Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XIV (2015), 767-817

Rigidity and regularity of codimension-one Sobolev isometric immersions

ZHUOMIN LIU AND MOHAMMAD REZA PAKZAD

Abstract. We prove the developability and $C^{1,1/2}_{\rm loc}$ regularity of $W^{2,2}$ isometric immersions of n-dimensional domains into $\mathbb{R}^{n+1}$. As a conclusion we show that any such Sobolev isometry can be approximated by smooth isometries in the $W^{2,2}$ strong norm, provided the domain is C^1 and convex. Both results fail to be true if the Sobolev regularity is weaker than $W^{2,2}$.

Mathematics Subject Classification (2010): 58D10 (primary); 46T10, 30L05, 53C24 (secondary).

1. Introduction

It has been known since at least the 19th century that any smooth surface with zero Gaussian curvature is locally ruled, *i.e.* passing through any point of the surface is a straight segment lying on the surface. Such surfaces were called developable there. This terminology was used as an indication that any such surface is in isometric equivalence with the plane, *i.e.* any piece of it can be *developed* on the flat plane without any stretching or compressing. Meanwhile, it was already suspected that there exist somewhat regular surfaces applicable to the plane, but yet not developable (see [4] for a review of this question). Nevertheless, it was not until the work of John Nash at the zenith of the last century that the existence of such unintuitive phenomena was rigorously established.

In his pioneering work, Nash settled several questions. He established that any Riemannian manifold can be isometrically embedded in a Euclidean space [19]. Moreover, if the dimension of the space is large enough, this embedding can be done in a manner that the diameter of the image is as small as one wishes. As for the lower dimensional embeddings, Nash [20] and Kuiper [17], established the existence of a C^1 isometric embedding of any Riemannian manifold into another manifold of dimension one higher. Their method, which is now famously re-cast in the framework of convex integration [8], involved iterated perturbations of a given short mapping of the manifold towards realizing an isometry.

Received February 1, 2013; accepted in revised version July 17, 2013.

A surprising corollary of these results is the existence of a C^1 flat torus in $\mathbb{R}^3$ [3]. Another one is that there are C^1 isometric embeddings of the two-dimensional unit sphere into three dimensional space with arbitrarily small diameter. By contrast, it was established by Hartman and Nirenberg that any flat C^2 surface in $\mathbb{R}^3$ must be developable [9], while Hilbert had already shown that any C^2 isometric immersion of the sphere must be a rigid motion. This latter result is a special case of a similar statement for any closed convex surface in $\mathbb{R}^3$, see [25, Chapter 12]. On the other hand, the former result was generalized by Pogorelov's for C^1 isometries with total zero curvature in [23, Chapter II] and [24, Chapter IX].

A natural question arises in this context for the analyst: What about isometric immersions of intermediate regularity, say of Hölder or Sobolev type? Regarding Hölder regularity, rigidity of $C^{1,\alpha}$ isometries of two-dimensional flat domains has been established for $\alpha \geq 2/3$, see [1,2], while their flexibility in the sense of Nash and Kuiper is known for $\alpha < 1/7$, see [2,6]. The critical value for α is conjectured to be $1/2$ in this case. As for the regularity of Sobolev isometries, following the results of Kirchheim in [16] on $W^{2,\infty}$ solutions to degenerate Monge-Ampère equations (see Proposition 1.3), the rigidity of $W^{2,2}$ isometries of a flat domain was established in [22]. More precisely, it was established that such mappings are developable in the classical sense, namely:

Theorem 1.1 (Pakzad [22]). *Let $v \in W^{2,2}(\Sigma, \mathbb{R}^3)$ be an isometric immersion, where Σ is a bounded Lipschitz domain in $\mathbb{R}^2$. Then $v \in C^{1,1/2}_{\mathrm{loc}}(\Sigma, \mathbb{R}^2)$. Furthermore, for every point x of Σ, there exists either a neighborhood of x, or a unique segment passing through x and joining $\partial\Sigma$ at both ends, on which ∇v is constant.*

Remark 1.2. It can be shown that this statement is actually valid for all bounded open sets $\Sigma \subset \mathbb{R}^2$, *i.e.* without any assumption on the regularity of the boundary. All one must prove is that the constancy segments, whose existence is locally established, can be extended all the way to the boundary one step at a time. Assuming the existence of any supposedly maximal constancy segment which does not reach the boundary, a contradiction could be achieved by creating a Lipschitz domain $\Sigma' \subset \Sigma$ including the closure of that segment and applying Theorem 1.1 to Σ'. In the same manner, the regularity assumption on $\partial\Omega$ in Theorem 1.4 can be removed.

To put this result in context, it is worth noting that a $W^{2,2}$ function on a two-dimensional domain barely fails to be C^1, but there is information available about weak second derivatives, and *e.g.* the Gaussian curvature of the image of a $W^{2,2}$ isometric immersion of a flat domain is identically zero as an L^1 function. This indicates that these isometries are far from the highly oscillatory solutions of Nash and Kuiper and hence possibly should behave in a rigid manner. Note that only the C^1 regularity result was stated in [22] and was a major ingredient of the proof, but the higher Hölder regularity announced here is an immediate consequence of the developability. In [18] it was established that the C^1 regularity can be extended to the boundary if the domain is of class $C^{1,\alpha}$. This does not hold true anymore for merely C^1 regular domains. Finally, the following proposition is a key step in establishing the above rigidity result and will be instrumental in proving Theorem 1.4:

Proposition 1.3 (Kirchheim [16], Pakzad [22]). *Let Σ be as above and let $f \in W^{1,2}(\Sigma, \mathbb{R}^3)$ be a map with almost everywhere symmetric and singular (i.e. of zero determinant) gradient. Then $f \in C^0(\Sigma)$ and for every point $x \in \Sigma$ there exists either a neighborhood U of x, or a segment passing through it and joining $\partial\Sigma$ at both ends, on which f is constant.*

It was proved furthermore in [22] that any $W^{2,2}$ isometry on a convex two-dimensional domain can be approximated in strong norm by smooth isometries. This is a nontrivial result, since the usual regularization techniques fail due to the non-linearity of the isometry constraint. The idea was to make use of the developability structure of these mappings and reduce the approximation problem to the one about mollifying the expressions $R^T R'$ for the Darboux moving frames $R(t)$ along the curves orthogonal to the rulings. The convexity assumption is a technical one, and as shown by Hornung [10,12], can be replaced by *e.g.* piece-wise C^1 regularity of the boundary, see also [11].

It is natural to ask whether these results can be generalized to higher dimensions. In [26], the authors showed the generalized developability of smooth isometric immersions of Euclidean domains into Euclidean spaces. We would like to pose the same problem for the same class of isometric immersions but considered only under sufficient Sobolev regularity assumptions. A first main result in this direction, presented in this paper, is the developability of $W^{2,2}$ codimension-one isometries of flat domains in $\mathbb{R}^n$:

Theorem 1.4. *Let $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ be an isometric immersion, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Then $u \in C^{1,1/2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n+1})$. Moreover, for every $x \in \Omega$, either ∇u is constant in a neighborhood of x, or there exists a unique $(n-1)$-dimensional hyperplane $P \ni x$ of $\mathbb{R}^n$ such that ∇u is constant on the connected component of x in $P \cap \Omega$.*

The interesting feature of this new result is that the Sobolev regularity $W^{2,2}$ is much below the required $W^{2,n+\varepsilon}$ for obtaining C^1 regularity. An extra difficulty which comes in the way of the proof in dimensions higher than 2 is that the argument used in [22, Lemma 2.1] to show the continuity of the derivatives of the given Sobolev isometry is no more generalizable to our case. Indeed, in [22], a very important first step of the proof of developability is to show the C^1 regularity. Here, on the other hand, we first show the developability of the mapping without having the C^1 regularity at hand. Our proof is based on induction on the dimension of slices of the domain, and on careful and detailed geometric arguments. Having established developability, the C^1 regularity (and better) follows in a straightforward manner.

The problem of regularity and developability of Sobolev isometric immersions of co-dimension $k > 1$ is more involved and could not be tackled through the methods discussed in this paper. In a forthcoming paper by Jerrard and the second author [15], another approach, more analytical in nature, is adapted to study this problem. It is based on the fact that the Hessian rank inequality

$$\operatorname{rank}(\nabla^2 v) \leq k \quad \text{a.e. in } \Omega \tag{1.1}$$

is satisfied by the components $v = u^j$ of such an isometry. Note that this equation becomes the degenerate Monge-Ampère equation when $k = n - 1$. Similarly as in [22], regularity and developability of the Sobolev solutions to (1.1) directly implies the same results for the corresponding isometries. However, one loses some natural advantages when working with (1.1) rather than with the isometries themselves as done in the present paper: the solution v is no more Lipschitz and it is just a scalar function, so one loses the extra information derived from the length-preserving properties of isometries. Methods of geometric measure theory applied to the class of Monge-Ampère functions developed by Jerrard in [13, 14] are used to overcome these obstacles.

The second main result of this paper concerns approximation of $W^{2,2}$ isometries by smooth ones:

Theorem 1.5. *Assume $\Omega \subset \mathbb{R}^n$ is a C^1 bounded convex domain and that $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$ is an isometric immersion. Then there is a sequence of isometric immersions $u_m \in C^\infty(\overline{\Omega}, \mathbb{R}^{n+1})$ converging to u in $W^{2,2}$ norm.*

The main idea of the proof, similar as in the two-dimension case, is to mollify the curves which pass orthogonally through the constancy hyperplanes of Theorem 1.4 both in the domain and on the image. This latter problem, framed within the general isometry mollification problem, is still nonlinear. However, identifying these curves with suitable orthonormal moving Darboux frames $R(t) \in \mathrm{SO}(n)$ and $\tilde{R}(t) = [(\nabla u)R(t), \mathbf{n}(t)] \in \mathrm{SO}(n+1)$, where $\mathbf{n}$ is the unit normal to the image of the isometry in $\mathbb{R}^{n+1}$, we could linearize the problem by considering the curvature matrices $R^T R'(t) \in \mathrm{so}(n)$ and $\tilde{R}^T \tilde{R}'(t) \in \mathrm{so}(n+1)$ and recover an approximating sequence of moving frames through their regularization. Many technical details must nevertheless be taken care of in this process; in particular one must make sure that the mollified curves can be used to define new smooth isometries. Also, the mapping as a whole cannot be described by one single pair of such curves and the domain must be partitioned into suitable subdomains.

Remark 1.6. Neither the C^1 regularity nor the convexity of the boundary seem to be absolutely necessary for the density result to hold true (see *e.g.* [12] for finer results in two-dimension), but omitting these assumptions goes beyond the scope of our paper. However, both of the results in Theorems 1.4 and 1.5 are sharp in the sense that they fail to be true if the isometric immersion is only of class $W^{2,p}$ for $p < 2$. An immediate counterexample is the following isometric immersion $u : B^2 \times (0,1)^{n-2} \to \mathbb{R}^{n+1}$, whose image can be visualized as a family of cones over a hyperplane of dimension $n - 2$:

$$u(r\cos\theta, r\sin\theta, x_3, \cdots, x_n) := \left(\frac{r}{2}\cos(2\theta), \frac{r}{2}\sin(2\theta), \frac{\sqrt{3}}{2}r, x_3, \cdots, x_n\right).$$

The paper is organized as follows: In Section 2 we will review some basic analytic properties of isometric immersions with second order derivatives. Section 3 is dedicated to the proof of Theorem 1.4. In Section 4 we will show that smooth isometric immersions are strongly dense in the space of $W^{2,2}$ isometric immersions from a domain of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$. The proof of Lemma 4.13, which is a crucial and difficult step in establishing the density result is postponed to the Appendix for the convenience of the reader.

ACKNOWLEDGEMENTS. The authors would like to thank Piotr Hajłasz for his interest in this problem and for many fruitful discussions. Z.L. was partially supported by NSF grants DMS-0900871 and DMS-0907844. M.R.P. was supported by the NSF grants DMS-0907844 and DMS-1210258. The authors thank the referee for a careful reading of the original manuscript and for useful comments and suggestions about the presentation.

2. Preliminaries

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$, with $n \geq 2$. We define the class of Sobolev isometric immersions from Ω to $\mathbb{R}^{n+1}$ as

$$I^{2,2}\big(\Omega, \mathbb{R}^{n+1}\big) := \big\{u \in W^{2,2}\big(\Omega, \mathbb{R}^{n+1}\big) : (\nabla u)^T \nabla u = \mathrm{I} \text{ a.e.}\big\}. \tag{2.1}$$

Note that the condition $(\nabla u)^T \nabla u = \mathrm{I}$ implies that u is Lipschitz continuous, thus,

$$I^{2,2}\big(\Omega, \mathbb{R}^{n+1}\big) \subset W^{2,2}\big(\Omega, \mathbb{R}^{n+1}\big) \cap W^{1,\infty}\big(\Omega, \mathbb{R}^{n+1}\big). \tag{2.2}$$

Given $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, let u^j, with $1 \leq j \leq n+1$, be the j-th component of u and let $u_{,i} = \partial u / \partial x_i$, $1 \leq i \leq n$, be the partial derivative of u in the $\mathbf{e}_i$ direction. Throughout the paper we will use the same notation for all functions.

For a.e. $x \in \Omega$, consider the cross product $\mathbf{n}(x) = u_{,1}(x) \times \cdots \times u_{,n}(x)$. That is, $\mathbf{n}(x)$ is the unique unit vector orthogonal to $u_{,i}(x)$ for all $1 \leq i \leq n$ such that $u_{,1}(x), \cdots, u_{,n}(x), \mathbf{n}(x)$ is a positive basis of $\mathbb{R}^{n+1}$.

Note that $\mathbf{n}$ can also be identified differential forms: consider the 1-form

$$\omega_i = \sum_{j=1}^{n+1} u^j_{,i} dx_j.$$

Then

$$\mathbf{n} = *(\omega_1 \wedge \cdots \wedge \omega_n), \tag{2.3}$$

because for any $\xi \in \bigwedge^1(\mathbb{R}^{n+1})$,

$$\langle \xi, \mathbf{n}\rangle = \langle \xi, *(\omega_1 \wedge \cdots \wedge \omega_n)\rangle = (-1)^n \xi \wedge \omega_1 \wedge \cdots \wedge \omega_n = (-1)^n \det[\xi, u_{,1}, \cdots, u_{,n}].$$

Since $u \in W^{2,2} \cap W^{1,\infty}(\Omega, \mathbb{R}^{n+1})$, it follows from (2.3) that $\mathbf{n} \in W^{1,2}(\Omega, \mathbb{R}^{n+1})$.

