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A classification theorem for hypersurfaces of Minkowski spaces

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Abstract. Let M^n be a compact hypersurface of a Minkowski space $(V^{n+1}, \overline{F})$. In this paper, using the Gauss formula of the Chern connection for Finsler submanifolds, we prove that if the second mean curvature H_2 of M is constant and the norm square S of the second fundamental form of M satisfies $S \leq \frac{n(n-1)}{n-2} H_2$, then M with the induced metric is isometric to the standard Euclidean sphere. This generalizes the result of [2] from the Euclidean to the Minkowski space.

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1. Introduction

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M$, $Y \in T_x M$ and let (x^i, Y^i) be local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, Y)$ is smooth on $TM\backslash 0$;
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;
(iii) Strong convexity: The fundamental quadratic form $g_Y = g_{ij}(x, Y) dx^i \otimes dx^j$ is positively definite, where $g_{ij} = \frac{1}{2}\partial^2(F^2)/\partial Y^i \partial Y^j$.

The simplest class of Finsler manifolds is Minkowski space. Let V^{n+1} be a real vector space. A Finsler metric $\overline{F} : TV^{n+1} \to [0, \infty)$ is called Minkowski if $\overline{F}$ is a function of $\overline{Y} \in V^{n+1}$ only. In this case $(V^{n+1}, \overline{F})$ is called a Minkowski space.

Riemannian submanifolds are important in modern differential geometry. There has been a long history for the study of Riemannian submanifolds. For a compact Riemannian hypersurface M of Euclidean space, the second fundamental form is $B = h_{ij}^{n+1} \omega^i \otimes \omega^j \otimes e_{n+1}$, where $\{\omega^i\}$ is the orthonormal coframe of M. The

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second mean curvature H_2 of M is defined by $H_2 = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \lambda_i \lambda_j$, where λ_i are the eigenvalues of the second fundamental tensor h_{ij}^{n+1} of M. For a compact hypersurface M of Euclidean space, the Gauss equation is $\sum_{i,j} R_{ijij} = n(n-1)H_2$, which implies that the scalar curvature is constant if and only if the second mean curvature H_2 is constant.

As well kown, using Cheng-Yau's self-adjoint operator $\Box$, Li [2] proved that if the second mean curvature H_2 is constant and the norm square S of the second fundamental form of M satisfies $S \le \frac{n(n-1)}{n-2} H_2$, then M is a Riemannian sphere. As far as we know, there are very few rigidity results on Finsler submanifolds. The main purpose of this paper is to generalize the above result of Li from the Euclidean to the Minkowski space. In this paper, using the Gauss formula for the Chern connection and defining a similar self-adjoint operator $\Box$ on Finsler manifolds, we study the hypersurfaces of Minkowski space $(V^{n+1}, \overline{F})$ and we obtain the following:

Main Theorem. *Let M^n be a compact hypersurface of Minkowski space $(V^{n+1}, \overline{F})$. If the second mean curvature H_2 is constant and the norm square S of the second fundamental form of M satisfies $S \le \frac{n(n-1)}{n-2} H_2$, then M with the induced metric is isometric to the standard Euclidean sphere.*

2. Preliminaries

Let (M^n, F) be an n-dimensional Finsler manifold. Then F inherits the *Hilbert* form, the fundamental tensor and the *Cartan* tensor as follows [1]:

$$\begin{aligned} \omega =& \frac{\partial F}{\partial Y^i} dx^i, \\ g_Y =& g_{ij}(x, Y) dx^i \otimes dx^j, \\ A_Y =& A_{ijk} dx^i \otimes dx^j \otimes dx^k, \\ A_{ijk} :=& \frac{F \partial g_{ij}}{2 \partial Y^k}. \end{aligned}$$

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to another one. We have [7]

$$F(Y) = \overline{F}(\varphi_*(Y)), \quad g_Y(U, V) = \overline{g}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V)), \tag{2.1}$$

$$A_Y(U, V, W) = \overline{A}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V), \varphi_*(W)), \tag{2.2}$$

where $Y, U, V, W \in TM$, $\overline{g}$ and $\overline{A}$ are the fundamental tensor and the *Cartan* tensor of $\overline{M}$, respectively. It can be seen from (2.1) that $\varphi^*(\overline{\omega}) = \omega$, where $\overline{\omega}$ is the *Hilbert* form of $\overline{M}$.

In the following we simplify A_Y and g_Y to A and g, respectively. Moreover any vector $U \in TM$ will be identified with the corresponding vector $\varphi_*(U) \in T\overline{M}$

and we will use the following convention:

$1 \le i, j, \cdots \le n, n+1 \le \alpha, \beta, \cdots \le n+p, 1 \le \lambda, \mu, \cdots \le n-1, 1 \le a, b, \cdots \le n+p.$

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to another one. Take a $\overline{g}$-orthonormal frame form $\{e_a\}$ for each fibre of $\pi^* T\overline{M}$ and let $\{\omega^a\}$ be its local dual coframe, such that $\{e_i\}$ is a frame field for each fibre of $\pi^* TM$ and ω^n is the $Hilbert$ form, where $\pi : TM \to M$ denotes the natural projection. Let θ^a_b and ω^i_j denote the Chern connection 1-form of $\overline{F}$ and F, respectively, *i.e.* $\overline{\nabla} e_a = \theta^b_a e_b$ and $\nabla e_i = \omega^j_i e_j$, where $\overline{\nabla}$ and ∇ are the Chern connections of $\overline{M}$ and M, respectively. We obtain that $A(e_i, e_j, e_n) = \overline{A}(e_a, e_b, e_n) = 0$, where $e_n = \frac{Y^i}{F}\frac{\partial}{\partial x^i}$ is the natural dual of the Hilbert form ω^n.

