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Quantitative uniqueness estimates for the shallow shell system and their application to an inverse problem

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Abstract. In this paper we derive some quantitative uniqueness estimates for the shallow shell equations. Our proof relies on appropriate Carleman estimates. For applications, we consider the size estimate inverse problem.

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1. Introduction

In this work we study a quantitative uniqueness for the shallow shell system and its application to the inverse problem of estimating the size of an embedded inclusion by boundary measurements. To begin, we let Ω be a bounded domain in $\mathbb{R}^2$. Without loss of generality, we assume $0 \in \Omega$. Let $\bar{\theta} : \overline{\Omega} \to \mathbb{R}$ satisfy an appropriate regularity assumption which will be specified later. For a shallow shell, its middle surface is described by $\{(x_1, x_2, \varepsilon\rho_0\bar{\theta}(x_1, x_2)) : (x_1, x_2) \in \overline{\Omega}\}$ for $\varepsilon > 0$, where $\rho_0 > 0$ is the characteristic length of Ω (see Section 3.1). From now on, we set $\theta = \rho_0\bar{\theta}$. Let $u = (u_1, u_2, u_3) = (u', u_3) : \Omega \to \mathbb{R}^3$ represent the displacement vector of the middle surface. Then u satisfies the following equations:

$$\begin{cases} -\partial_j n^\theta_{ij}(u) = 0 \quad \text{in} \quad \Omega, \\ \partial^2_{ij} m_{ij}(u_3) - \partial_j(n^\theta_{ij}(u)\partial_i\theta) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where

$$\begin{aligned} m_{ij}(u_3) &= \rho_0^2 \left\{ \frac{4\lambda\mu}{3(\lambda+2\mu)}(\Delta u_3)\delta_{ij} + \frac{4\mu}{3}\partial^2_{ij}u_3 \right\}, \\ n^\theta_{ij}(u) &= \frac{4\lambda\mu}{\lambda+2\mu} e^\theta_{kk}(u)\delta_{ij} + 4\mu e^\theta_{ij}(u), \\ e^\theta_{ij}(u) &= \frac{1}{2}(\partial_i u_j + \partial_j u_i + (\partial_i\theta)\partial_j u_3 + (\partial_j\theta)\partial_i u_3), \end{aligned} \tag{1.2}$$

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and λ, μ are Lamé coefficients. Hereafter, the Roman indices (except n) belong to $\{1, 2\}$ and the Einstein summation convention is used for repeated indices.

Assume that D is a measurable subdomain of Ω with $\overline{D} \subset \Omega$. We consider Lamé parameters

$$\widetilde{\lambda} = \lambda + \chi_D \lambda_0 \quad \text{and} \quad \widetilde{\mu} = \mu + \chi_D \mu_0,$$

where χ_D is the characteristic function of D. The domain D represents the inclusion inside of Ω. With such parameters $\widetilde{\lambda}, \widetilde{\mu}$, we denote the displacement field $\widetilde{u} = (\widetilde{u}', \widetilde{u}_3)^t$ satisfying (1.1) and the Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} \widetilde{n}^{\theta}_{ij} \nu_j = \rho_0^{-1} \widehat{T}_i, \\ \widetilde{m}_{ij} \nu_i \nu_j = \widehat{M}_\nu, \\ (\partial_i \widetilde{m}_{ij} - \widetilde{n}^{\theta}_{ij} \partial_i \theta) \nu_j + \partial_s (\widetilde{m}_{ij} \nu_i \tau_j) = -\partial_s \widehat{M}_\tau, \end{cases} \tag{1.3}$$

where $\widetilde{m}_{ij} = \widetilde{m}_{ij}(\widetilde{u}_3)$ and $\widetilde{n}^{\theta}_{ij} = \widetilde{n}^{\theta}_{ij}(\widetilde{u})$ are defined in (1.2) with λ, μ, u replaced by $\widetilde{\lambda}$, $\widetilde{\mu}$, $\widetilde{u}$. Hereafter, $\nu = (\nu_1, \nu_2)$, $\tau = (\tau_1, \tau_2)$ are, respectively, the normal and the tangent vectors along $\partial\Omega$, and s is the arclength parameter of $\partial\Omega$. Precisely, the tangent vector τ is obtained by rotating ν counterclockwise of angle $\pi/2$. The boundary field $\widehat{M} = \widehat{M}_\tau \nu + \widehat{M}_\nu \tau$, *i.e.*, $\widehat{M}_\tau = \widehat{M} \cdot \nu$ and $\widehat{M}_\nu = \widehat{M} \cdot \tau$. We remark that in the plate theory, $\widehat{M}_\tau$ and $\widehat{M}_\nu$ are the twisting and bending moments applied on $\partial\Omega$. The field $\widehat{T}$ satisfies the compatibility condition which will be specified in the following section. An interesting inverse problem is to determine geometric information on D from a pair $\{\widehat{T}, \widehat{M}; \widetilde{u}'|_{\partial\Omega}, (\widetilde{u}_3|_{\partial\Omega}, \partial_\nu \widetilde{u}_3|_{\partial\Omega})\}$, *i.e.*, from the Cauchy data of the solution $\widetilde{u}$. Despite its practical value, the fundamental global uniqueness, even for the scalar equation, is yet to be proved. For the development of the uniqueness issue for this kind of inverse problems, we refer to [13] and references therein for details.

In this paper we are interested in estimating the size of the area of D in terms of the Cauchy data of $\widetilde{u}$. This type of problem has been studied for the scalar equation and for systems of equations such as the isotropic elasticity and plate. We refer to the survey article [3] for the early developments and [20, 21] for the latest results on the plate equations. Specifically, the size of D is estimated by the following two quantities:

$$\widetilde{W} = \int_{\partial\Omega} \rho_0^{-1} \widehat{T} \cdot \widetilde{u}' + \widehat{M}_\nu \partial_\nu \widetilde{u}_3 + \partial_s \widehat{M}_\tau \widetilde{u}_3$$

and

$$W = \int_{\partial\Omega} \rho_0^{-1} \widehat{T} \cdot u' + \widehat{M}_\nu \partial_\nu u_3 + \partial_s \widehat{M}_\tau u_3,$$

where $u = (u', u_3)^t$ is the displacement vector satisfing (1.1) and (1.3) with $D = \emptyset$, *i.e.*, $\widetilde{\lambda} = \lambda$ and $\widetilde{\mu} = \mu$. Here we assume that λ, μ are given *a priori* , thus, both $\widetilde{W}$ and W are known. To be more precise, in this paper, we will show that under some *a priori* assumptions, there exist positive constants C_1, C_2 such that

$$C_1 \left| \frac{\widetilde{W} - W}{W} \right| \le \text{area}(D) \le C_2 \left| \frac{\widetilde{W} - W}{W} \right|, \tag{1.4}$$

where C_1, C_2 depend on the *a priori* data.

The derivation of the volume bounds on D relies on the following integral inequalities

$$\begin{aligned} \frac{1}{K}\int_D \sum_{ij} |e^{\theta}_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 &\le |W - \widetilde{W}| \\ &\le K \int_D \sum_{ij} |e^{\theta}_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2, \end{aligned} \tag{1.5}$$

where the constant K depends on the *a priori* data. The lower bound for area(D) is a consequence of the second inequality of (1.5) and the elliptic regularity estimate for u. To derive the upper bound for area(D), we shall use the first inequality of (1.5). As indicated in all previous related results, we need to estimate $\int_D \sum_{ij} |e^{\theta}_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2$ from below. This can be achieved by the quantitative uniqueness estimates of solutions u solving (1.1), which is one of the themes of the paper.

For the second order elliptic operator, using the Carleman or the frequency functions methods, quantitative estimates for the strong unique continuation under different assumptions on coefficients were derived in [8–11, 14, 16, 18]. For the isotropic elasticity, similar estimates can be found in [1, 4, 19]. Further, for the elastic plate, quantitative uniqueness estimates were derived in [20, 21]. Note that global versions of quantitative uniqueness estimates, in the form of doubling inequality, were given in [4] and [21], where their arguments rely on a local version for the power of Laplacian derived in [17].

In this paper, we will derive three-ball inequalities and doubling inequalities for the shallow shell system (1.1) with $\lambda, \mu \in C^{1,1}(\Omega)$. Since the first and the second equations in (1.1) have different orders, it seems that the Carleman method is the most efficient way to derive those quantitative uniqueness estimates for (1.1). We will give detailed derivations of quantitative uniqueness estimates based on the Carleman estimates in Section 4. The investigation of the inverse problem is given in Section 5. Since the Neumann boundary value problem for (1.1) is not standard, we will first study this forward problem in Section 3.

2. Notation

Definition 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \ge 2$. Given $k \in \mathbb{Z}^+$, we say that $\partial\Omega$ is of class $C^{k,1}$ with constants ρ_0, A_0, if, for any point $z \in \partial\Omega$, there exists a rigid coordinate transformation under which $z = 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n) \in B_{\rho_0}(0) : x_n > \varphi(x')\},$$

where $\varphi(x')$ is a $C^{k,1}$ function on $B'_{\rho_0}(0) = B_{\rho_0}(0) \cap \{x_n = 0\}$ satisfying $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$ if $k \geq 1$ and

$$\|\varphi\|_{C^{k,1}(B'_{\rho_0}(0))} \leq A_0 \rho_0.$$

Throughout the paper, we will normalize all norms such that they are dimensionally homogeneous and coincide with the standard definitions when the dimensional parameter is one. With this in mind, we define

$$\|\varphi\|_{C^{k,1}(B'_{\rho_0}(0))} = \sum_{j=0}^{k} \rho_0^j \|\nabla^j \varphi\|_{L^\infty(B'_{\rho_0}(0))} + \rho_0^{k+1} \|\nabla^{k+1}\varphi\|_{L^\infty(B'_{\rho_0}(0))}.$$

Similarly, when Ω with $\partial\Omega$ defined above and $w : \Omega \to \mathbb{R}$, we define

$$\|w\|_{C^{k,1}(\Omega)} = \sum_{j=0}^{k} \rho_0^j \|\nabla^j w\|_{L^\infty(\Omega)} + \rho_0^{k+1} \|\nabla^{k+1} w\|_{L^\infty(\Omega)},$$

$$\|w\|^2_{L^2(\Omega)} = \rho_0^{-n} \int_\Omega w^2,$$

$$\|w\|^2_{H^k(\Omega)} = \rho_0^{-n} \sum_{j=0}^{k} \rho_0^{2j} \int_\Omega |\nabla^j w|^2, \quad k \geq 1.$$

In particular, if $\Omega = B_\rho(0)$, then Ω satisfies Definition 2.1 with $\rho_0 = \rho$.

Let $\mathcal{A}$ be an open connected component of $\partial\Omega$. For any given point $z_0 \in \mathcal{A}$, we define the positive orientation of $\mathcal{A}$ associated with an arclength parametrization $\zeta(s) = (x_1(s), x_2(s))$, $s \in [0, \text{length}(\mathcal{A})]$ such that $\zeta(0) = z_0$ and $\zeta'(s) = \tau(\zeta(s))$. Finally, we define for any $h > 0$

$$\Omega_h = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > h\}.$$

3. The forward problem

3.1. The Neumann boundary value problem for the shallow shell equation

At this moment, we assume $\partial\Omega \in C^{1,1}$ with constants A_0, ρ_0. Also, let Ω satisfy

$$|\Omega| \leq A_1 \rho_0^2 \tag{3.1}$$

throughout the article, and

$$\|\nabla\theta\|_{L^\infty(\Omega)} = \rho_0 \|\nabla\bar{\theta}\|_{L^\infty(\Omega)} \leq A_2 \tag{3.2}$$

for some positive constants A_1 and A_2. We will investigate the Neumann boundary value problem, the forward problem, for the shallow shell system. To begin, let us assume that Lamé coefficients $\lambda, \mu \in L^\infty(\Omega)$ satisfying

$$0 < \delta_0 \le \mu(x), \quad \delta_0 \le \lambda(x), \quad \forall\, x \in \Omega. \tag{3.3}$$

We aim to find $u = (u_1, u_2, u_3) = (u', u_3)$ satisfying

$$\begin{cases} -\partial_j n^\theta_{ij}(u) = 0 \quad \text{in} \quad \Omega, \\ \partial^2_{ij} m_{ij}(u_3) - \partial_j (n^\theta_{ij}(u) \partial_i \theta) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{3.4}$$

with boundary conditions

$$\begin{cases} n^\theta_{ij}(u)\nu_j = \rho_0^{-1} \widehat{T}_i, \\ m_{ij}(u_3)\nu_i\nu_j = \widehat{M}_\nu, \\ (\partial_i m_{ij}(u_3) - n^\theta_{ij}(u)\partial_i\theta)\nu_j + \partial_s(m_{ij}(u_3)\nu_i\tau_j) = -\partial_s \widehat{M}_\tau. \end{cases} \tag{3.5}$$

Now assume that $u = (u', u_3)$ satisfies (3.4)-(3.5). Let $v = (v', v_3) \in (H^1(\Omega))^2 \times H^2(\Omega)$, then multiplying the first and second equations of (3.4) by v' and v_3, respectively, and using the standard integration by parts, we can obtain that

$$\int_\Omega \sum_{ij} (n^\theta_{ij}(u) e^\theta_{ij}(v) + m_{ij}(u_3)\partial^2_{ij} v_3) = \int_{\partial\Omega} \rho_0^{-1} \widehat{T} \cdot v' + \partial_s \widehat{M}_\tau v_3 + \widehat{M}_\nu \partial_\nu v_3. \tag{3.6}$$

The boundary field $\widehat{M} = \widehat{M}_\tau \nu + \widehat{M}_\nu \tau$ in the cartesian coordinates is written as

$$\widehat{M} = \widehat{M}_1 e_2 + \widehat{M}_2 e_1.$$

In view of the relation

$$\partial_s \widehat{M}_\tau v_3 = \partial_s(\widehat{M}_\tau v_3) - \widehat{M}_\tau \partial_s v_3,$$

one can see that the right-hand side of (3.6) becomes

$$\int_{\partial\Omega} \rho_0^{-1} \widehat{T} \cdot v' - \widehat{M}_\tau \partial_s v_3 + \widehat{M}_\nu \partial_\nu v_3.$$

Recall that $\partial_j v_3 = \partial_s v_3 \tau_j + \partial_\nu v_3 \nu_j$ for $j = 1, 2$. Using the relation $\tau = (-\nu_2, \nu_1)$ if $\nu = (\nu_1, \nu_2)$, we get that

$$\begin{aligned} \widehat{M}_1 \partial_1 v_3 - \widehat{M}_2 \partial_2 v_3 &= \widehat{M}_1(\partial_s v_3 \tau_1 + \partial_\nu v_3 \nu_1) - \widehat{M}_2(\partial_s v_3 \tau_2 + \partial_\nu v_3 \nu_2) \\ &= (\widehat{M}_1 \tau_1 - \widehat{M}_2 \tau_2)\partial_s v_3 + (\widehat{M}_1 \nu_1 - \widehat{M}_2 \nu_2)\partial_\nu v_3 \\ &= -\widehat{M}_\tau \partial_s v_3 + \widehat{M}_\nu \partial_\nu v_3 \end{aligned}$$

In view of the above computations, we deduce that

$$\int_\Omega \sum_{ij}(n^\theta_{ij}(u)e^\theta_{ij}(v)+m_{ij}(u_3)\partial^2_{ij}v_3)=\int_{\partial\Omega}\rho_0^{-1}\widehat{T}\cdot v'+\widehat{M}_1\partial_1 v_3-\widehat{M}_2\partial_2 v_3 \quad (3.7)$$

(see the similar derivation for the plate equation in [20]). Let $v'=a+W\cdot x+b\theta$ and $v_3=c-b\cdot x$, where $a=(a_1,a_2)$, $b=(b_1,b_2)$ are two-dimensional vectors, W is a 2×2 skew-symmetric matrix, and c is a scalar. Then $e^\theta_{ij}(v)=\partial^2_{ij}v_3=0$ for all $i,\ j$. Thus, to solve (3.4) and (3.5), the pair $(\widehat{T},\ \widehat{M})$ must satisfy the compatibility condition

$$\int_{\partial\Omega}\rho_0^{-1}\widehat{T}\cdot(a+W\cdot x+b\theta)-b_1\widehat{M}_1+b_2\widehat{M}_2=0. \quad (3.8)$$

Note that taking $b=0$, we have the usual compatibility condition for the traction of the elasticity equation, *i.e.*,

$$\int_{\partial\Omega}\widehat{T}\cdot(a+W\cdot x)=0.$$

On the other hand, to guarantee uniqueness for the forward problem, we impose the following normalization conditions

$$\int_\Omega u=0,\quad \int_\Omega \nabla u_3=0,\quad \int_\Omega(\partial_1u_2-\partial_2u_1)+(\partial_1\theta\partial_2u_3-\partial_2\theta\partial_1u_3)=0. \quad (3.9)$$

To solve the forward problem, the following Poincaré-Korn inequality is very important.

