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A drift homogenization problem revisited

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Abstract. This paper revisits a homogenization problem studied by L. Tartar related to a tridimensional Stokes equation perturbed by a drift (related to the Coriolis force). Here, a scalar equation and a two-dimensional Stokes equation with a L^2-bounded oscillating drift are considered. Under higher integrability conditions the Tartar approach based on the oscillations test functions method applies and leads to a limit equation with an extra zero-order term. When the drift is only assumed to be equi-integrable in L^2, the same limit behaviour is obtained. However, the lack of integrability makes difficult the direct use of the Tartar method. A new method in the context of homogenization theory is proposed. It is based on a parametrix of the Laplace operator which permits to write the solution of the equation as a solution of a fixed point problem, and to use truncated functions even in the vector-valued case. On the other hand, two counter-examples which induce different homogenized zero-order terms actually show the sharpness of the equi-integrability assumption.

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1. Introduction

At the end of the Seventies L. Tartar developed his method based on oscillating test functions to deal with the homogenization of PDE's. In the particular framework of hydrodynamics [14, 15] he studied the Stokes equation in a bounded domain Ω of $\mathbb{R}^3$, perturbed by an oscillating drift term, *i.e.*

$$\begin{cases} -\Delta u_\varepsilon + \operatorname{curl}(v_\varepsilon) \times u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the oscillations are produced by the sequence of vector-valued functions v_ε which weakly converges to some v in $L^3(\Omega)^3$. L. Tartar proved that the limit equa-

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tion of (1.1) is the Brinkman [5] type equation

$$\begin{cases} -\Delta u + \operatorname{curl}(v) \times u + \nabla p + Mu = f & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where M is a positive definite symmetric matrix-valued function. More precisely, M is defined by the convergences

$$(Dw_\varepsilon^\lambda)^T\, v_\varepsilon \rightharpoonup M\lambda \quad \text{weakly in } L^{\frac{3}{2}}(\Omega)^3, \quad \text{for any } \lambda \in \mathbb{R}^3, \tag{1.3}$$

where $w_\varepsilon^\lambda \in W^{1,3}(\Omega)^3$ solves the Stokes equation (1.1) in which the term $\operatorname{curl}(v_\varepsilon) \times u_\varepsilon$ is replaced by $\operatorname{curl}(v_\varepsilon) \times \lambda$. Then, the convergence (1.3) combined with the compactness of u_ε in $L^3(\Omega)^3$, yields the zero-order term Mu in (1.2). In [16] L. Tartar revisited this problem using the H-measures tool. On the other hand, the appearance of such a strange zero-order term in homogenization was also obtained from finely perforated domains by D. Cioranescu, F. Murat [6] for the Laplace equation, and by G. Allaire [2] for the Stokes equation, with zero Dirichlet boundary condition on the holes.

Since $\operatorname{curl}(v_\varepsilon) \times u_\varepsilon$ is orthogonal to u_ε, the energy associated with (1.1) is reduced to

$$\int_\Omega |Du_\varepsilon|^2\, dx, \tag{1.4}$$

and thus does not depend on the drift v_ε. Starting from this remark our aim is to study two drift homogenization problems associated with the same energy (1.4), and to specify the best integrability condition satisfied by the drift so that the Tartar approach holds. The first problem is scalar and the second problem is a two-dimensional equivalent of the Stokes problem (1.1). However, we have not succeeded in obtaining a similar result for the three-dimensional Stokes equation (1.1) since the best integrability assumption for v_ε is not clear.

In Section 2, we consider the following scalar equation in a bounded open set Ω of $\mathbb{R}^N$,

$$\begin{cases} -\Delta u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon + \operatorname{div}(b_\varepsilon\, u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $b_\varepsilon \in L^\infty(\Omega)^N$ is bounded in $L^2(\Omega)^N$. We obtain three different homogenization results:

In Section 2.1, assuming that the divergence of the drift b_ε is bounded in $W^{-1,q}(\Omega)$, with $q > N$, we prove (see Theorem 2.1) that the sequence u_ε weakly converges in $H_0^1(\Omega)$ to the solution u of the equation

$$\begin{cases} -\Delta u + b \cdot \nabla u + \operatorname{div}(b\, u) + \mu\, u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where μ is a nonnegative function. The proof follows the Tartar method using the oscillating test function

$$w_\varepsilon := \Delta^{-1}\big(\mathrm{div}\,(b_\varepsilon)\big) \in H^1_0(\Omega). \tag{1.7}$$

Then, in Section 2.2, assuming only the equi-integrability of the sequence ∇w_ε in $L^2(\Omega)^N$ (this is actually a weaker assumption than the equi-integrability of the whole sequence b_ε), we obtain (see Theorem 3.1) the limit problem (1.6) with

$$|\nabla w_\varepsilon - \nabla w|^2 \rightharpoonup \mu \ \text{ weakly in } L^1(\Omega) \quad \text{and} \quad \mu\, u^2 \in L^1(\Omega). \tag{1.8}$$

It seems intricate to apply directly the Tartar method with the test function w_ε, since we cannot control the terms $b_\varepsilon \cdot \nabla u_\varepsilon\, w_\varepsilon$ and $b_\varepsilon \cdot \nabla w_\varepsilon\, u_\varepsilon$. To this end, one should consider truncations of both w_ε and ∇w_ε. To overcome this difficulty we propose a new method, up to our knowledge, in the context of homogenization theory, based on a parametrix of the Laplace operator. It follows that u_ε reads as a solution of a fixed point problem, which allows us to estimate the sequence $\nabla w_\varepsilon \cdot \nabla u_\varepsilon$ only using a truncation of ∇w_ε. The equi-integrability of ∇w_ε then gives the thesis. Also assuming that $b \in L^q(\Omega)^N$, with $q > N$, (which ensures the uniqueness in (1.6)) we prove the following corrector result

$$u_\varepsilon - (1 + w_\varepsilon - w)\, u \longrightarrow 0 \quad \text{strongly in } W^{1,q}_{\mathrm{loc}}(\Omega), \quad \text{for any } q \in [1, N'). \tag{1.9}$$

Finally, in Section 2.3, we show the sharpness of the equi-integrability condition thanks to a counter-example in the periodic framework (see Theorem 2.6). Making a change of functions with $b_\varepsilon = \nabla w_\varepsilon$, equation (1.5) is shown to be equivalent to the following equation

$$-\Delta v_\varepsilon + \mu_\varepsilon\, v_\varepsilon = f_\varepsilon, \quad \text{with} \quad \mu_\varepsilon := |\nabla w_\varepsilon|^2, \tag{1.10}$$

the solution of which has the same limit as u_ε. G. Dal Maso, A. Garroni [7] proved that the class of equations of type (1.10) is stable under homogenization. Here, we do not use this general result, but we explicit an oscillating sequence w_ε so that the limit equation of (1.5), or equivalently (1.10), is

$$-\Delta u + \gamma\, u = f, \tag{1.11}$$

with an explicit constant γ which turns out to be $< \mu$. Therefore, the loss of equi-integrability for ∇w_ε violates the result of Section 2.2. Note that the vectorial character of the drift term in equation (1.5) makes difficult the derivation of a closure result similar to the one of [7] which is strongly based on the maximum principle.

In Section 3, we consider the following two-dimensional equivalent of the perturbed Stokes problem (1.1),

$$\begin{cases} -\Delta u_\varepsilon + \mathrm{curl}\,(v_\varepsilon)\, J u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega \\ \mathrm{div}\,(u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where J is the rotation matrix of angle 90°, and $v_\varepsilon \in L^\infty(\Omega)^2$ is bounded in $L^2(\Omega)^2$. Note that the weak formulation (3.4) of $\mathrm{curl}\,(v_\varepsilon)\,Ju_\varepsilon$ contains the drift term $(Du_\varepsilon)^T v_\varepsilon$, so that equation (1.12) can be also regarded as a drift problem. We follow the same scheme as in the scalar case:

In Section 3.1, assuming that the sequence v_ε is bounded in $L^r(\Omega)^2$ with $r > 2$, we show (see Theorem 3.1) that the sequence u_ε weakly converges in $H^1_0(\Omega)$ to the solution u of the Brinkman equation

$$\begin{cases} -\Delta u + \mathrm{curl}\,(v)\,Ju + \nabla p + Mu = f & \text{in } \Omega \\ \mathrm{div}\,(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where M is a symmetric positive definite matrix-valued function defined by the convergence (1.3) in $L^{\frac{2r}{r+2}}(\Omega)^2$.

In Section 3.2, assuming only the equi-integrability of the sequence v_ε in $L^2(\Omega)^2$, we prove (see Theorem 3.3) owing to the Tartar method that the sequence u_ε weakly converges in $H^1_0(\Omega)$ to the solution u of the Brinkman equation (1.13) with similarly to (1.8),

$$(Dw^\lambda_\varepsilon)^T\, v_\varepsilon \rightharpoonup M\lambda \quad \text{weakly in } L^1(\Omega)^2 \quad \text{and} \quad Mu \cdot u \in L^1(\Omega). \tag{1.14}$$

The proof is based on a double parametrix method carrying on both the velocity u_ε and the pressure p_ε. However, the proof of the last estimate of (1.14) is more delicate than the one of (1.8), since we cannot use a comparison principle as in the scalar case. We need to introduce a test function similar to w^λ_ε but associated with a truncation of v_ε. Moreover, if Ω has a regular boundary, $v \in L^r(\Omega)^2$ with $r > 2$, and $M \in L^m(\Omega)^{2\times 2}$ with $m > 1$, we get the corrector result

$$u_\varepsilon - u - W_\varepsilon\, u \longrightarrow 0 \text{ strongly in } W^{1,1}(\Omega)^2, \text{ where } W_\varepsilon\lambda := w^\lambda_\varepsilon, \text{ for } \lambda \in \mathbb{R}^2. \tag{1.15}$$

Finally, in Section 3.2, we construct an oscillating sequence v_ε which is not equi-integrable in $L^2(\Omega)^2$, which leads to the limit problem (1.13) involving a matrix Γ which is not symmetric and satisfies the strict inequality

$$\Gamma\lambda \cdot \lambda < M\lambda \cdot \lambda, \quad \text{for any } \lambda \neq 0,$$

which is inconsistent with the Tartar approach. This shows the sharpness of the equi-integrability condition as in the scalar case. It would be very interesting to find the closure of the family of problems (1.12) under the sole condition of L^2-boundedness of the sequences v_ε. This problem is far from being evident due to the absence of comparison principle for such a vector-valued equation.

Notations

- The space dimension is $N \geq 2$, and $2^* := \dfrac{2N}{N-2}$.

- The conjugate exponent of $p \geq 1$ is denoted by $p' := \dfrac{p}{p-1}$.
- For $u : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $Du := \left(\dfrac{\partial u_i}{\partial x_j}\right)_{1 \leq i,j \leq N}$.
- For $\Sigma : \mathbb{R}^N \longrightarrow \mathbb{R}^{N \times N}$, $\mathrm{Div}\,(\Sigma) := \left(\displaystyle\sum_{j=1}^{N} \frac{\partial \Sigma_{ij}}{\partial x_j}\right)_{1 \leq i \leq N}$.
- $H^1_\sharp(Y)$, with $Y := (0,1)^N$, denotes the space of the Y-periodic functions on $\mathbb{R}^N$ which belong to $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

2. A scalar equation with a drift term

Along this section Ω is a bounded regular open set of $\mathbb{R}^N$, with $N \geq 2$, and f is a distribution in $H^{-1}(\Omega)$.

2.1. The classical case

Let $q \in (N, \infty)$. Consider a sequence b_ε in $L^\infty(\Omega)^N$ such that

$$b_\varepsilon \rightharpoonup b \text{ weakly in } L^2(\Omega)^N \text{ and } \mathrm{div}\,(b_\varepsilon) \text{ is bounded in } W^{-1,q}(\Omega). \tag{2.1}$$

Let $w_\varepsilon \in W^{1,q}_0(\Omega)$ be the solution of the equation (see, *e.g.*, [9, Theorem 2.1])

$$\Delta w_\varepsilon = \mathrm{div}\,(b_\varepsilon) \quad \text{in } \mathcal{D}'(\Omega). \tag{2.2}$$

Up to a subsequence w_ε weakly converges in $W^{1,q}_0(\Omega)$ to the function w solution of

$$\Delta w = \mathrm{div}\,(b) \quad \text{in } \mathcal{D}'(\Omega). \tag{2.3}$$

We have the following result:

Theorem 2.1. *The solution $u_\varepsilon \in H^1_0(\Omega)$ of the equation*

$$-\Delta u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon + \mathrm{div}\,(b_\varepsilon\, u_\varepsilon) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{2.4}$$

weakly converges in $H^1_0(\Omega)$, up to a subsequence, to a solution $u \in H^1_0(\Omega)$ of the equation

$$-\Delta u + b \cdot \nabla u + \mathrm{div}\,(b\, u) + \mu\, u = f \quad \text{in } \mathcal{D}'(\Omega), \tag{2.5}$$

where μ is the function defined by the convergence

$$|\nabla w_\varepsilon - \nabla w|^2 \rightharpoonup \mu \quad \text{weakly in } L^{\frac{q}{2}}(\Omega). \tag{2.6}$$

Remark 2.2. The uniqueness for equation (2.4) is not evident under the sole assumption $b \in L^2(\Omega)^2$. Assuming a stronger integrability of b we will obtain in Theorem 2.4 the uniqueness for the limit equation.

Proof. The proof is based on the choice of appropriate oscillating test functions as Tartar did (see [13, Appendix], and [17]). The function w_ε of (2.2) will play the role of the oscillating test function. The variational formulation of (2.4) is

$$\begin{aligned}\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \int_\Omega b_\varepsilon \cdot \nabla u_\varepsilon\, \varphi\, dx - \int_\Omega b_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, dx \\ = \langle f, \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega).\end{aligned} \tag{2.7}$$

Then, by the Lax-Milgram theorem there exists a unique solution u_ε of (2.7) in $H_0^1(\Omega)$. In particular, for $v \in W^{1,\infty}(\Omega)$, putting $\varphi = v\, u_\varepsilon$ as test function in (2.7) we obtain the identity

$$\int_\Omega |\nabla u_\varepsilon|^2 v\, dx + \int_\Omega \nabla u_\varepsilon \cdot \nabla v\, u_\varepsilon\, dx - \int_\Omega b_\varepsilon \cdot \nabla v\, u_\varepsilon^2\, dx = \langle f, v\, u_\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \tag{2.8}$$

which will be used several times. So, choosing $v = 1$ in (2.8) the term with b_ε cancel so that we easily deduce that u_ε is bounded in $H_0^1(\Omega)$ and weakly converges, up to a subsequence, to a function u in $H_0^1(\Omega)$. Therefore, it follows from (2.7) the limit variational formulation

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx + \int_\Omega b \cdot \nabla u\, \varphi\, dx + \int_\Omega \varphi\, d\nu - \int_\Omega b \cdot \nabla\varphi\, u\, dx = \langle f, \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \tag{2.9}$$

which holds for any $\varphi \in W_0^{1,q}(\Omega)$ (due to the embedding of $W_0^{1,q}(\Omega)$ into $C(\bar\Omega)$ for $q > N$), where the measure ν is defined by the convergence

$$b_\varepsilon \cdot \nabla u_\varepsilon \rightharpoonup b \cdot \nabla u + \nu \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{2.10}$$

The limit equation associated with (2.9) is

$$-\Delta u + b \cdot \nabla u + \nu + \operatorname{div}(b\, u) = f \quad \text{in } \mathcal{D}'(\Omega). \tag{2.11}$$