The structure equations of $\overline{M}$ are given by

$$\begin{cases} d\theta^a = -\theta^a_b \wedge \theta^b, \\ d\theta^a_b = -\theta^a_c \wedge \theta^c_b + \frac{1}{2}\overline{R}^a_{bcd}\omega^c \wedge \omega^d + \overline{P}^a_{bcd}\omega^c \wedge \theta^d_n, \\ \theta^a_b + \theta^b_a = -2\overline{A}_{abc}\theta^c_n, \\ \theta^a_n + \theta^n_a = 0, \quad \theta^n_n = 0. \end{cases}$$

By $\theta^\alpha = 0$ and the structure equations of $\overline{M}$, we have that $\theta^\alpha_j \wedge \omega^j = 0$, which implies that $\theta^\alpha_j = h^\alpha_{ij}\omega^i, \quad h^\alpha_{ij} = h^\alpha_{ji}$. We obtain [3]

$$\omega^j_i = \theta^j_i - \Psi_{jik}\omega^k, \tag{2.3}$$

where

$$\begin{aligned} \Psi_{jik} = h^\alpha_{jn}\overline{A}_{ki\alpha} - h^\alpha_{kn}\overline{A}_{ji\alpha} - h^\alpha_{in}\overline{A}_{kj\alpha} - h^\alpha_{nn}\overline{A}_{iks}\overline{A}_{sj\alpha} + h^\alpha_{nn}\overline{A}_{ijs}\overline{A}_{sk\alpha} \\ + h^\alpha_{nn}\overline{A}_{jks}\overline{A}_{si\alpha}. \end{aligned} \tag{2.4}$$

In particular,

$$\omega^n_i = \theta^n_i - h^\alpha_{nn}\overline{A}_{ki\alpha}\omega^k. \tag{2.5}$$

Using the almost $\overline{g}$-compatibility, we have

$$\theta^j_\alpha = (-h^\alpha_{ij} - 2h^\beta_{ni}\overline{A}_{j\alpha\beta} + 2h^\beta_{nn}\overline{A}_{j\lambda\alpha}\overline{A}_{i\lambda\beta})\omega^i - 2\overline{A}_{j\alpha\lambda}\omega^\lambda_n. \tag{2.6}$$

In particular, $\theta^n_\alpha = -h^\alpha_{ni}\omega^i$. We quote the following results:

Proposition 2.1 ([3]). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Minkowski space. Then*

$$\begin{cases} (\overline{\nabla}_{e_i}\overline{A})(\bullet, \bullet, \bullet) = 0, \\ \overline{A}(\bullet, \ \bullet, \ \overline{\nabla}_{e_i}e_n) = 0. \end{cases}$$

Proposition 2.2 ([3] Gauss equations). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold another one. Then we have*

$$\begin{cases} P^j_{ik\lambda} = \overline{P}^j_{ik\lambda} + \Psi_{jik;\lambda} - 2\Psi_{sik}A_{js\lambda} - 2h^\alpha_{ik}\overline{A}_{j\lambda\alpha}, \\ R^j_{ikl} = \overline{R}^j_{ikl} - h^\alpha_{ik}h^\alpha_{jl} + h^\alpha_{il}h^\alpha_{jk} + \Psi_{jik|l} - \Psi_{jil|k} \\ \qquad +\Psi_{sik}\Psi_{jsl} - \Psi_{sil}\Psi_{jsk} - 2h^\alpha_{ik}h^\beta_{nl}\overline{A}_{j\alpha\beta} + 2h^\alpha_{il}h^\beta_{nk}\overline{A}_{j\alpha\beta} \\ \qquad +2h^\alpha_{ik}h^\beta_{nn}\overline{A}_{js\alpha}\overline{A}_{ls\beta} - 2h^\alpha_{il}h^\beta_{nn}\overline{A}_{js\alpha}\overline{A}_{ks\beta} - h^\alpha_{nn}\overline{A}_{sl\alpha}\overline{P}^j_{iks} \\ \qquad +h^\alpha_{nn}\overline{A}_{sk\alpha}\overline{P}^j_{ils} + h^\alpha_{nl}\overline{P}^j_{ik\alpha} - h^\alpha_{nk}\overline{P}^j_{il\alpha} \end{cases}$$

where "; " and "|" respectively denote the vertical and the horizontal covariant differentials with respect to the Chern connection ∇.