Proposition 3.1. *There exists an absolute constant $C>0$, depending on A_0,A_1,A_2, such that for all $u=(u',u_3)\in(H^1(\Omega))^2\times H^2(\Omega)$ satisfying* (3.9) *we have*

$$\|u'\|^2_{H^1(\Omega)}+\|u_3\|^2_{H^2(\Omega)}\le C\int_\Omega\sum_{ij}|e^\theta_{ij}(u)|^2+\rho_0^2|\partial^2_{ij}u_3|^2. \quad (3.10)$$

Proof. The inequality (3.10) is a combination of Poincaré's and Korn's inequalities. By abuse of notation, the variable x in our proof stands for $(x_1,x_2,x_3)=(x',x_3)$. Now let $\Gamma=\Omega\times(-\rho_0,\rho_0)\subset\mathbb{R}^3$ and introduce new variables $\widetilde{x}'=x'$ and $\widetilde{x}_3=x_3+\theta(x')$. Denote by $\widetilde{\Gamma}$ the domain of Ω under the coordinate transformation $x\mapsto\widetilde{x}$, *i.e.*, $\widetilde{\Gamma}=\Omega\times(-\rho_0+\theta,\rho_0+\theta)$. Both domains Γ and $\widetilde{\Gamma}$ are clearly Lipschitz. On $\widetilde{\Gamma}$, we have the standard Korn's inequality: there exists a constant $K_0>0$ such that for any 3 vector $v\in H^1(\widetilde{\Gamma})$ satisfying

$$\int_{\widetilde{\Gamma}}v\,d\widetilde{x}=0,\quad \int_{\widetilde{\Gamma}}\left(\nabla_{\widetilde{x}}v-(\nabla_{\widetilde{x}}v)^T\right)d\widetilde{x}=0, \quad (3.11)$$

we have

$$\rho_0^{-2}\|v\|^2_{L^2(\widetilde{\Gamma})}+\|\nabla_{\widetilde{x}}v\|^2_{L^2(\widetilde{\Gamma})}\le K_0\|\widehat{\nabla}_{\widetilde{x}}v\|^2_{L^2(\widetilde{\Gamma})}, \quad (3.12)$$

where $\widehat{\nabla}_{\widetilde{x}} v = \left(\nabla_{\widetilde{x}} v + (\nabla_{\widetilde{x}} v)^t\right)/2$ and K_0 depends on A_0, A_1, A_2. Let $w(x) = w(x_1, x_2, x_3) \in H^1(\Gamma)$, then $v(\widetilde{x}) := w(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3 - \theta(\widetilde{x}')) \in H^1(\widetilde{\Gamma})$. By observing that the Jacobian of the coordinate transformation $x \mapsto \widetilde{x}$ is 1, we can write (3.11), (3.12) in terms of x and get that for all $w \in H^1(\Gamma)$ satisfying

$$\begin{cases} \int_\Gamma w dx = 0, \\ \int_\Gamma (\partial_1 w_2 - \partial_1\theta\partial_3 w_2 - \partial_2 w_1 + \partial_2\theta\partial_3 w_1) dx = 0, \\ \int_\Gamma (\partial_1 w_3 - \partial_1\theta\partial_3 w_3 - \partial_3 w_1) dx = 0, \\ \int_\Gamma (\partial_2 w_3 - \partial_2\theta\partial_3 w_3 - \partial_3 w_2) dx = 0, \end{cases} \tag{3.13}$$

we have

$$\rho_0^{-2}\|w\|^2_{L^2(\Gamma)} + \|\nabla^\theta_x w\|^2_{L^2(\Gamma)} \le K_0 \|\widehat{\nabla}^\theta_x w\|^2_{L^2(\Gamma)}, \tag{3.14}$$

where

$$\nabla^\theta_x w = \begin{pmatrix} (\partial_1 - \partial_1\theta\partial_3)w_1 & (\partial_1 - \partial_1\theta\partial_3)w_2 & (\partial_1 - \partial_1\theta\partial_3)w_3 \\ (\partial_2 - \partial_2\theta\partial_3)w_1 & (\partial_2 - \partial_2\theta\partial_3)w_2 & (\partial_2 - \partial_2\theta\partial_3)w_3 \\ \partial_3 w_1 & \partial_3 w_2 & \partial_3 w_3 \end{pmatrix}$$

and the symmetric part $\widehat{\nabla}^\theta_x w$ of $\nabla^\theta_x w$ is defined similarly. In fact, by the form of $\nabla^\theta_x w$, (3.14) can be improved to

$$\rho_0^{-2}\|w\|^2_{L^2(\Gamma)} + \|\nabla_x w\|^2_{L^2(\Gamma)} \le K_1 \|\widehat{\nabla}^\theta_x w\|^2_{L^2(\Gamma)} \tag{3.15}$$

for some constant K_1, also depending on A_0, A_1, A_2. Now let $u = (u', u_3)^t \in (H^1(\Omega))^2 \times H^2(\Omega)$, we apply (3.13) and (3.15) to

$$w(x) = (u_1(x') - x_3\partial_1 u_3(x'), u_2(x') - x_3\partial_2 u_3(x'), u_3(x')),$$

where $(x', x_3) \in \Omega \times (-\rho_0, \rho_0)$. It is easy to check that the constraints (3.13) are reduced to the normalization conditions (3.9). On the other hand, easy computations show that (3.15) becomes

$$\rho_0 \int_\Omega \left(\rho_0^{-2}|u|^2 + |\nabla u|^2 + \rho_0^2 \sum_{ij} |\partial^2_{ij} u_3|^2\right) \le C \int_\Omega \sum_{ij} \left(\rho_0 |e^\theta_{ij}(u)|^2 + \rho_0^3 |\partial^2_{ij} u_3|^2\right)$$

with C only depending on A_0, A_1, A_2. □

3.2. Existence and uniqueness

We will use the variational method to solve the forward problem. This seems to be standard. But we could not find any literature discussing the Neumann boundary value problem for the shallow shell. For the sake of completeness, we give a proof of this forward problem. The arguments used here are adapted from [20]. To begin, let us introduce

$$H(u,v) = \int_\Omega \sum_{ij} n^\theta_{ij}(u)\partial_j v_i + m_{ij}(u_3)\partial^2_{ij} v_3 + n^\theta_{ij}(u)\partial_i\theta\partial_j v_3$$
$$= \int_\Omega \sum_{ij} n^\theta_{ij}(u)e^\theta_{ij}(v) + m_{ij}(u_3)\partial^2_{ij} v_3$$

and

$$L(v) = \int_{\partial\Omega} \rho_0^{-1}\widehat{T}\cdot v' + \partial_s\widehat{M}_\tau v_3 + \widehat{M}_\nu\partial_\nu v_3.$$

We now give a weak formulation of the Neumann boundary value problem (3.4)-(3.5).

Definition 3.2. A vector valued function $u = (u', u_3)^t \in (H^1(\Omega))^2 \times H^2(\Omega)$ is a weak solution to (3.4)-(3.5) if and only if

$$H(u,v) = L(v) \quad \text{for all } v = (v', v_3)^t \in H^1(\Omega)\times H^2(\Omega). \tag{3.16}$$

From the above computations, we know that

$$L(v) = \int_{\partial\Omega} \rho_0^{-1}\widehat{T}\cdot v' + \widehat{M}_1\partial_1 v_3 - \widehat{M}_2\partial_2 v_3 := \widetilde{L}(v).$$

In other words, (3.16) is equivalent to

$$H(u,v) = \widetilde{L}(v) \quad \text{for all } v = (v', v_3)^t \in (H^1(\Omega))^2\times H^2(\Omega). \tag{3.17}$$

Theorem 3.3. *Assume that* θ *satisfies* (3.2) *and* $\lambda, \mu \in L^\infty(\Omega)$ *satisfy* (3.3). *Given any boundary field* $(\widehat{T}, \widehat{M}) \in H^{-1/2}(\partial\Omega)$ *and the compatibility condition* (3.8) *holds. Then* (3.4)-(3.5) *admits a unique weak solution* $u = (u', u_3)^t$ *satisfying the conditions* (3.9) *and*

$$\|u'\|_{H^1(\Omega)} + \|u_3\|_{H^2(\Omega)} \le C\|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}, \tag{3.18}$$

where C *depends on* A_0, A_1, A_2, δ_0.

Proof. Let V be the subspace of $(H^1(\Omega))^2 \times H^2(\Omega)$ characterized by

$$V = \{w = (w', w_3) \in (H^1(\Omega))^2 \times H^2(\Omega) : \ w \text{ satisfies (3.9)}\}.$$

In view of (3.10), we have that

$$\begin{aligned}\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2|\partial^2_{ij}u_3|^2 &\le \|u'\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^2(\Omega)}\\ &\le C\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2|\partial^2_{ij}u_3|^2\end{aligned} \tag{3.19}$$

for all $u \in V$. We now define a functional $J : V \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} H(u, u) - \widetilde{L}(u).$$

We first want to prove that J has a unique minimizer on V. To this end, it suffices to show that J is coercive and strictly convex on V. It is easy to see that

$$\begin{aligned} H(u, u) &= \int_\Omega n^\theta_{ij}(u) e^\theta_{ij}(u) + m_{ij}(u_3)\partial^2_{ij} u_3 \\ &= \int_\Omega \frac{4\lambda\mu}{\lambda + 2\mu} \Big| \sum_k e^\theta_{kk}(u) \Big|^2 + 4\mu \sum_{ij} |e^\theta_{ij}(u)|^2 \\ &\qquad + \frac{4\lambda\mu\rho_0^2}{3(\lambda+2\mu)} |\Delta u_3|^2 + \frac{4\mu\rho_0^2}{3} \sum_{ij} |\partial^2_{ij} u_3|^2 \\ &\geq \frac{4\delta_0}{3} \int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2. \end{aligned}$$

Thus, (3.19) implies

$$H(u, u) \geq C(\|u'\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^2(\Omega)}) \tag{3.20}$$

with C depending only on A_0, A_1, A_2, δ_0. On the other hand, the trace inequality leads to

$$\widetilde{L}(u) \leq C \|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3} (\|u'\|_{H^1(\Omega)} + \|u_3\|_{H^2(\Omega)}).$$

Consequently, we obtain that

$$\begin{aligned} J(u) \geq C\Big(&\|u'\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^2(\Omega)} \\ &- \|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3} (\|u'\|_{H^1(\Omega)} + \|u_3\|_{H^2(\Omega)})\Big), \end{aligned}$$

which shows that J is coercive and bounded from below on V.

Now for $t \in [0, 1]$ and $u, v \in V$, we have that

$$\begin{aligned} &H(tu + (1-t)v, tu + (1-t)v) - tH(u, u) - (1-t)H(v, v) \\ &= -t(1-t)H(u - v, u - v) \leq 0 \end{aligned}$$

and for $t \in (0, 1)$

$$H(tu + (1-t)v, tu + (1-t)v) = tH(u, u) + (1-t)H(v, v)$$

if and only if

$$H(u - v, u - v) = 0.$$

Since $u, v \in V$, we see that $H(u-v, u-v) = 0$ if and only if $u = v$ in $(H^1(\Omega))^2 \times H^2(\Omega)$. In other words, we have shown that $H(u,u)$ is strictly convex on V. Taking into account that $\widetilde{L}(u)$ is linear, we have that $J(u)$ is strictly convex on V. Therefore, $J(u)$ has a unique minimizer, denoted by w, on V. In other words, $J'(w)[v] = 0$ for all $v \in V$, *i.e.*,

$$H(w, v) = \widetilde{L}(v) \tag{3.21}$$

for all $v \in V$. Now we need to show that (3.21) is valid for all $v \in (H^1(\Omega))^2 \times H^2(\Omega)$, that is, w indeed a weak solution. Given any $z = (z', z_3) \in H^1(\Omega) \times H^2(\Omega)$, one can easily check that $\widetilde{z}$ satisfies (3.9), where

$$\widetilde{z}' = z' - \frac{1}{|\Omega|}\int_\Omega z' - \left[\frac{1}{|\Omega|}\int_\Omega \frac{(\nabla z' - (\nabla z')^t)}{2}\right](x - x_\Omega) + (\theta - \theta_\Omega)\frac{1}{|\Omega|}\int_\Omega \nabla z_3,$$

$$\widetilde{z}_3 = z_3 - \frac{1}{|\Omega|}\int_\Omega z_3 - \left(\frac{1}{|\Omega|}\int_\Omega \nabla z_3\right)\cdot (x - x_\Omega),$$

and

$$\theta_\Omega = \frac{1}{|\Omega|}\int_\Omega \theta, \quad x_\Omega = \frac{1}{|\Omega|}\int_\Omega x.$$

Since $(\widehat{T}, \widehat{M})$ satisfies the compatibility condition (3.8), we conclude that

$$H(w, z) = H(w, \widetilde{z}) = \widetilde{L}(z) = \widetilde{L}(z) \quad \forall\, z \in (H^1(\Omega))^2 \times H^2(\Omega).$$

The estimate (3.18) is an easy consequence of (3.20) and the trace inequality. □

3.3. Global regularity

To study the inverse problem, we also need a global regularity theorem for the shallow shell equations. To simplify our presentation, we impose a technical assumption on $\bar{\theta}$ (or θ) in this section. Assume that $\bar{\theta}$ satisfies

$$\bar{\theta} = \nabla\bar{\theta} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.22}$$

We shall prove the following theorem.

Theorem 3.4. *Assume that Ω is a bounded domain in $\mathbb{R}^2$ satisfying* (3.1) *whose boundary $\partial\Omega$ is of class $C^{4,1}$ with constants A_0 and ρ_0. Let $\lambda, \mu \in C^{1,1}(\bar{\Omega})$ satisfy* (3.3) *and $\bar{\theta} \in C^{2,1}(\bar{\Omega})$ satisfy* (3.22) *and*

$$\|\lambda\|_{C^{1,1}(\bar{\Omega})} + \|\mu\|_{C^{1,1}(\bar{\Omega})} + \|\bar{\theta}\|_{C^{2,1}(\bar{\Omega})} \le A_2. \tag{3.23}$$

Let $u \in (H^1(\Omega))^2 \times H^2(\Omega)$ be the weak solution of (3.4), (3.5) *with Neumann boundary condition $(\widehat{T}, \widehat{M}) \in (H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)$ satisfying* (3.8). *Assume that u satisfies the normalization conditions* (3.9). *Then there exists a constant $C > 0$, depending on A_0, A_1, A_2, δ_0 such that*

$$\|u'\|_{H^2(\Omega)} + \|u\|_{H^4(\Omega)} \le C\|(\widehat{T}, \widehat{M})\|_{(H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)}. \tag{3.24}$$

Proof. To prove this theorem, it suffices to consider (3.4) with homogeneous Neumann boundary conditions. In view of (3.22), the boundary conditions (3.5) are simplified to

$$\begin{cases} n_{ij}(u')\nu_j = \rho_0^{-1}\widehat{T}_i, \\ m_{ij}(u_3)\nu_i\nu_j = \widehat{M}_\nu, \\ \partial_i m_{ij}(u_3)\nu_j + \partial_s(m_{ij}(u_3)\nu_i\tau_j) = -\partial_s\widehat{M}_\tau, \end{cases} \tag{3.25}$$

where

$$n_{ij}(u') = \frac{4\lambda\mu}{\lambda+2\mu}e_{kk}(u')\nu_i + 4\mu e_{ij}(u')\nu_j = \rho_0^{-1}\widehat{T}_i \quad \text{on} \quad \partial\Omega$$

with $e_{ij}(u') = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. It is clear that boundary conditions (3.25) are decoupled. Using the result in [20, Proposition 8.1], one can find $\tilde{w}_3$ satisfying

$$\begin{cases} m_{ij}(\tilde{w}_3)\nu_i\nu_j = \widehat{M}_\nu, \\ \partial_i m_{ij}(\tilde{w}_3)\nu_j + \partial_s(m_{ij}(\tilde{w}_3)\nu_i\tau_j) = -\partial_s\widehat{M}_\tau \end{cases}$$

on $\partial\Omega$ and the estimate

$$\|\tilde{w}_3\|_{H^4(\Omega)} \le C\|\widehat{M}\|_{H^{3/2}(\partial\Omega)}. \tag{3.26}$$

Similarly, we can choose $\tilde{w}'$ such that

$$n_{ij}(\tilde{w}')\nu_j = \rho_0^{-1}\widehat{T}_i \quad \text{on} \quad \partial\Omega$$

and

$$\|\tilde{w}'\|_{H^2(\Omega)} \le C\|\widehat{T}\|_{(H^{1/2}(\partial\Omega))^2}. \tag{3.27}$$

The constant C in (3.26) and (3.27) depend on A_0, A_1, A_2, δ_0. By setting

$$w' = \tilde{w}' - \frac{1}{|\Omega|}\int_\Omega \tilde{w}' - \left[\frac{1}{|\Omega|}\int_\Omega \frac{(\nabla\tilde{w}' - (\nabla\tilde{w}')^t)}{2}\right](x - x_\Omega) + (\theta - \theta_\Omega)\frac{1}{|\Omega|}\int_\Omega \nabla\tilde{w}_3$$

and

$$w_3 = \tilde{w}_3 - \frac{1}{|\Omega|}\int_\Omega \tilde{w}_3 - \left(\frac{1}{|\Omega|}\int_\Omega \nabla\tilde{w}_3\right)\cdot(x - x_\Omega),$$

we can see that (w', w_3) satisfies the boundary condition (3.25), the normalization conditions (3.9), and the estimate

$$\|w'\|_{H^2(\Omega)} + \|w_3\|_{H^4(\Omega)} \le C\|(\widehat{T}, \widehat{M})\|_{(H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)}, \tag{3.28}$$

where C depends on A_0, A_1, A_2, δ_0.

So now by letting $u = w + v$, we obtain that v satisfies

$$\begin{cases} -\partial_j n^\theta_{ij}(v) = f_i \quad \text{in} \quad \Omega, \\ \partial^2_{ij} m_{ij}(v_3) - \partial_j (n^\theta_{ij}(v)\partial_i\theta) = f_3 \quad \text{in} \quad \Omega, \end{cases} \tag{3.29}$$

with homogeneous Neumann boundary conditions on $\partial\Omega$

$$\begin{cases} n_{ij}(v')\nu_j = 0, \\ m_{ij}(v_3)\nu_i\nu_j = 0, \\ \partial_i m_{ij}(v_3)\nu_j + \partial_s(m_{ij}(v_3)\nu_i\tau_j) = 0, \end{cases} \tag{3.30}$$

where $f = (f_1, f_2, f_3) = (f', f_3)$ is given by

$$\begin{cases} f_i = \partial_j n^\theta_{ij}(w), i = 1, 2, \\ f_3 = -\partial^2_{ij} m_{ij}(w_3) + \partial_j(n^\theta_{ij}(w)\partial_i\theta). \end{cases}$$

Using the integration by parts, it is not hard to check that f satisfies the following compatibility conditions

$$\int_\Omega f = 0, \ \int_\Omega (f_1x_2 - f_2x_1) = 0, \ \int_\Omega (f_1\theta + f_3x_1) = 0, \ \int_\Omega (f_2\theta + f_3x_2) = 0. \tag{3.31}$$

Now to obtain a global estimate for v, we decouple (3.29) as follows

$$\begin{cases} -\partial_j n_{ij}(v') = f_i + \frac{1}{2}\partial_j(\partial_i\theta\partial_j v_3 + \partial_j\theta\partial_i v_3) := \tilde{f}_i \quad \text{in} \quad \Omega, \\ \partial^2_{ij} m_{ij}(v_3) = f_3 + \partial_j(n^\theta_{ij}(v)\partial_i\theta) := \tilde{f}_3 \quad \text{in} \quad \Omega. \end{cases} \tag{3.32}$$

By (3.31) and straightforward computations, we can deduce that $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = (\tilde{f}', \tilde{f}_3)$ satisfy

$$\int_\Omega \tilde{f} = 0, \ \int_\Omega (\tilde{f}_1x_2 - \tilde{f}_2x_1) = 0, \ \int_\Omega \tilde{f}_3x_1 = \int_\Omega \tilde{f}_3x_2 = 0, \tag{3.33}$$

which are the compatibility conditions for the existence of the boundary value problem (3.32) and (3.30). Recall the global estimate for the isotropic elasticity with homogeneous Neumann boundary condition, we have

$$\begin{aligned} \|v'\|_{H^2(\Omega)} &\le C(\rho_0^2\|\tilde{f}'\|_{L^2(\Omega)} + \|v'\|_{H^1(\Omega)}) \\ &\le C(\rho_0^2\|f'\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^1(\Omega)}), \end{aligned} \tag{3.34}$$

where C depend on A_0, A_1, A_2, δ_0. For v_3, we use [20, Proposition 8.2] to obtain that

$$\begin{aligned} \|v_3\|_{H^4(\Omega)} &\le C(\rho_0^2\|\tilde{f}_3\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)}) \\ &\le C(\rho_0^2\|\tilde{f}_3\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^1(\Omega)}). \end{aligned} \tag{3.35}$$

The dependence of C is the same as above. Putting (3.34) and (3.35) together yields

$$\|v'\|_{H^2(\Omega)} + \|v_3\|_{H^4(\Omega)} \le C(\rho_0^2\|f\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^1(\Omega)}). \quad (3.36)$$

Now using the weak formulation of the boundary value problem (3.32), (3.30), the Poincaré-Korn inequality (3.10), (3.28), we get from (3.36) that

$$\begin{aligned}\|v'\|_{H^2(\Omega)} + \|v_3\|_{H^4(\Omega)} &\le C(\rho_0^2\|f\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^1(\Omega)})\\ &\le C\rho_0^2\|f\|_{L^2(\Omega)}\\ &\le C\|(\widehat{T}, \widehat{M})\|_{(H^{1/2}(\partial\Omega))^2\times H^{3/2}(\partial\Omega)}. \end{aligned} \quad (3.37)$$

Finally, combining (3.28) and (3.37) gives (3.24). □

4. Quantitative uniqueness estimates

4.1. Main theorems

In this section, we would like to derive the three-ball inequalities for (1.1), which is a form of quantitative uniqueness estimate. The regularity of $\partial\Omega$ is irrelevant for the estimates derived here. But to make the paper consistent, we assume that Ω is at least a Lipschitz domain with constant A_0 and ρ_0. Let $\lambda(x)$, $\mu(x)$ satisfy (3.3) and $\lambda, \mu, \bar{\theta}$ satisfy estimate (3.23). We now first state the main results of this section. Assume that $B_{\rho_0\bar{R}_0} \subset \Omega$ with $\bar{R}_0 \le 1$. Let us denote $U_r = (ru', u_3) = (ru_1, ru_2, u_3)$. Then the following local estimates hold.

