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Extension of holomorphic functions defined on singular complex hypersurfaces with growth estimates

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Abstract. Let D be a strictly convex domain and X be a singular complex hypersurface in $\mathbb{C}^n$ such that $X \cap D \neq \emptyset$ and $X \cap bD$ is transverse. We first give necessary conditions for a function holomorphic on $D \cap X$ to admit a holomorphic extension belonging to $L^q(D)$, with $q \in [1, +\infty]$. When $n = 2$ and $q < +\infty$, we then prove that this condition is also sufficient. When $q = +\infty$ we prove that this condition implies the existence of a BMO-holomorphic extension. In both cases, the extensions are given by mean of integral representation formulas and new residue currents.

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1. Introduction

In the last few years, many classical problems in complex analysis have been investigated in the framework of singular spaces; for example the $\overline{\partial}$-Neumann operator has been studied in [34] by Ruppenthal, the Cauchy-Riemann equation in [6,17,21,32,33] by Andersson, Samuelsson, Diederich, Fornæss, Vassiliadou, Ruppenthal, ideals of holomorphic functions on analytic spaces in [5] by Andersson, Samuelsson and Sznajdman, problems of extensions and restrictions of holomorphic functions on analytic spaces in [18,20] by Diederich, Mazzilli and Duquenoy.

In this article we will be interested in problems of extension of holomorphic functions defined on a singular complex hypersurface. Let D be a bounded pseudoconvex domain of $\mathbb{C}^n$ with smooth boundary, let f be a holomorphic function in a neighbourhood of D and let $X = \{z : f(z) = 0\}$ be a singular complex hypersurface such that $D \cap X \neq \emptyset$. The first extension problem that one can consider is the following one: Is it true that a function g which is holomorphic on $D \cap X$ has a holomorphic extension to D?

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It is known by Cartan's theorem B that the answer to this question is affirmative and that any function g holomorphic on $X \cap D$ has a holomorphic extension G on the whole domain D if and only if D is pseudoconvex. More difficulties arise when we ask G to satisfy some growth conditions like being in $L^q(D)$ or in $BMO(D)$. This question has been widely studied by many authors under different assumptions on D or X. In [28], Ohsawa and Takegoshi proved when X is a hyperplane that any $g \in L^2(X \cap D) \cap \mathcal{O}(X \cap D)$ admits an extension $G \in L^2(D) \cap \mathcal{O}(D)$. This result was generalized to the case of manifolds of higher codimension in [29] by Ohsawa. In [8], Berndtsson investigated the case of singular varieties and obtained a condition on g which implies that it admits a holomorphic L^2 extension to D. However this condition requires that g vanishes on the singularities of X and thus $g \equiv 1$ does not satisfy this condition while it can trivially be extended holomorphically.

Assuming that D is strictly pseudoconvex and that X is a manifold, Henkin proved in [22] that any $g \in L^\infty(D \cap X) \cap \mathcal{O}(D \cap X)$ has an extension in $L^\infty(D) \cap \mathcal{O}(D)$, provided that bD, the boundary of D, and X are in general position. Cumenge in [12] generalized this result to the case of Hardy spaces and Amar in [3] removed the hypothesis of general position of bD and X assumed in [22]. The case of L^∞ extensions has also been investigated in the case of weak (pseudo)convexity. In [19] Diederich and Mazzilli proved that when D is convex of finite type and X is a hyperplane, any $g \in L^\infty(D \cap X) \cap \mathcal{O}(D \cap X)$ is the restriction of some $G \in L^\infty(D) \cap \mathcal{O}(D)$. In [1], again for D convex of finite type but for X a manifold, a sufficient and nearly necessary condition on X was given under which any function g which is bounded and holomorphic on $X \cap D$ is the restriction of a bounded holomorphic function on D. This restriction problem was also studied in [24] by Jasiczak for D a pseudoconvex domain of finite type in $\mathbb{C}^2$ and X a manifold.

In this article we consider a strictly convex domain D in $\mathbb{C}^n$ and a singular complex hypersurface X of $\mathbb{C}^n$ such that $X \cap D \neq \emptyset$ and $X \cap bD$ is transverse in the sense of tangent cones. We give necessary and, for $n = 2$, sufficient conditions under which a function g holomorphic on $X \cap D$ admits a holomorphic extension in the class $BMO(D)$ or $L^q(D)$, $q \in [1, +\infty)$.

Let us write D as $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ where ρ is a smooth strictly convex function defined on $\mathbb{C}^n$ such that the gradient of ρ does not vanish in a neighborhood $\mathcal{U}$ of bD. We denote by D_r, with $r \in \mathbb{R}$, the set $D_r = \{z \in \mathbb{C}^n,\ \rho(z) < r\}$, by η_ζ the outer unit normal to $bD_{\rho(\zeta)}$ at a point $\zeta \in \mathcal{U}$ and by v_ζ a smooth complex tangent vector field at ζ to $bD_{\rho(\zeta)}$. Our first result is the following:

Theorem 1.1. *For $n = 2$ there exist two integers $k, l \geq 1$ depending only on X such that if g is a holomorphic function on $X \cap D$ which has a C^∞ smooth extension $\tilde{g}$ on D which satisfies*

(i) *there exists $N \in \mathbb{N}$ such that $|\rho|^N \tilde{g}$ vanishes at order l on bD,*
(ii) *there exists $q \in [1, +\infty]$ such that $\left| \frac{\partial^{\alpha+\beta} \tilde{g}}{\partial \overline{\eta_\zeta}^\alpha \partial \overline{v_\zeta}^\beta} \right| |\rho|^{\alpha + \frac{\beta}{2}}$ belongs to $L^q(D)$ for all non-negative integers α and β with $\alpha + \beta \leq k$,*
(iii) $\frac{\partial^{\alpha+\beta} \tilde{g}}{\partial \overline{\eta_\zeta}^\alpha \partial \overline{v_\zeta}^\beta} = 0$ *on $X \cap D$ for all non-negative integers α and β with $0 < \alpha + \beta \leq k$*

then g has a holomorphic extension G in $L^q(D)$ when $q < +\infty$ and in $BMO(D)$ when $q = +\infty$. Moreover, up to a uniform multiplicative constant depending only on k, l and N, the norm of G is bounded by the supremum of the L^q-norm of $\zeta \mapsto \left|\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^{\alpha}\partial\overline{v_\zeta}^{\beta}}(\zeta)\right| |\rho(\zeta)|^{\alpha+\frac{\beta}{2}}$ for α, β with $\alpha+\beta \leq k$.

In Lemma 5.2, Corollary 5.3 and Theorem 5.5, we will give conditions under which a function g holomorphic on $X \cap D$ admits a smooth extension to D which satisfies the assumptions of Theorem 1.1.

Let us mention that the integer k in Theorem 1.1 can be taken equal to the maximum of the multiplicities of the singularities of X, and that the hypothesis of Theorem 1.1 can be relaxed a little in the following way. The theorem is still valid if for all the singularities $z_0 \in X \cap \overline{D}$ of X of multiplicity k_0 we check the hypotheses (ii) and (iii) with k replaced by k_0 and D replaced by $\mathcal{U}_0 \cap D$, where $\mathcal{U}_0$ is a neighbourhood of z_0.

The holomorphic extension of Theorem 1.1 is given by an integral operator combining the Berndtsson-Andersson reproducing kernel and a residue current. The classical residue current $\overline{\partial}\left[\frac{1}{f}\right]$ was defined in [23] by Herrera and Lieberman using Hironaka's Theorem on resolution of singularities. Its importance in the problem of extension was pointed out for the first time in [3] by Amar; and the extension used in [20] is given by an operator constructed by Passare, which uses this classical current (see [30]). However, as pointed out in [20], it is not so easy to handle the case of singularities of multiplicity greater than 2 and this current does not give a good extension in this case. This difficulty arises from the definition of the $\overline{\partial}\left[\frac{1}{f}\right]$ itself which uses Hironaka's Theorem. Hence the current $\overline{\partial}\left[\frac{1}{f}\right]$ is not explicit enough and it does not yield an extension with sufficiently precise growth estimates on the boundary.

To overcome this difficulty we have to adapt a construction due to the second author of new residue currents which will play the role of $\overline{\partial}\left[\frac{1}{f}\right]$ (see [25] and [26]). The extension given by Theorem 1.1 will be obtained via a linear operator which uses a Berndtsson-Andersson reproducing kernel and these new currents (see Section 3). We observe that these currents can also be defined in the case of higher codimension in $\mathbb{C}^n$, but the situation is more complicated: the currents are more difficult to define, less explicit and so more difficult to handle (see [26]).

Observe that in Theorem 1.1 we assume the existence of a smooth extension $\tilde{g}$ satisfying properties (i), (ii) and (iii), whereas no such assumption is made in the previous articles we quoted and which deal with extension problems. It should be pointed out that while boundedness is a sufficient hypothesis in order to obtain a bounded holomorphic extension when X is a manifold (see [1,3,12,19]), it is not possible to obtain L^∞ or even L^2 extensions when X has singularities if we only assume that g is bounded on $X \cap D$ (see [18]): a stronger condition is needed. Actually, even if in the manifold case no smooth extension is assumed to exist, a smooth extension, which satisfies (ii) and (iii), is constructed for example in [1, 12,19]. This is done as follows. When X is a manifold, let us locally write X as

$X = \{(z', \alpha(z')),\ z' \in \mathbb{C}^{n-1}\}$, with α holomorphic. If for $z = (z_1, \dots, z_n)$ we set $z' = (z_1, \dots, z_{n-1})$, then the function $\tilde{g}$ defined by $\tilde{g}(z) := g(z', \alpha(z'))$ is a local holomorphic extension of g. Gluing all these local extensions together we get a smooth extension which will satisfy (ii) and (iii). In some sense, the way the local holomorphic extension is constructed in the manifold case is a kind of interpolation: $\tilde{g}(z', \cdot)$ is the polynomial of degree 0 which interpolates $g(z', \alpha(z'))$ at the point $z_n = \alpha(z')$. Following this idea, we will construct in Section 5 a local holomorphic extension by interpolation. Provided we have a good control of the polynomials which interpolate g on the different sheets of X, gluing together these local extensions, we will obtain an appropriate smooth extension. The control of the interpolating polynomials will be achieved thanks to an assumption on the divided differences we can build with g between the different sheets of X. This will give us simple numerical conditions under which the function g has a smooth extension $\tilde{g}$ which satisfies (i), (ii) and (iii) from Theorem 1.1 (see Theorems 5.3 and 5.5). The divided differences are defined as follows:

For $z \in D$, a unit vector v in $\mathbb{C}^n$, and a positive real number ε we set $\Delta_{z,v}(\varepsilon) = \{z + \lambda v : |\lambda| < \varepsilon\}$ and

$$\tau(z, v, \varepsilon) = \sup\{\tau > 0 : \rho(z + \lambda v) - \rho(z) < \varepsilon \text{ for all } \lambda \in \mathbb{C},\ |\lambda| < \tau\}.$$

Therefore $\tau(z, v, \varepsilon)$ is the maximal radius $r > 0$ such that the disc $\Delta_{z,v}(r)$ is in $D_{\rho(z)+\varepsilon}$. It is also the distance from z to $bD_{\rho(z)+\varepsilon}$ in the direction v. For a small positive real number κ, to be chosen later on, we set

$$\Lambda_{z,v} = \{\lambda \in \mathbb{C} : |\lambda| < 3\kappa\tau(z, v, |\rho(z)|) \text{ and } z + \lambda v \in X\}.$$

The points $z+\lambda v$, for $\lambda \in \Lambda_{z,v}$, are the points of X which belong to $\Delta_{z,v}(3\kappa\tau(z, v, |\rho(z)|))$, thus they all belong to D provided $\kappa < \frac{1}{3}$.

For $\lambda \in \Lambda_{z,v}$ let us define $g_{z,v}[\lambda] = g(z + \lambda v)$ and if $g_{z,v}[\lambda_1, \dots, \lambda_k]$ is defined, let us set for $\lambda_1, \dots, \lambda_k, \lambda_{k+1}$ belonging to $\Lambda_{z,v}$ and pairwise distinct

$$g_{z,v}[\lambda_1, \dots, \lambda_{k+1}] = \frac{g_{z,v}[\lambda_1, \dots, \lambda_k] - g_{z,v}[\lambda_2, \dots, \lambda_{k+1}]}{\lambda_1 - \lambda_{k+1}}.$$

Let us notice that the divided differences can be defined in this way in the case of codimension 1 only and not in the case of varieties of higher codimension. Our approach therefore cannot be applied in this latter case. Now consider the quantity

$$c_\infty(g) = \sup |g_{z,v}[\lambda_1, \dots, \lambda_k]| \tau(z, v, |\rho(z)|)^{k-1}$$

where the supremum is taken over all $z \in D$, all $v \in \mathbb{C}^n$ with $|v| = 1$ and all $\lambda_1, \dots, \lambda_k \in \Lambda_{z,v}$ pairwise distinct. In Section 5, we will prove that the finiteness of $c_\infty(g)$ implies the existence of a smooth extension $\tilde{g}$ which satisfies the hypothesis of Theorem 1.1. We will then obtain the following:

Theorem 1.2. *In $\mathbb{C}^2$, any function g holomorphic on $X \cap D$ such that $c_\infty(g)$ is finite admits a holomorphic extension G which belongs to $BMO(D)$ such that $\|G\|_{BMO(D)}$ is bounded, up to a multiplicative uniform constant, by $c_\infty(g)$.*

Conversely, if we know that g admits a bounded holomorphic extension G on D and if λ_1, λ_2 belong to $\Lambda_{z,v}$, Montel in [27] proves that there exists a point a in the unit disc of $\mathbb{C}$ and μ in the segment $[\lambda_1, \lambda_2]$ such that $\frac{g_{z,v}(\lambda_1)-g_{z,v}(\lambda_2)}{\lambda_1-\lambda_2}$ can be written as $a\frac{\partial G}{\partial v}(z+\mu v)$. But since G is bounded, its derivative $\frac{\partial G}{\partial v}(z+\mu v)$, and therefore the divided difference $\frac{g_{z,v}(\lambda_1)-g_{z,v}(\lambda_2)}{\lambda_1-\lambda_2}$ as well, is bounded by $\|G\|_{L^\infty(D)}$ times the inverse of the distance from $z+\mu v$ to the boundary of D in the direction v, and this quantity is comparable to $\tau(z, v, |\rho(z)|)$. We will show in Section 5 that this necessary condition holds in fact in $\mathbb{C}^n$, $n \geq 2$, and for more than two points λ_1 and λ_2, and so we will prove the following:

Theorem 1.3. *In $\mathbb{C}^n$, with $n \geq 2$, if a function g holomorphic on $X \cap D$ admits an extension G which is bounded and holomorphic on D, then $c_\infty(g)$ is finite.*

In Section 5 we will also study the case of L^q extensions and, still using divided differences, we will give in $\mathbb{C}^n$, with $n \geq 2$, a necessary condition for a function g holomorphic on $X \cap D$ to admit a holomorphic extension to D which belong to $L^q(D)$. Then we will also prove that this condition is sufficient when $n = 2$ (see Theorems 5.4, 5.5 and 5.6 for precise statements). We will also see in Section 5, Theorems 5.10 and 5.11, that all these results can be generalized in a natural way to weakly holomorphic functions in the sense of Remmert.

A condition using divided differences was already used in [20] but only varieties with singularities of multiplicity 2 were considered there. Here we have no restriction on the multiplicity of the singularities, and our condition uses all the divided differences of degree at most the multiplicities of the singularities.

In Section 6, we illustrate these conditions by examples. Among other things, when D is the ball of center $(1,0)$ and radius 1 and $X = \{(z = (z_1, z_2) \in \mathbb{C}^2 : z_1^q = z_2^2\}$, with q a positive odd integer, we will prove that any function g holomorphic and bounded on $X \cap D$ has a L^2-holomorphic extension to D if and only if $q = 1$ or $q = 3$.

The article is organized as follows. In Section 2 we fix our notation and recall some results concerning the Berndtsson-Andersson kernel. In Section 3 we construct the new residue current adapted to our extension problem, and we prove Theorem 1.1 in Section 4. In Section 5 we prove Theorems 1.2 and 1.3 and we treat the case of L^q holomorphic extensions. We give examples of applications of our results in Section 6.

2. Notation and tools

As usually, when BMO questions or estimates of integral kernels arise in this context, the Koranyi balls or McNeal polydiscs, their generalization for convex domains of finite type, naturally appear (see [2,4,13] for example). This will be of course the case in this article, but here (and it seems to be the first time this happens) the Koranyi balls will appear directly in the construction of the residue current, and so in the construction of a good extension. These balls enable us to establish a connection between the geometric properties of the boundary of the domain and the geometric properties of the variety (see Section 3). The second classical tool we use is the Berndtsson-Andersson reproducing kernel which we also recall in this section.

2.1. Notation

Let us first fix our notation and adopt the following convention. We will often have estimates up to multiplicative constants. For readability convenience we introduce the following notation: We write $A \lesssim B$ if there exists some constant $c > 0$ such that $A \leq cB$. Each time we will mention on which parameters c depends. We will write $A \eqsim B$ if $A \lesssim B$ and $B \lesssim A$ both hold.

We write X as $X = \{z : f(z) = 0\}$ where f is a holomorphic function defined in a neighbourhood of $\overline{D}$. Without restriction we assume that f is minimal (see [10], Theorem 3, paragraph 50). We denote by η_ζ the outer unit normal to $bD_{\rho(\zeta)}$ at a point $\zeta \in \mathcal{U}$ and by v_ζ a smooth complex tangent vector field at ζ to $bD_{\rho(\zeta)}$.

