Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XIII (2014), 371-397

Formally reversible maps of $\mathbb{C}^2$

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Abstract. An element g of a group is called *reversible* if it is conjugate in the group to its inverse. This paper is about reversibles in the group $\mathfrak{G} = \mathfrak{G}_2$ of formally invertible pairs of formal power series in two variables, with complex coefficients. The main result is a description of the generic reversible elements of $\mathfrak{G}_2$. We list two explicit sequences of reversibles which between them represent all the conjugacy classes of such reversibles. We show that each such element is reversible by some element of finite order, and hence is the product of two elements of finite even order. The elements that may be reversed by an involution are called *strongly reversible*. We also characterise them.

We draw some conclusions about generic reversibles in the group $\mathcal{G} = \mathcal{G}_2$ of biholomorphic germs in two variables, and about the factorization of formal maps as products of reversibles. Specifically, we show that each product of reversibles reduces to the product of five.

Mathematics Subject Classification (2010): 30D05 (primary); 32A05, 32H02, 32H50, 37F10, 37F50 (secondary).

1. Introduction

An element g of a group is called *reversible* if it is conjugate to its inverse, *i.e.* if the conjugate $g^h := h^{-1}gh$ equals g^{-1} for some h in the group. We say that h *reverses* g or h *is a reverser of* g, in this case. Furthermore, if the reverser h can be chosen to be an involution, g is called *strongly reversible*. (Note that some literature uses the terminology "weakly reversible" and "reversible" instead of, respectively, "reversible" and "strongly reversible" used here.)

Reversible maps have their origin in problems of classical dynamics, such as the harmonic oscilator, the n-body problem or billiards, and Birkhoff [4] was one of the first to realize their significance. He observed that a Hamiltonian system with Hamiltonian quadratic in the momentum p (such as the n-body problem) or, more generally, any system in of the form

$$\begin{cases} \partial q/\partial t = Lp, \\ \partial p/\partial t = V(p,q), \end{cases} \qquad (q,p) \in \mathbb{R}^n \times \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{1.1}$$

Supported in part by the Science Foundation Ireland grant 10/RFP/MTH2878.
Received December 31, 2011; accepted in revised form May 31, 2012.

where L is linear, admits the so-called "time-reversal symmetry" $(t, q, p) \mapsto (-t, q, -p)$. In particular, the flow map $(q_0, p_0) \mapsto (q(t), p(t))$, where $(q(t), p(t))$ is the solution of (1.1) with $(q(0), p(0)) = (q_0, p_0)$, is reversed by the involution $(q, p) \mapsto (q, -p)$.

In CR geometry reversible maps played an important role in the celebrated work of Moser and Webster [11], arising as products of two involutions naturally associated to a CR singularity. Such a reversible map is called there "a discrete version of the Levi form" and plays a fundamental role in the proof of the convergence of the normal form for a CR singularity. More recently, this map has been used by Ahern and Gong [1] for the so-called parabolic CR singularities.

There is a class of diffeomorphisms of $\mathbb{R}^2$ that has received considerable attention: the so-called *standard maps*, or *Taylor-Chirikov maps*, which arise in many applications in physics, and are strongly-reversible in the group of real-analytic diffeomorphisms [8, Section 1.1.4]. The standard maps fix the origin and are area-preserving. When they are real-analytic, they thus give reversible elements of our power-series group.

This paper aims to classify formally reversible maps in two complex variables, for which the eigenvalues of the linear part are not roots of unity. The presence of roots of unity is a well-known obstruction leading to the presence of additional resonances.

Reversibility is already understood [12] in the group $\mathfrak{G}_1$ of formal power series maps of $\mathbb{C}$ without constant terms (see also [2] for reversibility in the group $\mathcal{G}_1$ of biholomorphic map germs of $\mathbb{C}$ fixing the origin):

Theorem 1.1. [12, Theorem 5] *A formal power series self-map of $\mathbb{C}$ without constant term is reversible if and only if it is formally conjugate to*

$$\varphi_{\mu,\lambda,k}(z) := \frac{\mu z}{(1+\lambda z^k)^{1/k}} \tag{1.2}$$

for some integer $k \geq 1$, $\mu = \pm 1$ and $\lambda \in \{0, 1\}$. The map (1.2) *is reversed by any rotation $z \mapsto \omega z$ with $\omega^k = -1$.*

Note that for $k = 1$, the map $\varphi_{1,1,1}$ is precisely the (unique up to conjugation) map $z \mapsto z + \mathrm{HOT}$ that is conjugate to a projective transformation, whereas $\varphi_{1,1,k}$ is obtained from $\varphi_{1,1,1}$ via "conjugation" under the non-invertible map $z \mapsto z^k$, *i.e.* $(\varphi_{1,1,k}(z))^k = \varphi_{1,1,1}(z^k)$.

In this paper, our main focus is on the group $\mathfrak{G} = \mathfrak{G}_2$ of formal power series self-maps of $\mathbb{C}^2$ without constant terms. We obtain the following formal normal form:

Theorem 1.2. *A formal power series map of $\mathbb{C}^2$ whose linear part has eigenvalue λ not a root of unity is reversible if and only if it is* (*formally*) *conjugate either to its linear part or to one of the following maps, which are all pairwise inequivalent* *under conjugation*:

$$(z_1, z_2) \mapsto \left(\lambda(1+p^k)z_1, \frac{1}{\lambda(1+p^k)}z_2\right), \quad \left(\frac{\lambda(1+p^k)^{\frac{1}{k}}}{(1+2p^k)^{\frac{1}{k}}}z_1, \frac{1}{\lambda(1+p^k)^{\frac{1}{k}}}z_2\right), \qquad p = z_1z_2, \tag{1.3}$$

where $k \geq 1$ is an integer. Furthermore, the following hold:

(1) *Each map in* (1.3) *is reversed by any $J_c(z_1, z_2) = c(z_2, z_1)$ such that $c^{2k} = 1$ for the first series of maps and $c^{2k} = -1$ for the second one*;
(2) *A map in* (1.3) (*with λ not a root of unity*) *is strongly reversible if and only if it is in the first series*;
(3) *Each reverser of a map in* (1.3) *has finite order.*

Similarly to those in (1.2), the maps (1.3) can be obtained from those corresponding to $k = 1$ by a "conjugation" under the non-invertible map $(z_1, z_2) \mapsto (z_1^k, z_2^k)$. The 'generic-type' condition that λ not be a root of unity is important as the following example shows. (Note that a similar condition also appears in the work of Moser-Webster [11], where it is also shown to be crucial for the existence of the formal normal form constructed there.)

Example 1.3. The assumption that λ is not a root of unity cannot be dropped in Theorem 1.2. Indeed, the map

$$F(z_1, z_2) = \left(\frac{z_1}{1+z_1}, \frac{z_2}{1+z_1}\right) \tag{1.4}$$

is reversed by the involution $(z_1, z_2) \mapsto (-z_1, z_2)$; in fact, it is the projectivization of the linear map $(z_0, z_1, z_2) \mapsto (z_0 + z_1, z_1, z_2)$ reversed by $(z_0, z_1, z_2) \mapsto (z_0, -z_1, z_2)$. However, it has the second order terms $(-z_1^2, -z_2z_1)$ that cannot be eliminated by conjugation and do not occur in a map as in (1.3). Hence, F is not conjugate to any map in (1.3).

We also obtain polynomial representatives conjugate to the maps of the second series:

Proposition 1.4. *For every k, the map from the second series in* (1.3) *is formally conjugate to the polynomial map*

$$(z_1, z_2) \mapsto \left(\lambda(1+p^k)z_1, \lambda^{-1}\left(1 + p^k + (2k+1)p^{2k}z_2\right)\right), \qquad p = z_1z_2.$$

Remark 1.5. Moser [10] showed that real-analytic area-preserving map germs on $(\mathbb{R}^2, 0)$ having a hyperbolic fixed point at the origin may be conjugated by a convergent coordinate change to the normal form

$$(\sigma \cdot x \cdot \exp(w(xy)), \sigma \cdot y \cdot \exp(-w(xy))), \tag{1.5}$$

where $\sigma = \pm 1$ and w is a convergent series. Thus they are strongly reversible and conjugate to an element of our first series in (1.3), with some real $\lambda \neq \pm 1$. On the other hand, a general map conjugate to (1.5) is always strongly reversible but need not be area-preserving, whereas maps of the second series in (1.3) are not area-preserving.

Theorem 1.2 and Proposition 1.4 are proved in Section 7.

Furthermore, following the arguments of Moser-Webster [11], we come to a biholomorphic classification for the maps conjugate to those in the first series:

Theorem 1.6. *If a biholomorphic map germ in $\mathbb{C}^2$ is formally conjugate to a map in the first series in* (1.3) *with $|\lambda| \neq 1$, then it is biholomorphically conjugate to it. On the other hand, there exist a biholomorphic map germ conjugate to a map in the second series in* (1.3)*, and hence formally reversible, that is not biholomorphically conjugate to its formal normal form and even not biholomorphically reversible.*

Note that any map in the first series in (1.3) is conjugated to itself by (*i.e.* commutes with) any map of the form $(z_1, z_2) \mapsto \big(z_1\varphi(p), z_2/\varphi(p)\big)$, where φ is a formal power series in $t \in \mathbb{C}$ with $\varphi(0) \neq 0$. Hence the biholomorphic conjugation map may differ from the original formal one.