Theorem 4.1. *There exists a positive number R_1, depending on δ_0, K_1, K_2, such that if $0 < r_1 < r_2 < r_3 \le \rho_0\bar{R}_0$ and $r_1/r_3 < r_2/r_3 < R_1$, then*

$$\int_{|x|<r_2} |U_{r_2}|^2 dx \le C_1 \left(\int_{|x|<r_1} |U_{r_1}|^2 dx\right)^{\tau} \left(\int_{|x|<r_3} |U_{r_3}|^2 dx\right)^{1-\tau} \quad (4.1)$$

for $(u', u_3) \in (H^1(B_{\rho_0\bar{R}_0}))^2 \times H^3(B_{\rho_0\bar{R}_0})$ satisfying (1.1) in $B_{\rho_0\bar{R}_0}$, where $C_1 > 0$ and $0 < \tau < 1$ depend on r_1/r_3, r_2/r_3, δ_0, A_2.

Remark 4.2. The estimate (4.1) is the three-ball inequality. Constants C_1 and τ appeared above can be explicitly written as $\tau = B/(E + B)$ and

$$C_1 = \max\{C_0[(\log(r_1/r_3))^2/(\log(r_2/r_3))^2](r_2/r_1)^2, \exp(B\beta_0)\}(r_3/r_1)^{2\tau},$$

where $C_0 > 1$ and β_0 are constants depending on δ_0, A_2 and

$$\begin{aligned} E &= E(r_1/r_3, r_2/r_3) = (\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2,\\ B &= B(r_2/r_3) = -1 - 2\log(r_2/r_3). \end{aligned}$$

Remark 4.3. If $r_3 \le 1$, then (4.1) is reduced to

$$\int_{|x|<r_2} |U|^2 dx \le \frac{C_1}{r_2^2} \left(\int_{|x|<r_1} |U|^2 dx \right)^{\tau} \left(\int_{|x|<r_3} |U|^2 dx \right)^{1-\tau}. \tag{4.2}$$

By abuse of notation, we denote $U = (u', u_3)$.

Using the three-ball inequality, we can prove

Theorem 4.4. *If* $(u', u_3) \in (H^1(B_{\rho_0 \bar{R}_0}))^2 \times H^3(B_{\rho_0 \bar{R}_0})$ *is a nontrivial solution to* (1.1)*, then we can find a constant* R_2 *depending on* δ_0, A_2 *and a constant* m_1 *depending on* δ_0, A_2 *and* $\|U_{R_2^2}\|_{L^2(|x|<\rho_0 R_2^2)} / \|U_{R_2^4}\|_{L^2(|x|<\rho_0 R_2^4)}$ *such that*

$$\int_{|x|<R} |U|^2 dx \ge K R^{m_1}, \tag{4.3}$$

where R *is sufficiently small and the constant* K *depends on* R_2 *and* U.

In view of the standard unique continuation property for (1.1) in a connected domain containing the origin, if u vanishes in a neighborhood of the origin then it vanishes identically in Ω. Theorem 4.4 provides an upper bound on the vanishing order of a nontrivial solution to (1.1). The following doubling inequality is another quantitative estimate of the strong unique continuation for (1.1).

Theorem 4.5. *Let* $(u', u_3) \in (H^1(B_{\rho_0 \bar{R}_0}))^2 \times H^3(B_{\rho_0 \bar{R}_0})$ *be a nonzero solution to* (1.1)*. Then there exist positive constants* R_3*, depending on* δ_0, A_2, $\|U_{R_2^2}\|_{L^2(|x|<\rho_0 R_2^2)} / \|U_{R_2^4}\|_{L^2(|x|<\rho_0 R_2^4)}$*, and* C_2*, depending on* δ_0, A_2, m_1*, such that if* $0 < r \le \rho_0 R_3$*, then*

$$\int_{|x|\le 2r} |U|^2 dx \le C_2 \int_{|x|\le r} |U|^2 dx, \tag{4.4}$$

where R_2 *and* m_1 *are the constants obtained in Theorem* 4.4.

The rest of this section is devoted to the proofs of Theorem 4.1, 4.4, and 4.5.

4.2. Preliminaries

From now on, it suffices to take $\rho_0 = 1$. The first step is to transform the system (1.1) into a new system with uncoupled principal parts. To simplify the notation in the following proofs, we denote $u = u' = (u_1, u_2)$ (suppress the prime), $w = u_3$, and $v = \operatorname{div} u' = \operatorname{div} u$. Putting (1.1) and the equation obtained by taking the divergence of the first system of (1.1) together, we come to the following new system

$$\begin{cases} \Delta u = P_1(Du, Dv) + P_2(D^2 w, Dw), \\ \Delta v = P_3(Du, Dv) + P_4(D^3 w, D^2 w, Dw), \\ \Delta^2 w = P_5(D^3 w, D^2 w, Dw) + P_6(Du), \end{cases} \tag{4.5}$$

where $P_1 - P_6$ are zeroth order operators with at least L^∞ coefficients which are bounded by a constant depending on δ_0, A_2.

To prove Theorem 4.1, the following interior estimate is useful. From now on, the notation $X \lesssim Y$ or $X \gtrsim Y$ means that $X \le CY$ or $X \ge CY$ with some constant C which could only depend on δ_0, A_2.

Lemma 4.6. *Let* $(u, w) \in (H^1_{\mathrm{loc}}(B_{\bar{R}_0}))^2 \times H^3_{\mathrm{loc}}(B_{\bar{R}_0})$ *be a solution of* (1.1). *Then for any* $0 < a_3 < a_1 < a_2 < a_4$*, there exists a constant* r_0 *with* $a_4 r_0 < \bar{R}_0 (< 1)$ *such that if* $r \le r_0$

$$\sum_{|\alpha|\le 2} \int_{a_1 r<|x|<a_2 r} |x|^{2|\alpha|}|D^\alpha u|^2 dx + \sum_{|\alpha|\le 4} \int_{a_1 r<|x|<a_2 r} |x|^{2|\alpha|}|D^\alpha w|^2 dx$$
$$\le C_3 \int_{a_3 r<|x|<a_4 r} (|u|^2 + |w|^2) dx, \tag{4.6}$$

where C_3 *is independent of* r *and* (u, w).

Proof. The proof here is motivated by the ideas used in [12, Corollary 17.1.4]. Let $X = B_{a_4 r}\backslash\overline{B_{a_3 r}}$ and $d(x)$ be the distant from $x \in X$ to $\mathbb{R}^2\backslash X$. We obtain from (1.1) that $u \in (H^2_{\mathrm{loc}}(B_{\bar{R}_0}\backslash\{0\}))^2$ and $w \in H^4_{\mathrm{loc}}(B_{\bar{R}_0}\backslash\{0\}$. Denote

$$\mathcal{L}(x, D)u := \frac{4\lambda\mu}{\lambda + 2\mu}\nabla(\mathrm{div} u) + 4\mu \mathrm{div}(\mathrm{Sym}(\nabla u)).$$

Since $\mathcal{L}(x, D)$ and Δ^2 are uniformly elliptic, it is obvious that

$$\begin{cases} \|f\|_{H^2(\mathbb{R}^n)} \lesssim \|\mathcal{L}(y, D)f\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)} \\ \|g\|_{H^4(\mathbb{R}^n)} \lesssim \|\Delta^2 g\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \end{cases} \tag{4.7}$$

for all $f \in H^2(\mathbb{R}^n)$, $g \in H^4(\mathbb{R}^n)$ and any fixed y in Ω. Note that the absolute constant appearing in the first estimate of (4.7) can be chosen to be uniformly in $y \in \Omega$. By changing variables $x \to B^{-1}x$ in (4.7), we will have

$$\begin{cases} \sum_{|\alpha|\le 1} B^{2-|\alpha|}\|f\|_{H^2(\mathbb{R}^n)} \lesssim \|\mathcal{L}(y, D)f\|_{L^2(\mathbb{R}^n)} + B^2\|f\|_{L^2(\mathbb{R}^n)} \\ \sum_{|\alpha|\le 3} B^{4-|\alpha|}\|D^\alpha g\|_{L^2(\mathbb{R}^n)} \lesssim \|\Delta^2 g\|_{L^2(\mathbb{R}^n)} + B^4\|g\|_{L^2(\mathbb{R}^n)} \end{cases} \tag{4.8}$$

for all $f \in H^2(\mathbb{R}^n)$ and $g \in H^4(\mathbb{R}^n)$. To apply (4.8) on (u, w), we need to cut-off (u, w). So let $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \le \xi(x) \le 1$ and

$$\xi(x) = \begin{cases} 1, & |x| < 1/4, \\ 0, & |x| \ge 1/2. \end{cases}$$

Let us denote $\xi_y(x) = \xi((x-y)/d(y))$. For $y \subset X$, we apply (4.8) to $\xi_y(x)u(x)$ and use the first equation of (1.1) to get that

$$\begin{aligned}
&\sum_{|\alpha|\le 2} B^{4-2|\alpha|}\int_{|x-y|\le d(y)/4}|D^\alpha u|^2dx\\
\lesssim &\sum_{|\alpha|\le 1}\int_{|x-y|\le d(y)/2} d(y)^{-4+2|\alpha|}|D^\alpha u|^2dx+\int_{|x-y|\le d(y)/2}|\mathcal{L}(x,D)u|^2dx\\
&+\int_{|x-y|\le d(y)/2}|\mathcal{L}(x,D)u-\mathcal{L}(y,D)u|^2\,dx+B^4\int_{|x-y|\le d(y)/2}|u|^2dx\\
&+\sum_{|\alpha|\le 2}\int_{|x-y|\le d(y)/2}|D^\alpha w|^2dx\\
\lesssim &\sum_{|\alpha|\le 1}\int_{|x-y|\le d(y)/2} d(y)^{-4+2|\alpha|}|D^\alpha u|^2dx+r\sum_{|\alpha|=2}\int_{|x-y|\le d(y)/2}|D^\alpha u|^2dx\\
&+B^4\int_{|x-y|\le d(y)/2}|u|^2dx+\sum_{|\alpha|\le 2}\int_{|x-y|\le d(y)/2}|D^\alpha w|^2dx. \qquad (4.9)
\end{aligned}$$

Now taking $B = Md(y)^{-1}$ for some positive constant M and multiplying $d(y)^4$ on both sides of (4.9), we have

$$\begin{aligned}
&\sum_{|\alpha|\le 2} M^{4-2|\alpha|}\int_{|x-y|\le d(y)/4}d(y)^{2|\alpha|}|D^\alpha u|^2dx\\
\lesssim &\sum_{|\alpha|\le 1}\int_{|x-y|\le d(y)/2} d(y)^{2|\alpha|}|D^\alpha u|^2dx+r\sum_{|\alpha|=2}\int_{|x-y|\le d(y)/2}d(y)^4|D^\alpha u|^2dx\\
&+M^4\int_{|x-y|\le d(y)/2}|u|^2dx+\sum_{|\alpha|\le 2}\int_{|x-y|\le d(y)/2}d(y)^4|D^\alpha w|^2dx. \qquad (4.10)
\end{aligned}$$

Integrating $d(y)^{-2}dy$ over X on both sides of (4.10) and using Fubini's Theorem, we get that

$$\begin{aligned}
&\sum_{|\alpha|\le 2} M^{4-2|\alpha|}\int_X\int_{|x-y|\le d(y)/4}d(y)^{2|\alpha|-2}|D^\alpha u|^2dydx\\
\lesssim &\sum_{|\alpha|\le 1}\int_X\int_{|x-y|\le d(y)/2} d(y)^{2|\alpha|-2}|D^\alpha u|^2dydx\\
&+M^4\int_X\int_{|x-y|\le d(y)/2}d(y)^{-2}|u|^2dydx+r\sum_{|\alpha|=2}\int_X\int_{|x-y|\le d(y)/2}d(y)^2|D^\alpha u|^2dydx\\
&+\sum_{|\alpha|\le 2}\int_X\int_{|x-y|\le d(y)/2}d(y)^2|D^\alpha w|^2dydx. \qquad (4.11)
\end{aligned}$$

Note that $|d(x) - d(y)| \le |x - y|$. If $|x - y| \le d(x)/3$, then

$$2d(x)/3 \le d(y) \le 4d(x)/3. \tag{4.12}$$

On the other hand, if $|x - y| \le d(y)/2$, then

$$d(x)/2 \le d(y) \le 3d(x)/2. \tag{4.13}$$

By (4.12) and (4.13), we have

$$\begin{cases} \int_{|x-y|\le d(y)/4} d(y)^{-2}dy \ge 9/16 \int_{|x-y|\le d(x)/6} d(x)^{-2}dy \ge 1/64 \int_{|y|\le 1} dy, \\ \int_{|x-y|\le d(y)/2} d(y)^{-2}dy \le 4 \int_{|x-y|\le 3d(x)/4} d(x)^{-2}dy \le 9/4 \int_{|y|\le 1} dy \end{cases} \tag{4.14}$$

Combining (4.11)–(4.14), we obtain

$$\begin{aligned} &\sum_{|\alpha|\le 2} M^{4-2|\alpha|} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx \\ \lesssim &\sum_{|\alpha|\le 1} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx + r \sum_{|\alpha|=2} \int_X d(x)^4 |D^\alpha u|^2 dx \\ &+ M^4 \int_X |u|^2 dx + \sum_{|\alpha|\le 2} \int_X d(x)^4 |D^\alpha w|^2 dx. \end{aligned} \tag{4.15}$$

We can take M large enough and r small enough to absorb the first two terms on the right-hand side of (4.15). Thus we conclude that

$$\begin{aligned} &\sum_{|\alpha|\le 2} M^{4-2|\alpha|} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx \\ \lesssim\ & M^4 \int_X |u|^2 dx + \sum_{|\alpha|\le 2} \int_X d(x)^4 |D^\alpha w|^2 dx. \end{aligned} \tag{4.16}$$

Similarly, we can apply (4.8) to $\xi_y(x)w(x)$ and use the second equation of (1.1) to get that

$$\begin{aligned} &\sum_{|\alpha|\le 4} M^{8-2|\alpha|} \int_X d(x)^{2|\alpha|} |D^\alpha w|^2 dx \\ \lesssim\ & M^8 \int_X |w|^2 dx + \sum_{|\alpha|\le 1} \int_X d(x)^8 |D^\alpha u|^2 dx. \end{aligned} \tag{4.17}$$

Combining (4.16), (4.17) and letting M be sufficiently large, we can eliminate the last terms of (4.16) and (4.17). After that we fix M and obtain

$$\begin{aligned} &\sum_{|\alpha|\le 2} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx + \sum_{|\alpha|\le 4} \int_X d(x)^{2|\alpha|} |D^\alpha w|^2 dx \\ \lesssim\ & \int_X |u|^2 dx + \int_X |w|^2 dx. \end{aligned} \tag{4.18}$$

We recall that $X = B_{a_4r}\backslash\overline{B_{a_3r}}$ and note that $d(x) \geq \widehat{C}r$ if $x \in B_{a_2r}\backslash B_{a_1r}$, where $\widehat{C}$ is independent of r. Hence, (4.6) is an easy consequence of (4.18). ☐

The next result follows from Lemma 4.6:

Corollary 4.7. *Let* $(u, w) \in (H^1_{\mathrm{loc}}(B_{\bar{R}_0}))^2 \times H^3_{\mathrm{loc}}(B_{\bar{R}_0})$ *be a solution of* (1.1) *and* $v = \operatorname{div} u$. *Then for any* $0 < a_3 < a_1 < a_2 < a_4$, *there exists a constant* r_0 *satisfying* $a_4r_0 < \bar{R}_0$ *such that if* $r \leq r_0$, *we have*

$$\sum_{|\alpha|\leq 1}\int_{a_1r<|x|<a_2r}|x|^{2|\alpha|+2}|D^\alpha v|^2dx \leq C_3\int_{a_3r<|x|<a_4r}(|u|^2+|w|^2)dx, \tag{4.19}$$

where the constant C_3 *is independent of* r *and* (u, w).

4.3. Proof of Theorem 4.1

To begin, we recall a Carleman estimate with weight $\varphi_\beta = \varphi_\beta(x) = \exp(\frac{\beta}{2}(\log|x|)^2)$ given in [15].

Lemma 4.8. [15, Corollary 3.2] *Given* $\sigma_1 \in \mathbb{Z}$ *and* $\sigma_2 \in \mathbb{Z}$ *there exist a sufficiently large number* $\beta_0 > 0$ *and a sufficiently small number* $r_0 > 0$ *depending on n, l,* σ_1 *and* σ_2 *such that for all* $u \in U_{r_0}$ *with* $0 < r_0 < e^{-1}$, $\beta \geq \beta_0$, *we have that*

$$\sum_{|\alpha|\leq 2l}\beta^{3l-2|\alpha|}\int\varphi_\beta^2|x|^{2\sigma_1+2|\alpha|-n}(\log|x|)^{2\sigma_2+2l-2|\alpha|}|D^\alpha u|^2dx \leq \widetilde{C}_0\int\varphi_\beta^2|x|^{2\sigma_1+4l-n}(\log|x|)^{2\sigma_2}|\Delta^l u|^2dx, \tag{4.20}$$

where $U_{r_0} = \{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \operatorname{supp}(u) \subset B_{r_0}\}$ *and* $\widetilde{C}_0$ *is a positive constant depending on n and l. Here* $e = \exp(1)$.