2.2. Koranyi balls in $\mathbb{C}^2$

We call the coordinate system centred at ζ of basis η_ζ, v_ζ the Koranyi coordinate system at ζ. We denote by (z_1^*, z_2^*) the coordinates of a point z in the Koranyi coordinates system centred at ζ. The Koranyi ball centred at ζ of radius r is the set $\mathcal{P}_r(\zeta) := \{\zeta + \lambda\eta_\zeta + \mu v_\zeta \ : \ |\lambda| < r, \ |\mu| < r^{\frac{1}{2}}\}$. These balls have the following properties:

Proposition 2.1. *There exists a neighbourhood $\mathcal{U}$ of bD and positive real numbers κ and c_1 such that:*

(i) *for all $\zeta \in \mathcal{U} \cap D$, $\mathcal{P}_{4\kappa|\rho(\zeta)|}(\zeta)$ is included in D;*
(ii) *for all $\varepsilon > 0$, all $\zeta, z \in \mathcal{U}$, $\mathcal{P}_\varepsilon(\zeta) \cap \mathcal{P}_\varepsilon(z) \neq \emptyset$ implies $\mathcal{P}_\varepsilon(z) \subset \mathcal{P}_{c_1\varepsilon}(\zeta)$;*
(iii) *for all $\varepsilon > 0$ sufficiently small, all $z \in \mathcal{U}$, all $\zeta \in \mathcal{P}_\varepsilon(z)$ we have $|\rho(z) - \rho(\zeta)| \leq c_1\varepsilon$;*
(iv) *For all $\varepsilon > 0$, all unit vector $v \in \mathbb{C}^n$, all $z \in \mathcal{U}$ and all $\zeta \in \mathcal{P}_\varepsilon(z)$, $\tau(z, v, \varepsilon) \eqsim \tau(\zeta, v, \varepsilon)$ uniformly with respect to ε, v , z and ζ.*

For $\mathcal{U}$ given by Proposition 2.1 and z and ζ belonging to $\mathcal{U}$, we set $\delta(z,\zeta) = \inf\{\varepsilon > 0, \zeta \in \mathcal{P}_\varepsilon(z)\}$. Proposition 2.1 implies that δ is a pseudo-distance in the following sense:

Proposition 2.2. *For $\mathcal{U}$ and c_1 given by Proposition 2.1 and for all z, ζ and ξ belonging to $\mathcal{U}$ we have*

$$\frac{1}{c_1}\delta(\zeta, z) \leq \delta(z, \zeta) \leq c_1\delta(\zeta, z)$$

and

$$\delta(z, \zeta) \leq c_1(\delta(z, \xi) + \delta(\xi, \zeta)).$$

2.3. Berndtsson-Andersson reproducing kernel in $\mathbb{C}^2$

We now recall the definition of the Berndtsson-Andersson kernel of D when D is a strictly convex domain of $\mathbb{C}^2$. We set $h_i(\zeta, z) = -\frac{\partial\rho}{\partial\zeta_i}(\zeta)$, $h = \sum_{i=1,2} h_i d\zeta_i$ and $\tilde{h} = \frac{1}{\rho}h$. For a $(1,0)$-form $\beta(\zeta, z) = \sum_{i=1,2}\beta_i d\zeta_i$ we set $\langle\beta(\zeta, z), \zeta - z\rangle = \sum_{i=1,2}\beta_i(\zeta, z)(\zeta_i - z_i)$. Then we define the Berndtsson-Andersson reproducing kernel by setting for an arbitrary positive integer N, $n = 1, 2$ and all $\zeta, z \in D$

$$P^{N,n}(\zeta, z) = C_{N,n}\left(\frac{1}{1 + \langle\tilde{h}(\zeta, z), \zeta - z\rangle}\right)^{N+n}\left(\overline{\partial}\tilde{h}\right)^n,$$

where $C_{N,n} \in \mathbb{C}$ is a constant. We also set $P^{N,n}(\zeta, z) = 0$ for all $z \in D$ and all $\zeta \notin D$.

In order to keep in mind an explicit example of a Berndtsson-Andersson's kernel during the computations, we give the expression of this kernel when D is the unit ball of $\mathbb{C}^2$. In this case $\rho(\zeta) = |\zeta|^2 - 1$,

$$P^{N,2}(\zeta, z) = \tilde{C}_{N,2}\frac{(1 - |\zeta|^2)^{N-1}}{(1 - \overline{\zeta}_1 z_1 - \overline{\zeta}_2 z_2)^{N+2}} d\zeta_1 \wedge d\overline{\zeta}_1 \wedge d\zeta_2 \wedge d\overline{\zeta}_2$$

and

$$P^{N,1}(\zeta, z) = \tilde{C}_{N,1}\frac{(1 - |\zeta|^2)^{N}}{(1 - \overline{\zeta}_1 z_1 - \overline{\zeta}_2 z_2)^{N+1}}\Bigg(d\zeta_1 \wedge d\overline{\zeta}_1 + d\zeta_2 \wedge d\overline{\zeta}_2 + \sum_{j,k=1,2}\frac{\zeta_j\overline{\zeta}_k d\overline{\zeta}_j \wedge d\zeta_k}{1 - |\zeta|^2}\Bigg).$$

The following representation formula holds (see [7]):

Theorem 2.3. *For all $g \in \mathcal{O}(D) \cap C^\infty(\overline{D})$ we have*

$$g(z) = \int_D g(\zeta) P^{N,2}(\zeta, z).$$

In the estimations of this kernel, we will need to write h in the Koranyi coordinates at some point ζ_0 belonging to D. We set for $i = 1, 2$ $h_i^* = -\frac{\partial\rho}{\partial\zeta_i^*}(\zeta)$. Then h is equal to $\sum_{i=1,2} h_i^* d\zeta_i^*$ and satisfies the following:

Proposition 2.4. *There exists a neighbourhood $\mathcal{U}$ of bD such that for all $\zeta \in D\cap\mathcal{U}$, all $\varepsilon > 0$ sufficiently small and all $z \in \mathcal{P}_\varepsilon(\zeta)$ we have*

(i) $|\rho(\zeta) + \langle h(\zeta, z), \zeta - z\rangle| \gtrsim \varepsilon + |\rho(\zeta)| + |\rho(z)|$,
(ii) $|h_1^*(\zeta, z)| \lesssim 1$,
(iii) $|h_2^*(\zeta, z)| \lesssim \varepsilon^{\frac{1}{2}}$,

and there exists $c > 0$ depending neither on ζ nor on ε such that for all $z \in \mathcal{P}_\varepsilon(\zeta) \setminus c\mathcal{P}_\varepsilon(\zeta)$ we have

$$|\langle h(\zeta, z), \zeta - z\rangle| \gtrsim \varepsilon + |\rho(z)| + |\rho(\zeta)|,$$

uniformly with respect to ζ, z and ε.

3. Construction of the extension operator

The holomorphic extension provided by Theorem 1.1 will be given by a linear integral operator. Its definition is based on the construction of Mazzilli in [25] which uses the Berndtsson-Andersson reproducing kernel and a current T such that $fT = 1$. The current T relies on a family of currents $T_{\mathcal{V}}$, where $\mathcal{V}$ is an open subset of D, such that $fT_{\mathcal{V}} = 1$ on $\mathcal{V}$. Then using a locally finite covering $\left(\mathcal{V}_j\right)_{j\in\mathbb{N}}$ of D and a partition of unity $\left(\chi_j\right)_{j\in\mathbb{N}}$ associated with this covering, Mazzilli glues together all the currents $T_{\mathcal{V}_j}$ and gets a current $T = \sum_{j\in\mathbb{N}} \chi_j T_{\mathcal{V}_j}$ such that $fT = 1$ on D. In [25], the only assumption on the covering $\left(\mathcal{V}_j\right)_j$ is to be locally finite.

In order to get very fine estimates of the operator, instead of an ordinary locally finite covering, we will use a covering of D by Koranyi balls $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ which will be more suited to the geometry of bD (see Subsection 3.1).

In [25], the local current $T_{\mathcal{V}}$ is constructed using the Weierstrass polynomial P_f of f in the open set $\mathcal{V}$. This means that every root of P_f, or equivalently every sheet of X intersecting $\mathcal{V}$, are used. We will modify the construction of $T_{\mathcal{V}}$ in order to use only the sheets of X which are meaningful for our purpose. In order to be able to choose the good sheets of X, we construct in Subsection 3.2 for z_0 near bD a parametrization of X in the Koranyi ball $\mathcal{P}_{\kappa|\rho(z_0)|}(z_0)$.

At last, we will have all the tools to define in Subsection 3.3 the current T such that $fT = 1$ and the extension operator.

3.1. Koranyi covering

In this subsection, for $\varepsilon_0 > 0$, we cover $D \setminus D_{-\varepsilon_0}$ with a family of Koranyi balls $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ where κ is a positive small real number. This construction uses classical ideas of the theory of homogeneous spaces and is analogous to the construction of the covering of [9].

Let ε_0, κ and c be sufficiently small positive real numbers. We construct a sequence of point of $D \setminus D_{-\varepsilon_0}$ as follows. Let k be a non-negative integer and choose $z_1^{(k)}$ in $bD_{-(1-c\kappa)^k\varepsilon_0}$ arbitrarily. When $z_1^{(k)}, \dots, z_j^{(k)}$ are chosen, there are two possibilities. Either for all $z \in bD_{-(1-c\kappa)^k\varepsilon_0}$ there exists $i \leq j$ such that $\delta(z, z_i^{(k)}) < c\kappa(1 - c\kappa)^k\varepsilon_0$ and the process ends here, or there exists $z \in bD_{-(1-c\kappa)^k\varepsilon_0}$ such that for all $i \leq j$ we have $\delta(z, z_i^{(k)}) \geq c\kappa(1 - c\kappa)^k\varepsilon_0$ and we chose $z_{j+1}^{(k)}$ among these points. Since $D_{-(1-c\kappa)^k\varepsilon_0}$ is bounded, this process stops at some rank n_k. We thus have constructed a sequence $(z_j^{(k)})_{k\in\mathbb{N},\, j\in\{1,\dots,n_k\}}$ such that:

(i) For all $k \in \mathbb{N}$, and all $j \in \{1,\dots,n_k\}$, $z_j^{(k)}$ belongs to $bD_{-(1-c\kappa)^k\varepsilon_0}$;
(ii) For all $k \in \mathbb{N}$, all $i, j \in \{1,\dots,n_k\}$, $i \neq j$, we have $\delta(z_i^{(k)}, z_j^{(k)}) \geq c\kappa(1-c\kappa)^k\varepsilon_0$;
(iii) For all $k \in \mathbb{N}$, all $z \in bD_{-(1-c\kappa)^k\varepsilon_0}$, there exists $j \in \{1,\dots,n_k\}$ such that $\delta(z, z_j^{(k)}) < c\kappa(1-c\kappa)^k\varepsilon_0$.

For such sequences, we prove the following:

Proposition 3.1. *For $\kappa > 0$ and $c > 0$ small enough, let $\left(z_j^{(k)}\right)_{k\in\mathbb{N},\, j\in\{1,\dots,n_k\}}$ be a sequence which satisfies* (i)*,* (ii) *and* (iii)*. Then:*

(a) *$D \setminus D_{-\varepsilon_0}$ is included in $\cup_{k=0}^{+\infty} \cup_{j=1}^{n_k} \mathcal{P}_{\kappa|\rho(z_j^{(k)})|}\left(z_j^{(k)}\right)$;*
(b) *there exists $M \in \mathbb{N}$ such that for $z \in D \setminus D_{-\varepsilon_0}$, $\mathcal{P}_{4\kappa|\rho(z)|}(z)$ intersect at most M Koranyi balls $\mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}\left(z_j^{(k)}\right)$.*

Proof. We first prove that (a) holds. For $z \in D \setminus D_{\varepsilon_0}$ let $k \in \mathbb{N}$ be such that

$$(1-c\kappa)^{k+1}\varepsilon_0 < |\rho(z)| < (1-c\kappa)^k\varepsilon_0$$

and let $\lambda \in \mathbb{C}$ be such that $\zeta = z + \lambda\eta_z$ belong to $bD_{-(1-c\kappa)^k\varepsilon_0}$. On the one hand the assumption (iii) implies that there exists $j \in \{1,\dots,n_k\}$ such that $\delta\left(\zeta, z_j^{(k)}\right) \leq c\kappa(1-c\kappa)^k\varepsilon_0$. On the other one hand we have $|\lambda| = \delta(z,\zeta) \leq Cc\kappa(1-c\kappa)^k\varepsilon_0$ where C depends neither on z nor on ζ nor on c. These two inequalities yield

$$\begin{aligned}
\delta\left(z, z_j^{(k)}\right) &\leq c_1(\delta(z,\zeta) + c_1\delta(\zeta, z_j^{(k)})\\
&\leq \kappa c c_1(1-c\kappa)^k\varepsilon_0(C\kappa+1)\\
&\leq \kappa\left|\rho\left(z_j^{(k)}\right)\right|
\end{aligned}$$

provided c is small enough. Therefore z belongs to $\mathcal{P}_{\kappa|\rho(z_j^{(k)})|}(z_j^{(k)})$ and (a) holds.

We now prove (b). Let z be a point of $D \setminus D_{-\varepsilon_0}$. For all $\zeta \in \mathcal{P}_{4\kappa|\rho(z)|}(z)$, if κ is small enough, Proposition 2.1 yields

$$\frac{1}{2}|\rho(z)| \leq |\rho(\zeta)| \leq 2|\rho(z)|.$$

The same inequalities hold for all $z_j^{(k)}$ and all $\zeta \in \mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}(z_j^{(k)})$.

Thus if $\mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}(z_j^{(k)}) \cap \mathcal{P}_{\kappa|\rho(z)|}(z) \neq \emptyset$ we have

$$\frac{1}{4}|\rho(z)| \leq (1-c\kappa)^k \leq 4|\rho(z)|.$$

Therefore k can take at most $\frac{4\ln 2}{|\ln(1-c\kappa)|}$ values.

For such a k, we set $I_k = \left\{ j \in \{1, \dots, n_k\}, \mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}(z_j^{(k)}) \cap \mathcal{P}_{4\kappa|\rho(z)|}(z) \neq \emptyset \right\}$. Assertion (b) will be proved provided we show that $\# I_k$, the cardinality of I_k, is bounded uniformly with respect to k and z.

We denote by σ the area measure on $bD_{-(1-c\kappa)^k\varepsilon_0}$. Since for all distinct $i, j \in I_k$ we have $\delta\left(z_i^{(k)}, z_j^{(k)}\right) \geq c\kappa(1-c\kappa)^k\varepsilon_0$, provided c is small enough, we have

$$\begin{aligned}
&\sigma\left(\cup_{j\in I_k}\mathcal{P}_{4\kappa\left|\rho\left(z_j^{(k)}\right)\right|}\left(z_j^{(k)}\right) \cap bD_{-(1-c\kappa)^k\varepsilon_0}\right)\\
&\geq \sigma\left(\cup_{j\in I_k}\mathcal{P}_{\frac{c}{c_1}\kappa(1-c\kappa)^k\varepsilon_0}\left(z_j^{(k)}\right) \cap bD_{-(1-c\kappa)^k\varepsilon_0}\right)\\
&\geq \# I_k\left(\frac{c}{c_1}\kappa(1-c\kappa)^k\varepsilon_0\right)^2.
\end{aligned}$$

Now we look for an upper bound of $\sigma\left(\cup_{j\in I_k}\mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}(z_j^{(k)}) \cap bD_{-(1-c\kappa)^k\varepsilon_0}\right)$. We fix $j_0 \in I_k$. For all $j \in I_k$, since $\mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}(z_j^{(k)}) \cap \mathcal{P}_{4\kappa|\rho(z)|}(z) \neq \emptyset$ and $\mathcal{P}_{4\kappa|\rho(z_{j_0}^{(k)})|}(z_{j_0}^{(k)}) \cap \mathcal{P}_{4\kappa|\rho(z)|}(z) \neq \emptyset$, we have

$$\begin{aligned}
\delta\left(z_{j_0}^{(k)}, z_j^{(k)}\right) &\lesssim \delta\left(z_{j_0}^{(k)}, z\right) + \delta\left(z, z_j^{(k)}\right)\\
&\lesssim 4\kappa\left(\left|\rho\left(z_{j_0}^{(k)}\right)\right| + \left|\rho\left(z_j^{(k)}\right)\right|\right)\\
&\lesssim \kappa(1-c\kappa)^k\varepsilon_0
\end{aligned}$$

uniformly with respect to k, j and j_0. Thus there exists K depending neither on z, nor on j, nor on j_0 nor on k such that $\mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}(z_j^{(k)}) \subset \mathcal{P}_{\kappa K|\rho(z_{j_0}^{(k)})|}(z_{j_0}^{(k)})$. Therefore

$$\begin{aligned}
\sigma\left(\cup_{j\in I_k}\mathcal{P}_{4\kappa|\rho(z_j^{(k)})|}(z_j^{(k)}) \cap bD_{-(1-c\kappa)^k\varepsilon_0}\right) &\leq \sigma\left(\mathcal{P}_{4K\kappa|\rho(z_{j_0}^{(k)})|}(z_{j_0}^{(k)}) \cap bD_{-(1-c\kappa)^k\varepsilon_0}\right)\\
&\lesssim (K\kappa(1-c\kappa)\varepsilon_0)^2
\end{aligned}$$

which yields $\# I_k \lesssim c^{-2}$. □

The covering property (a) allows us to settle the following definition

Definition 3.2. Let $\mathcal{U}$ be any subset of $\mathbb{C}^2$. If the sequence $(z_j)_{j\in\mathbb{N}}$ can be renumbered such that (i) and (ii) are satisfied and such that (iii) holds true for all $z \in \mathcal{U} \cap (D \setminus D_{-\varepsilon_0})$, the family $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ will be called a κ-covering of $\mathcal{U} \cap (D \setminus D_{-\varepsilon_0})$.

3.2. A family of parametrizations

In order to construct the current we use to define our extension operator, we will need some kind of parametrization for X over $\mathcal{P}_{\kappa|\rho(z_0)|}(z_0)$ when z_0 is near the boundary of the domain and when $\mathcal{P}_{\kappa|\rho(z_0)|}(z_0) \cap X \neq \emptyset$. Moreover, we will need some uniform estimates for this parametrization, which are achievable only in the case where the intersection of X and bD is transverse. Of course if we are near a regular point of X, such parametrizations do exist but the situation is more delicate when we are near a singularity of X.