Now, let us determine the measure ν of (2.10). Let $\varphi \in C_c^\infty(\Omega)$. Putting $\varphi\, w_\varepsilon$ as test function in (2.7) and $\varphi\, u_\varepsilon$ in (2.2), and taking the difference of the two equalities we get

$$\begin{aligned}&\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\, w_\varepsilon\, dx - \int_\Omega \nabla w_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, dx \\ &= \langle f, \varphi\, w_\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega b_\varepsilon \cdot \nabla u_\varepsilon\, \varphi\, w_\varepsilon\, dx + \int_\Omega b_\varepsilon \cdot \nabla w_\varepsilon\, \varphi\, u_\varepsilon\, dx \\ &\quad + \int_\Omega b_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, w_\varepsilon\, dx - \int_\Omega b_\varepsilon \cdot \nabla u_\varepsilon\, \varphi\, dx - \int_\Omega b_\varepsilon \cdot \nabla\varphi\, u_\varepsilon\, dx.\end{aligned} \tag{2.12}$$

Passing to the limit in (2.12) by using the strong convergence of u_ε in $L^p(\Omega)$, for $p < 2^*$, and the uniform convergence of w_ε in $C(\bar{\Omega})$ ($q > N$), we obtain

$$\begin{aligned}
&\int_\Omega \nabla u \cdot \nabla\varphi\, w\, dx - \int_\Omega \nabla w \cdot \nabla\varphi\, u\, dx \\
&= \langle f, \varphi\, w\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega b \cdot \nabla u\, \varphi\, w\, dx - \int_\Omega \varphi\, w\, d\nu + \int_\Omega \sigma\, \varphi\, u\, dx \\
&\quad + \int_\Omega b \cdot \nabla\varphi\, u\, w\, dx - \int_\Omega b \cdot \nabla u\, \varphi\, dx - \int_\Omega \varphi\, d\nu - \int_\Omega b \cdot \nabla\varphi\, u\, dx,
\end{aligned} \tag{2.13}$$

where the measure ν is defined by (2.10) and the function σ is defined, up to a subsequence, by the convergence

$$b_\varepsilon \cdot \nabla w_\varepsilon \rightharpoonup \sigma \quad \text{weakly in } L^{\frac{2q}{q+2}}(\Omega). \tag{2.14}$$

On the other hand, putting $\varphi\, w \in W_0^{1,q}(\Omega)$ in (2.9) and $\varphi\, u \in H_0^1(\Omega)$ in (2.3) we have

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla(\varphi\, w)\, dx = \langle f, \varphi\, w\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega b \cdot \nabla u\, \varphi\, w\, dx - \int_\Omega \varphi\, w\, d\nu \\
+ \int_\Omega b \cdot \nabla w\, \varphi\, u\, dx + \int_\Omega b \cdot \nabla\varphi\, u\, w\, dx,
\end{aligned} \tag{2.15}$$

$$\int_\Omega \nabla w \cdot \nabla(\varphi\, u)\, dx = \int_\Omega b \cdot \nabla u\, \varphi\, dx + \int_\Omega b \cdot \nabla\varphi\, u\, dx. \tag{2.16}$$

Equating the difference between (2.15) and (2.16) to the right-hand side of (2.13), it follows that

$$\int_\Omega \sigma\, \varphi\, u\, dx - \int_\Omega b \cdot \nabla w\, \varphi\, u\, dx - \int_\Omega \varphi\, d\nu = 0, \quad \text{for any } \varphi \in C_c^\infty(\Omega), \tag{2.17}$$

which implies that

$$\nu = \sigma\, u - b \cdot \nabla w\, u \quad \text{in } \mathcal{D}'(\Omega). \tag{2.18}$$

It thus remains to determine the limit equation (2.5). To this end, we pass to the limit by using $\varphi\, w_\varepsilon$ as test function in (2.2) and the definition (2.6) of μ, and we put $\varphi\, w$ in (2.3), which yields

$$\int_\Omega \left(\mu + |\nabla w|^2\right) \varphi\, dx + \int_\Omega \nabla w \cdot \nabla\varphi\, w\, dx = \int_\Omega \sigma\, \varphi\, dx + \int_\Omega b \cdot \nabla\varphi\, w\, dx, \tag{2.19}$$

$$\int_\Omega |\nabla w|^2\, \varphi\, dx + \int_\Omega \nabla w \cdot \nabla\varphi\, w\, dx = \int_\Omega b \cdot \nabla w\, \varphi\, dx + \int_\Omega b \cdot \nabla\varphi\, w\, dx. \tag{2.20}$$

Equating (2.19) and (2.20), we deduce that

$$\mu = \sigma - b \cdot \nabla w \quad \text{in } \mathcal{D}'(\Omega), \tag{2.21}$$

which combined with (2.18) implies that

$$\nu = \mu\, u \quad \text{in } \mathcal{D}'(\Omega). \tag{2.22}$$

Finally, the limit equation (2.11) and the relation (2.22) give the desired homogenized equation (2.5). □

Remark 2.3. It can be shown that

$$\mu(x) = \int_{S^{N-1}} \boldsymbol{\mu}\,(x, d\xi)\, \xi \cdot \xi\,, \tag{2.23}$$

where $\boldsymbol{\mu}$ denotes the matrix-valued H-measure (or micro-local defect measure) of the sequence b_ε (see [16] and [8]), and S^{N-1} the unit sphere of $\mathbb{R}^N$.

Assumption (2.1) is actually not sharp. In the next section we replace it by the boundedness of b_ε and the equi-integrability of ∇w_ε in $L^2(\Omega)^2$.

2.2. The case under an equi-integrability assumption

In this section Ω is a bounded open set of $\mathbb{R}^N$. Consider a sequence b_ε in $L^\infty(\Omega)^N$ the Hodge decomposition of which is

$$b_\varepsilon = \nabla w_\varepsilon + \xi_\varepsilon, \quad \text{with} \quad w_\varepsilon \in H_0^1(\Omega),\ \xi_\varepsilon \in L^2(\Omega)^N \text{ and } \operatorname{div}(\xi_\varepsilon) = 0, \tag{2.24}$$

such that

$$b_\varepsilon \rightharpoonup b \quad \text{weakly in } L^2(\Omega)^N. \tag{2.25}$$

Note that for a fixed $\varepsilon > 0$, $w_\varepsilon \in W^{1,p}(\Omega)$ and $\xi_\varepsilon \in L^p(\Omega)^N$ for any $p \in [2, \infty)$. But the essential point is the asymptotic behaviour of the sequences b_ε, ∇w_ε, ξ_ε. Our main assumption is the equi-integrability of the sequence ∇w_ε in $L^2(\Omega)^N$. By virtue of the Vitali-Saks theorem this is equivalent to the following convergence, up to an extraction of a subsequence,

$$|\nabla w_\varepsilon - \nabla w|^2 \rightharpoonup \mu \quad \text{weakly in } L^1(\Omega), \tag{2.26}$$

(Compare to (2.6) with $q > N$).

We have the following result:

Theorem 2.4.
i) *Under the equi-integrability assumption* (2.26) *the solution* u_ε *of* (2.4) *weakly converges in* $H_0^1(\Omega)$ *to a solution* u *of the equation*

$$-\Delta u + b \cdot \nabla u + \operatorname{div}(b\, u) + \mu\, u = f \quad \textit{in } \mathcal{D}'(\Omega), \tag{2.27}$$

with

$$\int_\Omega \mu\, u^2\, dx \le \langle f, u\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega |\nabla u|^2\, dx. \tag{2.28}$$

ii) *Also assume that* $b \in L^q(\Omega)^N$, *where* $q > 2$ *if* $N = 2$ *and* $q = N$ *if* $N > 2$. *Then, we have*

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega \mu\, u^2\,dx = \langle f, u\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \tag{2.29}$$

and there exists a unique solution $u \in H_0^1(\Omega)$ *of equation* (2.27), *with* $\mu\, u^2 \in L^1(\Omega)$.

Moreover, for any $p \in [1, 2)$ *if* $N = 2$ *and* $p = N'$ *if* $N > 2$, *we have the corrector result*

$$\nabla u_\varepsilon - \nabla u - (\nabla w_\varepsilon - \nabla w)\, u \longrightarrow 0 \quad \textit{strongly in } L^p_{\mathrm{loc}}(\Omega)^N, \tag{2.30}$$

and for any $r \in [1, p)$,

$$u_\varepsilon - (1 + w_\varepsilon - w)\, u \longrightarrow 0 \quad \textit{strongly in } W^{1,r}_{\mathrm{loc}}(\Omega). \tag{2.31}$$

Remark 2.5. No equi-integrability is required for the divergence free sequence ξ_ε. Actually, we can prove that the equi-integrability of the sequence b_ε in $L^2(\Omega)^N$ implies the equi-integrability of its two components ∇w_ε, ξ_ε in $L^2_{\mathrm{loc}}(\Omega)^N$. Therefore, condition (2.26) is really weaker than the equi-integrability of b_ε.

Moreover, the equi-integrability of ∇w_ε in $L^2(\Omega)^N$ is essential for deriving the limit equation with the zero-order term $\mu\, u$. When this condition is not satisfied we can obtain a similar limit equation but with a different zero-order term (see Section 2.3).

Proof of Theorem 2.4. The limit u of u_ε in $H_0^1(\Omega)$ solves the equation (2.11) where ν is defined by

$$b_\varepsilon \cdot \nabla u_\varepsilon - b \cdot \nabla u \rightharpoonup \nu \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{2.32}$$

We thus have

$$b_\varepsilon \cdot \nabla u_\varepsilon = (\xi_\varepsilon + \nabla w) \cdot \nabla u_\varepsilon + (\nabla w_\varepsilon - \nabla w) \cdot \nabla u_\varepsilon \rightharpoonup b \cdot \nabla u + \nu \quad \text{in } \mathcal{D}'(\Omega).$$

Moreover, by the Murat, Tartar div-curl lemma [11] the sequence $(\xi_\varepsilon + \nabla w) \cdot \nabla u_\varepsilon$ converges to $(\xi + \nabla w) \cdot \nabla u = b \cdot \nabla u$. This combined with the equi-integrability of ∇w_ε implies that ν is also given by the convergence

$$(\nabla w_\varepsilon - \nabla w) \cdot \nabla u_\varepsilon \rightharpoonup \nu \quad \text{weakly in } L^1(\Omega). \tag{2.33}$$

The proof of Theorem 2.4. is based on a parametrix method which allows us to express u_ε as a solution of a fixed point problem. As a consequence, we obtain a strong estimate of ∇u_ε in $L^p_{\mathrm{loc}}(\Omega)$ for some $p > 1$ close to 1. However, this estimate cannot provide directly the desired limit ν of (2.33) since $p < 2$. To overcome this difficulty we consider a truncation η_ε^k of ∇w_ε which is bounded by $k > 0$. Then, we

can pass to the limit as ε tends to zero in the product $\eta_\varepsilon^k \cdot \nabla u_\varepsilon$ for a fixed k. Hence, thanks to the equi-integrability of ∇w_ε we deduce the limit ν as k tends to infinity.

The proof is divided into four steps. In the first step we present the parametrix method which leads to a L^p-strong estimate of ∇u_ε. In the second step we determine the limit of the sequence $\eta_\varepsilon^k \cdot \nabla u_\varepsilon$ for a fixed $k > 0$. In the third step we determine the limit ν and the limit equation (2.27) together with (2.28). The fourth step is devoted to the proof of equality (2.29) and the corrector results (2.30) and (2.31).

First step. The parametrix method.

First, let us define a parametrix for the Laplace operator in Ω. To this end consider two sequences of functions φ_n, ψ_n in $C_c^\infty(\Omega)$, such that

$$\begin{cases} 0 \le \varphi_n,\, \psi_n \le 1 \text{ and } \varphi_n = 1 \text{ in } \operatorname{supp}(\psi_n)\,, \text{ for any } n \ge 1, \\ \big\{ n \ge 1 : \operatorname{supp}(\psi_n) \cap K \ne \emptyset \big\} \text{ is finite, for any compact subset } K \subset \Omega, \\ \displaystyle\sum_{n \ge 1} \psi_n = 1 \quad \text{in } \Omega. \end{cases} \tag{2.34}$$

Let E be the fundamental solution of the Laplace operator in $\mathbb{R}^N$. Then, the operator P defined in $\mathcal{D}'(\Omega)$ by

$$P(\zeta) := \sum_{n \ge 1} \psi_n\, E * (\varphi_n\, \zeta)\,, \quad \text{for } \zeta \in \mathcal{D}'(\Omega), \tag{2.35}$$

is a parametrix of the Laplace operator (see [1, Chapter I], for further details) which satisfies

$$P(\Delta\zeta) = \zeta - K(\zeta) \quad \text{and} \quad \Delta\big(P(\zeta)\big) = \zeta - K'(\zeta), \quad \text{for } \zeta \in \mathcal{D}'(\Omega), \tag{2.36}$$

where K, K' are two C^∞-kernel operators properly supported in Ω. Thanks to the Calderòn-Zygmund regularity for the Laplace operator (see, *e.g.*, [9, Theorem 2.1], and the references therein) we also have for any $p > 1$, and $s \in [0, 2]$ such that $s + \frac{1}{p}$ is not an integer,

$$P \text{ maps continuously } \mathcal{D}'(\Omega) \text{ to } \mathcal{D}'(\Omega), \text{ and } W^{-s,p}_{\mathrm{loc}}(\Omega) \text{ to } W^{2-s,p}_{\mathrm{loc}}(\Omega). \tag{2.37}$$

Then, applying (2.36) to the solution u_ε of (2.4) we have

$$\begin{aligned} u_\varepsilon &= P(\Delta u_\varepsilon) + K(u_\varepsilon) \\ &= P\left(\operatorname{div}\big[u\,(\nabla w_\varepsilon - \nabla w)\,\big]\right) + P\left(\operatorname{div}\big[\nabla w_\varepsilon\,(u_\varepsilon - u)\big]\right) \\ &\quad + P\big(\operatorname{div}(u \nabla w)\big) + P\big(\xi_\varepsilon \cdot \nabla u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon - f\big) + K(u_\varepsilon), \end{aligned} \tag{2.38}$$

Fix $p > 1$ close enough to 1 and $s \in (N/p', 1)$. Since $u_\varepsilon - u$ strongly converges to 0 in $L^q(\Omega)$ for any $q \in (2, 2^*)$, the sequence $\operatorname{div}\big(\nabla w_\varepsilon\,(u_\varepsilon - u)\big)$ strongly converges to 0 in $W^{-1,p}(\Omega)$, hence by (2.37) we have

$$P\left(\operatorname{div}\big[\nabla w_\varepsilon\,(u_\varepsilon - u)\big]\right) \longrightarrow 0 \quad \text{strongly in } W^{1,p}_{\mathrm{loc}}(\Omega).$$

Moreover, the sequence $\xi_\varepsilon \cdot \nabla u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon$ is bounded in $L^1(\Omega)$, thus in $W^{-s,p}(\Omega)$ since $s > N/p'$. Therefore, again by (2.37) the sequence $\nabla P\big(\xi_\varepsilon \cdot \nabla u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon - f\big)$ is bounded in $W^{1-s,p}(\Omega)^N$, and up to a subsequence strongly converges in $L^p_{\mathrm{loc}}(\Omega)^N$. Hence, since

$$\xi_\varepsilon \cdot \nabla u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon \rightharpoonup \xi \cdot \nabla u + \nu + b \cdot \nabla u \quad \text{in } \mathcal{D}'(\Omega),$$

we deduce from (2.38) the strong estimate

$$\begin{aligned}
&\nabla u_\varepsilon - \nabla P\left(\operatorname{div}\big[u\,(\nabla w_\varepsilon - \nabla w)\,\big]\right)\\
&= \nabla P\big(\operatorname{div}(u\nabla w) + \xi \cdot \nabla u + \nu + b \cdot \nabla u - f\big) + \nabla K(u) + o_{L^p_{\mathrm{loc}}(\Omega)^N}(1)\\
&= \nabla P\big(\nu + b \cdot \nabla u + \operatorname{div}(b\,u) - f\big) + \nabla K(u) + o_{L^p_{\mathrm{loc}}(\Omega)^N}(1)\\
&\qquad \big(\xi \cdot \nabla u = \operatorname{div}(u\,\xi)\big),
\end{aligned} \tag{2.39}$$

where $o_{L^p_{\mathrm{loc}}(\Omega)^N}(1)$ denotes a sequence which strongly converges to 0 in $L^p_{\mathrm{loc}}(\Omega)^N$. On the other hand, by (2.36) and (2.37) we have

$$\begin{aligned}
\nabla P\left(\operatorname{div}\big[u\,(\nabla w_\varepsilon - \nabla w)\,\big]\right) &= \nabla P\left(\Delta\big[u\,(w_\varepsilon - w)\,\big]\right) - \nabla P\left(\operatorname{div}\big[\nabla u\,(w_\varepsilon - w)\big]\right)\\
&= \nabla P\left(\Delta\big[u\,(w_\varepsilon - w)\,\big]\right) + o_{L^p_{\mathrm{loc}}(\Omega)^N}(1)\\
&= \nabla\big(u\,(w_\varepsilon - w)\,\big) + o_{L^p_{\mathrm{loc}}(\Omega)^N}(1)\\
&= u\,(\nabla w_\varepsilon - \nabla w) + o_{L^p_{\mathrm{loc}}(\Omega)^N}(1).
\end{aligned}$$

Therefore, this combined with (2.39) yields

$$\begin{aligned}
\nabla u_\varepsilon - u\,(\nabla w_\varepsilon - \nabla w) = \nabla P\big(\nu + b \cdot \nabla u + \operatorname{div}(b\,u) - f\big) + \nabla K(u)\\
+ o_{L^p_{\mathrm{loc}}(\Omega)^N}(1).
\end{aligned} \tag{2.40}$$

Second step. Estimate of the sequence $\eta^k_\varepsilon \cdot \nabla u_\varepsilon$.