Proposition 2.3 ([3] Codazzi equations). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold to another one. Then we have*

$$\begin{cases} h^\alpha_{ij;\lambda} = -\overline{P}^\alpha_{ij\lambda}, \\ h^\alpha_{ij|k} - h^\alpha_{ik|j} = -\overline{R}^\alpha_{ijk} + h^\beta_{nj}\overline{P}^\alpha_{ik\beta} - h^\beta_{nk}\overline{P}^\alpha_{ij\beta} \\ -h^\alpha_{lk}\Psi_{lij} + h^\alpha_{lj}\Psi_{lik} - h^\beta_{nn}\overline{A}_{lj\beta}\overline{P}^\alpha_{ikl} + h^\beta_{nn}\overline{A}_{lk\beta}\overline{P}^\alpha_{ijl}. \end{cases}$$

3. Hypersurfaces of a Minkowski space

Let (M^n, F) be a compact hypersurfaces of a Minkowski space $(V^{n+1}, \overline{F})$. Then we have

$$h^{n+1}_{ij|k}\omega^k + h^{n+1}_{ij;\lambda}\omega^\lambda_n = dh^{n+1}_{ij} - h^{n+1}_{kj}\omega^k_i - h^{n+1}_{ik}\omega^k_j + h^{n+1}_{ij}\theta^{n+1}_{n+1}. \tag{3.1}$$

Exterior differentiation of the left-hand side terms of (3.1), gives

$$\begin{aligned} &dh^{n+1}_{ij|k} \wedge \omega^k + h^{n+1}_{ij|k}d\omega^k + dh^{n+1}_{ij;\lambda} \wedge \omega^\lambda_n + h^{n+1}_{ij;\lambda}d\omega^\lambda_n \\ &= \Big\{h^{n+1}_{ij|k|l}\omega^l + h^{n+1}_{ij|k;\mu}\omega^\mu_n + h^{n+1}_{lj|k}\omega^l_i + h^{n+1}_{il|k}\omega^l_j + h^{n+1}_{ij|l}\omega^l_k - h^{n+1}_{ij|k}\theta^{n+1}_{n+1}\Big\} \wedge \omega^k \\ &\quad + h^{n+1}_{ij|k}\{-\omega^k_l \wedge \omega^l\} \\ &\quad + \Big\{h^{n+1}_{ij;\lambda|l}\omega^l + h^{n+1}_{ij;\lambda;\mu}\omega^\mu_n + h^{n+1}_{lj;\lambda}\omega^l_i + h^{n+1}_{il;\lambda}\omega^l_j + h^{n+1}_{ij;\mu}\omega^\mu_\lambda - h^{n+1}_{ij;\lambda}\theta^{n+1}_{n+1}\Big\} \wedge \omega^\lambda_n \\ &\quad + h^{n+1}_{ij;\lambda}\Big\{ -\omega^\lambda_\mu \wedge \omega^\mu_n + \tfrac{1}{2}R^\lambda_{nls}\omega^l \wedge \omega^s + P^\lambda_{nl\mu}\omega^l \wedge \omega^\mu_n\Big\} \\ &= \Big\{ -h^{n+1}_{ij|k|l} + \tfrac{1}{2}h^{n+1}_{ij;\lambda}R^\lambda_{nkl}\Big\}\omega^k \wedge \omega^l - h^{n+1}_{ij;\lambda;\mu}\omega^\lambda_n \wedge \omega^\mu_n \\ &\quad + \Big\{ -h^{n+1}_{ij|k;\lambda} + h^{n+1}_{ij;\lambda|k} + h^{n+1}_{ij;\mu}P^\mu_{nkl} + h^{n+1}_{ij|k}\overline{A}_{n+1n+1\lambda}\Big\}\omega^k \wedge \omega^\lambda_n \\ &\quad + h^{n+1}_{lj|k}\omega^l_i \wedge \omega^k + h^{n+1}_{il|k}\omega^l_j \wedge \omega^k + h^{n+1}_{lj;\lambda}\omega^l_i \wedge \omega^\lambda_n + h^{n+1}_{il;\lambda}\omega^l_j \wedge \omega^\lambda_n \\ &\quad - h^{n+1}_{ij|k}\theta^{n+1}_{n+1} \wedge \omega^k - h^{n+1}_{ij;\lambda}\theta^{n+1}_{n+1} \wedge \omega^\lambda_n. \end{aligned} \tag{3.2}$$