The second part of Theorem 1.6 is established by Example 8.1 below. The statement of Theorem 1.6 also does not hold without the assumption $|\lambda| \neq 1$ due to the remarkable theorem of Gong [6, Theorem 1.1] stating the existence of a biholomorphic map $F(z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2) + \mathrm{HOT}$ with λ not a root of unity and $|\lambda| = 1$ such that F is formally reversible by an involution (and hence F is formally conjugate to a map in the first series in (1.3)) but is not reversible by a biholomorphic involution (and hence F is not biholomorphically conjugate to its formal normal form).

We shall further discuss reversibility and centralisers in some related groups, along the way. We draw some conclusions about generic reversibles in the group $\mathcal{G}_2$ of biholomorphic germs in two variables (Theorem 10.1), and about the factorization of maps as products of reversibles. Since a reversible element of $\mathfrak{G}_2$ has linear part with determinant ± 1 (see Subsection 2.4), any product of them has the same property. Conversely we show:

Theorem 1.7. *Each element of $\mathfrak{G}_2$ whose linear part has determinant ± 1 is the product of at most four reversibles and an involution.*

Theorem 1.7 is proved in Section 9.

2. Notation and preliminaries

2.1. The map L

A typical element $F \in \mathfrak{G}$ takes the form

$$F(z) = (F_{,1}(z), F_{,2}(z)) = (F_{,1}(z_1, z_2), F_{,2}(z_1, z_2))$$

where each $F_{,j}(z)$ is a power series in two variables having complex coefficients, and no constant term. We shall refer to such series F as maps, even though they may be just 'formal', *i.e.* the series may fail to converge at any $z \neq 0$.

We usually write the formal composition of two maps $F, G \in \mathfrak{G}$ as FG. We also write the product of two complex numbers a and b as ab, but in cases where there might be some ambiguity we use $a \cdot b$.

The series F may be expressed as a sum

$$F = \sum_{k=1}^{\infty} L_k(F),$$

where $L_k(F)$ is homogeneous of degree k. We abbreviate $L_1(F)$ to $L(F)$. This term, the *linear part of* F, belongs to the group $\mathsf{GL} = \mathsf{GL}(2, \mathbb{C})$. We have inclusions $\mathsf{GL} \to \mathfrak{G}$, and $L : \mathfrak{G} \to \mathsf{GL}$ is a group homomorphism.

The elements of the kernel of L are said to be *tangent to the identity*.

2.2. Elements of finite order

We note the following, in which $\mathfrak{G}_n$ denotes the group of all invertible formal power series self-maps of $\mathbb{C}^n$ without constant coefficients. (Various cases of this lemma, and the idea of its simple proof, are well-known. The case $n = 1$ is very classical. For holomorphic maps in n variables, see [3, page 298]. In the differentiable category, Montgomery and Zippin [9] proved the local conjugacy of any involution to its derivative at a fixed point. The equivalent global question is more delicate.)

Lemma 2.1. *Let $n \in \mathbb{N}$ and let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}_n$ such that*

(1) *$L(F) \in \mathfrak{H}$ whenever $F \in \mathfrak{H}$, and*
(2) *$\mathfrak{H} \cap \ker L$ is closed under convex combinations, i.e. if $F_1, F_2 \in \mathfrak{H}$, $L(F_1) = L(F_2) = \mathsf{id}$ and $0 < \alpha < 1$, then $\alpha F_1 + (1-\alpha) F_2 \in \mathfrak{H}$.*

Suppose $\Theta \in \mathfrak{H}$ has finite order. Then Θ is conjugated by an element of $\mathfrak{H} \cap \ker L$ to its linear part $L(\Theta)$.

Proof. Suppose $\Theta^k = \mathsf{id}$. Let $T = L(\Theta)$, and form

$$H = \frac{1}{k}\left(\mathsf{id} + T^{-1}\Theta + T^{-2}\Theta^2 + \cdots + T^{-k-1}\Theta^{k-1}\right). \tag{2.1}$$

Then the assumptions imply that $H \in \mathfrak{H}$, and a short calculation shows that $T^{-1}H\Theta = H$, so that $T^H = \Theta$, as required. $\square$

This applies to $\mathfrak{H} = \mathfrak{G}$, $\mathfrak{G} \cap \ker L$, $\mathfrak{G} \cap \ker(\det \circ L) = L^{-1}(\mathsf{SL}(2, \mathbb{C}))$, $L^{-1}(\mathsf{U}(2, \mathbb{C}))$ (and, more generally to $L^{-1}(H)$ for any subgroup $H \leq \mathsf{GL}$), to the corresponding subgroups of biholomorphic germs (*i.e.* series that converge on a neighbourhood of the origin) and to other subgroups introduced below. It applies to the intersection of any two groups to which it applies.

In particular, in any $\mathfrak{H}$ to which the lemma applies, each involution is conjugate to one of the linear involutions in the group. In $\mathsf{GL} = \mathsf{GL}(n, \mathbb{C})$, a matrix is an involution if and only if it is diagonalizable with eigenvalues ± 1.

2.3. Reversibles in one variable

Here we collect facts about reversibles in one variable that will be used throughout the paper.

Lemma 2.2. *A map $h \in \mathfrak{G}_1$ is a reverser of the map* (1.2) *with $\mu = 1$ and $\lambda \neq 0$ if and only if it is of the form*

$$h_{\omega,\nu}(z) = \frac{\omega z}{(1+\nu z^k)^{1/k}}, \tag{2.2}$$

where $\nu, \omega \in \mathbb{C}$ are arbitrary with $\omega^k = -1$.

Proof. Since $\varphi_{1,\lambda,k}$ is the inverse of $\varphi_{1,-\lambda,k}$, the map $h_{\omega,0}(z) = \omega z$ reverses $\varphi_{1,\lambda,k}$ for any ω with $\omega^k = -1$. Furthermore, since any map $h_{1,\nu}$ commutes with $\varphi_{1,\lambda,k}$, we have

$$h^{-1}_{\omega,\nu}\varphi_{1,\lambda,k}h_{\omega,\nu} = h^{-1}_{1,\nu}h^{-1}_{\omega,0}\varphi_{1,\lambda,k}h_{\omega,0}h_{1,\nu} = h^{-1}_{1,\nu}\varphi^{-1}_{1,\lambda,k}h_{1,\nu} = \varphi^{-1}_{1,\lambda,k}$$

and therefore any $h_{\omega,\nu}$ reverses $\varphi_{1,\lambda,k}$. Vice versa, if h reverses $\varphi_{1,\lambda,k}$, comparing the coefficients of z^{k+1} in the equation

$$h\varphi_{1,-\lambda,k} = \varphi_{1,\lambda,k}h \tag{2.3}$$

yields $h(z) = \omega z + \mathrm{O}(z^2)$ for some $\omega^k = -1$. Furthermore, we can choose ν such that $g := h^{-1}_{\omega,\nu}h$ has no coefficient of z^{p+1}. Then it follows from (2.3) that g commutes with $\varphi_{1,-\lambda}$. We claim that $g = \mathsf{id}$. Suppose on the contrary, that $g(z) = z + az^\beta + \cdots$ with $a \neq 0$ and $\beta \neq p+1$. Then comparing the coefficients of $z^{\beta+k+1}$ in the identity $g\varphi_{1,\lambda,k} = \varphi_{1,\lambda,k}g$ yields a contradiction. Hence $g = \mathsf{id}$ proving that h is of the form (2.3). □

Corollary 2.3. *Any reverser of the map* (1.2) *with $\mu = 1$ and $\lambda \neq 0$ is of finite order at most $2k$.*

Proof. Since $h_{\omega,0}$ reverses $\varphi_{1,\lambda,k}$, we have

$$h^{2k}_{\omega,\nu} = (h_{\omega,0}\varphi_{1,\nu,k}h_{\omega,0}\varphi_{1,\nu,k})^k = (h_{\omega,0}h_{\omega,0}\varphi^{-1}_{1,\nu,k}\varphi_{1,\nu,k})^k = h^{2k}_{\omega,0} = \mathsf{id}.$$ □

2.4. Linear reversibles

Reversibility is preserved by homomorphisms, so a map $F \in \mathfrak{G}_n$ is reversible only if $L(F)$ is reversible in $\mathsf{GL}(n,\mathbb{C})$. Classification of linear reversible maps is simple. Suppose $F \in \mathsf{GL}(n,\mathbb{C})$ is reversible. Since the Jordan normal form of F^{-1} consists of blocks of the same size as F with inverse eigenvalues, the eigenvalues of F that are not ± 1 must split into groups of pairs λ, λ^{-1}. Furthermore, we must have the same number of Jordan blocks of each size for λ as for λ^{-1}. Vice versa, if the eigenvalues of F are either ± 1 or split into groups of pairs λ, λ^{-1} with the same number of Jordan blocks of each size, then both F and F^{-1} have the same Jordan normal form and are therefore conjugate to each other.

2.5. The Group $\mathsf{GL}(2, \mathbb{C})$

In particular, a linear map is reversible in $\mathsf{GL}(2, \mathbb{C})$ if and only if it is an involution or is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ or to a matrix of the form

$$\begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{2.4}$$

for some $\mu \in \mathbb{C}^\times$. Thus each reversible F is conjugate in $\mathfrak{G}$ (by a linear conjugacy) to a map having one of these types as its linear part.

The collection of maps (2.4) forms an abelian subgroup, which we denote by D. The element (2.4) has infinite order precisely when μ is not a root of unity, and this is what we regard as the generic situation. In this paper, we are going to concentrate on the following question:

Which elements $F \in \mathfrak{G}$ for which $L(F)$ has an eigenvalue that is not a root of unity are reversible?