Set $\eta^k_\varepsilon := \nabla w_\varepsilon\, 1_{\{|\nabla w_\varepsilon| < k\}}$, for a positive integer k. Let us determine the limit of $\eta^k_\varepsilon \cdot \nabla u_\varepsilon$ in $L^2_{\mathrm{loc}}(\Omega)^N$. Using a diagonal extraction, there exists a subsequence of ε, still denoted by ε, such that η^k_ε weakly converges to some η^k in $L^\infty(\Omega)^N$ for any k. By the strong convergence (2.40) combined with the weak convergence of $u\,(\nabla w_\varepsilon - \nabla w)$ to 0 in $L^p(\Omega)^N$ (for p close to 1) we have

$$\begin{aligned}
&\eta^k_\varepsilon \cdot \nabla u_\varepsilon - \big(\eta^k_\varepsilon - \eta^k\big) \cdot (\nabla w_\varepsilon - \nabla w)\,u\\
&\rightharpoonup \eta^k \cdot \nabla P\big(\nu + b \cdot \nabla u + \operatorname{div}(b\,u) - f\big) + \eta^k \cdot \nabla K(u) \quad \text{weakly in } L^p_{\mathrm{loc}}(\Omega).
\end{aligned}$$

Hence, we get that

$$\sigma^k = \mu^k\,u + \eta^k \cdot \nabla P\big(\nu + b \cdot \nabla u + \operatorname{div}(b\,u) - f\big) + \eta^k \cdot \nabla K(u) \quad \text{in } \Omega, \tag{2.41}$$

where

$$\begin{cases} \sigma^k := \lim\limits_{\varepsilon\to 0}\Big[\eta^k_\varepsilon\cdot\nabla u_\varepsilon\Big] & \text{weakly in } L^2(\Omega),\\ \mu^k := \lim\limits_{\varepsilon\to 0}\Big[\Big(\eta^k_\varepsilon-\eta^k\Big)\cdot(\nabla w_\varepsilon-\nabla w)\Big] & \text{weakly in } L^2(\Omega). \end{cases} \tag{2.42}$$

Third step. Determination of ν and the limit equation (2.27).

Starting from the limit equation (2.11) we have by (2.36)

$$u = P\big(\nu + b\cdot\nabla u + \operatorname{div}(b\,u) - f\big) + K(u) \quad \text{in } \Omega,$$

hence

$$\eta^k\cdot\nabla u = \eta^k\cdot\nabla P\big(\nu + b\cdot\nabla u + \operatorname{div}(b\,u) - f\big) + \eta^k\cdot\nabla K(u) \quad \text{in } \Omega.$$

Equating this with (2.41) we obtain

$$\sigma^k = \mu^k\,u + \eta^k\cdot\nabla u \quad \text{in } \Omega. \tag{2.43}$$

Now, let us pass to the limit as $k\to\infty$. By virtue of the equi-integrability of ∇w_ε in $L^2(\Omega)^N$ and by definition (2.42) the sequence μ^k strongly converges in $L^1(\Omega)$ to the function μ of (2.26), η^k strongly converges to ∇w in $L^2(\Omega)^N$, and σ^k strongly converges to $\nu+\nabla w\cdot\nabla u$ in $L^1(\Omega)$. Then, up to a subsequence μ^k converges to μ a.e. in Ω, and by the Fatou lemma combined with equality (2.43) we get

$$\begin{aligned} \int_\Omega |\mu\,u|\,dx &\le \liminf_{k\to\infty}\int_\Omega |\mu^k\,u|\,dx\\ &\le \liminf_{k\to\infty}\int_\Omega |\sigma^k-\eta^k\cdot\nabla u|\,dx = \int_\Omega |\nu|\,dx. \end{aligned} \tag{2.44}$$

We deduce from (2.44) and (2.43) that $\mu\,u\in L^1(\Omega)$ and

$$\nu = \mu\,u \quad \text{in } \Omega, \tag{2.45}$$

which yields the limit equation (2.27).

It remains to prove the inequality of (2.28). Let $v\in L^\infty(\Omega)$ and $t\in\mathbb{R}$. By (2.26), (2.33) and (2.45) we have

$$\begin{aligned} &\int_\Omega \big|\nabla u_\varepsilon - \nabla u - (\nabla w_\varepsilon - \nabla w)\,t\,v\big)\big|^2\,dx\\ &= \int_\Omega |\nabla u_\varepsilon - \nabla u|^2\,dx + t^2\int_\Omega |\nabla w_\varepsilon - \nabla w|^2\,v^2\,dx\\ &\quad - 2\,t\int_\Omega \nabla u_\varepsilon\cdot(\nabla w_\varepsilon - \nabla w)\,v\,dx + o(1)\\ &= \langle f, u\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} - \int_\Omega |\nabla u|^2\,dx + t^2\int_\Omega \mu\,v^2\,dx\\ &\quad - 2\,t\int_\Omega \mu\,u\,v\,dx + o(1), \end{aligned} \tag{2.46}$$

hence

$$t^2 \int_\Omega \mu\, v^2\, dx - 2\, t \int_\Omega \mu\, u\, v\, dx + \langle f, u\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega |\nabla u|^2\, dx \geq 0, \quad \forall\, t \in \mathbb{R}.$$

This implies that

$$\left(\int_\Omega \mu\, u\, v\, dx\right)^2 \leq \left(\langle f, u\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega |\nabla u|^2\, dx\right) \int_\Omega \mu\, v^2\, dx. \tag{2.47}$$

Let T_k, $k > 0$, be a function in $C^1(\mathbb{R})$ such that

$$0 \leq T_k' \leq 1 \quad \text{and} \quad \begin{cases} T_k(t) = t & \text{if } |t| \leq k \\ |T_k(t)| = k+1 & \text{if } |t| \geq k+2. \end{cases} \tag{2.48}$$

Putting $v = T_k(u)$ as test function in (2.47) and using that $T_k(u)^2 \leq u\, T_k(u)$, we get

$$\left(\int_\Omega \mu\, u\, T_k(u)\, dx\right)^2 \leq \left(\langle f, u\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega |\nabla u|^2\, dx\right) \int_\Omega \mu\, u\, T_k(u)\, dx,$$

hence

$$\int_\Omega \mu\, u\, T_k(u)\, dx \leq \langle f, u\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_\Omega |\nabla u|^2\, dx. \tag{2.49}$$

Since $u\, T_k(u)$ is a nondecreasing nonnegative sequence which converges to u^2 a.e. in Ω, the Beppo-Levi theorem applied to (2.49) thus gives inequality (2.28).

Fourth step. Proof of equality (2.29) and of the corrector results (2.30), (2.31).

Assume that $b \in L^q(\Omega)^N$, where $q > 2$ if $N = 2$ and $q = N$ if $N > 2$. Let φ_n be a sequence in $C_0^1(\mathbb{R})$ which strongly converges to u in $H_0^1(\Omega)$ and a.e. in Ω, and such that $|\nabla \varphi_n|$ is dominated by a fixed function in $L^2(\Omega)$. Putting the truncation function $T_k(\varphi_n)$ (2.48) in the limit equation (2.27) we have

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla T_k(\varphi_n)\, dx + \int_\Omega b \cdot \nabla u\, T_k(\varphi_n)\, dx - \int_\Omega b \cdot \nabla T_k(\varphi_n)\, u\, dx \\ &\quad + \int_\Omega \mu\, u\, T_k(\varphi_n) dx \\ &= \left\langle f, T_k(\varphi_n)\right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned}$$

Since $b \cdot \nabla u$, $\mu\, u \in L^1(\Omega)$ and $b\, u \in L^2(\Omega)^N$ (as a consequence of $b \in L^q(\Omega)^N$), we can pass to the limit as $n \to \infty$ in the previous equality owing to the Lebesgue dominated convergence theorem, which yields

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla T_k(u)\, dx + \int_\Omega b \cdot \nabla u\, T_k(u)\, dx - \int_\Omega b \cdot \nabla T_k(u)\, u\, dx \\ &\quad + \int_\Omega \mu\, u\, T_k(u)\, dx \\ &= \left\langle f, T_k(u)\right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{aligned} \tag{2.50}$$

Then, using that $|T_k(u)| \leq |u|$, $0 \leq T_k'(u) \leq 1$, $T_k(u)$ strongly converges to u in $H_0^1(\Omega)$, and that $b\,u \in L^2(\Omega)^N$, $\mu\,u^2 \in L^1(\Omega)$, and passing to the limit as $k \to \infty$ owing to the Lebesgue dominated convergence theorem we get

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega b\cdot\nabla u\,u\,dx - \int_\Omega b\cdot\nabla u\,u\,dx + \int_\Omega \mu\,u^2\,dx = \langle f, u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,,$$

which is (2.29). Moreover, the proof of equality (2.29) with $f = 0$ shows that there exists a unique solution $u \in H_0^1(\Omega)$ of equation (2.27), with $\mu\,u^2 \in L^1(\Omega)$.

It remains to prove the corrector results. By the estimate (2.46) with $v = T_k(u)$ and $t = 1$, combined with equality (2.29) we have

$$\begin{aligned}&\lim_{k\to\infty}\lim_{\varepsilon\to 0}\left(\int_\Omega \left|\nabla u_\varepsilon - \nabla u - (\nabla w_\varepsilon - \nabla w)\,T_k(u)\right|^2 dx\right)\\ &= \lim_{k\to\infty}\left(\int_\Omega \mu\big(u - T_k(u)\big)^2\,dx\right) = 0.\end{aligned} \tag{2.51}$$

On the other hand, let $p \in [1, 2)$ if $N = 2$ and $p = N'$ if $N > 2$, and consider an open set $\omega \Subset \Omega$. By the Hölder inequality we have

$$\begin{aligned}&\int_\omega \left|\nabla u_\varepsilon - \nabla u - (\nabla w_\varepsilon - \nabla w)\,u\right|^p dx\\ &\leq 2^{p-1}\left(\int_\Omega \left|\nabla u_\varepsilon - \nabla u - (\nabla w_\varepsilon - \nabla w)\,T_k(u)\right|^p\right.\\ &\qquad\qquad \left. + \int_\omega |\nabla w_\varepsilon - \nabla w|^p\,\left|u - T_k(u)\right|^p dx\right)\\ &\leq c\left(\int_\Omega \left|\nabla u_\varepsilon - \nabla u - (\nabla w_\varepsilon - \nabla w)\,T_k(u)\right|^2\right)^{\frac{p}{2}}\\ &\quad + c\left(\int_\omega \left|u - T_k(u)\right|^{\frac{2p}{2-p}}\,dx\right)^{1-\frac{p}{2}}\\ &\leq c\left(\int_\Omega \left|\nabla u_\varepsilon - \nabla u - (\nabla w_\varepsilon - \nabla w)\,T_k(u)\right|^2\right)^{\frac{p}{2}}\\ &\quad + c\left(\int_{\{|u|>k\}\cap\omega} |u|^{\frac{2p}{2-p}}\,dx\right)^{1-\frac{p}{2}}.\end{aligned} \tag{2.52}$$

Since $u \in L^{\frac{2p}{2-p}}(\omega)$ by the Sobolev embedding, passing successively to the limits $\varepsilon \to 0$ and $k \to \infty$ in (2.52) owing to convergence (2.51) we obtain the strong convergence (2.30).

Let $r \in [1, p)$. Since $w_\varepsilon - w$ strongly converges to 0 in $L^{\frac{2r}{2-r}}(\omega)$, by the Hölder inequality the sequence $(w_\varepsilon - w)\,\nabla u$ strongly converges to 0 in $L^r(\omega)^N$. Finally, this combined with (2.30) implies the corrector result (2.31). □

2.3. A counter-example

In this section Ω is a regular bounded open set of $\mathbb{R}^2$, and $Y := (-\frac{1}{2}, \frac{1}{2})^2$. For fixed $R \in (0, \frac{1}{2})$ and $\mu > 0$, let $r_\varepsilon \in (0, R)$ be defined by the equality

$$\frac{2\pi}{\varepsilon^2 \, |\ln r_\varepsilon|} = \mu. \tag{2.53}$$

Let W_ε be the Y-periodic function and w_ε be the εY-periodic function defined by

$$W_\varepsilon(y) := \begin{cases} \dfrac{\ln r - \ln r_\varepsilon}{\ln R - \ln r_\varepsilon} & \text{if } r := |y| \in (r_\varepsilon, R) \\ 0 & \text{si } r \le r_\varepsilon \\ 1 & \text{si } r \ge R, \end{cases} \qquad y \in Y, \;\; w_\varepsilon(x) := W_\varepsilon\left(\frac{x}{\varepsilon}\right), \;\; x \in \mathbb{R}^2. \tag{2.54}$$

Note that by (2.53) we have

$$\frac{1}{\varepsilon^2} \int_Y |\nabla W_\varepsilon|^2 \, dy = \frac{2\pi}{\varepsilon^2 \ln (R/r_\varepsilon)} \underset{\varepsilon \to 0}{\longrightarrow} \mu. \tag{2.55}$$

We then consider the drift b_ε defined by

$$b_\varepsilon(x) := \nabla w_\varepsilon(x) = \frac{1}{\varepsilon} \nabla W_\varepsilon\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \mathbb{R}^2. \tag{2.56}$$

Taking into account (2.53) it is easy to check that

$$w_\varepsilon \rightharpoonup 1 \quad \text{weakly in } H^1(\Omega) \text{ and weakly-}* \text{ in } L^\infty(\Omega). \tag{2.57}$$

Let f be a non-zero function in $L^2(\Omega)$. We study the asymptotic behavior of the equation (2.4) with the drift b_ε of (2.56), *i.e.*

$$-\Delta u_\varepsilon + \nabla w_\varepsilon \cdot \nabla u_\varepsilon + \operatorname{div}(\nabla w_\varepsilon \, u_\varepsilon) = f \quad \text{in } \mathcal{D}'(\Omega). \tag{2.58}$$

We have the following result:

Theorem 2.6. *The solution u_ε of* (2.58) *weakly converges in $H_0^1(\Omega)$ to the solution u of the equation*

$$-\Delta u + \gamma \, u = f \quad \textit{in } \mathcal{D}'(\Omega), \quad \textit{where} \quad \gamma := \frac{3\left(e^2 - 1\right)}{4\left(e^2 + 1\right)} \mu < \mu. \tag{2.59}$$

Remark 2.7. Using the periodicity we can check that the sequence $|b_\varepsilon|^2 = |\nabla w_\varepsilon|^2$ converges in the weak-$*$ sense of measures on Ω – but not weakly in $L^1(\Omega)$ – to the constant μ defined by (2.53). Theorem 2.6 can thus be regarded as a counter-example to the statement of Theorem 2.4 without the equi-integrability assumption on the drift b_ε in $L^2(\Omega)^2$. Indeed, the conclusion of Theorem 2.4 would give a limit equation (2.59), with $\gamma = \mu$.