Remark 4.9. The estimate (4.20) in Lemma 4.8 remains valid if we assume $u \in H^{2l}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$ with compact support. This can be easily seen by cutting off u for small $|x|$ and regularizing.

We first consider the case where $0 < r_1 < r_2 < R < 1/e$ and $B_R \subset \Omega$. The constant R will be chosen later. To use the estimate (4.20), we need to cut-off u. So let $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \xi(x) \leq 1$ and

$$\xi(x) = \begin{cases} 0, & |x| \leq r_1/e,\\ 1, & r_1/2 < |x| < er_2,\\ 0, & |x| \geq 3r_2. \end{cases}$$

It is easy to check that for all multi-index α

$$\begin{cases} |D^\alpha \xi| = O(r_1^{-|\alpha|}) \text{ for all } r_1/e \le |x| \le r_1/2 \\ |D^\alpha \xi| = O(r_2^{-|\alpha|}) \text{ for all } er_2 \le |x| \le 3r_2. \end{cases} \tag{4.21}$$

Noting that the commutator $[\Delta^l, \xi]$ is a $2l-1$ order differential operator and using the estimate (4.20) on ξu with parameters $\sigma_1 = 0$, $\sigma_2 = 0$, $l = 1$, $n = 2$, we can derive from the first equations of (4.5) and (4.21) that

$$\begin{aligned} &\sum_{|\alpha|\le 1} \beta^{3-2|\alpha|} \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{2|\alpha|-2} (\log|x|)^{2-2|\alpha|} |D^\alpha u|^2 dx \\ &\lesssim \sum_{|\alpha|\le 2} \beta^{3-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-2} (\log|x|)^{2-2|\alpha|} |D^\alpha (\xi u)|^2 dx \\ &\lesssim \int \varphi_\beta^2 |x|^2 |\Delta(\xi u)|^2 dx \\ &\lesssim \int \varphi_\beta^2 |x|^2 \left(|\Delta u|^2 + \sum_{|\alpha|\le 1} |[\Delta, \xi] u|^2 \right) dx \\ &\lesssim \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^2 \left[\sum_{|\alpha|\le 1} (|D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha|\le 2} |D^\alpha w|^2 \right] dx \\ &\quad + \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 |x|^2 \left[\sum_{|\alpha|\le 1} (|x|^{2|\alpha|-4} |D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha|\le 2} |D^\alpha w|^2 \right] dx \\ &\quad + \int_{er_2<|x|<3r_2} \varphi_\beta^2 |x|^2 \left[\sum_{|\alpha|\le 1} (|x|^{2|\alpha|-4} |D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha|\le 2} |D^\alpha w|^2 \right] dx. \end{aligned} \tag{4.22}$$

Similarity, applying (4.20) to ξv with parameters $\sigma_1 = 1, \sigma_2 = 0, l = 1, n = 2$, we can derive from the second equation of (4.5) and (4.21) that

$$\begin{aligned} &\sum_{|\alpha|\le 1} \beta^{3-2|\alpha|} \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{2|\alpha|} (\log|x|)^{2-2|\alpha|} |D^\alpha v|^2 dx \\ &\lesssim \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^4 \left[\sum_{|\alpha|\le 1} (|D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha|\le 3} |D^\alpha w|^2 \right] dx \\ &\quad + \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 |x|^4 \left[\sum_{|\alpha|\le 1} (|x|^{2|\alpha|-4} |D^\alpha v|^2 + |D^\alpha u|^2) + \sum_{|\alpha|\le 3} |D^\alpha w|^2 \right] dx \\ &\quad + \int_{er_2<|x|<3r_2} \varphi_\beta^2 |x|^4 \left[\sum_{|\alpha|\le 1} (|x|^{2|\alpha|-4} |D^\alpha v|^2 + |D^\alpha u|^2) + \sum_{|\alpha|\le 3} |D^\alpha w|^2 \right] dx. \end{aligned} \tag{4.23}$$

Finally, applying (4.20) to ξw with parameters $\sigma_1 = 0, \sigma_2 = 1, l = 2, n = 2$, we obtain from the third equation of (4.5) and (4.21) that

$$\begin{aligned}
&\sum_{|\alpha|\le 3} \beta^{6-2|\alpha|} \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{2|\alpha|-2} (\log|x|)^{6-2|\alpha|} |D^\alpha w|^2 dx \\
&\lesssim \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^6 (\log|x|)^2 \left[\sum_{|\alpha|\le 1} |D^\alpha u|^2 + \sum_{|\alpha|\le 3} |D^\alpha w|^2\right] dx \\
&\quad + \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 |x|^6 (\log|x|)^2 \left[\sum_{|\alpha|\le 1} |D^\alpha u|^2 + \sum_{|\alpha|\le 3} |x|^{2|\alpha|-8} |D^\alpha w|^2\right] dx \\
&\quad + \int_{er_2<|x|<3r_2} \varphi_\beta^2 |x|^6 (\log|x|)^2 \left[\sum_{|\alpha|\le 1} |D^\alpha u|^2 + \sum_{|\alpha|\le 3} |x|^{2|\alpha|-8} |D^\alpha w|^2\right] dx.
\end{aligned} \tag{4.24}$$

Putting (4.22), (4.23), (4.24) together, we can take $\beta \ge \widetilde{\beta}_0 \gg 1$ and $R \le \widetilde{R}_0 \ll 1$ such that the terms $\int_{r_1/2<|x|<er_2} (\cdots) dx$ on the right-hand side are absorbed by the corresponding terms on the left-hand side. In other words, for $\beta \ge \beta_0$ and $R \le \widetilde{R}_0$, we have that

$$\begin{aligned}
&\sum_{|\alpha|\le 1} \beta^{3-2|\alpha|} \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{2|\alpha|-2} (\log|x|)^{2-2|\alpha|} |D^\alpha u|^2 dx \\
&\quad + \sum_{|\alpha|\le 1} \beta^{3-2|\alpha|} \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{2|\alpha|} (\log|x|)^{2-2|\alpha|} |D^\alpha v|^2 dx \\
&\quad + \sum_{|\alpha|\le 3} \beta^{6-2|\alpha|} \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{2|\alpha|-2} (\log|x|)^{6-2|\alpha|} |D^\alpha w|^2 dx \\
&\lesssim \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 |x|^{-2} \left(\sum_{|\alpha|\le 1} |x|^{2|\alpha|} |D^\alpha u|^2 + |x|^{2|\alpha|+2} |D^\alpha v|^2\right) dx \qquad (4.25) \\
&\quad + \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 (\log|x|)^2 |x|^{-2} \sum_{|\alpha|\le 3} |x|^{2|\alpha|} |D^\alpha w|^2 dx \\
&\quad + \int_{er_2<|x|<3r_2} \varphi_\beta^2 |x|^{-2} \left(\sum_{|\alpha|\le 1} |x|^{2|\alpha|} |D^\alpha u|^2 + |x|^{2|\alpha|+2} |D^\alpha v|^2\right) dx \\
&\quad + \int_{er_2<|x|<3r_2} \varphi_\beta^2 (\log|x|)^2 |x|^{-2} \sum_{|\alpha|\le 3} |x|^{2|\alpha|} |D^\alpha w|^2 dx.
\end{aligned}$$

Now using (4.6) and (4.19) in (4.25) leads to

$$
\begin{aligned}
&(\log r_2)^2 r_2^{-2}\varphi_\beta^2(r_2)\int_{r_1/2<|x|<r_2}|U|^2dx\\
\lesssim\ &(\log r_1)^2 r_1^{-2}\varphi_\beta^2(r_1/e)\int_{r_1/4<|x|<r_1}|U|^2dx \qquad (4.26)\\
&+(\log r_2)^2 r_2^{-2}\varphi_\beta^2(er_2)\int_{2r_2<|x|<4r_2}|U|^2dx.
\end{aligned}
$$

Here $U=(u_1,u_2,w)=(u_1,u_2,u_3)$. Note that we have used the restriction $r_1<r_2<1/e$ in the above computations. Dividing by $(\log r_2)^2 r_2^{-2}\varphi_\beta^2(r_2)$ on both sides of (4.26) implies

$$
\begin{aligned}
&\int_{r_1/2<|x|<r_2}|U|^2dx\\
\lesssim\ &[(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]\int_{r_1/4<|x|<r_1}|U|^2dx\\
&+[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]\int_{2r_2<|x|<4r_2}|U|^2dx \qquad (4.27)\\
\lesssim\ &[(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]\int_{|x|<r_1}|U|^2dx\\
&+[(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]\int_{|x|<1}|U|^2dx.
\end{aligned}
$$

Adding $\int_{|x|<r_1/2}|U|^2dx$ to both sides of (4.27), we get for $\beta\ge\beta_0$ that

$$
\begin{aligned}
&\int_{|x|<r_2}|U|^2dx\\
\lesssim\ &[(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]\int_{|x|<r_1}|U|^2dx \qquad (4.28)\\
&+[(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]\int_{|x|<1}|U|^2dx.
\end{aligned}
$$

By denoting

$$
\begin{aligned}
E&=\beta^{-1}\,\log[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]=(\log r_1-1)^2-(\log r_2)^2>0,\\
B&=-\beta^{-1}\,\log[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]=-1-2\log r_2>0,
\end{aligned}
$$

(4.28) becomes

$$\begin{aligned}&\int_{|x|<r_2}|U|^2dx\\ &\lesssim [(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2\times \\ &\left(\exp(E\beta)\int_{|x|<r_1}|U_{r_1}|^2dx+\exp(-B\beta)\int_{|x|<1}|U|^2dx\right).\end{aligned}\tag{4.29}$$

To further simplify the terms on the right-hand side of (4.29), we consider two cases. If

$$\int_{|x|<r_1}|U|^2dx\neq 0$$

and

$$\exp(E\beta_0)\int_{|x|<r_1}|U|^2dx<\exp(-B\beta_0)\int_{|x|<1}|U|^2dx,$$

then we can pick a $\beta>\beta_0$ such that

$$\exp(E\beta)\int_{|x|<r_1}|U|^2dx=\exp(-B\beta)\int_{|x|<1}|U|^2dx.$$

Using such β, we obtain from (4.29) that

$$\begin{aligned}&\int_{|x|<r_2}|U|^2dx\\ &\lesssim [(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2\exp(E\beta)\int_{|x|<r_1}|U|^2dx\\ &\lesssim [(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2\left(\int_{|x|<r_1}|U|^2dx\right)^{\frac{B}{E+B}}\left(\int_{|x|<1}|U|^2dx\right)^{\frac{E}{E+B}}.\end{aligned}\tag{4.30}$$

If

$$\int_{|x|<r_1}|U|^2dx=0,$$

then it follows from (4.29) that

$$\int_{|x|<r_2}|U|^2dx=0$$

since we can take β arbitrarily large. The three-sphere inequality obviously holds.

On the other hand, if

$$\exp(-B\beta_0)\int_{|x|<1}|U|^2dx\leq\exp(E\beta_0)\int_{|x|<r_1}|U|^2dx,$$

then we have

$$\begin{aligned}
&\int_{|x|<r_2} |U|^2 dx \\
&\le \left(\int_{|x|<1} |U|^2 dx\right)^{\frac{B}{E+B}} \left(\int_{|x|<1} |U|^2 dx\right)^{\frac{E}{E+B}} \\
&\le \exp(B\beta_0) \left(\int_{|x|<r_1} |U|^2 dx\right)^{\frac{B}{E+B}} \left(\int_{|x|<1} |U|^2 dx\right)^{\frac{E}{E+B}} .
\end{aligned} \tag{4.31}$$

Putting together (4.30), (4.31), we arrive at

$$\int_{|x|<r_2} |U|^2 dx \le \tilde{C}_3 \left(\int_{|x|<r_1} |U|^2 dx\right)^{\frac{B}{E+B}} \left(\int_{|x|<1} |U|^2 dx\right)^{\frac{E}{E+B}}, \tag{4.32}$$

where $\tilde{C}_3 = \max\{\tilde{C}_2[(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2, \exp(B\beta_0)\}$ for some positive constant $\tilde{C}_2$, depending on δ_0, A_2.

Now for the general case, we take $R_1 = \widetilde{R}_0$ and consider $0 < r_1 < r_2 < r_3 \le \rho_0 \bar{R}_0 < 1$ with $r_1/r_3 < r_2/r_3 \le R_1$. By defining $\widehat{u}(y) := r_3 u(r_3 y)$, $\widehat{w}(y) := w(r_3 y)$, $\widehat{\lambda}(y) := \lambda(r_3 y)$, $\widehat{\mu}(y) := \mu(r_3 y)$, $\widehat{\theta}(y) = \theta(r_3 y)$, we can see that the system (1.1) is invariant under this scaling. On the other hand, $\widehat{\lambda}(y)$, $\widehat{\mu}(y)$ and $\widehat{\theta}(y)$ satisfy (3.23), respectively, with the same constants. Therefore, from (4.32), we get that

$$\int_{|y|<r_2/r_3} |\widehat{U}|^2 dy \le \tilde{C}_1 \left(\int_{|y|<r_1/r_3} |\widehat{U}|^2 dy\right)^{\tau} \left(\int_{|y|<1} |\widehat{U}|^2 dy\right)^{1-\tau} \tag{4.33}$$

where $|\widehat{U}|^2 = |\widehat{u}|^2 + |\widehat{w}|^2$, $\tau = B/(E+B)$ with

$$\begin{aligned}
E &= E(r_1/r_3, r_2/r_3) = (\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2, \\
B &= B(r_2/r_3) = -1 - 2\log(r_2/r_3),
\end{aligned}$$

and $\tilde{C}_1 = \max\{\tilde{C}_2[(\log r_1/r_3)^2/(\log r_2/r_3)^2](r_2/r_1)^2, \exp(B\beta_0)\}$. Rewriting (4.33) with the original variables yields

$$\int_{|x|<r_2} |U_{r_2}|^2 dx \le C_1 \left(\int_{|x|<r_1} |U_{r_1}|^2 dx\right)^{\tau} \left(\int_{|x|<r_3} |U_{r_3}|^2 dx\right)^{1-\tau}$$

with $C_1 = \tilde{C}_1 (r_3/r_1)^{2\tau}$. □

4.4. Proof of Theorems 4.4 and 4.5

In this section we prove Theorem 4.4 and 4.5. We begin with another Carleman estimate derived in [15, Lemma 2.1]: for any $f \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ and for any $m \in$

$\{j+\frac{1}{2},\, j\in\mathbb{N}\}$ we have

$$\sum_{|\alpha|\le 2l}\int m^{2l-2|\alpha|}|x|^{-2m+2|\alpha|-n}|D^\alpha f|^2dx\le C\int |x|^{-2m+4l-n}|\triangle^l f|^2dx, \quad (4.34)$$

where C depends only on the dimension n and the power l.

Remark 4.10. Using the cut-off function and regularization, estimate (4.34) remains valid for any fixed m if $f\in H^{2l}_{\text{loc}}(\mathbb{R}^n\backslash\{0\})$ with compact support.

In view of Remark 4.10, we define $\chi(x)\in C_0^\infty(\mathbb{R}^2\backslash\{0\})$ such that

$$\chi(x)=\begin{cases}0 & \text{if} \quad |x|\le\delta/3,\\ 1 & \text{in} \quad \delta/2\le|x|\le(R_0+1)R_0R/4=r_4R,\\ 0 & \text{if} \quad 2r_4R\le|x|,\end{cases}$$

where $\delta\le R_0^2R/4$, $R_0>0$ is a small number which will be chosen later and R is given by $R=(\gamma m)^{-1}$, where $\gamma>0$ is a large constant which will be chosen later. In view of the definition of χ, it is easy to see that for all multi-index α

$$\begin{cases}|D^\alpha\chi|=O(\delta^{-|\alpha|}) \text{ for all } \delta/3<|x|<\delta/2,\\ |D^\alpha\chi|=O((r_4R)^{-|\alpha|}) \text{ for all } r_4R<|x|<2r_4R.\end{cases} \quad (4.35)$$

Using the estimate (4.34) to χu with parameters $l=1$, $n=2$ and the equations (4.5), (4.35), the same arguments as (4.22) arrive that

$$\begin{aligned}&\sum_{|\alpha|\le 2}m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-2}|D^\alpha u|^2dx\\ &\lesssim\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2}\Big[\sum_{|\alpha|\le1}(|D^\alpha u|^2+|D^\alpha v|^2)+\sum_{|\alpha|\le 2}|D^\alpha w|^2\Big]dx\\ &\quad+\int_{\delta/3<|x|<\delta/2}|x|^{-2m+2}\Big[\sum_{|\alpha|\le1}(|x|^{2|\alpha|-4}|D^\alpha u|^2+|D^\alpha v|^2)+\sum_{|\alpha|\le 2}|D^\alpha w|^2\Big]dx\\ &\quad+\int_{r_4R<|x|<2r_4R}|x|^{-2m+2}\Big[\sum_{|\alpha|\le1}(|x|^{2|\alpha|-4}|D^\alpha u|^2+|D^\alpha v|^2)+\sum_{|\alpha|\le 2}|D^\alpha w|^2\Big]dx.\end{aligned} \quad (4.36)$$

Similarity, applying (4.34) to χv with parameters $m = m-1, l = 1, n = 2$, we can derive from (4.5) and (4.35) that

$$\sum_{|\alpha|\le 2}(m-1)^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|}|D^\alpha v|^2dx$$

$$\lesssim \int_{\delta/2\le|x|\le r_4R}|x|^{-2m+4}\left[\sum_{|\alpha|\le 1}(|D^\alpha u|^2+|D^\alpha v|^2)+\sum_{|\alpha|\le 3}|D^\alpha w|^2\right]dx \tag{4.37}$$

$$+\int_{\delta/3<|x|<\delta/2}|x|^{-2m+4}\left[\sum_{|\alpha|\le 1}(|x|^{2|\alpha|-4}|D^\alpha v|^2+|D^\alpha u|^2)+\sum_{|\alpha|\le 3}|D^\alpha w|^2\right]dx$$

$$+\int_{r_4R<|x|<2r_4R}|x|^{-2m+4}\left[\sum_{|\alpha|\le 1}(|x|^{2|\alpha|-4}|D^\alpha v|^2+|D^\alpha u|^2)+\sum_{|\alpha|\le 3}|D^\alpha w|^2\right]dx.$$

Next applying (4.34) to χw with parameters $l = 2, n = 2$, we get from (4.5) and (4.35) that

$$\sum_{|\alpha|\le 3}m^{4-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-2}|D^\alpha w|^2dx$$

$$\lesssim \int_{\delta/2\le|x|\le r_4R}|x|^{-2m+6}\left[\sum_{|\alpha|\le 1}|D^\alpha u|^2+\sum_{|\alpha|\le 3}|D^\alpha w|^2\right]dx \tag{4.38}$$

$$+\int_{\delta/3<|x|<\delta/2}|x|^{-2m+6}\left[\sum_{|\alpha|\le 1}|D^\alpha u|^2+\sum_{|\alpha|\le 3}|x|^{2|\alpha|-8}|D^\alpha w|^2\right]dx$$

$$+\int_{r_4R<|x|<2r_4R}|x|^{-2m+6}\left[\sum_{|\alpha|\le 1}|D^\alpha u|^2+\sum_{|\alpha|\le 3}|x|^{2|\alpha|-8}|D^\alpha w|^2\right]dx.$$