To approach this, we are going to begin by studying the centraliser of D in $\mathfrak{G}$. In fact, the classical Poincaré-Dulac Theorem [7, Section 4.8, Theorem 4.22] implies that any map $F \in \mathfrak{G}_n$ is conjugate to a map in the centralizer of the linear part $L(F)$. In case $L(F) = \mathsf{diag}(\mu, \mu^{-1}) \in D$ and μ is not a root of unity, it is easy to see that the centralizer of $L(F)$ coincides with the centralizer of D. We shall, incidentally, discover some classes of reversibles F that have $L(F) = \mathsf{id}$, although our focus is not on these non-generic examples. Apart from the maps given in (1.3), which are tangent to the identity when $\lambda = 1$, we also meet the maps given in Equation (7.9) below.

2.6. The Groups $\mathfrak{C} = C_D(\mathfrak{G})$ and $\mathfrak{E} = E_D(\mathfrak{G})$

A map $F \in \mathfrak{G}$ is in the *centralizer* of D if and only if it commutes with any particular element of D of infinite order, and if and only if it takes the form $F = M(\varphi, \psi)$, given by

$$M(\varphi, \psi)(z) = (z_1\varphi(z_1z_2), z_2\psi(z_1z_2)), \tag{2.5}$$

where $\varphi(t)$ and $\psi(t)$ are series in one variable such that $\varphi(0) \neq 0 \neq \psi(0)$ (*i.e.* they are series having nonzero constant term). We denote by $\mathfrak{C}$ the group of all such maps:

$$\mathfrak{C} := C_D(\mathfrak{G}) = \{M(\varphi, \psi) : \varphi, \psi \text{ as above}\}.$$

In the following we shall adopt the notation

$$p := z_1z_2, \quad J(z) = \tilde{z} := (z_2, z_1).$$

Then J reverses every $\Lambda \in D$, *i.e.* $J^{-1}\Lambda J = \Lambda^{-1}$. Furthermore, a map $\Theta \in \mathfrak{G}$ reverses each (or any fixed infinite-order) element $\Lambda \in D$ if and only if $J\Theta$ commutes with Λ and hence, if and only if

$$\Theta(z) = (z_2\varphi(p), z_1\psi(p)), \tag{2.6}$$

where $\varphi(t)$ and $\psi(t)$ are as before. This may be written as $\Theta = M(\varphi, \psi)J = JM(\psi, \varphi)$. We denote the collection of such Θ by $\mathfrak{R}$:

$$\mathfrak{R} := \{JF : F \in \mathfrak{C}\} = \{FJ : F \in \mathfrak{C}\}$$

and note that it is both a left and a right coset of $\mathfrak{C}$.

The *extended centraliser* $E_S(G)$ of a subset S in a group G is the set of all elements of the group that either commute with all the elements of the subset or reverse them all. We denote the extended centraliser of D in $\mathfrak{G}$, by $\mathfrak{E}$:

$$\mathfrak{E} := E_D(\mathfrak{G}) = \mathfrak{C} \cup \mathfrak{R}.$$

This is a group, in which $\mathfrak{C}$ has index 2. Lemma 2.1 applies to $\mathfrak{H} = \mathfrak{C}$ and to $\mathfrak{H} = \mathfrak{E}$:

Lemma 2.4. *If $F \in \mathfrak{E}$ has finite order, then there exists $H \in \mathfrak{C}$ such that F^H is linear.*

2.7. The homomorphisms P, H, and Φ

To $F \in \mathfrak{E}$ we associate the one variable power series

$$P(F) = \rho(t) \in \mathfrak{G}_1, \quad \rho(t) := t \cdot \varphi(t) \cdot \psi(t),$$

where F is given by the right-hand side of either (2.5) or (2.6). Note that, following our convention, we use $\cdot$ for the (formal) pointwise multiplication of power series. Denoting

$$p = \pi(z_1, z_2) := z_1 z_2,$$

we have the basic property

$$P(F) \circ \pi = \pi \circ F, \tag{2.7}$$

i.e. $P(F)$ is "semi-conjugate" to F via π. Property (2.7) determines $P(F)$ uniquely. A routine calculation using (2.7) proves:

Lemma 2.5. $P : \mathfrak{E} \to \mathfrak{G}_1$ *is a group homomorphism.*

The map P and its homomorphic property have been fundamentally used in [5] in a more general context of one-resonant maps. The kernel of P is the set of maps F of the form

$$F(z) = \left(z_1\varphi(p), \frac{z_2}{\varphi(p)}\right) \text{ or } \left(z_2\varphi(p), \frac{z_1}{\varphi(p)}\right),$$

and $\mathfrak{C} \cap \ker P$ is abelian.

Define the "lifting" homomorphism map $H : \mathfrak{G}_1 \to \mathfrak{C}$ by

$$H(\chi)(z) = \left(\frac{z_1\chi(p)}{p}, z_2\right), \tag{2.8}$$

whose basic property is $P(H(\chi)) = \chi$. In particular, the restriction $P|\mathfrak{C}$ is surjective. We also consider the "lifting" to the second argument given by

$$(JH(\chi)J)(z) = \left(z_1, \frac{z_2\chi(p)}{p}\right)$$

that also has the basic property $P(JH(\chi)J) = \chi$.

We denote by $\mathfrak{F}_1^\times$ the multiplicative group of the ring of formal power series $\varphi(t) = a_0 + a_1 t + \cdots$ in one variable having $a_0 \neq 0$, (in which the group operation corresponds to convolution on the coefficients, the formal equivalent of pointwise multiplication, as opposed to composition). The map

$$\Phi : \begin{cases} \mathfrak{F}_1^\times \to \mathfrak{C}, \\ \varphi \mapsto \left(z_1\varphi(p), \dfrac{z_2}{\varphi(p)}\right), \end{cases}$$

is a group isomorphism onto its image, which is equal to $\mathfrak{C} \cap \ker P$.

We note that each $F \in \mathfrak{C}$ has a unique factorization in the form $H(\rho)\Phi(\psi)$, and another in the form $JH(\rho)J\Phi(\varphi)$. However, $\mathfrak{C}$ is not the *direct* product of $\operatorname{im} H$ and $\operatorname{im}\Phi$.

3. Centralisers in $\mathfrak{E}$

3.1. Centralisers in $\mathfrak{C}$

A routine calculation gives:

Lemma 3.1. *Let $F_j(z) = (z_1\varphi_j(p), z_2\psi_j(p))$, and $\rho_j = P(F_j)$, for $j = 1, 2$. Then*

$$F_1(F_2(z)) = (z_1\varphi_2(p)\varphi_1(\rho_2(p)), z_2\psi_2(p)\psi_1(\rho_2(p))). \tag{3.1}$$

This immediately yields:

Lemma 3.2. *Let $F_j(z) = (z_1\varphi_j(p), z_2\psi_j(p))$, and $\rho_j = P(F_j)$, for $j = 1, 2$. Then $F_1F_2 = F_2F_1$ if and only if*

$$\begin{cases} \varphi_2(p)\varphi_1(\rho_2(p)) = \varphi_1(p)\varphi_2(\rho_1(p)) \\ \psi_2(p)\psi_1(\rho_2(p)) = \psi_1(p)\psi_2(\rho_1(p)) \end{cases}. \tag{3.2}$$

Lemma 3.3. *Let $F_1 \in \operatorname{im}\Phi$ $(= \mathfrak{C} \cap \ker P)$ and $F_2 \in \mathfrak{C}$. Suppose $F_1F_2 = F_2F_1$. Then either F_1 is linear* (*and hence is in the group D of matrices* (2.4)) *or $P(F_2)$ has finite order.*

The proof is based on the following useful fact in one variable:

Lemma 3.4. *Let $\varphi(t)$ be a formal power series in one variable that is invariant under a formal change of variables $t' = \rho(t) \in \mathfrak{G}_1$ with $\rho(0) = 0$, i.e.*

$$\varphi(\rho(t)) = \varphi(t). \tag{3.3}$$

Then either $\varphi = \text{const}$ or ρ has finite order.

Proof. Suppose that $\varphi \neq \text{const}$, *i.e.*

$$\varphi(t) = \varphi(0) + \alpha t^k + \sum_{j=1}^{\infty} \alpha_j t^{k+j}, \quad \alpha \neq 0.$$

Since $\rho(t) = ct + \cdots$, comparing the coefficients of t^k in (3.3) gives $c^k = 1$. Then, replacing ρ with ρ^k, we may assume $c = 1$. It clearly suffices to show that $\rho = \mathsf{id}$. Assuming the contrary, we have

$$\rho(t) = t(1 + \beta t^r + \text{HOT}), \quad \beta \neq 0,$$

where HOT denotes terms of degree greater than r. Now

$$\begin{aligned}\varphi(\rho(t)) &= 1 + \alpha\rho(t)^k + \sum_{j=1}^{\infty} \alpha_j \rho(t)^{k+j} \\ &= 1 + \alpha t^k(1 + k\beta t^r) + \sum_{j=1}^{r} \alpha_j t^{k+j} + \text{HOT}\end{aligned}$$

where HOT denotes terms of degree greater than $k + r$, so comparing coefficients of t^{k+r} in (3.3), we get $\alpha\beta k = 0$, a contradiction. $\square$

Proof of Lemma 3.3. Replacing F_2 by F_2^2 if necessary, we may assume that $F_2 \in \mathfrak{C}$. In the notation of Lemma 3.2 we have $\rho_1(t) = t$ and therefore the first identity in (3.2) implies $\varphi_1(\rho_2(t)) = \varphi_1(t)$. Then by Lemma 3.4, either $\varphi_1 = \text{const}$ and then $F_1 \in D$ or $\rho_2 = P(F_2)$ has finite order. $\square$

Corollary 3.5. *If commuting elements $F_1 \in \operatorname{im}\Phi$ and $F_2 \in \mathfrak{C}$ are tangent to the identity, then $F_1 = \mathsf{id}$ or $F_2 \in \operatorname{im}\Phi$. In fact, it suffices to assume that F_1 and $P(F_2)$ are tangent to the identity.*

Proof. By Lemma 3.3, either F_1 is linear and hence $F_1 = \mathsf{id}$ or $P(F_2)$ has finite order and hence is the identity. $\square$

Corollary 3.6. *The centraliser of* $\operatorname{im}\Phi$ *in* $\mathfrak{C}$ *is* $\operatorname{im}\Phi$.