Proof of Theorem 2.6. The proof is divided into two steps. In the first step we construct an oscillating test function z_ε which solves equation (2.64) below. In the second step we determine the limit equation (2.59).

First step. Construction of an oscillating test function.

Denote by Q_r the disk of radius r centered at the origin. Consider the unique solution Z_ε in $H^1(Q_R)$ of the equation

$$\begin{cases} -\dfrac{1}{\varepsilon^2}\,\Delta Z_\varepsilon + \dfrac{1}{\varepsilon^2}\,|\nabla W_\varepsilon|^2\, Z_\varepsilon = \dfrac{1_{Q_R}}{|Q_R|} & \text{in } Q_R \\ \dfrac{\partial Z_\varepsilon}{\partial n} = 0 & \text{on } \partial Q_R. \end{cases} \tag{2.60}$$

The function Z_ε is radial and can be computed explicitly. Using the Laplace operator in polar coordinates and $|\nabla W_\varepsilon|^2 = \alpha_\varepsilon^2\, r^{-2}\, 1_{Q_R\setminus\bar{Q}_{r_\varepsilon}}$, we get

$$Z_\varepsilon(r) = \begin{cases} -\dfrac{\varepsilon^2}{4\pi R^2}\, r^2 + c_\varepsilon & \text{if } r\in(0,r_\varepsilon] \\ a_\varepsilon\, r^{\alpha_\varepsilon} + b_\varepsilon\, r^{-\alpha_\varepsilon} + \dfrac{\varepsilon^2}{\pi R^2\left(\alpha_\varepsilon^2-3\right)}\, r^2 & \text{if } r\in(r_\varepsilon, R], \end{cases} \tag{2.61}$$

$$\text{where } \alpha_\varepsilon := \frac{1}{\ln(R/r_\varepsilon)}.$$

The constants a_ε, b_ε, c_ε are determined owing to the boundary condition on ∂Q_R and to the transmission conditions on $\partial Q_{r_\varepsilon}$, *i.e.*

$$Z'_\varepsilon(R) = 0 \quad \text{and} \quad Z_\varepsilon(r_\varepsilon^+) = Z_\varepsilon(r_\varepsilon^-), \quad Z'_\varepsilon(r_\varepsilon^+) = Z'_\varepsilon(r_\varepsilon^-). \tag{2.62}$$

We extend Z_ε by the constant value $Z_\varepsilon(R)$ in $Y\setminus\bar{Q}_R$, and by Y-periodicity in the whole space $\mathbb{R}^2$. The Y-periodic extension is still denoted by Z_ε. An explicit computation combined with (2.53) yields

$$Z_\varepsilon \longrightarrow \bar{Z} := \frac{4\left(e^2+1\right)}{3\left(e^2-1\right)}\,\frac{1}{\mu} \quad \text{strongly in } H^1_\sharp(Y). \tag{2.63}$$

As a consequence of (2.60), (2.61) the rescaled function $z_\varepsilon(x) := Z_\varepsilon(\frac{x}{\varepsilon})$ is solution of the equation

$$-\Delta z_\varepsilon + |\nabla w_\varepsilon|^2\, z_\varepsilon = \chi^\sharp_{Q_R}\left(\frac{x}{\varepsilon}\right) \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{2.64}$$

where $\chi^\sharp_{Q_R}$ is the Y-periodic function agreeing with $\frac{1_{Q_R}}{|Q_R|}$ in the period cell Y. Moreover, the following convergences hold

$$z_\varepsilon \rightharpoonup \bar{Z} \ \text{ weakly in } H^1(\Omega) \text{ and } \chi^\sharp_{Q_R}\left(\frac{x}{\varepsilon}\right) \rightharpoonup 1 \ \text{ weakly-}*\text{ in } L^\infty(\Omega), \tag{2.65}$$

where the constant $\bar{Z}$ is defined by (2.63).

Second step. Determination of the limit equation (2.59).

Define the function $v_\varepsilon := e^{1-w_\varepsilon}\, u_\varepsilon$. Then, equation (2.58) is equivalent to

$$-\Delta v_\varepsilon + |\nabla w_\varepsilon|^2\, v_\varepsilon = e^{1-w_\varepsilon}\, f \quad \text{in } \mathcal{D}'(\Omega). \tag{2.66}$$

G. Dal Maso, A. Garroni [7] proved that this class of equations is stable under homogenization. In the present case, the use of the oscillating test function z_ε will allow us to obtain the limit equation (2.59).

On the one hand, choosing $v = w_\varepsilon$ in (2.8) we get

$$\int_\Omega |\nabla w_\varepsilon|^2\, u_\varepsilon^2\, dx - \int_\Omega \nabla w_\varepsilon \nabla u_\varepsilon\, u_\varepsilon\, dx = \int_\Omega |\nabla u_\varepsilon|^2\, w_\varepsilon\, dx - \int_\Omega f\, w_\varepsilon\, u_\varepsilon\, dx \le c, \tag{2.67}$$

since u_ε is bounded in $H_0^1(\Omega)$ and $0 \le w_\varepsilon \le 1$. Then, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_\Omega |\nabla w_\varepsilon|^2\, u_\varepsilon^2\, dx &\le c + c\left(\int_\Omega |\nabla u_\varepsilon|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w_\varepsilon|^2\, u_\varepsilon^2\, dx\right)^{\frac{1}{2}} \\ &\le c + c'\left(\int_\Omega |\nabla w_\varepsilon|^2\, u_\varepsilon^2\, dx\right)^{\frac{1}{2}}, \end{aligned} \tag{2.68}$$

hence $u_\varepsilon \nabla w_\varepsilon$ is bounded in $L^2(\Omega)^2$. This combined with convergence (2.57) implies that v_ε weakly converges to u in $H_0^1(\Omega)$.

On the other hand, for $\varphi \in C_c^\infty(\Omega)$, putting the functions $\varphi\, z_\varepsilon$ in (2.66) and $\varphi\, v_\varepsilon$ in (2.64), taking the difference of the two equalities, and passing to the limit owing to convergences (2.65) we obtain the equality

$$\int_\Omega \nabla u \cdot \nabla \varphi\, \bar{Z}\, dx + \int_\Omega \varphi\, u\, dx = \int_\Omega f\, \varphi\, \bar{Z}\, dx, \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{2.69}$$

which is the variational formulation of equation (2.59), with $\gamma = \bar{Z}^{-1}$. □

3. A Stokes equation with a drift term

3.1. The classical case

In [14, 15] L. Tartar noted that the nonlinear term of the three-dimensional Navier-Stokes equation for the divergence free velocity u reads as

$$(u \cdot \nabla)\, u = \mathrm{Div}\,(u \otimes u) = \mathrm{curl}\,(u) \times u + \nabla\left(\tfrac{1}{2}\, |u|^2\right). \tag{3.1}$$

This led him to study the perturbed Stokes equation

$$-\Delta u + \mathrm{curl}\,(v) \times u + \nabla p = f, \tag{3.2}$$

where a given vector-valued function v replaced the velocity u of the Navier-Stokes equation. The equivalent of transformation (3.1) in two-dimension is

$$\begin{aligned} &\operatorname{Div}(u \otimes u) = \operatorname{curl}(u)\, Ju + \nabla\left(\tfrac{1}{2}|u|^2\right), \\ &\text{where } \operatorname{curl}(u) := \partial_1 u_2 - \partial_2 u_1 \quad \text{and} \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

More generally equality (3.3) extends for any divergence free functions u, v to the following one

$$\operatorname{curl}(v)\, Ju = \operatorname{Div}(v \otimes u) + (Du)^T v - \nabla(v \cdot u). \tag{3.4}$$

Similarly to (3.2) this leads us to the two-dimensional perturbed Stokes equation

$$-\Delta u + \operatorname{curl}(v)\, Ju + \nabla p = f. \tag{3.5}$$

Let Ω be a bounded domain of $\mathbb{R}^2$. Let v_ε be a sequence in $L^\infty(\Omega)^2$ and let f be a distribution in $H^{-1}(\Omega)^2$. Consider the perturbed Stokes equation

$$\begin{cases} -\Delta u_\varepsilon + \operatorname{curl}(v_\varepsilon)\, Ju_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

In the three-dimensional case where $\operatorname{curl}(v_\varepsilon) \times u_\varepsilon$ replaces $\operatorname{curl}(v_\varepsilon)\, Ju_\varepsilon$, L. Tartar [15] derived a Stokes equation with a Brinkman law under the assumption that v_ε is bounded in $L^3(\Omega)^3$ (see Introduction). Mimicking the Tartar approach in dimension two we can derive a similar homogenized equation using the test function w_ε^λ, for $\lambda \in \mathbb{R}^2$, solution of the Stokes equation

$$\begin{cases} -\Delta w_\varepsilon^\lambda + \operatorname{Div}\big((v_\varepsilon - v) \otimes \lambda\big) + \nabla q_\varepsilon^\lambda = 0 & \text{in } \Omega \\ \operatorname{div}\left(w_\varepsilon^\lambda\right) = 0 & \text{in } \Omega \\ w_\varepsilon^\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Then, we have the following result:

Theorem 3.1. *Assume that v_ε is bounded in $L^r(\Omega)^2$, with $r > 2$. Then, the solution u_ε of* (3.6) *weakly converges in $H_0^1(\Omega)$ to the solution u of the Brinkman equation*

$$\begin{cases} -\Delta u + \operatorname{curl}(v)\, Ju + \nabla p + Mu = f & \textit{in } \Omega \\ \operatorname{div}(u) = 0 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.8}$$

where M is the positive definite symmetric matrix-valued function defined by

$$\begin{cases} (Dw_\varepsilon^\lambda)^T v_\varepsilon \longrightarrow M\lambda & \textit{weakly in } L^{\frac{2r}{2+r}}(\Omega)^2 \textit{ and in } L^{\frac{r}{2}}_{\mathrm{loc}}(\Omega)^2 \\ Dw_\varepsilon^\lambda \cdot Dw_\varepsilon^\mu \rightharpoonup M\lambda \cdot \mu & \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega)^2 \textit{ and in } L^{\frac{r}{2}}_{\mathrm{loc}}(\Omega)^2, \end{cases} \quad \textit{for } \lambda, \mu \in \mathbb{R}^2. \tag{3.9}$$

Moreover, the zero-order term of (3.8) *is given by the convergences*

$$\begin{cases} (Du_\varepsilon)^T (v_\varepsilon - v) \rightharpoonup Mu & \textit{weakly in } L^{\frac{2r}{2+r}}(\Omega)^2 \\ Du_\varepsilon : Dw_\varepsilon^\lambda \rightharpoonup Mu \cdot \lambda & \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega) \textit{ and in } L^{\frac{2r}{2+r}}_{\mathrm{loc}}(\Omega)^2. \end{cases} \tag{3.10}$$

Proof. By the representation formula (3.4) we have

$$\operatorname{curl}(v_\varepsilon)\, Ju_\varepsilon = (Du_\varepsilon)^T v_\varepsilon + \operatorname{Div}(v_\varepsilon \otimes u_\varepsilon) - \nabla(v_\varepsilon \cdot u_\varepsilon). \tag{3.11}$$

Hence, the variational formulation of (3.6) reads as

$$\int_\Omega Du_\varepsilon : D\varphi\, dx + \int_\Omega (Du_\varepsilon)^T v_\varepsilon \cdot \varphi\, dx - \int_\Omega (v_\varepsilon \otimes u_\varepsilon) : D\varphi\, dx = \langle f, u_\varepsilon \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2},$$
$$\text{for any } \varphi \in H_0^1(\Omega)^2,\ \operatorname{div}(\varphi) = 0. \tag{3.12}$$

By the Lax-Milgram theorem there exists a unique divergence free function $u_\varepsilon \in H_0^1(\Omega)^2$ solution of (3.12). Then, putting the velocity u_ε as test function in (3.12) it follows that

$$\int_\Omega |Du_\varepsilon|^2\, dx = \langle f, u_\varepsilon \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}, \tag{3.13}$$

which implies that u_ε is bounded in $H_0^1(\Omega)^2$. Let ω be a regular domain of Ω. Applying (3.12) to divergence free functions in $H_0^1(\omega)^2$, there exists a unique p_ε in $L^2(\omega)/\mathbb{R}$ such that equation (3.6) holds in $\mathcal{D}'(\omega)^2$. Moreover, by (3.11) and the boundedness of v_ε in $L^r(\Omega)^2$ the sequence ∇p_ε is bounded in $H^{-1}(\omega)^2$. Hence, due to the regularity of ω the sequence p_ε is bounded in $L^2(\omega)$. Then, considering an exhaustive sequence of regular domains the union of which is Ω, we can construct in Ω a pressure p_ε which is bounded in $L^2_{\mathrm{loc}}(\Omega)$. Therefore, up to a subsequence the following convergences hold

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly in } H_0^1(\Omega)^2 \\ p_\varepsilon \rightharpoonup p & \text{weakly in } L^2_{\mathrm{loc}}(\Omega)/\mathbb{R}, \end{cases} \tag{3.14}$$

Now, in view of (3.11) it is enough to determine the limit of the term $(Du_\varepsilon)^T v_\varepsilon$. By the regularity results for the Stokes equation (see, *e.g.*, [10] Theorem 2, p. 67) the sequences w_ε^λ and q_ε^λ satisfy

$$\begin{cases} w_\varepsilon^\lambda \rightharpoonup 0 \text{ weakly in } H^1(\Omega)^2 \text{ and in } W^{1,r}_{\mathrm{loc}}(\Omega)^2 \\ q_\varepsilon^\lambda \rightharpoonup 0 \text{ weakly in } L^2(\Omega)/\mathbb{R} \text{ and in } L^r_{\mathrm{loc}}(\Omega)/\mathbb{R}. \end{cases} \tag{3.15}$$

which imply convergence (3.9). Let $\varphi \in C_c^\infty(\Omega)$. Following the Tartar method we put $\varphi\, w_\varepsilon^\lambda$ in equation (3.6) and $\varphi\, u_\varepsilon$ in equation (3.7). Then, from the representation

(3.11), the convergences (3.14), (3.15) and the boundedness of v_ε in $L^r(\Omega)$ we deduce that

$$\begin{cases} \displaystyle\int_\Omega Du_\varepsilon : Dw_\varepsilon^\lambda \, \varphi \, dx - \int_\Omega (v_\varepsilon \otimes u_\varepsilon) : Dw_\varepsilon^\lambda \, \varphi \, dx = o(1) \\ \displaystyle\int_\Omega Dw_\varepsilon^\lambda : Du_\varepsilon \, \varphi \, dx - \int_\Omega \big((v_\varepsilon - v) \otimes \lambda\big) : Du_\varepsilon \, \varphi \, dx = o(1), \end{cases} \quad \text{for any } \varphi \in C_c^\infty(\Omega),$$

hence

$$\begin{cases} Du_\varepsilon : Dw_\varepsilon^\lambda - (Dw_\varepsilon^\lambda)^T v_\varepsilon \cdot u_\varepsilon \rightharpoonup 0 \\ (Du_\varepsilon)^T (v_\varepsilon - v) \cdot \lambda - (Dw_\varepsilon^\lambda)^T v_\varepsilon \cdot u_\varepsilon \rightharpoonup 0 \end{cases} \quad \text{in } \mathcal{D}'(\Omega). \tag{3.16}$$

By virtue of the strong convergence of u_ε in any $L^s(\Omega)^2$ space for $s \in (1, \infty)$, convergences (3.16) and (3.9) imply (3.10). This combined with (3.11) yields finally the limit problem (3.8). □

Remark 3.2. It can be shown that

$$M(x) = \int_{S^1} \big[\operatorname{tr}(\boldsymbol{\mu}(x, d\xi)) - \boldsymbol{\mu}(x, d\xi)\, \xi \cdot \xi\big]\, \xi \otimes \xi \tag{3.17}$$

where $\boldsymbol{\mu}$ is the matrix-valued H-measure of the sequence v_ε (see [16, 17]).