Given a point z_0 near a singularity ζ_0 of X which belongs to bD, we denote by $(\zeta^*_{0,1}, \zeta^*_{0,2})$ the coordinates of ζ_0 in the Koranyi coordinates at z_0. We denote by Δ the unit disc of $\mathbb{C}$ and by $\Delta_z(r)$ the disc of $\mathbb{C}$ centred at z of radius r. Our goal in this subsection is to prove the following results:

Proposition 3.3. *If the intersection of bD and X is transverse at ζ_0, then there exist a neighbourhood $\mathcal{U}$ of ζ_0 and $\kappa > 0$ sufficiently small such that the following property holds: for all $z_0 \in D\cap\mathcal{U}$, if $X\cap\mathcal{P}_{2\kappa|\rho(z_0)|}(z_0) \neq \emptyset$, then $|\zeta^*_{0,1}| \geq 2\kappa|\rho(z_0)|$.*

Proposition 3.4. *If the intersection of bD and X is transverse at ζ_0, there exist κ and r positive real numbers sufficiently small, a positive integer p_0 and a neighbourhood $\mathcal{U}$ of ζ_0 such that for all $z_0 \in \mathcal{U}$, if $|\zeta^*_{0,1}| \geq \kappa|\rho(z_0)|$ then there exist $\alpha^*_1, \ldots, \alpha^*_{p_0}$ holomorphic functions in $\Delta_0(2\kappa|\rho(z_0)|)$ which satisfy the following:*

(i) *α^*_j and $\frac{\partial\alpha^*_j}{\partial z^*_1}$ are bounded on $\Delta_0(2\kappa|\rho(z_0)|)$ uniformly with respect to z_0;*
(ii) *if there exists j and z^*_1 such that $(z^*_1, \alpha^*_j(z^*_1))$ belong to $\mathcal{P}_{2\kappa|\rho(z_0)|}(z_0)$ then for all $\zeta^*_1 \in \Delta_0(2\kappa|\rho(z_0)|)$ we have $|\alpha^*_j(\zeta^*_1)| \leq (3\kappa|\rho(z_0)|)^{\frac{1}{2}}$;*
(iii) *There exists u_0 holomorphic in $\Delta_{z_0}(r)^2$ such that $|u_0| \eqsim 1$ uniformly with respect to z_0 and $f(\zeta) = u_0(\zeta)\prod_{i=1}^{p_0}(\zeta^*_2 - \alpha^*_i(\zeta^*_1))$ for all $\zeta \in \mathcal{P}_{2\kappa|\rho(z_0)|}(z_0)$.*

The proof relies on two preliminary results:

Lemma 3.5. *Let (A, d) be a metric space, let α_0 be an element of A and let $(f_\alpha)_{\alpha\in A}$ be a family of holomorphic function on Δ^2 such that*

- *$(f_\alpha)_{\alpha\in A}$ converges uniformly to f_{α_0} when α tends to α_0,*
- *$f_{\alpha_0}(0, \cdot) \neq 0$ and $f_{\alpha_0}(0) = 0$.*

Then there exist positive real numbers $r_1, r_2, \eta > 0$, a positive integer p such that, for all $\alpha \in A$ with $d(\alpha, \alpha_0) < \eta$, there exist p functions $a^{(\alpha)}_1, \ldots, a^{(\alpha)}_p$ holomorphic on $\Delta_0(r_1)$ and a function u_α holomorphic in $\Delta_0(r_1) \times \Delta_0(r_2)$ which satisfy the following:

(i) $f_\alpha(z) = u_\alpha(z)\left(z_2^p + a_1^{(\alpha)}(z_1)z_2^{p-1} + \ldots + a_p^{(\alpha)}(z_1)\right)$;
(ii) $|u_\alpha(z)| \eqsim 1$ *for all* $z \in \Delta_0(r_1) \times \Delta_0(r_2)$ *uniformly with respect to* z *and* α.

Proof. We first want to apply Rouché's theorem to $f_\alpha(z_1, \cdot) - f_{\alpha_0}(0, \cdot)$, z_1 fixed in $\Delta_0(r_1)$ where $r_1 > 0$ is to be chosen in a moment. Since $f_{\alpha_0}(0, \cdot)$ is not identically zero, there exists $r_2 > 0$ such that $f_{\alpha_0}(0, z_2) \neq 0$ for all $z_2 \in \Delta_0(r_2) \setminus \{0\}$. We denote by a the positive real number $a = \inf_{|z_2|=r_2} |f_{\alpha_0}(0, z_2)|$ and by p the multiplicity of the root 0 of $f_{\alpha_0}(0, \cdot)$. Since $(f_\alpha)_\alpha$ converges uniformly to f_{α_0} on $\Delta_0(1)$, there exists $\eta > 0$ such that for all $\alpha \in A$, with $d(\alpha_0, \alpha) < \eta$, and all $z \in \Delta_0(1)^2$ the following inequality holds: $\sup_{z\in\Delta_0(1)^2} |f_\alpha(z) - f_{\alpha_0}(z)| < \frac{a}{4}$. By Cauchy's inequalities, there exists $r_1 > 0$ such that for all $z \in \Delta_0(r_1) \times \Delta_0(r_2)$ we have $|f_{\alpha_0}(z_1, z_2) - f_{\alpha_0}(0, z_2)| < \frac{a}{4}$. Thus $|f_\alpha(z_1, z_2) - f_{\alpha_0}(0, z_2)| \leq |f_{\alpha_0}(0, z_2)|$ and by Rouché's theorem, $f_\alpha(z_1, \cdot)$ has exactly p zeros in $\Delta_0(r_2)$ for all z_1 fixed in $\Delta_0(r_1)$. Therefore by the Weierstrass preparation theorem there exist p functions $a_1^{(\alpha)}, \ldots, a_p^{(\alpha)}$ holomorphic on $\Delta_0(r_1)$ and a function u_α holomorphic on $\Delta_0(r_1) \times \Delta_0(r_2)$ zero free such that

$$f_\alpha(z) = u_\alpha(z)\left(z_2^p + a_1^{(\alpha)}(z_1)z_2^{p-1} + \ldots + a_p^{(\alpha)}(z_1)\right).$$

We set $P_\alpha(z_1, z_2) = z_2^p + a_1^{(\alpha)}(z_1)z_2^{p-1} + \ldots + a_p^{(\alpha)}(z_1)$. In order to finish the proof of the lemma we have to prove that $1 \lesssim |u_\alpha| \lesssim 1$. We prove the lower uniform boundedness.

For all $z_1 \in \Delta_0(r_1)$, $\frac{1}{u_\alpha(z_1,\cdot)}$ is holomorphic and

$$\frac{1}{|u_\alpha(z_1, z_2)|} \leq \max_{|\zeta_2|=r_2} \left|\frac{P_\alpha(z_1, \zeta_2)}{f_\alpha(z_1, \zeta_2)}\right|.$$

On the one hand, for all $\alpha \in A$ such that $d(\alpha, \alpha_0) < \eta$, all $(z_1, z_2) \in \Delta_0(r_1) \times b\Delta_0(r_2)$, we have

$$\begin{aligned}|f_\alpha(z)| &\geq |f_{\alpha_0}(0, z_2)| - |f_{\alpha_0}(z) - f_{\alpha_0}(0, z_2)| - |f_\alpha(z) - f_{\alpha_0}(z)|\\ &\geq a - \frac{a}{4} - \frac{a}{4} = \frac{a}{2}.\end{aligned}$$

On the other hand, since $(f_\alpha)_{\alpha\in A}$ converges uniformly to f_{α_0} when α tends to α_0 and since $f_\alpha(z)$ is uniformly bounded away from 0 for $(z_1, z_2) \in \Delta_0(r_1) \times b\Delta_0(r_2)$, $(a_j^{(\alpha)})_{\alpha\in A}$ converges uniformly to $a_j^{(\alpha_0)}$ for all j when α tends to α_0. This implies that $(P_\alpha)_{\alpha\in A}$ converges uniformly to P_{α_0} and therefore $\sup_{\Delta_0(r_1)\times\Delta_0(r_2)} |P_\alpha|$ is uniformly bounded for α near α_0. This yields $|u_\alpha(z)| \gtrsim 1$ uniformly with respect to $z \in \Delta_0(r_1) \times \Delta_0(r_2)$ and $\alpha \in A$ such that $d(\alpha, \alpha_0) < \eta$. The upper boundedness can be proved in the same way. □

The following result does not hold true if the intersection $X \cap bD$ is not transverse.

Lemma 3.6. *Let $\zeta_0 \in bD$ be a singularity of X, let $z_0 \in D$ be a point near enough ζ_0. There exist $r > 0$ not depending on z_0 and a parametric representation of X in the Koranyi coordinates system centred at z_0 of the form $(t^{*p} + \zeta^*_{0,1}, \phi(t^*) + \zeta^*_{0,2})$, such that $|\phi^*(t^*)| \lesssim |t^*|^p$, $t^* \in \Delta_0(r)$, uniformly with respect to z_0.*

Proof. Without restriction we assume that ζ_0 is the origin of $\mathbb{C}^2$. Maybe after a unitary linear change of coordinates if needed, there exists $r_0 > 0$, $p, q \in \mathbb{N}$, $q > p > 1$, and u holomorphic and bounded on $\Delta_0(r_0)$, $u(0) \neq 0$ such that $\phi : t \mapsto (t^p, t^q u(t))$ is a parametric representation of X over $\Delta_0(r_0)$.

We consider z_0 such that $|\zeta_0 - z_0| < r_0$ and we denote by (α, β) the coordinates of η_{z_0} and by $(-\overline{\beta}, \overline{\alpha})$ the coordinates of v_{z_0}. In the Koranyi coordinates centred at z_0, X is parametrized by $t \mapsto (\overline{\alpha} t^p + \overline{\beta} t^q u(t) + \zeta^*_{0,1}, -\beta t^p + \alpha t^q u(t) + \zeta^*_{0,2})$. Let (α_0, β_0) denote the coordinates of η_{ζ_0}. The transversality hypothesis implies that $\alpha_0 \neq 0$ so there exists $r_1 > 0$ and a p-th determination of the root ϕ_1 in $\Delta_{\overline{\alpha_0}}(r_1)$. If $r_0 > 0$ is sufficiently small, α belongs to $\Delta_{\alpha_0}(r_1)$ and $\overline{\alpha} t^p + \overline{\beta} t^q u(t) = (\phi_1(\overline{\alpha}) t)^p \left(1 + \frac{\overline{\beta}}{\overline{\alpha}} t^{q-p} u(t)\right)$. Since $q > p$, there exists $r_2 \in]0, r_1[$ such that for all $t \in \Delta_0(r_2)$, all $\beta \in \Delta_{\beta_0}(r_2)$ and all $\alpha \in \Delta_{\alpha_0}(r_2)$, we have $\left|1 + \frac{\overline{\beta}}{\overline{\alpha}} t^{q-p} u(t)\right| \geq \frac{1}{2}$ and so there exists ϕ_2 holomorphic for $t \in \Delta_0(r_2)$, C^∞-smooth for $\alpha \in \Delta_{\alpha_0}(r_2)$ and $\beta \in \Delta_{\beta_0}(r_2)$ such that $\phi_2(t, \alpha, \beta)^p = 1 + \frac{\overline{\beta}}{\overline{\alpha}} t^{q-p} u(t)$.

We apply the implicit functions theorem to $\Psi : (t, t^*, \alpha, \beta) \mapsto t^* - \phi_1(\overline{\alpha}) \phi_2(t, \alpha, \beta) t$. Since $\Psi(0, 0, \alpha_0, \beta_0) = 0$ and $\frac{\partial \Psi}{\partial t}(0, 0, \alpha_0, \beta_0) \neq 0$, there exist $r > 0$ and $\tilde{\psi} : \Delta_0(r) \times \Delta_{\alpha_0}(r) \times \Delta_{\beta_0}(r) \to V(0)$, $V(0)$ neighbourhood of $0 \in \mathbb{C}$ such that $\tilde{\psi}$ is holomorphic in t, and C^∞-smooth in α and β and which satisfies $t^{*p} = \overline{\alpha} t^p + \overline{\beta} t^q u(t)$ if and only if $t = \tilde{\psi}(t^*, \alpha, \beta)$. We now finish the proof of the lemma by setting

$$\phi^*(t^*) = -\beta \tilde{\psi}(t^*, \alpha, \beta)^p + \alpha \tilde{\psi}(t^*, \alpha, \beta)^q u\left(\tilde{\psi}(t^*, \alpha, \beta)\right). \qquad \square$$

Proof of Proposition 3.3. We first choose $\kappa > 0$ such that $2\kappa |\rho(z_0)| \leq r$, with r given by Lemma 3.6 and we write $\zeta \in X \cap \mathcal{P}_{2\kappa|\rho(z_0)|}(z_0)$ as $\zeta = (t^{*p_0} + \zeta^*_{0,1}, \phi^*(t^*) + \zeta^*_{0,2})$ for some t^* belonging to $\Delta_0(r)$. Now, if we assume that $\left|\zeta^*_{0,1}\right| < 2\kappa|\rho(z_0)|$ we get $|\zeta^*_1 - \zeta^*_{0,1}| \leq 4\kappa|\rho(z_0)|$ and therefore $|t^*| \leq (4\kappa|\rho(z_0)|)^{\frac{1}{p_0}}$. This yields

$$\begin{aligned} |\zeta^*_{0,2}| &\leq |\zeta^*_{0,2} - \zeta^*_2| + |\zeta^*_2| \\ &\leq |\phi^*(t^*)| + |\zeta^*_2| \\ &\lesssim \kappa|\rho(z_0)| + (\kappa|\rho(z_0)|)^{\frac{1}{2}} \\ &\lesssim (\kappa|\rho(z_0)|)^{\frac{1}{2}} \end{aligned}$$

uniformly with respect to z_0. Thus there exists $K > 0$ depending neither on z_0 nor on κ such that ζ_0 belongs to $\mathcal{P}_{\kappa K|\rho(z_0)|}(z_0)$. Moreover, if κ is chosen sufficiently small, for all $\xi \in \mathcal{P}_{\kappa K|\rho(z_0)|}(z_0)$ Proposition 2.1 gives $|\rho(\xi)| \geq \frac{1}{2}|\rho(z_0)|$. This gives a contradiction because $|\rho(\zeta_0)| = 0 < |\rho(z_0)|$ whereas ζ_0 belongs to $\mathcal{P}_{\kappa K|\rho(z_0)|}(z_0)$. Therefore we can choose $\kappa > 0$ not depending on z_0 such that $|\zeta^*_{0,1}| \geq 2\kappa|\rho(z_0)|$. $\square$

Proof of Proposition 3.4. Let p_0 be the multiplicity of the singularity ζ_0 of X and let ψ be a p_0-th determination of the root holomorphic in $\Delta_{\zeta^*_{0,1}}(2\kappa|\rho(z_0)|)$. We set $\alpha^*_j(z^*_1) = \phi^*\left(\psi(z^*_1 - \zeta^*_{0,1})e^{\frac{2i\pi}{p_0}j}\right) + \zeta^*_{0,2}$, $j = 1, \dots, p_0$. For all j, α^*_j is holomorphic on $\Delta_0(2\kappa|\rho(z_0)|)$ and is uniformly bounded on $\Delta_0(2\kappa|\rho(z_0)|)$. We have

$$\frac{\partial\alpha^*_j}{\partial z^*_1}(z^*_1) = \psi'(z^*_1 - \zeta^*_{0,1})\frac{\partial\phi^*}{\partial t^*}\left(\psi(z^*_1 - \zeta^*_{0,1})e^{\frac{2i\pi}{p_0}j}\right)e^{\frac{2i\pi}{p_0}j}.$$

Since $|\phi^*(t^*)| \lesssim |t^*|^{p_0}$, this yields $\left|\frac{\partial\alpha^*_j}{\partial z^*_1}(z^*_1)\right| \lesssim 1$ which proves (i).

We now prove that (ii) holds. We denote by K a uniform bound of the derivative of α^*_j. If $z^*_1 \in \Delta_0(2\kappa|\rho(z_0)|)$ is such that $|\alpha^*_j(z^*_1)| \leq (2\kappa|\rho(z_0)|)^{\frac{1}{2}}$, we have for all $\zeta^*_1 \in \Delta(2\kappa|\rho(z_0)|)$:

$$\begin{aligned}
|\alpha^*_j(\zeta^*_1)| &\leq |\alpha^*_j(z^*_1)| + \left|\alpha^*_j(z^*_1) - \alpha^*_j(\zeta^*_1)\right| \\
&\leq (2\kappa|\rho(z_0)|)^{\frac{1}{2}} + K|\zeta^*_1 - z^*_1| \\
&\leq (2\kappa|\rho(z_0)|)^{\frac{1}{2}} + 4K\kappa|\rho(z_0)|.
\end{aligned}$$

Therefore choosing again κ small enough, uniformly with respect to z_0, we get $|\alpha^*_j(\zeta^*_1)| \leq (3\kappa|\rho(z_0)|)^{\frac{1}{2}}$.

Only (iii) is left to be shown. For z near ζ_0 we set $f_z(\lambda, \mu) = f(\zeta_0 + \lambda\eta_z + \mu v_z)$ and we apply Lemma 3.5 to the family $(f_z)_z$ which gives u_0 and P_0 such that $f_{z_0} = u_0 P_0$ where $|u_0| \eqsim 1$ uniformly with respect to z_0 and where $P_0(\lambda\eta_{z_0} + \mu v_{z_0})$ is a polynomial of the variable μ with coefficients holomorphic with respect to λ. We have $f_{z_0}(z_0 - \zeta_0 + \zeta^*_1\eta_{z_0} + \alpha^*_i(\zeta^*_1)v_{z_0}) = 0$ for all i. Hence, for all ζ such that $|\zeta^*_1| < 2\kappa|\rho(z_0)|$, we get

$$P_0(\zeta^*_1 - \zeta^*_{0,1}, \zeta^*_2 - \zeta^*_{0,2}) = \prod_{i=1}^{p_0}(\zeta^*_2 - \alpha^*_i(\zeta^*_1)). \qquad \square$$

3.3. Definition of the operator

We now come to the definition of the current T such that $fT = 1$ and of the extension operator. Our construction is a refinement of [25]. We choose a positive real number κ so that Propositions 3.1 and 3.4 hold true for such a κ and such that Proposition 2.1 implies that $2\rho(z) \leq \rho(\zeta) \leq \frac{1}{2}\rho(z)$ for all $z \in D$ near bD.

For $\varepsilon_0 > 0$ and $z_0 \in \overline{D_{-\varepsilon_0}}$, that is when z_0 is far from the boundary, we do not modify the construction except that we require that $\mathcal{U}_0$ is included in $D_{-\frac{\varepsilon_0}{2}}$. We get a covering $\mathcal{U}_{-m}, \dots, \mathcal{U}_{-1}$ of $\overline{D_{-\varepsilon_0}}$ and the corresponding currents $T_{-m}, \dots, T_{-1}$ such that $fT_j = 1$ on $\mathcal{U}_j$ for all $j = -m, \dots, -1$.