Since u is isometric immersion, $\langle u_{,i}, u_{,j}\rangle = \delta_{ij}$ for all $1 \le i, j \le n$. Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, we can differentiate using the product rule to obtain

$$\big\langle u_{,ik}, u_{,j}\big\rangle + \big\langle u_{,i}, u_{,jk}\big\rangle = 0 \quad \text{a.e.} \tag{2.4}$$

Permutation of indices i, j, k yields

$$\big\langle u_{,ij}, u_{,k}\big\rangle + \big\langle u_{,i}, u_{,kj}\big\rangle = 0 \quad \text{a.e.} \tag{2.5}$$
$$\big\langle u_{,ki}, u_{,j}\big\rangle + \big\langle u_{,k}, u_{,ji}\big\rangle = 0 \quad \text{a.e.} \tag{2.6}$$

Using the fact that $u_{,ij} = u_{,ji}$ for all i, j, we add (2.4) and (2.5), then subtract (2.6) to obtain,

$$\big\langle u_{,i}, u_{,jk}\big\rangle = 0 \quad \text{a.e.} \quad \text{for all} \quad 1 \le i, j, k \le n. \tag{2.7}$$

Since for a.e. point in the domain, $\mathbf{n}$, $u_{,1}, \cdots, u_{,j}$ is an orthonormal basis of $\mathbb{R}^{n+1}$, we can write

$$u_{,jk} = \sum_{i=1}^{n} \big\langle u_{,jk}, u_{,i}\big\rangle u_{,i} + \big\langle u_{,jk}, \mathbf{n}\big\rangle \mathbf{n}.$$

Then (2.7) gives

$$u_{,jk} = \big\langle u_{,jk}, \mathbf{n}\big\rangle \mathbf{n} \quad \text{a.e. for all} \quad 1 \le j, k \le n. \tag{2.8}$$

Note that $A_{jk} := \langle u_{,jk}, \mathbf{n}\rangle$ is the element in row j and column k of the second fundamental form A, which is a symmetric $n \times n$ matrix. In particular, (2.8) holds for each component of $u_{,jk}$ and $\mathbf{n}$, *i.e.*,

$$u^{\ell}_{,jk} = A_{jk}\mathbf{n}^{\ell} \quad \text{for all} \quad 1 \le \ell \le n+1, \quad 1 \le j, k \le n.$$

Thus, the Hessian of u^{ℓ} satisfies

$$\nabla^2 u^{\ell} = \mathbf{n}^{\ell} A, \quad 1 \le \ell \le n+1. \tag{2.9}$$

Lemma 2.1. *The second fundamental form $A \in M^{n\times n}$ has the following properties:*

$$\frac{\partial A_{ij}}{\partial x_k} = \frac{\partial A_{ik}}{\partial x_j} \quad \textit{in distributional sense for all} \quad 1 \le i, j, k \le n; \tag{2.10}$$
$$A_{ij}A_{kl} - A_{il}A_{kj} = 0 \quad \textit{for all} \quad 1 \le i, j, k, l \le n. \tag{2.11}$$

Proof. For a smooth immersion $v : \Omega \to \mathbb{R}^{n+1}$, not necessarily isometric, let $g_{ij} = \langle v_{,i}, v_{,j}\rangle$ be the first fundamental form. Differentiating g_{ij} twice we get

$$g_{ij,kl} = \big\langle v_{,ikl}, v_{,j}\big\rangle + \big\langle v_{,ik}, v_{,jl}\big\rangle + \big\langle v_{,il}, v_{,jk}\big\rangle + \big\langle v_{,i}, v_{,jkl}\big\rangle.$$

Summation over the proper permutations of i, j, k, l yields

$$g_{ij,kl} + g_{kl,ij} - g_{il,kj} - g_{kj,il} = -2\langle v_{,ij}, v_{,kl}\rangle + 2\langle v_{,il}, v_{,kj}\rangle. \tag{2.12}$$

Given any other smooth immersion $w : \Omega \to \mathbb{R}^{n+1}$, the following identity is also obvious:

$$\langle v_{,ij}, w\rangle_{,k} - \langle v_{,ik}, w\rangle_{,j} = \langle v_{,ij}, w_{,k}\rangle - \langle v_{,ik}, w_{,j}\rangle. \tag{2.13}$$

Now we consider a sequence of smooth immersions $u_m \to u$ in $W^{2,2}(\Omega, \mathbb{R}^{n+1})$ with $\mathbf{n}_m \to \mathbf{n}$ in $W^{1,2}(\Omega, \mathbb{R}^{n+1})$. Writing the left-hand sides of (2.12) and (2.13) as distributional derivatives and passing to the limit we get

$$0 = -2\langle u_{,ij}, u_{,kl}\rangle + 2\langle u_{,il}, u_{,kj}\rangle \tag{2.14}$$

because $\langle u_{,i}, u_{,j}\rangle = \delta_{ij}$ for all i, j. In addition, since $\mathbf{n}$ is a unit vector, $\langle \mathbf{n}_{,k}, \mathbf{n}\rangle = 0$. Then by (2.8), $\langle u_{,ij}, \mathbf{n}_{,k}\rangle = 0$ for all i, j, k, thus

$$\langle u_{,ij}, \mathbf{n}\rangle_{,k} - \langle u_{,ik}, \mathbf{n}\rangle_{,j} = 0. \tag{2.15}$$

The two identities in the lemma easily follow from $A_{ij} = \langle u_{,ij}, \mathbf{n}\rangle$, (2.14), and (2.15). The proof is complete. □

Corollary 2.2. *The second fundamental form A satisfies* rank A ≤ 1 *and A is symmetric a.e. in Ω. Moreover, the Hessian of each component of u satisfies* $\operatorname{rank}\nabla^2 u^\ell \leq 1$ *for all* $1 \leq \ell \leq n+1$ *a.e. on Ω.*

Proof. By identity (2.11), all 2×2 minors of A vanish, hence the rank of A is less than or equal to 1. By (2.9), $\operatorname{rank} \nabla^2 u^\ell \leq \operatorname{rank} A \leq 1$ and A is symmetric a.e. since $\nabla^2 u^\ell$ is symmetric a.e. The proof is complete. □

3. Developability and regularity

Our first main result (Theorem 1.4) follows from:

Proposition 3.1. *Let $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let A be the second fundamental form of u. Let P_k be a k-dimensional plane of $\mathbb{R}^n$, with $k \leq n$. Suppose that on $P_k \cap \Omega$ we have the following properties:*

(1) *There exists a sequence of smooth functions u^ϵ defined in the domain Ω such that*

$$\int_{P_k\cap\Omega} \left|u^\epsilon - u\right|^2 + \left|\nabla u^\epsilon - \nabla u\right|^2 + \left|\nabla^2 u^\epsilon - \nabla^2 u\right|^2 d\mathcal{H}^k \to 0.$$

Here ∇u^ϵ, ∇u, $\nabla^2 u^\epsilon$ and $\nabla^2 u$ denote the first and second full gradients with respect to the domain Ω.

(2) *The full gradient* ∇u *satisfies* $\nabla u^T \nabla u = \mathrm{I}\ \mathcal{H}^k$*-a.e. on* $P_k \cap \Omega$.
(3) $\nabla^2 u^\ell = \mathbf{n}^\ell A$ *for each* $1 \le \ell \le n+1$ $\mathcal{H}^k$*-a.e. on* $P_k \cap \Omega$.
(4) rank $A \le 1$ *and* A *is symmetric* $\mathcal{H}^k$*-a.e. on* $P_k \cap \Omega$.

Then $u \in C^{1,1/2}_{\mathrm{loc}}(P_k, \mathbb{R}^{n+1})$. *Moreover, for every* $x \in P_k \cap \Omega$, *either* ∇u *is constant on a neighborhood in* $P_k \cap \Omega$ *of* x, *or there exists a unique* $(k-1)$*-dimensional hyperplane* $P^x_{k-1} \ni x$ *of* P_k *such that* ∇u *is constant on the connected component of* x *in* $P^x_{k-1} \cap \Omega$.

The proof of this proposition is based on induction on lower dimensional slices. Before we prove Proposition 3.1, we will show that it implies Theorem 1.4.

Proof of Theorem 1.4. We simply take $k = n$ in Proposition 3.1, in which case $P_n \cap \Omega = \Omega$. Since $u \in W^{2,2}(\Omega, \mathbb{R}^{n+1})$, the convolution of u with the standard mollifier u^ϵ obviously satisfies assumption (1). By the fact that $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, we have $\nabla u^T \nabla u = \mathrm{I}$ a.e. in Ω, which is property (2). Property (3) follows from equation (2.9), and property (4) follows from Corollary 2.2 . Therefore, all the assumptions of Proposition 3.1 are satisfied, and hence the conclusion of Theorem 1.4 follows from the conclusion of Proposition 3.1. The proof is complete. □

Corollary 3.2. *Let* $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$, *where* Ω *is a Lipschitz domain in* $\mathbb{R}^n$. *Then for every* k*-dimensional slice* $P_k \cap \Omega$, ∇u *is constant either on* k*-dimensional neighborhoods of* $P_k \cap \Omega$, *or constant on* $(k-1)$*-dimensional slice of* $P_k \cap \Omega$.

Proof. Since assumptions (1)-(4) of Proposition 3.1 are satisfied a.e. in Ω, by Fubini's theorem, assumptions (1)-(4) also holds in a.e. k-dimensional slice. Thus the conclusion of Proposition 3.1 holds for a.e. k-dimensional slices. Since ∇u is continuous, by a simple approximation argument, it holds on *every* k-dimensional slice. The proof is complete. □

Assumptions (2), (3) and (4) regard the properties of isometric immersions, while (1) can be formulated for any general Sobolev function. This latter assumption is necessary for allowing the use of the chain rule which involves the full gradient even in lower dimensional slices. To be precise, we prove the following lemma which will play an important role everywhere in the proof of Proposition 3.1:

Lemma 3.3 (Chain Rule). *Let* $\Psi \in W^{1,2}(\Omega, \mathbb{R}^N)$, *with* $N \ge 1$. *Let* $\Sigma \subset \Omega$ *be a* k*-dimensional flat domain. Suppose that there exists a sequence of smooth functions* $\Psi^\epsilon \in C^\infty(\Omega, \mathbb{R}^N)$ *such that*

$$\int_\Sigma \left(|\Psi^\epsilon - \Psi|^2 + |\nabla\Psi^\epsilon - \nabla\Psi|^2\right) d\mathcal{H}^k \to 0, \tag{3.1}$$

where $\nabla\Psi$ *denotes the full gradient with respect to the domain* Ω. *Let* $\mathbf{v}$ *be any directional vector tangent to* Σ. *Then the chain rule*

$$\frac{d}{dt}\Big|_{t=0} \Psi(\cdot + t\mathbf{v}) = \nabla\Psi\mathbf{v}$$

holds in the weak sense over the domain Σ. *In particular,*

$$\Psi \in W^{1,2}\big(\Sigma, \mathbb{R}^N\big).$$

Proof. Let $\phi \in C_0^\infty(\Sigma)$; then

$$\int_\Sigma \frac{d}{dt}\Big|_{t=0} \Psi^\epsilon(x+t\mathbf{v})\phi(x)d\mathcal{H}^k = -\int_\Sigma \Psi^\epsilon(x)\frac{d}{dt}\Big|_{t=0}\phi(x+t\mathbf{v})d\mathcal{H}^k.$$

Since Ψ^ϵ is smooth in Ω, we have

$$\int_\Sigma \frac{d}{dt}\Big|_{t=0} \Psi^\epsilon(x+t\mathbf{v})\phi(x)d\mathcal{H}^k = \int_\Sigma \nabla\Psi^\epsilon(x)\mathbf{v}\phi(x)d\mathcal{H}^k.$$

By (3.1) we pass to the limit to conclude that

$$\int_\Sigma \nabla\Psi(x)\mathbf{v}\phi(x)d\mathcal{H}^k = -\int_\Sigma \Psi(x)\frac{d}{dt}\Big|_{t=0}\phi(x+t\mathbf{v})d\mathcal{H}^k.$$

Thus the chain rule as stated in the lemma holds in the weak sense over the domain Σ. The proof is complete. $\square$

Remark 3.4. Note that the above lemma involves the *full* gradient of Ψ. The assumption $\Psi \in W^{1,2}(\Sigma, \mathbb{R}^N)$ by itself is not enough to deduce the chain rule.

3.1. Base case: 2-dimensional slices

Suppose for a 2-dimensional plane P_2 all the assumptions (1)-(4) in Proposition 3.1 are satisfied. Without loss of generality, we can assume P_2 is parallel to the space spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. Indeed, it is easy to see that assumption (1)-(4) in Proposition 3.1 are invariant under rotating the coordinate system. We denote P_2 by $P_{\mathbf{e}_1\mathbf{e}_2}$ to remind ourselves this fact.

Let $f = \nabla u^\ell \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for some arbitrary $1 \le \ell \le n+1$. Define

$$g := \big(f^1, f^2\big)|_{P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega} \in W^{1,2}\big(P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega, \mathbb{R}^2\big).$$

Lemma 3.5. *Let* $f^\epsilon : \Omega \to \mathbb{R}^n$ *be a smooth sequence converging strongly to* f *in* $W^{1,2}(\Omega, \mathbb{R}^n)$. *Let* C *be a line segment in* $P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega$ *such that*

$$\int_C \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big)d\mathcal{H}^1 \to 0, \tag{3.2}$$

rank $\nabla f \le 1$ *and* ∇f *is symmetric for* $\mathcal{H}^1$*-a.e. point on* C. *Then if* g *is constant on* C, *so is* f.

Proof. Let $\mathbf{v}$ be the unit directional vector of C. Since $\mathbf{v}$ is a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$\mathbf{v} = \big(\mathbf{v}_1, \mathbf{v}_2, 0, \cdots, 0\big).$$

Let $\tilde{\mathbf{v}} = (\mathbf{v}_1, \mathbf{v}_2)$; then the first two components of f satisfy $\nabla f^1 \cdot \mathbf{v} = \nabla g^1 \cdot \tilde{\mathbf{v}}$ a.e. on C and $\nabla f^2 \cdot \mathbf{v} = \nabla g^2 \cdot \tilde{\mathbf{v}}$ a.e. on C.

Since f satisfies the assumption of Lemma 3.3, the chain rule

$$\frac{d}{dt}\Big|_{t=0} f(\cdot + t\mathbf{v}) = (\nabla f)\mathbf{v}$$

holds in the weak sense on C. In particular, it holds for it first two component f^1 and f^2 and, of course, g.