Exterior differentiation of the right-hand side terms of (3.1), gives also

$$
\begin{aligned}
&-dh^{n+1}_{kj}\wedge\omega^k_i - h^{n+1}_{kj}d\omega^k_i - dh^{n+1}_{ik}\wedge\omega^k_j - h^{n+1}_{ik}d\omega^k_j\\
&+dh^{n+1}_{ij}\wedge\theta^{n+1}_{n+1} + h^{n+1}_{ij}d\theta^{n+1}_{n+1}\\
=&-\Big\{h^{n+1}_{kj|l}\omega^l + h^{n+1}_{kj;\lambda}\omega^\lambda_n + h^{n+1}_{lj}\omega^l_k + h^{n+1}_{kl}\omega^l_j - h^{n+1}_{kj}\theta^{n+1}_{n+1}\Big\}\wedge\omega^k_i\\
&-h^{n+1}_{kj}\Big\{-\omega^k_l\wedge\omega^l_i + \tfrac{1}{2}R^k_{ils}\omega^l\wedge\omega^s + P^k_{il\lambda}\omega^l\wedge\omega^\lambda_n\Big\}\\
&-\Big\{h^{n+1}_{ik|l}\omega^l + h^{n+1}_{ik;\lambda}\omega^\lambda_n + h^{n+1}_{lk}\omega^l_i + h^{n+1}_{il}\omega^l_k - h^{n+1}_{ik}\theta^{n+1}_{n+1}\Big\}\wedge\omega^k_j\\
&-h^{n+1}_{ik}\Big\{-\omega^k_l\wedge\omega^l_j + \tfrac{1}{2}R^k_{jls}\omega^l\wedge\omega^s + P^k_{jl\lambda}\omega^l\wedge\omega^\lambda_n\Big\}\\
&+\Big\{h^{n+1}_{ij|k}\omega^k + h^{n+1}_{ij;\lambda}\omega^\lambda_n + h^{n+1}_{kj}\omega^k_i + h^{n+1}_{ik}\omega^k_j\Big\}\wedge\theta^{n+1}_{n+1}\\
&+h^{n+1}_{ij}\Big\{2h^{n+1}_{sk}h^{n+1}_{nl}\overline{A}_{sn+1n+1} - 2h^{n+1}_{sk}h^{n+1}_{nn}\overline{A}_{stn+1}\overline{A}_{tln+1}\Big\}\omega^k\wedge\omega^l\\
&+2h^{n+1}_{ij}h^{n+1}_{sk}\overline{A}_{sn+1\lambda}\omega^k\wedge\omega^\lambda_n\\
=&\Big\{-\tfrac{1}{2}h^{n+1}_{sj}R^s_{ikl} - \tfrac{1}{2}h^{n+1}_{is}R^k_{jkl} + 2h^{n+1}_{ij}h^{n+1}_{sk}h^{n+1}_{nl}\overline{A}_{sn+1n+1}\\
&-2h^{n+1}_{ij}h^{n+1}_{sk}h^{n+1}_{nn}\overline{A}_{stn+1}\overline{A}_{tln+1}\Big\}\omega^k\wedge\omega^l\\
&+\Big\{-h^{n+1}_{sj}P^s_{ik\lambda} - h^{n+1}_{sj}P^k_{jk\lambda} + 2h^{n+1}_{ij}h^{n+1}_{sk}\overline{A}_{sn+1\lambda}\Big\}\omega^k\wedge\omega^\lambda_n\\
&-h^{n+1}_{kj|l}\omega^l\wedge\omega^k_i - h^{n+1}_{kj;\lambda}\omega^\lambda_n\wedge\omega^k_i - h^{n+1}_{ik|l}\omega^l\wedge\omega^k_j - h^{n+1}_{ik;\lambda}\omega^\lambda_n\wedge\omega^k_j\\
&-h^{n+1}_{ij|k}\theta^{n+1}_{n+1}\wedge\omega^k - h^{n+1}_{ij;\lambda}\theta^{n+1}_{n+1}\wedge\omega^\lambda_n.
\end{aligned}
\tag{3.3}
$$

It can be seen from (3.2) and (3.3) that

$$
\begin{aligned}
&\Big\{h^{n+1}_{ij|k|l} - \tfrac{1}{2}h^{n+1}_{sj}R^s_{ikl} - \tfrac{1}{2}h^{n+1}_{is}R^s_{jkl} - \tfrac{1}{2}h^{n+1}_{ij;\lambda}R^\lambda_{nkl}\\
&\quad+2h^{n+1}_{ij}h^{n+1}_{sk}h^{n+1}_{nl}\overline{A}_{sn+1n+1} - 2h^{n+1}_{ij}h^{n+1}_{sk}h^{n+1}_{nn}\overline{A}_{stn+1}\overline{A}_{tln+1}\Big\}\omega^k\wedge\omega^l\\
&+\Big\{h^{n+1}_{ij|k;\lambda} - h^{n+1}_{ij;\lambda|k} + h^{n+1}_{ij;\mu}P^\mu_{nk\lambda} + h^{n+1}_{is}P^s_{jkl} + h^{n+1}_{sj}P^s_{ikl} - h^{n+1}_{ij|k}\overline{A}_{n+1n+1\lambda}\\
&\quad+2h^{n+1}_{ij}h^{n+1}_{sk}\overline{A}_{sn+1\lambda}\Big\}\omega^k\wedge\omega^\lambda_n + h^{n+1}_{ij;\lambda;\mu}\omega^\lambda_n\wedge\omega^\mu_n\\
&=0.
\end{aligned}
\tag{3.4}
$$

From (3.4) we immediately obtain the following:

Proposition 3.1. *If M^n be a hypersurface of Minkowski space $(V^{n+1}, \overline{F})$, then*

$$\begin{cases} h^{n+1}_{ij;\lambda;\mu} - h^{n+1}_{ij;\mu;\lambda} = 0, \\ h^{n+1}_{ij|k;\lambda} - h^{n+1}_{ij;\lambda|k} = -h^{n+1}_{sj} P^s_{ikl} - h^{n+1}_{is} P^s_{jkl} - h^{n+1}_{ij;\mu} P^{\mu}_{nk\lambda} + h^{n+1}_{ij|k} \overline{A}_{n+1n+1\lambda} \\ \qquad\qquad -2h^{n+1}_{ij} h^{n+1}_{sk} \overline{A}_{sn+1\lambda}, \\ h^{n+1}_{ij|k|l} - h^{n+1}_{ij|l|k} = h^{n+1}_{sj} R^s_{ikl} + h^{n+1}_{is} R^s_{jkl} + h^{n+1}_{ij;\lambda} R^{\lambda}_{nkl} \\ \qquad\qquad -2h^{n+1}_{ij} h^{n+1}_{sk} h^{n+1}_{nl} \overline{A}_{sn+1n+1} + 2h^{n+1}_{ij} h^{n+1}_{sl} h^{n+1}_{nk} \overline{A}_{sn+1n+1} \\ \qquad\qquad +2h^{n+1}_{ij} h^{n+1}_{sk} h^{n+1}_{nn} \overline{A}_{stn+1} \overline{A}_{tln+1} \\ \qquad\qquad -2h^{n+1}_{ij} h^{n+1}_{sl} h^{n+1}_{nn} \overline{A}_{stn+1} \overline{A}_{tkn+1}. \end{cases}$$

The form $B = h^{n+1}_{ij} \omega^i \otimes \omega^j \otimes e_{n+1}$ is called the second fundamental form of M and $H = \frac{1}{n} tr B = \frac{1}{n} \sum_i h^{n+1}_{ii} e_{n+1}$ is called the mean curvature vector. The norm square S of the second fundamental form of M is $S = \sum_{ij} (h^{n+1}_{ij})^2$. Let λ_i be the eigenvalues of the second fundamental tensor h^{n+1}_{ij} of M. The second mean curvature $H_2 = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \lambda_i \lambda_j = \frac{1}{n(n-1)} [(nH)^2 - S]$.

From the second formula of Proposition 2.1, we obtain that

$$\overline{A}(\bullet, \bullet, \nabla_{e_i} e_n) + \overline{A}(\bullet, \bullet, e_{\lambda}) \Psi_{\lambda ni} + \overline{A}(\bullet, \bullet, e_{n+1}) h^{n+1}_{ni} = 0. \tag{3.6}$$

Let P be an arbitrary point in M. There exists a local coordinate system $\{x^i\}$ such that $\frac{Y^n}{F} \frac{\partial}{\partial x^n} = e_n$. Let $\gamma(t)$ be a curve in M with $\gamma(0) = P$ and tangent vector field $\dot{\gamma}(t) = e_n$. Let $X_i(t)$ be parallel vector fields along $\gamma(t)$ with $X_i(0) = \frac{\partial}{\partial x^i}|_p$. We have that $\nabla_{e_n} \frac{\partial}{\partial x^i} = \frac{Y^n}{F} \nabla_{\frac{\partial}{\partial x^n}} \frac{\partial}{\partial x^i} = 0$ at P, *i.e.* $\Gamma^k_{ni} = 0$ at P, then we can obtain that at P

$$\begin{aligned} \overline{A}(\bullet, \bullet, \nabla_{e_i} e_n) &= \overline{A}(\bullet, \bullet, \tfrac{Y^n}{F} \nabla_{u^j_i \frac{\partial}{\partial x^j}} \tfrac{\partial}{\partial x^n}) \\ &= \overline{A}(\bullet, \bullet, \tfrac{Y^n}{F} u^j_i \Gamma^k_{jn} \tfrac{\partial}{\partial x^k}) = 0. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6) yields that $\overline{A}(\bullet, \bullet, e_j) \Psi_{jni} + \overline{A}(\bullet, \bullet, e_{n+1}) h^{n+1}_{ni} = 0$ at P, which together with (2.4) yields that $-h^{n+1}_{nn} \overline{A}_{jin+1} \overline{A}_{stj} + h^{n+1}_{ni} \overline{A}_{stn+1} = 0$ at P, so by (2.4) we have that

$$\Psi_{ijk} = 0 \; and \; h^{n+1}_{ni} \overline{A}(\bullet, \bullet, e_{n+1}) = 0, \quad \forall i, j, k, at \; P. \tag{3.8}$$

It follows from the first formula of Proposition 2.1 and $\overline{A}(\bullet, \bullet, \bullet)_{|i} = 0$ that

$$\overline{A}(\bullet, \bullet, \bullet)_{;\lambda} \Psi_{\lambda ni} + \overline{A}(\bullet, \bullet, \bullet)_{;n+1} h^{n+1}_{ni} = 0 \;\; at \;\; P. \tag{3.9}$$

It can be seen from (3.8) and (3.9) that

$$\overline{A}(\bullet, \bullet, \bullet)_{;n+1} h^{n+1}_{ni} = 0 \;\; at \;\; P. \tag{3.10}$$