Near the boundary, we have to be more precise and we use a κ-covering $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ of $D \setminus D_{-\varepsilon_0}$ constructed in Section 3.1. In the Koranyi coordinates centred at z_j, the fiber of X above $(z_1^*, 0) \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ is given by $\{(z_1^*, \alpha_i^*(z_1^*)),\ i = 1, \dots, p_j\}$ where p_j and $\alpha_1^*, \dots, \alpha_{p_j}^*$ are given by Proposition 3.4. In [25], Mazzilli actually considered the Weierstrass polynomial in a neighbourhood of z_j but this neighbourhood may be smaller than $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ or the Weierstrass polynomial may include all the α_i^*. However, in order to make a good link between the geometry of the boundary of D and X, we need to have a polynomial in all $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ and we have to take into account only the sheets of X which intersect $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ or equivalently the α_i^* such that for some $z_1^* \in \Delta_0(\kappa|\rho(z_j)|)$, the point $z_j + z_1^*\eta_{z_j} + \alpha_i^*(z_1^*)v_{z_j}$ belongs to $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$. So we put $I_j = \left\{i : \exists z_1^* \in \Delta_0(\kappa|\rho(z_j)|) \text{ such that } |\alpha_i^*(z_1^*)| \leq (2\kappa|\rho(z_j)|)^{\frac{1}{2}}\right\}$, $q_j = \# I_j$, the cardinal of I_j, and for any C^∞-smooth $(2,2)$-form ϕ compactly supported in $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ we set

$$\tilde{T}_j[\phi] = \int_{\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)} \frac{\prod_{i\in I_j} \overline{\zeta_2^* - \alpha_i^*(\zeta_1^*)}}{f(\zeta)} \frac{\partial^{q_j}\phi}{\partial \overline{\zeta_2^*}^{q_j}}(\zeta).$$

As in [25], integrating by parts q_j times gives $f\tilde{T}_j = c_j$ where $|c_j| = q_j!$

Now let $\left(\chi_j\right)_{j\geq -m}$ be a partition of unity subordinated to the covering $\mathcal{U}_{-m}, \dots, \mathcal{U}_{-1}$, $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ of D. We assume that χ_j has been chosen so that $\left|\frac{\partial^{\alpha+\overline{\alpha}+\beta+\overline{\beta}}\chi_j}{\partial\zeta_1^{*\alpha}\partial\overline{\zeta_1^*}^{\overline{\alpha}}\partial\zeta_2^{*\beta}\partial\overline{\zeta_2^*}^{\overline{\beta}}}(\zeta)\right| \lesssim \frac{1}{|\rho(z_j)|^{\alpha+\overline{\alpha}+\frac{\beta+\overline{\beta}}{2}}}$ for all $j \in \mathbb{N}$, $\zeta \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$, $\alpha, \beta, \overline{\alpha}, \overline{\beta} \in \mathbb{N}$, uniformly with respect to z_j and ζ. We set as in [25]: $T_j = \frac{1}{c_j}\tilde{T}_j$ for $j \in \mathbb{N}$ and $T = \sum_{j=-m}^{\infty} \chi_j T_j$.

Therefore we have $fT = 1$ on D. Moreover, since T is supported in $\overline{D}$ which is compact, T is of finite order (see [35]) and we can apply T to smooth forms vanishing to a sufficient order l on bD. Therefore if the function $\tilde{g}$ is such that $|\rho|^N\tilde{g}$ belongs to $C^l(\overline{D})$, we can apply T to $\tilde{g}P^{N,2}$. This gives us the integer l of Theorem 1.1.

Let $b(\zeta, z) = \sum_{j=1,2} b_j(\zeta, z)d\zeta_j$ be the holomorphic $(1,0)$-form defined by $b_j(\zeta, z) = \int_0^1 \frac{\partial f}{\partial\zeta_j}(\zeta + t(z-\zeta))dt$ so that for all z and ζ we have $f(z) - f(\zeta) = \sum_{i=1,2} b_i(\zeta, z)(z_i - \zeta_j)$. Let g be a holomorphic function admitting a smooth extension $\tilde{g}$ which satisfies the assumptions of Theorem 1.1. Following the construction of [25], we define the extension $E_N(g)$ of g by setting

$$E_N(g)(z) = C_1\overline{\partial}T[\tilde{g}b(\cdot, z) \wedge P^{N,1}(\cdot, z)] \qquad \forall z \in D,$$

where C_1 is a suitable constant (see [25]). We have to check that $E_N(g)$ is indeed an extension of g.

We have the two following facts:

Fact 1: Mazzilli proved in [25] that if $\tilde{g}$ is holomorphic on D and of class C^l on $\overline{D}$ then $E_N\tilde{g} = \tilde{g}$ on $X \cap D$.

Fact 2: We have $E_N\tilde{g}_1 = E_N\tilde{g}_2$ when $\tilde{g}_1$ and $\tilde{g}_2$ are any smooth functions such that $\frac{\partial^{\alpha+\beta}\tilde{g}_1}{\partial\overline{\zeta_1^*}^{\alpha}\partial\overline{\zeta_2^*}^{\beta}} = \frac{\partial^{\alpha+\beta}\tilde{g}_2}{\partial\overline{\zeta_1^*}^{\alpha}\partial\overline{\zeta_2^*}^{\beta}}$ on $X \cap D$ for all integers α, β with $\alpha + \beta \le k$, where k is the supremum of the multiplicities of the singularities of X. Indeed, since f is assumed to be minimal, using [36, Theorem I, Paragraph 11.2 and the theorem of Paragraph 14.2], for any function $\tilde{g}$ we can write $E_N\tilde{g}$ as a sum of integrals over $X \cap D$ where only the derivatives $\frac{\partial^{\alpha+\beta}\tilde{g}P^{N,1}}{\partial\overline{\zeta_1^*}^{\alpha}\partial\overline{\zeta_2^*}^{\beta}}$ with $\alpha + \beta \le k$ appear. Applying this formula to $\tilde{g} = \tilde{g}_1$ and $\tilde{g} = \tilde{g}_2$ we get $E_N\tilde{g}_1 = E_N\tilde{g}_2$. We notice that this gives us the integer k of Theorem 1.1.

Now let g be a holomorphic function on $X \cap D$ which admits a smooth extension $\tilde{g}$ which satisfies the assumptions of Theorem 1.1. We prove that $E_N(g)(z_0) = g(z_0)$ for all $z_0 \in X \cap D$. For $\varepsilon > 0$ small enough we construct $P_\varepsilon^{N,n}$, the Berndtsson-Andersson kernel of the domain $D_{-\varepsilon}$ which has the defining function $\rho_\varepsilon = \rho + \varepsilon$. We set $P_\varepsilon^{N,n}(\zeta, z) = 0$ for $\zeta \notin D_{-\varepsilon}$. For a fixed z_0 in D, the kernel $P_\varepsilon^{N,n}(\cdot, z_0)$ converges to $P^{N,n}(\cdot, z_0)$ when ε tends to 0 in $C^k(\overline{D})$ for all $k \in \mathbb{N}$, $C^k(\overline{D})$ being endowed with its usual topology.

Now let g_ε be an holomorphic extension of g to $D_{-\frac{\varepsilon}{2}}$ given by Cartan's Theorem B. Fact 1 yields

$$\begin{aligned} g(z_0) &= g_\varepsilon(z_0) \\ &= \int_D g_\varepsilon(\zeta) \wedge P_\varepsilon^{N,2}(\zeta, z_0) \\ &= T\left[f g_\varepsilon \wedge P_\varepsilon^{n,2}(\cdot, z_0)\right] \\ &= C_1\overline{\partial}T\left[g_\varepsilon b(\cdot, z_0) \wedge P_\varepsilon^{N,1}(\cdot, z_0)\right]. \end{aligned}$$

Then, since $P_\varepsilon^{N,1}$ is supported in $D_{-\varepsilon}$, since $\tilde{g} = g_\varepsilon$ on $X \cap D_{-\frac{\varepsilon}{2}}$, and since $\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\zeta_1^*}^{\alpha}\partial\overline{\zeta_2^*}^{\beta}} = 0$ on $D_{-\frac{\varepsilon}{2}} \cap X$, fact 2 gives

$$g(z_0) = C_1\overline{\partial}T\left[\tilde{g}b(\cdot, z_0) \wedge P_\varepsilon^{N,1}(\cdot, z_0)\right]. \tag{3.1}$$

Finally, since $P_\varepsilon^{N,1}(\cdot, z_0)$ converges to $P^{N,1}(\cdot, z_0)$ in $C^k(\overline{D})$ for all $k \in \mathbb{N}$ when ε tends to 0 and since $\overline{\partial}T$ is a current of finite order supported in $\overline{D}$, letting ε goes to 0 in (3.1) yields $g(z_0) = E_N(g)(z_0)$ and thus $E_N(g)$ is an extension of g.

4. Estimate of the extension operator

We prove in this section that $E_N(g)$ satisfies the conclusion of Theorem 1.1. For this purpose we write b in the Koranyi coordinates at z_j, as $b(\zeta,z)=\sum_{l=1,2} b_l^*(\zeta, z)d\zeta_l^*$ where $b_l^*(\zeta, z) = \int_0^1 \frac{\partial f}{\partial \zeta_l^*}(\zeta+t(z-\zeta))dt$. We recall that for any non-negative integer j, p_j is the integer given by Proposition 3.4 and

$$I_j = \left\{ i : \exists z_1^* \in \Delta_0(\kappa|\rho(z_j)|) \text{ such that } |\alpha_i^*(z_1^*)| \le (2\kappa|\rho(z_j)|)^{\frac{1}{2}} \right\}.$$

We prove the following estimates:

Proposition 4.1. *For all positive integers j, all z in D and all ζ in $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$, we have uniformly in z, ζ and j*

$$\left| \frac{\prod_{i\in I_j} \overline{\zeta_2^* - \alpha_i^*(\zeta_1)}}{f(\zeta)} b_1(\zeta, z) \right| \lesssim \sum_{0\le\alpha+\beta\le p_j} \delta(\zeta, z)^{\alpha+\frac{\beta}{2}} |\rho(\zeta)|^{-1-\alpha+\frac{\#I_j-\beta}{2}},$$

$$\left| \frac{\prod_{i\in I_j} \overline{\zeta_2^* - \alpha_i^*(\zeta_1)}}{f(\zeta)} b_2(\zeta, z) \right| \lesssim \sum_{0\le\alpha+\beta\le p_j} \delta(\zeta, z)^{\alpha+\frac{\beta}{2}} |\rho(\zeta)|^{-\frac{1}{2}-\alpha+\frac{\#I_j-\beta}{2}},$$

$$\left| \frac{\prod_{i\in I_j} \overline{\zeta_2^* - \alpha_i^*(\zeta_1)}}{f(\zeta)} d_z b_1(\zeta, z) \right| \lesssim \sum_{0\le\alpha+\beta\le p_j} \delta(\zeta, z)^{\alpha+\frac{\beta}{2}} |\rho(\zeta)|^{-2-\alpha+\frac{\#I_j-\beta}{2}},$$

$$\left| \frac{\prod_{i\in I_j} \overline{\zeta_2^* - \alpha_i^*(\zeta_1)}}{f(\zeta)} d_z b_2(\zeta, z) \right| \lesssim \sum_{0\le\alpha+\beta\le p_j} \delta(\zeta, z)^{\alpha+\frac{\beta}{2}} |\rho(\zeta)|^{-\frac{3}{2}-\alpha+\frac{\#I_j-\beta}{2}}.$$

Proof. We prove the first inequality, the others are analogous. For $A \subset \{1, \dots, p_j\}$ we denote by A^c the complementary of A in $\{1, \dots, p_j\}$. Proposition 3.4 yields:

$$\left| \frac{\prod_{i\in I_j} \overline{\zeta_2^* - \alpha_i^*(\zeta_1^*)}}{f(\zeta)} \right| \lesssim \frac{1}{\prod_{i\in I_j^c} |\zeta_2^* - \alpha_i^*(\zeta_1^*)|}$$

uniformly with respect to ζ and j. We estimate b_1^*. We have

$$\frac{\partial f}{\partial \zeta_1^*}(\zeta + t(z-\zeta)) = \sum_{0\le\alpha+\beta\le p_j} \frac{\partial^{\alpha+\beta+1} f}{\partial \zeta_1^{*\alpha+1} \partial \zeta_2^{*\beta}}(\zeta)(z^* - \zeta^*)^{\alpha+\beta} + o(|\zeta^* - z^*|^{p_j})$$

and

$$\left| \frac{\partial^{\alpha+\beta+1} f}{\partial \zeta_1^{*\alpha+1} \partial \zeta_2^{*\beta}}(\zeta) \right| = \left| \sum_{\substack{n_1+\ldots+n_{p_j}=\alpha+1 \\ F_1 \dot\cup F_2 \dot\cup F_3 = \{1,\ldots,p_j\}}} \prod_{i\in F_1} \frac{\partial^{n_i} \alpha_i^*}{\partial \zeta_1^{*n_i}}(\zeta_1^*) \prod_{i\in F_3} (\zeta_2^* - \alpha_i^*(\zeta_1^*)) \right|$$

where $\dot\cup$ means that the union is disjoint, $F_1 = \{i,\ n_i \neq 0\}$ and $\#F_2 = \beta$.

Since $\frac{\partial \alpha_i^*}{\partial \zeta_1^*}$ is uniformly bounded and holomorphic on $\Delta_0(2\kappa|\rho(z_j)|)$, we have $\left|\frac{\partial^{n_i}\alpha_i^*}{\partial {\zeta_1^*}^{n_i}}\right| \lesssim |\rho(z_j)|^{-n_i+1}$ on $\Delta_0(\kappa|\rho(z_j)|)$. Moreover Proposition 2.1 gives $|\rho(z_j)| \eqsim |\rho(\zeta)|$ for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ so

$$\left|\frac{\partial^{\alpha+\beta+1} f}{\partial {\zeta_1^*}^{\alpha+1}\partial {\zeta_2^*}^{\beta}}(\zeta)\right| \lesssim \sum_{\substack{n_1+\ldots+n_{p_j}=\alpha+1\\ F_1\dot\cup F_2\dot\cup F_3=\{1,\ldots,p_j\}\\ \#F_2=\beta}} |\rho(\zeta)|^{-\alpha-1+\#F_1}\prod_{i\in F_3}|\zeta_2^*-\alpha_i^*(\zeta_1^*)|$$

and so

$$|b_1^*(\zeta,z)| \lesssim \sum_{0\le\alpha+\beta\le p_j}\ \sum_{\substack{F_1\dot\cup F_2\dot\cup F_3=\{1,\ldots,p_j\}\\ \#F_2=\beta}} |\rho(\zeta)|^{-1-\alpha+\#F_1}\delta(\zeta,z)^{\alpha+\frac{\beta}{2}}\prod_{i\in F_3}|\zeta_2^*-\alpha_i^*(\zeta_1^*)|.$$

Therefore $\frac{\prod_{i\in I_j}\overline{\zeta_2^*-\alpha_i^*(\zeta_1^*)}}{f(\zeta)}b_1^*(\zeta,z)$ is bounded by a sum for $0 \le \alpha+\beta \le p_j$, $F_1\dot\cup F_2\dot\cup F_3 = \{1,\ldots,p_j\}, \#F_2 = \beta$ of

$$S^{\alpha,\beta}_{F_1,F_2,F_3} := \frac{\prod_{i\in F_3}|\zeta_2^*-\alpha_i^*(\zeta_1^*)|}{\prod_{i\in I_j^c}|\zeta_2^*-\alpha_i^*(\zeta_1^*)|}|\rho(\zeta)|^{-1-\alpha+\#F_1}\delta(\zeta,z)^{\alpha+\frac{\beta}{2}}.$$

On the one hand for $i \in I_j^c$ and $\zeta \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ we have $|\zeta_2^*-\alpha_i^*(\zeta_1^*)| \gtrsim |\rho(z_j)|^{\frac12} \eqsim |\rho(\zeta)|^{\frac12}$. On the other hand for $i \in I_j$ and $\zeta \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$ we have $|\zeta_2^*-\alpha_i^*(\zeta_1^*)| \lesssim |\rho(\zeta)|^{\frac12}$. Therefore, writing $\frac{\prod_{i\in F_3}(\zeta_2^*-\alpha_i^*(\zeta_1^*))}{\prod_{i\in I_j^c}(\zeta_2^*-\alpha_i^*(\zeta_1^*))}$ as $\frac{\prod_{i\in F_3\cap I_j}(\zeta_2^*-\alpha_i^*(\zeta_1^*))}{\prod_{i\in I_j^c\cap F_3^c}(\zeta_2^*-\alpha_i^*(\zeta_1^*))}\cdot \frac{\prod_{i\in F_3\cap I_j^c}(\zeta_2^*-\alpha_i^*(\zeta_1^*))}{\prod_{i\in I_j^c\cap F_3}(\zeta_2^*-\alpha_i^*(\zeta_1^*))}$ we get

$$S^{\alpha,\beta}_{F_1,F_2,F_3} \lesssim \delta(\zeta,z)^{\alpha+\frac{\beta}{2}}|\rho(\zeta)|^{-1-\alpha+\#F_1+\frac{\#F_3\cap I_j-\#F_3^c\cap I_j^c}{2}}.$$

The equality $\#F_3\cap I_j - \#F_3^c\cap I_j^c = \#I_j - \#F_3^c$ implies that $\#F_1 + \frac{\#F_3\cap I_j-\#F_3^c\cap I_j^c}{2} \ge \frac{\#I_j-\beta}{2}$.