The case where v_ε is only bounded in $L^2(\Omega)^2$ is much more delicate. On the one hand, under additional assumptions we will extend the Tartar result when v_ε is bounded and equi-integrable in $L^2(\Omega)^2$. On the other hand, we will give an example of a sequence v_ε for which the homogenized Brinkman equation is not the one obtained by the Tartar procedure.

3.2. The case under an equi-integrability condition

In this section we make the following weaker assumption on the drift,

$$v_\varepsilon \rightharpoonup v \ \text{ weakly in } L^2(\Omega)^2 \quad \text{and} \quad v_\varepsilon \text{ is equi-integrable in } L^2(\Omega)^2. \tag{3.18}$$

Then, we have the following extension of Theorem 3.1:

Theorem 3.3.
i) *Under the equi-integrability assumption* (3.18) *the solution* u_ε *of* (2.4) *weakly converges in* $H_0^1(\Omega)$ *to the solution* u *of equation* (3.8) *with*

$$\int_\Omega Mu \cdot u \, dx \le \langle f, u \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} - \int_\Omega |Du|^2 \, dx, \tag{3.19}$$

where M *is the positive definite symmetric matrix-valued function defined by*

$$\begin{cases} (Dw_\varepsilon^\lambda)^T v_\varepsilon \rightharpoonup M\lambda & \textit{weakly in } L^1(\Omega)^2 \\ Dw_\varepsilon^\lambda : Dw_\varepsilon^\mu \rightharpoonup M\lambda \cdot \mu & \textit{weakly-}* \textit{ in } \mathcal{M}(\Omega)^2, \end{cases} \quad \textit{for } \lambda, \mu \in \mathbb{R}^2. \tag{3.20}$$

ii) *Also assume that* Ω *has a Lipschitz boundary,* $v \in L^r(\Omega)^2$*, with* $r > 2$*, and* $M \in L^m(\Omega)^{2\times 2}$*, with* $m > 1$*. Then, we have the equality*

$$\int_\Omega |Du|^2\,dx + \int_\Omega Mu \cdot u\,dx = \langle f, u\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}, \tag{3.21}$$

and there exists a unique solution $u \in H_0^1(\Omega)^2$ *of equation* (3.8)*, with* $Mu \cdot u \in L^1(\Omega)$*.*

Moreover, we have the corrector result

$$u_\varepsilon - u - W_\varepsilon u \longrightarrow 0 \quad \textit{strongly in } W^{1,1}(\Omega)^2, \tag{3.22}$$

where W_ε *is the matrix-valued function defined by*

$$W_\varepsilon \lambda := w_\varepsilon^\lambda, \quad \textit{for } \lambda \in \mathbb{R}^2. \tag{3.23}$$

Remark 3.4. Contrary to Theorem 2.4, in the part ii) of Theorem 3.3 we need to assume a higher integrability for the matrix-valued M. Indeed, we cannot apply a truncation principle on $Mu \cdot u$. Moreover, the regularity of Ω is necessary to obtain the density of the smooth divergence free functions in the space of the divergence free functions of $H_0^1(\Omega)^2$.

Proof of Theorem 3.3. As in the proof of Theorem 3.3 the sequence u_ε is bounded in $H_0^1(\Omega)^2$, and thus in any $L^s(\Omega)^2$ space. Then, in view of (3.11) and (3.6) together with the boundedness of u_ε and v_ε the sequence ∇p_ε is bounded in $L^1(\Omega)^2 + W^{-1,r}(\Omega)^2$ for any $r \in (1,2)$. Hence, thanks to the embedding of $L^1_{\mathrm{loc}}(\Omega)$ into $W^{-\sigma,r}_{\mathrm{loc}}(\Omega)$ for any $r > 1$ and $\sigma > 2/r'$, the sequence p_ε is bounded in $L^r_{\mathrm{loc}}(\Omega)/\mathbb{R}$ for any $r \in (1,2)$. Therefore, up to a subsequence we have the convergences

$$\begin{cases} u_\varepsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega)^2 \\ p_\varepsilon \rightharpoonup p \text{ weakly in } L^r_{\mathrm{loc}}(\Omega)/\mathbb{R}, \text{ for any } r \in (1,2). \end{cases} \tag{3.24}$$

The problem is to determine the vector-valued distribution ν defined by

$$\mathrm{curl}\,(v_\varepsilon)\,Ju_\varepsilon - \mathrm{curl}\,(v)\,Ju \rightharpoonup \nu \quad \text{in } \mathcal{D}'(\Omega)^2. \tag{3.25}$$

Taking into account the representation formula (3.11) and the equi-integrability of v_ε in $L^2(\Omega)^2$, ν is actually in $L^1(\Omega)^2$, and is given by

$$(Du_\varepsilon)^T(v_\varepsilon - v) \rightharpoonup \nu \quad \text{weakly in } L^1(\Omega)^2, \tag{3.26}$$

so that u is solution of the equation

$$-\Delta u + \nu + \mathrm{curl}\,(v)\,Ju + \nabla p = f \quad \text{in } \mathcal{D}'(\Omega). \tag{3.27}$$

From now on the proof follows the same scheme as the one of Theorem 2.4 using a representation of the velocity and the pressure owing to the parametrix P of (2.35).

The proof is divided into five steps. The first step deals with a double parametrix method for both u_ε and p_ε, which allows us to derive a strong approximation of Du_ε. In the second step we compute the limit σ^k of the sequence $(Du_\varepsilon)^T v_\varepsilon^k$, where v_ε^k is a truncation of v_ε for a fixed $k > 0$. In the third step we obtain the limit equation (3.8). In the fourth step we prove inequality (3.19). The fifth step is devoted to the proof of equality (3.21) and the corrector result (3.22).

First step. The double parametrix method.

Consider the parametrix P (2.35) for the Laplace operator. Abusively we denote by Δ the vector-valued Laplace operator as well as by P the associated vector-valued parametrix each component of which is defined by (2.35). Taking the divergence of equation (3.6) we have

$$\Delta p_\varepsilon = \operatorname{div}(f) - \operatorname{div}\big(\operatorname{curl}(v_\varepsilon)\, J u_\varepsilon\big) \quad \text{in } \Omega,$$

hence by (2.36)

$$p_\varepsilon = P\left(\operatorname{div}(f) - \operatorname{div}\big[\operatorname{curl}(v_\varepsilon)\, J u_\varepsilon\big]\right) + K(p_\varepsilon) \quad \text{in } \Omega. \tag{3.28}$$

Substituting p_ε by the right-hand side of (3.28) in (3.6) it follows that

$$\Delta u_\varepsilon = \operatorname{curl}(v_\varepsilon)\, J u_\varepsilon - \nabla P\big(\operatorname{div}\big[\operatorname{curl}(v_\varepsilon)\, J u_\varepsilon\big]\big) + \nabla P\big(\operatorname{div}(f)\big) - f + \nabla K(p_\varepsilon) \quad \text{in } \Omega,$$

hence again by (2.36) we have in Ω

$$\begin{aligned} u_\varepsilon = {} & P\left(\operatorname{curl}(v_\varepsilon)\, J u_\varepsilon - \nabla P\big(\operatorname{div}\big[\operatorname{curl}(v_\varepsilon)\, J u_\varepsilon\big]\big)\right) \\ & + P\left(\nabla P\big(\operatorname{div}(f)\big) - f\right) + L(u_\varepsilon, p_\varepsilon), \end{aligned} \tag{3.29}$$

where L is a C^∞-kernel operator acting on the pair $(u_\varepsilon, p_\varepsilon)$. Using the representation (3.11) of $\operatorname{curl}(v_\varepsilon)\, J u_\varepsilon$, and setting

$$g_\varepsilon := \operatorname{Div}\big((v_\varepsilon - v) \otimes u_\varepsilon\big) - \nabla\big((v_\varepsilon - v) \cdot u_\varepsilon\big), \tag{3.30}$$

we get

$$u_\varepsilon = P\left((Du_\varepsilon)^T v_\varepsilon + g_\varepsilon - \nabla P\left(\operatorname{div}\big[(Du_\varepsilon)^T v_\varepsilon + g_\varepsilon\big]\right)\right) + F(u_\varepsilon, p_\varepsilon), \tag{3.31}$$

where

$$\begin{aligned} F(\zeta,\theta) := {} & P\big(\operatorname{Div}(v \otimes \zeta) - \nabla(v \cdot \zeta) - f - \nabla P\big(\operatorname{div}\big[\operatorname{Div}(v \otimes \zeta) - \nabla(v \cdot \zeta) - f\big]\big)\big) \\ & + L(\zeta, \theta). \end{aligned} \tag{3.32}$$

Note that by (2.37) we have

$$F(u_\varepsilon, p_\varepsilon) \longrightarrow F(u, p) \quad \text{strongly in } W^{1,r}_{\mathrm{loc}}(\Omega), \quad \text{for any } r \in (1, 2).$$

Moreover, by (3.26) the sequence $(Du_\varepsilon)^T v_\varepsilon$ weakly converges to $\nu + (Du)^T v$ in $L^1(\Omega)^2$ which is compactly embedded in $W^{-1,r}_{\mathrm{loc}}(\Omega)^2$ for any $r \in (1,2)$. Hence, as in the first step of the proof of Theorem 2.4, from (3.31) and the two previous convergences we deduce, for any $r \in (1,2)$, the strong convergence

$$u_\varepsilon - P\left(g_\varepsilon - \nabla P\big(\mathrm{div}\,(g_\varepsilon)\big)\right) \longrightarrow P\left(\nu + (Du)^T v - \nabla P\big(\mathrm{div}\left[\nu + (Du)^T v\right]\big)\right) + F(u,p) \quad \text{strongly in } W^{1,r}_{\mathrm{loc}}(\Omega)^2. \tag{3.33}$$

Second step. Determination of the limit σ^k of $(Du_\varepsilon)^T v^k_\varepsilon$.

Fix $r \in (1,2)$ such that (3.33) holds. Set

$$z_\varepsilon := P\left(g_\varepsilon - \nabla P\big(\mathrm{div}\,(g_\varepsilon)\big)\right) \quad \text{and} \quad q_\varepsilon := P\big(\mathrm{div}\,(g_\varepsilon)\big). \tag{3.34}$$

In view of (3.30) the sequence g_ε weakly converges to 0 in $W^{-1,r}(\Omega)^2$, hence by (2.37) we have

$$z_\varepsilon \rightharpoonup 0 \ \text{ weakly in } W^{1,r}_{\mathrm{loc}}(\Omega)^2 \quad \text{and} \quad q_\varepsilon \rightharpoonup 0 \ \text{ weakly in } L^r_{\mathrm{loc}}(\Omega)/\mathbb{R}. \tag{3.35}$$

Moreover, by (2.36) we have

$$\Delta z_\varepsilon = g_\varepsilon - \nabla q_\varepsilon - K'(g_\varepsilon) \quad \text{and} \quad \Delta q_\varepsilon = \mathrm{div}\,(g_\varepsilon) - K'(q_\varepsilon) \quad \text{in } \Omega, \tag{3.36}$$

hence

$$\Delta\big(\mathrm{div}\,(z_\varepsilon)\big) = K'(q_\varepsilon) - \mathrm{div}\left(K'(g_\varepsilon)\right) \longrightarrow 0 \quad \text{strongly in } L^r_{\mathrm{loc}}(\Omega)^2, \text{ say.}$$

This combined with the first convergence of (3.35) and (2.37) yields

$$\mathrm{div}\,(z_\varepsilon) \longrightarrow 0 \quad \text{strongly in } W^{2,r}_{\mathrm{loc}}(\Omega)^2. \tag{3.37}$$

On the other hand, set $v^k_\varepsilon := v_\varepsilon\, 1_{\{|v_\varepsilon|<k\}}$, for a positive integer k. Up to a subsequence of ε still denoted by ε, v^k_ε weakly converges to some function v^k in $L^2(\Omega)^2$ for any k. Consider for $\lambda \in \mathbb{R}^2$, the solutions $w^{\lambda,k}_\varepsilon$ and $q^{\lambda,k}_\varepsilon$ of the Stokes problem

$$\begin{cases} -\Delta w^{\lambda,k}_\varepsilon + \mathrm{Div}\left((v^k_\varepsilon - v^k)\otimes\lambda\right) + \nabla q^{\lambda,k}_\varepsilon = 0 & \text{in } \Omega \\ \mathrm{div}\left(w^{\lambda,k}_\varepsilon\right) = 0 & \text{in } \Omega \\ w^{\lambda,k}_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{3.38}$$

which consists in an approximation of equation (3.7). By the regularity results for the Stokes equation (see, *e.g.*, [10]) we have

$$\begin{cases} w^{\lambda,k}_\varepsilon \rightharpoonup 0 \ \text{ weakly in } W^{1,s}_{\mathrm{loc}}(\Omega)^2 \\ q^{\lambda,k}_\varepsilon \rightharpoonup 0 \ \text{ weakly in } L^s_{\mathrm{loc}}(\Omega)/\mathbb{R}, \end{cases} \quad \text{for any } s \in (1,\infty). \tag{3.39}$$

Take $s := r'$ and choose good oscillating test functions as Tartar did (see [13, Appendix]). Let $\varphi \in C_c^\infty(\Omega)$. Putting $\varphi\, w_\varepsilon^{\lambda,k}$ in the first equation of (3.36) and $\varphi\, z_\varepsilon$ in equation (3.38), and using the definition (3.30) of g_ε and the convergences (3.35), (3.37), (3.39) we have

$$(Dz_\varepsilon)^T (v_\varepsilon^k - v^k) \cdot \lambda - \big(Dw_\varepsilon^{\lambda,k}\big)^T (v_\varepsilon - v) \cdot u_\varepsilon \rightharpoonup 0 \quad \text{weakly in } \mathcal{D}'(\Omega)^2.$$

Hence, since Dz_ε weakly converges to 0 in $L^r(\Omega)^{2\times 2}$, we deduce that

$$(Dz_\varepsilon)^T v_\varepsilon^k \rightharpoonup M^k u \quad \text{weakly in } \mathcal{D}'(\Omega)^2, \tag{3.40}$$

where the matrix-valued function M^k is defined by

$$\big(Dw_\varepsilon^{\lambda,k}\big)^T v_\varepsilon \rightharpoonup M^k \lambda \quad \text{weakly in } L^s_{\text{loc}}(\Omega)^2, \quad \text{for any } s \in [1, 2). \tag{3.41}$$

Now, we are able to determine the limit σ^k of the sequence $(Du_\varepsilon)^T v_\varepsilon^k$ in $L^2(\Omega)^2$. With the definition (3.34) of z_ε the strong convergence (3.33) implies that

$$\begin{aligned}(Du_\varepsilon)^T v_\varepsilon^k - (Dz_\varepsilon)^T v_\varepsilon^k \rightharpoonup \Big[DP\Big(\nu + (Du)^T v - \nabla P\Big(\operatorname{div}\big[\nu + (Du)^T v\big]\Big)\Big)\Big]^T v^k \\ + \big(DF(u, p)\big)^T v^k \quad \text{weakly in } L^r_{\text{loc}}(\Omega)^2.\end{aligned}$$

This combined with (3.40) thus yields

$$\begin{aligned}\sigma^k = M^k u + \Big[DP\Big(\nu + (Du)^T v - \nabla P\Big(\operatorname{div}\big[\nu + (Du)^T v\big]\Big)\Big)\Big]^T v^k \\ + \big(DF(u, p)\big)^T v^k.\end{aligned} \tag{3.42}$$

Third step. Determination of the limit equation (3.8).

