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On the genus of curves in a Jacobian variety

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Abstract. We prove that the geometric genus p of a curve in a very generic Jacobian of dimension $g > 3$ satisfies either $p = g$ or $p > 2g - 3$. This gives a positive answer to a conjecture of Naranjo and Pirola. For small values of g the second inequality can be further improved to $p > 2g - 2$.

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1. Introduction

In this paper we deal with curves in Abelian varieties and, more precisely, in Jacobian varieties. This topic is classically known as *theory of correspondences*: the $\mathbb{Z}$-module of the equivalence classes of correspondences between two curves is, in fact, canonically isomorphic to the group of homomorphisms between their Jacobians (see *e.g.* [5, Theorem 11.5.1]). In [7] it is proved that all curves of genus g lying on a very generic Jacobian variety of dimension $g \geq 4$ are birationally equivalent to each other. We recall that a Jacobian $J(C)$ is said to be *very generic* if $[J(C)]$ lies outside a countable union of proper analytic subvarieties of the Jacobian locus.

We give conditions on the possible genus of a curve in a Jacobian variety. Namely, we show:

Theorem A. *Let D be a curve lying on a very generic Jacobian variety $J(C)$ of dimension greater than or equal to 4. Then the geometric genus $g(D)$ satisfies one of the following*

$$(i)\ g(D) \geq 2g(C) - 2, \qquad (ii)\ g(D) = g(C).$$

Theorem A gives a positive answer to a conjecture stated in [13], where the authors prove an analogous statement for Prym varieties. An equivalent formulation of Theorem A is the following:

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Theorem B. *Given two smooth projective curves C and D, where C is very generic, $g(C) \geq 4$ and $g(D) < 2g(C) - 2$, then either the Néron-Severi group of $C \times D$ has rank 2, or it has rank 3 and $C \simeq D$.*

Theorem B is implied by the fact that, in the previous hypotheses, if $f\colon J(D) \to J(C)$ is a surjective map, then $J(D)$ is isomorphic to $J(C)$ and f is the multiplication by a non-zero integer n.

We briefly outline the strategy of the proof: first we factorize the map $f\colon J(D) \to J(C)$ into a surjective map $g\colon J(D) \to B$ of Abelian varieties with connected kernel and an isogeny $h\colon B \to J(C)$. Then we study independently the two maps by a degeneration argument. The key point is the analysis of the limit $f_0\colon J(D_0) \to J(C_0)$ of f, when C degenerates to the Jacobian of an irreducible stable curve with one node.

By a rigidity result (see [17]), if we let C_0 vary by keeping its normalization fixed, then also the normalization of D_0 does not change. The comparison of the relations between the extension classes in $\mathrm{Pic}^0(J(\widetilde{D}_0))$ and $\mathrm{Pic}^0(J(\widetilde{C}_0))$ of the two generalized Jacobians shows that the image of the map $H^1(J(D_0), \mathbb{Z}) \to H^1(J(C_0), \mathbb{Z})$, between the cohomology groups, is $nH^1(J(C_0), \mathbb{Z})$ for some non-zero integer n (see Section 4.1). This argument is an adaptation of the proof for the case $g(D) = g(C)$ (see [7]), but our setting requires a more careful analysis of the relations between the extension classes. For example, it is not sufficient to consider only the limit f_0, but we have to take into account also the relations coming from the limit $\widehat{f_0}$ of the dual map.

The comparison of two independent degenerations of the previous type allows us to prove that $B \simeq J(C)$ and $h\colon B \to J(C)$ is the multiplication by n (see Section 4.3). To conclude the proof, we notice that, since $J(C)$ is very generic, the polarization Ξ, induced by $J(D)$ on $B \simeq J(C)$, is an integral multiple of the standard principal polarization of $J(C)$. The analysis of the behavior of the map $g\colon J(D) \to B \simeq J(C)$ at the boundary shows that Ξ is principal. From the irreducibility of $J(D)$ it follows that g is an isomorphism (see Section 4.4).

It seems natural to suppose that strict inequality holds in case (i) of Theorem A, that is: there are no curves of genus $2g(C) - 2$ on a very generic Jacobian $J(C)$ of dimension $g(C) \geq 4$ (see Conjecture 5.1). However, the previous argument cannot be applied because it is no longer possible to use the rigidity result. In the last part of the paper we prove a weaker statement which supports our conjecture: given a map $f\colon J(D) \to J(C)$, from a Jacobian of dimension $2g(C) - 2$ to a very generic Jacobian of dimension $g(C) \geq 4$, the induced map $\varphi\colon D \to A$, where A is the identity component of the kernel of f, is birational on the image (see Proposition 5.2). The proof is based on a result on Prym varieties stated in [12], which asserts that a very generic Prym variety, of dimension greater or equal to 4, of a ramified double covering is not isogenous to a Jacobian variety. When $g(C) = 4$ or $g(C) = 5$, the analysis of the possible deformations of D in A shows that φ is constant and thus that f is the zero map.

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2. Notation and preliminaries

We work over the field $\mathbb{C}$ of complex numbers. Each time we have a family of objects parameterized by a scheme X (respectively by a subset $Y \subset X$) we say that the *generic* element of the family has a certain property $\mathfrak{p}$ if $\mathfrak{p}$ holds on a non-empty Zariski open subset of X (respectively of Y). Moreover, we say that a *very generic* element of X (respectively of Y) has the property $\mathfrak{p}$ if $\mathfrak{p}$ holds on the complement of a union of countably many proper subvarieties of X (respectively of Y).

We denote by M_g the moduli space of smooth projective curves of genus g and by $\overline{M}_g$ the Deligne-Mumford compactification of M_g, that is the moduli space of stable curves of genus g. Let M_g^0 be the open set of M_g whose points correspond to curves with no non-trivial automorphisms and let $\overline{M}_g^0$ be the analogous open set in $\overline{M}_g$. We denote by δ_0 the divisor of $\overline{M}_g$ whose generic point parameterizes the isomorphism class of an irreducible stable curve with one node.

Given a projective curve C, we denote by $g(C)$ its geometric genus. Given an Abelian variety A we will denote sometimes by $\widehat{A}$ its dual Abelian variety $\mathrm{Pic}^0(A)$. We recall that if J is a very generic Jacobian, then $\mathrm{rk}\,(\mathrm{NS}\,(J)) = 1$; in particular J has no non-trivial Abelian subvarieties (*cf.* [15]). We will also need the following results:

Theorem 2.1 ([17, Remark 2.7]). *Let J be a very generic Jacobian of dimension $n \geq 2$, D be a smooth projective curve and $f\colon D \to J$ be a non-constant map. If $g(D) < 2n - 1$, then the only deformations of (D, f), with J fixed, are obtained by composing f with translations.*

Corollary 2.2. *Let J be a very generic Jacobian of dimension $n \geq 2$, $\mathcal{D}$ be a family of smooth projective curves over a smooth connected scheme B and*

$$F\colon (J \times B)/B \to J(\mathcal{D})/B$$

be a non-constant map of families of Jacobians. If

$$\dim J(\mathcal{D}) - \dim B < 2n - 1,$$

then $\mathcal{D}$ is a trivial family.

2.1. Semi-Abelian varieties

Throughout the paper we will use some properties of degenerations of Abelian varieties. Standard references for this topic are [10] and [8] or, for Jacobian varieties, [2,6,14,18]. Here we recall some basic facts.

Definition 2.3. Given an Abelian variety A, its *Kummer variety* $\mathcal{K}(A)$ is the quotient of A by the involution $x \mapsto -x$. We denote by $\mathcal{K}^0(A)$ the Kummer variety of $\mathrm{Pic}^0(A)$.

Definition 2.4. A *semi-Abelian variety S* of rank n is an extension

$$0 \to T \to S \to A \to 0 \tag{2.1}$$

of an Abelian variety A by an algebraic torus $T = \prod^n \mathbb{G}_m$.