This gives $S^{\alpha,\beta}_{F_1,F_2,F_3} \lesssim \delta(\zeta,z)^{\alpha+\frac{\beta}{2}}|\rho(\zeta)|^{-1-\alpha+\frac{\#I_j-\beta}{2}}$ which finally yields

$$\left|\frac{\prod_{i\in I_j}\overline{\zeta_2^*-\alpha_i^*(\zeta_1)}}{f(\zeta)}b_1(\zeta,z)\right| \lesssim \sum_{0\le\alpha+\beta\le p_j}\delta(\zeta,z)^{\alpha+\frac{\beta}{2}}|\rho(\zeta)|^{-1-\alpha+\frac{\#I_j-\beta}{2}}. \qquad \square$$

As usual in the estimates of the Berndtsson-Andersson kernel, the main difficulty appears when we integrate for ζ near z and z near bD. Therefore we choose $\varepsilon_0 > 0$

arbitrarily small and we divide the domain of integration in two parts : $\mathcal{P}_{\frac{\varepsilon_0}{2c_1}}(z)$ and $D \setminus \mathcal{P}_{\frac{\varepsilon_0}{2c_1}}(z)$ where c_1 is given by Proposition 2.1. In order to estimate the integral over $\mathcal{P}_{\frac{\varepsilon_0}{2c_1}}(z)$, we prove the following:

Lemma 4.2. *For all z such that $0 > \rho(z) > -\frac{\varepsilon_0}{2}$, let j_0 be an integer such that $(1-c\kappa)^{-j_0}\varepsilon_0 < |\rho(z)| \leq (1-c\kappa)^{-j_0-1}\varepsilon_0$ and let $z_1^{i,j}, \dots, z_{m_{i,j}}^{i,j}$, $i \in \mathbb{N}$, $j \in \mathbb{Z}$, be the points of the covering such that*

- $\rho(z_m^{i,j}) = -(1-c\kappa)^{j-j_0}\varepsilon_0$,
- $\delta(z_m^{i,j}, z) \in [i\kappa(1-c\kappa)^{j-j_0}\varepsilon_0, (i+1)\kappa(1-c\kappa)^{j-j_0}\varepsilon_0[$,
- $\delta(z_m^{i,j}, z) \leq \varepsilon_0$.

For $j \geq j_0$ let $i_0(j)$ be the non-negative integer such that $i_0(j)\kappa(1-c\kappa)^{j-j_0} < 1 \leq (1+i_0(j))\kappa(1-c\kappa)^{j-j_0}$.
Then

(i) $\mathcal{P}_{\frac{\varepsilon_0}{2c_1}}(z) \cap D \subset \cup_{j=j_0}^{+\infty} \cup_{i=0}^{i_0(j)} \cup_{m=1}^{m_{i,j}} \mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j})$,

(ii) $m_{i,j} \lesssim i^2$ *uniformly with respect to z_0, z, i and j.*

Proof. We first prove (i). Let ζ be a point in $\mathcal{P}_{\frac{\varepsilon_0}{2c_1}}(z) \cap D$. Proposition 2.1 implies that ζ belongs to $D \setminus D_{-\varepsilon_0}$ so there exists a point ζ_0 of the covering such that ζ belongs to $\mathcal{P}_{\kappa|\rho(\zeta_0)|}(\zeta_0)$. The point ζ_0 belongs to $D \setminus D_{-\varepsilon_0}$ thus there exists $j \geq j_0$ such that $|\rho(\zeta_0)| = (1-c\kappa)^{j-j_0}\varepsilon_0$. Moreover if κ is small enough

$$\begin{aligned}
\delta(\zeta_0, z) &\leq c_1(\delta(\zeta, \zeta_0) + \delta(\zeta, z)) \\
&\leq c_1 \left(\kappa(1-c\kappa)^{j-j_0}\varepsilon_0 + \frac{\varepsilon_0}{2c_1} \right) \\
&< \varepsilon_0.
\end{aligned}$$

So there exists $i \in \mathbb{N}$ such that $\delta(\zeta_0, z)$ belongs to $[i\kappa(1-c\kappa)^{j-j_0}\varepsilon_0, (i+1)\kappa(1-c\kappa)^{j-j_0}\varepsilon_0[$ and $(i+1)\kappa(1-c\kappa)^{j-j_0}\varepsilon_0 \leq \varepsilon_0$ which means that $i \leq i_0(j)$. Thus ζ_0 is one the points $z_1^{i,j}, \dots, z_{m_{i,j}}^{i,j}$ and (i) holds.

In order to prove that $m_{i,j} \lesssim i^2$ we introduce the set

$$E_{i,j} = \{\zeta \in D : \rho(\zeta) = -(1-c\kappa)^{j-j_0}\varepsilon_0 \text{ and } \delta(\zeta, z) \leq c_1\kappa(i+2)(1-c\kappa)^j|\rho(z)|\}.$$

On the one hand we have

$$\begin{aligned}
\sigma(E_{i,j}) &= \sigma\left(bD_{-(1-c\kappa)^{j-j_0}\varepsilon_0} \cap \mathcal{P}_{c_1\kappa(i+2)(1-c\kappa)^j|\rho(z)|}(z)\right) \\
&\leq \left(c_1\kappa(i+2)(1-c\kappa)^j|\rho(z)|\right)^2 \qquad (4.1) \\
&\lesssim \left(c_1\kappa(i+2)(1-c\kappa)^{j-j_0-1}\varepsilon_0\right)^2
\end{aligned}$$

On the other one hand for all m, all $\zeta \in \mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j})$ we have

$$\begin{aligned}\delta(\zeta, z) &\leq c_1(\delta(\zeta, z_m^{i,j}) + \delta(z_m^{i,j}, z))\\ &\leq c_1(\kappa(1-c\kappa)^{j-j_0}\varepsilon_0 + \kappa(i+1)(1-c\kappa)^{j-j_0}\varepsilon_0)\\ &\leq c_1\kappa(i+2)(1-c\kappa)^{j-j_0}\varepsilon_0.\end{aligned}$$

This implies that $\mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j}) \cap bD_{-(1-c\kappa)^{j-j_0}\varepsilon_0} \subset E_{i,j}$ for all m and so

$$\sigma(E_{i,j}) \geq \sigma\left(\cup_{m=1}^{m_{i,j}} \mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j}) \cap bD_{-(1-c\kappa)^{j-j_0}\varepsilon_0}\right).$$

Now, the construction of a κ-covering and Proposition 2.1 imply that the intersection of $\mathcal{P}_{\frac{c\kappa}{c_1}|\rho(z_m^{i,j})|}(z_m^{i,j})$ and $\mathcal{P}_{\frac{c\kappa}{c_1}|\rho(z_l^{i,j})|}(z_l^{i,j})$ is empty for $l \neq m$. Therefore we have

$$\begin{aligned}\sigma(E_{i,j}) &\geq \sum_{m=1}^{m_{i,j}} \sigma\left(\mathcal{P}_{\frac{c\kappa}{c_1}|\rho(z_m^{i,j})|}(z_m^{i,j}) \cap bD_{-(1-c\kappa)^{j-j_0}\varepsilon_0}\right),\\ &\geq m_{i,j}\left(\frac{c\kappa}{c_1}(1-c\kappa)^{j-j_0}\varepsilon_0\right)^2.\end{aligned} \tag{4.2}$$

Inequalities (4.1) and (4.2) together imply that $m_{i,j} \lesssim i^2$, uniformly with respect to z, i and j. □

In order to prove the BMO-estimates of Theorem 1.1 we apply the following classical lemma:

Lemma 4.3. *Let h be a function of class C^1 on D. If there exists $C > 0$ such that $\mathrm{d}h(\zeta) \leq C|\rho(\zeta)|^{-1}$ then h belongs to $BMO(D)$ and $\|h\|_{BMO(D)} \leq C$.*

Proof of Theorem 1.1 *for $q = +\infty$*. Let g be a holomorphic function on $X \cap D$ which have a smooth extension $\tilde{g}$ which satisfies the assumptions (i), (ii) and (iii) of Theorem 1.1. We put $\gamma_\infty = \sup_{\substack{\zeta\in D\\ \alpha+\beta\leq k}} \left|\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^\alpha\partial\overline{v_\zeta}^\beta}(\zeta)\right| |\rho(\zeta)|^{\alpha+\frac{\beta}{2}}$. In order to prove Theorem 1.1 when $q = +\infty$, we have to prove that $E_N(g)$ is in $BMO(D)$ and $\|E_N(g)\|_{BMO(D)} \lesssim \gamma_\infty$.

Since the Berndtsson-Andersson kernel is regular when ζ and z are far from each other or when z is far from bD, we only have to estimate the integral over $\mathcal{P}_{\frac{\varepsilon_0}{2c_1}}(z) \cap D$ for z near bD and $\varepsilon_0 > 0$ not depending on z. We keep the notation of Lemma 4.2 and use the covering $\cup_{j=j_0}^{+\infty} \cup_{i=0}^{i_0(j)} \cup_{m=1}^{m_{i,j}} \mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j})$ of $\mathcal{P}_{\frac{\varepsilon_0}{2c_1}}(z)$ given by Lemma 4.2. We denote by $p_m^{i,j}$ the number of sheets given by Proposition 3.4 for $z_m^{i,j}$, $I_m^{i,j}$ is the set $I_m^{i,j} = \left\{k : \exists z_1^* \in \Delta_0(\kappa|\rho(z_m^{i,j})|) \text{ such that } |\alpha_k^*(z_1^*)| \leq (2\kappa|\rho(z_m^{i,j})|)^{\frac{1}{2}}\right\}$ and $q_m^{i,j}$ denotes its cardinal.

From Proposition 2.4 and 4.1 we get for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j})$

$$\left| d_z \left(\frac{\prod_{i \in I_m^{i,j}} \overline{\zeta_2^* - \alpha_i^*(\zeta_1)}}{f(\zeta)} b(\zeta, z) \wedge \overline{\partial} \frac{\partial^{q_m^{i,j}}}{\partial \overline{\zeta_2^*}^{q_m^{i,j}}} \left(\tilde{g}(\zeta) P^{N,n}(\zeta, z) \right) \right) \right|$$

$$\lesssim \gamma_\infty \sum_{0 \le \alpha + \beta \le p_m^{i,j}} \left(\frac{\delta(\zeta, z)}{|\rho(\zeta)|} \right)^{\alpha + \frac{\beta}{2}} \frac{|\rho(\zeta)|^N}{(|\rho(\zeta)| + |\rho(z)| + \delta(z, \zeta))^{N+4}}$$

$$\lesssim \gamma_\infty \frac{|\rho(\zeta)|^{N'}}{(|\rho(\zeta)| + |\rho(z)| + \delta(z, \zeta))^{N'+4}}$$

where $N' = N - \max_{i,j} p_{i,j}$. We have for all $\zeta \in \mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j})$, $|\rho(\zeta)| \ge \frac{1}{2}|\rho(z_m^{i,j})|$ and thus:

$$\begin{aligned} |\rho(\zeta)| + \delta(\zeta, z) &\ge \frac{1}{2}|\rho(z_m^{i,j})| + \frac{1}{c_1}\delta(z, z_m^{i,j}) - \delta(z_m^{i,j}, \zeta) \\ &\ge |\rho(z_m^{i,j})| \left(\frac{1}{2} - \kappa \right) + \frac{1}{c_1}\delta(z, z_m^{i,j}) \\ &\gtrsim |\rho(z_m^{i,j})| + \delta(z, z_m^{i,j}). \end{aligned}$$

Therefore

$$\left| d_z \left(\frac{\prod_{i \in I_m^{i,j}} \overline{\zeta_2^* - \alpha_i^*(\zeta_1)}}{f(\zeta)} b(\zeta, z) \wedge \overline{\partial} \frac{\partial^{q_m^{i,j}}}{\partial \overline{\zeta_2^*}^{q_m^{i,j}}} \left(\tilde{g}(\zeta) P^{N,n}(\zeta, z) \right) \right) \right|$$

$$\lesssim \gamma_\infty \frac{|\rho(z_m^{i,j})|^{N'}}{(|\rho(z)| + |\rho(z_m^{i,j})| + \delta(z, z_m^{i,j}))^{N'+4}}.$$

Now, integrating over $\mathcal{P}_{\kappa|\rho(z_m^{i,j})|}(z_m^{i,j})$ and summing over m, i and j we have to prove that the sum

$$\sum_{j=j_0}^{\infty} \sum_{i=0}^{i_0(j)} \sum_{m=1}^{m_{i,j}} \frac{|\rho(z_m^{i,j})|^{N'}}{\left((i+1)|\rho(z_m^{i,j})| + |\rho(z)| \right)^{N'+1}}$$

is uniformly bounded by $\frac{1}{|\rho(z)|}$. We have:

$$\sum_{j=j_0}^{\infty}\sum_{i=0}^{i_0(j)}\sum_{m=1}^{m_{i,j}} \frac{|\rho(z_m^{i,j})|^{N'}}{\Big((i+1)|\rho(z_m^{i,j})|+|\rho(z)|\Big)^{N'+1}}$$

$$\leq \sum_{j=j_0}^{\infty}\sum_{i=0}^{i_0(j)}\sum_{m=1}^{m_{i,j}} \left(\frac{(1-c\kappa)^j}{(i+1)(1-c\kappa)^j+1}\right)^{N'}\cdot\frac{1}{((i+1)(1-c\kappa)^j+1)|\rho(z)|}$$

$$\leq \frac{1}{|\rho(z)|}\left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(1-c\kappa)^j}{(i+1)^{N'-3}}+\sum_{j=j_0}^{-1}\sum_{i=0}^{\infty}\frac{1}{(i+1)^{N'-2}(1-c\kappa)^j}\right)$$

$$\lesssim \frac{1}{|\rho(z)|}.$$

So $E_N(g)$ belongs to $BMO(D)$ and $\|E_N(g)\|_{BMO(D)} \lesssim \sup_{\substack{\zeta\in D\\ \alpha+\beta\leq k}} \left|\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^{\alpha}\partial\overline{v_\zeta}^{\beta}}(\zeta)\right| \times |\rho(\zeta)|^{\alpha+\frac{\beta}{2}}$. □

The L^q-estimates of Theorem 1.1 are left to be shown. For $q\in(1,+\infty)$ we will apply the following (see [31]):

Lemma 4.4. *Suppose the kernel $k(\zeta,z)$ is defined on $D\times D$ and the operator K is defined by $Kf(z)=\int_{\zeta\in D}k(\zeta,z)f(\zeta)d\lambda(\zeta)$. If for every $\varepsilon\in]0,1[$ there exists a constant c_ε such that*

$$\int_{\zeta\in D}|\rho(\zeta)|^{-\varepsilon}|k(\zeta,z)|d\lambda(\zeta)\leq c_\varepsilon|\rho(z)|^{-\varepsilon},\quad \forall z\in D,$$
$$\int_{z\in D}|\rho(z)|^{-\varepsilon}|k(\zeta,z)|d\lambda(z)\leq c_\varepsilon|\rho(\zeta)|^{-\varepsilon},\quad \forall \zeta\in D$$

then for all $q\in]1,+\infty[$, there exists $c_q>0$ such that $\|Kf\|_{L^q(D)}\leq\|f\|_{L^q(D)}$.

Proof of Theorem 1.1 *for $q\in(1,+\infty)$.* Applying Lemma 4.4 and Propositions 2.4 and 4.1, it suffices to prove that for all $\varepsilon\in(0,1)$ there exists $c_\varepsilon>0$ such that

$$\int_{\zeta\in D}\frac{|\rho(\zeta)|^{N'-\varepsilon}}{(|\rho(\zeta)|+|\rho(z)|+\delta(\zeta,z))^{N'+3}}d\lambda(\zeta)\leq c_\varepsilon|\rho(z)|^{-\varepsilon},\ \forall z\in D, \tag{4.3}$$
$$\int_{z\in D}\frac{|\rho(\zeta)|^{N'}|\rho(z)|^{-\varepsilon}}{(|\rho(\zeta)|+|\rho(z)|+\delta(\zeta,z))^{N'+3}}d\lambda(z)\leq c_\varepsilon|\rho(\zeta)|^{-\varepsilon},\ \forall \zeta\in D. \tag{4.4}$$

Inequality (4.3) can be shown as in the proof of Theorem 1.1 for $q=\infty$. In order to prove that inequality (4.4) holds true we cover D with the Koranyi balls $\mathcal{P}_{\kappa|\rho(\zeta)|}(\zeta)$

and $\big(\mathcal{P}_{2^{j+1}\kappa|\rho(\zeta)|}(\zeta)\setminus\mathcal{P}_{2^{j}\kappa|\rho(\zeta)|}(\zeta)\big)$, $j\in\mathbb{N}$. For $z\in\mathcal{P}_{\kappa|\rho(\zeta)|}(\zeta)$, $|\rho(z)|\eqsim|\rho(\zeta)|$ and thus

$$\int_{z\in\mathcal{P}_{\kappa|\rho(\zeta)|}(\zeta)}\frac{|\rho(\zeta)|^{N'}|\rho(z)|^{-\varepsilon}}{(|\rho(\zeta)|+|\rho(z)|+\delta(\zeta,z))^{N'+3}}d\lambda(z)\lesssim|\rho(\zeta)|^{-\varepsilon}. \tag{4.5}$$

When we integrate on $\mathcal{P}_{2^{j+1}\kappa|\rho(\zeta)|}(\zeta)\setminus\mathcal{P}_{2^{j}\kappa|\rho(\zeta)|}(\zeta)$ we get

$$\begin{aligned}\int_{\mathcal{P}_{2^{j+1}\kappa|\rho(\zeta)|}(\zeta)\setminus\mathcal{P}_{2^{j}\kappa|\rho(\zeta)|}(\zeta)}&\frac{|\rho(\zeta)|^{N'}|\rho(z)|^{-\varepsilon}}{(|\rho(\zeta)|+|\rho(z)|+\delta(\zeta,z))^{N'+3}}d\lambda(z)\\ &\lesssim\int_{\substack{|x_1|,|y_1|\le 2^{j+1}\kappa|\rho(\zeta)|\\ |x_2|,|y_2|\le\sqrt{2^{j+1}\kappa|\rho(\zeta)|}}}\frac{|\rho(\zeta)|^{N'}x_1^{-\varepsilon}}{\big(|\rho(\zeta)|+2^j\kappa|\rho(\zeta)|\big)^{N'+3}}d\lambda(z)\\ &\lesssim(2^{j+1}\kappa|\rho(\zeta)|)^{-\varepsilon+3}\frac{|\rho(\zeta)|^{N'}}{\big(|\rho(\zeta)|+2^j\kappa|\rho(\zeta)|\big)^{N'+3}}\\ &\lesssim|\rho(\zeta)|^{-\varepsilon}2^{-j(N'+\varepsilon)}.\end{aligned}\tag{4.6}$$

Summing (4.5) and (4.6) for all non-negative integer j we prove inequality (4.4). Theorem 1.1 is therefore proved for $q\in(1,+\infty)$. □

Proof of Theorem 1.1 *for $q=1$.* We prove directly that $E_N(g)$ belongs to $L^1(D)$. Propositions 2.4 and 4.1 yield

$$\begin{aligned}\int_D|E_Ng(z)|d\lambda(z)\lesssim\sum_{j=0}^{\infty}\sum_{0\le\alpha+\beta\le q_j+1}\int_{\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)}|\rho(z_j)|^{\alpha+\frac{\beta}{2}}\left|\frac{\partial^{\alpha+\beta}\tilde g}{\partial\overline{\zeta_1^*}^{\alpha}\partial\overline{\zeta_2^*}^{\beta}}(\zeta)\right|\\ \cdot\left(\int_D\frac{|\rho(\zeta)|^{N'}}{(|\rho(\zeta)|+|\rho(z)|+\delta(\zeta,z))^{N'+3}}d\lambda(z)\right)d\lambda(\zeta).\end{aligned}$$

As for the proof of (4.4), we cover D using Koranyi corona and get

$$\begin{aligned}\int_D|Eg(z)|d\lambda(z)&\lesssim\sum_{j=0}^{\infty}\sum_{0\le\alpha+\beta\le q_j+1}\int_{\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)}|\rho(z_j)|^{\alpha+\frac{\beta}{2}}\left|\frac{\partial^{\alpha+\beta}\tilde g}{\partial\overline{\zeta_1^*}^{\alpha}\partial\overline{\zeta_2^*}^{\beta}}(\zeta)\right|d\lambda(\zeta)\\ &\lesssim\sum_{0\le\alpha+\beta\le k}\left\|\zeta\mapsto\frac{\partial^{\alpha+\beta}\tilde g}{\partial\overline{\eta}_\zeta^{\alpha}\partial\overline{v}_\zeta^{\beta}}(\zeta)\rho(\zeta)^{\alpha+\frac{\beta}{2}}\right\|_{L^1(D)}.\end{aligned}$$

□

5. Smooth extension and divided differences

In this section we give necessary conditions in $\mathbb{C}^n$ that a function g holomorphic on $X \cap D$ has to satisfy in order to have an L^q-holomorphic extension to D, $q \in [1, +\infty]$. We also prove that these conditions are sufficient in $\mathbb{C}^2$ for g to have a L^q-holomorphic extension to D when q belongs to $[1, +\infty)$ or a BMO-holomorphic extension when $q = +\infty$.