Now taking the exterior differentiation of $A_{ijk} = \overline{A}_{ijk}$, we obtain that

$$\begin{aligned} A_{ijk|l} = & \overline{A}_{ijk;\lambda}\Psi_{\lambda nl} + \overline{A}_{ijk;n+1}h^{n+1}_{nl} \\ & + \overline{A}_{sjk}\Psi_{sil} + \overline{A}_{isk}\Psi_{sjl} + \overline{A}_{ijs}\Psi_{skl} \\ & + \overline{A}_{n+1jk}h^{n+1}_{il} + \overline{A}_{in+1k}h^{n+1}_{jl} + \overline{A}_{ijn+1}h^{n+1}_{kl}. \end{aligned} \tag{3.11}$$

By (3.8) and (3.11), we obtain that

$$A_{ijk|n} = 0 \quad at \quad P. \tag{3.12}$$

Define $\delta Y^i = dY^i + N^i_j dx^j$. The pull-back of the Sasaki metric $g_{ij}dx^i \otimes dx^j + g_{ij}\delta Y^i \otimes \delta Y^j$ from $TM\backslash\{0\}$ to the sphere bundle SM is a Riemannian metric $\widehat{g} = g_{ij}dx^i \otimes dx^j + \delta_{ab}\omega^a_n \otimes \omega^b_n$.

We quote the following results:

Lemma 3.2 ([5]). *For* $X = \sum_i x_i\omega^i \in \Gamma(\pi^*T^*M)$, $\mathrm{div}_{\widehat{g}}X = \sum_i x_{i|i} + \sum_{\mu,\lambda} x_\mu P^n_{\lambda\lambda\mu}$.

Lemma 3.3 ([6]). *Let B be a real symmetric matrix with $tr\, B = 0$. Then*

$$|tr\, B^3| \leq \frac{n-2}{\sqrt{n(n-1)}}(tr\, B^2)^{\frac{3}{2}}.$$

Lemma 3.4 ([4]). *All Landsberg spaces of nonzero constant flag curvature must be Riemannian.*

Let $\phi = \sum_{i,j}\phi_{ij}\omega^i \otimes \omega^j$ be a symmetric tensor defined on the sphere bundle SM and $\psi = \sum_i \psi_i\omega^i \in \Gamma(\pi^*T^*M)$. Now we can define an operator $\Box$ associated to ϕ by

$$\Box f = \sum_{i,j}\phi_{ij}f_{|i|j} + \sum_{i,\lambda,\mu}\phi_{i\lambda}f_{|i}P^n_{\mu\mu\lambda} + \sum_i \psi_i f_{|i}, \quad \forall f \in C^\infty(SM). \tag{3.13}$$

Proposition 3.5. *Let (M, F) be a compact manifold. Then the operator $\Box$ is self-adjoint if and only if* $\sum_j \phi_{ij|j} - \psi_i = 0$.

Proof. Let $X = \sum_{i,j} g\phi_{ij}f_{|i}\omega^j \in \Gamma(\pi^*T^*M), \quad \forall f, g \in C^\infty(SM)$. Then we have from Lemma 3.2

$$\begin{aligned} \mathrm{div}_{\widehat{g}}X = & \sum_{i,j}\{g_{|j}\phi_{ij}f_{|i} + g\phi_{ij|j}f_{|i} + g\phi_{ij}f_{|i|j}\} + \sum_{i,\lambda,\mu} g\phi_{i\lambda}f_{|i}P^n_{\mu\mu\lambda} \\ = & \sum_{i,j} g_{|j}\phi_{ij}f_{|i} + \sum_i g\psi_i f_{|i} + \sum_{i,j} g\phi_{ij}f_{|i|j} + \sum_{i,\lambda,\mu} g\phi_{i\lambda}f_{|i}P^n_{\mu\mu\lambda}. \end{aligned} \tag{3.14}$$

Integrating (3.14) yields

$$\int_{SM}(\Box f)g\,dV_{SM} = -\int_{SM}\sum_{i,j} g_{|j}\phi_{ij}f_{|i}\,dV_{SM}. \tag{3.15}$$

Hence we get

$$\int_{SM}(\Box f)g\,dV_{SM} = \int_{SM}(\Box g)f\,dV_{SM}, \tag{3.16}$$

which completes the proof. $\Box$

4. Proof of the main theorem

Substituting (3.8) into the second formula of Proposition 2.2, by Proposition 2.3 and Proposition 3.1, we obtain that at P

$$\begin{cases} h^{n+1}_{ij|k|l} = h^{n+1}_{ik|j|l} + h^{n+1}_{sk}\Psi_{sij|l} + h^{n+1}_{sj}\Psi_{sik|l} \\ h^{n+1}_{ij|k|l} = h^{n+1}_{ij|l|k} + h^{n+1}_{sj}R^{s}_{ikl} + h^{n+1}_{is}R^{s}_{jkl} \\ R^{j}_{ikl} = -h^{\alpha}_{ik}h^{\alpha}_{jl} + h^{\alpha}_{il}h^{\alpha}_{jk} + \Psi_{jik|l} - \Psi_{jil|k}. \end{cases} \tag{4.1}$$