Note that (2.1) induces a long exact sequence

$$1 \to H^0\big(A, \mathcal{O}_A^*\big) \to H^0\big(S, \mathcal{O}_S^*\big) \to H^0\big(T, \mathcal{O}_T^*\big) \xrightarrow{\delta} H^1\big(A, \mathcal{O}_A^*\big),$$

and thus a group homomorphism

$$h\colon \mathbb{Z}^n \simeq \mathrm{Hom}\,(T, \mathbb{G}_m) \to \mathrm{Pic}^0\,(A). \tag{2.2}$$

Viceversa, each homomorphism h determines an exact sequence as in (2.1) (see [8, Chapter II, Section 2]). In particular, the classes of isomorphism of semi-Abelian varieties of rank 1 with compact part A are parameterized (up to multiplication by -1) by the homomorphisms $\mathbb{Z} \to \mathrm{Pic}^0\,(A)$ and, consequently, by the points of $\mathcal{K}^0(A)$. By the previous argument we have the following proposition.

Proposition 2.5. *Consider two semi-Abelian varieties*

$$0 \to T \to S \to A \to 0, \qquad 0 \to T' \to S' \to A' \to 0.$$

A map $f\colon S \to S'$ is determined by a map of Abelian varieties $g\colon A \to A'$ and by a morphism of groups

$$\chi\colon \mathrm{Hom}\,\big(T', \mathbb{G}_m\big) \to \mathrm{Hom}\,(T, \mathbb{G}_m)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}^m \simeq \mathrm{Hom}\,\big(T', \mathbb{G}_m\big) & \longrightarrow & \mathrm{Pic}^0\,\big(A'\big) \\ \Big\downarrow{\scriptstyle \chi} & & \Big\downarrow{\scriptstyle g^*} \\ \mathbb{Z}^n \simeq \mathrm{Hom}\,(T, \mathbb{G}_m) & \longrightarrow & \mathrm{Pic}^0\,(A). \end{array}$$

Definition 2.6. Let D be a projective curve having only nodes (ordinary double points) as singularities. The *generalized Jacobian variety* of D is defined as $J(D) := \mathrm{Pic}^0\,(D)$.

Notice that, if C is the normalization of D, we have a surjective morphism

$$\mathrm{Pic}^0(D) \to \mathrm{Pic}^0(C)$$

whose kernel is an algebraic torus; thus $\mathrm{Pic}^0(D)$ is a semi-Abelian variety.

We recall the following result (see *e.g.* [2, Theorem 2.3] or [14, Proposition 10.2]):

Proposition 2.7. *The group homomorphism* (2.2) *corresponding to* $J(D)$ *can be identified to a map*

$$h\colon H_1(\Gamma, \mathbb{Z}) \to \mathrm{Pic}^0(J(C)),$$

where Γ *is the (oriented) dual graph of* D*, defined as follows. Given* $e = \sum_i n_i e_i \in H_1(\Gamma, \mathbb{Z})$*, where* e_i *is the edge of* Γ *corresponding to the node* N_i *of* D*, then* $h(e) = \sum n_i[p_i - q_i]$*, where* p_i *and* q_i *are mapped to* N_i *by the natural morphism* $C \to D$.

Given a non-singular projective curve C, we denote by $C_{p,q}$ the nodal curve obtained from C by pinching the points $p, q \in C$. By Proposition 2.7, the semi-Abelian variety $J(C_{p,q})$ is the extension of $J(C)$ by $\mathbb{G}_m$ determined by $\pm[p-q] \in \mathcal{K}^0(J(C))$.

Given a smooth projective curve C of genus greater than 1, we denote by Γ_C the image of the difference map

$$\begin{aligned} C \times C &\to J(C) \xrightarrow{\sim} \mathrm{Pic}^0(J(C)) \\ (p, q) &\mapsto [p - q]. \end{aligned} \tag{2.3}$$

The image Γ'_C of Γ_C through the projection $\sigma_C\colon \mathrm{Pic}^0(J(C)) \to \mathcal{K}^0(J(C))$ is a surface, in the Kummer variety, that parameterizes generalized Jacobian varieties of rank 1 with Abelian part $J(C)$. Given an integer $n \in \mathbb{Z}$, we denote by $n\Gamma_C$ the image of Γ_C under the multiplication by n and by $n\Gamma'_C$ the projection of $n\Gamma_C$ in $\mathcal{K}^0(J(C))$.

Proposition 2.8. *Let* C *be a non-hyperelliptic curve and* n *be a non-zero integer. Then*

(1) $n\Gamma_C$ *is birational to* $C \times C$.
(2) $n\Gamma'_C$ *is birational to the double symmetric product* C_2 *of* C.

In particular, Γ_C *is birational to* $C \times C$ *and* Γ'_C *is birational to* C_2.

Proof. Arguing as in [7, Lemma 3.1.1, Proposition 3.2.1], we can assume $n = 1$. First we prove (2.1): given a generic point $(a, b) \in C \times C$, if there exists $(c, d) \in C \times C$ such that $[a - b] = [c - d]$, then C is hyperelliptic. Statement (2.2) follows from (2.1). □

Notation. Given an Abelian (or a semi-Abelian) variety A and a non-zero integer n, we denote by $\mathfrak{m}_n\colon A \to A$ the map $x \mapsto nx$.

Proposition 2.9. *Let C be a generic curve of genus $g \geq 3$ and $[a-b]$ be a generic point of Γ_C. If there exist a point $[c-d] \in \Gamma_C$ and two positive integers m, n such that $n[a-b] = m[c-d]$, then $n = m$ and $[a-b] = \pm[c-d]$.*

Proof. Assume, by contradiction, that for each $a, b \in C$ there are $c, d \in C$ such that $na + md \equiv mc + nb$. It follows that C has a two-dimensional family of maps of degree $m+n$ to $\mathbb{P}^1$ with a ramification of order $n+m-2$ over two distinct points. By a count of moduli we get a contradiction. □

3. Local systems

In this section we introduce the monodromy representation of a local system. A standard reference for this topic is [19, Chapter 3].

Definition 3.1. Let X be an arcwise connected and locally simply connected topological space and G be an Abelian group. A *local system* $\mathcal{G}$, of stalk G, on X is a sheaf on X which is locally isomorphic to the constant sheaf of stalk G.

Given two local systems $\mathcal{G}$ and $\mathcal{F}$ on X, we say that $\varphi\colon \mathcal{G} \to \mathcal{F}$ is a morphism of local systems if φ is a map of sheaves.

We recall that (see [19, Corollary 3.10]), given a point $x \in X$, a local system $\mathcal{G}$ induces a representation

$$\begin{aligned} \rho\colon \pi_1(X,x) &\to \operatorname{Aut}(\mathcal{G}_x) = \operatorname{Aut}(G) \\ \gamma &\mapsto \rho_\gamma \end{aligned}$$

of the fundamental group that is called *monodromy representation*. The group $\rho(\pi_1(X,x))$ is the *monodromy group*. It is possible to define a functor, the *monodromy functor*, from the category of local systems on X to the category of Abelian representations of $\pi_1(X,x)$, which associates ρ to $\mathcal{G}$. It holds:

Proposition 3.2. *The monodromy functor induces an equivalence of categories between the category of local systems on X and the category of Abelian groups with an action on $\pi_1(X,x)$.*

Remark 3.3. We recall that, given a local system $\mathcal{G}$ on X, a map $\phi\colon Y \to X$ and a point $y \in Y$, the monodromy representation of $\pi_1(Y,y)$ corresponding to the local system $\phi^{-1}\mathcal{G}$ is the composition of the natural morphism $\pi_1(Y,y) \to \pi_1(X,\phi(y))$ with the monodromy representation of $\pi_1(X,\phi(y))$.