The function u solves the equation (3.27) which, by (3.4) and similarly to (3.31), can read as

$$u = P\Big(\nu + (Du)^T v - \nabla P\Big(\operatorname{div}\big[\nu + (Du)^T v\big]\Big)\Big) + F(u, p).$$

This implies that

$$(Du)^T v^k = \Big[DP\Big(\nu + (Du)^T v - \nabla P\Big(\operatorname{div}\big[\nu + (Du)^T v\big]\Big)\Big)\Big]^T v_k + \big(DF(u, p)\big)^T v^k.$$

Therefore, equating the previous equation with (3.42) yields

$$\sigma^k = (Du)^T v^k + M^k u \quad \text{in } \Omega. \tag{3.43}$$

It remains to pass to the limit as k tends to infinity. Due to the equi-integrability of v_ε in $L^2(\Omega)^2$ and by convergence (3.26) the sequence σ^k strongly converges to

$\nu + (Du)^T v$ in $L^1(\Omega)$. On the other hand, putting the function $w_\varepsilon^{\lambda,k} - w_\varepsilon^\lambda$ both in equations (3.7) and (3.38) we get the equality

$$\int_\Omega \left|Dw_\varepsilon^{\lambda,k} - Dw_\varepsilon^\lambda\right|^2 dx = \int_\Omega \left(Dw_\varepsilon^{\lambda,k} - Dw_\varepsilon^\lambda\right)^T (v_\varepsilon^k - v_\varepsilon - v^k + v) \cdot \lambda \, dx,$$

which, again by the equi-integrability of v_ε, yields

$$\lim_{k\to\infty} \sup_{\varepsilon>0} \left(\int_\Omega \left|Dw_\varepsilon^{\lambda,k} - Dw_\varepsilon^\lambda\right|^2 dx \right) = 0. \tag{3.44}$$

Estimate (3.44) implies that the sequence M^k defined by (3.41) strongly converges in $L^1(\Omega)^{2\times 2}$ to the matrix-valued function M defined by (3.20). In particular, up to a subsequence M^k converges to M a.e. in Ω. Then, by the Fatou lemma combined with (3.43) and the strong convergences of σ^k in $L^1(\Omega)^2$ and v_k in $L^2(\Omega)^2$, we get that the function Mu belongs to $L^1(\Omega)^2$. Finally, passing to the limit in (3.43) we obtain the equality

$$\nu = Mu \quad \text{in } \Omega,$$

which gives the limit equation (3.8).

Fourth step. Proof of inequality (3.19).

Similarly to (3.23) let W_ε^k, $k > 0$, be the matrix-valued function defined by $W_\varepsilon^k \lambda := w_\varepsilon^{\lambda,k}$, where $w_\varepsilon^{\lambda,k}$ solves (3.38). We simply denote $w_\varepsilon^{i,k}$ when $\lambda = e_i := (2-i, i-1)$, for $i = 1, 2$. Let $\varphi \in C_c^1(\Omega)^2$, and let $t \in \mathbb{R}$. Using (3.13) we have

$$\begin{aligned} \int_\Omega \left|Du_\varepsilon - Du - t\, D(W_\varepsilon^k \varphi)\right|^2 dx = {} & \langle f, u\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} - \int_\Omega |Du|^2 \, dx \\ & - 2t \int_\Omega Du_\varepsilon : D(W_\varepsilon^k \varphi)\, dx \\ & + t^2 \int_\Omega \left|D(W_\varepsilon^k \varphi)\right|^2 dx + o(1). \end{aligned} \tag{3.45}$$

Moreover, similarly to the second convergences of (3.9) and (3.10), we have for $i, j = 1, 2$,

$$\begin{cases} Du_\varepsilon : Dw_\varepsilon^{i,k} \rightharpoonup M^k u \cdot e_i \\ Dw_\varepsilon^{i,k} \cdot Dw_\varepsilon^{j,k} \rightharpoonup \hat{M}^k e_i \cdot e_j, \end{cases} \quad \text{weakly in } L^s_{\mathrm{loc}}(\Omega), \text{ for any } s \in [1, 2), \tag{3.46}$$

where (compare to the definition (3.41) of M^k) the matrix-valued $\hat{M}^k$ is defined by

$$\left(Dw_\varepsilon^{i,k}\right)^T v_\varepsilon^k \rightharpoonup \hat{M}^k e_i \quad \text{weakly in } L^2(\Omega)^2. \tag{3.47}$$

Then, from convergences (3.39) and (3.46) we deduce that

$$\int_\Omega Du_\varepsilon : D(W_\varepsilon^k\varphi)dx = \sum_{i=1}^2 \int_\Omega Du_\varepsilon : Dw_\varepsilon^{i,k}\,\varphi_i\,dx + o(1) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega M^k u\cdot\varphi\,dx,$$

$$\int_\Omega \Big|D(W_\varepsilon^k\varphi)\Big|^2 dx = \sum_{i,j=1}^2 \int_\Omega Dw_\varepsilon^{i,k} : Dw_\varepsilon^{j,k}\,\varphi_i\,\varphi_j\,dx + o(1) \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \hat{M}^k\varphi\cdot\varphi\,dx.$$

This combined with (3.45) implies that

$$\begin{aligned}\int_\Omega \Big|Du_\varepsilon - Du - t\,D(W_\varepsilon^k\varphi)\Big|^2 dx &= \langle f,u\rangle_{H^{-1}(\Omega)^2,H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx \\ &\quad - 2t\int_\Omega M^k u\cdot\varphi\,dx \\ &\quad + t^2\int_\Omega \hat{M}^k\varphi\cdot\varphi\,dx + o(1).\end{aligned} \tag{3.48}$$

Therefore, we have for any $t\in\mathbb{R}$,

$$t^2\int_\Omega \hat{M}^k\varphi\cdot\varphi\,dx - 2t\int_\Omega M^k u\cdot\varphi\,dx + \langle f,u\rangle_{H^{-1}(\Omega)^2,H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx \ge 0,$$

hence

$$\left(\int_\Omega M^k u\cdot\varphi\,dx\right)^2 \le \left(\langle f,u\rangle_{H^{-1}(\Omega)^2,H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx\right)\int_\Omega \hat{M}^k\varphi\cdot\varphi\,dx. \tag{3.49}$$

Let $\delta>0$, and let ω be an open set such that $\omega\Subset\Omega$. Since by (3.41) and (3.47) M^k and $\hat{M}^k$ belong to $L^s(\omega)^{2\times 2}$ for $s\in[1,2)$, putting in (3.49) strong approximations φ of $\frac{1_\omega u}{1+\delta|u|}$ in $L^{2s'}(\Omega)^2$, we get

$$\begin{aligned}\left(\int_\omega \frac{M^k u\cdot u}{1+\delta|u|}\,dx\right)^2 &\le \left(\langle f,u\rangle_{H^{-1}(\Omega)^2,H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx\right)\int_\omega \frac{\hat{M}^k u\cdot u}{(1+\delta|u|)^2}\,dx \\ &\le \left(\langle f,u\rangle_{H^{-1}(\Omega)^2,H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx\right)\int_\Omega \frac{\hat{M}^k u\cdot u}{(1+\delta|u|)^2}\,dx,\end{aligned}$$

which by the arbitrariness of ω yields the inequality

$$\left(\int_\Omega \frac{M^k u\cdot u}{1+\delta|u|}\,dx\right)^2 \le \left(\langle f,u\rangle_{H^{-1}(\Omega)^2,H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx\right)\int_\Omega \frac{\hat{M}^k u\cdot u}{(1+\delta|u|)^2}\,dx. \tag{3.50}$$

Recall that, by virtue of the equi-integrability of v_ε in $L^2(\Omega)^2$, the sequences M^k and $\hat{M}^k$ strongly converge to M in $L^1(\Omega)^{2\times 2}$, thus converge, up to a subsequence

of k, a.e. in Ω and in a dominated way. Therefore, passing to the limit as $k \to \infty$ owing to the Fatou lemma for the left-hand side of (3.50) and owing to the Lebesgue dominated convergence theorem for the right-hand side of (3.50), it follows that

$$\left(\int_\Omega \frac{Mu \cdot u}{1+\delta\,|u|}\,dx\right)^2 \le \left(\langle f, u\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx\right)\int_\Omega \frac{Mu \cdot u}{(1+\delta\,|u|)^2}\,dx < \infty,$$

which implies the inequality

$$\int_\Omega \frac{Mu \cdot u}{1+\delta\,|u|}\,dx \le \langle f, u\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} - \int_\Omega |Du|^2\,dx. \tag{3.51}$$

Finally, applying the Fatou lemma in (3.51) as $\delta \to 0$ we obtain the desired inequality (3.19).

Fifth step. Proof of equality (3.21) and of the corrector result (3.22).

Assume that Ω has a Lipschitz boundary, $v \in L^r(\Omega)^N$, with $r > 2$, and $M \in L^m(\Omega)^{2\times 2}$, with $m > 1$. Let φ be a divergence free function in $C_c^\infty(\Omega)^2$. Putting φ as test function in the limit Stokes equation (3.8) and using the representation formula (3.4) we have

$$\begin{aligned}\int_\Omega Du : D\varphi\,dx + \int_\Omega (Du)^T v \cdot \varphi\,dx - \int_\Omega (v \otimes u) : D\varphi\,dx + \int_\Omega Mu \cdot \varphi\,dx \\ = \langle f, u\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2}.\end{aligned} \tag{3.52}$$

Due to the regularity of Ω the set of divergence free functions is known to be dense in the space of divergence free functions in $H_0^1(\Omega)^2$ (see, *e.g.*, [18]). Moreover, by the higher integrability of v and M the mapping

$$\varphi \longmapsto \int_\Omega (Du)^T v \cdot \varphi\,dx - \int_\Omega (v \otimes u) : D\varphi\,dx + \int_\Omega Mu \cdot \varphi\,dx$$

is continuous in $H_0^1(\Omega)^2$. Therefore, considering in (3.52) a divergence free strong approximation φ of u in $H_0^1(\Omega)^2$ we get

$$\begin{aligned}\int_\Omega |Du|^2\,dx + \int_\Omega (Du)^T v \cdot u\,dx - \int_\Omega (v \otimes u) : Du\,dx + \int_\Omega Mu \cdot u\,dx \\ = \langle f, u\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2},\end{aligned}$$

which is (3.21). This equality clearly implies the uniqueness of a solution $u \in H_0^1(\Omega)^2$ of (3.8), with $Mu \cdot u \in L^1(\Omega)$.

It remains to prove the corrector result (3.22). Let $\varphi \in C_c^\infty(\Omega)$. Applying successively the triangle inequality and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, u)\big|\, dx \\
&\leq \int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, \varphi)\big|\, dx + \int_\Omega \big|D\big(W_\varepsilon\, (u-\varphi)\big)\big|\, dx \\
&\leq \int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, \varphi)\big|\, dx + \int_\Omega |DW_\varepsilon|\, |u-\varphi|\, dx + \int_\Omega |W_\varepsilon|\, |Du - D\varphi|\, dx \\
&\leq \int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, \varphi)\big|\, dx + c\, \|W_\varepsilon\|_{H^1(\Omega)^{2\times 2}}\, \|u-\varphi\|_{H_0^1(\Omega)^2}\,,
\end{aligned}$$

hence by the boundedness of W_ε in $H_0^1(\Omega)^{2\times 2}$,

$$\int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, u)\big|\, dx \leq \int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, \varphi)\big|\, dx + c\, \|u-\varphi\|_{H_0^1(\Omega)^2}. \tag{3.53}$$

On the other hand, proceeding as in fourth step owing to the second convergences of (3.20) and (3.10) (which hold in the weak-$*$ sense of measures on Ω) we get similarly to (3.48) the equality

$$\begin{aligned}
\int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, \varphi)\big|^2\, dx = \langle f, u\rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} - \int_\Omega |Du|^2\, dx \\
- 2\int_\Omega Mu\cdot\varphi\, dx + \int_\Omega M\,\varphi\cdot\varphi\, dx + o(1).
\end{aligned}$$

Hence, taking into account equality (3.21) and using the Hölder inequality combined with the embedding of $H_0^1(\Omega)$ in any $L^s(\Omega)$ space, it follows that

$$\begin{aligned}
\int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, \varphi)\big|^2\, dx &= \int_\Omega M\,(u-\varphi)\cdot(u-\varphi)\, dx + o(1) \\
&\leq c\, \|M\|_{L^m(\Omega)^{2\times 2}}\, \|u-\varphi\|^2_{H_0^1(\Omega)^2} + o(1).
\end{aligned} \tag{3.54}$$

Therefore, by (3.53) and (3.54) we obtain the inequality

$$\limsup_{\varepsilon\to 0} \int_\Omega \big|Du_\varepsilon - Du - D(W_\varepsilon\, u)\big|\, dx \leq c\, \|u-\varphi\|_{H_0^1(\Omega)^2}\,, \quad \text{for any } \varphi \in C_c^\infty(\Omega)^2, \tag{3.55}$$

which implies the desired convergence (3.22) and concludes the proof of Theorem 3.3. □

As in the scalar case we show in the next section that the equi-integrability condition is crucial to derive the limit Brinkman equation (3.8) with the matrix-valued function M introduced by L. Tartar [15, 17].

3.3. A counter-example

Let Ω be a regular bounded domain of $\mathbb{R}^2$. For $\varepsilon > 0$, let ω_ε be the intersection of Ω with the periodic lattice of disks of center $2\varepsilon\,\kappa$, $\kappa \in \mathbb{Z}^2$, and of radius $\varepsilon\, r_\varepsilon$ such that

$$\frac{4\pi}{\varepsilon^2\,|\ln r_\varepsilon|} \underset{\varepsilon\to 0}{\longrightarrow} \gamma \in (0,\infty). \tag{3.56}$$

This geometry was used by Cioranescu, Murat [6] for the Laplace equation and by Allaire [2] for the Stokes equation, in order to derive a "strange term" of zero-order from the homogenization of the Dirichlet boundary conditions on the small disks.