5.1. L^∞-BMO extension

We first prove the following lemma for functions defined on $X \cap D$ which have holomorphic extension to D. We use the notation defined in the introduction.

Lemma 5.1. *If g defined on $X \cap D$ has a holomorphic extension G on D then uniformly with respect to g, G, $z \in D$, v unit vector of $\mathbb{C}^n$ and positive integer k such that $k \leq \#\Lambda(z, v)$:*

$$\sup_{\substack{\lambda_1,\dots,\lambda_k\in\Lambda_{z,v}\\ \lambda_i\neq\lambda_j \text{ for } i\neq j}} |g_{z,v}[\lambda_1, \dots, \lambda_k]| \tau(z, v, |\rho(z)|)^{k-1} \lesssim \sup_{b\Delta_{z,v}(4\kappa\tau(z,v,|\rho(z)|))} |G|.$$

Proof. For $\lambda_1, \dots, \lambda_k \in \Lambda_{\zeta,v}$ pairwise distinct, we have by Cauchy's formula

$$g_{z,v}[\lambda_1, \dots, \lambda_k] = \frac{1}{2i\pi} \int_{|\lambda|=4\tau(z,v,|\rho(z)|)} \frac{G(z + \lambda v)}{\prod_{l=1}^{k} (\lambda - \lambda_i)} d\lambda$$

since for all λ_i we have $|\lambda_i| \leq 3\tau(z, v, |\rho(z)|)$, we get

$$|g_{z,v}[\lambda_1, \dots, \lambda_k] \lesssim \left(\frac{1}{\tau\,(z, v, |\rho(z)|)}\right)^{k-1} \sup_{b\Delta_{z,v}(4\kappa\tau(z,v,|\rho(z)|))} |G|. \qquad \square$$

Proof of Theorem 1.3. Lemma 5.1 implies directly that $c_\infty(g) \lesssim \|G\|_{L^\infty(D)}$. $\square$

Now we prove that an even weaker assumption than $c_\infty(g) < \infty$ is actually sufficient in $\mathbb{C}^2$ for g to have a smooth extension which satisfies the hypothesis of Theorem 1.1 for $q = \infty$ and thus for g to have a holomorphic BMO extension to D. We define for κ and ε_0 positive real number

$$c^{(\infty)}_{\kappa,\varepsilon_0}(g) = \sup |g_{\zeta+z_1^*\eta_\zeta,v_\zeta}[\lambda_1, \dots, \lambda_k]| \tau(\zeta, v_\zeta, |\rho(\zeta)|)^{k-1}$$

where the supremum is taken over $\zeta \in D \setminus D_{-\varepsilon_0}$, $z_1^* \in \mathbb{C}$ such that $|z_1^*| \leq \kappa|\rho(\zeta)|$, $\lambda_1, \dots, \lambda_k \in \Lambda_{\zeta+z_1^*\eta_\zeta,v_\zeta}$ pairwise distinct. Of course, $c^{(\infty)}_{\kappa,\varepsilon_0}(g) \leq c_\infty(g)$ and it may be simpler to check that $c^{(\infty)}_{\kappa,\varepsilon_0}(g)$ is finite than to check that $c_\infty(g)$ is finite. Moreover, as told by the following lemma, when $c^{(\infty)}_{\kappa,\varepsilon_0}(g)$ is finite, g admits a smooth extension which satisfies the assumptions of Theorem 1.1.

Lemma 5.2. *In $\mathbb{C}^2$, let $g \in \mathcal{O}(X \cap D)$ be such that $c^{(\infty)}_{\kappa,\varepsilon_0}(g) < \infty$. Then there exist a neighbourhood $\mathcal{U}$ of bD and $\tilde{g} \in C^\infty(D \cap \mathcal{U})$ such that:*

(i) *for all non-negative integer N, $|\rho|^{N+1}\tilde{g}$ vanishes to order N on bD;*

(ii) *for all α and β non-negative integer, $\left|\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^\alpha \partial\overline{v_\zeta}^\beta}\right| |\rho|^{\alpha+\frac{\beta}{2}}$ is bounded up to a uniform multiplicative constant on $D \cap \mathcal{U}$ by $c^{(\infty)}_{\kappa,\varepsilon_0}(g)$;*

(iii) *for all α and β non-negative integer such that $\alpha + \beta > 0$, $\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^\alpha \partial\overline{v_\zeta}^\beta} = 0$ on $X \cap D \cap \mathcal{U}$.*

Proof. For $\varepsilon_0 > 0$, we cover $D \setminus D_{-\varepsilon_0}$ with a κ-covering $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ constructed in Subsection 3.1. For a fixed non-negative integer j, we set $w_1^* = \eta_{z_j}$ and $w_2^* = v_{z_j}$. Let $\alpha_1, \dots, \alpha_{p_j}$ be the parametrization given by Proposition 3.4, $I_j = \{i : \exists z_1^* \in \mathbb{C} \text{ with } |z_1^*| < \kappa|\rho(z_j)| \text{ and } |\alpha_i(z_1^*)| \le 2\kappa|\rho(z_j)|\}$, $q_j = \#I_j$.

If $I_j = \emptyset$ we put $\tilde{g}_j = 0$ on $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$. Otherwise, without restriction we assume that $I_j = \{1, \dots, q_j\}$ and for $z = z_j + z_1^* w_1^* + z_2^* w_2^* \in \mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$, we put

$$\tilde{g}_j(z) = \sum_{k=1}^{q_j} g_{z_j+z_1^*w_1^*,w_2^*}[\alpha_1(z_1^*), \dots, \alpha_k(z_1^*)] \prod_{l=1}^{k-1}(\zeta_2^* - \alpha_l(z_1^*)).$$

Proposition 3.4 implies for all $z_1^* \in \Delta_0(2\kappa|\rho(z_j)|)$ that $\alpha_j(z_1^*)$ belongs to $\Lambda_{z_j+z_1^*w_1^*,w_2^*}$ thus $\tilde{g}_j$ is well-defined on $\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$. The function $\zeta \mapsto \tilde{g}_j(z_j + z_1^* w_1^* + \zeta w_2^*)$ is the polynomial which interpolates $\zeta \mapsto g(z_j + z_1^* w_1^* + \zeta w_2^*)$ at the points $\alpha_1(z_1^*), \dots, \alpha_{q_j}(z_1^*)$ and thus $\tilde{g}_j$ is a holomorphic extension of g to $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$. For all $z = z_j + z_1^* w_1^* + z_2^* w_2^* \in \mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$, we have

$$|z_2^* - \alpha_l(z_1^*)| \le \tau(z_j, w_2^*, 2\kappa|\rho(z_j)|) \lesssim \tau(z, w_2^*, 2\kappa|\rho(z)|).$$

Hence it follows that $|\tilde{g}_j(z)| \lesssim c^{(\infty)}_{\kappa,\varepsilon_0}(g)$ on $\mathcal{P}_{2\kappa|\rho(z_j)|}(z_j)$ and $|\rho(z_j)|^{\alpha+\frac{\beta}{2}}\left|\frac{\partial^{\alpha+\beta}\tilde{g}_j}{\partial w_1^{*\alpha}\partial w_2^{*\beta}}(z)\right| \lesssim c^{(\infty)}_{\kappa,\varepsilon_0}(g)$ on $\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$. Now we glue together all the $\tilde{g}_j$ using a suitable partition of unity and get our extension to $D \setminus D_{-\varepsilon_0}$. Let $(\chi_j)_{j\in\mathbb{N}}$ be a partition of unity subordinated to $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ such that for all j and all $\zeta \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$, we have $\left|\frac{\partial^{\alpha+\overline{\alpha}+\beta+\overline{\beta}}\chi_j}{\partial w_1^{*\alpha}\partial w_2^{*\beta}\partial\overline{w_1^*}^{\overline{\alpha}}\partial\overline{w_2^*}^{\overline{\beta}}}(\zeta)\right| \lesssim \frac{1}{|\rho(z_j)|^{\alpha+\overline{\alpha}+\frac{\beta+\overline{\beta}}{2}}}$, uniformly with respect to z_j and ζ. We set $\tilde{g}_{\varepsilon_0} = \sum_j \chi_j \tilde{g}_j$. By construction for all $N \in \mathbb{N}$, $\rho^{N+1}\tilde{g}_{\varepsilon_0}$ is of class C^N on $\overline{D} \setminus D_{-\varepsilon_0}$ and vanishes to order N on bD. Moreover, since for all j the function $\tilde{g}_j$ is holomorphic, $\frac{\partial^{\alpha+\beta}\tilde{g}_{\varepsilon_0}}{\partial\overline{z}_1^\alpha\partial\overline{z}_2^\beta} = 0$ on $X \cap (D \setminus D_{-\varepsilon})$ and, by our choice of χ_j, $\left|\frac{\partial^{\alpha+\beta}\tilde{g}_{\varepsilon_0}}{\partial\overline{\eta_\zeta}^\alpha\partial\overline{v_\zeta}^\beta}(\zeta)\right| \lesssim |\rho(\zeta)|^{-\left(\alpha+\frac{\beta}{2}\right)}$ for all $\zeta \in D \setminus D_{-\varepsilon_0}$. $\square$

As a direct consequence of Lemma 5.2, we have:

Corollary 5.3. *In $\mathbb{C}^2$, let $g \in \mathcal{O}(X \cap D)$ be such that $c_\infty(g) < \infty$. Then there exist a neighbourhood $\mathcal{U}$ of bD and $\tilde{g} \in C^\infty(D \cap \mathcal{U})$ such that:*

(i) *for all non-negative integer N, $|\rho|^{N+1}\tilde{g}$ vanishes to order N on bD;*
(ii) *for all α and β non-negative integer, $\left|\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^\alpha \partial\overline{v_\zeta}^\beta}\right| |\rho|^{\alpha+\frac{\beta}{2}}$ is bounded up to a uniform multiplicative constant on $D \cap \mathcal{U}$ by $c_\infty(g)$;*
(iii) *for all α and β non-negative integer such that $\alpha+\beta>0$, $\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^\alpha \partial\overline{v_\zeta}^\beta} = 0$ on $X \cap D \cap \mathcal{U}$.*

Theorem 1.2 now follows from Theorem 1.1 and Corollary 5.3:

Proof of Theorem 1.2. We use Corollary 5.3 to get an extension $\tilde{g}$ of g which satisfies the hypothesis of Theorem 1.1 on $\mathcal{U} \cap D$. Cartan's Theorem B gives us a bounded holomorphic extension to $D \setminus \mathcal{U}$. Gluing these two extensions together, we get a smooth extension of g which satisfies the hypothesis of Theorem 1.1 in the whole domain D and thus, Theorem 1.1 ensure the existence of a BMO holomorphic extension of g. □

5.2. $L^q(D)$-extension

The case of L^q-extensions is a bit harder to handle because it is not a punctual estimate but an average estimate. Therefore the assumption under which a function g holomorphic on $X \cap D$ admits a L^q-holomorphic extension to D uses a κ-covering $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ in addition to the divided differences.

By transversality of X and bD, for all j there exists w_j in the complex tangent plane to $bD_{\rho(z_j)}$ such that π_j, the orthogonal projection on the hyperplane orthogonal to w_j passing through z_j, is a p_j sheeted covering of X. We denote by $w_1^*, \dots, w_n^*$ an orthonormal basis of $\mathbb{C}^n$ such that $w_1^* = \eta_{z_j}$ and $w_n^* = w_j$ and we set $\mathcal{P}'_\varepsilon(z_j) = \{z' = z_j + z_1^* w_1^* + \dots + z_{n-1}^* w_{n-1}^* : |z_1^*| < \varepsilon \text{ and } |z_k^*| < \varepsilon^{\frac{1}{2}},\ k = 2, \dots, n-1\}$. We put

$$c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) = \sum_{j=0}^{\infty} \int_{z'\in\mathcal{P}'_{2\kappa|\rho(z_j)|}(z_j)} \sum_{\substack{\lambda_1,\dots,\lambda_k\in\Lambda_{z',w_n^*} \\ \lambda_i\neq\lambda_l \text{ for } i\neq l}} |\rho(z_j)|^{q\frac{k-1}{2}+1} \left|g_{z',w_n^*}[\lambda_1,\dots,\lambda_k]\right| dV_{n-1}(z')$$

where dV_{n-1} is the Lebesgue measure in $\mathbb{C}^{n-1}$.

Theorem 5.4. *In $\mathbb{C}^n$, with $n \geq 2$, let $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ be a κ-covering of $D \cap X$. If $g \in \mathcal{O}(X \cap D)$ has a holomorphic extension $G \in L^q(D)$ then $c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) \lesssim \|G\|^q_{L^q(D)}$ uniformly with respect to g, G and the covering $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$.*

Proof. For all $j \in \mathbb{N}$ all $z' \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$, all $r \in \mathbb{R}$ such that $\frac{7}{2}\kappa|\rho(z_j)|^{\frac{1}{2}} \leq r \leq 4\kappa|\rho(z_j)|^{\frac{1}{2}}$, all $\lambda_1, \dots, \lambda_k \in \Lambda_{z',w_n^*}$ pairwise distinct we have by Cauchy's formula

$$g_{z',w_j}[\lambda_1, \dots, \lambda_k] = \frac{1}{2i\pi} \int_{|\lambda|=r} \frac{G(z' + \lambda w_j)}{\prod_{l=1}^k (\lambda - \lambda_i)} d\lambda.$$

After integration for $r \in [7/2\kappa|\rho(z_j)|^{\frac{1}{2}}, 4\kappa|\rho(z_j)|^{\frac{1}{2}}]$, Jensen's inequality yields

$$\left|g_{z',w_j}[\lambda_1, \dots, \lambda_k]\right|^q \lesssim |\rho(z_j)|^{\frac{1-k}{2}q-1} \int_{|\lambda| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}} |G(z' + \lambda w_j)|^q dV_1(\lambda)$$

and thus

$$\begin{aligned} &\int_{z' \in \mathcal{P}'_{\kappa|\rho(z_j)|}(z_j)} \left|g_{z',w_j}[\lambda_1, \dots, \lambda_k]\right|^q |\rho(z_j)|^{\frac{k-1}{2}q+1} dV_{n-1} \\ &\lesssim \int_{z \in \mathcal{P}_{4\kappa|\rho(z_j)|}(z_j)} |G(z)|^q dV_n(\lambda). \end{aligned}$$

Since $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ is a κ-covering, we deduce from this inequality that $c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) \lesssim \|G\|^q_{L^q(D)}$. □

Now we come back to $\mathbb{C}^2$ and prove that the condition $c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) < \infty$ is indeed sufficient for g to have an L^q extension.