As g is constant on C,

$$0 = \frac{d}{dt}\Big|_{t=0} g(\cdot + t\tilde{\mathbf{v}}) \tag{3.3}$$

in the weak sense. Hence $(\nabla g)\tilde{\mathbf{v}} = 0 \quad$ a.e. on C. This implies

$$\nabla f^1 \cdot \mathbf{v} = 0 \quad \text{and} \quad \nabla f^2 \cdot \mathbf{v} = 0 \quad \text{a.e. on } C.$$

For $z \in C$ such that $\nabla f^1(z) \cdot \mathbf{v} = 0$ and $\nabla f^2(z) \cdot \mathbf{v} = 0$, $\operatorname{rank} \nabla f(z) \le 1$ and $\nabla f(z)$ is symmetric, so we have two cases: $(1)\nabla f^1(z) \neq 0$ or $\nabla f^2(z) \neq 0$; $(2)\nabla f^1(z) = \nabla f^2(z) = 0$. In the first case, we can assume with loss of generality that $\nabla f^1(z) \neq 0$. Therefore, $\operatorname{rank} \nabla f(z) = 1$ and

$$\nabla f^i(z) = a_z^i \nabla f^1(z) \quad \text{for all } i \ge 1.$$

It then follows that

$$\nabla f^i(z) \cdot \mathbf{v} = a_z^i\big(\nabla f^1(z) \cdot \mathbf{v}\big) = 0 \quad \text{for all } i \ge 1.$$

In the second case, by symmetry,

$$f^i_{,j}(z) = f^j_{,i}(z) = 0, \quad \text{for } j = 1, 2, \text{ and } i = 1, \cdots, n.$$

As $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, 0, \cdots, 0)$,

$$\nabla f^i(z) \cdot \mathbf{v} = 0 \quad \text{for all } i = 1, \cdots, n.$$

Therefore, in either cases, we have proved

$$\nabla f^i \cdot \mathbf{v} = 0 \quad \text{a.e. on } C \quad \text{for all } i = 1, \cdots, n.$$

Hence, f is constant on C by the chain rule (3.3). The proof is complete. $\square$

Corollary 3.6. *If g is constant on a 2-dimensional region U in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, f is constant on U as well.*

Proof. Observe that if U is a 2-dimensional region of $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, which has strictly positive 2-dimensional Hausdorff measure, then the assumptions (1) and (4) of Proposition 3.1 imply

$$\int_U \left|f^\epsilon - f\right|^2 + \left|\nabla f^\epsilon - \nabla f\right|^2 d\mathcal{H}^2 \to 0,$$

rank $\nabla f \leq 1$ and ∇f is symmetric for $\mathcal{H}^2$ a.e. points on U. Thus the same argument for line segments in Lemma 3.5 gives for any directional vector $\mathbf{v}$ of U that $\nabla f^i \cdot \mathbf{v} = 0$ a.e. on U for all $i = 1, \cdots, n$, hence the chain rule implies that f is constant on U. The proof is complete. □

Lemma 3.7. *Suppose assumptions* (1)-(4) *of Proposition* 3.1 *are satisfied on the two-dimensional region $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Let $f = \nabla u^\ell \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for some arbitrary $1 \leq \ell \leq n+1$. Then the restriction $f \in C^{0,1/2}_{\mathrm{loc}}(P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega, \mathbb{R}^n)$. Moreover, for every point $x \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, either there exists a neighborhood in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ of x, or a unique line segment in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through x and joining $\partial\Omega$ at both ends, on which f is constant.*

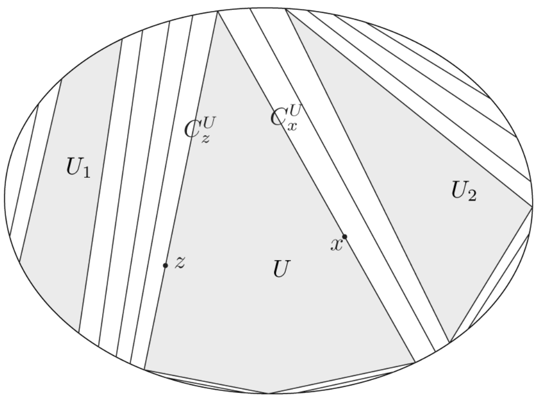

Figure 3.1. Inverse image of g in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$

Proof. The proof is divided into seven steps.

Step 0. Preliminary set up: by assumption (4) of Proposition 3.1, ∇f satisfies rank $\nabla f \leq 1$ and $\nabla f = \nabla^2 u^\ell$ is symmetric a.e. on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Therefore, $g := (f^1, f^2)|_{P_{\mathbf{e}_1\mathbf{e}_2}\cap\Omega} \in W^{1,2}(P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega, \mathbb{R}^2)$ also satisfies rank $\nabla g \leq 1$ and ∇g is symmetric a.e. on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. We employ [22, Proposition 1], which is cited above as Proposition 1.3. The function g satisfies the assumption of this proposition on the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ and hence the conclusions holds true for g. Suppose g is constant on some maximal connected neighborhood $U \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, by continuity of g, it is also constant on its closure $\overline{U} \cap \Omega$.

Step 1. We claim that the boundary of U only consists of line segments joining the boundary and none of these line segments intersect inside Ω. Indeed, if $x \in \partial U \cap \Omega$, then x is not contained in a constancy neighborhood of g, therefore by Proposition 1.3, there exists a unique line segment $C_x^U \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through x and joining $\partial\Omega$ at both ends on which g is constant, which implies $\partial U \cap \Omega \subset \bigcup_{x \in \partial U \cap \Omega} C_x^U$. Moreover, for $x, z \in \partial U \cap \Omega$, $C_x^U = C_z^U$ if $z \in C_x^U$ and $C_x^U \cap C_z^U \cap \Omega = \emptyset$ if $z \notin C_x^U$ (Figure 3.1). This follows from the fact that if g is constant on two such intersecting segments, it must be constant on their convex hull inside Ω too. On the other hand, suppose g is constant on some line segment C_x^U passing through $x \in \partial U \cap \Omega$ and joining $\partial\Omega$ at both end, since g is constant on $\overline{U}$ and C_x^U, which intersect at x, it must be constant on the convex hull of $\overline{U}$ and C_x^U inside Ω. But U is maximal, hence $\bigcup_{x \in \partial U \cap \Omega} C_x^U \subset \partial U \cap \Omega$. Therefore,

$$\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} C_x^U.$$

Step 2. We claim that we can choose small enough $\delta > 0$ so that for any region U on which g is constant, the 2-dimensional ball $B^2(x_0, \delta) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ intersects ∂U at no more than *two* line segments belonging to ∂U. Indeed, let $x_0 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ be such that g is not constant in a neighborhood of x_0. We use the fact that for any maximal constant region U, line segments in ∂U do not intersect inside Ω. If x_0 is at a positive distance of all constancy regions, the conclusion is trivial. The same is true if it lies on the boundary of one of the constancy regions and yet is positively distant from all others. Suppose therefore that there is a sequence of maximal constancy regions U_m converging to x_0 in distance, in which case there are two line segments $C_{x_1}^{U_m}$ and $C_{x_2}^{U_m}$ in ∂U_m whose angle (if they are nonparallel) or distance (if they are parallel) converges to zero, since both of these sequences of segments must converge to the same constancy segment passing through x_0. Then since all the other line segments in ∂U_m must be arbitrarily close to $\partial\Omega$, we can again choose δ small enough that $B^2(x_0, \delta)$ is away from $\partial\Omega$ and hence it does not intersect a third line segment in ∂U_m (Figure 3.2).

Step 3. We construct a foliation of the ball $B^2(x_0, \delta) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Firstly, for any $x \in B^2(x_0, \delta)$, we will construct a line segment C_x in $B^2(x_0, \delta)$ passing through x and joining $\partial B^2(x_0, \delta)$ at both ends on which g is constant and $C_x \cap C_z \cap B^2(x_0, \delta) = \emptyset$ if $z \notin C_x$. The construction is as follows: for those x not contained in a constant region of g, this line segment is given automatically by Proposition 1.3. If x is contained in a constant maximal region U of g, then it is constant on every line segment in U that passes through it so we have to choose the appropriate one: 1) If $B^2(x_0, \delta)$ intersect only one line segment C^U in Ω that belongs to ∂U, then we define C_x to be the line segments inside $B^2(x_0, \delta)$ passing through x and parallel to C^U; 2) If $B^2(x_0, \delta)$ intersects two line segments

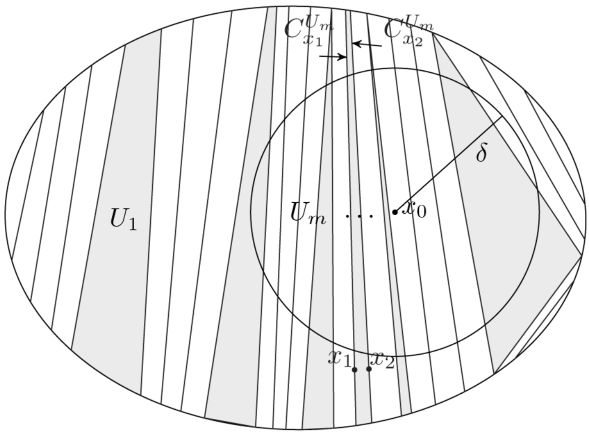

Figure 3.2. $B^2(x_0,\delta)$ intersects ∂U_m at two line segments

.

C_1^U, C_2^U in Ω that belongs to ∂U, let L_1 and L_2 be the two lines that contain C_1^U and C_2^U. If L_1 and L_2 are not parallel, let $O := L_1 \cap L_2$ and let C_x be the segment given by the intersection with $B^2(x_0,\delta)$ of the line passing through O and x. If L_1 and L_2 are parallel, then we let C_x be the line segment inside $B^2(x_0,\delta)$ passing through x and parallel to L_1. (Figure 3.3). In this way, we have constructed a family of line segments $\{C_x\}_{x\in B^2(x_0,\delta)}$ in $B^2(x_0,\delta)$ on which g is constant and $C_x \cap C_z \cap B^2(x_0,\delta) = \emptyset$ if $z \notin C_x$.

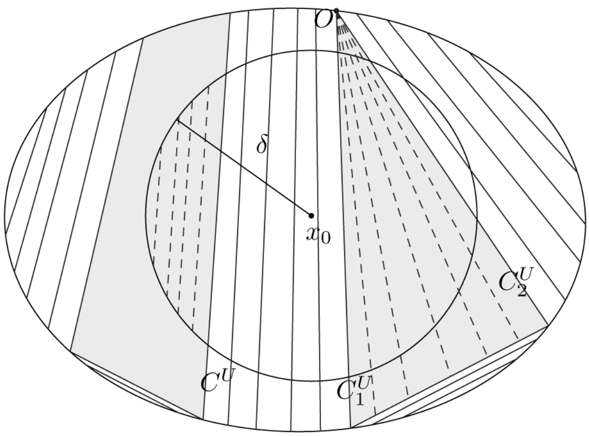

Figure 3.3. Constructin of foliations.

Secondly, for every $x \in B^2(x_0,\delta)$, let $\mathbf{N}(x)$ be the vector field orthogonal to C_x. By making δ smaller we can make sure that none of the C_x's intersect inside $B^k(x_0,2\delta)$, and therefore we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector field inside the ball of radius δ. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0,$$

then has a unique solution $\gamma : (a, b) \to B^2(x_0, \delta)$ for some interval $(a, b) \subset \mathbb{R}$ containing 0. Moreover, if necessary by making δ smaller, $\cup\{C_{\gamma(t)}\}_{t\in(a,b)} = B^2(x_0, \delta)$. Therefore, $\{C_{\gamma(t)}\}_{t\in(a,b)}$ is a foliation of $B^2(x_0, \delta)$ (Figure 3.4).

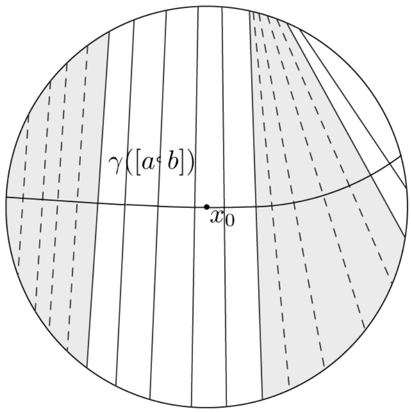

Figure 3.4. Foliations of $B^2(x_0, \delta)$.

Step 4. We now want to show the assumptions of Lemma 3.5 are satisfied along $C_{\gamma(t)}$ for a.e. $t \in (a, b)$. We define the function $h : B^2(x_0, \delta) \to B^2(x_0, \delta)$ as

$$h(x) = \gamma(t) \quad \text{if } x \in C_{\gamma(t)}.$$

Since none of the $C_{\gamma(t)}$ intersect inside $B^2(x_0, \delta)$, h is well defined and h is constant along each $C_{\gamma(t)}$, *i.e.* $h^{-1}(\gamma(t)) = C_{\gamma(t)}$. Since γ is Lipschitz, h is Lipschitz as well. Moreover, since $|\gamma''(t)|$ is uniformly bounded, we have the one-dimensional Jacobian $J_h > C > 0$. (For the definition of general k-dimensional Jacobian see [7, page 88].)

Let E_0 be the set of all $x \in B^2(x_0, \delta)$ such that $\operatorname{rank} \nabla f(x) > 1$ or $\nabla f(x)$ is not symmetric. By assumption (4) of Proposition 3.1 on f, $|E_0| = 0$. As h is Lipschitz, we can apply the general co-area formula [7, page 112] to h to obtain,

$$\begin{aligned} 0 = \int_{E_0} J_h(x)dx &= \int_\gamma \mathcal{H}^1\big(E_0 \cap h^{-1}(w)\big)d\mathcal{H}^1(w) \\ &= \int_a^b \mathcal{H}^1\big(E_0 \cap h^{-1}(\gamma(t))\big)|\gamma'(t)|\,dt \\ &= \int_a^b \mathcal{H}^1\big(E_0 \cap C_{\gamma(t)}\big)|\gamma'(t)|\,dt. \end{aligned}$$

Therefore, for a.e. $t \in (a,b)$, $\mathcal{H}^1\big(E_0 \cap C_{\gamma(t)}\big) = 0$ since $|\gamma'| = 1$. Moreover, by change of variable formula, if f^ϵ is a smooth approximation sequence,

$$\begin{aligned}
\int_{B^2(x_0,\delta)} &\big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) J_h \\
&= \int_\gamma \int_{h^{-1}(w)} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1 d\mathcal{H}^1(w) \\
&= \int_a^b \int_{h^{-1}(\gamma(t))} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1 |\gamma'(t)|\, dt \\
&= \int_a^b \int_{C_{\gamma(t)}} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1\, dt.
\end{aligned}$$

Since J_h is bounded, together with assumption (1) in Proposition 3.1, we then have for a.e. $t \in (a,b)$,

$$\int_{C_{\gamma(t)}} \big(|f^\epsilon - f|^2 + |\nabla f^\epsilon - \nabla f|^2\big) d\mathcal{H}^1 \to 0.$$

Therefore, the assumptions of Lemma 3.5 are satisfied along $C_{\gamma(t)}$ for a.e. $t \in (a,b)$.