Let $\omega = dS = S_{|i}\omega^i + S_{;i}\omega^i_n$. Then ω is a global section of π^*T^*M. By the first formula of (2.24), we have $S_{;i} = 0$, *i.e.* $\omega = dS = S_{|i}\omega^i$. In the following, the computation is pointwisely estimated. Using the first formula of (4.1) and Lemma 3.2, we have that

$$\begin{aligned} \operatorname{div}_{\hat{g}}\omega =& 2\left[\sum_{i,j,k} h^{n+1}_{ij}h^{n+1}_{ij|k}\right]_{|k} + 2\sum_{i,j,k} h^{n+1}_{ij}h^{n+1}_{ij|\lambda}P^{n}_{\mu\mu\lambda} \\ =& 2\sum_{i,j,k}(h^{n+1}_{ij|k})^2 + 2\sum_{i,j,k} h^{n+1}_{ij}h^{n+1}_{ik|j|k} + 2\sum_{i,j,k,s} h^{n+1}_{ij}h^{n+1}_{sk}\Psi_{sij|k} \\ &+ 2\sum_{i,j,k,s} h^{n+1}_{ij}h^{n+1}_{sj}\Psi_{sik|k} + 2\sum_{i,j,k} h^{n+1}_{ij}h^{n+1}_{ij|\lambda}P^{n}_{\mu\mu\lambda}. \end{aligned} \tag{4.2}$$

It can be seen from (4.1) and (4.2) that

$$\begin{aligned} \operatorname{div}_{\hat{g}}\omega =& 2\sum_{i,j,k}(h^{n+1}_{ij|k})^2 + 2\sum_{i,j,k,s} h^{n+1}_{ij}\left\{h^{n+1}_{ki|k|j} + h^{n+1}_{si}R^{s}_{kjk} + h^{n+1}_{ks}R^{s}_{ijk}\right\} \\ &+ 2\sum_{i,j,k,s} h^{n+1}_{ij}h^{n+1}_{sk}\Psi_{sij|k} + 2\sum_{i,j,k,s} h^{n+1}_{ij}h^{n+1}_{sj}\Psi_{sik|k} + 2\sum_{i,j,k} h^{n+1}_{ij}h^{n+1}_{ij|\lambda}P^{n}_{\mu\mu\lambda} \\ =& \sum_{i,j,k}(h^{n+1}_{ij|k})^2 + 2\sum_{i,j,k,s} h^{n+1}_{ij}h^{n+1}_{kk|i|j} + 2\sum_{i,j,k} h^{n+1}_{ij}h^{n+1}_{ij|\lambda}P^{n}_{\mu\mu\lambda} \\ &+ nH\sum_{i}(\lambda_i - H)^3 + 3nH^2S - 2n^2H^4 - S^2 \\ \geq& \sum_{i,j,k}(h^{n+1}_{ij|k})^2 + \sum_{i,j,k,s} h^{n+1}_{ij}h^{n+1}_{kk|i|j} + \sum_{i,j,k} h^{n+1}_{ij}h^{n+1}_{ij|\lambda}P^{n}_{\mu\mu\lambda} \\ &+ \frac{n-1}{n}[S - nH_2]\Big\{2(n-1)H_2 - \frac{n-2}{n}S \\ &\qquad - \frac{n-2}{n}\sqrt{(n(n-1)H_2 + S)(S - nH_2)}\Big\}, \end{aligned} \tag{4.3}$$

where the second mean curvature $H_2 = \frac{1}{n(n-1)}[(nH)^2 - S]$. Let

$$\Box f = \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})f_{|i|j} + \sum_{i,\lambda,\mu}(nH\delta_{i\lambda} - h_{i\lambda}^{n+1})f_{|i}P^n_{\mu\mu\lambda} + \sum_{i,j,k}(h_{kj}^{n+1}\Psi_{kji} - h_{ki}^{n+1}\Psi_{kjj})f_{|i}.$$

By the second formula of Proposition 2.3, we can obtain that

$$\sum_j (nH\delta_{ij} - h_{ij}^{n+1})_{|j} - \sum_{j,k}(h_{kj}^{n+1}\Psi_{kji} - h_{ki}^{n+1}\Psi_{kjj}) = 0,$$

which together with Proposition 3.5 implies that the operator $\Box$ is self-adjoint. When H_2 is constant, we have the following computation by (4.3)

$$\begin{aligned}
\Box(nH) = {} & \sum_i nH(nH)_{|i|i} - \sum_{i,j,k} h_{ij}^{n+1}h_{kk|i|j}^{n+1} \\
& + \sum_{i,\lambda,\mu}(nH\delta_{i\lambda} - h_{i\lambda}^{n+1})(nH)_{|i}P^n_{\mu\mu\lambda} \\
& + \sum_{i,j,k}(h_{kj}^{n+1}\Psi_{kji} - h_{ki}^{n+1}\Psi_{kjj})(nH)_{|i} \\
\geq {} & \sum_{i,j,k}(h_{ij|k}^{n+1})^2 - \sum_i (nH_{|i})^2 \\
& + \frac{(n-1)}{n}[S - nH_2]\Big\{2(n-1)H_2 - \frac{n-2}{n}S \\
& \qquad - \frac{n-2}{n}\sqrt{(n(n-1)H_2 + S)(S - nH_2)}\Big\} \\
& + \sum_{i,\lambda,\mu}(nH\delta_{i\lambda} - h_{i\lambda}^{n+1})(nH)_{|i}P^n_{\mu\mu\lambda} \\
& + \sum_{i,j,k}(h_{kj}^{n+1}\Psi_{kji} - h_{ki}^{n+1}\Psi_{kjj})(nH)_{|i}.
\end{aligned} \tag{4.4}$$