In the square $Y := (-1,1)^2$, let Q be the disk centered at the origin and of radius 1, and let Q_{r_ε} be the disk of same center and of radius r_ε with measure $|Q_{r_\varepsilon}| = \pi\, r_\varepsilon^2$. Then, for $f \in H^{-1}(\Omega)^2$, we consider the Stokes equation

$$\begin{cases} -\Delta u_\varepsilon + \dfrac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|}\, J u_\varepsilon + \nabla p_\varepsilon = f & \text{in } \Omega \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.57}$$

Note that, in view of the definition of ω_ε, we have $|\omega_\varepsilon| \approx |\Omega|\,|Q_{r_\varepsilon}|$. Moreover, if $z_\varepsilon \in H_0^1(\Omega)$ is the solution of the Laplace equation

$$\Delta z_\varepsilon = \frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} \quad \text{in } \mathcal{D}'(\Omega), \tag{3.58}$$

we have

$$\frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} = \operatorname{curl}(v_\varepsilon) \quad \text{in } \mathcal{D}'(\Omega), \quad \text{where} \quad v_\varepsilon := J\nabla z_\varepsilon. \tag{3.59}$$

Hence, the Stokes problem (3.57) is of the same type as (3.6). On the other hand, using successively the Cauchy-Schwarz inequality and the estimate (3.67) below combined with (3.56) we have

$$\int_\Omega |\nabla z_\varepsilon|^2\,dx = -\frac{|\omega_\varepsilon|}{|Q_{r_\varepsilon}|} \fint_{\omega_\varepsilon} z_\varepsilon\,dx \le \frac{|\omega_\varepsilon|}{|Q_{r_\varepsilon}|}\left(\fint_{\omega_\varepsilon} z_\varepsilon^2\,dx\right)^{\frac12} \le c\,\|\nabla z_\varepsilon\|_{L^2(\Omega)^2},$$

which implies that z_ε is bounded in $H_0^1(\Omega)$. Therefore, the sequence v_ε is bounded in $L^2(\Omega)^2$. Moreover, since by periodicity the sequence $\frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|}$ converges weakly-$*$ to $\frac14$ in $\mathcal{M}(\Omega)$, we get

$$v_\varepsilon \rightharpoonup v \ \text{ weakly in } L^2(\Omega)^2, \quad \text{with} \quad \operatorname{curl}(v) = \frac14 \ \text{ in } \mathcal{D}'(\Omega). \tag{3.60}$$

On the other hand, it is not difficult to check that v_ε is not equi-integrable in $L^2(\Omega)^2$. In fact, the following result shows that Theorem 3.3 does not hold for this particular sequence v_ε:

Theorem 3.5. *The sequence u_ε weakly converges in $H^1_0(\Omega)^2$ to the solution u of the Brinkman equation*

$$\begin{cases} -\Delta u + \dfrac{1}{4}\, J u + \nabla p + \Gamma u = f & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.61}$$

where the extra zero-order term Γu is given by

$$\left(\frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} - \frac{1}{4} \right) J u_\varepsilon \rightharpoonup \Gamma u \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2, \tag{3.62}$$

and Γ is the constant matrix defined by

$$\Gamma := \frac{1}{4\left(\gamma^2 + 1\right)} \left(\gamma\, I - J\right). \tag{3.63}$$

Moreover, the matrix obtained from convergence (3.20) *according to the Tartar approach is given by*

$$M = \frac{1}{4\,\gamma}\, I. \tag{3.64}$$

Remark 3.6. The matrix Γ of the Brinkman equation (3.61) is not symmetric contrary to the matrix M arising in the Tartar approach. Moreover, we have

$$\Gamma u \cdot u < M u \cdot u \quad \text{if } u \neq 0.$$

The gap between the two previous energies (which are the energies dissipated by viscosity according to [14]) is due to the loss of equi-integrability of the sequence v_ε defined by (3.59). Therefore, the equi-integrability of v_ε can be regarded as the best condition to ensure the result of Theorem 3.3.

Remark 3.7. It is worth to mention that the pathology displayed in Theorem 3.5 is not due to the absence of correctors. Indeed, with the oscillating sequences v^1_ε, v^2_ε defined by (3.68), (3.69) below, the following corrector result holds:

Proposition 3.8. *Assume that $u \in W^{1,r}(\Omega)^2$ for some $r > 2$. Then, we have*

$$\begin{gathered} u_\varepsilon - u - v_1\, v^1_\varepsilon - v_2\, v^2_\varepsilon \longrightarrow 0 \quad \text{strongly in } H^1(\Omega), \\ \text{where } v = (v_1, v_2) := \frac{1}{\gamma^2 + 1} \left(-u_1 + \gamma u_2, -u_2 - \gamma u_1\right). \end{gathered} \tag{3.65}$$

Remark 3.9. If the right-hand side f belongs to $W^{-1,r}(\Omega)^2$ for some $r > 2$, then using the regularity results for the Stokes equation (see, *e.g.*, [10]) the solution u of the Stokes equation (3.61) belongs to $W^{1,r}(\Omega)^2$. This provides a quite general condition under which the strong convergence (3.65) holds.

The proof of Theorem 3.5 is partially based on the properties of the test functions v_ε^1, v_ε^2 defined by (3.68), (3.69) below, and introduced by Allaire [2]. They were also used in [5] to derive a homogenized Brinkman type equation but, contrary to (3.6), from a Stokes equation without zero-order term. More precisely, in [2] the velocity is assumed to be zero in the set ω_ε. In [5] the viscosity is assumed to be very high in cylinders of section ω_ε, which leads to a three-dimensional nonlocal Brinkman equation. In the perturbed Stokes equation (3.57) a highly oscillating zero-order term is concentrated on ω_ε.

On the one hand, the sets Q_{r_ε} and ω_ε satisfy the following estimates:

Lemma 3.10. *There exists a constant $C > 0$ such that*

$$\forall\, V \in H^1(Y), \qquad \left| \fint_{Q_{r_\varepsilon}} V\, dy - \fint_Y V\, dy \right| \le C \sqrt{|\ln r_\varepsilon|}\, \|\nabla V\|_{L^2(Y)^2}, \tag{3.66}$$

$$\forall\, v \in H_0^1(\Omega), \qquad \fint_{\omega_\varepsilon} |v|^2\, dx \le C \left(1 + \varepsilon^2 |\ln r_\varepsilon|\right) \|\nabla v\|^2_{L^2(\Omega)^2}. \tag{3.67}$$

Proof. Estimate (3.66) can be easily proved using the polar coordinates. Estimate (3.67) is an immediate consequence of the Lemma 3 of [12], and can also be deduced from (3.66). □

On the other hand, consider the εY-periodic functions v_ε^i and p_ε^i, for $i = 1, 2$, defined by

$$v_\varepsilon^i(x) := V_\varepsilon^i\left(\frac{x}{\varepsilon}\right), \quad p_\varepsilon^i(x) := \frac{1}{\varepsilon} P_\varepsilon^i\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \mathbb{R}^2, \tag{3.68}$$

where $V_\varepsilon^i \in H_\#^1(Y)$ are $P_\varepsilon^i \in L^2(Y)$ are the Y-periodic functions defined by

$$V_\varepsilon^i := \begin{cases} e_i & \text{in } Q_{r_\varepsilon} \\ 0 & \text{in } Y \setminus Q, \end{cases} \qquad P_\varepsilon^i = 0 \ \text{ in } Q_{r_\varepsilon} \cup (Y \setminus Q), \qquad \int_Y P_\varepsilon^i\, dy = 0, \tag{3.69}$$

which solve the Stokes equation

$$-\Delta V_\varepsilon^i + \nabla P_\varepsilon^i = 0 \quad \text{in } Q_{r_\varepsilon} \setminus \bar{Q}. \tag{3.70}$$

Moreover, the sequences V_ε^i and P_ε^i satisfy the following estimates:

Lemma 3.11. *There exists a constant $C > 0$ such that*

$$\begin{cases} \|V_\varepsilon^i\|_{L^2(Y)^2} + \|DV_\varepsilon^i\|^2_{L^2(Y)^{2\times 2}} + \|P_\varepsilon^i\|^2_{L^2(Y)} \le \dfrac{C}{|\ln r_\varepsilon|} \\ \|V_\varepsilon^i\|_{L^\infty(Y)^2} \le C, \end{cases} \tag{3.71}$$

and for any function $V \in H^1(Y)$,

$$\left| \int_Y DV_\varepsilon^i : DV\, dy - \int_Y P_\varepsilon^i \operatorname{div}(V)\, dy - \gamma_\varepsilon^i\, e_i \cdot \left(\fint_{Q_{r_\varepsilon}} V - \fint_{Y \setminus Q} V \right) \right| \leq \frac{C}{|\ln r_\varepsilon|} \|DV\|_{L^2(Y)^{2\times 2}}, \tag{3.72}$$

where $\fint$ denotes the average value and

$$\gamma_\varepsilon^i \underset{\varepsilon \to 0}{\approx} \frac{4\pi}{|\ln r_\varepsilon|}. \tag{3.73}$$

Proof. Estimate (3.71) can be proved using the polar coordinates (see also [2]). Estimate (3.72) is a straightforward consequence of the Lemma 3.3 of [4] (with a refinement for the right-hand side of the inequality). □

Proof of Theorem 3.5. The proof is divided into two steps. In the first step we determine the homogenized Brinkman equation (3.61). The second step is devoted to the computation of the matrix M defined in the Tartar approach.

First step. Determination of the homogenized equation.

Using u_ε as test function we have

$$\int_\Omega |Du_\varepsilon|^2 = \langle f, u_\varepsilon \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} \leq c\, \|f\|_{H^{-1}(\Omega)^2}\, \|Du_\varepsilon\|_{L^2(\Omega)^{2\times 2}},$$

which implies that u_ε is bounded in $H_0^1(\Omega)^2$. On the other hand, let $\varphi \in C_c^\infty(\Omega)$ with zero Ω-average. There exists (see, *e.g.*, [3]) a vector-valued function $\Phi \in C_c^\infty(\Omega)^2$ such that

$$\operatorname{div}(\Phi) = \varphi \ \text{ in } \Omega \quad \text{and} \quad \|\Phi\|_{H_0^1(\Omega)^2} \leq c\, \|\varphi\|_{L^2(\Omega)},$$

where the constant c is independent of φ, Φ. Using Φ as test function in equation (3.57) and applying successively the Cauchy-Schwarz inequality, estimates (3.67) and (3.56) we get

$$\begin{aligned}
\left| \int_\Omega p_\varepsilon\, \varphi\, dx \right| &\leq \left| \langle f, \Phi \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} \right| + \left| \int_\Omega Du_\varepsilon : D\Phi \right| + \left| \int_\Omega \frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|}\, Ju_\varepsilon \cdot \Phi \right| \\
&\leq c\, \|D\Phi\|_{L^2(\Omega)^{2\times 2}} + c \left(\fint_{\omega_\varepsilon} |u_\varepsilon|^2 \right)^{\frac{1}{2}} \left(\fint_{\omega_\varepsilon} |\Phi|^2 \right)^{\frac{1}{2}} \\
&\leq c\, \|D\Phi\|_{L^2(\Omega)^{2\times 2}} + c\, \varepsilon^2\, |\ln r_\varepsilon|\, \|Du_\varepsilon\|_{L^2(\Omega)^{2\times 2}}\, \|D\Phi\|_{L^2(\Omega)^{2\times 2}} \\
&\leq c\, \|\varphi\|_{L^2(\Omega)}.
\end{aligned}$$

This combined with the regularity of Ω implies that p_ε is bounded in $L^2(\Omega)/\mathbb{R}$. Therefore, up to a subsequence the following convergences hold

$$\begin{cases} u_\varepsilon \rightharpoonup u \text{ weakly in } H^1_0(\Omega)^2 \\ p_\varepsilon \rightharpoonup p \text{ weakly in } L^2(\Omega)/\mathbb{R}. \end{cases} \tag{3.74}$$

Now, we have to determine the limit of the sequence $\frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} J u_\varepsilon$. On the one hand, re-scaling inequality (3.72) we obtain that the functions v^i_ε and p^i_ε, $i = 1, 2$, of (3.68) and any function $v \in H^1_0(\Omega)^2$ satisfy the inequality

$$\begin{aligned} &\left| \int_\Omega Dv^i_\varepsilon : Dv - \int_\Omega p^i_\varepsilon \operatorname{div}(v) - \frac{\gamma^i_\varepsilon}{\varepsilon^2} e_i \cdot \left(\int_\Omega \frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} v - \int_\Omega \frac{1_{Y\setminus Q}}{|Y \setminus Q|} \left(\frac{x}{\varepsilon}\right) v \right) \right| \\ &\quad \leq \frac{c}{\varepsilon\, |\ln r_\varepsilon|} \|Dv\|_{L^2(\Omega)^{2\times 2}}. \end{aligned} \tag{3.75}$$

Moreover, by (3.71) and (3.56) the following convergences hold

$$\begin{cases} v^i_\varepsilon \rightharpoonup 0 \text{ weakly in } H^1(\Omega)^2 \\ p^i_\varepsilon \rightharpoonup 0 \text{ weakly in } L^2(\Omega)/\mathbb{R}. \end{cases} \tag{3.76}$$

Then, applying inequality (3.75) with $v = \varphi\, u_\varepsilon$, $\varphi \in C^\infty_c(\Omega)$, we deduce from (3.73) and (3.56) that

$$\int_\Omega Dv^i_\varepsilon : Du_\varepsilon\, \varphi - (\gamma + o(1))\, e_i \cdot \left(\int_\Omega \frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} \varphi\, u_\varepsilon - \int_\Omega \frac{1}{4} \varphi\, u \right) = o(1). \tag{3.77}$$

On the other hand, putting $\varphi\, v^i_\varepsilon$ as test function in (3.57), using that $v^i_\varepsilon = e_i$ in ω_ε and the convergences (3.76), (3.74), we have

$$\int_\Omega Du_\varepsilon : Dv^i_\varepsilon\, \varphi + \int_\Omega \frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} J u_\varepsilon \cdot e_i\, \varphi = o(1). \tag{3.78}$$

Denote

$$\nu := \lim_{\varepsilon\to 0} \frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} J u_\varepsilon \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2,$$

where the limit holds up to a subsequence by virtue of the estimate (3.67) combined with the Cauchy-Schwarz inequality. Then, equating (3.77) and (3.78) and passing to the limit we get for $i = 1, 2$,

$$\int_\Omega \varphi\, e_i \cdot \nu = \gamma \int_\Omega \varphi\, e_i \cdot J\nu + \frac{\gamma}{4} \int_\Omega \varphi\, e_i \cdot u, \quad \text{for any } \varphi \in C^\infty_c(\Omega),$$

which implies the equality $\nu = \gamma\, J\nu + \dfrac{\gamma}{4} u$. Hence, we deduce the convergence

$$\frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|} J u_\varepsilon \rightharpoonup \nu = \frac{\gamma}{4} (I - \gamma\, J)^{-1} u = \frac{\gamma}{4\,(\gamma^2+1)} (I + \gamma\, J)\, u \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2. \tag{3.79}$$

Therefore, passing to the limit in (3.57) with (3.79) we obtain the homogenized equation

$$-\Delta u + \frac{1}{4}\, Ju + \nabla p + \frac{1}{4\,(\gamma^2+1)}\,(\gamma\, I - J)\, u = f \quad \text{in } \mathcal{D}'(\Omega), \tag{3.80}$$

which yields the desired Brinkman equation (3.61) with the matrix Γ of (3.63).

Second step. Derivation of the matrix M.