Proof. Recall that $\mathfrak{E} = \mathfrak{C} \cup \mathfrak{R}$ (see Section 2.6 for the notation). Suppose $F \in \mathfrak{E}$ commutes with all elements of $\operatorname{im}\Phi$. If $F \in \mathfrak{R}$, then it does not commute with, for instance, $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, so F must belong to $\mathfrak{C}$, in particular, $\Lambda := L(F) \in D$. Since Λ commutes with each element of $\operatorname{im}\Phi$, it follows that $F_2 = \Lambda^{-1}F$ commutes with each element of $\operatorname{im}\Phi$, and is tangent to the identity. Taking F_1 to be any element of $\operatorname{im}\Phi$ that is tangent to the identity but not equal to the identity, and applying the last corollary, we conclude that $F_2 \in \operatorname{im}\Phi$. □

Corollary 3.7. $\operatorname{im}\Phi$ *is a maximal Abelian subgroup of* $\mathfrak{E}$.

4. Reversibility in $\mathfrak{E}$

Involutions are trivially reversible (by the identity), in any group. By Lemma 2.4, each involution in $\mathfrak{C}$ is conjugate to $(\pm z_1, \pm z_2)$, and each proper involution in $\mathfrak{R}$ is conjugate to $\pm(z_2, z_1)$. It is not altogether clear what other elements of $\mathfrak{R}$ are reversible in $\mathfrak{E}$, but we do not need to know this, for our purposes. We concentrate on describing the elements of $\mathfrak{C}$ reversible in $\mathfrak{E}$.

4.1. Elements of $\operatorname{im}\Phi$

Lemma 4.1. *Each element of* $\operatorname{im}\Phi$ *is strongly-reversible in* $\mathfrak{E}$.

Proof. Such maps are obviously reversed by the involution J. □

4.2. Elements of $\mathfrak{C}$

We remark that if $F \in \mathfrak{C}$ is reversible, then $\det(L(F)) = \pm 1$, so $P(F)(t) = \pm t + \cdots$. Thus $L(F)$ takes the form $\begin{pmatrix} \pm\mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ for some nonzero $\mu \in \mathbb{C}$, so $L(F^2) \in D$.

Theorem 4.2. *Let* $F \in \mathfrak{C}$ *be reversed by* $\Theta \in \mathfrak{E}$, *i.e.*

$$\Theta^{-1}F\Theta = F^{-1}. \tag{4.1}$$

Then either $P(F)$ *has finite order or some power of* Θ *belongs to the matrix group* D.

We use the following simple relation between reversers and commuting maps:

Lemma 4.3. *If* F *is reversed by* Θ, *then* F *is reversed by any* Θ^m *with* $m \in \mathbb{Z}$ *odd and commutes with any* Θ^m *with* $m \in \mathbb{Z}$ *even*.

Proof. The statement is straightforward for $m \geq 0$. Taking the inverse of (4.1) yields $\Theta^{-1}F^{-1}\Theta = F$ showing the statement for $m = -1$ and therefore for any $m < 0$. □

Proof of Theorem 4.2. Replacing F by F^2, if need be, we may assume $L(F) \in D$. Suppose $P(F)$ has infinite order. Since $P(F)$ (which takes the form $t + \cdots$) is reversed by $P(\Theta)$, it follows from Corollary 2.3 that $P(\Theta)$ has finite order, so we may choose $k \in \mathbb{N}$ such that $\Theta^{2k} \in \operatorname{im}\Phi$. Since Θ^{2k} also commutes with F by Lemma 4.3, we may apply Lemma 3.3 with $F_1 = \Theta^{2k}$ and $F_2 = F$, and conclude that $\Theta^{2k} \in D$. $\square$

Corollary 4.4. *Suppose $F \in \mathfrak{C}$ has linear part $L(F) \in D$, with $L(F) \neq \pm\mathsf{id}$, and is reversible in $\mathfrak{E}$. Then* (1) *each reverser of F lies in $\mathfrak{R}$, and* (2) *F may be reversed in $\mathfrak{E}$ by some element of finite order.*

Proof. Suppose $\Theta \in \mathfrak{E}$ reverses F. Then $L(\Theta)$ reverses $L(F)$, which takes the form (2.4), with $\mu \neq \pm 1$. Thus $L(\Theta)$ has to interchange the eigenvectors of μ and $1/\mu$, and must take the form (az_2, bz_1) for some nonzero a and b. In particular, the reverser cannot belong to $\mathfrak{C}$, so part (1) is proved. Composing with an element of D, we may assume that the reverser Θ has linear part (cz_2, cz_1). Then Theorem 4.2 tells us that either $P(F)$ has finite order or some power $\Theta^{2k} \in D$.

In the first case, $P(F)$ is linearizable by Lemma 2.1. Furthermore, since $L(F) \in D$, the map $P(F)$ is also tangent to the identity. Hence we must have $P(F) = \mathsf{id}$ and therefore $F \in \operatorname{im}\Phi$, which is reversed by J.

In the second case, $\Theta^{2k} = L(\Theta)^{2k} = c^{2k} \cdot \mathsf{id}$, but the only multiples of the identity belonging to D are $\pm\mathsf{id}$, so Θ has order at most $4k$. $\square$

Corollary 4.5. *Suppose $F \in \mathfrak{C}$ has linear part $L(F) \in D$ with $L(F) \neq \pm\mathsf{id}$, and is reversible in $\mathfrak{E}$. Then either F is linear or each reverser of F in $\mathfrak{E}$ has finite order.*

Proof. We have seen this in the second case of Proof of Corollary 4.4. In the first case, $F \in \operatorname{im}\Phi$ is reversed by J, so each other reverser of F takes the form JG, where G commutes with F, and G belongs to $\mathfrak{C}$ since $L(F)$ has two different eigenvalues. Then for $\Lambda := L(G) \in D$, the map $\Lambda^{-1}G$ is tangent to the identity and commutes with $L(F)^{-1}F$. If $F \neq L(F)$, then Corollary 3.5 tells us that $\Lambda^{-1}G \in \operatorname{im}\Phi$, hence $G \in \operatorname{im}\Phi$, so $(JG)^2 = JGG^{-1}J = \mathsf{id}$. $\square$

We draw a further corollary from the proof of Corollary 4.4:

Corollary 4.6. *Suppose $F \in \mathfrak{C}$ is reversible in $\mathfrak{E}$ and has linear part $L(F) \neq \pm\mathsf{id}$. Then F is conjugate in $\mathfrak{E}$ to a map reversed by one of the maps $J_c(z) = c\tilde{z}$, where c is a root of unity.*

Proof. In the first case, we may take $c = 1$. In the second case, multiplying by a suitable element $\Lambda \in D$, we see that Θ can be assumed to be of finite order and to satisfy $L(\Theta) = J_c$, where c is a root of unity. The statement now follows from Lemma 2.1. $\square$

5. Maps reversed by $J_c(z) = c\tilde{z}$

Here we fix one of the linear maps $J_c \in \mathfrak{R}$ identified in the last subsection, and we describe all the elements of $\mathfrak{C}$ that it reverses. We assume that c is a root of unity, and we set $\omega = \bar{c}^2$. Let k be the least natural number with $\omega^{2k} = 1$ (*i.e.* k is the order of ω^2).

5.1 If a map Θ reverses two commuting maps F and G, then it also reverses their composition FG. Thus if J_c reverses a map $F \in \mathfrak{C}$ having linear part Λ, then, since J_c reverses Λ, it also reverses the map $G = \Lambda^{-1}F$, which is tangent to the identity. Thus each $F \in \mathfrak{C}$ reversed by J_c factors as ΛG, where $G \in \mathfrak{C}$ is tangent to the identity, and is reversed by J_c.

5.2 If J_c reverses an $F \in \mathfrak{C}$ then $\rho = P(F)$ is reversed by $c^2 t = \bar{\omega} t$ (and hence by the inverse ωt, see Lemma 4.3), *i.e.*

$$\omega^{-1}\rho(\omega\rho(t)) = t. \tag{5.1}$$

In particular, Lemma 4.3 implies

$$\rho(\omega^2 t) = \omega^2 \rho(t). \tag{5.2}$$

Also, reversibility implies $\rho(t) = \pm t + \text{HOT}$ (see Subsection 2.4). Furthermore, since J_c interchanges the eigenspaces of $L(F)$, we must have $L(F) \in D$ and therefore

$$\rho(t) = t + \text{HOT}. \tag{5.3}$$

5.3 Consider an arbitrary $F \in \mathfrak{C}$, with $L(F) \in D$, of the form $F = (z_1\varphi(p), z_2\psi(p))$, with $\rho = P(F)$. For any complex c, we calculate

$$(F J_c^{-1} F J_c)(z)$$
$$= \Big(z_1 \cdot \psi(c^2 p) \cdot \varphi\left(p \cdot \varphi(c^2 p) \cdot \psi(c^2 p)\right), z_2 \cdot \varphi(c^2 p) \cdot \psi\left(p \cdot \varphi(c^2 p) \cdot \psi(c^2 p)\right)\Big).$$

Thus J_c reverses F if and only if

$$\begin{cases} \psi(c^2 t) \cdot \varphi\left(t\varphi(c^2 t)\psi(c^2 t)\right) = 1, \\ \varphi(c^2 t) \cdot \psi\left(t\varphi(c^2 t)\psi(c^2 t)\right) = 1, \end{cases} \tag{5.4}$$

or, equivalently,

$$\begin{cases} \psi(c^2 t) \cdot \varphi\left(\omega\rho(c^2 t)\right) = 1, \\ \varphi(c^2 t) \cdot \psi\left(\omega\rho(c^2 t)\right) = 1, \end{cases} \tag{5.5}$$

where as before $\rho(t) = t\varphi(t)\psi(t)$.