Adding (4.36), $K_1\times$(4.37) and $K_2 m^2\times$(4.38) together, we obtain that

$$\begin{aligned}
&\sum_{|\alpha|\le 2} m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-2}|D^\alpha u|^2dx\\
&+K_1\sum_{|\alpha|\le 2}(m-1)^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|}|D^\alpha v|^2dx\\
&+K_2\sum_{|\alpha|\le 3} m^{4-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-2}|D^\alpha w|^2dx\\
\lesssim&\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2}\Big[\sum_{|\alpha|\le 1}(|D^\alpha u|^2+|D^\alpha v|^2)+\sum_{|\alpha|\le 2}|D^\alpha w|^2\Big]dx \qquad (4.39)\\
&+K_1\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+4}\Big[\sum_{|\alpha|\le 1}(|D^\alpha u|^2+|D^\alpha v|^2)+\sum_{|\alpha|\le 3}|D^\alpha w|^2\Big]dx\\
&+K_2\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+6}\Big[\sum_{|\alpha|\le 1}|D^\alpha u|^2+\sum_{|\alpha|\le 3}|D^\alpha w|^2\Big]dx\\
&+\int_{\delta/3<|x|<\delta/2}(\cdots)+\int_{r_4R<|x|<2r_4R}(\cdots).
\end{aligned}$$

Now we choose K_1 sufficiently large such that the terms

$$\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2}\sum_{|\alpha|=1}|D^\alpha v|^2dx$$

on the right-hand side of (4.39) are absorbed by its left-hand side. After that the constant K_1 is fixed. We continue to choose K_2 large enough such that

$$K_1\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+4}\sum_{|\alpha|=3}|D^\alpha w|^2dx$$

on the right-hand side of (4.39) are absorbed by its left-hand side. Then we fix K_2. To eliminate other terms inside the integral $\int_{\delta/2\le|x|\le r_4R}$ on the right-hand side of (4.39), we recall that $R=(\gamma m)^{-1}$. So by choosing $\gamma\ge\gamma_0$ and $m\ge m_0'$ with large

γ_0 and m_0', we have that

$$\sum_{|\alpha|\le 2} m^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-2}|D^\alpha u|^2dx$$
$$+\sum_{|\alpha|\le 2}(m-1)^{2-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|}|D^\alpha v|^2dx$$
$$+\sum_{|\alpha|\le 3} m^{6-2|\alpha|}\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+2|\alpha|-2}|D^\alpha w|^2dx$$
$$\lesssim \int_{\delta/3<|x|<\delta/2}|x|^{-2m-2}\left(\sum_{|\alpha|\le 1}|x|^{2|\alpha|}|D^\alpha u|^2+|x|^{2|\alpha|+2}|D^\alpha v|^2\right)dx \quad (4.40)$$
$$+\int_{\delta/3<|x|<\delta/2}|x|^{-2m-2}m^2\sum_{|\alpha|\le 3}|x|^{2|\alpha|}|D^\alpha w|^2dx$$
$$+\int_{r_4R<|x|<2r_4R}|x|^{-2m-2}\left(\sum_{|\alpha|\le 1}|x|^{2|\alpha|}|D^\alpha u|^2+|x|^{2|\alpha|+2}|D^\alpha v|^2\right)dx$$
$$+\int_{r_4R<|x|<2r_4R}|x|^{-2m-2}m^2\sum_{|\alpha|\le 3}|x|^{2|\alpha|}|D^\alpha w|^2dx.$$

Note that $R_0^2 \le r_4$ provided $R_0 \le 1/3$. Also, if $R_0 \le 1/3$, it is obvious that $2r_4 \le R_0$. Dividing m^2 on both sides of (4.40) and using (4.19) and (4.6) in (4.40), it obtains that

$$(2\delta)^{-2m-2}\int_{\delta/2<|x|\le 2\delta}|U|^2dx+(R_0^2R)^{-2m-2}\int_{2\delta<|x|\le R_0^2R}|U|^2dx$$
$$\lesssim \int_{\delta/2\le|x|\le r_4R}|x|^{-2m-2}|U|^2dx \quad (4.41)$$
$$\le C'(\delta/3)^{-2m-2}\int_{|x|\le\delta}|U|^2dx+C''(r_4R)^{-2m-2}\int_{|x|\le R_0R}|U|^2dx,$$

where C' and C'' absolute constants. From now on, we need to trace the constants to make the estimates more clearly. Adding $(2\delta)^{-2m-2}\int_{|x|\le\delta/2}|U|^2dx$ to both sides of (4.41), we have that

$$
\begin{aligned}
&\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\le 2\delta}|U|^2dx+(R_0^2R)^{-2m-2}\int_{|x|\le R_0^2R}|U|^2dx\\
=\ &\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\le 2\delta}|U|^2dx+(R_0^2R)^{-2m-2}\int_{|x|\le 2\delta}|U|^2dx\\
&+(R_0^2R)^{-2m-2}\int_{2\delta<|x|\le R_0^2R}|U|^2dx\\
\le\ &\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\le 2\delta}|U|^2dx+\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\le 2\delta}|U|^2dx\\
&+(R_0^2R)^{-2m-2}\int_{2\delta<|x|\le R_0^2R}|U|^2dx\\
\le\ &(C'+1)(\delta/3)^{-2m-2}\int_{|x|\le\delta}|U|^2dx+C''(r_4R)^{-2m-2}\int_{|x|\le R_0R}|U|^2dx\\
=\ &(C'+1)(\delta/3)^{-2m-2}\int_{|x|\le\delta}|U|^2dx\\
&+(R_0^2R)^{-2m-2}C''(\frac{R_0^2}{r_4})^{2m+2}\int_{|x|\le R_0R}|U|^2dx.
\end{aligned}
\tag{4.42}
$$

We now observe that

$$
\begin{aligned}
C''(\frac{R_0^2}{r_4})^{2m+2}&=C''\left(\frac{4R_0}{R_0+1}\right)^{2m+2}\\
\le\ C''(4R_0)^{2m+2}&\ \le\exp(-2m)
\end{aligned}
$$

for all $R_0<e^{-1}/4$ and $m\ge m_0$, where m_0 depends on C'' and R_0. Thus, we obtain that

$$
\begin{aligned}
&\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\le 2\delta}|U|^2dx+(R_0^2R)^{-2m-2}\int_{|x|\le R_0^2R}|U|^2dx\\
\le\ &(C'+1)(\delta/3)^{-2m-2}\int_{|x|\le\delta}|U|^2dx\\
&+(R_0^2R)^{-2m-2}\exp(-2m)\int_{|x|\le R_0R}|U|^2dx.
\end{aligned}
\tag{4.43}
$$

It should be noted that (4.43) is valid for all $m=j+\frac{1}{2}$ with $j\in\mathbb{N}$ and $j\ge j_0$, where j_0 depends on R_0. Setting $R_j=(\gamma(j+\frac{1}{2}))^{-1}$ and using the relation $m=$

$(\gamma R)^{-1}$, we get from (4.43) that

$$\begin{aligned} &\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\leq 2\delta}|U|^2dx+(R_0^2R_j)^{-2m-2}\int_{|x|\leq R_0^2R_j}|U|^2dx \\ \leq\ &(C'+1)(\delta/3)^{-2m-2}\int_{|x|\leq\delta}|U|^2dx \\ &+(R_0^2R_j)^{-2m-2}\exp(-2cR_j^{-1})\int_{|x|\leq R_0R_j}|U|^2dx \end{aligned} \tag{4.44}$$

for all $j\geq j_0$ and $c=\gamma^{-1}$. We now observe that

$$R_{j+1}<R_j<2R_{j+1}\quad\text{for all}\quad j\in\mathbb{N}.$$

Thus, if $R_{j+1}<r\leq R_j$, we can conclude that

$$\begin{cases} \displaystyle\int_{|x|\leq R_0^2r}|U|^2dx\leq\int_{|x|\leq R_0^2R_j}|U|^2dx, \\ \displaystyle\exp(-2cR_j^{-1})\int_{|x|\leq R_0R_j}|U|^2dx\leq\exp(-cr^{-1})\int_{|x|\leq r}|U|^2dx, \end{cases} \tag{4.45}$$

where we have used the inequality $R_0R_j<2R_0R_{j+1}\leq R_{j+1}/(2e)<R_{j+1}$ to derive the second inequality above. Namely, we have from (4.44) and (4.45) that

$$\begin{aligned} &\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\leq 2\delta}|U|^2dx+(R_0^2R_j)^{-2m-2}\int_{|x|\leq R_0^2r}|U|^2dx \\ \leq\ &(C'+1)(\delta/3)^{-2m-2}\int_{|x|\leq\delta}|U|^2dx \\ &+(R_0^2R_j)^{-2m-2}\exp(-cr^{-1})\int_{|x|\leq r}|U|^2dx. \end{aligned} \tag{4.46}$$

If there exists $s\in\mathbb{N}$ such that

$$R_{j+1}<R_0^{2s}\leq R_j\quad\text{for some}\quad j\geq j_0, \tag{4.47}$$

then replacing r by R_0^{2s} in (4.46) leads to

$$\begin{aligned} &\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\leq 2\delta}|U|^2dx+(R_0^2R_j)^{-2m-2}\int_{|x|\leq R_0^{2s+2}}|U|^2dx \\ \leq\ &(C'+1)(\delta/3)^{-2m-2}\int_{|x|\leq\delta}|U|^2dx \\ &+(R_0^2R_j)^{-2m-2}\exp(-cR_0^{-2s})\int_{|x|\leq R_0^{2s}}|U|^2dx. \end{aligned} \tag{4.48}$$

Here s and R_0 are yet to be determined. The trick now is to find suitable s and R_0 satisfying (4.47) and the inequality

$$\exp(-cR_0^{-2s})\int_{|x|\le R_0^{2s}}|U|^2dx \le \frac{1}{2}\int_{|x|\le R_0^{2s+2}}|U|^2dx \tag{4.49}$$

holds with such choices of s and R_0.

It is time to use the three-ball inequality (4.1). To this end, we choose $r_1 = R_0^{2k+2}$, $r_2 = R_0^{2k}$ and $r_3 = R_0^{2k-2}$ for $k \ge 1$ and require $R_0^2 \le \min\{(1/4e)^2, R_1\}$. Thus (4.1) implies

$$\begin{aligned}&\int_{|x|<R_0^{2k}}|U_{R_0^{2k}}|^2dx\Big/\int_{|x|<R_0^{2k+2}}|U_{R_0^{2k+2}}|^2dx\\ &\le C^{1/\tau}\left(\int_{|x|<R_0^{2k-2}}|U_{R_0^{2k-2}}|^2dx\Big/\int_{|x|<R_0^{2k}}|U_{R_0^{2k}}|^2dx\right)^a,\end{aligned} \tag{4.50}$$

where

$$C=\max\{4C_0R_0^{-4}, \exp(\beta_0(-1-4\log R_0))\}R_0^{-8\tau}$$

and

$$\begin{aligned}a=\frac{1-\tau}{\tau}=\frac{A}{B}&=\frac{(\log(r_1/r_3)-1)^2-(\log(r_2/r_3))^2}{-1-2\log(r_2/r_3)}\\ &=\frac{(4\log R_0-1)^2-(2\log R_0)^2}{-1-4\log R_0}.\end{aligned}$$

It is not hard to see that

$$\begin{cases}1 < C \le C_0R_0^{-\beta_1},\\ 2 < a \le -4\log R_0,\end{cases} \tag{4.51}$$

where $\beta_1 = 32\max\{1, \beta_0\}$ (note $\tau < 1$). Combining (4.51) and using (4.50) recursively, we have that

$$\begin{aligned}&\int_{|x|\le R_0^{2s}}|U_{R_0^{2s}}|^2dx\Big/\int_{|x|\le R_0^{2s+2}}|U_{R_0^{2s+2}}|^2dx\\ &\le C^{1/\tau}(\int_{|x|<R_0^{2s-2}}|U_{R_0^{2s-2}}|^2dx\Big/\int_{|x|<R_0^{2s}}|U_{R_0^{2s}}|^2dx)^a\\ &\le C^{\frac{a^{s-1}-1}{\tau(a-1)}}(\int_{|x|<R_0^{2}}|U_{R_0^{2}}|^2dx\Big/\int_{|x|<R_0^{4}}|U_{R_0^{4}}|^2dx)^{a^{s-1}}\end{aligned} \tag{4.52}$$

for all $s \ge 1$. Now from the definition of a, we have $\tau = 1/(a+1)$ and thus

$$\frac{a^{s-1}-1}{\tau(a-1)}=\frac{a+1}{a-1}(a^{s-1}-1)\le 3a^{s-1}.$$

Then it follows from (4.52) that

$$
\begin{aligned}
&\int_{|x|\le R_0^{2s}} |U_{R_0^{2s}}|^2 dx \Big/ \int_{|x|\le R_0^{2s+2}} |U_{R_0^{2s+2}}|^2 dx \\
&\le C^{3(-4\log R_0)^{s-1}} \left(\int_{|x|<R_0^2} |U_{R_0^2}|^2 dx \Big/ \int_{|x|<R_0^4} |U_{R_0^4}|^2 dx\right)^{a^{s-1}} \\
&\le (C_0^3 (R_0)^{-3\beta_1})^{(-4\log R_0)^{s-1}} \left(\int_{|x|<R_0^2} |U_{R_0^2}|^2 dx \Big/ \int_{|x|<R_0^4} |U_{R_0^4}|^2 dx\right)^{a^{s-1}}.
\end{aligned} \tag{4.53}
$$

Note that

$$
\begin{aligned}
&\int_{|x|\le R_0^{2s}} |U|^2 dx \le R_0^{-4s} \int_{|x|\le R_0^{2s}} |U_{R_0^{2s}}|^2 dx, \\
&\int_{|x|\le R_0^{2s+2}} |U_{R_0^{2s+2}}|^2 dx \le \int_{|x|\le R_0^{2s+2}} |U|^2 dx.
\end{aligned} \tag{4.54}
$$

Thus, by (4.53) and (4.54), we can get that

$$
\begin{aligned}
&\exp(-cR_0^{-2s}) \int_{|x|\le R_0^{2s}} |U|^2 dx \\
&\le \exp(-cR_0^{-2s}) R_0^{-4s} (C_0^3 (R_0)^{-3\beta_1})^{(-4\log R_0)^{s-1}} \\
&\left(\int_{|x|<R_0^2} |U_{R_0^2}|^2 dx \Big/ \int_{|x|<R_0^4} |U_{R_0^4}|^2 dx\right)^{a^{s-1}} \int_{|x|\le R_0^{2s+2}} |U|^2 dx.
\end{aligned} \tag{4.55}
$$

Let $\mu = -\log R_0$, then if R_0 is sufficiently small, *i.e.*, μ is sufficiently large, we can see that

$$
\frac{c}{4}\exp(2\mu t) > 4t\mu + (4\mu)^{t-1}(\log C_0^3 + 3\beta_1\mu)
$$

for all $t \in \mathbb{N}$. In other words, we have that for R_0 small

$$
R_0^{-4t}(C_0^3 R_0^{-3\beta_1})^{(-4\log R_0)^{t-1}} < \exp(cR_0^{-2t}/4) < (1/2)\exp(cR_0^{-2t}/2) \tag{4.56}
$$

for all $t \in \mathbb{N}$. We now fix a $R_0 \le \min\{1/4e, \sqrt{R_1}\}$ so that (4.56) holds. The constants $m_0(R_0)$ and $j_0(R_0)$ are then fixed as well. It is a key step in our proof that we can find a universal constant R_0. After fixing R_0, we then define a number t_0, depending on R_0 and U, by

$$
\begin{aligned}
t_0 = \inf\Bigg\{ t \in \mathbb{R} : t \ge \Bigg(&\log 2 - \log(ac) \\
&+ \log\log\left(\int_{|x|<R_0^2} |U_{R_0^2}|^2 dx \Big/ \int_{|x|<R_0^4} |U_{R_0^4}|^2 dx\right)\Bigg)(-2\log R_0 - \log a)^{-1} \Bigg\}.
\end{aligned}
$$

By (4.51), one can easily check that $-2\log R_0 - \log a > 0$ for all $R_0 \le 1/16$. With the choice of t_0, we can see that

$$\left(\int_{|x|<R_0^2} |U_{R_0^2}|^2 dx \Big/ \int_{|x|<R_0^4} |U_{R_0^4}|^2 dx\right)^{a^{t-1}} \le \exp(cR_0^{-2t}/2) \tag{4.57}$$

for all $t \ge t_0$.

Let s_1 be the smallest positive integer such that $s_1 \ge t_0$. If

$$R_0^{2s_1} \le R_{j_0} = (\gamma(j_0+1/2))^{-1}, \tag{4.58}$$

then we can find a $j_1 \in \mathbb{N}$ with $j_1 \ge j_0$ such that (4.47) holds, *i.e.*,

$$R_{j_1+1} < R_0^{2s_1} \le R_{j_1}.$$

On the other hand, if

$$R_0^{2s_1} > R_{j_0}, \tag{4.59}$$

then we pick the smallest positive integer $s_2 > s_1$ such that $R_0^{2s_2} \le R_{j_0}$ and thus we can also find a $j_1 \in \mathbb{N}$ with $j_1 \ge j_0$ for which (4.47) holds. We now define

$$s = \begin{cases} s_1 & \text{if} \quad (4.58) \quad \text{holds,} \\ s_2 & \text{if} \quad (4.59) \quad \text{holds.} \end{cases}$$

It is important to note that with such an s, (4.47) is satisfied for some j_1 and (4.56), (4.57) hold. Now we set $m_1 = 2 + 2(j_1 + 1/2)$ and $m = (m_1 - 2)/2$. Combining (4.55), (4.56) and (4.57) yields that

$$\begin{aligned}
&\exp(-cR_0^{-2s}) \int_{|x|\le R_0^{2s}} |U|^2 dx \\
&\le \exp(-cR_0^{-2s}) R_0^{-4s} (C_0^3 (R_0)^{-3\beta_1})^{(-3\log R_0)^{s-1}} \\
&\quad \left(\int_{|x|<R_0^2} |U_{R_0^2}|^2 dx \Big/ \int_{|x|<R_0^4} |U_{R_0^4}|^2 dx\right)^{a^{(s-1)}} \int_{|x|\le R_0^{2s+2}} |U|^2 dx. \\
&\le \frac{1}{2} \int_{|x|\le R_0^{2s+2}} |U|^2 dx
\end{aligned}$$

which is (4.49). Using (4.49) in (4.48), we have that

$$\begin{aligned}
&\frac{1}{2}(2\delta)^{-2m-2} \int_{|x|\le 2\delta} |U|^2 dx + \frac{1}{2}(R_0^2 R_{j_1})^{-2m-2} \int_{|x|\le R_0^{2s+2}} |U|^2 dx \\
&\le (C'+1)(\delta/3)^{-2m-2} \int_{|x|\le\delta} |U|^2 dx.
\end{aligned} \tag{4.60}$$

It follows from (4.60) that

$$\frac{1}{2(C'+1)}(3R_0^2R_{j_1})^{-m_1}\int_{|x|\le R_0^{2s+2}}|U|^2dx \le \delta^{-m_1}\int_{|x|\le\delta}|U|^2dx \tag{4.61}$$

and

$$\frac{1}{2}(2\delta)^{-2m-2}\int_{|x|\le 2\delta}|U|^2dx \le (C'+1)(\delta/3)^{-2m-2}\int_{|x|\le\delta}|U|^2dx$$

which implies

$$\int_{|x|\le 2\delta}|U|^2dx \le \frac{1}{2(C'+1)}6^{m_1}\int_{|x|\le\delta}|U|^2dx. \tag{4.62}$$

The estimates (4.61) and (4.62) are valid for all $\delta \le R_0^{2s+2}/4$. Now we choose $R_2 = R_0$ in Theorem 4.4 and $R_3 = R_0^{2s+2}/4$ in Theorem 4.5. The proof is complete. □

4.5. Lipschitz propagation of smallness

To study our inverse problem, we need to obtain three-ball inequalities in terms of $\sum_{ij}|e_{ij}^\theta(u)|^2+\rho_0^2|\partial_{ij}^2u_3|^2$ instead of $|u'|^2+|u_3|^2$. To this end, the following Caccioppoli-type inequality is useful.