Let $\lambda \in \mathbb{R}^2$. Consider the solutions $W^\lambda_{\sharp,\varepsilon} \in H^1_\sharp(Y)$ (the set of the Y-periodic functions in $H^1_{\text{loc}}(\mathbb{R}^2)$) and $Q^\lambda_{\sharp,\varepsilon} \in L^2_\sharp(Y)/\mathbb{R}$ of the perturbed Stokes problem

$$\begin{cases} -\Delta W^\lambda_{\sharp,\varepsilon} + \varepsilon\left(\dfrac{1_{Q_{r_\varepsilon}}}{|Q_{r_\varepsilon}|} - \dfrac{1}{4}\right) J\lambda + \nabla Q^\lambda_{\sharp,\varepsilon} = 0 & \text{in } \mathbb{R}^2 \\ \operatorname{div}\left(W^\lambda_{\sharp,\varepsilon}\right) = 0 & \text{in } \mathbb{R}^2 \\ W^\lambda_{\sharp,\varepsilon} & \text{is } Y\text{-periodic} \\ \displaystyle\int_Y W^\lambda_{\sharp,\varepsilon} = 0. \end{cases} \tag{3.81}$$

Note that the first equation of (3.81) is equivalent to the variational formulation in the torus,

$$\forall\, V \in H^1_\sharp(Y),\ \int_Y DW^\lambda_{\sharp,\varepsilon} : DV\, dy + \varepsilon\left(\fint_{Q_{r_\varepsilon}} V - \fint_Y V\right)\cdot J\lambda - \int_Y Q^\lambda_{\sharp,\varepsilon} \operatorname{div}(V)\, dy = 0. \tag{3.82}$$

Hence, the re-scaled functions $w^\lambda_{\sharp,\varepsilon}$ and $q^\lambda_{\sharp,\varepsilon}$ defined by

$$w^\lambda_{\sharp,\varepsilon}(x) := \varepsilon\, W^\lambda_{\sharp,\varepsilon}\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad q^\lambda_{\sharp,\varepsilon}(x) := Q^\lambda_{\sharp,\varepsilon}\left(\frac{x}{\varepsilon}\right), \quad \text{for } x \in \Omega, \tag{3.83}$$

are εY-periodic solutions of the problem

$$\begin{cases} -\Delta w^\lambda_{\sharp,\varepsilon} + \left(\dfrac{1_{Q_{r_\varepsilon}}}{|Q_{r_\varepsilon}|}\left(\dfrac{x}{\varepsilon}\right) - \dfrac{1}{4}\right) J\lambda + \nabla q^\lambda_{\sharp,\varepsilon} = 0 & \text{in } \mathbb{R}^2 \\ \operatorname{div}\left(w^\lambda_{\sharp,\varepsilon}\right) = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{3.84}$$

First of all, let us determine a priori estimates satisfied by the sequences $W^\lambda_{\sharp,\varepsilon}$, $w^\lambda_{\sharp,\varepsilon}$, $Q^\lambda_{\sharp,\varepsilon}$, and $q^\lambda_{\sharp,\varepsilon}$. Putting $W^\lambda_{\sharp,\varepsilon}$ as test function in equation (3.82) we have

$$\int_Y \left|DW^\lambda_{\sharp,\varepsilon}\right|^2 dy + \varepsilon \fint_{Q_{r_\varepsilon}} J\lambda \cdot W^\lambda_{\sharp,\varepsilon}\, dy = 0, \tag{3.85}$$

hence by the estimates (3.66) of Lemma 3.10 and (3.56)

$$\begin{aligned}\|DW^\lambda_{\sharp,\varepsilon}\|^2_{L^2(Y)^{2\times 2}} &= \varepsilon\left|\fint_{Q_{r_\varepsilon}} W^\lambda_{\sharp,\varepsilon}\,dy\right| \\ &\le C\,\varepsilon\sqrt{|\ln r_\varepsilon|}\,\|DW^\lambda_{\sharp,\varepsilon}\|_{L^2(Y)^{2\times 2}} \\ &\le c\,\|DW^\lambda_{\sharp,\varepsilon}\|_{L^2(Y)^{2\times 2}}.\end{aligned}$$

Therefore, $W^\lambda_{\sharp,\varepsilon}$ is bounded in $H^1_\sharp(Y)^2$, and there exists a constant vector $\bar{W}^\lambda \in \mathbb{R}^2$ such that up to a subsequence we have

$$\lim_{\varepsilon\to 0}\left(\varepsilon\fint_{Q_{r_\varepsilon}} W^\lambda_{\sharp,\varepsilon}\,dy\right) = \bar{W}^\lambda. \tag{3.86}$$

On the other hand, let $\varphi \in C^\infty_\sharp(Y)$ with zero Y-average. There exists $\Phi \in C^\infty_\sharp(Y)^2$ with zero Y-average such that

$$\operatorname{div}(\Phi) = \varphi \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad \|\Phi\|_{L^2(Y)^{2\times 2}} \le c\,\|\varphi\|_{L^2(Y)},$$

where c is a constant independent of φ, Φ. Putting Φ as test function in (3.82) we have by (3.66) and (3.56)

$$\begin{aligned}\left|\int_Y Q^\lambda_{\sharp,\varepsilon}\,\varphi\,dy\right| &\le \left|\int_Y DW^\lambda_{\sharp,\varepsilon} : D\Phi\,dy\right| + \varepsilon\left|\fint_{Q_{r_\varepsilon}} \Phi\,dy\right| \\ &\le c\,\|D\Phi\|_{L^2(Y)^{2\times 2}} + c\,\varepsilon\sqrt{|\ln r_\varepsilon|}\,\|D\Phi\|_{L^2(Y)^{2\times 2}} \\ &\le c\,\|D\Phi\|_{L^2(Y)^{2\times 2}} \le c\,\|\varphi\|_{L^2(Y)},\end{aligned}$$

hence $Q^\lambda_{\sharp,\varepsilon}$ is bounded in $L^2_\sharp(Y)/\mathbb{R}$. From the boundedness and the Y-periodicity of $W^\lambda_{\sharp,\varepsilon}$ and $Q^\lambda_{\sharp,\varepsilon}$ we thus deduce that the sequences $w^\lambda_{\sharp,\varepsilon}$ and $q^\lambda_{\sharp,\varepsilon}$ of (3.83) satisfy the convergences

$$\begin{cases} w^\lambda_{\sharp,\varepsilon} \rightharpoonup 0 & \text{weakly in } H^1(\Omega)^2 \\ q^\lambda_{\sharp,\varepsilon} \rightharpoonup 0 & \text{weakly in } L^2(\Omega)/\mathbb{R}. \end{cases} \tag{3.87}$$

Now, let us check that the periodic function $w^\lambda_{\sharp,\varepsilon}$ of (3.84) gives the same matrix M (3.20) as the function w^λ_ε of (3.7) which satisfies a Dirichlet boundary condition. Since M is symmetric, this is equivalent to prove that for any $\lambda \in \mathbb{R}^2$,

$$(Dw^\lambda_{\sharp,\varepsilon})^T v_\varepsilon \cdot \lambda - (Dw^\lambda_\varepsilon)^T v_\varepsilon \cdot \lambda \rightharpoonup 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.88}$$

where v_ε is defined by (3.59). Let $\varphi \in C^\infty_c(\Omega)$. Putting $\varphi\, w^\lambda_{\sharp,\varepsilon}$ in the equation (3.7) satisfied by w^λ_ε and $\varphi\, w^\lambda_\varepsilon$ in the equation (3.84) satisfied by $w^\lambda_{\sharp,\varepsilon}$, and using

the convergences (3.87) satisfied by $w^\lambda_{\sharp,\varepsilon}$, $q^\lambda_{\sharp,\varepsilon}$ as well as the similar ones satisfied by w^λ_ε, q^λ_ε, we get

$$\left\{\begin{array}{ll} \displaystyle\int_\Omega Dw^\lambda_\varepsilon : Dw^\lambda_{\sharp,\varepsilon}\,\varphi - \int_\Omega \operatorname{curl}(v_\varepsilon)\, J w^\lambda_{\sharp,\varepsilon}\cdot\lambda\,\varphi & \underset{\varepsilon\to 0}{\longrightarrow} 0 \\ \displaystyle\int_\Omega Dw^\lambda_{\sharp,\varepsilon} : Dw^\lambda_\varepsilon\,\varphi - \int_\Omega \operatorname{curl}(v_\varepsilon)\, J w^\lambda_\varepsilon\cdot\lambda\,\varphi & \underset{\varepsilon\to 0}{\longrightarrow} 0. \end{array}\right. \tag{3.89}$$

Moreover, by the representation formula (3.11) we have

$$\left\{\begin{array}{ll} \operatorname{curl}(v_\varepsilon)\, J w^\lambda_{\sharp,\varepsilon} - (Dw^\lambda_{\sharp,\varepsilon})^T\, v_\varepsilon & \rightharpoonup 0 \\ \operatorname{curl}(v_\varepsilon)\, J w^\lambda_\varepsilon - (Dw^\lambda_\varepsilon)^T\, v_\varepsilon & \rightharpoonup 0 \end{array}\right. \quad \text{in } \mathcal{D}'(\Omega)^2. \tag{3.90}$$

Therefore, combining (3.89) and (3.90) we obtain the desired convergence (3.88).

It remains to determine the matrix M. On the one side putting $w^\lambda_{\sharp,\varepsilon}$ as test function in (3.84) and using the convergences (3.88), (3.20), and on the other side using the εY-periodicity of $w^\lambda_{\sharp,\varepsilon}$ (3.83), we get similarly to (3.9) and up to a subsequence

$$\left|Dw^\lambda_{\sharp,\varepsilon}\right|^2 \rightharpoonup M\lambda\cdot\lambda \text{ and } \left|Dw^\lambda_{\sharp,\varepsilon}\right|^2 \rightharpoonup \lim_{\varepsilon\to 0}\left(\fint_Y \left|DW^\lambda_{\sharp,\varepsilon}\right|^2 dy\right) \text{ weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{3.91}$$

This combined with (3.85) and (3.86) gives

$$M\lambda\cdot\lambda = \frac{1}{4}\, J\bar{W}^\lambda\cdot\lambda. \tag{3.92}$$

Let us compute the constant vector $\bar{W}^\lambda$. To this end, putting the divergence free function $W^\lambda_{\sharp,\varepsilon}$ in the inequality (3.72) satisfied by V^i_ε, $i = 1, 2$, and taking into account the estimates (3.71), (3.56) and the boundedness of $W^\lambda_{\sharp,\varepsilon}$ in $H^1(Y)^2$, we have

$$\int_Y DV^i_\varepsilon : DW^\lambda_{\sharp,\varepsilon}\, dy = \gamma^i_\varepsilon\, e_i\cdot\left(\fint_{Q_{r_\varepsilon}} W^\lambda_{\sharp,\varepsilon}\, dy\right) + o(\varepsilon). \tag{3.93}$$

Moreover, putting the divergence free function V^i_ε in (3.82) with $V^i_\varepsilon = e_i$ in Q_{r_ε}, we get

$$\int_Y DW^\lambda_{\sharp,\varepsilon} : DV^i_\varepsilon\, dy = -\,\varepsilon\left(\fint_{Q_{r_\varepsilon}} V^i_\varepsilon - \fint_Y V^i_\varepsilon\right)\cdot J\lambda = \varepsilon\, Je_i\cdot\lambda + o(\varepsilon), \tag{3.94}$$

since by (3.71) V^i_ε strongly converges to zero in $L^2(Y)^2$. The estimates (3.93) and (3.94) divided by ε together with (3.86), (3.73) and (3.56) imply that

$$\gamma\, e_i\cdot\bar{W}^\lambda = Je_i\cdot\lambda \quad \text{or equivalently} \quad \bar{W}^\lambda = -\frac{1}{\gamma}J\lambda. \tag{3.95}$$

This combined with (3.92) yields the value (3.20) of the symmetric matrix M. □

Proof of Proposition 3.8. Let $v = (v_1, v_2) \in W^{1,r}(\Omega)^2$. Considering the functions v_ε^i, $i = 1, 2$, which are defined by (3.68) and satisfy the convergences (3.76), we have

$$\begin{aligned} E_\varepsilon &:= \int_\Omega \left| Du_\varepsilon - Du - v_1\, Dv_\varepsilon^1 - v_2\, Dv_\varepsilon^2 \right|^2 dx \\ &= \int_\Omega |Du_\varepsilon|^2\, dx - \int_\Omega |Du|^2\, dx + \int_\Omega \left(v_1^2\, |Dv_\varepsilon^1|^2 + v_2^2\, |Dv_\varepsilon^2|^2 \right) dx \\ &\quad - 2 \int_\Omega \left(v_1\, Du_\varepsilon : Dv_\varepsilon^1 + v_2\, Du_\varepsilon : Dv_\varepsilon^2 \right) dx + o(1). \end{aligned} \tag{3.96}$$

Putting u_ε in equation (3.57) and u in equation (3.61) we get

$$\begin{aligned} \int_\Omega |Du_\varepsilon|^2\, dx - \int_\Omega |Du|^2\, dx &= \langle f, u \rangle_{H^{-1}(\Omega)^2, H_0^1(\Omega)^2} - \int_\Omega |Du|^2\, dx + o(1) \\ &= \int_\Omega \Gamma u \cdot u\, dx + o(1) \\ &= \frac{\gamma}{4\,(\gamma^2+1)} \int_\Omega |u|^2\, dx + o(1). \end{aligned} \tag{3.97}$$

Moreover, putting V_ε^i in estimate (3.72) together with $V_\varepsilon^i = e_i$ in Q_{r_ε}, (3.71), (3.73), (3.56), and using the εY-periodicity of Dv_ε^i, we get

$$|Dv_\varepsilon^i|^2 \rightharpoonup \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon^2} \fint_Y |DV_\varepsilon^i|^2\, dy \right) = \frac{\gamma}{4} \quad \text{weakly-}* \text{ in } \mathcal{M}(\bar{\Omega}),$$

hence since $v_i \in C(\bar{\Omega})$,

$$\int_\Omega \left(v_1^2\, |Dv_\varepsilon^1|^2 + v_2^2\, |Dv_\varepsilon^2|^2 \right) dx = \frac{\gamma}{4} \int_\Omega |v|^2\, dx + o(1). \tag{3.98}$$

Estimates (3.96), (3.97) and (3.98) thus imply that

$$\begin{aligned} E_\varepsilon &= \frac{\gamma}{4\,(\gamma^2+1)} \int_\Omega |u|^2\, dx + \frac{\gamma}{4} \int_\Omega |v|^2\, dx \\ &\quad - 2 \int_\Omega \left(v_1\, Du_\varepsilon : Dv_\varepsilon^1 + v_2\, Du_\varepsilon : Dv_\varepsilon^2 \right) dx + o(1). \end{aligned} \tag{3.99}$$

On the other hand, applying the estimate (3.75) with the function $v = v_i\, u_\varepsilon$, $i = 1, 2$, and using the convergences (3.76), (3.73), (3.56) and (3.79), we obtain

$$\begin{aligned} \int_\Omega Dv_\varepsilon^i : Du_\varepsilon\, v_i\, dx &= \int_\Omega Dv_\varepsilon^i : D(v_i\, u_\varepsilon)\, dx + o(1) \\ &= \frac{\gamma_\varepsilon^i}{\varepsilon^2} e_i \cdot \left(\int_\Omega \frac{1_{\omega_\varepsilon}}{|Q_{r_\varepsilon}|}\, u_\varepsilon v_i\, dx - \int_\Omega \frac{1_{Y \setminus Q}}{|Y \setminus Q|} \left(\frac{x}{\varepsilon} \right) u_\varepsilon v_i dx \right) + o(1) \\ &= \frac{\gamma^2}{4\,(\gamma^2+1)} \int_\Omega e_i \cdot (\gamma\, I - J)\, u\, v_i\, dx - \frac{\gamma}{4} \int_\Omega u_i\, v_i\, dx + o(1). \end{aligned}$$

This combined with (3.99) yields

$$\begin{aligned} E_\varepsilon = & \frac{\gamma}{4\,(\gamma^2+1)} \int_\Omega |u|^2\,dx + \frac{\gamma}{4} \int_\Omega |v|^2\,dx \\ & + \frac{\gamma}{2\,(\gamma^2+1)} \int_\Omega u \cdot v\,dx + \frac{\gamma^2}{2\,(\gamma^2+1)} \int_\Omega Ju \cdot v\,dx + o(1). \end{aligned} \tag{3.100}$$

Putting the function

$$v := -\frac{1}{\gamma^2+1}\,(I + \gamma\,J)\,u$$

in (3.100) we get

$$E_\varepsilon = \int_\Omega \left| Du_\varepsilon - Du - v_1\,Dv_\varepsilon^1 - v_2\,Dv_\varepsilon^2 \right|^2 dx \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{3.101}$$

Finally, since the sequences v_ε^i strongly converge to zero in $L^{\frac{2r}{r-2}}(\Omega)^2$ by (3.76) and $u \in W^{1,r}(\Omega)^2$, the strong convergence (3.65) is a straightforward consequence of (3.101). □

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