Now define

$$\sigma(t) := \omega\rho(t), \tag{5.6}$$

which satisfies

$$\sigma(t) = \omega t + \text{HOT} \tag{5.7}$$

in view of (5.3). Then the reversibility equation (5.1) for ρ becomes

$$\sigma^2(t) = \omega^2 t, \tag{5.8}$$

and the reversibility equations (5.5) for F become

$$\begin{cases} \psi(t) \cdot \varphi(\sigma(t)) = 1, \\ \varphi(t) \cdot \psi(\sigma(t)) = 1, \end{cases} \tag{5.9}$$

where we replaced $c^2 t$ by t. Denoting

$$g(t) := \frac{\varphi(t)}{\psi(t)}, \tag{5.10}$$

we obtain from (5.9) that

$$g(\sigma(t)) = \frac{\varphi(\sigma(t))}{\psi(\sigma(t))} = \frac{\varphi(t)}{\psi(t)} = g(t). \tag{5.11}$$

In view of (5.8), it follows that

$$g(\omega^2 t) = g(\sigma^2(t)) = g(t). \tag{5.12}$$

Equation (5.8) admits two possibilities, *a priori*:

I: σ may be the linear map $\sigma(t) = \omega t$. In this case, equation (5.6) yields $\psi(t) = 1/\varphi(t)$, so $F \in \operatorname{im}\Phi$. Equations (5.9) then yield $\varphi(\omega t) = \varphi(t)$, *i.e.* $\varphi(t)$ takes the form $\varphi_1(t^k)$, where k is the order of ω.

Conversely, J_c reverses $F(z) = (z_1\varphi(p), z_2/\varphi(p))$ whenever $\varphi(c^2 t) = \varphi(t)$, *i.e.* $\varphi(t)$ is a function of t^k and the order of c divides $2k$.

We note that each of these F's is reversed by the involution $J = J_1$.

II: The more interesting possibility is that σ is a nonlinear solution of (5.8). Since ω is a root of unity, σ has finite even order. Then Lemma 2.1 implies the existence of some $h(t) = t + \cdots$ such that $\sigma^h(t) = \omega t$, where σ^h denotes the conjugate $h^{-1}\sigma h$, *i.e.* $h^{-1}(\sigma(h(t))) = \omega t$ or, equivalently,

$$h(\omega t) = \sigma(h(t)). \tag{5.13}$$

Since $\sigma(\omega^2 t) = \omega^2\sigma(t)$, it follows from formula (2.1) that $h(\omega^2 t) = \omega^2 h(t)$. Furthermore, setting $g_1(t) = g(h(t))$ we obtain, using (5.13) and (5.11):

$$g_1(\omega t) = g(h(\omega t)) = g(\sigma(h(t))) = g(h(t)) = g_1(t). \tag{5.14}$$

The equations

$$\begin{cases} \varphi(t)\psi(t) = \dfrac{\rho(t)}{t}, \\ \dfrac{\varphi(t)}{\psi(t)} = g(t), \end{cases}$$

are clearly equivalent to

$$\begin{aligned} \varphi(t) &= \left(\frac{\rho(t)g(t)}{t}\right)^{\frac{1}{2}}, \\ \psi(t) &= \left(\frac{\rho(t)}{tg(t)}\right)^{\frac{1}{2}}, \end{aligned} \tag{5.15}$$

where the branches of the square roots are chosen to make

$$\varphi(0) = \frac{1}{\psi(0)} = \lambda \tag{5.16}$$

the first eigenvalue of F. It follows from (5.15), (5.2) and (5.12) that

$$\varphi(\omega^2 t) = \varphi(t), \quad \psi(\omega^2 t) = \psi(t), \tag{5.17}$$

so that these functions, also, depend only on t^k.

Conversely, suppose c is a $4k$-th root of unity, for some k, and take any invertible $h \in \mathfrak{G}_1$ with $h(t) = t + \text{HOT}$ and $h(\omega^2 t) = \omega^2 h(t)$, *i.e.* h is any power series $h(t) = t(1 + \sum_{j\geq 1} h_j t^{kj})$. Define

$$\sigma(t) = h(\omega h^{-1}(t)), \quad \rho(t) = \omega^{-1}\sigma(t) = \omega^{-1}h(\omega h^{-1}(t)), \tag{5.18}$$

in particular, $\sigma^h(t) = \omega t$. Then clearly both σ and ρ commute with ω^2, and

$$\sigma^2(t) = h(\omega h^{-1}(h(\omega h^{-1}(t)))) = \omega^2 t,$$

i.e. (5.8) holds, which is equivalent to ρ being reversed by ωt. Note that σ has order dividing $2k$. Take any $\lambda \neq 0$ and $g_1(t) = \lambda^2 + \text{HOT}$ satisfying $g_1(\omega t) = g_1(t)$, and define

$$g(t) = g_1(h^{-1}(t)), \tag{5.19}$$

so (5.14) holds. Since h and g_1 commute with ωt, also g does, *i.e.* $g(\omega^2 t) = g(t)$. Furthermore,

$$g(\sigma(t)) = g_1((h^{-1}(h(\omega h^{-1}(t))))) = g(t).$$

Finally define holomorphic germs φ and ψ by (5.15). Then

$$\psi(t)\varphi(\sigma(t)) = \left(\frac{\omega^{-1}\sigma^2(t)}{\sigma(t)}\right)^{\frac{1}{2}} \left(\frac{\omega^{-1}\sigma(t)}{t}\right)^{\frac{1}{2}} = 1,$$

so the first equation in (5.9) holds. The second equation follows from the first one, using (5.10) and (5.11), and may also be verified by direct calculation. Thus J_c reverses F.

Example 5.1. An explicit example of F in case II, reversed by $J_i(z) = i\widetilde{z}$, and corresponding to the choice

$$h(t) = \frac{t}{1-t}, \quad g_1(t) = g(t) = \lambda^2,$$

and hence

$$\rho(t) = -\sigma(t) = \frac{t}{1+2t}, \quad \varphi(t) = \frac{\lambda}{(1+2t)^{1/2}}, \quad \psi(t) = \frac{\lambda^{-1}}{(1+2t)^{1/2}},$$

is given by

$$F(z_1, z_2) = \left(\frac{\lambda z_1}{(1+2z_1z_2)^{1/2}}, \frac{\lambda^{-1}z_2}{(1+2z_1z_2)^{1/2}}\right).$$

Alternatively, choosing

$$g_1(t) = \frac{4\lambda^2}{1+t},$$

and hence

$$g(t) = g_1(h^{-1}(t)) = \lambda^2\frac{(1+t)^2}{1+2t}, \quad \varphi(t) = \lambda\frac{1+t}{1+2t}, \quad \psi(t) = \lambda^{-1}\frac{1}{1+t},$$

we obtain an example of a rational map

$$F(z_1, z_2) = \left(\frac{\lambda(1+z_1z_2)z_1}{(1+2z_1z_2)}, \frac{z_2}{\lambda(1+z_1z_2)}\right).$$

These maps are not reversed by J or any other involution. Indeed, if an involution T reverses F, then $L(T) = \pm J$ and hence $P(T)(t) = t + \mathrm{HOT}$ and therefore $P(T)(t) = \mathsf{id}$ since T is an involution. But then $P(T)$ cannot reverse $P(F)$.

Assembling the cases, we have identified all the series $F \in \mathfrak{C}$ reversed by a given J_c, and we can state the following:

Theorem 5.2.

(1) *Let $c^{2k} = 1$. Then each $F \in \operatorname{im}\Phi$ reversed by J_c takes the form*

$$(z_1\varphi(p), z_2/\varphi(p)),$$

where $\varphi(c^2 t) = \varphi(t)$.

(2) *Let $c^{4k} = 1$, and $\omega = \bar{c}^2$. Then each $F \in \mathfrak{C}$ with $L(F) = \mathsf{diag}(\lambda, \lambda^{-1})$ that is reversed by J_c and does not belong to* $\operatorname{im}\Phi$ *takes the form $(z_1\varphi(p), z_2\psi(p))$, where φ and ψ are defined by the equations* (5.15) *and* (5.16)*, ρ is defined by* (5.18) *for some $h \in \mathfrak{G}_1$ tangent to the identity that satisfies $h(\omega^2 t) = \omega^2 h(t)$, and g is defined by* (5.19) *for some $g_1(t) = \lambda^2 +$* HOT *that satisfies $g_1(\omega t) = g_1(t)$. In particular, both φ and ψ depend only on t^k. Moreover all these maps are reversed by J_c.*

In part (2), the map F may also be written with the notation of Subsection 2.7 in the form

$$F(z) = JH(\rho)J\Phi(\varphi) = H(\rho)\Phi(1/\psi), \tag{5.20}$$

where $\varphi(\omega^2 t) = \varphi(t)$ by (5.17), and $\rho = P(F)$ is reversed by $t \mapsto \omega t$ (see Subsection 5), so that both $\varphi(t)$ and $\rho(t)/t$ depend only on t^k. However, such maps F are not reversed by J_c in general, unless φ is constructed as above from h and some g_1.