Lemma 4.11. *Assume that $\lambda(x),\mu(x)\in L^\infty(B_\rho)$ satisfying* (3.3) *and there exists $K_3>0$ such that*

$$\|\lambda\|_{L^\infty(B_\rho)}+\|\mu\|_{L^\infty(B_\rho)}+\|\nabla\theta\|_{L^\infty(B_\rho)}\le K_3.$$

Let $(u',u_3)\in(H^1(B_\rho))^2\times H^2(B_\rho)$ be a solution of (1.1) *in B_ρ. Then there exists a constant $C>0$, depending on δ_0, K_3 such that*

$$\int_{B_{\rho/2}}\sum_{ij}|e_{ij}^\theta(u)|^2+\rho_0^2|\partial_{ij}^2u_3|^2 \le \frac{C}{\rho^2}\int_{B_\rho}|u'|^2+C\left(\frac{1}{\rho^4}+\frac{1}{\rho^2}\right)\int_{B_\rho}|u_3|^2. \tag{4.63}$$

Proof. The proof of this lemma is adopted from [21]. Let $\eta\in C_0^4(B_\rho)$ with $0\le\eta\le 1$ and $\eta\equiv 1$ on $B_{\rho/2}$ satisfying

$$\sum_{|\alpha|\le 3}\rho^{|\alpha|}|\partial^\alpha\eta|\le C_1 \quad \text{in} \quad B_\rho \tag{4.64}$$

for some positive constant C_1. Multiplying the first equation of (1.1) by η^4u' and the second equation of (1.1) by η^4u_3 and performing integration by parts, we can get that

$$\int_{B_\rho}n_{ij}^\theta(u)\partial_j(\eta^4u_i)+\int_{B_\rho}m_{ij}(u_3)\partial_{ij}^2(\eta^4u_3)+\int_{B_\rho}n_{ij}^\theta(u)\partial_i\theta\partial_j(\eta^4u_3)=0. \tag{4.65}$$

It is easy to see that (4.65) is equivalent to

$$\begin{aligned}\int_{B_\rho} & n_{ij}^{\theta}(u)[\eta^4\partial_j u_i + 4(\partial_j\eta)\eta^3 u_i] \\ & + m_{ij}(u_3)[\eta^4\partial_{ij}^2 u_3 + 8(\partial_j\eta)\eta^3\partial_i u_3 + (\partial_{ij}\eta^4)u_3] \\ & + n_{ij}^{\theta}(u)\partial_i\theta[\eta^4\partial_j u_3 + 4(\partial_j\eta)\eta^3 u_3] = 0.\end{aligned} \tag{4.66}$$

It follows from (4.66) that

$$\begin{aligned}&\int_{B_\rho}\eta^4\sum_{ij}\Big(n_{ij}^{\theta}(u)e_{ij}^{\theta}(u) + m_{ij}(u_3)\partial_{ij}^2 u_3\Big) \\ &= -4\int_{B_\rho} n_{ij}^{\theta}(u)(\partial_j\eta)\eta^3 u_i - 8\int_{B_\rho} m_{ij}(u_3)(\partial_j\eta)\eta^3\partial_i u_3 \\ &\quad - \int_{B_\rho} m_{ij}(u_3)(\partial_{ij}^2\eta^4)u_3 \\ &\quad - 4\int_{B_\rho} n_{ij}^{\theta}(u)(\partial_i\theta)(\partial_j\eta)\eta^3 u_3.\end{aligned} \tag{4.67}$$

Observe that

$$\left|\int_{B_\rho} n_{ij}^{\theta}(u)(\partial_j\eta)\eta^3 u_i\right| \le \frac{\varepsilon}{2}\int_{B_\rho}\eta^4\sum_{ij}|n_{ij}^{\theta}|^2 + \frac{C_2}{\varepsilon\rho^2}\int_{B_\rho}|u'|^2 \tag{4.68}$$

for some $C_2 > 0$, depending on C_1. Likewise, we can obtain that

$$\left|\int_{B_\rho} m_{ij}(u_3)(\partial_{ij}^2\eta^4)u_3\right| \le \frac{\varepsilon}{2}\int_{B_\rho}\eta^4\sum_{ij}|m_{ij}(u_3)|^2 + \frac{C_3}{\varepsilon\rho^4}\int_{B_\rho}|u_3|^2 \tag{4.69}$$

for some $C_3 > 0$, and

$$\left|\int_{B_\rho} n_{ij}^{\theta}(u)(\partial_i\theta)(\partial_j\eta)\eta^3 u_3\right| \le \frac{\varepsilon}{2}\int_{B_\rho}\eta^4\sum_{ij}|n_{ij}^{\theta}(u)|^2 + \frac{C_4}{\varepsilon\rho^2}\int_{B_\rho}|u_3|^2 \tag{4.70}$$

for some $C_4 > 0$, also depending on K_3. Finally, we have

$$\begin{aligned}\left|\int_{B_\rho} m_{ij}(u_3)(\partial_j\eta)\eta^3\partial_i u_3\right| &\le \frac{\varepsilon}{2}\int_{B_\rho}\eta^4\sum_{ij}|m_{ij}(u_3)|^2 \\ &\quad + \frac{1}{2\varepsilon}\int_{B_\rho}\eta^2\sum_{ij}(\partial_j\eta\partial_i u_3)^2.\end{aligned} \tag{4.71}$$

Using the same computations on page 10-11 of [21], we get that

$$\int_{B_\rho} \eta^2 \sum_{ij} (\partial_j \eta \partial_i u_3)^2 \le \frac{C_5}{\rho^4}\left(1+\frac{1}{\varepsilon^2}\right)\int_{B_\rho} |u_3|^2 + \frac{\varepsilon^2}{2}\int_{B_\rho} \eta^4 \sum_{ij} |\partial^2_{ij} u_3|^2 \quad (4.72)$$

for some $C_5 > 0$. Putting (4.67)-(4.72) together and taking ε sufficiently small, we immediately arrive at the desired estimate (4.63). □

Remark 4.12. If $\rho \le 1$, then (4.63) can be written as

$$\int_{B_{\rho/2}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \le \frac{C}{\rho^4}\int_{B_\rho} |u'|^2 + |u_3|^2. \quad (4.73)$$

Our aim here is to derive another version of the three sphere inequality. We will make use of arguments introduced in [5]. For any scalar or vector valued function f, we denote $(f)_r = \frac{1}{|B_r|}\int_{B_r} f$. Then we define an operator $T : (H^1(B_R))^2 \times H^2(B_R) \to (H^1(B_R))^2 \times H^2(B_R)$ by $Tu = T(u', u_3) = (v'(x;r), v_3(x;r))$, where

$$\begin{aligned} v'(x;r) = (u')_r &+ \frac{1}{2}(\nabla u' - (\nabla u')^t)_r x \\ &+ \frac{1}{2}[(\nabla\theta)_r \otimes (\nabla u_3)_r - ((\nabla\theta)_r \otimes (\nabla u_3)_r)^t]x \\ &- (\theta - (\theta)_r)(\nabla u_3)_r \end{aligned} \quad (4.74)$$

and

$$v_3(x;r) = (u_3)_r + (\nabla u_3)_r \cdot x. \quad (4.75)$$

Here the tensor product of two vectors ξ and η is defined as

$$\xi \otimes \eta = \begin{pmatrix} \xi_1\eta_1 & \xi_1\eta_2 \\ \xi_2\eta_1 & \xi_2\eta_2 \end{pmatrix}.$$

Note that $(x)_r = 0$.

We now denote the space

$$\mathcal{R} = \{w = (w', w_3) \mid w' = a + Wx - \theta c,\ w_3 = b + c\cdot x\},$$

where a, c are two-dimensional vectors, b is a scalar, and W is a 2×2 skew-symmetric matrix. It is readily seen that $Tu = u$ for all $u \in \mathcal{R}$. Denote $Lu = ((e^\theta_{ij}(u))_{1\le i\le 2, 1\le j\le 2}, (\partial^2_{ij} u_3)_{1\le i\le 2, 1\le j\le 2})$. It is also easy to check that $\mathcal{R}$ is the null space of L. We need to compute the norm of T. Recall that

$$\begin{aligned} &\|v'\|^2_{H^1(B_R)} + \|v_3\|^2_{H^2(B_R)} \\ &= \frac{1}{R^2}\int_{B_R} |v'|^2 + \int_{B_R} |\nabla v'|^2 + \frac{1}{R^2}\int_{B_R} |v_3|^2 + \int_{B_R} |\nabla v_3|^2 + R^2\int_{B_R} |\nabla^2 v_3|^2. \end{aligned}$$

In view of (4.74), (4.75) and the assumption on θ, we obtain that

$$\|T\| \le C\left(1+\frac{R^2}{r^2}+\frac{1}{r^2}\right)^{1/2}$$

with an absolute constant $C > 0$. Assume that $B_R \subset \Omega$ and so (3.2) holds on B_R. Now if we take $r = R$, then $(u' - v'(\cdot;\, R), u_3 - v_3(\cdot;\, R))$ satisfies the normalization conditions (3.9) with Ω be replaced by B_R and therefore (3.10) becomes

$$\|u' - v'(\cdot;\, R)\|^2_{H^1(B_R)} + \|u_3 - v_3(\cdot;\, R)\|^2_{H^2(B_R)} \le C\int_{B_R}\sum_{ij}|e^\theta_{ij}(u)|^2 + R^2|\partial^2_{ij}u_3|^2,$$

where C depends on A_2. Using Lemma 2.1 in [5], we conclude that

$$\begin{aligned}
&\|u' - v'(\cdot;r)\|^2_{H^1(B_R)} + \|u_3 - v_3(\cdot;r)\|^2_{H^2(B_R)}\\
&\le C(1+\|T\|)^2\int_{B_R}\sum_{ij}|e^\theta_{ij}(u)|^2 + R^2|\partial^2_{ij}u_3|^2\\
&\le C(1+\frac{R^2}{r^2}+\frac{1}{r^2})\int_{B_R}\sum_{ij}|e^\theta_{ij}(u)|^2 + R^2|\partial^2_{ij}u_3|^2.
\end{aligned}$$

In particular, we have that

$$\begin{aligned}
&\int_{B_R}|u' - v'(\cdot;r)|^2 + |u_3 - v_3(\cdot;r)|^2\\
&\le CR^2\left(1+\frac{R^2}{r^2}+\frac{1}{r^2}\right)\left(1+\frac{R^2}{\rho_0^2}\right)\int_{B_R}\sum_{ij}|e^\theta_{ij}(u)|^2 + \rho_0^2|\partial^2_{ij}u_3|^2.
\end{aligned} \tag{4.76}$$

We now prove the following three-ball inequalities.

Theorem 4.13. *Assume that $\bar{R}_0$ and R_1 are given in Theorem* 4.1. *If* $0 < r_1 < r_2 < 2r_2 < r_3 \le \min\{\rho_0\bar{R}_0, 1\}$ *and* $r_1/r_3 < 2r_2/r_3 < R_1$, *then*

$$\begin{aligned}
&\int_{B_{r_2}}\sum_{ij}|e^\theta_{ij}(u)|^2 + \rho_0^2|\partial^2_{ij}u_3|^2\\
&\le \frac{C}{r_2^6}\left(\frac{r_3}{r_1}\right)^{2-2\tau}\left(\int_{B_{r_1}}\sum_{ij}|e^\theta_{ij}(u)|^2 + \rho_0^2|\partial^2_{ij}u_3|^2\right)^{\tau}\\
&\quad\times\left(\int_{B_{r_3}}\sum_{ij}|e^\theta_{ij}(u)|^2 + \rho_0^2|\partial^2_{ij}u_3|^2\right)^{1-\tau}
\end{aligned} \tag{4.77}$$

for $(u', u_3) \in (H^1(B_{\rho_0\bar{R}_0}))^2 \times H^3(B_{\rho_0\bar{R}_0})$ *satisfying* (1.1), *where* $C > 0$ *and* $0 < \tau < 1$ *depend on* r_1/r_3, r_2/r_3, δ_0, A_2.

Proof. Let $\tilde{u}' = u' - v'(x; r_1)$, $\tilde{u}_3 = u_3 - v_3(x; r_1)$. Note that $(\tilde{u}', \tilde{u}_3)$ satisfies the normalization condition (3.9) on B_{r_1}. Recall that $\bar{R}_0 \le 1$. Now combining (4.73), (4.76), (4.2), and (3.10) implies that

$$\begin{aligned}
&\int_{B_{r_2}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \\
&= \int_{B_{r_2}} \sum_{ij} |e^\theta_{ij}(\tilde{u})|^2 + \rho_0^2 |\partial^2_{ij} \tilde{u}_3|^2 \\
&\le \frac{C}{r_2^4} \int_{B_{2r_2}} |\tilde{u}'|^2 + |\tilde{u}_3|^2 \\
&\le \frac{C'}{r_2^6} \left(\int_{B_{r_1}} |\tilde{u}'|^2 + |\tilde{u}_3|^2 \right)^\tau \left(\int_{B_{r_3}} |\tilde{u}'|^2 + |\tilde{u}_3|^2 \right)^{1-\tau} \\
&\le \frac{C''}{r_2^6} \left(r_1^2 (1 + \frac{1}{r_1^2})(1 + \frac{r_1^2}{\rho_0^2}) \right)^\tau \left(r_3^2 (1 + \frac{r_3^2}{r_1^2} + \frac{1}{r_1^2})(1 + \frac{r_3^2}{\rho_0^2}) \right)^{1-\tau} \\
&\quad \times \left(\int_{B_{r_1}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^\tau \left(\int_{B_{r_3}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^{1-\tau} \\
&\le C''' \frac{1}{r_2^6} \left(\frac{r_3}{r_1} \right)^{2-2\tau} \\
&\quad \times \left(\int_{B_{r_1}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^\tau \left(\int_{B_{r_3}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^{1-\tau}.
\end{aligned}$$

□

A key ingredient in solving our inverse problem is a continuation estimate from the interior for the solution u of (3.4), (3.5). To do this, we need some assumptions the coupled field $(\widehat{T}, \widehat{M})$. We assume that $(\widehat{T}, \widehat{M})$ satisfies

$$\operatorname{supp}(\widehat{T}, \widehat{M}) \subset \Gamma_0, \tag{4.78}$$

where Γ_0 is an open subarc of $\partial\Omega$ with

$$|\Gamma_0| \le (1 - \gamma_0)|\partial\Omega| \tag{4.79}$$

for some $\gamma_0 > 0$. We first prove the following lemma.

Lemma 4.14. *Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{2,1}$ boundary $\partial\Omega$ characterized by constants A_0 and ρ_0. Assume that $\lambda, \mu \in L^\infty(\Omega)$ satisfy* (3.3), *$\nabla\theta \in L^\infty(\Omega)$, and*

$$\|\lambda\|_{L^\infty(\Omega)} + \|\mu\|_{L^\infty(\Omega)} + \|\nabla\theta\|_{L^\infty(\Omega)} \le A_2$$

for some $A_2 > 0$. *Let* $(u', u_3) \in (H^1(\Omega))^2 \times H^2(\Omega)$ *be the unique weak solution of* (3.4), (3.5) *satisfying* (3.9), *with* $(\widehat{T}, \widehat{M}) \in H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ *satisfying* (4.78), (4.79), *and* (3.8). *Then we have*

$$\|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3} \le C \left(\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^{1/2}, \tag{4.80}$$

where C *depends on* $\delta_0, A_0, A_1, A_2, \gamma_0$.