Step 5. We now prove the Lemma for the ball $B^2(x_0,\delta)$. Step 4 and Lemma 3.5 imply that f is constant on $C_{\gamma(t)}$ for a.e. $t \in (a,b)$. By choosing an initial value for γ arbitrary close to x_0 and applying the general co-area formula in a similar manner we can make sure that f is of class $W^{1,2}$ on γ. Hence we conclude that f is $C^{0,1/2}$ on γ by the Sobolev embedding theorem. Let F be the set of $t \in (a,b)$ such that f is not constant along $C_{\gamma(t)}$, then $\mathcal{H}^1(F) = 0$. We modify f to be constant along $C_{\gamma(t)}$ for each $t \in F$. Note that,

$$\mathcal{H}^2\left(\bigcup\{C_{\gamma(t)} : t \in F\}\right) \le 2\delta \sup J_h^{-1} \mathcal{H}^1(\{\gamma(t) : t \in F\}) = 2\delta \sup J_h^{-1} \mathcal{H}^1(F) = 0.$$

Hence f is $C^{0,1/2}$ up to modification of a set of measure zero in $B^2(x_0,\delta)$. Moreover, f is constant on $C_{\gamma(t)}$ for all t, which foliates $B^2(x_0,\delta)$. In addition, by Corollary 3.6, f is constant on every 2-dimensional region in $B^2(x_0,\delta)$ on which g is constant. Therefore, f is either constant on a line segment joining $\partial B^2(x_0,\delta)$ at both ends, or constant on a 2-dimensional region in $B^2(x_0,\delta)$. This proves Lemma 3.7 for the ball $B^2(x_0,\delta)$.

Step 6. Finally, we prove the lemma for the entire domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Suppose there is some $x \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ that is not contained in a constant region of f. Then by what we have proved, f is constant on a line segment passing through x and joining the boundary of $B^2(x,\delta_x) \subset P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ for some $\delta_x > 0$. Let $\overline{y_1 y_2}$ be the largest line segment containing this segment on which f is constant. Suppose $y_1 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, then from what we have proved, f is either constant on 2-dimensional regions or line segments passing through y_1 and joining the boundary of $B^2(y_1,\delta_{y_1}) \subset$

$P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ for some $\delta_{y_1} > 0$. Firstly, y_1 cannot be contained in a constant region of f, otherwise we can prolong the segment $[y_1, y_2]$. Thus, there must be a line segment $\overline{z_1 z_2}$ passing through y_1 and joining the boundary of $B^2(y_1, \delta_{y_1})$ at both end on which f is constant. Secondly, $\overline{z_1 z_2}$ cannot have the same direction as $\overline{y_1 y_2}$, otherwise, we can again prolong the segment $\overline{y_1 y_2}$. Then we consider the region Δ bounded by $\overline{y_2 z_1}$, $\overline{z_1 z_2}$ and $\overline{z_2 y_2}$. Since g is constant on $\overline{y_1 y_2}$ and $\overline{z_1 z_2}$, by Proposition 1.3, g must be constant on Δ because no line segment can join the boundary of $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ passing through a point inside Δ without intersecting either $\overline{y_1 y_2}$ or $\overline{z_1 z_2}$ (Figure 3.5). Hence by Corollary 3.17, f is constant on Δ as well, contradiction to our assumption x is not contained in a constant region of f. The proof is complete. □

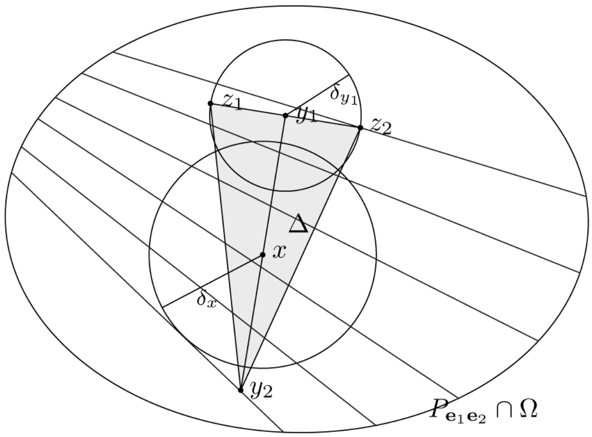

Figure 3.5.

Now we are ready to prove Proposition 3.1 for the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$. Since we take $f = \nabla u^\ell$ for arbitrary $1 \le \ell \le n+1$, Lemma 3.7 gives all ∇u^ℓ are continuous on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ and constant either on 2-dimensional neighborhoods or line segments in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ joining $\partial\Omega$ at both ends. Therefore, what is left is to prove that they are constant on the *same* neighborhoods or line segments in $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$.

Recall from equation (2.3) that $\mathbf{n}$ is the wedge product of entries of ∇u, hence is continuous. Let

$$\Delta_\ell = \{x \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega : \mathbf{n}^\ell(x) \neq 0\}.$$

Apparently each Δ_ℓ is open by continuity. Moreover, since $|\mathbf{n}| = 1$ everywhere,

$$\bigcup_{1 \le \ell \le n+1} \Delta_\ell = P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega.$$

Let $x_0 \in P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, then $x_0 \in \Delta_\ell$ for some ℓ. Without loss of generality, we assume $x_0 \in \Delta_1$. Then by the same argument as in the proof of Lemma 3.7, there exist $B^2(x_0, \delta) \subset \Delta_1$ for some $\delta > 0$, on which we can construct a foliation $\{C_{\gamma(t)}\}_{t\in(a,b)}$, *i.e.* $\cup\{C_{\gamma(t)}\}_{t\in(a,b)} = B^2(x_0, \delta)$ and $C_\gamma(t) \cap C_\gamma(t') \cap B^2(x_0, \delta) = \emptyset$ for $t' \neq t$. Moreover, $\nabla^2 u^1$ is constant on $C_\gamma(t)$ for every $t \in (a, b)$. Assumption (1) and (3) in Proposition 3.1, together with the same argument using co-area and change of variable formulas as in the proof of Lemma 3.7 yield for a.e. $t \in (a, b)$

$$\int_{C_{\gamma(t)}} \left|\nabla u^\epsilon - \nabla u\right|^2 + \left|\nabla^2 u^\epsilon - \nabla^2 u\right|^2 d\mathcal{H}^1 \to 0$$

and $\nabla^2 u^\ell = (\mathbf{n}^\ell/\mathbf{n}^1)\nabla^2 u^1, 2 \le \ell \le n+1, \mathcal{H}^1$ a.e on $C_{\gamma(t)}$.

Let $\mathbf{v}$ be the directional vector of one such $C_{\gamma(t)}$, then the chain rule in Lemma 3.3 and the fact that ∇u^1 is constant on $C_{\gamma(t)}$ imply

$$0 = \frac{d}{dt}\Big|_{t=0} \nabla u^1(\cdot + t\mathbf{v}) = \left(\nabla^2 u^1\right)\mathbf{v}$$

in the weak sense in $C_{\gamma(t)}$. Therefore,

$$\left(\nabla^2 u^\ell\right)\mathbf{v} = \frac{\mathbf{n}^\ell}{\mathbf{n}^1}\left(\nabla^2 u^1\right)\mathbf{v} = 0, \quad 2 \le \ell \le n+1 \text{ a.e. on } C_{\gamma(t)}.$$

Hence again by the chain rule in Lemma 3.3, $\nabla u^\ell, 2 \le \ell \le n+1$, is constant on $C_{\gamma(t)}$. Therefore, each ∇u^ℓ is constant on $C_{\gamma(t)}$ for a.e. $t \in (a, b)$. Furthermore, since for each $1 \le \ell \le n+1$, ∇u^ℓ is continuous on $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$, we conclude that ∇u^ℓ for all $1 \le \ell \le n+1$ are constant on all $C_{\gamma(t)}$ that foliates $B^2(x_0, \delta)$. On the other hand, each 2-dimensional region U of $B^2(x_0, \delta)$ automatically satisfies all the assumptions (1) and (3) in Proposition 3.1, hence the same argument for each $C_{\gamma(t)}$ gives that ∇u^ℓ for all $2 \le \ell \le n+1$ is constant on the same region on which ∇u^1 is constant. This proves ∇u is either constant on 2-dimensional regions or constant on line segments in $B^2(x_0, \delta)$ joining the boundary. The proof of Proposition 3.1 for the domain $P_{\mathbf{e}_1\mathbf{e}_2} \cap \Omega$ follows from exactly the same argument as the last step of the proof of Lemma 3.7. The proof for the base case is complete. □

3.2. Inductive step: k-dimensional slices

In this subsection, we will prove that Proposition 3.1 holds true for k if it holds true for $k-1$ when $2 < k \le n$. This, combined with the base case $k = 2$ established in the previous step, completes the proof of Proposition 3.1.

3.2.1. *Developability*

Based on the induction hypothesis for $k-1$, we first prove a weaker result in k-dimensional slices of Ω than Proposition 3.1. That is, we prove that u is developable on all k-dimensional slices satisfying assumptions (1)-(4) of Proposition 3.1 in the following sense:

Proposition 3.8. *Suppose Proposition* 3.1 *is true for any* $(k-1)$*-dimensional slice of* Ω *on which assumptions* (1)-(4) *are satisfied. Let* P_k *be any* k*-dimensional plane such that assumptions* (1)-(4) *for* u *holds on* $P_k \cap \Omega$*, then for every* $x \in \Omega$*, either* u *is* affine *in a neighborhood in* $P_k \cap \Omega$ *of* x*, or there exists a unique* $(k-1)$*-dimensional hyperplane* $P^x_{k-1} \ni x$ *of* P_k *such that* u *is* affine *on the connected component of* x *in* $P^x_{k-1} \cap \Omega$.

Proof. We first need to define a terminology that is the higher dimensional version of "line segments joining the boundary of some domain at both ends".

Definition 3.9. By a *k-plane P in Σ* we mean a connected component of a k-dimensional plane $P \cap \Sigma$, where Σ is any N-dimensional region with $N \geq k \geq 1$.

Remark 3.10. We emphasize here that such a k-plane P in Σ does *not* refer to the entire plane, but just to the part inside a region. On the other hand, it refers to the *entire* connected part inside this region.

Let $\mathbf{v}$ be any unit directional vector of P_k, let $\mathbf{v}_1, \cdots, \mathbf{v}_{k-1}$ be a set of *linearly independent unit* vectors of P_k perpendicular to $\mathbf{v}$. We parametrize the family of $(k-1)$-dimensional planes parallel to the space spanned by these vectors as follows:

$$P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} = \left\{ z : z = y + \sum_{i=1}^{k-1} s_i \mathbf{v}_i,\ s_i \in \mathbb{R} \right\}, \quad y \in \operatorname{span}\langle \mathbf{v} \rangle.$$

Lemma 3.11. *Given the direction* $\mathbf{v}$*, for a.e.* $y \in \operatorname{span}\langle \mathbf{v} \rangle$*,* u *is* $C^{1,1/2}_{\mathrm{loc}}$ *and is an isometry on* $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$*. Moreover for every* $x \in P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$*,* u *is either affine on a* $(k-1)$*-dimensional region in* $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ *containing* x*, or affine on a* $(k-2)$*-plane in* $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ *passing through* x.

Proof. Since u satisfies assumptions (1)-(4) on $P_k \cap \Omega$, by Fubini's theorem, for a.e. $y \in \operatorname{span}\langle \mathbf{v} \rangle$, assumptions (1)-(4) are also satisfied on $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$. Hence by our induction hypothesis on $(k-1)$-slices of Ω, ∇u is $C^{0,1/2}_{\mathrm{loc}}$ on $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$. By assumption (2) $\nabla u^T \nabla u = \mathrm{I}$ a.e., and hence everywhere in $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ by continuity. Therefore, by assumption (1) and the chain rule in Lemma 3.3, u is an isometry on $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$.

Moreover by our induction hypothesis, for every $x \in P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$, ∇u is either constant on a $(k-1)$-dimensional region in $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ containing x, or constant on an $(k-2)$-plane in $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$ passing through x. Hence by the the chain rule in Lemma 3.3, u is either affine on $(k-1)$ dimensional regions in $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$, or affine on $(k-2)$-plane in $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$. The proof is complete. □

Now we want to show that a substantial part of Lemma 3.11 is true for *every* rather than a.e. $(k-1)$-dimensional plane in Ω.

Lemma 3.12. *Given a direction* $\mathbf{v}$*, for all* $y \in \operatorname{span}\langle \mathbf{v} \rangle$ *and for all* $x \in P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$*,* u *is either an* affine isometry *on a* $(k-1)$*-dimensional region in* $P^y_{\mathbf{v}_1 \cdots \mathbf{v}_{k-1}} \cap \Omega$

containing x, or an affine isometry *on a $(k-2)$-plane in $P^{y}_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}}\cap\Omega$ passing through x.*

Remark 3.13. We obtain from the proof of Lemma 3.11 that u is C^1 on a.e. plane. However, Lemma 3.11 does *not* imply u is C^1 on *every* plane because even though ∇u is continuous on a.e. plane, we cannot conclude from here that ∇u is continuous in Ω, so we cannot pass to the limit to conclude as in Lemma 3.11.

Proof. Given $y\in\operatorname{span}\langle\mathbf{v}\rangle$, Lemma 3.11 guarantees a sequence $y_m\in\operatorname{span}\langle\mathbf{v}\rangle$, $y_m\to y$ such that Lemma 3.11 is true on $P^{y_m}_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}}\cap\Omega$ for every m.

Let $x\in P^{y}_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}}\cap\Omega$, we divide the proof into the following two cases:

(1) There is a sequence of $(k-2)$-planes P_m in $P^{y_m}_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}}\cap\Omega$ on which u is an affine isometry and P_m converges to x in distance.
(2) There does not exist such a sequence of $(k-2)$-planes.

Suppose we are in case (1); then the limit of P_m must also be a $(k-2)$-plane P in $P^{y}_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}}\cap\Omega$ passing through x. Also since u is Lipschitz continuous, u must also be an affine isometry on P, which proves the lemma in this case (Figure 3.6).

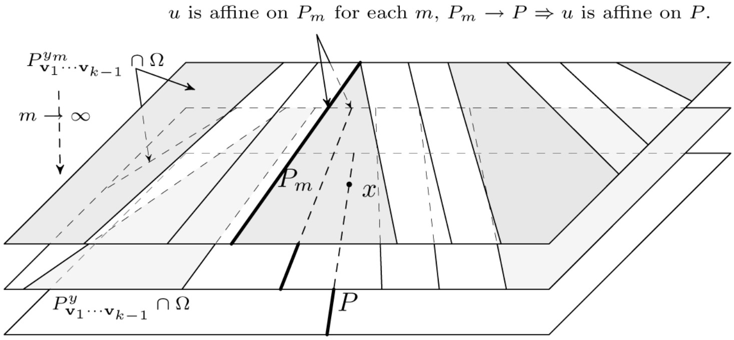

Figure 3.6. Case (1).