Using the fact that $P^n_{ij\lambda} = -A_{ij\lambda|n}$, by (3.12) and (3.8) we have

$$\begin{aligned}
\Box(nH) \geq {} & \sum_{i,j,k}(h_{ij|k}^{n+1})^2 - \sum_i (nH_{|i})^2 \\
& + \frac{(n-1)}{n}[S - nH_2]\Big\{2(n-1)H_2 - \frac{n-2}{n}S \\
& \qquad - \frac{n-2}{n}\sqrt{(n(n-1)H_2 + S)(S - nH_2)}\Big\}.
\end{aligned} \tag{4.5}$$

On the other hand, let $x = \overline{x}^a \frac{\partial}{\partial \overline{x}^a}$ be the position vector field of the Minkowski space V^{n+1} with respect to the origin. By a direct simple computation, we get

$\overline{\nabla}_Z x = Z, \quad \forall Z = z^a \frac{\partial}{\partial \overline{x}^a}$ on V^{n+1}. This together with the second formula of Proposition 2.1 implies that $\nabla_{e_i} x^2 = 2\langle e_i, x\rangle$ and $\nabla_{e_i}\langle e_i, x\rangle = \theta_i^j(e_i)\langle e_j, x\rangle + h_{ii}^{n+1}\langle e_{n+1}, x\rangle + 1$. Then, when M is compact, there exists a point $P \in M$ such that $h_{ii}^{n+1}(P) > 0, \forall i$, hence we have that $n(n-1)H_2(P) = \sum_{1\le i<j\le n}\lambda_i(P)\lambda_j(P) > 0$, thus the constant $H_2 > 0$, which yields $(nH)^2 > S$. On the other hand, we have

$$\sum_i (nH)^2(nH_{|i})^2 = \sum_i \left[\sum_{j,k} h_{jk}^{n+1} h_{jk|i}^{n+1}\right]^2 \le S \sum_{i,j,k}[h_{jk|i}^{n+1}]^2. \tag{4.6}$$

Hence

$$(nH_{|i})^2 \le (h_{jk|i}^{n+1})^2. \tag{4.7}$$

It can be seen that our assumption $S \le \frac{n(n-1)}{n-2}H_2$ is equivalent to

$$2(n-1)H_2 - \frac{n-2}{n}S - \frac{n-2}{n}\sqrt{(n(n-1)H_2 + S)(S - nH_2)} \ge 0, \tag{4.8}$$

therefore the right-hand side of (4.5) is non-negative by (4.7) and (4.8). Because of the compactness of M, we get that h_{ij}^{n+1} is constant and $h_{ij}^{n+1} = 0, \forall i \ne j$ on M. Exterior differentiation of $h_{na}^{n+1} = 0$ yields $h_{aa}^{n+1} = h_{nn}^{n+1}, \ \forall a = 1, \cdots, n-1$, *i.e.*, $h_{ii}^{n+1} = H, \forall i$. Since $h_{ij|k}^{n+1} = 0$, it can be seen from the second formula of Proposition 2.3 that

$$h_{lk}^{n+1}\Psi_{lij} - h_{lj}^{n+1}\Psi_{lik} = 0. \tag{4.9}$$

Set $j = n, k = \lambda$ in (4.9); by $h_{ij}^{n+1} = 0, \forall i \ne j$ we obtain that

$$h_{\lambda\lambda}^{n+1}\Psi_{\lambda in} - h_{nn}^{n+1}\Psi_{ni\lambda} = 0, \tag{4.10}$$

which together with (2.4) yields

$$h_{nn}^{n+1}\overline{A}_{ijn+1} = 0, \forall i, j. \tag{4.11}$$

So we get that $\Psi_{ijk} = 0$ on M. Using the first formula of Proposition 2.2, we have that $P_{ij\lambda}^n = 0$ on M, thus M is a Landsberg space. It is easy to see from the second formula of Proposition 2.2 and $h_{ij}^{n+1} = H\delta_{ij}$ that

$$R_{ikl}^j = H(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{4.12}$$

It can be seen from $h_{ii}^{n+1} = H$ and the constant $H_2 = \frac{1}{n(n-1)}[(nH)^2 - S] > 0$ that $H \ne 0$ is constant, hence we get that M is a Landsberg space with the nonzero constant flag curvature H, which together with Lemma 3.4 finishes the proof of Theorem 4.2.

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