Proof. We follow the arguments used in the proof of [20, Lemma 7.1]. For any $(f, g) \in (H^{1/2}(\partial\Omega))^3$, one can find $(v', v_3) \in (H^1(\Omega))^2 \times H^2(\Omega)$ such that $v'|_{\partial\Omega} = f$, $v_3|_{\partial\Omega} = 0$, $\partial_\nu v_3|_{\partial\Omega} = g$ and

$$\|(v', v_3)\|_{(H^1(\Omega))^2 \times H^2(\Omega)} \le C \|(f, \rho_0 g)\|_{(H^{1/2}(\partial\Omega))^3}, \tag{4.81}$$

where C depends on A_0 and A_1. In view of the weak formulation of the solution, we can compute

$$\begin{aligned}
&\int_{\partial\Omega} \frac{1}{\rho_0} \widehat{T} \cdot f + \widehat{M}_\nu g = \frac{1}{\rho_0} \int_{\partial\Omega} \widehat{T} \cdot v' + \widehat{M}_\nu \rho_0 g \\
&= \frac{1}{\rho_0} \int_{\partial\Omega} \widehat{T} \cdot v' + \rho_0 \widehat{M}_\nu \partial_\nu v_3 + \rho_0 \partial_s \widehat{M}_\tau v_3 \\
&= \frac{1}{\rho_0^2} \int_\Omega \sum_{ij} (\rho_0^2 n^\theta_{ij}(u) e^\theta_{ij}(v) + \rho_0^4 m_{ij}(u_3) \partial^2_{ij} v_3) \\
&\le C \left(\frac{1}{\rho_0^2} \int_\Omega \sum_{ij} \rho_0^2 |e^\theta_{ij}(u)|^2 + \rho_0^4 |\partial^2_{ij} u_3|^2 \right)^{1/2} \\
&\quad \times \left(\frac{1}{\rho_0^2} \int_\Omega \sum_{ij} \rho_0^2 |e^\theta_{ij}(v)|^2 + \rho_0^4 |\partial^2_{ij} v_3|^2 \right)^{1/2} \\
&\le C \left(\frac{1}{\rho_0^2} \int_\Omega \sum_{ij} \rho_0^2 |e^\theta_{ij}(u)|^2 + \rho_0^4 |\partial^2_{ij} u_3|^2 \right)^{1/2} \|(v', v_3)\|_{(H^1(\Omega))^2 \times H^2(\Omega)} \\
&\le C \left(\frac{1}{\rho_0^2} \int_\Omega \sum_{ij} \rho_0^2 |e^\theta_{ij}(u)|^2 + \rho_0^4 |\partial^2_{ij} u_3|^2 \right)^{1/2} \|(f, \rho_0 g)\|_{(H^{1/2}(\partial\Omega))^3},
\end{aligned}$$

where C depends on δ_0, A_0, A_1, A_2. Consequently, we obtain

$$\|(\widehat{T}, \widehat{M}_\nu)\|_{(H^{-1/2}(\partial\Omega))^3} \le C \left(\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^{1/2}. \tag{4.82}$$

On the other hand, for any given $g \in H^{1/2}(\partial\Omega)$, let $h \in H^{3/2}(\Gamma_0)$ satisfy $\partial_s h = g$ on Γ_0 and $\|h\|_{H^{3/2}(\Gamma_0)} \le C\|\rho_0 g\|_{H^{1/2}(\Gamma_0)} \le C\|\rho_0 g\|_{H^{1/2}(\partial\Omega)}$, where C here depends on A_0 and A_1. Now let $\tilde{h} \in H^{3/2}(\partial\Omega)$ be such that $\tilde{h} = h$ on Γ_0 and $\|\tilde{h}\|_{H^{3/2}(\partial\Omega)} \le C\|h\|_{H^{3/2}(\Gamma_0)}$, where C depends on A_0, A_1, γ_0. Moreover, let $v_3 \in H^2(\Omega)$ satisfy $v_3 = \tilde{h}$, $\partial_\nu v_3 = 0$ on $\partial\Omega$ and $\|v_3\|_{H^2(\Omega)} \le C\|\tilde{h}\|_{H^{3/2}(\partial\Omega)}$, where C depends on A_0 and A_1. Let $f \in (H^{1/2}(\partial\Omega))^2$ be the same function given above. Now we can derive that

$$\begin{aligned}
&\int_{\partial\Omega} \frac{1}{\rho_0}\widehat{T}\cdot(-f) + \widehat{M}_\tau g = \frac{1}{\rho_0}\int_{\Gamma_0} \widehat{T}\cdot(-f) + \rho_0 \widehat{M}_\tau \partial_s h \\
&= -\frac{1}{\rho_0}\int_{\Gamma_0} \widehat{T}\cdot f + \rho_0 \partial_s \widehat{M}_\tau \tilde{h} \\
&= -\frac{1}{\rho_0}\int_{\partial\Omega} \widehat{T}\cdot f + \rho_0 \partial_s \widehat{M}_\tau v_3 + \rho_0 \widehat{M}_\nu \partial_\nu v_3 \\
&\le C\left(\frac{1}{\rho_0^2}\int_\Omega \sum_{ij} \rho_0^2 |e^\theta_{ij}(u)|^2 + \rho_0^4 |\partial^2_{ij} u_3|^2\right)^{1/2} \|(f, \rho_0 g)\|_{(H^{1/2}(\partial\Omega))^3}
\end{aligned}$$

with C depending on $\delta_0, A_0, A_1, A_2, \gamma_0$, which implies

$$\|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3} \le C\left(\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2\right)^{1/2}. \tag{4.83}$$

Finally, combining (4.82) and (4.83) leads to (4.80). ☐

We are now ready to prove the following theorem.

Theorem 4.15 (Lipschitz propagation of smallness). *Assume that Ω is a bounded domain having boundary $\partial\Omega \in C^{4,1}$ with constants A_0, ρ_0. Let $\lambda, \mu \in C^{1,1}(\bar{\Omega})$ satisfy* (3.3) *and $\bar{\theta} \in C^{2,1}(\bar{\Omega})$ satisfy* (3.22) *and* (3.23) *hold. Let $u \in (H^1(\Omega))^2 \times H^2(\Omega)$ be the weak solution of* (3.4), (3.5) *satisfying* (3.9) *with Neumann boundary condition $(\widehat{T}, \widehat{M}) \in (H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)$ satisfying* (3.8), (4.78), (4.79). *Then for every $\rho > 0$ and every $x \in \Omega_{\frac{7}{\vartheta}\rho\rho_0}$, we have*

$$\int_{B_{\rho\rho_0}(x)} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \ge C_\rho \int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2, \tag{4.84}$$

where C_ρ depends on $A_0, A_1, A_2, \delta_0, \gamma_0, \rho$, and $\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2 \times H^{1/2}(\partial\Omega)} / \|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}$. Here $\vartheta = R_1$ and R_1 is the constant given in Theorem 4.1.

Proof. There is no restriction to take $\rho_0 = 1$. The general case can be proved by a simple scaling argument. Note that $\Omega_{\frac{7\rho}{\vartheta}}$ is connected for all $0 < \rho \le \zeta$, where ζ

depends on δ_0, A_0, A_2. It suffices to prove the result for small ρ. We now choose a ρ such that $7\rho/\vartheta \le \bar{R}_0$. Let $y \in \Omega_{\frac{7\rho}{\vartheta}}$ and $\gamma(t)$ be an arc in $\Omega_{\frac{7\rho}{\vartheta}}$ joining y and x. We now define $\{x_i\}$, $i = 1, \cdots, L$, as follows: $x_1 = x$, $x_{i+1} = \gamma(t_i)$ with $t_i = \max\{t \mid |\gamma(t) - x_i| = 2\rho\}$ if $|x_i - y| > 2\rho$, otherwise, let $i = L$ and stop the process. By construction, we can see that the spheres $B_\rho(x_i)$ are pairwise disjoint and $|x_{i+1} - x_i| = 2\rho$ for $i = 1, \cdots, L-1$, $|x_L - y| \le 2\rho$.

Since $x_i \in \Omega_{\frac{7\rho}{\vartheta}}$, we use the three-sphere inequality (4.77) with $x = x_i, r_1 = \rho$, $r_2 = 3\rho, r_3 = \frac{7\rho}{\vartheta} \le \bar{R}_0 < 1$ for $i = 1, \cdots, L-1$ to obtain

$$\frac{\int_{B_\rho(x_{i+1})} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2} \le C \left(\frac{1}{\rho}\right)^6 \left(\frac{\int_{B_\rho(x_i)} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}\right)^\tau,$$

where $C > 0$ depends on δ_0, A_2. Induction on i implies

$$\frac{\int_{B_\rho(y)} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2} \le C^{\frac{1}{1-\tau}} \left(\frac{1}{\rho}\right)^{\frac{6}{1-\tau}} \left(\frac{\int_{B_\rho(x)} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}\right)^{\tau^L}. \tag{4.85}$$

Note that $L \le |\Omega|/(\pi\rho^2) \le A_1/\pi$.

Let us now cover $\Omega_{\frac{8\rho}{\vartheta}}$ with internally nonoverlapping closed cubes of side $\ell = \sqrt{2}\rho/\vartheta$. It is clear that any such cube is contained in a sphere of radius ρ with center in $\Omega_{\frac{7\rho}{\vartheta}}$ and the number of such cube is controlled by $N = 2|\Omega|\vartheta^2/(4\rho^2)$. It follows from (4.85) that

$$\frac{\int_{\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2} \le C \left(\frac{1}{\rho}\right)^{\frac{8-2\tau}{1-\tau}} \left(\frac{\int_{B_\rho(x)} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}\right)^{\tau^L}. \tag{4.86}$$

Here C depends on δ_0, A_2, $|\Omega|$.

Now we want to estimate the left-hand side of (4.86) from below by a positive constant. Obviously, we have

$$\frac{\int_{\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2} = 1 - \frac{\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}. \tag{4.87}$$

It suffices to show that there exists $\overline{\rho} > 0$ such that

$$\frac{\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_\Omega \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij} u_3|^2} \le \frac{1}{2}, \tag{4.88}$$

for every ρ, $0 < \rho \le \overline{\rho}$. The proposition, then, follows from (4.86) and (4.88).

By Hölder's inequality and Sobolev's inequality

$$\|w\|^2_{L^4(\Omega)} \leq C\|w\|^2_{H^{1/2}(\Omega)}$$

with C depending on A_0, A_1, we have

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij}u_3|^2 \\
&\leq |\Omega\setminus\Omega_{8\rho/\vartheta}|^{1/2} \sum_{ij}\left(\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} \left(|e^\theta_{ij}(u)|^2 + |\partial^2_{ij}u_3|^2\right)^2\right)^{1/2} \\
&\leq \sqrt{2}|\Omega\setminus\Omega_{8\rho/\vartheta}|^{1/2} \sum_{ij}\left(\left(\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} |e^\theta_{ij}(u)|^4\right)^{1/2} + \left(\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} |\partial^2_{ij}u_3|^4\right)^{1/2}\right) \\
&\leq C|\Omega\setminus\Omega_{8\rho/\vartheta}|^{1/2}\|(u', u_3)\|^2_{(H^{3/2}(\Omega))^2\times H^{5/2}(\Omega)}.
\end{aligned} \tag{4.89}$$

Interpolating the global estimates (3.18) and (3.24) yields

$$\begin{aligned}
&\|(u', u_3)\|_{(H^{3/2}(\Omega))^2\times H^{5/2}(\Omega)} \\
&\leq \|(u', u_3)\|_{(H^{3/2}(\Omega))^2\times H^{3}(\Omega)} \\
&\leq C\|(\widehat{T}, \widehat{M})\|_{(L^2(\partial\Omega))^2\times H^{1/2}(\partial\Omega)}
\end{aligned} \tag{4.90}$$

where C depends on A_0, A_1, A_2, δ_0.

Following the argument of [6] (see [6, A.3] for details), there exists a positive constant C, depending on A_0, A_1, A_2, δ_0, such that

$$|\Omega\setminus\Omega_{8\rho/\vartheta}| \leq C\rho. \tag{4.91}$$

It follows from (4.89), (4.90), and (4.91) that

$$\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij}u_3|^2 \leq C\rho^{1/2}\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2\times H^{1/2}(\partial\Omega)}. \tag{4.92}$$

From (4.80), we can obtain that

$$\frac{\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij}u_3|^2}{\int_{\Omega} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij}u_3|^2} \leq C\rho^{1/2}\frac{\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2\times H^{1/2}(\partial\Omega)}}{\|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}},$$

where C depends on A_0, A_1, A_2, δ_0, γ_0. Finally, we can choose $\bar{\rho}$, depending on A_0, A_1, A_2, δ_0, γ_0, and $\|(\widehat{T}, \widehat{M})\|^2_{(L^2(\partial\Omega))^2\times H^{1/2}(\partial\Omega)}/\|(\widehat{T}, \widehat{M})\|_{(H^{-1/2}(\partial\Omega))^3}$, such that

$$\frac{\int_{\Omega\setminus\Omega_{8\rho/\vartheta}} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij}u_3|^2}{\int_{\Omega} \sum_{ij} |e^\theta_{ij}(u)|^2 + |\partial^2_{ij}u_3|^2} \leq \frac{1}{2}$$

for all $0 < \rho < \bar{\rho}$. The proof now is complete. □

5. The inverse problem

In this section, we would like to study the problem of estimating the size of inclusion embedded in a shallow shell by one boundary measurement. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with boundary $\partial\Omega$, which is of class $C^{4,1}$ with constants A_0, ρ_0. Assume that (3.1) holds. Now let D be a possibly disconnected measurable subdomain of Ω satisfying

$$\operatorname{dist}(D, \partial\Omega) \geq d_0\rho_0 \tag{5.1}$$

for some given constant d_0. Let $\lambda, \mu \in C^{1,1}(\bar{\Omega})$ satisfy (3.3) and $\bar{\theta} \in C^{2,1}(\bar{\Omega})$ satisfy (3.22). Besides, assume that the estimate (3.23) holds. For measurable functions λ_0, μ_0, we define

$$\tilde{\lambda} = \lambda + \chi_D\lambda_0 \qquad \text{and} \qquad \tilde{\mu} = \mu + \chi_D\mu_0,$$

where χ_D is the characteristic function of D. To guarantee the well-posedness of the forward problem, we assume

$$0 < \tilde{\delta}_0 \leq \tilde{\lambda} \quad \text{and} \quad \tilde{\delta}_0 \leq \tilde{\mu} \quad \forall\, x \in \Omega.$$

To describe the jump condition, we introduce some shorthand notations. We set

$$a = \frac{4\lambda\mu}{\lambda + 2\mu},\ b = 4\mu,\ c = \frac{4\lambda\mu}{3(\lambda + 2\mu)},\ d = \frac{4\mu}{3} \tag{5.2}$$

and the corresponding $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ replacing λ, μ with $\tilde{\lambda}, \tilde{\mu}$ respectively. We assume the following condition on the jump at the interface ∂D. There exists a constant $k_0 > 0$ such that

$$(\tilde{f} - f) \leq k_0 f \qquad \forall\, x \in \partial D, \tag{5.3}$$

where $f = a, b, c, d$ and $\tilde{f} = \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. On the prescribed boundary field $(\widehat{T}, \widehat{M})$, we assume

$$(\widehat{T}, \widehat{M}) \in (H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega) \quad \text{and} \quad \operatorname{supp}(\widehat{T}, \widehat{M}) \subset \Gamma_0, \tag{5.4}$$

where Γ_0 is an open subarc of $\partial\Omega$ with

$$|\Gamma_0| \leq (1 - \gamma_0)|\partial\Omega| \tag{5.5}$$

for some $\gamma_0 > 0$ and satisfies the compatibility condition (3.8). We consider two boundary value problems. Let $u = (u', u_3)$ satisfy

$$\begin{cases} \partial_j n^{\theta}_{ij}(u) = 0 \quad \text{in} \quad \Omega, \\ \partial^2_{ij} m_{ij}(u_3) - \partial_j(n^{\theta}_{ij}(u)\partial_i\theta) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{5.6}$$

with boundary conditions

$$\begin{cases} n^{\theta}_{ij}(u)\nu_j = \rho_0^{-1}\widehat{T}_i, \\ m_{ij}(u_3)\nu_i\nu_j = \widehat{M}_\nu, \\ (\partial_i m_{ij}(u_3) - n^{\theta}_{ij}(u)\partial_i\theta)\nu_j + \partial_s(m_{ij}(u_3)\nu_i\tau_j) = -\partial_s\widehat{M}_\tau. \end{cases} \tag{5.7}$$

Next we let $\widetilde{u} = (\widetilde{u}', \widetilde{u}_3)$ satisfy

$$\begin{cases} \partial_j \widetilde{n}^{\theta}_{ij}(\widetilde{u}) = 0 \quad \text{in} \quad \Omega, \\ \partial^2_{ij}\widetilde{m}_{ij}(\widetilde{u}_3) - \partial_j(\widetilde{n}^{\theta}_{ij}(\widetilde{u})\partial_i\theta) = 0 \quad \text{in} \quad \Omega, \end{cases} \tag{5.8}$$

with boundary conditions

$$\begin{cases} \widetilde{n}^{\theta}_{ij}(\widetilde{u})\nu_j = \rho_0^{-1}\widehat{T}_i, \\ \widetilde{m}_{ij}(\widetilde{u}_3)\nu_i\nu_j = \widehat{M}_\nu, \\ (\partial_i \widetilde{m}_{ij}(\widetilde{u}_3) - \widetilde{n}^{\theta}_{ij}(\widetilde{u})\partial_i\theta)\nu_j + \partial_s(\widetilde{m}_{ij}(\widetilde{u}_3)\nu_i\tau_j) = -\partial_s\widehat{M}_\tau. \end{cases} \tag{5.9}$$

To ensure the uniqueness of the solution, we impose the normalization conditions (3.9). Let

$$\begin{aligned} \widetilde{W} &= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}\cdot\widetilde{u}' + \widehat{M}_\nu\partial_\nu\widetilde{u}_3 + \partial_s\widehat{M}_\tau\widetilde{u}_3, \\ W &= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}\cdot u' + \widehat{M}_\nu\partial_\nu u_3 + \partial_s\widehat{M}_\tau u_3 \\ &= \int_\Omega \sum_{ij} n^{\theta}_{ij}(u)e^{\theta}_{ij}(u) + m_{ij}(u_3)\partial^2_{ij}u_3, \end{aligned} \tag{5.10}$$

represent the work exerted by the boundary field when the inclusion is present or absent, respectively. For $r > 0$ we shall use the notation

$$D_r = \{x \in D : \operatorname{dist}(x, \partial D) > r\}.$$

We can now state our main result.

Theorem 5.1. *Suppose that all the hypotheses stated in this section hold. Moreover, we assume $\rho_0 < 1$. Let D be an inclusion satisfying the following fatness condition*

$$|D_{h_1\rho_0}| \geq \frac{1}{2}|D| \tag{5.11}$$

for a given positive constant h_1. Then we have the estimate

$$C_1\rho_0^2\left|\frac{W-\widetilde{W}}{W}\right| \leq |D| \leq C_2\rho_0^2\left|\frac{W-\widetilde{W}}{W}\right|, \tag{5.12}$$

where C_1 depends on A_0, A_1, A_2, d_0, k_0 and δ_0 and C_2 depends on A_0, A_1, A_2, δ_0, γ_0, d_0, h_1 and the ratio

$$\|(\widehat{M},\widehat{T})\|_{(L^2(\partial\Omega))^2\times H^{1/2}(\partial\Omega)}/\|(\widehat{M},\widehat{T})\|_{(H^{-1/2}(\partial\Omega))^3}.$$

The key ingredients in the proof of Theorem 5.1 are the energy estimate for the Neumann problems (5.6)-(5.7), (5.8)-(5.9), and the Lipschitz propagation of smallness.

Lemma 5.2. *Let* $(\widehat{M}, \widehat{T}) \in L^2(\partial\Omega)$ *satisfy* (5.4)*,* (5.5) *and* (3.8)*. Let* $\lambda, \mu, \tilde{\lambda}, \tilde{\mu} \in L^\infty(\Omega)$ *satisfy* (3.3) *and* (5.3)*. Let* $u, \tilde{u} \in (H^1(\Omega))^2 \times H^2(\Omega)$ *solutions to* (5.6)*–*(5.7) *and* (5.8)*–*(5.9) *respectively. Then there exist positive constants* $\tilde{C}_1, \tilde{C}_2$ *depending on* $A_0, A_1, A_2, \delta_0, k_0$ *such that*

$$\tilde{C}_1 \int_D \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \le W - \widetilde{W} \le \tilde{C}_2 \int_D \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2. \quad (5.13)$$

With the help of Lemma 5.2 and Theorem 4.15, we first prove Theorem 5.1.