Theorem 5.5. *In $\mathbb{C}^2$, let $\left(\mathcal{P}_{\kappa|\rho(z_j)|}(z_j)\right)_{j\in\mathbb{N}}$ be a κ-covering of $D \cap X$. If the function g is holomorphic on $X \cap D$ and satisfies $c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) < \infty$, then there exist a neighbourhood $\mathcal{U}$ of bD and a smooth extension $\tilde{g} \in C^\infty(D \cap \mathcal{U})$ of g such that:*

(i) *for all $N \in \mathbb{N}$, $|\rho|^{N+4}\tilde{g}$ vanishes to order N on bD;*
(ii) *for all non-negative integers α and β the function $\zeta \mapsto \left|\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^\alpha \partial\overline{v_\zeta}^\beta}(\zeta)\right| |\rho(\zeta)|^{\alpha+\frac{\beta}{2}}$ has a L^q norm on $D \cap \mathcal{U}$ bounded by $c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g)$ up to a uniform multiplicative constant;*
(iii) *for all non-negative integer α and β such that $\alpha + \beta > 0$, $\frac{\partial^{\alpha+\beta}\tilde{g}}{\partial\overline{\eta_\zeta}^\alpha \partial\overline{v_\zeta}^\beta} = 0$ on $X \cap D \cap \mathcal{U}$.*

Proof. We proceed as in the proof of Lemma 5.2. Let ε_0 be a positive real number. On $D \setminus D_{-\varepsilon_0}$ we define, for any non-negative integer j, χ_j and $\tilde{g}_j$ and $\tilde{g}_{\varepsilon_0}$ as in the proof of Lemma 5.2 and we prove that it satisfies the wanted estimates. As in the proof of Lemma 5.2, $\rho^{N+4}\tilde{g}_{\varepsilon_0}$ vanishes at order N on bD and $\frac{\partial^{\alpha+\beta}\tilde{g}_{\varepsilon_0}}{\partial\overline{z_1}^{\alpha}\partial\overline{z_2}^{\beta}} = 0$ on $X \cap D$. Moreover we have for $z \in \mathcal{P}_{\kappa|\rho(z_j)|}(z_j)$

$$\begin{aligned}\left|\tilde{g}_j(z)\frac{\partial^{\alpha+\beta}\chi_j}{\partial\overline{\eta_z}^{\alpha}\partial\overline{v_z}^{\beta}}(z)\right| &\lesssim |\rho(z_j)|^{-\alpha-\frac{\beta}{2}}\left|\tilde{g}_j(z)\right| \\ &\lesssim |\rho(z_j)|^{-\alpha-\frac{\beta}{2}}\sum_{k=1}^{q_j}\left|g_{z_j,v_{z_j}}[\alpha_1(z_1^*),\dots,\alpha_k(z_1^*)]\right||\rho(z_j)|^{\frac{k-1}{2}} \\ &\lesssim |\rho(z)|^{-\alpha-\frac{\beta}{2}}\sum_{k=1}^{q_j}\left|g_{z_j,v_{z_j}}[\alpha_1(z_1^*),\dots,\alpha_k(z_1^*)]\right||\rho(z)|^{\frac{k-1}{2}}\end{aligned}$$

and thus $z \mapsto |\rho(z)|^{\alpha+\frac{\beta}{2}}\frac{\partial^{\alpha+\beta}\tilde{g}_{\varepsilon_0}}{\partial\overline{\eta_z}^{\alpha}\partial\overline{v_z}^{\beta}}(z)$ is in $L^q(D)$ for all α and β. □

As a corollary of Theorem 1.1 and Theorem 5.5 we get:

Theorem 5.6. *In $\mathbb{C}^2$, if the function g holomorphic in $X \cap D$ is such that $c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) < \infty$, then g has a holomorphic extension G which belongs to $L^q(D)$.*

Proof. Theorem 5.5 and Cartan's Theorem B give a smooth extension to which we can apply Theorem 1.1 and get a holomorphic extension in $L^q(D)$. □

5.3. Extension and weakly holomorphic functions

One may notice that each time the smooth extension near the boundary is controlled only by the values of g on $X\cap D$. Moreover we have never used the strong holomorphy of g excepted when we involved Cartan's Theorem B in order to get a bounded extension far from the boundary. Actually, we can use only weak holomorphy and get a smooth extension and then apply theorem 1.1 in order to get a holomorphic extension with BMO or L^q norm controlled only by the values of g on $X \cap D$. Let us first recall the definition of weak holomorphy we shall use

Definition 5.7. Let $\mathcal{U}$ be an open set of $\mathbb{C}^n$. A function g defined on X is said to be weakly holomorphic on $X \cap \mathcal{U}$ if it is locally bounded on $X \cap \mathcal{U}$ and holomorphic on the regular set of $X \cap \mathcal{U}$.

The following theorem is a direct corollary of Lemma 5.1:

Theorem 5.8. *In $\mathbb{C}^n$, for $q \in [1,+\infty)$, if the function g, defined on $X \cap D$, has a holomorphic extension $G \in L^q(D)$ then*

$$\sup\left|g_{z,v}[\lambda_1,\dots,\lambda_k]\right|\tau(z,v,|\rho(z)|)^{k-1}\left(\operatorname{Vol}\mathcal{P}_{\kappa|\rho(z)|}(z)\right)^{\frac{1}{2}} \le \|G\|_{L^q(\mathcal{P}_{\kappa|\rho(z)|}(z))}$$

where the supremum is taken over all $z \in D$, all unit vector v in $\mathbb{C}^n$, all positive integer k such that $k \le \#\Lambda_{z,v}$ and all $\lambda_1,\dots,\lambda_k \in \Lambda_{z,v}$ pairwise distinct.

When z is far from bD, Theorem 5.8 essentially says that the divided differences have to be bounded even in the case of L^q extensions, $q < \infty$. This is sufficient when $n = 2$ to construct a smooth bounded extension in $D_{-\varepsilon}$ for $\varepsilon > 0$.

Lemma 5.9. *For X and D in $\mathbb{C}^2$, let ε be a positive real number. Let g be a weakly holomorphic function on $X \cap D$ such that $c_\varepsilon(g) = \sup \left|g_{z,v}[\lambda_1, \ldots, \lambda_k]\right| < \infty$ where the supremum is taken over $z \in D_{-\frac{\varepsilon}{2}}$, all unit vector v in $\mathbb{C}^n$, all positive integer k such that $k \leq \#\Lambda_{z,v}$, all $\lambda_1, \ldots, \lambda_k \in \Lambda_{z,v}$ pairwise distinct. Then g has a smooth extension to $D_{-\varepsilon}$ bounded by c_ε up to a multiplicative constant uniform with respect to g.*

Proof. We proceed locally and glue all the extension. Since the only problems occur when we are near a singularity, we consider a singularity z_0 of X and we choose an orthonormal basis w_1, w_2 such that π_0, the orthogonal projection on the hyperplane orthogonal to w_2 passing through z_0, is a k_0 sheeted covering of X in a neighbourhood $\mathcal{U}_0 \subset D$ of z_0.

For $z_1 \neq 0$, we denote by $\lambda_1(z_1), \ldots, \lambda_{k_0}(z_1)$ the pairwise distinct complex number such that for $k = 1, \ldots, k_0$, $z_0 + z_1 w_1 + \lambda_k(z_1) w_2$ belongs to X. We set for $z = z_0 + z_1 w_1 + z_2 w_2, z_1 \neq 0$:

$$\tilde{g}_0(z) = \tilde{g}_0(z_0 + z_1 w_1 + z_2 w_2) = \sum_{k=1}^{k_0} \prod_{\substack{l=1 \\ l \neq k}}^{k_0} \frac{z_2 - \lambda_l(z_1)}{\lambda_k(z_1) - \lambda_l(z_1)} g(z_0 + z_1 w_1 + \lambda_k(z_1) w_2).$$

By construction, $\tilde{g}_0(z) = g(z)$ for all $z \in X \cap \mathcal{U}_0$, $z \neq z_0$. We denote by Δ_0 the complex line passing through z_0 and supported by w_2. Since z_0 is an isolated singularity of X, away from 0, the λ_j's depend locally holomorphicaly on z_1 and thus $\tilde{g}_0$ is holomorphic on $\mathcal{U}_0 \setminus \Delta_0$. Since the divided differences are bounded on $D_{-\frac{\varepsilon}{2}}$ by c_ε, $\tilde{g}_0$ is bounded on $\mathcal{U}_0 \setminus \Delta_0$ by c_ε up to a uniform multiplicative constant and thus $\tilde{g}_0$ is holomorphic and bounded on $\mathcal{U}_0$. □

Combining Theorems 1.1, 5.5, Lemma 5.9 and Corollary 5.3 we get the two following results:

Theorem 5.10. *For X and D in $\mathbb{C}^2$, let g be a weakly holomorphic function on $X \cap D$ such that $c_\infty(g) < \infty$. Then g has a holomorphic extension G which belong to $BMO(D)$ such that $\|G\|_{BMO(D)} \lesssim c_\infty(g)$.*

Theorem 5.11. *For X and D in $\mathbb{C}^2$, let g be a weakly holomorphic function in $X \cap D$ such that $c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) < \infty$ and $c_\varepsilon(g) < \infty$. Then g has a holomorphic extension G which belongs to $L^q(D)$ and such that $\|G\|_{L^q(D)} \lesssim c^{(q)}_{\kappa,(z_j)_{j\in\mathbb{N}}}(g) + c_\varepsilon(g)$.*

6. Examples

Example 6.1 (BMO extension). Let D be the ball of radius 1 and center $(1,0)$ in $\mathbb{C}^2$. We choose $\rho(z) = |z_1-1|^2+|z_2|^2-1$ as a defining function for D. For $\alpha_1,\alpha_2,\dots,\alpha_k \in \mathbb{C}$ pairwise distinct we set $v_i = (-\overline{\alpha_i}, 1)$. We denote by P_i the plane orthogonal to v_i passing through the origin and we set $\Delta_i = P_i \cap D$ and $X = \cup_{i=1}^k P_i$. Let also $g_1,\dots,g_k$ be k bounded holomorphic functions on Δ, the unit disc in $\mathbb{C}$. Since $\Delta_i = \{(z_1,z_2)\in\mathbb{C}^2 : z_2=\alpha_i z_1 \text{ and } |z_1-(1+|\alpha_i|^2)^{-1}| < (1+|\alpha_i|^2)^{-1}\}$, the function

$$g : \begin{cases} X\cap D \longrightarrow \mathbb{C} \\ (z_1,z_2) \longmapsto g_i(z_1(1+|\alpha_i|^2)-1) \end{cases}$$

is well-defined, bounded and holomorphic on $X\cap D$. Question: Under which conditions does g have a BMO holomorphic extension to the domain D?

In order to answer this question, we will try to find an upper bound for $c^{(\infty)}_{\kappa,\varepsilon_0}(g)$. Let $\zeta = (\zeta_1,\zeta_2)$ be a point in $D\setminus D_{-\varepsilon_0}$, let $z_1^*\in\mathbb{C}$ be such that $|z_1^*| < \kappa|\rho(\zeta)|$ and let $\lambda_1,\dots,\lambda_l$ be complex numbers pairwise distinct belonging to $\Lambda_{\zeta+z_1^*\eta_\zeta,v_\zeta}$. Perhaps after renumbering, we assume that $\zeta+z_1^*\eta_\zeta+\lambda_i v_\zeta$ belongs to Δ_i for all i. Moreover, if ζ is sufficiently near the origin, we can also assume that v_ζ does not belong to any of the plane P_i. We have

$$\begin{aligned} &g_{\zeta+z_1^*\eta_\zeta,v_\zeta}[\lambda_1,\dots,\lambda_l] \\ &= \sum_{i=1}^{l} \frac{1}{\prod_{\substack{j=1\\ j\neq i}}^{l}(\lambda_i-\lambda_j)} g_i\Big((\zeta_1+z_1^*\eta_{\zeta,1}+\lambda_i v_{\zeta,1})(1+|\alpha_i|^2)-1\Big). \end{aligned}$$

For $m=i,j$, λ_m satisfies the following equalities

$$\zeta_2+z_1^*\eta_{\zeta,2}+\lambda_m v_{\zeta,2} = \alpha_m(\zeta_1+z_1^*\eta_{\zeta,1}+\lambda_l v_{\zeta,1}), \qquad m=i,j$$

which yield $(\lambda_i-\lambda_j)v_{\zeta,2} = (\alpha_i-\alpha_j)(\zeta_1+z_1^*\eta_{\zeta,1}+\lambda_i v_{\zeta,1})+\alpha_j(\lambda_i-\lambda_j)v_{\zeta,1}$ and so

$$|\lambda_i-\lambda_j|\cdot|v_{\zeta,2}-\alpha_j v_{\zeta,1}| = |\alpha_i-\alpha_j|\cdot|\zeta_1+z_1^*\eta_{\zeta,1}+\lambda_i v_{\zeta,1}|.$$

We show that $|\zeta_1+z_1^*\eta_{\zeta,1}+\lambda_i v_{\zeta,1}| \eqsim |\zeta_1|$. First, we have $|z_1^*|\le\kappa|\rho(\zeta)|$ and since ζ belongs to D, $|\rho(\zeta)| \lesssim |\zeta_1|$ so $|z_1^*| \lesssim \kappa|\zeta_1|$. Secondly, $|v_{\zeta,1}| \eqsim \left|\frac{\partial\rho}{\partial\zeta_2}(\zeta)\right| \eqsim |\zeta_2|$ and since ζ belongs to D, $|\zeta_2| \lesssim |\zeta_1|^{\frac12}$. Since $|\lambda_i| \le 3\kappa|\rho(\zeta)|^{\frac12} \le |\zeta_1|^{\frac12}$, we get $|\lambda_i v_{\zeta,1}| \lesssim \kappa|\zeta_1|$ and $|\zeta_1+z_1^*\eta_{\zeta,1}+\lambda_i v_{\zeta,1}| \eqsim |\zeta_1|$. Hence provided κ is small enough, $|\lambda_i-\lambda_j| \gtrsim |\zeta_1|$ and

$$\left|g_{\zeta+z_1^*\eta_\zeta,v_\zeta}[\lambda_1,\dots,\lambda_l]\right| \lesssim \frac{1}{|\zeta_1|^{l-1}}\sum_{i=1}^{l}\left|g_i\Big((\zeta_1+z_1^*\eta_{\zeta,1}+\lambda_i v_{\zeta,1})(1+|\alpha_i|^2)-1\Big)\right|.$$

Since $\tau(\zeta, v_\zeta, |\rho(\zeta)|) \lesssim |\zeta_1|^{\frac{1}{2}}$, if we assume that there exists $c \in \mathbb{C}$ and $C > 0$ such that for all i, $|g_i(z+1) - c| \leq C|z|^{\frac{l-1}{2}}$ for all z near the origin of $\mathbb{C}$, we get

$$\tau(\zeta, v_\zeta, |\rho(\zeta)|)^{l-1} \left| g_{\zeta + z_1^* \eta_\zeta, v_\zeta}[\lambda_1, \ldots, \lambda_l] \right| \lesssim C.$$

So $c^{(\infty)}_{\kappa,\varepsilon_0}(g)$ is finite and Lemma 5.2 and Theorem 1.1 implies that g admits a BMO-holomorphic extension to D.

This is in general the best result we can get. For example, let α be a real number and let g_i be the function defined on the unit disc of $\mathbb{C}$ by $g_i(z) = (1+z)^\alpha$, $i = 1, \ldots, k$. Let x be a small positive real number and let ζ in D be the point $(x, 0)$. We have $\eta_\zeta = (1, 0)$, $v_\zeta = (0, 1)$, $\tau(\zeta, v_\zeta, |\rho(\zeta)|) \eqsim x^{\frac{1}{2}}$, $(x, \alpha_i x)$ belongs to Δ_i if x is sufficiently small, and

$$g_{\zeta, v_\zeta}[\alpha_1 x, \ldots, \alpha_k x] = \sum_{i=1}^{k} \frac{1}{x^{k-1} \prod_{\substack{j=1 \\ j \neq i}} (\alpha_i - \alpha_j)} \left(x(1 + |\alpha_i|^2) \right)^\alpha.$$

Therefore if $\alpha < \frac{k-1}{2}$, $\tau(\zeta, v_\zeta, |\rho(\zeta)|)^{k-1} |g_{\zeta, v_\zeta}[\alpha_1 x, \ldots, \alpha_k x]|$ is unbounded when x goes to 0. So $c_\infty(g)$ is not finite and Theorem 1.3 implies that g does not admit a holomorphic extension bounded on D.

Example 6.2 (L^2-extension in $\mathbb{C}^2$). Again let D be the ball of radius 1 and center $(1, 0)$ in $\mathbb{C}^2$ and for any positive odd integer q, let X be the singular complex hypersurface $X = \{z \in \mathbb{C}^2,\ z_1^q = z_2^2\}$. Then all g holomorphic and bounded on $X \cap D$ has a L^2 holomorphic extension to D if and only if $q = 1$ or $q = 3$.

When $q = 1$, X is a manifold and there is nothing to do.

When $q = 3$, X has a singularity at the origin. We will prove that the assumptions of Theorem 5.5 are satisfied for any κ-covering provided κ is small enough. To check these hypothesis, we set $\rho(z) = |z_1 - 1|^2 + |z_2|^2 - 1$, we fix a holomorphic square root α in $\mathbb{C} \setminus (-\infty, 0]$ and we prove the two following facts. The first one gives a relation between the distance from $z \in X \cap D$ to $z + \lambda v \in X \cap D$ and the coordinates of z.

Fact 6.3. Let κ be a sufficiently small positive real number, let K be a large positive real number, let $z = (z_1, z_2)$ be a point in $D \cap X$ near the origin, let $v = (v_1, v_2)$ be a unit vector of $\mathbb{C}^2$ such that $|v_1| \leq K|z_1|^{\frac{1}{2}}$ and let λ be a complex number such that $z + \lambda v$ belongs to $X \cap D$ and $|\lambda| \leq 4\kappa|\tau(z, v, |\rho(z)|)$.

Then, if κ is small enough, we have $|\lambda| \gtrsim |z_1|^{\frac{q}{2}}$, $|z_1| \lesssim |\rho(z)|^{\frac{1}{q}}$ and $|z_2| \lesssim |\rho(z)|^{\frac{1}{2}}$ each time uniformly with respect to z, κ and v.

Remark 6.4. The assumption $|v_1| \leq K|z_1|^{\frac{1}{2}}$ means that v is "nearly" tangential to $bD_{\rho(z)}$.

Proof. We first prove that $|\lambda| \gtrsim |\rho(z)|^{\frac{q}{2}}$. Since $|v_1|$ is small, v is transverse to X and without restriction we can assume that $z = (z_1, \alpha(z_1)^q)$ and that $z + \lambda v = (z_1 + \lambda v_1, -\alpha(z_1 + \lambda v_1)^q)$. Therefore we have

$$|\lambda| \geq |\alpha^q(z_1) + \alpha^q(z_1 + \lambda v_1)| \geq 2|z_1|^{\frac{q}{2}} - |\alpha^q(z_1) - \alpha^q(z_1 + \lambda v_1)|.$$

The mean value theorem gives

$$|\alpha^q(z_1) - \alpha^q(z_1 + \lambda v_1)| \lesssim |\lambda||v_1| \sup_{\zeta \in [z_1, z_1 + \lambda v_1]} \left| \frac{\partial \alpha^q}{\partial \zeta}(\zeta) \right|.$$

For all $\zeta \in [z_1, z_1 + \lambda v_1]$, we have $|\zeta| \lesssim |z_1|$, and so, provided κ is small enough, we get $|\lambda| \geq |z_1|^{\frac{q}{2}}$. Now, since $|\lambda| \leq 4\kappa|\rho(z)|^{\frac{1}{2}}$, we get $|z_1| \lesssim |\rho(z)|^{\frac{1}{q}}$ and $|z_2| \lesssim |\rho(z)|^{\frac{1}{2}}$. □

As previously, we denote by η_ζ the outer unit normal to $bD_{\rho(\zeta)}$ at ζ and by v_ζ a tangent vector to $bD_{\rho(\zeta)}$ at ζ. The second fact gives some kind of uniformity of Fact 6.3 on a Koranyi ball.