6. Reversibility in $\mathfrak{G}$

6.1. Resonances

Lemma 6.1. *If $F \in \mathfrak{G}$ is reversible in $\mathfrak{G}$, and has an eigenvalue that is not a root of unity, then F is conjugate to some element of $\mathfrak{C}$ having $L(F) \in D$ of infinite order. Moreover, the conjugating map can be chosen to be tangent to the identity.*

Proof. By a linear conjugation, we may convert $L(F)$ to the form $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in D$. Since λ is not a root of unity, the only resonance relations are of the form

$$\lambda = \lambda^{k+1}(1/\lambda)^k, \quad 1/\lambda = \lambda^k(1/\lambda)^{k+1},$$

so the Poincaré-Dulac Theorem [7, Section 4.8, Theorem 4.22] tells us that F may be conjugated to the resonant form (2.5). □

Lemma 6.2. *Suppose that $F \in \mathfrak{G}$ has $L(F) \in D$, with $L(F) \neq \pm\mathsf{id}$, and is reversible. Then each reverser $\Theta \in \mathfrak{G}$ has linear part of the form $L(\Theta)(z_1, z_2) = (az_2, bz_1)$. Also, it is possible to choose a reverser with linear part J_c.*

Proof. The first assertion follows from the fact that $L(\Theta)$ must interchange the eigenspaces of $L(F)$. In view of Lemma 6.1, we may assume that $F \in \mathfrak{C}$. In general, the composition of a reverser of F and an element of the centraliser $C_F(\mathfrak{G})$ is another reverser of F. Since $D \le C_F$, we may compose Θ with an element of D to convert its linear part to the form J_c. □

6.2. Terminology

It is usual to say that a map is in *resonant form* if the homogeneous terms in its expansion commute with the linear part. For maps F that belong to $\mathfrak{G}$ and have $L(F) \in D$ of infinite order, this just means that they belong to $\mathfrak{C}$, and it is independent of the particular F. For maps with $L(F)$ of finite order, resonance amounts to a less restrictive condition. Since we are concentrating on the generic case, we shall use the term *D-resonant map* to mean an element of $\mathfrak{C}$. Similarly, we shall refer to all the elements of $\mathfrak{R}$ as *D-inverse-resonant maps*.

More generally, we extend this terminology to maps that may not be invertible:

Definition 6.3. A formal power series map $G\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is called *D-resonant* (respectively, *D-inverse-resonant*) if $G \circ M = M \circ G$ (respectively, $G \circ M = M^{-1} \circ G$), whenever $M \in D$.

Remark 6.4. So a series $G\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with $L(G) \in D$ of infinite order is D-resonant (respectively D-inverse-resonant) if and only if it is the sum of monomials $p^k az$ (respectively $p^k a\widetilde{z}$), where a is some diagonal matrix, $p = z_1z_2$ and $\widetilde{z} = (z_2, z_1)$.

We have the following obvious properties:

Lemma 6.5. *Let G_1, G_2 be D-resonant and H_1, H_2 be D-inverse-resonant. Then $G_1 \circ G_2$ and $H_1 \circ H_2$ are D-resonant, whereas $G_1 \circ H_1$ and $H_1 \circ G_1$ are D-inverse-resonant.*

6.3. Reversers of resonant maps

Proposition 6.6. *Suppose that $F \in \mathfrak{C}$, with $\Lambda = L(F)$ of infinite order, and F is reversed by some $\Theta \in \mathfrak{G}$. Then $\Theta \in \mathfrak{R}$.*

Proof. By Lemma 6.2, the linear part of Θ belongs to $\mathfrak{R}$. Assume by induction that all the terms of order less than k in the expansion of Θ are D-inverse-resonant. Identifying homogeneous components of order k in the basic reversibility equation

$$F \circ \Theta \circ F = \Theta, \tag{6.1}$$

we obtain an identity $\Lambda L_k(\Theta)(\Lambda z) = L_k(\Theta)(z) + \dots$ (using the notation introduced in Subsection 2.1), where the dots contain expressions involving only $L_m(\Theta)$ with $m < k$. It then follows from the inductive assumption and Lemma 6.5 that these terms are D-inverse-resonant. Hence $\Lambda L_k(\Theta)(\Lambda z) - L_k(\Theta)(z)$ is D-inverse-resonant, which is only possible when $L_k(\Theta)$ is D-inverse-resonant, as is readily seen by using the fact that Λ has infinite order. The proof is complete. □

Combining Lemmas 6.1 and 6.6, we have:

Theorem 6.7. *Let $F \in \mathfrak{G}$ and suppose $L(F)$ has an eigenvalue that is not a root of unity. Then F is reversible in $\mathfrak{G}$ if and only if it is conjugate in $\mathfrak{G}$ to some D-resonant element $G \in \mathfrak{C}$, having $L(G) \in D$, that is reversed by some D-inverse-resonant element $\Theta \in \mathfrak{R}$.*

7. Conjugacy classes of reversibles

By Theorem 6.7, each generic reversible of $\mathfrak{G}$ is conjugate in $\mathfrak{G}$ to a map of the form ΛF, where $\Lambda \in D$ and $F \in \mathfrak{C}$ is tangent to the identity and is reversible in $\mathfrak{C}$. By Corollary 4.4, ΛF, and hence F, may be reversed by some element of finite order in $\mathfrak{R}$, and by a further conjugation (using an element of $\mathfrak{C}$, which does not disturb the factorization ΛF), we may arrange that F is reversed by some linear map, which may be taken to be a J_c, for some root of unity c. By Theorem 5.2, F is of one of two kinds. Now we turn to the question of cataloging the conjugacy classes in $\mathfrak{G}$ of the maps of these two kinds.

7.1 First we consider their conjugacy classes in $\mathfrak{C}$. The key idea is to use the conjugacy actions induced by the homomorphisms $H : \mathfrak{G}_1 \to \mathfrak{C}$ and $\Phi : \mathfrak{F}_1^\times \to \mathfrak{C}$ on the group $\mathfrak{C}$ (as introduced in Subsection 2.7).

Consider general $F = M(\varphi, \psi) \in \mathfrak{C}$ reversible or not, given by (2.5). Let $\rho = P(F)$. For $\chi \in \mathfrak{G}_1$, letting $K_1 = H(\chi)$ and $K_2 = JH(\chi)J$, we calculate

$$K_1^{-1} F K_1(z) = \left(\frac{z_1 \rho^\chi(p)}{p\psi(\chi(p))}, z_2\psi(\chi(p)) \right), \tag{7.1}$$

and, similarly,

$$K_2^{-1} F K_2(z) = \left(z_1\varphi(\chi(p)), \frac{z_2 \rho^\chi(p)}{p\varphi(\chi(p))} \right). \tag{7.2}$$

Also, for any $\varphi \in \mathfrak{F}_1^\times$, we calculate

$$\Phi(\varphi_1)^{-1} F \Phi(\varphi_1)(z) = \left(\frac{z_1\varphi(p)\varphi_1(p)}{\varphi_1(\rho(p))}, \frac{z_2\psi(p)\varphi_1(\rho(p))}{\varphi_1(p)} \right). \tag{7.3}$$

Now consider the two kinds of reversible $F \in \mathfrak{C}$, tangent to, but not equal to, the identity, reversed by J_c, as in Theorem 5.2:

I: $F \in \operatorname{im}\Phi$, so $\rho = \mathsf{id}$.

We have $\varphi(t) = 1 + \alpha t^k + \text{HOT}$, for some $k \in \mathbb{N}$ and $\alpha \neq 0$. Then we may choose $\chi \in \mathfrak{G}_1$ so that $\varphi(\chi(t)) = 1 + t^k$, and then

$$K_2^{-1} F K_2(z) = \left(z_1(1 + p^k), \frac{z_2}{(1 + p^k)} \right). \tag{7.4}$$

Thus F is represented, up to conjugacy in $\mathfrak{C}$, by one of the maps (7.4), for some $k \in \mathbb{N}$.

II: $\rho \neq \mathsf{id}$. F takes the form

$$\left(z_1\varphi(p), \frac{z_2\rho(p)}{p\varphi(p)}\right) = JH(\rho)J\Phi(\varphi), \tag{7.5}$$

for some reversible $\rho \in \mathfrak{G}_1$, with $\rho = t + \text{HOT}$, but $\rho \neq \mathsf{id}$, (reversed by ωt, where $\omega^k = -1$, for some $k \in \mathbb{N}$) and some $\varphi(t)$ that depends only on t^k (see Subsection 5). Then [12] ρ is conjugate to

$$f_k(t) = \frac{t}{(1-kt^k)^{\frac{1}{k}}} = t + t^{k+1} + \binom{k+1}{2} t^{2k+1} + \text{HOT}, \tag{7.6}$$

which is also reversed by all odd powers of ωt (where $\omega = \bar{c}^2$, as before). Let $\chi \in \mathfrak{G}_1$ conjugate ρ to f_k. Then conjugating F by $JH(\chi)J$, we may assume that $\rho = f_k$. Then the formal iterate ρ^α is given by

$$\rho^\alpha(t) = f_k^\alpha(t) = \frac{t}{(1-k\alpha t^k)^{\frac{1}{k}}} = t\left(1 + \alpha t^k + \text{HOT}\right), \tag{7.7}$$

whenever $\alpha \in \mathbb{C}$.