Proof of Theorem 5.1. By the interior regularity theorem and the Sobolev embedding, we have that

$$\Big\| \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \Big\|_{L^\infty(D)} \le \frac{C}{\rho_0^2} \left(\|u'\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^2(\Omega)} \right), \quad (5.14)$$

with C depending on A_2, d_0, δ_0. From (5.14), Proposition 3.1, (5.10), we obtain that

$$\Big\| \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \Big\|_{L^\infty(D)} \le \frac{C}{\rho_0^2} W, \quad (5.15)$$

where C depends on $A_0, A_1, A_2, d_0, \delta_0$. The lower bound on $|D|$ in (5.12) follows from the right-hand side of (5.13) and from (5.15).

Let us prove the upper bound for $|D|$. Let $\varepsilon = \min\{2d_0\vartheta/7, h_1/\sqrt{2}\}$ and let us cover $D_{h_1\rho_0}$ with internally non overlapping closed squares Q_l of side $\varepsilon\rho_0$ for $l = 1, \ldots, L$. By choice of ε the squares Q_l are contained in D. Let $\bar{l}$ be such that

$$\int_{Q_{\bar{l}}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 = \min_l \int_{Q_l} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2.$$

Noticing that $|D_{h_1\rho_0}| \le L\varepsilon^2\rho_0^2$, we have

$$\begin{aligned} \int_D \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 &\ge \int_{\cup_{l=1}^L Q_l} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \\ &\ge L \int_{Q_{\bar{l}}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \\ &\ge \frac{|D_{h_1\rho_0}|}{\rho_0^2 \varepsilon^2} \int_{Q_{\bar{l}}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2. \end{aligned} \quad (5.16)$$

Let $\overline{x}$ be the center of $Q_{\overline{l}}$. From (5.16), estimate (4.84) with $x = \overline{x}$ and $\rho = \varepsilon/2$, and hypothesis (5.11), we conclude that

$$\int_D \sum_{ij} |e^{\theta}_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \geq \frac{K|D|}{\rho_0^2} W, \tag{5.17}$$

where K is a positive constant depending on A_0, A_1, A_2, d_0, δ_0, γ_0, h_1 and $\|(\widehat{M}, \widehat{T})\|_{(L^2(\partial\Omega))^2 \times H^{1/2}(\partial\Omega)} / \|(\widehat{M}, \widehat{T})\|_{(H^{-1/2}(\partial\Omega))^3}$. The upper bound for D follows from the left-hand side of (5.13) and from (5.17). □

To end this section, we give a proof of Lemma 5.2.

Proof of Lemma 5.2. Let $w = (w', w_3)$. From the first equation of (5.6), we have that

$$\int_{\Omega} n^{\theta}_{ij}(u) \partial_j w'_i = \int_{\partial\Omega} \rho_0^{-1} \widehat{T}_i w'_i. \tag{5.18}$$

On the other hand, by the second equation of (5.6), the integration by parts leads to

$$\begin{aligned} &\int_{\Omega} m_{ij}(u_3) \partial^2_{ij} w_3 + n^{\theta}_{ij}(u) \partial_i \theta \partial_j w_3 \\ &= \int_{\partial\Omega} -\partial_i m_{ij}(u_3) \nu_j w_3 + m_{ij}(u_3) \nu_i \partial_j w_3 + n^{\theta}_{ij}(u) \partial_i \theta \nu_j w_3. \end{aligned} \tag{5.19}$$

Replacing $\partial_j w_3$ by $\nu_j \partial_\nu w_3 + \tau_j \partial_s w_3$ in (5.19) gives

$$\begin{aligned} &\int_{\Omega} m_{ij}(u_3) \partial^2_{ij} w_3 + n^{\theta}_{ij}(u) \partial_i \theta \partial_j w_3 \\ &= \int_{\partial\Omega} -\partial_i m_{ij}(u_3) \nu_j w_3 + m_{ij}(u_3) \nu_i (\nu_j \partial_\nu w_3 + \tau_j \partial_s w_3) \\ &\qquad + n^{\theta}_{ij}(u) \partial_i \theta \nu_j w_3 \\ &= \int_{\partial\Omega} -\partial_i m_{ij}(u_3) \nu_j w_3 - \partial_s (m_{ij}(u_3) \nu_i \tau_j) w_3 + n^{\theta}_{ij}(u) \partial_i \theta \nu_j w_3 \\ &\qquad + m_{ij}(u_3) \nu_i \nu_j \partial_\nu w_3 \\ &= \int_{\partial\Omega} \partial_s \widehat{M}_\tau w_3 + \widehat{M}_\nu \partial_\nu w_3. \end{aligned} \tag{5.20}$$

Thus, by combining (5.18) and (5.19), we get

$$\begin{aligned} &\int_{\Omega} n^{\theta}_{ij}(u) \partial_j w'_i + m_{ij}(u_3) \partial^2_{ij} w_3 + n^{\theta}_{ij}(u) \partial_i \theta \partial_j w_3 \\ &= \int_{\partial\Omega} \rho_0^{-1} \widehat{T}_i w'_i + \partial_s \widehat{M}_\tau w_3 + \widehat{M}_\nu \partial_\nu w_3. \end{aligned} \tag{5.21}$$

Likewise, we can deduce

$$\begin{aligned}&\int_\Omega \widetilde{n}^\theta_{ij}(\widetilde{u})\partial_j w'_i + \widetilde{m}_{ij}(\widetilde{u}_3)\partial^2_{ij}w_3 + \widetilde{n}^\theta_{ij}(\widetilde{u})\partial_i\theta\partial_j w_3\\ &= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}_i w'_i + \partial_s \widehat{M}_\tau w_3 + \widehat{M}_\nu \partial_\nu w_3.\end{aligned} \tag{5.22}$$

and therefore,

$$\begin{aligned}&\int_\Omega n^\theta_{ij}(u)\partial_j w'_i + m_{ij}(u_3)\partial^2_{ij}w_3 + n^\theta_{ij}(u)\partial_i\theta\partial_j w_3\\ &= \int_\Omega \widetilde{n}^\theta_{ij}(\widetilde{u})\partial_j w'_i + \widetilde{m}_{ij}(\widetilde{u}_3)\partial^2_{ij}w_3 + \widetilde{n}^\theta_{ij}(\widetilde{u})\partial_i\theta\partial_j w_3.\end{aligned}$$

In turn, we obtain

$$\begin{aligned}&\int_\Omega \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_j w'_i + \widetilde{m}_{ij}(\widetilde{u}_3-u_3)\partial^2_{ij}w_3 + \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_i\theta\partial_j w_3\\ &= \int_\Omega (n^\theta_{ij}-\widetilde{n}^\theta_{ij})(u)\partial_j w'_i + (m_{ij}-\widetilde{m}_{ij})(u)\partial^2_{ij}w_3\\ &\quad + (n^\theta_{ij}-\widetilde{n}^\theta_{ij})(u)\partial_i\theta\partial_j w_3.\end{aligned} \tag{5.23}$$

Substituting $w = \widetilde{u}$ into (5.23), we get that

$$\begin{aligned}&\int_\Omega \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_j \widetilde{u}'_i + \widetilde{m}_{ij}(\widetilde{u}_3-u_3)\partial^2_{ij}\widetilde{u}_3 + \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_i\theta\partial_j \widetilde{u}_3\\ &= \int_\Omega (n^\theta_{ij}-\widetilde{n}^\theta_{ij})(u)\partial_j \widetilde{u}'_i + (m_{ij}-\widetilde{m}_{ij})(u)\partial^2_{ij}\widetilde{u}_3\\ &\quad + (n^\theta_{ij}-\widetilde{n}^\theta_{ij})(u)\partial_i\theta\partial_j \widetilde{u}_3.\end{aligned} \tag{5.24}$$

By straightforward computations, we can see that

$$\begin{aligned}&\int_\Omega \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_j \widetilde{u}'_i + \widetilde{m}_{ij}(\widetilde{u}_3-u_3)\partial^2_{ij}\widetilde{u}_3 + \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_i\theta\partial_j \widetilde{u}_3\\ &= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}_i(\widetilde{u}'_i-u'_i) + \partial_s\widehat{M}_\tau(\widetilde{u}_3-u_3) + \widehat{M}_\nu\partial_\nu(\widetilde{u}_3-u_3)\end{aligned}$$

and it follows from (5.24) that

$$\begin{aligned}&\int_\Omega (n^\theta_{ij}-\widetilde{n}^\theta_{ij})(u)\partial_j \widetilde{u}'_i + (m_{ij}-\widetilde{m}_{ij})(u)\partial^2_{ij}\widetilde{u}_3 + (n^\theta_{ij}-\widetilde{n}^\theta_{ij})(u)\partial_i\theta\partial_j \widetilde{u}_3.\\ &= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}_i(\widetilde{u}'_i-u'_i) + \partial_s\widehat{M}_\tau(\widetilde{u}_3-u_3) + \widehat{M}_\nu\partial_\nu(\widetilde{u}_3-u_3).\end{aligned} \tag{5.25}$$

Now replacing $w = \widetilde{u} - u$ in (5.23) and using (5.25), we obtain that

$$\begin{aligned}
&\int_\Omega \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_j(\widetilde{u}'_i - u'_i) + \widetilde{m}_{ij}(\widetilde{u}_3 - u_3)\partial^2_{ij}(\widetilde{u}_3 - u_3) \\
&\quad + \widetilde{n}^\theta_{ij}(\widetilde{u}-u)\partial_i\theta\partial_j(\widetilde{u}_3 - u_3) \\
&= \int_\Omega (n^\theta_{ij} - \widetilde{n}^\theta_{ij})(u)\partial_j(\widetilde{u}'_i - u'_i) + (m_{ij} - \widetilde{m}_{ij})(u)\partial^2_{ij}(\widetilde{u}_3 - u_3) \\
&\quad + (n^\theta_{ij} - \widetilde{n}^\theta_{ij})(u)\partial_i\theta\partial_j(\widetilde{u}_3 - u_3) \qquad (5.26) \\
&= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}_i(\widetilde{u}'_i - u'_i) + \partial_s\widehat{M}_\tau(\widetilde{u}_3 - u_3) + \widehat{M}_\nu\partial_\nu(\widetilde{u}_3 - u_3) \\
&\quad + \int_D (\widetilde{n}^\theta_{ij} - n^\theta_{ij})(u)\partial_j u'_i + (\widetilde{m}_{ij} - m_{ij})(u)\partial^2_{ij}u_3 \\
&\quad + (\widetilde{n}^\theta_{ij} - n^\theta_{ij})(u)\partial_i\theta\partial_j u_3.
\end{aligned}$$

Exchanging the role of $\widetilde{u}$ and u, we can deduce that

$$\begin{aligned}
&\int_\Omega n^\theta_{ij}(u-\widetilde{u})\partial_j(u'_i - \widetilde{u}'_i) + m_{ij}(u_3 - \widetilde{u}_3)\partial^2_{ij}(u_3 - \widetilde{u}_3) \\
&\quad + n^\theta_{ij}(u-\widetilde{u})\partial_i\theta\partial_j(u_3 - \widetilde{u}_3) \\
&= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}_i(u'_i - \widetilde{u}'_i) + \partial_s\widehat{M}_\tau(u_3 - \widetilde{u}_3) + \widehat{M}_\nu\partial_\nu(u_3 - \widetilde{u}_3) \qquad (5.27) \\
&\quad + \int_D (n^\theta_{ij} - \widetilde{n}^\theta_{ij})(\widetilde{u})\partial_j\widetilde{u}'_i \\
&\quad + (m_{ij} - \widetilde{m}_{ij})(\widetilde{u})\partial^2_{ij}\widetilde{u}_3 + (n^\theta_{ij} - \widetilde{n}^\theta_{ij})(\widetilde{u})\partial_i\theta\partial_j\widetilde{u}_3.
\end{aligned}$$

Finally, plugging $w = \widetilde{u}$ into (5.21) and $w = u$ into (5.22), respectively, we have that

$$\begin{aligned}
&\int_D (\widetilde{n}^\theta_{ij} - n^\theta_{ij})(\widetilde{u})\partial_j u_i + (\widetilde{m}_{ij} - m_{ij})(\widetilde{u}_3)\partial^2_{ij}u_3 + (\widetilde{n}^\theta_{ij} - n^\theta_{ij})(\widetilde{u})\partial_i\theta\partial_j u_3 \\
&= \int_{\partial\Omega} \rho_0^{-1}\widehat{T}_i(u'_i - \widetilde{u}'_i) + \partial_s\widehat{M}_\tau(u_3 - \widetilde{u}_3) + \widehat{M}_\nu\partial_\nu(u_3 - \widetilde{u}_3).
\end{aligned} \qquad (5.28)$$

The following identity is useful in our further arguments. Let a, b, c and d be any functions. It is easy to compute that

$$\begin{aligned}
&\Big(ae^\theta_{kk}(w)\delta_{ij} + be^\theta_{ij}(w)\Big)\partial_j w_i + \Big(c(\Delta w_3)\delta_{ij} + d\partial^2_{ij}w_3\Big)\partial^2_{ij}w_3 \\
&\quad + \Big(ae^\theta_{kk}(w)\delta_{ij} + be^\theta_{ij}(w)\Big)\partial_i\theta\partial_j w_3 \qquad (5.29) \\
&= a|\nabla\cdot w' + \nabla\theta\cdot\nabla w_3|^2 + b\sum_{ij}|e^\theta_{ij}(w)|^2 + c|\Delta w_3|^2 + d\sum_{ij}|\partial^2_{ij}w_3|^2
\end{aligned}$$

Now let a, b, c, d be given in (5.2) and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ be defined similarly. Putting $w = \widetilde{u} - u$, using (5.29), (5.26), we get that

$$\begin{aligned} W-\widetilde{W} &\le \int_D (\widetilde{n}^\theta_{ij}-n^\theta_{ij})(u)\partial_j u'_i+(\widetilde{m}_{ij}-m_{ij})(u)\partial^2_{ij}u_3+(\widetilde{n}^\theta_{ij}-n^\theta_{ij})(u)\partial_i\theta\partial_j u_3 \\ &= \int_D (\widetilde{a}-a)|\nabla\cdot u'+\nabla\theta\cdot\nabla u_3|^2+(\widetilde{b}-b)\sum_{ij}|e^\theta_{ij}(u)|^2 \\ &\quad + (\widetilde{c}-c)|\Delta u_3|^2+(\widetilde{d}-d)\sum_{ij}|\partial^2_{ij}u_3|^2. \end{aligned} \tag{5.30}$$

On the other hand, for any $\varepsilon > 0$, one can easily compute that

$$\begin{aligned} &\int_D (\widetilde{a}-a)|\nabla\cdot u'+\nabla\theta\cdot\nabla u_3|^2+(\widetilde{b}-b)\sum_{ij}|e^\theta_{ij}(u)|^2+(\widetilde{c}-c)|\Delta u_3|^2 \\ &\quad +(\widetilde{d}-d)\sum_{ij}|\partial^2_{ij}u_3|^2 \\ &\le (1+\varepsilon^{-1})\int_D (\widetilde{a}-a)|\nabla\cdot(\widetilde{u}'-u')+\nabla\theta\cdot\nabla(\widetilde{u}_3-u_3)|^2 \\ &\quad +(\widetilde{b}-b)\sum_{ij}|e^\theta_{ij}(\widetilde{u}-u)|^2+(\widetilde{c}-c)|\Delta(\widetilde{u}_3-u_3)|^2 \\ &\quad +(\widetilde{d}-d)\sum_{ij}|\partial^2_{ij}(\widetilde{u}_3-u_3)|^2+(1+\varepsilon)\int_D(\widetilde{a}-a)|\nabla\cdot\widetilde{u}'+\nabla\theta\cdot\nabla\widetilde{u}_3|^2 \\ &\quad +(\widetilde{b}-b)\sum_{ij}|e^\theta_{ij}(\widetilde{u})|^2+(\widetilde{c}-c)|\Delta\widetilde{u}_3|^2+(\widetilde{d}-d)\sum_{ij}|\partial^2_{ij}\widetilde{u}_3|^2. \end{aligned} \tag{5.31}$$

By (5.3) we can choose $\varepsilon_* > 0$ such that

$$\frac{1+\varepsilon_*^{-1}}{1+\varepsilon_*}=\frac{1}{k_0}.$$

Therefore, from (5.31) and (5.29), we deduce that

$$\begin{aligned} &\int_D (\widetilde{a}-a)|\nabla\cdot u'+\nabla\theta\cdot\nabla u_3|^2+(\widetilde{b}-b)\sum_{ij}|e^\theta_{ij}(u)|^2+(\widetilde{c}-c)|\Delta u_3|^2 \\ &\quad +(\widetilde{d}-d)\sum_{ij}|\partial^2_{ij}u_3|^2 \\ &\le (1+\varepsilon_*)\Bigg(\int_D n^\theta_{ij}(u-\widetilde{u})\partial_j(u'_i-\widetilde{u}'_i)+m_{ij}(u_3-\widetilde{u}_3)\partial^2_{ij}(u_3-\widetilde{u}_3) \\ &\quad +n^\theta_{ij}(u-\widetilde{u})\partial_i\theta\partial_j(u_3-\widetilde{u}_3) \\ &\quad +\int_D(\widetilde{n}^\theta_{ij}-n^\theta_{ij})(\widetilde{u})\partial_j\widetilde{u}'_i+(\widetilde{m}_{ij}-m_{ij})(\widetilde{u})\partial^2_{ij}\widetilde{u}_3+(\widetilde{n}^\theta_{ij}-n^\theta_{ij})(\widetilde{u})\partial_i\theta\partial_j\widetilde{u}_3\Bigg). \end{aligned} \tag{5.32}$$

Now combining (5.27) and (5.32) immediately yields

$$\frac{1}{1+\varepsilon_*}\int_D (\widetilde{a}-a)|\nabla\cdot u' + \nabla\theta\cdot\nabla u_3|^2 + (\widetilde{b}-b)\sum_{ij}|e^{\theta}_{ij}(u)|^2 + (\widetilde{c}-c)|\Delta u_3|^2 + (\widetilde{d}-d)\sum_{ij}|\partial^2_{ij}u_3|^2 \le W - \widetilde{W} \tag{5.33}$$

and we obtain (5.13). □

Remark 5.3. It is tempting to estimate the size of D without the *a priori* fatness condition (5.11) as in [2,7], and [21]. The important tool in these papers is a global doubling inequality. It seems possible to derive the size estimate without the fatness condition for the shallow shell system since we have derived local doubling inequalities (4.4). Like Theorem 4.13, to investigate this inverse problem, we actually need global doubling inequalities in terms of $\sum_{ij}|e^{\theta}_{ij}(u)|^2+\rho_0^2|\partial^2_{ij}u_3|^2$ instead of $|u'|^2+|u_3|^2$. However, attempts to derive such global doubling inequalities were unsuccessful. The difficulty is due to the fact that u' and u_3 in (3.4) have different scalings.

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