Suppose now we are in case (2). If we cannot find such a sequence of $(k-2)$-planes, then we can find $x_m\in P^{y_m}_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}}\cap\Omega$, $x_m\to x$ with the property that there is $\epsilon>0$ such that u is an affine isometry on $B^{k-1}(x_m,\epsilon)\subset P^{y_m}_{\mathbf{v}_1\cdots\mathbf{v}_{k-1}}\cap\Omega$. Otherwise, there will again be a sequence of $(k-2)$-planes (*i.e.* the boundaries of the maximal affine regions containing x_m) converging to x in distance, contradiction to the fact that we are in case (2). Continuity of u then must force u to be an affine isometry on $B^{k-1}(x,\epsilon)\subset P^{y}_{\mathbf{v}_1\cdots\mathbf{v}_{n-1}}\cap\Omega$, which again proves the lemma in this case (Figure 3.7). The proof is complete. □

Lemma 3.14. *Suppose u is an affine isometry on two line segments C_1 and C_2 in $P_k\cap\Omega$ intersecting at a point x in the interior of both C_1 and C_2. Let H be the convex hull of the line segments C_1 and C_2, then u is an affine isometry on $H\cap\Omega$.*

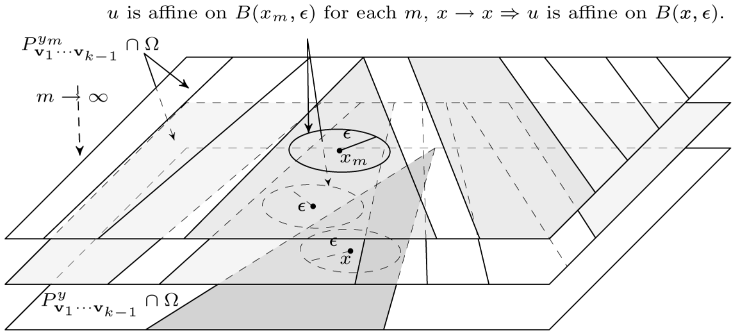

Figure 3.7. Case (2).

Proof. We parametrize C_1 and C_2 by $\{x + t\mathbf{v}_1, t \in [-a, b]\}$ and $\{x + s\mathbf{v}_2, s \in [-c, d]\}$, respectively, with both $\mathbf{v}_1$ and $\mathbf{v}_2$ unit vectors. We can assume $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, otherwise, the conclusion of the lemma is obvious. Since u is affine on both C_1 and C_2, $u(C_1)$ and $u(C_2)$ are both line segments in $\mathbb{R}^{n+1}$. We can again parametrize the lines that contain the line segments $u(C_1)$ and $u(C_2)$ by $u(x)+t\tilde{\mathbf{v}}_1$ and $u(x)+s\tilde{\mathbf{v}}_2$, where both $\tilde{\mathbf{v}}_1$ and $\tilde{\mathbf{v}}_2$ are unit vectors due to the isometry assumption.

Let $y \in H \cap \Omega$, we can of course assume that y is neither in C_1 nor C_2, otherwise, there is nothing to prove. In this way, we can find a line L_3 passing through y and intersecting C_1 at only one point, denoted x_{13}; and C_2 at only one point, denoted x_{23}, where the segment $\overline{x_{13}x_{23}}$ lies inside Ω. Since $x_{13} \in C_1$, $x_{13} = x + t_0\mathbf{v}_1$ for some $t_0 \in [-a, b]$. Similarly $x_{23} = x + s_0\mathbf{v}_2$ for some $s_0 \in [-c, d]$. Then since

$$y = wx_{13} + (1 - w)x_{23} \quad \text{for some } w \in [0, 1], \tag{3.4}$$

it follows

$$y = x + wt_0\mathbf{v}_1 + (1 - w)s_0\mathbf{v}_2.$$

To prove that u is an affine isometry on H, we need to prove

$$u(y) = u(x) + wt_0\tilde{\mathbf{v}}_1 + (1 - w)s_0\tilde{\mathbf{v}}_2. \tag{3.5}$$

We first claim that the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between C_1 and C_2. Since x is in the interior of C_1 and C_2, we can construct a parallelogram $ABCD$ centered at x, with $A, C \in C_1$ and $B, D \in C_2$. Since u is an affine isometry on C_1 and C_2, $|u(A)-u(x)| = |A-x|$, $|u(B)-u(x)| = |B-x|$, $|u(C)-u(x)| = |C-x|$ and $|u(D)-u(x)| = |D-x|$. On the other hand, $|u(A) - u(B)| \le |A - B|$ and $|u(B) - u(C)| \le |B - C|$ since u is 1-Lipschitz (Figure 3.8).

This implies the angle α_2 between the line segments $\overline{u(x)u(A)}$ and $\overline{u(x)u(B)}$ must be smaller than or equal to the angle α_1 between $\overline{xA}$ and $\overline{xB}$, and the angle β_2 between the line segments $\overline{u(x)u(B)}$ and $\overline{u(x)u(C)}$ must be smaller than or equal

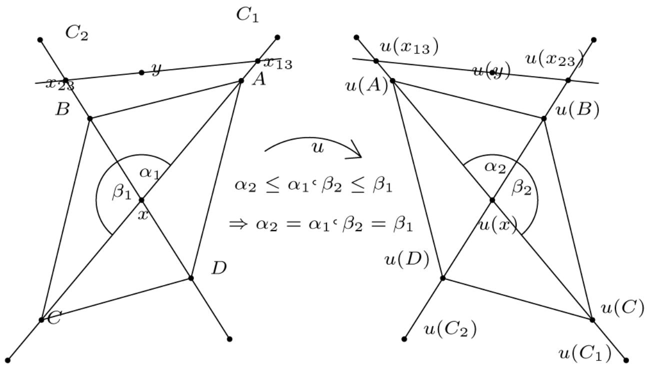

Figure 3.8.

to the angle β_1 between $\overline{xB}$ and $\overline{xC}$. Hence $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. This proves our claim.

Since by assumption, u is an affine isometry on $\overline{x_{13}x}$ and $\overline{x_{23}x}$, we have

$$u\big(x_{13}\big) - u(x) = t_0\tilde{\mathbf{v}}_1 \quad \text{and} \quad u\big(x_{23}\big) - u(x) = s_0\tilde{\mathbf{v}}_2$$

for the same t_0, s_0 and unit vector $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2$ as defined before. In particular, $|u(x_{13}) - u(x)| = |x_{13} - x|$ and $|u(x_{23}) - u(x)| = |x_{23} - x|$. Moreover, since the angle between line segments $u(C_1)$ and $u(C_2)$ is the same as the angle between C_1 and C_2, we have $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$.

On the other hand, $u(\overline{x_{13}x_{23}})$ is a 1-Lipschitz curve, hence the length the the curve $u(\overline{x_{13}x_{23}})$, denoted by $|u(\overline{x_{13}x_{23}})|$, satisfies $|u(\overline{x_{13}x_{23}})| \leq |x_{13} - x_{23}|$. Altogether we have

$$\big|u\big(x_{13}\big) - u\big(x_{23}\big)\big| \leq \big|u\big(\overline{x_{13}x_{23}}\big)\big| \leq \big|x_{13} - x_{23}\big| = \big|u\big(x_{13}\big) - u\big(x_{23}\big)\big|.$$

This implies

$$\big|u\big(\overline{x_{13}x_{23}}\big)\big| = \big|u\big(x_{13}\big) - u\big(x_{23}\big)\big|.$$

Hence the curve $u(\overline{x_{13}x_{23}})$ must coincide with line segment $\overline{u(x_{13})u(x_{23})}$. Therefore, u also maps the line segment $\overline{x_{13}x_{23}}$ onto a line segment $\overline{u(x_{13})u(x_{23})}$, which means u is affine on $\overline{x_{13}x_{23}}$.

Finally, since u is 1-Lipschitz, $|u(x_{13}) - u(y)| \leq |x_{13} - y|$ and $|u(x_{23}) - u(y)| \leq |x_{23} - y|$. However, since u is affine on $\overline{x_{13}x_{23}}$,

$$\begin{aligned}\big|u\big(x_{13}\big) - u\big(x_{23}\big)\big| &= \big|u\big(x_{13}\big) - u(y)\big| + \big|u(y) - u\big(x_{23}\big)\big| \\ &\leq \big|x_{13} - y\big| + \big|y - x_{23}\big| = \big|x_{13} - x_{23}\big|.\end{aligned}$$

But we already showed that $|x_{13} - x_{23}| = |u(x_{13}) - u(x_{23})|$. Hence $|u(x_{13}) - u(y)| = |x_{13} - y|$ and $|u(x_{23}) - u(y)| = |x_{23} - y|$. Therefore,

$$u(y) = wu\big(x_{13}\big) + (1 - w)u\big(x_{23}\big)$$

for the same w as (3.4), which yields (3.5). The proof is complete. □

Corollary 3.15. *Given a ℓ-dimensional ($\ell \leq k$) region U in $P_k \cap \Omega$, and a line segment C in $P_k \cap \Omega$ for which there exists $x \in C \cap U$ that lies in the interior of both U and C, if u is an affine isometry on both U and C, then u is an affine isometry on the convex hull H of U and C inside Ω*

Proof. Let $y \in H \cap \Omega$. We need to show that $u(y) = u(x) + t\tilde{\mathbf{v}}$ for some $\tilde{\mathbf{v}}$ given by a linear combination of directional vectors in $u(U)$ and $u(C)$ and $|t\tilde{\mathbf{v}}| = |y - x|$. Let P_y be a 2-dimensional plane that contains y and C. Then P_y intersects U at some line segment C_y. Since u is an affine isometry on both C and C_y, by Lemma 3.14, u is an affine isometry on the convex hull of C and C_y (Figure 3.9).

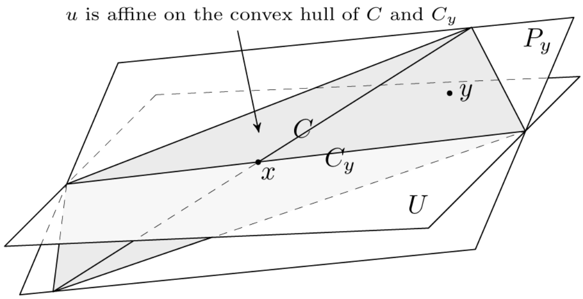

Figure 3.9.

Since this convex hull contains both y and x, this implies $u(y) = u(x) + t\tilde{\mathbf{v}}$ for some vector $\tilde{\mathbf{v}}$, $|t\tilde{\mathbf{v}}| = |y - x|$, and $\tilde{\mathbf{v}}$ is a linear combination of directional vectors of $u(C)$ and $u(C_y)$. Our claim then follows because $C_y \subset U$ and u is an affine isometry on U, so any vectors of $u(C_y)$ is a linear combination of vectors in $u(U)$. The proof is complete. □

By obvious induction we then have:

Corollary 3.16. *Suppose U_1 and U_2 are k_1 and k_2-dimensional regions ($k_1, k_2 \leq k$) in $P_k \cap \Omega$ with nonempty intersections. Moreover, there exists a point $x \in U_1 \cap U_2$ belonging to the interior of both U_1 and U_2. If u is an affine isometry on both U_1 and U_2, then u is an affine isometry on the convex hull of U_1 and U_2 inside Ω.*

Now we are ready to prove Proposition 3.8. Given $x \in P_k \cap \Omega$, we first claim that there is a $(k-1)$-dimensional hyperplane P_0^x in P_k and a $(k-1)$-dimensional neighborhood $U_0^x \subset P_0^x \cap \Omega$ containing x on which u is an affine isometry. Otherwise, for all $(k-1)$-dimensional hyperplanes in $P_k \cap \Omega$ that pass through x, x is not contained in any $(k-1)$-dimensional neighborhood on which u is an affine isometry. In particular, let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be linearly independent vectors of P_k and let $P^x_{\mathbf{v}_1 \cdots \hat{\mathbf{v}}_i \cdots \mathbf{v}_k}$, $i = 1, \ldots, k$ be the $(k-1)$-dimensional hyperplanes in Ω passing through x and parallel to the space spanned by $\mathbf{v}_1, \cdots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \cdots, \mathbf{v}_k$. Since x is not contained in any $(k-1)$-dimensional neighborhood in $P^x_{\mathbf{v}_1\cdots\hat{\mathbf{v}}_i\cdots\mathbf{v}_k} \cap \Omega$ on which u is an affine isometry, by Lemma 3.12 there exists $(k-2)$-planes $P_{\hat{i}} \ni x$ in $P^x_{\mathbf{v}_1\cdots\hat{\mathbf{v}}_i\cdots\mathbf{v}_k} \cap \Omega$ such that u is an affine isometry on $P_{\hat{i}}$. By Corollary 3.16, u is an affine isometry on the convex hull of $P_{\hat{i}}$ for all $1 \le i \le k$ (Figure 3.10 Case 1). Let $\mathbf{v}_{\hat{i}}$ be a directional vector of $P_{\hat{i}}$. Since $P_{\hat{i}} \subset P^x_{\mathbf{v}_1\cdots\hat{\mathbf{v}}_i\cdots\mathbf{e}_k}$, which is orthogonal to $\mathbf{v}_i$, at least $k-1$ out of these k vectors are linearly independent. This convex hull has $k-1$ linearly independent directional vectors, hence it must contain a $(k-1)$-dimensional neighborhood of x, contradiction to our assumption, which proves our claim.

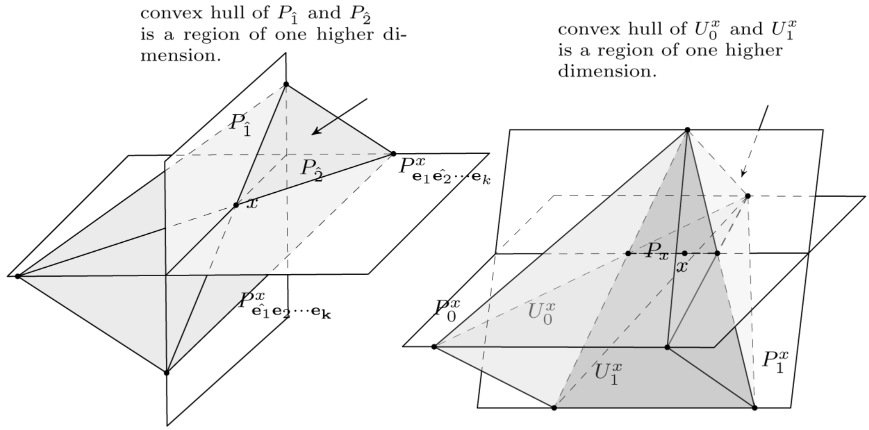

Figure 3.10. Case 1 (left) and Case 2 (right).