In the following we assume G to be a *lattice*, that is an Abelian finitely generated free group of even rank. The *rank of the local system* $\mathcal{G}$ is the rank of the group G.

Definition 3.4. Let $G \simeq \mathbb{Z}^{2g}$ be a lattice of rank $2g$. A *polarization* of G is a non-degenerate, antisymmetric, bilinear form $\theta\colon G \times G \to \mathbb{Z}$. If the induced map $G \to \operatorname{Hom}(G,\mathbb{Z})$ is an isomorphism, we say that θ is a *principal polarization*.

Given a principal polarization θ of G, a *symplectic basis* for G (with respect to θ) is a minimal system of generators $a_1, \dots, a_g, b_1, \dots, b_g$ such that

$$\theta(a_i, b_i) = 1, \quad \theta(a_i, a_j) = 0, \quad \theta(b_i, b_j) = 0, \quad \theta(a_i, b_j) = 0, \quad \forall i, \forall j \neq i.$$

We denote by Aut (G, θ) the group of *symplectic automorphisms* of G, that are the automorphisms $T : G \to G$ satisfying

$$\theta(a, b) = \theta(T(a), T(b)) \qquad \forall a, b \in G.$$

Let $g \in G$; the map $T_g : G \to G$, defined as

$$T_g(a) = a + \theta(g, a)g \qquad \forall a \in G, \tag{3.1}$$

is a symplectic automorphism of G. We notice that, for each $N \in \mathbb{Z}$, we have $T_g^N = T_{Ng}$; in particular, T_g has finite order in Aut (G, θ) if and only if $g = 0$ and $T_g = \mathrm{Id}_G$.

Definition 3.5. Let $\mathcal{G}$ be a local system on X. A *polarization* (respectively a *principal polarization*) Θ of $\mathcal{G}$ is a map of local systems $\Theta : \mathcal{G} \times \mathcal{G} \to \mathcal{Z}$, where $\mathcal{Z}$ is the constant sheaf $\mathbb{Z}$, such that, for each $x \in X$, $\Theta_x : \mathcal{G}_x \times \mathcal{G}_x \to \mathbb{Z}$ is a polarization (respectively a principal polarization) of $\mathcal{G}_x$.

Now we state a result that will be useful later in the paper.

Notation. Given a lattice automorphism $T : G \to G$, we denote by

$$\mathrm{Inv}\,(T) := \{x \in G \text{ so that } T(x) = x\}$$

the subgroup of the elements of G that are fixed by T.

Proposition 3.6. *Let $\mathcal{H} \hookrightarrow \mathcal{G}$ be an injective map of local systems on X of the same rank. Given $x \in X$, denote by $\rho : \pi_1(X, x) \to \mathcal{H}_x$ the monodromy representation associated to $\mathcal{H}$ and by $\sigma : \pi_1(X, x) \to \mathcal{G}_x$ the monodromy representation associated to $\mathcal{G}$. Consider two elements $\gamma_1, \gamma_2 \in \pi_1(X, x)$ and set*

$$G_i := \mathrm{Inv}\,(\sigma_{\gamma_i}), \qquad H_i := \mathrm{Inv}\,(\rho_{\gamma_i}) \qquad \forall\, i = 1, 2.$$

Assume that

(1) $G_1 + G_2 = \mathcal{G}_x$;
(2) $G_1 \cap G_2 \neq \{0\}$;
(3) *for each $i = 1, 2$, $H_i = n_i G_i$ for some $n_i \in \mathbb{N}$.*

Then $n_1 = n_2$ and $\mathcal{H}_x = n_1 \mathcal{G}_x$.

Proof. Let $a \in G_1 \cap G_2$ be such that a is not zero and it is not a multiple. It holds

$$n_2 a \in G_1 \cap H_2 \subset G_1 \cap \mathcal{H}_x = H_1 = n_1 G_1,$$

and consequently $n_1 \leq n_2$. In the same way, we find $n_2 \leq n_1$. For the second statement, observe that

$$\mathcal{H}_x = H_1 + H_2 = n_1 G_1 + n_1 G_2 = n_1 \mathcal{G}_x.$$

□

3.1. A family of curves

We conclude this section by recalling that, given a holomorphic, submersive and projective morphism $\phi\colon Y \to X$ between complex manifolds, $R^k\phi_*\mathbb{Z}$ is a local system on X and the monodromy representation on the cohomology of the fibre $H^*(Y_x, \mathbb{Z})$ is compatible with the cup product (see [19, Section 3.1.2]). In particular, if we consider a family of curves $\omega\colon \mathcal{C} \to X$ and the corresponding family of Jacobian varieties $\phi\colon J(\mathcal{C}) \to X$, then $R^1\phi_*\mathbb{Z} \simeq \big(R^1\omega_*\mathbb{Z}\big)^*$ has a natural polarization $\langle\cdot,\cdot\rangle$ induced by the intersection form.

Let Y be a complex surface and $\phi\colon Y \to \Delta$ be a *Lefschetz degeneration*. This means that ϕ is a holomorphic projective morphism with non-zero differential over the punctured disk Δ^* and that the fibre Y_0 is an irreducible curve with an ordinary double point as its only singularity (see [19, Section 3.2.1]). Given a point $x \in \Delta^*$, the *vanishing cocycle* $\delta \in H^1(Y_x, \mathbb{Z})$ is a generator for

$$\ker\Big(H^1(Y_x, \mathbb{Z}) \simeq H_1(Y_x, \mathbb{Z}) \to H_1(Y, \mathbb{Z})\Big).$$

We recall the following fact (see [19, Theorem 3.16]).

Proposition 3.7 (Picard-Lefchetz formula). *The image of the monodromy representation*

$$\pi_1\big(\Delta^*, x\big) \to \operatorname{Aut}\Big(H^1(Y_x, \mathbb{Z}), \langle\cdot,\cdot\rangle\Big)$$

is generated by an element T_δ defined as in (3.1). *In particular, given a symplectic basis $a_1, \dots, a_g, b_1, \dots b_g$ of $H^1(Y_x, \mathbb{Z})$ such that $a_1 = \delta$, then*

$$\operatorname{Inv}(T_\delta) := \Big\{a \in H^1(Y_x, \mathbb{Z}) \text{ so that } T_\delta(a) = a\Big\}$$

is generated by $a_1, \dots, a_g, b_2, \dots, b_g$.

Remark 3.8. Notice that, after a finite base change $z \mapsto z^k$ of Δ, the new generator of the monodromy group is $T_{k\delta}$.

4. The main theorem

The present section is devoted to prove the following result.

Theorem 4.1. *Let J be a very generic Jacobian variety of dimension $g \geq 4$ and Ω be a curve lying on J. Then either $g(\Omega) \geq 2g - 2$ or $g(\Omega) = g$.*

Assume that for a very generic Jacobian variety J of dimension $g \geq 4$ there exists a smooth projective curve D of genus $p < 2g - 2$ and a non-constant map $D \to J$. We want to prove that $p = g$.

Arguing as in [7], one finds a map of families

$$\mathcal{D}/\mathcal{V} \to J(\mathcal{C})/\mathcal{V}, \tag{4.1}$$

verifying:

- $\mathcal{V}$ is a finite étale covering of a dense open subset $\mathcal{U}$ of M_g^0;
- $J(\mathcal{C})$ is the Jacobian bundle of the family of curves $\mathcal{C}/\mathcal{V}$ obtained as the pullback of the universal family $\mathcal{C}/M_g^0$;
- for all $t \in \mathcal{V}$,
  - $J(\mathcal{C}_t)$ is a Jacobian variety of dimension g;
  - $\mathcal{D}_t$ is a smooth projective curve of genus $p < 2g-2$;
  - $\mathcal{D}_t \to J(\mathcal{C}_t)$ is a non-constant map.