Choose $\alpha \in \mathbb{C}$ such that $\varphi(t) = 1 + \alpha t^k + \text{HOT}$ (note that F is tangent to the identity). Then $\frac{\rho^\alpha(t)}{\varphi(t)} = t(1+\text{HOT})$, where the higher terms involve only monomials $b_j t^{jk}$ with $j \geq 2$, so we may choose $\varphi_1(t)$, depending only on t^k, such that

$$\frac{\varphi_1(t)}{\varphi_1(\rho(t))} = \frac{\rho^\alpha(t)}{t\varphi(t)}.$$

The latter fact follows by writing $\varphi_1(t) = 1 + \sum a_j t^{jk}$, expanding the formula

$$1 + \sum a_j t^{jk} = \left(1 + \sum_{j\geq 2} b_j t^{jk}\right)\left(1 + \sum a_j (t(1 + t^k + \text{HOT}))^{jk}\right), \tag{7.8}$$

identifying coefficients of t^{jk} and solving inductively for a_{j-1}.

Then the conjugation (7.3) converts F to the form

$$F = \left(\frac{z_1\rho^\alpha(p)}{p}, \frac{z_2\rho(p)}{\rho^\alpha(p)}\right), \tag{7.9}$$

and this is reversed by

$$\Phi(\varphi_1)^{-1} J_c \Phi(\varphi_1) = J_c \Phi(\varphi_1^2).$$

We calculate (using the fact that $z \mapsto \omega z$ reverses ρ^α)

$$\begin{aligned}
F^{J_c}(z) &= J_{\bar{c}} F(cz_2, cz_1) \\
&= J_{\bar{c}} \left(\frac{cz_2 \rho^\alpha(\bar{\omega} p)}{\bar{\omega} p}, \frac{cz_1 \rho(\bar{\omega} p)}{\rho^\alpha(\bar{\omega} p)} \right) \\
&= \left(\frac{cz_2 \bar{\omega} \rho^{-\alpha}(p)}{\bar{\omega} p}, \frac{cz_1 \bar{\omega} \rho^{-1}(p)}{\bar{\omega} \rho^{-\alpha}(p)} \right) \\
&= \left(\frac{z_1 \rho^{-1}(p)}{\rho^{-\alpha}(p)}, \frac{z_2 \rho^{-\alpha}(p)}{p} \right), \\
H(\rho) \Phi\left(\frac{\rho^\alpha(t)}{\rho(t)} \right)(z) &= H(\rho) \left(\frac{z_1 \rho^\alpha(p)}{\rho(p)}, \frac{z_2 \rho(p)}{\rho^\alpha(p)} \right) \\
&= \left(\frac{z_1 \rho^\alpha(p)}{\rho(p)} \frac{\rho(p)}{p}, \frac{z_2 \rho(p)}{\rho^\alpha(p)} \right) = F(z), \\
F^{-1}(z) &= \Phi\left(\frac{\rho^\alpha(t)}{\rho(t)} \right)^{-1} H(\rho)^{-1}(z) \\
&= \Phi\left(\frac{\rho(t)}{\rho^\alpha(t)} \right) H(\rho^{-1})(z) \\
&= \Phi\left(\frac{\rho(t)}{\rho^\alpha(t)} \right) \left(\frac{z_1 \rho^{-1}(p)}{p}, z_2 \right) \\
&= \left(\frac{z_1 \rho^{-1}(p)}{p} \cdot \frac{\rho(\rho^{-1}(p)}{\rho^\alpha(\rho^{-1}(p))}, \frac{z_2 \rho^\alpha(\rho^{-1}(p))}{\rho(\rho^{-1}(p))} \right) \\
&= \left(\frac{z_1 \rho^{-1}(p)}{\rho^{\alpha-1}(p)}, \frac{z_2 \rho^{\alpha-1}(p)}{p} \right).
\end{aligned}$$

But conjugation of $M(\varphi, \psi)$ by $\Phi(\varphi_1)$, and hence by $\Phi(\varphi_1^2)$, does not change the coefficient of t^k in φ, so comparing this coefficient in the maps F^{J_c} and F^{-1}, we obtain $\alpha - 1 = -\alpha$, or $\alpha = \frac{1}{2}$. Thus F takes the form

$$F(z) = \left(\frac{z_1 \rho^{\frac{1}{2}}(p)}{p}, \frac{z_2 \rho(p)}{\rho^{\frac{1}{2}}(p)} \right). \tag{7.10}$$

Conjugating with $\nu(z) = \widetilde{z}$ we obtain

$$F(z) = \left(\frac{z_1 \rho(p)}{\rho^{\frac{1}{2}}(p)}, \frac{z_2 \rho^{\frac{1}{2}}(p)}{p} \right) = \left(\frac{(1 - \frac{k}{2} p^k)^{\frac{1}{k}}}{(1 - k p^k)^{\frac{1}{k}}} z_1, \frac{1}{(1 - \frac{k}{2} p^k)^{\frac{1}{k}}} z_2 \right). \tag{7.11}$$

Conjugating further by a suitable scaling, $\widetilde{F}$ takes the form

$$F(z) = \left(\frac{(1+p^k)^{\frac{1}{k}}}{(1+2p^k)^{\frac{1}{k}}} z_1, \frac{1}{(1+p^k)^{\frac{1}{k}}} z_2 \right). \tag{7.12}$$

The alternative forms

$$\left(\frac{(1-\frac{1}{2}p^k)}{(1-kp^k)^{\frac{1}{k}}} z_1, \frac{1}{(1-\frac{1}{2}p^k)} z_2 \right), \quad \left(\frac{1}{(1-kp^k)^{\frac{1}{k}}(1+\frac{1}{2}p^k)} z_1, \left(1+\frac{1}{2}p^k\right) z_2 \right) \tag{7.13}$$

may be obtained from (7.11) by a conjugation of the type (7.3) with φ_1 satisfying respectively

$$\frac{\varphi_1(t)}{\varphi_1(\rho(t))} = \frac{(1-\frac{1}{2}p^k)}{(1-\frac{k}{2}p^k)^{1/k}} \quad \text{or} \quad \frac{\varphi_1(t)}{\varphi_1(\rho(t))} = \frac{1}{(1-\frac{k}{2}p^k)^{1/k}(1+\frac{1}{2}p^k)}.$$

The latter fact follows by an argument analogous to that preceeding (7.8).

Vice versa, we have the following lemma that can be verified by direct calculation:

Lemma 7.1. *Let $\rho(t) = t + \mathrm{HOT}$ be reversed by the rotation $t \mapsto \omega t$ with $\omega = c^{-2}$. Then*

$$F(z) = \left(\frac{z_1 \rho(p)}{\rho^{\frac{1}{2}}(p)}, \frac{z_2 \rho^{\frac{1}{2}}(p)}{p} \right), \quad p = z_1 z_2, \tag{7.14}$$

is reversed by J_c.

Proof of Theorem 1.2. Summarizing, we obtain that any reversible map is formally conjugate either to a linear map or to a map (7.4) or to a map (7.12), which proves the first assertion of Theorem 1.2.

To show that these map are pairwise inequivalent under conjugation, note that the maps F in (7.4) have $P(F) = \mathrm{id}$, whereas the ones in (7.12) have $P(F)$ conjugate to f_k. As consequence of Poincaré-Dulac, any conjugation map between those maps must be D-resonant, *i.e.* in the centralizer $\mathfrak{C}$. Consequently, the corresponding one-variable maps $P(F)$ must be conjugate. This shows that the maps in (7.12) are pairwise inequivalent under conjugation and are not conjugate to any map in (7.4). To see that also the maps in (7.4) are pairwise inequivalent, since any map in $\mathfrak{C}$ splits as $H(\chi)\Phi(\varphi)$, it suffices to observe any conjugation by $\Phi(\varphi)$ is trivial, whereas a conjugation by $H(\chi)$ is given by (7.2) and hence cannot change the integer k.

Statement (1) of Theorem 1.2 is evident for maps as in (7.4) and it follows from Lemma 7.1 for maps as in (7.12).

To show the statement (2), observe that any map in (7.4) is reversed by the involution $J(z) = \widetilde{z}$ and hence is strongly reversible. On the other hand, suppose a map as F in (7.12) is reversed by an involution Θ. Then we know from Proposition 6.6 that Θ must be D-inverse-resonant, *i.e.* it reverses the linear part of F. Since Θ is an involution and $\lambda \neq \lambda^{-1}$, we must have $L(\Theta) = \pm J$ and therefore $P(\Theta)(t) = t + \text{HOT}$. But then, since $P(\Theta)$ is also an involution, it follows that $P(\Theta) = \mathsf{id}$. Consequently $P(\Theta)$ cannot reverse $P(F) \neq P(F)^{-1}$ and hence Θ cannot reverse F.