Therefore, we have proved that x must be contained in a $(k-1)$-dimensional neighborhood $U_0^x \subset P_0^x \cap \Omega$ for some $(k-1)$-dimensional hyperplane P_0^x and u is an affine isometry on U_0^x. If U_0^x is the entire connected component containing x in $P_0^x \cap \Omega$, then the conclusion of the proposition is achieved. Otherwise, we can find a maximal $(k-2)$-plane P_x in U_0^x, which is *not* a $(k-2)$-plane in $P_0^x \cap \Omega$, *i.e.*, it is away from $\partial\Omega$, on which u is an affine isometry. Let P_1^x be any other $(k-1)$-dimensional hyperplane containing the region P_x. We have $P_x = U_0^x \cap P_1^x$ and since the maximal affine region $P_x \subset P_1^x \cap \Omega$ is not a $(k-2)$-plane in $P_1^x \cap \Omega$, by Lemma 3.12, x must be contained in a $(k-1)$-dimensional neighborhood $U_1^x \subset P_1^x \cap \Omega$ on which u is an affine isometry (Figure 3.10 Case 2). By Corollary 3.16, u is an affine isometry on the convex hull of U_0^x and U_1^x, whose interior is a k-dimensional region, which also achieves the conclusion of Proposition 3.8. The proof is complete. □

3.2.2. *Regularity and the conclusion of the inductive step*

In out last step, we will essentially show that Proposition 3.8 combined with assumptions (1)-(4) of Proposition 3.1 for a k-dimensional slice P_k, implies the conclusion of the latter proposition. This will hence conclude the inductive step. The key point is to show that if u is affine on a $(k-1)$-plane, then its full gradient must be constant on the same region. The arguments are very similar to what we used in the proofs of Lemmas 3.5-3.7. We will first prove the following:

Lemma 3.17. *Suppose on a k-plane P $(1 \le k \le n)$ in Ω we have the following:*

(1) *There is a sequence of smooth functions $u^\epsilon \in C^\infty(\Omega, \mathbb{R}^{n+1})$ such that*

$$\int_P |u^\epsilon - u|^2 + |\nabla u^\epsilon - \nabla u|^2 + |\nabla^2 u^\epsilon - \nabla^2 u|^2 d\mathcal{H}^k \to 0.$$

(2) Rank $\nabla^2 u^\ell \le 1$ *and* $\nabla^2 u^\ell$ *is symmetric a.e. on P for all* $1 \le \ell \le n+1$.

Then if u is affine on P, ∇u is constant on P.

Proof. Let $\mathbf{v}$ be any unit directional vector in P. By assumption (1) and the chain rule in Lemma 3.3, u is affine on P implies

$$\nabla u(x)\mathbf{v} = \text{constant} \quad \text{for a.e. } x \in P.$$

Take the directional derivative one more time, together with assumption (1) we obtain,

$$(\mathbf{v})^T \nabla^2 u^\ell \mathbf{v} = 0 \quad \text{for a.e. } x \in P \tag{3.6}$$

for all $1 \le \ell \le n+1$. However, to show that ∇u is constant on P, we need a conclusion stronger than (3.6), *i.e.*,

$$\nabla^2 u^\ell \mathbf{v} = 0 \quad \text{for a.e. } x \in P \tag{3.7}$$

for all $1 \le \ell \le n+1$. Indeed, by assumption (2), we can write $\nabla^2 u^\ell$ as

$$\nabla^2 u^\ell(x) = \lambda(x)\mathbf{b}(x) \otimes \mathbf{b}(x) \quad \text{a.e.}$$

for some scalar function λ and $\mathbf{b} \in \mathbb{S}^{n-1}$. Then (3.6) implies,

$$(\mathbf{v})^T \lambda(x)\mathbf{b}(x) \otimes \mathbf{b}(x)\mathbf{v} = \lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle^2 = 0 \quad \text{a.e.}$$

This then implies

$$\lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle = 0 \quad \text{a.e.}$$

Therefore,

$$\nabla^2 u^\ell \mathbf{v} = \lambda(x)\langle \mathbf{v}, \mathbf{b}(x)\rangle \mathbf{b}(x) = 0 \quad \text{a.e.}$$

which is exactly (3.7). The proof of the lemma is complete. $\square$

Let P_k be any k-dimensional plane such that assumptions (1)-(4) in Proposition 3.1 hold on $P_k \cap \Omega$.

By means of Lemma 3.7 and making use of Proposition 3.8 and Corollary 3.16, similarly as before we get

$$\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} P_x^U$$

where P_x^U is some $(n-1)$-plane in Ω containing x with the property that for $x, z \in \partial U \cap \Omega$, $P_x^U = P_z^U$ if $z \in P_x^U$ and $P_x^U \cap P_z^U \cap \Omega = \emptyset$ if $z \notin P_x^U$.

Similarly as in the proof of Lemma 3.7 (Figure 3.5), it suffices to show that the conclusions holds true locally. If $x_0 \in P_k \cap \Omega$ is a point lying in an affine neighborhood for u in P_k, then Lemma 3.17 and the assumptions of Proposition 3.1 immediately imply that ∇u must be constant in the same neighborhood, which is the desired conclusion. Otherwise, we may and do choose a small $\delta > 0$ so that for any region U on which u is affine, the k-dimensional ball $B^k(x_0, \delta) \subset P_k \cap \Omega$ intersects ∂U at no more than *two* $(k-1)$-planes belonging to ∂U.

We now focus on $B^k(x_0, \delta) \subset P_k \cap \Omega$. For any $x \in B^k(x_0, \delta)$, as in Lemma 3.7, we construct a $(k-1)$-plane P_x in $B^k(x_0, \delta)$ passing through x on which u is affine and $P_x \cap P_z \cap B^k(x_0, \delta) = \emptyset$ if $z \notin P_x$, see Figure 3.3. We then construct a foliation of $B^k(x_0, \delta)$, see Figure 3.4 and obtain that the assumptions of Lemma 3.7 are satisfied along $P_{\gamma(t)}$ for a.e. $t \in (a, b)$ by the same argument as Step 4 and Step 5 of Lemma 3.7. It then follows that ∇u is constant on $P_{\gamma(t)}$ for a.e. $t \in (a, b)$.

By choosing an initial value for γ arbitrary close to x_0 and applying the co-area formula in a similar manner we can make sure that ∇u is of class $W^{1,2}$ on γ. Hence we conclude that ∇u is $C^{0,1/2}$ on γ by the Sobolev embedding theorem. Let F be the set of $t \in (a, b)$ such that ∇u is not constant along $P_{\gamma(t)}$, then $\mathcal{H}^1(F) = 0$. We modify ∇u to be constant along $P_{\gamma(t)}$ for each $t \in F$. Note that,

$$\mathcal{H}^k\left(\bigcup\left\{P_{\gamma(t)} : t \in F\right\}\right) \leq c(2\delta)^{k-1}\mathcal{H}^1(\{\gamma(t) : t \in F\}) = c(2\delta)^{k-1}\mathcal{H}^1(F) = 0$$

for some constant c. Hence ∇u is $C^{0,1/2}$ up to modification of a set of measure zero in $B^k(x_0, \delta)$. Moreover, ∇u is constant on $P_{\gamma(t)}$ for all t, which foliates $B^k(x_0, \delta)$. Thus ∇u is constant on any region on which u is affine. Therefore, ∇u is constant either on a $(k-1)$-plane or k-dimensional region in $B^k(x_0, \delta)$. This implies that the conclusions of Proposition 3.1 under the induction hypothesis are true and hence the inductive step is established. As a conclusion the proofs of Proposition 3.1 and Theorem 1.4 are complete. □

4. Density: proof of Theorem 1.5

In this section we show that isometric immersions smooth up to the boundary are strongly dense in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ if $\Omega \subset \mathbb{R}^n$ is a convex C^1 domain. Note that it is sufficient to prove that $I^{2,2} \cap C^\infty(\Omega, \mathbb{R}^{n+1})$ is strongly dense in $I^{2,2}(\Omega, \mathbb{R}^{n+1})$. Having this result at hand, and since Ω is assumed convex, the approximating sequence can be easily rescaled to be smooth up to the boundary.

4.1. Foliations of the domain

We have argued in the proof of Theorem 1.4 in section 3.2.2 that for every maximal region $U \subset \Omega$ on which u is affine, $\partial U \cap \Omega = \bigcup_{x \in \partial U \cap \Omega} P_x^U$, where P_x^U is some $(n-1)$-plane in Ω containing x with the property that for $x_1, x_2 \in \partial U \cap \Omega$, $P_{x_1}^U = P_{x_2}^U$ if $x_2 \in P_{x_1}^U$ and $P_{x_1}^U \cap P_{x_2}^U \cap \Omega = \emptyset$ if $x_2 \notin P_{x_1}^U$

We say a maximal region on which u is affine is a *body* if its boundary contains more than *two* different $(n-1)$-planes in Ω.

Lemma 4.1. *It is sufficient to prove Theorem* 1.5 *for a function in* $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ *with a finite number of bodies*.

Proof. The proof is similar to the proof of [22, Lemma 3.8] and is omitted for brevity. □

Now we can just assume $u \in I^{2,2}(\Omega, \mathbb{R}^{n+1})$ has finite number of bodies. Each body is closed and so is therefore their union, whose complement we denote by $\widetilde{\Omega}$. Note that now for every n-dimensional maximal-affine region $U \subset \widetilde{\Omega}$, $\partial U \cap \widetilde{\Omega}$ consists of at most two $(n-1)$-planes.

Similarly as in the proof of Lemma 3.7, for every $x \in \widetilde{\Omega}$, we will construct an $(n-1)$-plane P_x in $\widetilde{\Omega}$ passing through it on which ∇u is constant and $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$. To apply the same construction in Lemma 3.7, we makes use of Theorem 1.4 and the fact that $\partial U \cap \widetilde{\Omega}$ consists of at most two $(n-1)$-planes, see in Figure 4.1.

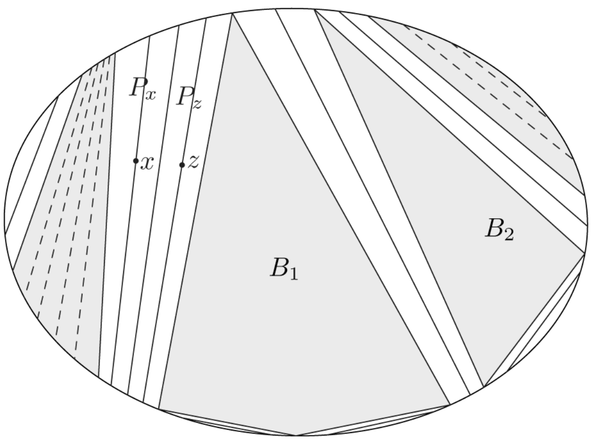

Figure 4.1. Construction of global foliations in $\widetilde{\Omega}$.

For every $x \in \widetilde{\Omega}$, we define the normal vector field $\mathbf{N}(x)$ as the unit vector orthogonal to the family P_x constructed above. Since none of the P_x's intersect inside $\widetilde{\Omega}$ we can choose an orientation such that $\mathbf{N}$ is a Lipschitz vector fields. The ODE,

$$\gamma'(t) = \mathbf{N}(\gamma(t)) \quad \gamma(0) = x_0 \tag{4.1}$$

has a unique solution $\gamma : (a, b) \to \widetilde{\Omega}$ for some interval $(a, b) \subset \mathbb{R}$ containing 0. Note that $P_x = P_{\gamma(t)}$ if $x \in P_{\gamma(t)}$, therefore, $\{P_{\gamma(t)}\}_{t\in(a,b)}$ is a local foliation of $\widetilde{\Omega}$ such that ∇u is constant on $P_{\gamma(t)}$ for all $t \in (a, b)$ (Figure 4.2).

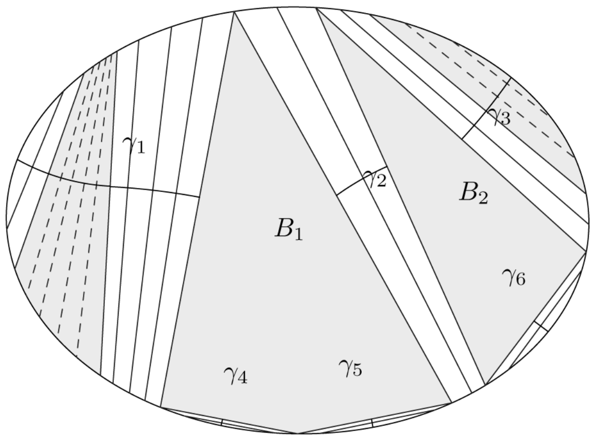

Figure 4.2.

4.2. Leading curves in the domain

Definition 4.2. Let $\{P_x\}_{x\in\widetilde{\Omega}}$ be a family of $(n-1)$-planes in $\widetilde{\Omega}$ passing through x on which ∇u is constant, satisfying $P_x \cap P_z \cap \widetilde{\Omega} = \emptyset$ if $z \notin P_x$ and $P_x = P_z$ if $z \in P_x$. We say that a curve $\gamma \in C^{1,1}([0, \ell], \widetilde{\Omega})$ parametrized by arclength is a *leading curve* if it is orthogonal at any possible point of intersection $z \in \gamma([0, \ell]) \cap P_x$ to $P_x = P_z$ for all $x \in \widetilde{\Omega}$ (Figure 4.3).

It is easy to see that γ constructed in Subsection 4.1 when restricted to the interval $[0, \ell]$ is a leading curve, since by the ODE (4.1), $|\gamma'| = 1$ and $|\gamma''|$ is bounded as $\mathbf{N}$ is Lipschitz.

Definition 4.3. The $(n - 1)$-dimensional hyperplane $F_\gamma(t)$ orthogonal to $\gamma(t)$ at $t \in [0, \ell]$ is called the *leading front* of γ at $t \in [0, \ell]$ (Figure 4.3).

Remark 4.4. It then follows from the definition of the leading curve that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \widetilde{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0, \ell]$ such that $t \neq \tilde{t}$. Moreover, $F_\gamma(t) \subset \widetilde{\Omega}$, otherwise, $F_\gamma(t) \cap B \neq \emptyset$ where B is one of the bodies in $\Omega \setminus \widetilde{\Omega}$. Since ∇u, being continuous,

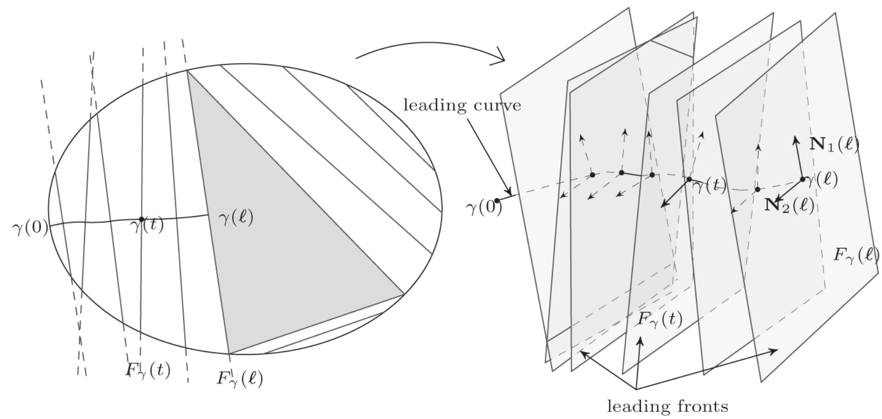

Figure 4.3. Leading curve and leading fronts.

is constant on $F_\gamma(t) \cap \widetilde{\Omega}$ and B, it must be constant on their convex hull, which is again a body, contradiction to that a body is a maximal region.