Passing to the Jacobian bundle of $\mathcal{D}/\mathcal{V}$ we get a map

$$F\colon J(\mathcal{D})/\mathcal{V} \to J(\mathcal{C})/\mathcal{V}, \tag{4.2}$$

of families of Jacobians, which we can assume to be surjective on each fibre.

Notice that, for each $t \in \mathcal{V}$, the map

$$F_t\colon J(\mathcal{D}_t) \to J(\mathcal{C}_t)$$

factorizes canonically into a surjective homomorphism with connected kernel followed by an isogeny. This is sometimes called Stein factorization for Abelian varieties (see *e.g.* [5, Section 1.2]). Namely we have

$$F_t\colon J(\mathcal{D}_t) \to J(\mathcal{D}_t)/(\ker F_t)^\circ \to J(\mathcal{C}_t), \tag{4.3}$$

where $(\ker F_t)^\circ$ is the identity component of the kernel of F_t. As complex tori, the following identifications hold:

$$J(\mathcal{D}_t) = H^1\big(\mathcal{D}_t, \mathcal{O}_{\mathcal{D}_t}\big)/H_1(\mathcal{D}_t, \mathbb{Z}), \qquad J(\mathcal{C}_t) = H^1\big(\mathcal{C}_t, \mathcal{O}_{\mathcal{C}_t}\big)/H_1(\mathcal{C}_t, \mathbb{Z}),$$
$$J(\mathcal{D}_t)/(\ker F_t)^\circ = H^1\big(\mathcal{C}_t, \mathcal{O}_{\mathcal{C}_t}\big)/\operatorname{Im}\left(H_1(\mathcal{D}_t, \mathbb{Z}) \to H_1(\mathcal{C}_t, \mathbb{Z})\right).$$

The factorization (4.3) can be performed globally. Denote by

$$\pi: \mathcal{D} \to \mathcal{V}, \qquad \rho: \mathcal{C} \to \mathcal{V}$$

the projections of the two families of curves on the basis. We recall that

$$J(\mathcal{D}) = R^1\pi_*\mathcal{O}_{\mathcal{D}}/\Big(R^1\pi_*\mathbb{Z}\Big)^*, \qquad J(\mathcal{C}) = R^1\rho_*\mathcal{O}_{\mathcal{C}}/\Big(R^1\rho_*\mathbb{Z}\Big)^*.$$

Set

$$\mathcal{B} := R^1\rho_*\mathcal{O}_{\mathcal{C}}/\operatorname{Im}\left(\Big(R^1\pi_*\mathbb{Z}\Big)^* \to \Big(R^1\rho_*\mathbb{Z}\Big)^*\right) \tag{4.4}$$

and denote by

$$G\colon J(\mathcal{D}) \to \mathcal{B}, \qquad H: \mathcal{B} \to J(\mathcal{C}) \tag{4.5}$$

the natural projections. The following conditions are satisfied:

- $\mathcal{B}$ is a family of g-dimensional Abelian varieties on $\mathcal{V}$;
- $F = H \circ G$;
- $\mathcal{A} := \ker G$ is a family of Abelian varieties on $\mathcal{V}$;
- $\mathcal{A}_t = (\ker F_t)^\circ$ and $\mathcal{B}_t = J(\mathcal{D}_t)/(\ker F_t)^\circ$, for each $t \in \mathcal{V}$;

In the following we prove that $\mathcal{B} \simeq J(\mathcal{C})$, $H\colon \mathcal{B} \to J(\mathcal{C})$ is the multiplication by a non-zero integer (Corollary 4.8) and $G\colon J(\mathcal{D}) \to \mathcal{B}$ is an isomorphism (Section 4.4).

Before starting with the proof of Theorem 4.1 in the general case, we notice that, for $g = 4$, the statement is a direct consequence of the following proposition.

Proposition 4.2. *There are no curves of genus $g+1$ lying on a very generic Jacobian of dimension $g \geq 4$.*

Proof. With the previous notation, observe that, if $p = g+1$, $\mathcal{A}$ is a family of elliptic curves. Thus, for each $t \in \mathcal{V}$, we have a non-constant map of curves $\mathcal{D}_t \to \mathcal{A}_t$. Since the moduli space of coverings of genus $g+1$ of elliptic curves has dimension $2g$, it follows that $J(\mathcal{D})$ is a trivial family and we get a contradiction. □

4.1. Comparison of the extension classes

Let C be a very generic smooth curve of genus $g-1$, in particular assume that $J(C)$ is simple. Given a generic point $[p-q] \in \Gamma'_C$ (see (2.3) in Section 2.1), consider the nodal curve $C_{p,q}$ obtained from C by pinching p and q and a non-constant map $\tau\colon \Delta \to \overline{M}_g^0$ such that $\tau(\Delta^*) \subset M_g^0$ and $\tau(0)$ is the class of $C_{p,q}$. Let us restrict our initial families of Jacobian varieties $J(\mathcal{D})$ and $J(\mathcal{C})$, defined in (4.2), to Δ^* (we suppose that $\tau(\Delta^*) \subset \mathcal{U}$). Changing base, if necessary, we get a map of families of semi-Abelian varieties

$$F\colon J(\mathcal{D})/\Delta \to J(\mathcal{C})/\Delta,$$

satisfying the following conditions:

- $F|_{\Delta^*}$ coincides with the map defined in (4.2);
- $\mathcal{C}_0 = C_{p,q}$;
- $\mathcal{D}_0$ is a nodal curve of arithmetic genus $p < 2g-2$.

Given the map of families

$$\widehat{F}\colon J(\mathcal{C})/\Delta \to J(\mathcal{D})/\Delta,$$

obtained from F by dualization, we denote by $f\colon J(C) \to J(\widetilde{\mathcal{D}_0})$ the map induced by the map of semi-Abelian varieties $\widehat{F_0}$ on the Abelian quotients.

The aim of the following propositions is to describe the limit $J(\mathcal{D}_0)$ of the family of Jacobian varieties $J(\mathcal{D})$ and the map $\widehat{F_0}\colon J(\mathcal{C}_0) \to J(\mathcal{D}_0)$.

Proposition 4.3. *The smooth curve C is isomorphic to a connected component of $\widetilde{\mathcal{D}}_0$ and $f = i \circ \mathfrak{m}_n$, where $\mathfrak{m}_n\colon J(C) \to J(C)$ is the multiplication by a non-zero integer n and $i\colon J(C) \to J(\widetilde{\mathcal{D}}_0)$ is the natural inclusion.*

Proof. Since $J(C)$ is simple, f has finite kernel and $f(J(C))$ is an irreducible Abelian subvariety, of dimension $g-1$, that does not contain curves of geometric genus lower than $g-1$. From the inequality $g(\mathcal{D}_0) \le p - 1 < 2g - 3$, we can conclude that there is only a connected component X of $\widetilde{\mathcal{D}}_0$ such that $g(X) \ge g-1$. It follows $f(J(C)) \subset J(X)$. We want to prove that X is isomorphic to C and that

$$f\colon J(C) \to J(X) \tag{4.6}$$

is the multiplication by a non-zero integer n. Now we procede by steps:

Step I: rigidity.
By varying the point $[p-q]$ in Γ'_C, and consequently the curve $C_{p,q}$, we can perform different degenerations of the family of curves $\mathcal{D}/\mathcal{V}$ (defined in (4.1)). By the previous construction, to each family of degenerations corresponds a family of curves $\mathcal{X}$, over a smooth connected scheme B, such that

$$g-1 = g(C) \le g(\mathcal{X}_t) < 2g(C) - 1 = 2g - 3$$

for each $t \in B$, and there is a non-constant map of families of Jacobians

$$(J(C) \times B)/B \to J(\mathcal{X})/B.$$

By Corollary 2.2, the family $J(\mathcal{X})$ is trivial. This means that, though the limit $F_0\colon J(\mathcal{D}_0) \to J(\mathcal{C}_0) = J\big(C_{p,q}\big)$ depends on $[p-q] \in \Gamma'_C$, the map $f\colon J(C) \to J(X)$ on the Abelian parts is independent of the choice of the point $[p-q]$.