Finally the statement (3) follows from Corollary 4.5. □

Proof of Proposition 1.4. In the foregoing argument, the specific form f_k may be replaced by any element $\rho(t) \in \mathfrak{G}_1$ that only has powers t^{1+jk} and takes the form

$$\rho(t) = t\left(1 + t^k + \left(\frac{k+1}{2}\right)t^{2k} + \text{HOT}\right),$$

and we still obtain the normal form (7.10) with the new ρ. To see this, note first that each such ρ is conjugate in $\mathfrak{G}_1$ to f_k, and that the conjugating map, say χ, only has powers t^{1+jk} and may be chosen equal to be equal to the identity up to the terms of order $2k+1$. It follows that J_c commutes with $H(\chi)$ up to order $4k+1$ and therefore it reverses $F^{H(\chi)}$ up to terms of degree $4k+1$ in z, and, arguing as before, we can conjugate $F^{H(\chi)}$ to the form (7.9), and then we still get $\alpha = \frac{1}{2}$. Hence $F^{H(\chi)}$ is conjugate to (7.13). Now, noting that

$$\frac{\rho(t)}{t(1+\frac{1}{2}t^k)} = 1 + \frac{1}{2}t^k + \left(\frac{2k+1}{4}\right)t^{2k} + \text{HOT},$$

we may choose

$$\rho(t) = t\left(1 + \frac{1}{2}t^k + \left(\frac{2k+1}{4}\right)t^{2k}\right)\left(1 + \frac{1}{2}t^k\right)$$

and we get the form

$$F(z) = \left(z_1\left(1 + \frac{1}{2}p^k\right), z_2\left(1 + \frac{1}{2}p^k + \left(\frac{2k+1}{4}\right)p^{2k}\right)\right).$$

Conjugating by $z \mapsto \alpha z$, with $\alpha^{2k} = 2$, we get the tidier polynomial form

$$F(z) = \left(z_1(1+p^k), z_2\left(1 + p^k + (2k+1)p^{2k}\right)\right). \tag{7.15}$$

□

The alternative forms (7.10)/(7.15) each have their advantages. The second has the simplest form, but the first is reversed by the simple map J_c.

Remark 7.2. We have seen that maps of the form (7.9) are only reversed by some J_c when $\alpha = \frac{1}{2}$. However, it is worth remarking that for every $\alpha \in \mathbb{C}$ they are reversed by the *D-resonant* maps $c \cdot H(\rho^\alpha)$, as is readily checked. This does not mean, of course, that the original map is ΛF is reversible for $\alpha \neq \frac{1}{2}$ but might be of interest for the study of the reversible $F \in \mathfrak{G}$ that are tangent to the identity.

8. Convergence properties of the normal form

The convergence of the normal form as stated in Theorem 1.6 is obtained by closely following the arguments of the [11, proof of Theorem 4.1, page 283]. In fact, assume that F is biholomorphic and admits a formal Poincaré-Dulac normal form

$$\xi' = M\xi, \quad \eta' = M^{-1}\eta, \tag{8.1}$$

(written in the form similar to (3.6) in [11]), where M is a formal power series in the product $\xi\eta$. Changing the notation $(x, y) = (z_1, z_2)$ as in [11] and writing F as

$$x' = \lambda x + f(x, y), \quad y' = \lambda^{-1}y + g(x, y) \tag{8.2}$$

with f and g convergent of order at least 2, analogously to (3.2) in [11], and the conjugation map ψ into the normal form as

$$x = U(\xi, \eta) = \xi + u(\xi, \eta), \quad y = V(\xi, \eta) = \eta + v(\xi, \eta) \tag{8.3}$$

with u and v of order at least 2, analogously to (3.3) in [11], the fact that ψ conjugates F to (8.1) can be written as

$$U(M\xi,M^{-1}\eta)-\lambda U(\xi,\eta)=f(U,V), \quad V(M\xi,M^{-1}\eta)-\lambda V(\xi,\eta)=g(U,V), \tag{8.4}$$

analogously to (4.1) in [11].

As in [11] a formal power series $p(\xi, \eta)$ is said to have type s if it can be written

$$p(\xi, \eta) = \sum_{i-j=s} p_{ij}\xi^i\eta^j$$

and any power series p admits an unique decomposition

$$p(\xi, \eta) = \sum p_s(\xi, \eta)$$

with p_s having type s. According to the proof of Poincaré-Dulac, we may assume that the conjugating map ψ has no resonant terms, which as in [11, (3.4)] can be written using the type decomposition as

$$u_1 = 0, \quad v_{-1} = 0. \tag{8.5}$$

As in [11, (4.2)], taking terms of the same type in (8.4) yields

$$(M^s - \lambda)U_s = [f(U, V)]_s, \quad (M^s - \lambda^{-1})V_s = [g(U, V)]_s. \tag{8.6}$$

Then proceeding as in [11], pages 284–286, we obtain a convergent series $W(\xi)$ majorizing $u(\xi, \xi)$ and $v(\xi, \xi)$ and hence proving the convergence of the conjugating map ψ. Note that this proof gives the convergence of the (unique) map ψ satisfying (8.5) and conjugating F to a normal form (8.1).

Finally, once F is biholomorphically conjugate to (8.1) with convergent M, its further biholomorphic conjugation to the form (7.4) is obtained by a suitable $K_2 = H(\chi)$ associated to the convergent χ. This ends the proof of Theorem 1.6.

The following example shows that, in contrast to the first series in (1.3), biholomorphic reversible maps conjugate to a map in the second series in (1.3) may fail to be biholomorphically conjugate to their formal normal form.

Example 8.1. Let $\rho(t) = t + \mathrm{HOT}$ be any one-variable biholomorphic map that is formally but not biholomorphically reversible, see [2, Example 4.8] for the existence of such maps. Let θ be a formal map reversing ρ. Then any two-variable map

$$F := \Lambda H(\rho) = \left(\frac{\rho(p)}{p}\lambda z_1, \lambda^{-1} z_2\right)$$

is formally reversible by $H(\theta)$. On the other hand, any biholomorphic reverser Θ of F with λ not a root of unity, would be inverse-resonant by Proposition 6.6. But then $\rho = P(F)$ would be reversible by the biholomorphic map $P(\Theta)$, which is impossible due to the choice of ρ. Hence F cannot be biholomorphically reversible. In particular, it cannot be biholomorhically conjugate to any map in (1.3).

9. Factorization in $\mathfrak{G}$

The homomorphisms Φ and H may also be applied to resolve another question. It is an interesting fact that in many very large groups each element may be factored as the product of a fixed small number of elements of a handful of conjugacy classes. For instance, any permutation of a finite set is a product of transpositions, and also the product of two involutions. Of particular interest are products of involutions, and, more generally, products of reversibles. In the present case, we have the following:

Theorem 9.1. *If $F \in \mathfrak{G}$ has* $\det L(F) = 1$, *then it may be factorized as $F = g_1 g_2 g_3 g_4$, where each g_j is reversible in $\mathfrak{G}$.*

Remark 9.2. Each product $F = f_1 \cdots f_n$ of reversible f_j's has $\det L(F) = \pm 1$, so (multiplying if necessary by a suitable linear involution) it follows from the theorem that each product of reversibles reduces to the product of five. It also follows that the elements that are products of reversibles are precisely those with $\det L(F) = \pm 1$.

Proof of Theorem 9.1. In fact, if $\det L(F) = 1$, then conjugating to a Jordan normal form and multiplying by some (reversible) $\Lambda \in D$ we can arrange that $L(F\widetilde{\Lambda})$ is conjugate to an infinite-order element of D, where $\widetilde{\Lambda}$ is conjugate to Λ and therefore reversible. Then by Poincaré-Dulac, $F\widetilde{\Lambda}$ is conjugate (say by $K \in \mathfrak{G}$) to some element of the centralizer $\mathfrak{C}$, so that $(F\widetilde{\Lambda})^K$ is resonant and hence may be factored as $H(\chi)\Phi(\varphi)$, where $\chi(t) = t + \mathrm{HOT}$. Now $\Phi(\varphi)$ is reversible, and we know [12, Theorem 9(2)] that χ is the product of two reversibles in $\mathfrak{G}_1$, so $H(\chi)$ is the product of two reversibles, say $H(\chi_1)$ and $H(\chi_2)$. Thus

$$F^K = H(\chi_1)H(\chi_2)\Phi(\varphi)(\widetilde{\Lambda}^{-1})^K$$

is the product of four reversibles, and conjugating with K^{-1} we obtain the result. $\square$

Theorem 1.7 now follows from Theorem 9.1. Indeed, if $\det L(F) = 1$, it is the product of 4 reversibles with the involution equal to the identity. Otherwise if $\det L(F) = -1$, consider the involution $\nu(z_1, z_2) = (-z_1, z_2)$. Then $\det L(F\nu) = 1$ and hence $F\nu$ is the product of 4 reversibles, implying the result.

10. Reversible biholomorphic maps

For series in one variable, the single formal conjugacy class of f_k in $\mathfrak{G}_1$, intersected with the subgroup $\mathcal{G}_1$ of biholomorphic maps splits into uncountably many conjugacy classes. Functional moduli for these classes have been provided by Écalle and Voronin [7]. It is not necessarily true that every formally reversible biholomorphic map is biholomorphically-reversible, but because of the fact that all reversers are of finite order it is true that every reversible biholomorphic map is conjugate to one that is reversed by a rational rotation. The same principle carries over to our present context:

Theorem 10.1. *Let $F \in \mathcal{G}$ be an invertible biholomorphic germ on $(\mathbb{C}^2, 0)$, and suppose that $L(F)$ has an eigenvalue that is not a root of unity. Then F is reversible in $\mathcal{G}$ if and only if it is conjugate in $\mathcal{G}$ to a map that is reversed by linear map of finite order.*

Proof. Suppose $\Theta \in \mathcal{G}$ reverses $F \neq \mathsf{id}$ in $\mathcal{G}$. Then Θ reverses F in $\mathfrak{G}$ by Theorem 1.2, and hence has finite order. Thus, by Lemma 2.1, Θ is conjugate in $\mathcal{G}$ to a linear map. Applying the same conjugation to F, we obtain the result. $\square$

Remark 10.2. It remains open, even for one-variable maps, whether results such as Theorems 9.1 or 1.7 hold for biholomorphic maps.

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