We say that a curve γ covers the domain $A \subset \Omega$ if

$$A \subset \bigcup \{F_\gamma(t) : t \in [0, \ell]\}.$$

By $\Omega(\gamma)$ we refer to the *biggest set covered by γ in Ω*. We now restrict our attention to the covered domain $\Omega(\gamma)$. It is obvious that $\Omega(\gamma)$ is convex since it is bounded by $F_\gamma(0)$, $F_\gamma(\ell)$ and $\partial\Omega$.

From the construction in Subsection 4.1, the $(n-1)$-planes $P_{\gamma(t)}$ in $\widetilde{\Omega}$, $t \in [0, \ell]$ which constitute a local foliation of $\widetilde{\Omega}$ are global foliations of $\Omega(\gamma)$. Moreover, $P_{\gamma(t)} = F_\gamma(t) \cap \Omega(\gamma) = F_\gamma(t) \cap \Omega$ for all $t \in [0, \ell]$. We relabel them $P_\gamma(t)$ to be in consistence of notation and we name them:

Definition 4.5. The component $P_\gamma(t) := F_\gamma(t) \cap \Omega$ is called the leading $(n-1)$-plane in Ω of γ at $t \in [0, \ell]$.

Let $\{\mathbf{N}_i(t)\}_{i=1}^{n-1}$ be an orthonormal basis for the leading front $F_\gamma(t)$ (Figure 4.3) such that $\mathbf{N}_i$ is Lipschitz for all $1 \le i \le n-1$ and $\det[\gamma'(t), \mathbf{N}_1(\tilde{t}), \cdots, \mathbf{N}_{n-1}(\tilde{t})] = 1$. It is obvious that such orthonormal basis exists because we can pick $\{\mathbf{N}_i(0)\}_{i=1}^{n-1}$ as an orthonormal basis for $F_\gamma(0)$ that forms a positive orientation with $\gamma'(0)$ and then move this frame along γ in an orientation preserving way (note that γ is not a closed curve so this is possible). Let $\Phi : [0, \ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as

$$\Phi(t, s) := \gamma(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_i(t), \tag{4.2}$$

where $s = (s_1, \cdots, s_{n-1})$. Then we can represent the leading front at $t \in [0, \ell]$ as

$$F_\gamma(t) = \left\{\Phi(t, s), s = \left(s_1, \cdots, s_{n-1}\right) \in \mathbb{R}^{n-1}\right\}. \tag{4.3}$$

For each $t \in [0, \ell]$, define the open set

$$\Sigma^{\gamma}(t) = \left\{ s = \left(s_1, \cdots, s_{n-1}\right) \in \mathbb{R}^{n-1} : \Phi(t, s) \in \Omega \right\}. \tag{4.4}$$

It is obvious that $0 \in \Sigma^{\gamma}(t)$, hence it is non-empty. Then we can also parametrize the leading planes as

$$P_{\gamma}(t) = \left\{ \Phi(t, s), s = \left(s_1, \cdots, s_{n-1}\right) \in \Sigma^{\gamma}(t) \right\}. \tag{4.5}$$

Now define

$$\Sigma^{\gamma} := \{(t, s), \Phi(t, s) \in \Omega\}. \tag{4.6}$$

Of course we can also write

$$\Sigma^{\gamma} = \left\{ (t, s), t \in [0, \ell], s = \left(s_1, \cdots, s_{n-1}\right) \in \Sigma^{\gamma}(t) \right\}.$$

We will focus on the restriction of Φ in Σ^{γ}. However, if no confusion is caused, we still denote such restriction Φ. It is easy to see that Φ maps Σ^{γ} into $\Omega(\gamma)$. Indeed, if $x = \Phi(t, s)$ for some $(t, s) \in \Sigma^{\gamma}$, by definition of Σ^{γ}, $\Phi(t, s) \in \Omega$. On the other hand, $\Phi(t, s) \in F_{\gamma}(t)$, thus, $x = \Phi(t, s) \in F_{\gamma}(t) \cap \Omega \subset \Omega(\gamma)$.

Lemma 4.6. $\Phi : \Sigma^{\gamma} \to \Omega(\gamma)$ *is one-to-one and onto. In particular,*

$$\Omega(\gamma) = \left\{ \Phi(t, s), (t, s) \in \Sigma^{\gamma} \right\} = \bigcup \left\{ P_{\gamma}(t) : t \in [0, \ell] \right\}.$$

Proof. We first show Φ is one-to-one. Suppose $\Phi(t_1, s_1) = \Phi(t_2, s_2)$ while $(t_1, s_1) \neq (t_2, s_2)$. Since $s \mapsto \Phi(t, s)$ is obviously one-to-one by the definition of Φ, it must be $t_1 \neq t_2$. We have argued in Remark 4.4 that $F_{\gamma}(t_1) \cap F_{\gamma}(t_2) \cap \Omega = \emptyset$. Therefore, $F_{\gamma}(t_1) \cap F_{\gamma}(t_2) \cap \Omega(\gamma) = \emptyset$ since $\Omega(\gamma) \subset \Omega$. However, $\Phi(t_1, s_1) \in F_{\gamma}(t_1)$ and $\Phi(t_2, s_2) \in F_{\gamma}(t_2)$, contradiction to $\Phi(t_1, s_1) = \Phi(t_2, s_2)$.

We will now show Φ is onto. Let $x \in \Omega(\gamma)$; then $x = \Phi(t, s)$ for some $t \in [0, \ell]$ and $s \in \mathbb{R}^{n-1}$. Since $x \in \Omega(\gamma)$, $\Phi(t, s) \in \Omega(\gamma) \subset \Omega$, hence $(t, s) \in \Sigma^{\gamma}$. The proof is complete. □

Obviously we can rewrite $\Phi(t, s) := \gamma(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$, $t \in [0, \ell]$, $s \in \mathbb{R}^{n-1}$ as

$$\Phi(t, S \cdot s) = \gamma(t) + S \left(\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t) \right), t \in [0, \ell], s \in \mathbb{S}^{n-2}, S \geq 0.$$

We then rewrite the representation of leading front in (4.3) in an equivalent way:

$$F_{\gamma}(t) = \left\{ \Phi(t, S \cdot s), S \geq 0, s = \left(s_1, \cdots, s_{n-1}\right) \in \mathbb{S}^{n-2} \right\}. \tag{4.7}$$

For each $t \in [0, \ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$, define the scalar function,

$$S_s^{\gamma}(t) := \sup\{S \geq 0 : \Phi(t, S \cdot s) \in \Omega\}. \tag{4.8}$$

That is, $S_s^\gamma(t)$ is the distance from $\gamma(t)$ to $\partial\Omega$ in the direction $\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$. From the definition of $\Sigma^\gamma(t)$ and Σ^γ,

$$\Sigma^\gamma(t) = \Big\{(S \cdot s) : s = \big(s_1, \cdots, s_{n-1}\big) \in \mathbb{S}^{n-2}, 0 < S < S_s^\gamma(t)\Big\}, \tag{4.9}$$

and

$$\Sigma^\gamma = \Big\{(t, S \cdot s), t \in [0, \ell], s = \big(s_1, \cdots, s_{n-1}\big) \in \mathbb{S}^{n-2}, 0 < S < S_s^\gamma(t)\Big\}. \tag{4.10}$$

Since $|\gamma'(t)| = 1$, $\gamma''(t) \cdot \gamma'(t) = 0$, we can then write $\gamma''(t) = \sum_{i=1}^{n-1} \kappa_i(t)\mathbf{N}_i(t)$. Similarly we can also write

$$\mathbf{N}_i' = \kappa_{i_0}\gamma' + \sum_{j=1}^{n-1} \kappa_{i_j}\mathbf{N}_j. \tag{4.11}$$

It is easy to see that $\kappa_{i_0} = -\kappa_i$, $\kappa_{i_i} = 0$ and $\kappa_{i_j} = -\kappa_{j_i}$. These equations can be written as the matrix equation

$$\begin{pmatrix} \gamma' \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 & \cdots & \kappa_{n-1} \\ -\kappa_1 & 0 & \kappa_{1_2} & \cdots & \kappa_{1_{n-1}} \\ -\kappa_2 & -\kappa_{1_2} & 0 & \cdots & \kappa_{2_{n-1}} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\kappa_{n-1} & -\kappa_{1_{n-1}} & -\kappa_{2_{n-1}} & \cdots & 0 \end{pmatrix} \begin{pmatrix} \gamma' \\ \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{n-1} \end{pmatrix}. \tag{4.12}$$

Given two *non-parallel* leading fronts $F_\gamma(t)$ and $F_\gamma(\tilde{t})$, denote their $(n-2)$-plane of intersection by $F(t, \tilde{t})$. Given $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$, define $L_s(t, \tilde{t})$ as the distance from $\gamma(t)$ to $F(t, \tilde{t})$ along the direction $\sum_{i=1}^{n-1} s_i\mathbf{N}_i(t)$ (we set $L_s(t, \tilde{t}) = +\infty$ if it does not hit $F(t, \tilde{t})$ along this direction: see Figure 4.4). We then define

$$L_s^\gamma(t) := \inf\big\{L_s(t, \tilde{t}) : \tilde{t} \neq t\big\}. \tag{4.13}$$

Since all $F(t, \tilde{t})$ are outside Ω, $L_s^\gamma(t) \geq S_s^\gamma(t)$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$.

Lemma 4.7. $L_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i\kappa_i(t)\Big) \leq 1$ *for all* $t \in [0, \ell]$ *and* $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$.

Proof. Suppose $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are not parallel. Solving for their intersection yields

$$\gamma(t) + \sum_{i=1}^{n-1} s_i\mathbf{N}_i(t) = \gamma(\tilde{t}) + \sum_{i=1}^{n-1} r_i\mathbf{N}_i(\tilde{t}).$$

This is a linear system of n equations and $2n-2$ unknowns $(s_i)_{i=1}^{n-1}$ and $(r_i)_{i=1}^{n-1}$. A solution for this system of equations exists because the two leading front are not

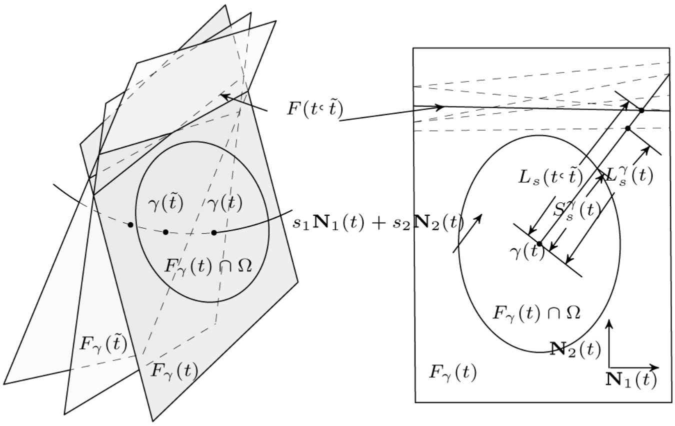

Figure 4.4.

parallel. Then direct computation using Cramer's rule gives the formula for $F(t, \tilde{t})$ explicitly,

$$F(t, \tilde{t}) = \left\{ x \in F_\gamma(t) : (x - \gamma(t)) \cdot \left(-\sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, \tilde{t})} \mathbf{N}_i(t) \right) = 1 \right\},$$

where

$$h_i(t, \tilde{t}) := \det \left[\mathbf{N}_1(\tilde{t}), \cdots, \mathbf{N}_{n-1}(\tilde{t}), \mathbf{N}_i(t) \right]$$

for $1 \le i \le n-1$, and

$$H(t, \tilde{t}) = \det \left[\mathbf{N}_1(\tilde{t}), \cdots, \mathbf{N}_{n-1}(\tilde{t}), \gamma(t) - \gamma(\tilde{t}) \right].$$

Note that $H(t, \tilde{t}) \neq 0$ since $\gamma(t) - \gamma(\tilde{t})$ is not parallel to $F_\gamma(\tilde{t})$.

We firstly claim that

$$L_s(t, \tilde{t}) \left(-\sum_{i=1}^{n-1} \frac{h_i(t, \tilde{t})}{H(t, \tilde{t})} s_i \right) \le 1. \tag{4.14}$$

Indeed, we divide the situation into two cases. In the first case, suppose we travel from $\gamma(t)$ along a given direction $\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$ and hit $F(t, \tilde{t})$, then for $x \in F(t, \tilde{t})$,

$$x - \gamma(t) = L_s(t, \tilde{t}) \left(\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t) \right).$$

Therefore,

$$\begin{aligned} &L_s(t,\tilde{t})\left(\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)\right)\cdot\left(-\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}\mathbf{N}_i(t)\right) \\ &= L_s(t,\tilde{t})\left(-\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i\right) = 1. \end{aligned} \tag{4.15}$$

Suppose for a certain direction $\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$ we do not hit $F(t,\tilde{t})$, in which case we set $L_s(t,\tilde{t}) = +\infty$; then we must hit $F(t,\tilde{t})$ through the direction $-\sum_{i=1}^{n-1} s_i \mathbf{N}_i(t)$, therefore, by (4.15),

$$L_{-s}(t,\tilde{t})\left(\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i\right) = 1.$$

In particular, since $L_{-s}(t,\tilde{t}) > 0$,

$$\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i > 0.$$

We then must have

$$L_s(t,\tilde{t})\left(-\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i\right) < 0. \tag{4.16}$$

Now (4.15) and (4.16) together give that in either case (4.14) holds true, which proves our claim.

We secondly claim that

$$L_s^{\gamma}(t)\left(-\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i\right) \le 1 \tag{4.17}$$

for all $t,\tilde{t} \in [0,\ell]$ and $s \in \mathbb{S}^{n-2}$. Indeed, if for a given $t,\tilde{t}$ and $s \in \mathbb{S}^{n-2}$, $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are not parallel, and

$$-\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i \ge 0, \tag{4.18}$$

then,

$$L_s^{\gamma}(t)\left(-\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i\right) \le L_s(t,\tilde{t})\left(-\sum_{i=1}^{n-1}\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})}s_i\right) = 1$$

which gives (4.17) for this case. If for a certain $t, \tilde{t}$ and $s \in \mathbb{S}^{n-2}$, (4.18) fails to hold, then (4.17) is obviously satisfied. Finally, if $F_\gamma(t)$ and $F_\gamma(\tilde{t})$ are parallel, then $h_i(t,\tilde{t}) = 0$ for all $1 \le i \le n-1$, hence the (4.17) is again satisfied.