Step II: $f^*(\Gamma_X) = n\Gamma_C$ for some non-zero integer n.
By Propositions 2.5 and 2.7, for each $[p-q] \in \Gamma_C \subset \operatorname{Pic}^0(J(C))$ there exists $[y-z] \in \Gamma_X \subset \operatorname{Pic}^0(J(X))$ and a non-zero integer n such that $f^*([y-z]) = n[p-q]$. It follows that

$$\Gamma_C \subset \bigcup_{n\ge 1} \mathfrak{m}_n^{-1}\big(f^*(\Gamma_X)\big).$$

This implies that Γ_C is contained in $\mathfrak{m}_n^{-1}(f^*(\Gamma_X))$ for some n. By the irreducibility of $f^*(\Gamma_X)$, it holds $f^*(\Gamma_X) = n\Gamma_C$.

Step III: X is isomorphic to C and $f\colon J(C) \to J(X)$ is the multiplication by n.
By the previous step and Proposition 2.8, we can define a rational dominant map

$$X \times X \dashrightarrow \Gamma_X \xrightarrow{f^*} n\Gamma_C \dashrightarrow C \times C.$$

Consequently we have a non-constant morphism $h\colon X \to C$. By the Riemann-Hurwitz formula we get

$$4g - 6 > 2p - 2 > 2g(X) - 2 \ge \deg h(2g(C) - 2) = \deg h(2g - 4);$$

thus $\deg h = 1$ and $C \simeq X$. Since we can assume $\mathrm{End}\,(J(C)) = \mathbb{Z}$ (see *e.g.* [9]), $f = \mathfrak{m}_k$ for some $k \in \mathbb{Z}$ (notice that, by Proposition 2.9, $n = \pm k$). □

Denote by $D_1, \ldots, D_k$ the connected components of $\widetilde{\mathcal{D}_0}$. By Proposition 4.3, since C has no non-trivial automorphisms, we can assume $D_1 = C$. We have the following result:

Proposition 4.4. *The map of semi-Abelian varieties $\widehat{F_0}\colon J(\mathcal{C}_0) \to J(\mathcal{D}_0)$ corresponds to the following morphism of extensions*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & J(\mathcal{C}_0) & \longrightarrow & J(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \hat{F}_0} & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & \Pi\mathbb{G}_m & \longrightarrow & J(\mathcal{D}_0) & \longrightarrow & J(C) \times J(D_2) \times \ldots \times J(D_k) & \longrightarrow & 0
\end{array}$$

where $\chi(1) = (n, \ldots, 0)$ and η is the composition of the multiplication by n with the inclusion in the first factor.

Proof. We recall (see Section 2.1) that the semi-Abelian variety $J(\mathcal{D}_0)$ is determined by a morphism

$$\alpha\colon \mathbb{Z}^m \simeq \mathrm{Hom}\left(\prod \mathbb{G}_m, \mathbb{G}_m\right) \to \mathrm{Pic}^0\,(J(C)) \times \mathrm{Pic}^0\,(J(D_2)) \times \cdots \times \mathrm{Pic}^0\,(J(D_k)),$$

where $m := p - g(\widetilde{\mathcal{D}_0})$. By Proposition 2.5, $\widehat{F_0}\colon J(\mathcal{C}_0) = J\big(C_{p,q}\big) \to J(\mathcal{D}_0)$ corresponds to a diagram

$$\begin{array}{ccc}
\mathbb{Z}^m & \xrightarrow{\alpha} & \mathrm{Pic}^0\,(J(C)) \times \mathrm{Pic}^0\,(J(D_2)) \times \cdots \times \mathrm{Pic}^0\,(J(D_k)) \\
{\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathbb{Z} & \xrightarrow{\beta} & \mathrm{Pic}^0\,(J(C))
\end{array}$$

where, f^* is the composition of the first projection with $\mathfrak{m}_n$ (see Proposition 4.3). We claim that there exists a basis $e_1, \ldots, e_m$ of $\mathbb{Z}^m$ such that for each $i = 1, \ldots, m$, either $\chi(e_i) = \pm n$, or $\chi(e_i) = 0$. Given a basis $e_1, \ldots, e_m$, it holds

$$f^*(\alpha(e_i)) = \beta(\chi(e_i)) = j[p - q],$$

for some $j \in \mathbb{Z}$. Then, by Proposition 2.7, if $j \neq 0$,

$$n[a - b] = f^*(\alpha(e_i)) = j[p - q]$$

for some non-zero $[a - b] \in \Gamma_C$. From Proposition 2.9 it follows

$$[p - q] = \pm[a - b]$$

and $j = n$. We recall that, since $\widehat{F_0}$ is a limit of maps with finite kernel, $\chi \not\equiv 0$. Up to a suitable choice of the basis $e_1, \ldots, e_m$, we can assume $\chi(e_1) = n$ and $\chi(e_i) = 0$ for $i = 2, \ldots, m$. □

Corollary 4.5. *The image of the map*

$$H^1(J(\mathcal{D}_0),\mathbb{Z}) \to H^1(J(\mathcal{C}_0),\mathbb{Z}),$$

induced by $\widehat{F}_0\colon J(\mathcal{C}_0)\to J(\mathcal{D}_0)$ *on the cohomology groups, is* $nH^1(J(\mathcal{C}_0),\mathbb{Z})$.

4.2. Local systems

Let consider again the families of Abelian varieties $J(D)/\mathcal{V}$, $\mathcal{B}/\mathcal{V}$ and $J(\mathcal{C})/\mathcal{V}$, defined in (4.2), and (4.4) and denote by

$$\mu\colon J(\mathcal{D})\to\mathcal{V},\qquad \varphi\colon \mathcal{B}\to\mathcal{V},\qquad \psi\colon J(\mathcal{C})\to\mathcal{V}$$

the projections on the base. Given a point $t\in\mathcal{V}$, we denote by

$$\begin{aligned}\nu&\colon \pi_1(\mathcal{V},t)\to H^1(J(\mathcal{D}_t),\mathbb{Z}),\\ \rho&\colon \pi_1(\mathcal{V},t)\to H^1(\mathcal{B}_t,\mathbb{Z}),\\ \sigma&\colon \pi_1(\mathcal{V},t)\to H^1(J(\mathcal{C}_t),\mathbb{Z})\end{aligned}$$

the monodromy representations corresponding to the local systems $R^1\mu_*\mathbb{Z}$, $R^1\varphi_*\mathbb{Z}$ and $R^1\psi_*\mathbb{Z}$. We recall that $\mathcal{V}$ has a finite map $\pi:\mathcal{V}\to\mathcal{U}$ on an open dense set of M_g^0 and, by construction (see Section 4),

$$R^1\psi_*\mathbb{Z}=\pi^{-1}R^1\omega_*\mathbb{Z} \tag{4.7}$$

where $\omega\colon\mathcal{C}\to\mathcal{U}$ is the restriction of the universal family of curves to $\mathcal{U}$.

The map $F\colon J(\mathcal{D})\to J(\mathcal{C})$ (or, more precisely, its dual $\widehat{F}\colon J(\mathcal{C})\to J(\mathcal{D})$) induces a map of local systems

$$R^1\mu_*\mathbb{Z}\overset{\mathfrak{F}}{\to}R^1\psi_*\mathbb{Z} \tag{4.8}$$

which factorizes in a surjective map $\mathfrak{G}\colon R^1\mu_*\mathbb{Z}\to R^1\varphi_*\mathbb{Z}$ followed by an injective map $\mathfrak{H}\colon R^1\varphi_*\mathbb{Z}\to R^1\psi_*\mathbb{Z}$.

