V be a projective Kähler manifold. If D and V are weakly relatives then D and V are also relatives.*

Proof. Let $L : S_1 \to S_2$ be the local isometry between the two Kähler manifolds S_1 and S_2 which makes D and V weakly relatives and let $h_1 : S_1 \to D$ and $h_2 : S_2 \to V$ be the corresponding Kähler immersions. Since all the concepts involved are of local nature, we can assume that L is a global isometry and, with the aid of De Rham decomposition theorem, that the Riemannian manifold $S_1 = S_2$ decompose as

$$S_1 = S_2 = F \times I_1 \times \cdots \times I_k,$$

where F is an open subset of the Euclidean space with the flat metric and I_j, $j = 1, \ldots k$ are irreducible Riemannian manifolds. Observe that the factor F is indeed a flat Kähler manifold of S_2. Since S_2 is a projective Kähler manifold it follows by the above mentioned theorem of Calabi (see the introduction) that F is trivial. We also claim that the above decomposition does not contain Ricci flat factors. Indeed, assume for example that I_j is such a factor. Then, as a consequence of the Gauss equation and the non-positivity of the curvature of (D, g), it follows that the map $h_1 : I_j \to D$ is totally geodesic. Since a totally geodesic submanifold of a locally homogeneous Riemannian manifold is also locally homogeneous, a well-known theorem of Alekseevsky–Kimel'fel'd–Spiro (see [2] and [13]) implies that I_j is actually flat, which proves our claim. Finally, observe that the isometry L above takes an irreducible factor I of S_1 into an irreducible factor $L(I)$ of S_2. Since these factors are not Ricci flat it is well-known that, $L : I \to L(I)$ or its conjugate $\bar{L}$ is holomorphic and so D and V are relatives since they share the same Kähler manifold I. □

Remark 2.5. The above lemma is valid (the proof follows the same line) when D is a homogeneous bounded domain of non-positive holomorphic bisectional curvature. Notice that the celebrated example of Pyatetski-Shapiro [12] shows that

Hermitian symmetric spaces of non-compact type are strictly contained in such domains (see also [7]).

Proof of Theorem 1.4. Let $\mathcal{H}_k$ be the Hilbert space consisting of holomorphic functions f on D such that $\int_D \frac{|f|^2}{K_B^{kc}} dz < +\infty$, where dz is the Lebesgue measure on D and k is a positive integer. Let F_j^k be an orthonormal basis for $\mathcal{H}_k$. It is not hard to see that $\frac{\sum_{j=0}^{+\infty} |F_j^k(z)|^2}{K_B^{kc}(z,z)}$ is invariant under the action of the group of isometric biholomorphisms of $(D, g = cg_B)$. Since this group acts transitively on D we have that

$$\sum_{j=0}^{+\infty} |F_j^k(z)|^2 = c_k K_B^{kc}(z, z), c_k > 0. \tag{2.4}$$

It can be shown that for k sufficiently large, $\mathcal{H}_k$ contains all polynomials (see [10] and reference therein). Fix such an k. As in the proof of the previous theorem one can built a holomorphic map

$$\Phi_k : D \to \mathbb{C}P^\infty, z = (z_1, \dots, z_n) \mapsto [P^k(z_1), F^k(z)], \tag{2.5}$$

where $P^k(z_1) = (P_0^k(z_1), P_1^k(z_1), \dots)$ is an infinite sequence of linearly independent polynomials in the variable z_1 and $P_0^k(z_1)$ is a non-zero complex number. Moreover, it follows by (2.4) that $\Phi_k^*(g_{FS}^\infty) = kg$. Observe now that there exists a holomorphic immersion $V_k : \mathbb{C}P^m \to \mathbb{C}P^{\binom{m+k}{m}}$ (obtained by a suitable rescaling of the Veronese embedding) satisfying $V_m^*(g_{FS}) = mg_{FS}$ (see Calabi [6]). Assume, by a contradiction, that the Hermitian symmetric space of noncompact type (D, g) and $\mathbb{C}P^m$ are weakly relatives. Then by Lemma 2.4 they are also relatives and so there exists an open subset $S \subset \mathbb{C}$ and two Kähler immersions $f : S \to D$ and $h : S \to \mathbb{C}P^m$. Then, obviously, (D, kg) and $(\mathbb{C}P^m, kg_{FS})$ would be relatives. Hence, as in the proof of Theorem 1.2, we get the desired contradiction, by applying Lemma 2.2 to the Kähler immersions $\Phi_k \circ f : S \to \mathbb{C}P^\infty$ and $i \circ V_k \circ h : S \to \mathbb{C}P^\infty$, (where $i : \mathbb{C}P^{\binom{m+k}{m}} \hookrightarrow \mathbb{C}P^\infty$ is the natural inclusion) which are respectively non-degenerate and degenerate. □

Remark 2.6. Observe that Theorem 1.4 when D is a rank one Hermitian symmetric space of non-compact type (*i.e.* $D = \mathbb{C}H^n$) was proven in [14] by Masaaki Umehara. Umehara's proof is based on the Calabi embedding $\mathbb{C}H^n \hookrightarrow l^2(\mathbb{C})$. Since such (Kählerian) embeddings do not exist for higher rank Hermitian symmetric space of non-compact type (see [8] for a proof) Umehara's approach cannot be used to give an alternative proof of our theorem.

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