Fact 6.5. Let κ be a sufficiently small positive real number, let ζ be a point in D, let $z = \zeta + z_1^* \eta_\zeta + z_2^* v_\zeta$ be a point in $\mathcal{P}_{4\kappa|\rho(\zeta)|}(\zeta) \cap D \cap X$ and let λ be a complex number such that $z + \lambda v_\zeta$ belongs to $X \cap D \cap \mathcal{P}_{4\kappa|\rho(\zeta)|}(\zeta)$.
Then $|\lambda| \gtrsim |\zeta_1|^{\frac{q}{2}}$, $|\zeta_2| \lesssim |\rho(\zeta)|^{\frac{1}{2}}$ and $|\zeta_1| \lesssim |\rho(\zeta)|^{\frac{1}{q}}$ uniformly with respect to z, ζ and λ.

Proof. We want to apply Fact 6.3, so we first have to check that $|v_{\zeta,1}| \lesssim |z_1|^{\frac{1}{2}}$, uniformly with respect to z and ζ. On the one hand we have $|v_{\zeta,1}| \eqsim \left|\frac{\partial \rho}{\partial \zeta_2}(\zeta)\right| \eqsim |\zeta_2| \lesssim |\zeta_1|^{\frac{1}{2}}$. On the other hand $z_1 = \zeta_1 + z_1^* \eta_{\zeta,1} + z_2^* v_{\zeta,1}$ thus

$$\begin{aligned}
|\zeta_1| &\leq |z_1^*| + |z_2^*||v_{\zeta,1}| + |z_1| \\
&\lesssim \kappa|\rho(z)| + \kappa|\zeta_1| + |z_1| \\
&\lesssim |z_1| + \kappa|\zeta_1|.
\end{aligned}$$

Therefore, if κ is small enough, $|\zeta_1| \lesssim |z_1|$ and $|v_{\zeta,1}| \lesssim |z_1|^{\frac{1}{2}}$. Therefore we can apply Fact 6.3 which gives $|\lambda| \gtrsim |z_1|^{\frac{q}{2}}$ and since $|z_1| \gtrsim |\zeta_1|$ the first inequality is proved. The third inequality follows from the first one and from the fact that $|\lambda| \lesssim |\rho(\zeta)|^{\frac{1}{2}}$.

Fact 6.3 also gives $|z_2| \lesssim |\rho(z)|^{\frac{1}{2}}$ and since $|\rho(\zeta)| \eqsim |\rho(z)|$, we have

$$|\zeta_2| \lesssim |\zeta_2 - z_2| + |z_2| \lesssim |\rho(\zeta)|^{\frac{1}{2}} + |\rho(z)|^{\frac{1}{2}} \lesssim |\rho(\zeta)|^{\frac{1}{2}}.$$

□

Now we check the assumptions of Theorem 5.5 and for any κ-covering, $\kappa > 0$ sufficiently small, and any function g bounded on $X \cap D$ we prove that $c^{(2)}_{\kappa,(\zeta_j)_{j\in\mathbb{N}}}(g) \lesssim \|g\|_{L^\infty(D\cap X)}$, uniformly with respect to g.

Let $\mathcal{U}_0$ be a neighbourhood of the origin, let c, ε_0 and κ be small positive real numbers and let $\mathcal{P}_{\kappa|\rho(\zeta_j^{(k)})|}(\zeta_j^{(k)}), k \in \mathbb{N}, j \in \{1,\dots,n_k\}$ be a κ-covering of $D \cap \mathcal{U}_0$ such that for all k and all j, the point $\zeta_j^{(k)}$ belongs to $bD_{-(1-c\kappa)^k\varepsilon_0}$. We assume that κ is so small that Fact 6.5 holds true and we set $\tilde\kappa = 1 - c\kappa$.

For all $\zeta \in D$, the following inequality holds and is optimal in general:

$$|\rho(\zeta)| \int_{|z_1^*|<4\kappa|\rho(\zeta)|} \sum_{\lambda\in\Lambda_{\zeta+z_1^*\eta_\zeta,v_\zeta}} \left| g_{\zeta+z_1^*\eta_\zeta,v_\zeta}[\lambda] \right|^2 dV(z_1) \lesssim \|g\|^2_{L^\infty(X\cap D)} |\rho(\zeta)|^3.$$

This means that the corresponding estimate for $\zeta_j^{(k)}$ does not depend on j and since we will add these bound for all k and $j = 1,\dots,n_k$, we will also need an upper bound for n_k. For any non-negative integer k, we denote by σ_k the area measure on $bD_{-\tilde\kappa^k\varepsilon_0}$. Since $\mathcal{P}_{\kappa|\rho(\zeta_j^{(k)})|}(\zeta_j^{(k)})$ is a κ-covering, for all k we have as in the proof of Proposition 3.1

$$\begin{aligned}
\sigma_k\left(bD_{\tilde\kappa^k\varepsilon_0}\right) &\geq \sigma_k\left(bD_{\tilde\kappa^k\varepsilon_0} \cap \cup_{j=1}^{n_k} \mathcal{P}_{\kappa|\rho(\zeta_j^{(k)})|}((\zeta_j^{(k)}))\right)\\
&\geq \sum_{j=1}^{n_k} \sigma_k\left(bD_{\tilde\kappa^k\varepsilon_0} \cap \mathcal{P}_{\frac{c}{c_1}\kappa|\rho(\zeta_j^{(k)})|}((\zeta_j^{(k)}))\right)\\
&\gtrsim n_k\left(\tilde\kappa^k\varepsilon_0\right)^2.
\end{aligned}$$

Therefore $n_k \lesssim (\tilde\kappa^k\varepsilon_0)^{-2}$ and we have uniformly with respect to g

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{j=1}^{n_k} |\rho(\zeta_j^{(k)})| \int_{|z_1^*|<4\kappa|\rho(\zeta_j^{(k)})|} \sum_{\lambda\in\Lambda_{\zeta+z_1^*\eta_{\zeta_j^{(k)}},v_{\zeta_j^{(k)}}}} \left| g_{\zeta_j^{(k)}+z_1^*\eta_{\zeta_j^{(k)}},v_{\zeta_j^{(k)}}}[\lambda] \right|^2 dV(z_1^*)\\
&\qquad\lesssim \|g\|^2_{L^\infty(X\cap D)} \sum_{k=0}^{\infty} n_k \left(\tilde\kappa^k\varepsilon_0\right)^3\\
&\qquad\lesssim \|g\|^2_{L^\infty(X\cap D)}.
\end{aligned}$$

Now we handle the case of divided differences of order 2. We set

$$I(\zeta) = |\rho(\zeta)|^2 \int_{|z_1^*|<4\kappa|\rho(\zeta)|} \sum_{\substack{\lambda_1,\lambda_2\in\Lambda_{\zeta+z_1^*\eta_\zeta,v_\zeta}\\ \lambda_1\neq\lambda_2}} \left| g_{\zeta+z_1^*\eta_\zeta,v_\zeta}[\lambda_1,\lambda_2] \right|^2 dV(z_1^*)$$

and we aim to prove that $\sum_{k=0}^{+\infty}\sum_{j=1}^{n_k} I(\zeta_j^{(k)}) \lesssim \|g\|_{L^\infty(X\cap D)}$. Let ζ be a point in $bD_{-\tilde{\kappa}^k\varepsilon_0}$. If for all complex number z_1^* such that $|z_1^*| \leq \kappa|\rho(\zeta)|$ we have $\#\Lambda_{\zeta+z_1^*\eta_\zeta,v_\zeta} < 2$, then $I(\zeta)=0$. Otherwise Fact 6.5 implies that $|\zeta_2| \leq K(\tilde{\kappa}^k\varepsilon_0)^{\frac{1}{2}}$ for some $K>0$ and that $|\lambda_1-\lambda_2| \gtrsim |\zeta_1|^{\frac{3}{2}}$ for all λ_1,λ_2 distinct in $\Lambda_{\zeta+z_1^*\eta_\zeta,v_\zeta}$, $z_1^*\in\mathbb{C}$ such that $|z_1^*|\leq\kappa|\rho(\zeta)|$. Therefore, for all such ζ, we have

$$I(\zeta) \lesssim |\rho(\zeta)|^2 \int_{|z_1^*|<4\kappa|\rho(\zeta)|} \frac{\|g\|_{L^\infty(D\cap X)}}{|\zeta_1|^3} dV(z_1^*) \lesssim \|g\|_{L^\infty(X\cap D)} \frac{|\rho(\zeta)|^4}{|\zeta_1|^3}. \quad (6.1)$$

Thus, when we denote by $Z^{(k)}$ the set

$$Z^{(k)} = \{j\in\mathbb{N} : \exists z_1^*\in\mathbb{C},\ |z_1^*|<\kappa|\rho(\zeta_j^{(k)})| \text{ and } \#\Lambda_{\zeta_j^{(k)}+z_1^*\eta_{\zeta_j^{(k)}},v_{\zeta_j^{(k)}}}=2\},$$

we have to estimate the sum $\sum_{k=0}^{+\infty}\sum_{j\in Z^{(k)}} \frac{(\tilde{\kappa}^k\varepsilon_0)^4}{|\zeta_{j,1}^{(k)}|^3}$.

We use the inclusion $Z^{(k)} \subset \cup_{i=1}^{\infty} Z_i^{(k)}$ where

$$Z_i^{(k)} = \{j\in Z^{(k)} : i\tilde{\kappa}^k\varepsilon_0 \leq |\zeta_{j,1}^{(k)}| < (i+1)\tilde{\kappa}^k\varepsilon_0 \text{ and } |\zeta_{j,2}^{(k)}| \leq K(\tilde{\kappa}^k\varepsilon_0)^{\frac{1}{2}}\}$$

and we look for an upper bound of $\#Z_i^{(k)}$. We have

$$\sigma_k(bD_{-\tilde{\kappa}^k\varepsilon_0}\cap\{z,\ \frac{1}{2}i\tilde{\kappa}^k\varepsilon_0 \leq |z_1| \leq 2(i+1)\tilde{\kappa}^k\varepsilon_0 \text{ and } |z_2|\leq 2K(\tilde{\kappa}^k\varepsilon_0)^{\frac{1}{2}}\}) \eqsim (\tilde{\kappa}^k\varepsilon_0)^2$$

and, if κ is small enough,

$$\begin{aligned}
&\sigma_k(bD_{-\tilde{\kappa}^k\varepsilon_0}\cap\{z,\ \frac{1}{2}i\tilde{\kappa}^k\varepsilon_0 \leq |z_1| \leq 2(i+1)\tilde{\kappa}^k\varepsilon_0 \text{ and } |z_2|\leq 2K(\tilde{\kappa}^k\varepsilon_0)^{\frac{1}{2}}\})\\
&\quad\gtrsim \sigma_k(\cup_{j\in Z_i^{(k)}}\mathcal{P}_{\kappa|\rho(\zeta_j^{(k)})|}(\zeta_j^{(k)})\cap bD_{-\tilde{\kappa}^k\varepsilon_0})\\
&\quad\gtrsim \#Z_i^{(k)}\cdot(\tilde{\kappa}^k\varepsilon_0)^2.
\end{aligned}$$

These last two inequalities imply that $\#Z_i^{(k)}$ is bounded by a constant which depends neither on i nor on k.

For $j\in Z_0^{(k)}$, since $|\zeta_{j,1}^{(k)}| \gtrsim |\rho(\zeta_j^{(k)})|$, Inequality (6.1) yields $I(\zeta_j^{(k)}) \lesssim \tilde{\kappa}^k\varepsilon_0\|g\|_{L^\infty(X\cap D)}$ thus

$$\sum_{k=0}^{+\infty}\sum_{j\in Z_0^{(k)}} I(\zeta_j^{(k)}) \lesssim \|g\|_{L^\infty(X\cap D)}.$$

For $i > 0$, we use directly (6.1) which gives

$$\sum_{i=1}^{+\infty}\sum_{k=0}^{+\infty}\sum_{j\in Z_i^{(k)}} I(\zeta_j^{(k)}) \lesssim \|g\|_{L^\infty(X\cap D)} \sum_{k=0}^{+\infty}\sum_{i=1}^{+\infty} \frac{(\tilde{\kappa}^k\varepsilon_0)^4}{(i\tilde{\kappa}^k\varepsilon_0)^3} \lesssim \|g\|_{L^\infty(X\cap D)}.$$

This finishes to prove that $c^{(2)}_{\kappa,(\zeta_j^{(k)})_{k\in\mathbb{N},j\in\{1,\dots,n_k\}}}$ is finite and Theorem 5.5 now implies that g admits a L^2-holomorphic extension to D.

Now, for $q \geq 5$, we consider g defined for z in X by $g(z) = \frac{z_2}{z_1^{\frac{q}{2}}}$. The function g is holomorphic and bounded on X because $|z_2| = |z_1|^{\frac{q}{2}}$ for all $(z_1, z_2) \in X$ but we will see that g does not admits a L^2-holomorphic extension to D.

For $\varepsilon_0, \kappa, c > 0$ small enough we set $\tilde{\kappa} = 1 - c\kappa$ and we denote by $\zeta_0^{(k)} = (x_k, 0)$ the point of $\mathbb{C}^2$ such $\rho(\zeta_0^{(k)}) = -\tilde{\kappa}^k\varepsilon_0$. We have $x_k \eqsim \tilde{\kappa}^k\varepsilon_0$ uniformly with respect to k, κ and ε_0. We complete the sequence $(\zeta_0^{(k)})_{k\in\mathbb{N}}$ so as to get a κ-covering $\mathcal{P}_{\kappa|\rho(\zeta_j^{(k)})|}(\zeta_j^{(k)})$, $k \in \mathbb{N}$ and $j \in \{0, \dots, n_k\}$, of a neighbourhood of the origin. We set $w_1 = (1, 0)$ and $w_2 = (0, 1)$. For all k, $\eta_{\zeta_0^{(k)}} = w_1$, $v_{\zeta_0^{(k)}} = w_2$ and, for all z_1, we have $\Lambda_{\zeta_0^{(k)}+z_1w_1,w_2} = \{(z_1 + \tilde{\kappa}^k\varepsilon_0)^{\frac{q}{2}}, -(z_1 + \tilde{\kappa}^k\varepsilon_0)^{\frac{q}{2}}\}$. So, if κ is small enough, for all k we have

$$\begin{aligned}
&|\rho(\zeta_0^{(k)})|^2 \int_{|z_1|<4\kappa|\rho(\zeta_0^{(k)})|} \left| g_{\zeta_0^{(k)}+z_1w_1,w_2}\left[(z_1 + \tilde{\kappa}^k\varepsilon_0)^{\frac{q}{2}}, -(z_1 + \tilde{\kappa}^k\varepsilon_0)^{\frac{q}{2}}\right]\right|^2 dV(z_1)\\
&\quad\gtrsim (\tilde{\kappa}^k\varepsilon_0)^2 \int_{|z_1|<4\kappa|\rho(\zeta_0^{(k)})|} \frac{1}{|z_1 + \tilde{\kappa}^k\varepsilon_0|^q} dV(z_1)\\
&\quad\gtrsim (\tilde{\kappa}^k\varepsilon_0)^{4-q}.
\end{aligned}$$

Since for $q \geq 5$ the series $\sum_{k\geq 0}(\tilde{\kappa}^k\varepsilon_0)^{4-q}$ diverges $c^{(2)}_{\kappa,(\zeta_j^{(k)})_{k\in\mathbb{N},j\in\{0,\dots,n_k\}}}(g)$ is not finite and so Theorem 5.4 implies that g does not have a L^2 holomorphic extension to D.

Example 6.6 (The example of Diederich-Mazzilli). Let B_3 be the unit ball of $\mathbb{C}^3$, $X = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 \;:\; z_1^2 + z_2^q = 0\}$ where $q \geq 10$ is an uneven integer, and define the holomorphic function f on $\mathbb{C}^3$ by

$$f(z) = \frac{z_1}{(1 - z_3)^{\frac{q}{4}}}.$$

Then f is bounded on $X \cap B_3$ and has no L^2 holomorphic extension to B_3.

This was shown in [18] by Diederich and the second author. We will prove this result here with Theorem 5.4.

We set $\rho(\zeta) = |\zeta_1|^2 + |\zeta_2|^2 + |\zeta_3|^2 - 1$, and we denote by w_1, w_2, w_3 the canonical basis of $\mathbb{C}^3$. For all non-negative integer j and ε_0, c and κ small suitable constants for X and B_3, we define $\tilde{\kappa} = (1 - c\kappa)$. For any integer j, we denote by $\zeta_j = (0, 0, \zeta_{j,3})$ the point of $\mathbb{C}^3$ such that $\zeta_{j,3}$ is real and satisfies $\rho(\zeta_j) = -\tilde{\kappa}^j \varepsilon_0$. The point ζ_j can be chosen at the first step of the construction of a κ-covering of $X \cap D$ in a neighbourhood of $(0, 0, 1)$ and so the Koranyi balls $\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$, $j \in \mathbb{N}$, are extract from a κ-covering. For all j we have

$$|\rho(\zeta_j)|^2 \int_{\substack{|z_2|<(4\kappa|\rho(\zeta_j)|)^{\frac{1}{2}} \\ |z_3-\zeta_{j,3}|<4\kappa|\rho(\zeta_j)|}} \left| f_{\zeta_j+z_2w_2+z_3w_3,w_1}\left[z_2^{\frac{q}{2}}, -z_2^{\frac{q}{2}}\right] \right|^2 dV(z_2, z_3) \gtrsim \tilde{\kappa}^{j(5-\frac{q}{2})}$$

and thus when $q \geq 5$,

$$\sum_{j=0}^{+\infty} |\rho(\zeta_j)|^2 \int_{\substack{|z_2|<(4\kappa|\rho(\zeta_j)|)^{\frac{1}{2}} \\ |z_3-\zeta_{j,3}|<4\kappa|\rho(\zeta_j)|}} \left| f_{\zeta_j+z_2w_2+z_3w_3,w_1}\left[z_2^{\frac{q}{2}}, -z_2^{\frac{q}{2}}\right] \right|^2 dV(z_1, z_3) = +\infty.$$

Theorem 5.4 then implies that f does not have an L^2 holomorphic extension to B_3.

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