We restrict the local systems introduced before to the pointed disk Δ^* considered in Section 4.1. We have the following result:

Proposition 4.6. *Given a generator* γ *of* $\pi_1(\Delta^*,t)$, *we have*

$$\operatorname{Inv}\left(\rho_\gamma\right)=n\operatorname{Inv}\left(\sigma_\gamma\right).$$

Proof. By Proposition 3.7, Remark 3.8 and (4.7), it follows that

$$\operatorname{Inv}\left(\sigma_\gamma\right)=\operatorname{Im}\Big(\zeta: H^1(J(\mathcal{C}_0),\mathbb{Z})=H^1(J(\mathcal{C}),\mathbb{Z})\to H^1(J(\mathcal{C}_t),\mathbb{Z})\Big),$$

where ζ is the map induced by the inclusion $J(\mathcal{C}_t) \hookrightarrow J(\mathcal{C})$. This implies

$$\begin{aligned}
\mathrm{Inv}\left(\rho_\gamma\right) &= \mathrm{Im}\left(H^1(\mathcal{B}_t,\mathbb{Z}) \xrightarrow{\mathfrak{H}_t} H^1(J(\mathcal{C}_t),\mathbb{Z})\right) \cap \mathrm{Inv}\left(\sigma_\gamma\right) \\
&= \mathrm{Im}\left(H^1(J(\mathcal{D}_t),\mathbb{Z}) \xrightarrow{\mathfrak{F}_t} H^1(J(\mathcal{C}_t),\mathbb{Z})\right) \cap \mathrm{Inv}\left(\sigma_\gamma\right) \\
&= \mathrm{Im}\left(H^1(J(\mathcal{D}_t),\mathbb{Z}) \xrightarrow{\mathfrak{F}_t} H^1(J(\mathcal{C}_t),\mathbb{Z})\right) \cap \mathrm{Im}\,(\zeta) \\
&= \mathrm{Im}\left(H^1(J(\mathcal{D}_0),\mathbb{Z}) \to H^1(J(\mathcal{C}_0),\mathbb{Z}) \xrightarrow{\zeta} H^1(J(\mathcal{C}_t),\mathbb{Z})\right) \\
&= n\,\mathrm{Im}\left(H^1(J(\mathcal{C}_0),\mathbb{Z}) \xrightarrow{\zeta} H^1(J(\mathcal{C}_t),\mathbb{Z})\right) \\
&= n\,\mathrm{Inv}\left(\sigma_\gamma\right),
\end{aligned}$$

where the next to last identity follows from Corollary 4.5. □

4.3. Double degeneration

Let P be a very generic irreducible stable curve of arithmetic genus $g \geq 4$, with exactly two nodes N_1, N_2 as singularities. Consider an open analytic neighborhood $U \subset \overline{M}_g^0$ of $[P] \in \overline{M}_g$ biholomorphic to a $(3g-3)$-dimensional polydisk. Assume that U has local coordinates $z_1, \dots, z_{3g-3}$ centered at $[P]$ and such that the local equation of $\delta_0 \cap U$ is $z_1 \cdot z_2 = 0$. Furthermore, for $i = 1, 2$, $z_i = 0$ is the local equation in U of the locus δ_0^i where the singularity N_i persists. We set $U' := U \setminus \delta_0$.

Let consider our initial families of isogenies $H\colon \mathcal{B}/\mathcal{V} \to J(\mathcal{C})/\mathcal{V}$, defined in (4.5). We recall that $\pi\colon \mathcal{V} \to \mathcal{U}$ is a finite étale covering of a dense open subset $\mathcal{U}$ of M_g^0. Set $U^* := U' \cap \mathcal{U}$, let V^* be a connected component of $\pi^{-1}(U^*)$ and consider the restriction

$$H\colon \mathcal{B}/V^* \to J(\mathcal{C})/V^*$$

of H to V^*. Then the following holds:

Proposition 4.7. *For each $t \in V^*$, the Abelian variety B_t is isomorphic to $J(\mathcal{C}_t)$ and the map $H_t\colon B_t \to J(C_t)$ is the multiplication by a non-zero integer n.*

Proof. Let us restrict the local systems $R^1\varphi_*\mathbb{Z}$ and $R^1\psi_*\mathbb{Z}$, defined in Section 4.2, and their monodromy representations ρ and σ, to V^*. The statement will follow from Proposition 3.6.

Let $\omega\colon \mathcal{C} \to U'$ be the restriction of the universal family of curves to U' and denote by ℓ the inclusion $U^* \hookrightarrow U'$. It holds $\pi^{-1}\ell^{-1}R^1\omega_*\mathbb{Z} = R^1\psi_*\mathbb{Z}$, where we recall that π is the finite covering $\pi\colon V^* \to U^*$. Set $u := \pi(t)$ and denote by $M \leq \mathrm{Aut}\left(H^1(J(\mathcal{C}_t),\mathbb{Z})\right)$ the monodromy group of the local system $R^1\psi_*\mathbb{Z}$ and by $L \leq \mathrm{Aut}\left(H^1(\mathcal{C}_u,\mathbb{Z})\right) = \mathrm{Aut}\left(H^1(J(\mathcal{C}_t),\mathbb{Z})\right)$ the monodromy group of the local system $R^1\omega_*\mathbb{Z}$. By Remark 3.3, since $\pi_1(U^*, u) \to \pi_1\big(U', u\big)$ is surjective, M is a subgroup of finite index of L.

We consider a symplectic basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ of $H^1(\mathcal{C}_u, \mathbb{Z})$ such that, for $i = 1, 2$, a_i is the vanishing cocycle for the Lefschetz degeneration centered in a point of δ_0^i. We recall that, by Proposition 3.7, L is generated by T_{a_1} and T_{a_2} (see (3.1) in Section 3). This implies that, for each $i = 1, 2$, there is an element $\gamma_i \in \pi_1(V^*, t)$ such that $\sigma_{\gamma_i} = T_{k_i a_i}$ for some non-zero $k_i \in \mathbb{N}$. Thus, $\mathrm{Inv}\,(\sigma_{\gamma_i})$ is generated by $\{a_1, \dots, a_g, b_1, \dots, b_g\} \setminus \{b_i\}$ and, by Proposition 4.6,

$$\mathrm{Inv}\,(\rho_{\gamma_i}) = n_i\,\mathrm{Inv}\,(\sigma_{\gamma_i}),$$

for some non-zero $n_i \in \mathbb{N}$. Proposition 3.6 implies $n_1 = n_2$ and $H^1(\mathcal{B}_t, \mathbb{Z}) = n_1 H^1(J(\mathcal{C}_t), \mathbb{Z})$. □

By considering a covering of M_g^0 of analytic open sets as U', we have the following corollary.

Corollary 4.8. *The Abelian scheme $\mathcal{B}/\mathcal{V}$ is isomorphic to $J(\mathcal{C})/\mathcal{V}$ and the map $H\colon \mathcal{B}/\mathcal{V} \to J(\mathcal{C})/\mathcal{V}$ is on each fibre the multiplication by a non-zero integer n.*

4.4. Conclusion of the proof of Theorem 4.1

Let consider the surjective map of Abelian schemes

$$G\colon J(\mathcal{D})/\mathcal{V} \to \mathcal{B}/\mathcal{V}$$

defined in (4.5). The aim of this section is to show that G is an isomorphism. This concludes the proof of Theorem 4.1.

Remark 4.9. By Corollary 4.8, $\ker G = 0$ implies that $J(\mathcal{D})$ is isomorphic to $J(\mathcal{C})$ and $F\colon J(\mathcal{D}) \to J(\mathcal{C})$ is the multiplication by n. We recall that, given a very generic point $t \in \mathcal{V}$, we can assume $\mathrm{NS}\,(J(\mathcal{C}_t)) = \mathbb{Z}$. This implies that $J(\mathcal{D}_t)$ is isomorphic to $J(\mathcal{C}_t)$ as a principally polarized Abelian variety and, by Torelli theorem (see [3]) $\mathcal{D}_t \simeq \mathcal{C}_t$. It follows that the two families of curves $\mathcal{D}$ and $\mathcal{C}$ are isomorphic. Thus, when $g(\mathcal{D}_t) = g(\mathcal{C}_t)$, we recover the result in [7].

The surjective map $G\colon J(\mathcal{D}) \to \mathcal{B} \simeq J(\mathcal{C})$ induces an injective map of local systems $R^1\psi_*\mathbb{Z} \hookrightarrow R^1\mu_*\mathbb{Z}$ (*cf.* Section 4.2). Thus $R^1\psi_*\mathbb{Z}$ has two different polarizations: the natural polarization Θ induced by the intersection of cycles on $\mathcal{C}$, and the polarization Ξ inherited by $R^1\mu_*\mathbb{Z}$ endowed with the polarization induced by the intersection of cycles on $\mathcal{D}$ (see Section 3.1).

In the following proposition we show that the two polarizations coincide. This implies that, given $t \in \mathcal{V}$, the exact sequence of polarized Abelian varieties

$$0 \to J(\mathcal{C}_t) \to J(\mathcal{D}_t) \to J(\mathcal{D}_t)/J(\mathcal{C}_t) \to 0,$$

induced by the dual map $\widehat{G}_t\colon J(\mathcal{C}_t) \to J(\mathcal{D}_t)$, splits (see *e.g.* [5, Corollary 5.3.13]). Since $\mathcal{D}_t$ is an irreducible curve, the theta divisor of $J(\mathcal{D}_t)$ is irreducible and $\ker G_t = 0$.

Proposition 4.10. *The polarizations* Θ *and* Ξ *of* $R^1\psi_*\mathbb{Z}$ *coincide.*

Proof. Let t be a very generic point in $\mathcal{V}$ and let $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ be a basis of $H^1(J(\mathcal{C}_t), \mathbb{Z})$ that is symplectic with respect to Θ_t. Consider a Lefschetz degeneration $\mathcal{C}/\Delta$ of $\mathcal{C}_t$ such that a_1 is the vanishing cocycle. We restrict $J(\mathcal{D})$ to Δ^* and, arguing as in Section 4.1, up to a base change of Δ, we get the limit

$$G\colon J(\mathcal{D})/\Delta \to J(\mathcal{C})/\Delta$$

of the map G when $\mathcal{C}_t$ degenerates to a nodal curve. Notice that, by Corollary 4.8, the map of semi-Abelian varieties $F_0\colon J(\mathcal{D}_0) \to J(\mathcal{C}_0)$ is the composition of the map $G_0\colon J(\mathcal{D}_0) \to J(\mathcal{C}_0)$ with the multiplication by n.

We identify $H^1(\widetilde{\mathcal{C}_0}, \mathbb{Z})$ to the sub-lattice L of $H^1(J(\mathcal{C}_t), \mathbb{Z})$ generated by the elements $\{a_2, \dots, a_g, b_2, \dots, b_g\}$. In this way, $\Theta_t|_L$ is the polarization induced by the intersection of cycles on $\widetilde{\mathcal{C}_0}$ and $\Xi_t|_L$ is the polarization inherited by $H^1(\widetilde{\mathcal{D}_0}, \mathbb{Z})$ (endowed with the polarization induced by the intersection of cycles on $\widetilde{\mathcal{D}_0}$) through the inclusion $H^1(\widetilde{\mathcal{C}_0}, \mathbb{Z}) \hookrightarrow H^1(\widetilde{\mathcal{D}_0}, \mathbb{Z})$. By Proposition 4.3,

$$J(\widetilde{\mathcal{D}_0}) \simeq J(C) \times J(D_2) \times \dots \times J(D_k)$$

and the map $J(\widetilde{\mathcal{D}_0}) \to J\left(\widetilde{\mathcal{C}_0}\right)$, induced by $G_0\colon J(\mathcal{D}_0) \to J(\mathcal{C}_0)$ on the compact quotient, is the first projection. Thus $\Theta_t|_L = \Xi_t|_L$. For a very generic $t \in \mathcal{V}$, we can assume $\mathrm{NS}\,(J(C_t)) = \mathbb{Z}$ and $\Xi_t = n\Theta_t$ for some $n \in \mathbb{N}$. This implies $\Theta_t = \Xi_t$. □

5. Curves of genus $2g-2$ on a Jacobian of dimension g

It is natural to ask whether the bound given in Theorem 4.1 is sharp. Notice that, if $p > 2g-2$, it is possible to give examples of non-constant maps $\Omega \to J$, from a curve of genus p to a Jacobian of dimension g. Namely, it is always possible to find a finite covering of genus p of a curve of genus g.

The following conjecture was suggested to us by Gian Pietro Pirola.

Conjecture 5.1. There are no curves of geometric genus $2g-2$ lying on a very generic Jacobian of dimension $g \geq 4$.

If the conjecture were false, as in the previous case (see (4.2) in Section 4), we would find a map of families

$$F\colon J(\mathcal{D})/\mathcal{V} \to J(\mathcal{C})/\mathcal{V},$$

and a family of Abelian varieties $\mathcal{B}/\mathcal{V}$ such that F can be factorized in two morphisms (see (4.5))

$$G\colon J(\mathcal{D}) \to \mathcal{B} \qquad H\colon \mathcal{B} \to J(\mathcal{C}),$$

where G has connected fibres and H is a finite morphism. In this case, the Abelian scheme $\mathcal{A} := \ker G$ would be a family of Abelian varieties of dimension $g-2$ with a natural inclusion $I\colon \mathcal{A} \hookrightarrow J(\mathcal{D})$.

Let us consider the dual map

$$\widehat{I}\colon J(\mathcal{D}) \simeq \widehat{J(\mathcal{D})} \to \widehat{\mathcal{A}}$$

and its composition with the Abel map

$$L\colon \mathcal{D} \to \widehat{\mathcal{A}}. \tag{5.1}$$

Proposition 5.2. *For a generic $t \in \mathcal{V}$, the map $L_t\colon \mathcal{D}_t \to \widehat{\mathcal{A}}_t$ is birational on its image.*

Proof. Set $W_t := L_t(\mathcal{D}_t)$. The map of curves $L_t\colon \mathcal{D}_t \to W_t$ factors through $\ell_t\colon \mathcal{D}_t \to \widetilde{W}_t$. Up to a restriction of $\mathcal{V}$, we can suppose that the degree of ℓ_t, the total ramification order of ℓ_t and the geometric genus of W_t (denoted, respectively, by d, r and q) do not depend on $t \in \mathcal{V}$. We recall that $g(\mathcal{D}_t) = 2g-2$, $g(\mathcal{C}_t) = g$ and $\widehat{I}$ is surjective on each fibre. By Riemann-Hurwitz formula, either L_t is birational on the image or $g-2 \leq q \leq g-1$.

Assume $q = g-2$. By a count of moduli, either the family $\mathcal{D}/\mathcal{V}$ is trivial, or $d = 2$ and $r = 6$. In the second case $\widehat{\mathcal{B}}_t = \ker \widehat{I}$, and consequently $J(\mathcal{C}_t)$, is isogenous to the Prym variety of the ramified double covering $\ell_t\colon \mathcal{D}_t \to \widetilde{W}_t$. The dimension of the moduli space of the double coverings of a curve of genus $g-2$ with 6 branch points is $3g-3$. It follows that, in this case, the dimension of the Prym locus is equal to the dimension of the Jacobian locus. By [12, Theorem 1.2], a very generic Prym variety of dimension at least 4 is not isogenous to a Jacobian. This yields a contradiction.

If $q = g-1$, then $\ker\big(J(\mathcal{D}_t) \to J\big(\widetilde{W}_t\big)\big)$ contains an Abelian subvariety S_t of $J(\mathcal{D}_t)$ of dimension $g-1$. Since S_t is contained in $\widehat{G}(\widehat{\mathcal{B}}_t)$, then $\widehat{\mathcal{B}}_t$, and consequently $J(\mathcal{C}_t)$, is not simple. Thus we get a contradiction. □

We want to show that the conjecture is true when $g = 4, 5$. To this end, we need the following result on the number of parameters of curves lying on an Abelian variety. When the dimension of the Abelian variety is greater than 2 this is an improvement of the estimate in [9, Proposition 2.4].

Proposition 5.3. *Let X be an irreducible subvariety of M_g whose points parameterize normalizations of curves of geometric genus $g \geq 3$ lying on an Abelian variety A of dimension $n \geq 2$. Then* $\dim X \leq g-2$ *if A is a surface and* $\dim X \leq g-3$ *if $n \geq 3$.*

Proof. Let C be an irreducible smooth curve of genus g and $\varphi\colon C \to A$ be a morphism birational on the image. The space of the first-order deformations of φ is $H^0(C, N)$, where N is the sheaf defined by the exact sequence

$$0 \to T_C \to \varphi^* T_A \to N \to 0. \tag{5.2}$$

The sheaf N is usually not locally free but there is an exact sequence

$$0 \to S \to N \to N' \to 0, \tag{5.3}$$

where S is the skyscraper sheaf, with support in the points of C in which $d\varphi$ vanishes, and N' is a locally free sheaf. Arbarello and Cornalba (see [1]) proved that the infinitesimal deformations of φ corresponding to sections of S induce trivial deformations of the curve $\varphi(C)$. It follows that the dimension of an irreducible subvariety of the Hilbert scheme of curves on A, whose points parameterize curves of geometric genus g, has dimension less or equal than $h^0(N')$. This implies that X has dimension at most $h^0(N') - n$.

From (5.2) and (5.3) we get

$$c_1(N') = c_1(N) - c_1(S) = c_1(\varphi^* T_A) - c_1(T_C) - c_1(S) = \omega_C(-D),$$

where D is the divisor associated to the support of S.

If A is an Abelian surface, N' is a line bundle and it follows that $h^0(N') \leq g$. Otherwise, N' is a locally free sheaf of rank $n-1$, generated by the global sections (see (5.2)). Up to a suitable choice of $n-2$ independent global sections of N', we have the following exact sequence

$$0 \to \mathcal{O}_C^{n-2} \to N' \to \omega_C(-D) \to 0. \tag{5.4}$$

We want to prove that $\varphi\colon C \to A$ depends on, at most, $g-3+n$ parameters. If not, $h^0(N') > g-3+n$ and the following inequality holds

$$g - 3 + n < h^0(N') \leq n - 2 + h^0(\omega_C(-D)).$$

Thus $\deg D = 0$, $N' = N$ and the exact sequence (5.4) becomes

$$0 \to \mathcal{O}_C^{n-2} \to N \to \omega_C \to 0.$$

By the rigidity of hyperelliptic curves in Abelian varieties (see [16, Section 2]), we can assume C to be not hyperelliptic; it follows that the previous exact sequence splits and $N = \mathcal{O}_C^{n-2} \oplus \omega_C$. This implies that the exact sequence (5.2) becomes

$$0 \to T_C \to \mathcal{O}_C^n \to \mathcal{O}_C^{n-2} \oplus \omega_C \to 0$$

and $T_C \subset \mathcal{O}_C^2$. Thus we get a contradiction. □

Theorem 5.4. *There are no curves of genus $2g-2$ on a very generic Jacobian of dimension $g = 4, 5$.*

Proof. By Proposition 5.2, it is sufficient to show that, for $g = 4$ (respectively $g = 5$) the map $L_t\colon \mathcal{D}_t \to \widehat{\mathcal{A}_t}$ (see (5.1)) is not birational on its image. Set $n := \dim \widehat{\mathcal{A}_t} = 2$ (respectively $n = 3$) and $w := (3g-3)-n(n+1)/2 = 6$ (respectively $w = 6$). If $L_t\colon \mathcal{D}_t \to \widehat{\mathcal{A}_t}$ would be birational on its image for all $t \in \mathcal{V}$, there would be an Abelian variety A, of dimension n, and an irreducible subvariety $X \subset M_{2g-2}$ of dimension w, whose points parameterize curves of geometric genus $2g-2 = 6$ (respectively $2g-2 = 8$) on A. By Proposition 5.3 we get a contradiction. □

Remark 5.5. Notice that for $g = 4$ the bound is sharp since a very generic principally polarized Abelian fourfold contains curves of geometric genus 7 (see [4, Section 2]).

References

[1] E. ARBARELLO and M. CORNALBA, *On a conjecture of Petri*, Comment. Math. Helv. **56** (1981), 1–38.

[2] V. ALEXEEV, *Compactified Jacobians and Torelli map*, Publ. Res. Inst. Math. Sci. **40** (2004), 1241–1265.

[3] A. ANDREOTTI, *On a theorem of Torelli*, Amer. J. Math. **80** (1958), 801–828.

[4] F. BARDELLI, C. CILIBERTO and A. VERRA, *Curves of minimal genus on a general Abelian variety*, Compos. Math. **96** (1995), 115–147.

[5] C. BIRKENHAKE and H. LANGE, "Complex Abelian Varieties", Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 302, Springer-Verlag, Berlin, second edition, 2004.

[6] S. BOSCH, W. LÜTKEBOHMERT and M. RAYNAUD, "Néron Models", Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol. 21, Springer-Verlag, Berlin, 1990.

[7] F. BARDELLI and G. P. PIROLA, *Curves of genus g lying on a g-dimensional Jacobian variety*, Invent. Math. **95** (1989), 263–276.

[8] C.-L. CHAI, "Compactification of Siegel Moduli Schemes" London Mathematical Society Lecture Note Series, Vol. 107, Cambridge University Press, Cambridge, 1985.

[9] C. CILIBERTO, G. VAN DER GEER and M. TEIXIDOR I BIGAS, *On the number of parameters of curves whose Jacobians possess nontrivial endomorphisms*, J. Algebr. Geom. **1** (1992), 215–229.

[10] G. FALTINGS and C.-L. CHAI, "Degeneration of Abelian Varieties", Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol. 22, with an appendix by David Mumford, Springer-Verlag, Berlin, 1990.

[11] R. HARTSHORNE, "Algebraic Geometry", Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York, 1977.

[12] V. MARCUCCI and G. P. PIROLA, *Generic Torelli theorem for Prym varieties of ramified coverings*, Compos. Math. **148** (2012), 1147–1170.

[13] J. C. NARANJO and G. P. PIROLA, *On the genus of curves in the generic Prym variety*, Indag. Math. (N.S.) **5** (1994), 101–105.

[14] T. ODA and C. S. SESHADRI, *Compactifications of the generalized Jacobian variety*, Trans. Amer. Math. Soc. **253** (1979), 1–90.

[15] G. P. PIROLA, *Base number theorem for Abelian varieties. An infinitesimal approach*, Math. Ann. **282** (1988), 361–368.

[16] G. P. PIROLA, *Curves on generic Kummer varieties*, Duke Math. J. **59** (1989), 701–708.

[17] G. P. PIROLA, *On a conjecture of Xiao*, J. Reine Angew. Math. **431** (1992), 75–89.
[18] J.-P. SERRE, "Algebraic Groups and Class Fields", Graduate Texts in Mathematics, Vol. 117, translated from the French, Springer-Verlag, New York, 1988.
[19] C. VOISIN, "Hodge Theory and Complex Algebraic Geometry. II", Cambridge Studies in Advanced Mathematics, Vol. 77, translated from the French by Leila Schneps, Cambridge University Press, Cambridge, 2003.

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