We thirdly claim that

$$-\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} \to \kappa_i(t), \quad 1 \le i \le n-1, \tag{4.19}$$

as $\tilde{t} \to t$. Indeed, since $\det[\gamma'(t), \mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t)] = 1$ for all $t \in [0,\ell]$,

$$H(t,\tilde{t}) \approx \det\big[\mathbf{N}_1(\tilde{t}), \cdots, \mathbf{N}_{n-1}(\tilde{t}), \gamma'(\tilde{t})(t-\tilde{t})\big] = (-1)^{n-1}(t-\tilde{t})$$

as $\tilde{t} \to t$. Moreover,

$$h_i(t,t) = \det\big[\mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)\big] = 0.$$

Then,

$$\begin{aligned} &-\frac{h_i(t,\tilde{t})}{H(t,\tilde{t})} \approx -\frac{h_i(t,\tilde{t}) - h_i(t,t)}{(-1)^{n-1}(t-\tilde{t})} \to \\ &(-1)^{n-1}\bigg(\det\big[\mathbf{N}_1'(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)\big] + \cdots \\ &+ \det\big[\mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}'(t), \mathbf{N}_i(t)\big]\bigg). \end{aligned} \tag{4.20}$$

Recalling (4.11) and plugging this expression into (4.20), it is easy to see that all other terms vanish except

$$\begin{aligned} \det[\mathbf{N}_1(t), \cdots, \mathbf{N}_i'(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)]& \\ &= -\kappa_i \det\big[\mathbf{N}_1(t), \cdots, \gamma'(t), \cdots, \mathbf{N}_{n-1}(t), \mathbf{N}_i(t)\big] \\ &= \kappa_i \det\big[\mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t), \gamma'(t)\big] = (-1)^{n-1}\kappa_i \end{aligned}$$

because $\det[\gamma'(t), \mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t)] = 1$. This proves (4.19).

Passing in (4.17) to the limit $\tilde{t} \to t$ we obtain the lemma. The proof is complete. □

Recall that $S_s^\gamma(t)$ as defined in (4.8) satisfies $0 \le S_s^\gamma(t) \le L_s^\gamma(t)$ for all $s \in \mathbb{S}^{n-2}$ due to the fact that $F_\gamma(t) \cap F_\gamma(\tilde{t}) \cap \Omega = \emptyset$ for all $t, \tilde{t} \in [0,\ell]$, $\tilde{t} \ne t$. We then have:

Corollary 4.8. $S_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i\kappa_i(t)\Big) \le 1$ *for all* $t \in [0,\ell]$ *and* $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$.

Proof. If $\sum_{i=1}^{n-1} s_i\kappa_i(t) \geq 0$, then $S_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i\kappa_i(t)\Big) \leq L_s^\gamma(t)\Big(\sum_{i=1}^{n-1} s_i\kappa_i(t)\Big) \leq 1$. If $\sum_{i=1}^{n-1} s_i\kappa_i(t) < 0$, then the result is obviously true. □

From the definition of Φ in (4.2), Φ is Lipschitz, hence its Jacobian $J_\Phi = \det D\Phi$ exists a.e. on Σ^γ, where Σ^γ has two equivalent representations (4.6) and (4.10). We will show the Corollary 4.8 implies $J_\Phi > 0$ a.e. on Σ^γ, namely:

Lemma 4.9. $J_\Phi(t, s) = 1 - \sum_{i=1}^{n-1} s_i\kappa_i(t) > 0$ *for all* $(t, s) \in \Sigma^\gamma$.

Proof. Differentiating $\Phi(t, s)$ with respect to $(t, s_1, \cdots, s_{n-1})$ gives

$$J_\Phi(t, s) = \det\left[\gamma'(t) + \sum_{i=1}^{n-1} s_i\mathbf{N}_i'(t), \mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t)\right]. \tag{4.21}$$

Substituting (4.11) into (4.21) we obtain, after Gaussian elimination, that

$$J_\Phi(t, s) = 1 - \sum_{i=1}^{n-1} s_i\kappa_i(t). \tag{4.22}$$

If $\sum_{i=1}^{n-1} s_i\kappa_i(t) \leq 0$, then obviously $J_\Phi(t, s) > 0$. Suppose now $\sum_{i=1}^{n-1} s_i\kappa_i(t) > 0$. By (4.8) and (4.10) we have

$$\sum_{i=1}^{n-1} s_i\kappa_i(t) = |s|\left(\sum_{i=1}^{n-1} \frac{s_i}{|s|}\kappa_i(t)\right) < S_s^\gamma(t)\left(\sum_{i=1}^{n-1} \frac{s_i}{|s|}\kappa_i(t)\right) \leq 1$$

by Corollary 4.8. Therefore, $J_\Phi(t, s) > 0$ for all $(t, s) \in \Sigma^\gamma$. The proof is complete. □

4.3. Moving frames in the target space

We are now in a position to define the moving frame in the target space $\mathbb{R}^{n+1}$. Let $\mathbf{N}_i(t)$, $1 \leq i \leq n-1$ be as in Subsection 4.2. Define the leading curve corresponding to γ in $u(\Omega(\gamma))$ to be

$$\tilde{\gamma} := u \circ \gamma.$$

We also recall from Subsection 4.1 the definitions (4.2), (4.5), and that ∇u is constant on $P_\gamma(t)$ for each $t \in [0, \ell]$. Hence for each $t \in [0, \ell]$, $\nabla u \circ \Phi$ is constant on $\Sigma^\gamma(t)$.

Consider the Darboux frame $(\tilde{\gamma}', \mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{n})$ where $\mathbf{v}_i(t) = \nabla u(\gamma(t))\mathbf{N}_i(t)$, $i = 1, \ldots, n-1$ and $\mathbf{n}(t) = \tilde{\gamma}'(t) \times \mathbf{v}_1(t) \times \cdots \times \mathbf{v}_{n-1}(t)$. Since u is an isometric affine map along $P_\gamma(t)$ for each $t \in [0, \ell]$ we obtain

$$u(\Phi(t, s)) = \tilde{\gamma}(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_i(t) \tag{4.23}$$

for all $t \in [0, \ell]$ and $s \in \Sigma^\gamma(t)$. Differentiating with respect to t, by (4.2) we get

$$\nabla u(\Phi(t, s)) \left(\gamma'(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_i'(t) \right) = \tilde{\gamma}'(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_i'(t), \tag{4.24}$$

and differentiating with respect to s_i, $1 \leq i \leq n-1$ we obtain for each i,

$$\nabla u(\Phi(t, s))\mathbf{N}_i(t) = \mathbf{v}_i(t). \tag{4.25}$$

By the linear expansion of N_i' in (4.11) and (4.12), together with (4.24) and (4.25) we get

$$\begin{aligned} &\tilde{\gamma}'(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_i'(t) \\ &= \nabla u(\Phi(t, s)) \left(1 - \sum_{i=1}^{n-1} s_i \kappa_i(t) \right) \gamma'(t) + \sum_{i=1}^{n-1} s_i \left(\sum_{j=1}^{n-1} \kappa_{i_j}(t) \mathbf{v}_j(t) \right) \end{aligned} \tag{4.26}$$

with $\kappa_{i_i} = 0$ and $\kappa_{i_j} = -\kappa_{j_i}$. Also, by (4.24), for $s = 0$ we have

$$\nabla u(\Phi(t, 0))\gamma'(t) = \tilde{\gamma}'(t).$$

Since $\nabla u \circ \Phi$ is constant on $\Sigma^\gamma(t)$ for each $t \in [0, \ell]$, we obtain

$$\nabla u(\Phi(t, s))\gamma'(t) = \nabla u(\Phi(t, 0))\gamma'(t) = \tilde{\gamma}'(t) \quad \text{for all } s \in \Sigma^\gamma(t). \tag{4.27}$$

Alongside (4.25), this shows that at each point in $\Omega(\gamma)$, ∇u maps an orthonormal frame to another orthonormal frame and this orthonormal frame only depends on t. Finally, using (4.26) and matching coefficients yields for all $1 \leq i \leq n-1$,

$$\mathbf{v}_i' = -\kappa_i \tilde{\gamma}' + \sum_{j=1}^{n-1} \kappa_{i_j} \mathbf{v}_j, \quad \kappa_{i_i} = 0 \text{ and } \kappa_{i_j} = -\kappa_{j_i}. \tag{4.28}$$

In other words, the following system of ODEs is satisfied by the Darboux frame of $\tilde{\gamma}$:

$$\begin{pmatrix} \tilde{\gamma}' \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} \\ \mathbf{n} \end{pmatrix}' = \mathcal{K} \begin{pmatrix} \tilde{\gamma}' \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_{n-1} \\ \mathbf{n} \end{pmatrix}. \tag{4.29}$$

Here the skew-symmetric curvature matrix $\mathcal{K}$ is given by

$$\mathcal{K} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 & \cdots & \kappa_{n-1} & \kappa_{\mathbf{n}} \\ -\kappa_1 & 0 & \kappa_{1_2} & \cdots & \kappa_{1_{n-1}} & 0 \\ -\kappa_2 & -\kappa_{1_2} & 0 & \cdots & \kappa_{2_{n-1}} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\kappa_{n-1} & -\kappa_{1_{n-1}} & -\kappa_{2_{n-1}} & \cdots & 0 & 0 \\ -\kappa_{\mathbf{n}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

4.4. Change of variable formula

Recall that $\Phi : \Sigma^\gamma \to \Omega(\gamma)$ is one-to-one and onto, where Σ^γ was defined in (4.6), For $(t, s) \in \Sigma^\gamma$, let $u_i(t, s) := (\frac{\partial}{\partial x_i} u) \circ \Phi(t, s)$, note that u_i is the ith column of $\nabla u \circ \Phi$. The following holds for all $(t, s) \in \Sigma^\gamma$: since $\nabla u^T \mathbf{n} \cdot \gamma' = \mathbf{n} \cdot \nabla u \gamma' = \mathbf{n} \cdot \tilde{\gamma}' = 0$ and $\nabla u^T \mathbf{n} \cdot \mathbf{N}_j = \mathbf{n} \cdot \nabla u \mathbf{N}_j = \mathbf{n} \cdot \mathbf{v}_j = 0$ for all $1 \le j \le n-1$, we have $\nabla u^T \mathbf{n} = 0$, *i.e.* $u_i \cdot \mathbf{n} = 0$ for all $1 \le i \le n$. Thus,

$$\begin{aligned} u_i &= \big(u_i \cdot \tilde{\gamma}'\big)\tilde{\gamma}' + \sum_j \big(u_i \cdot \mathbf{v}_j\big)\mathbf{v}_j + \big(u_i \cdot \mathbf{n}\big)\mathbf{n} \\ &= \big(u_i \cdot \tilde{\gamma}'\big)\tilde{\gamma}' + \sum_j \big(u_i \cdot \mathbf{v}_j\big)\mathbf{v}_j \\ &= \big(u_i \cdot \nabla u \gamma'\big)\tilde{\gamma}' + \sum_j \big(u_i \cdot \nabla u \mathbf{N}_j\big)\mathbf{v}_j \\ &= \big(\nabla u^T u_i \cdot \gamma'\big)\tilde{\gamma}' + \sum_j \big(\nabla u^T u_i \cdot \mathbf{N}_j\big)\mathbf{v}_j \\ &= \big(\mathbf{e}_i \cdot \gamma'\big)\tilde{\gamma}' + \sum_j \big(\mathbf{e}_i \cdot \mathbf{N}_j\big)\mathbf{v}_j. \end{aligned} \tag{4.30}$$

Note that the right-hand side of (4.30) is independent of s. Differentiating with respect to s_j, $0 \le j \le n-1$, by (4.2) we get for all $1 \le i \le n$ and $1 \le j \le n-1$,

$$\left(\nabla \frac{\partial}{\partial x_i} u\right)(\Phi(t, s))\mathbf{N}_j(t) = 0. \tag{4.31}$$

Differentiating u_i with respect to t we obtain

$$\left(\nabla\frac{\partial}{\partial x_i}u\right)(\Phi(t,s))\left(\gamma'(t)+\sum_{j=1}^{n-1}s_j\mathbf{N}_j'(t)\right)$$

$$= \big(\mathbf{e}_i\cdot\gamma''(t)\big)\tilde{\gamma}'(t)+(\mathbf{e}_i\cdot\gamma'(t))\tilde{\gamma}''(t) \tag{4.32}$$

$$+\sum_{j=1}^{n-1}\big(\mathbf{e}_i\cdot\mathbf{N}_j'(t)\big)\mathbf{v}_j(t)+\sum_{j=1}^{n-1}\big(\mathbf{e}_i\cdot\mathbf{N}_j(t)\big)\mathbf{v}_j'(t).$$

If we write $\mathbf{N}_i'$ as a linear combination of γ' and $\mathbf{N}_j$, $j=1,\cdots,n-1$ as in (4.11) and (4.12), the left-hand side of (4.32) becomes

$$\left(1-\sum_{j=1}^{N-1}s_j\kappa_j(t)\right)\left(\nabla\frac{\partial}{\partial x_i}u\right)(\Phi(t,s))\gamma'(t).$$

For the right-hand side of (4.32), if we write out γ'', $\tilde{\gamma}''$, $\mathbf{N}_j'$ and $\mathbf{v}_j'$ as linear combinations of γ', $\tilde{\gamma}'$, $\mathbf{N}_\ell$ and $\mathbf{v}_\ell$, $\ell=1,\cdots,n-1$ and $\mathbf{n}$ as in (4.11) and (4.28), we obtain,

$$\begin{aligned}
&\big(\mathbf{e}_i\cdot\gamma''\big)\tilde{\gamma}'+\big(\mathbf{e}_i\cdot\gamma'\big)\tilde{\gamma}''+\sum_j\big(\mathbf{e}_i\cdot\mathbf{N}_j'\big)\mathbf{v}_j+\sum_j\big(\mathbf{e}_i\cdot\mathbf{N}_j\big)\mathbf{v}_j'\\
&=\left(\mathbf{e}_i\cdot\sum_j\kappa_j\mathbf{N}_j\right)\tilde{\gamma}'+\big(\mathbf{e}_i\cdot\gamma'\big)\left(\sum_j\kappa_j\mathbf{v}_j+\kappa_n\mathbf{n}\right)\\
&\quad+\sum_j\left(\mathbf{e}_i\cdot\left(-\kappa_j\gamma'+\sum_\ell\kappa_{j_\ell}\mathbf{N}_\ell\right)\right)\mathbf{v}_j+\sum_j\big(\mathbf{e}_i\cdot\mathbf{N}_j\big)\left(-\kappa_j\tilde{\gamma}'+\sum_\ell\kappa_{j_\ell}\mathbf{v}_\ell\right)\\
&=\sum_j\big(\mathbf{e}_i\cdot\kappa_j\mathbf{N}_j\big)\tilde{\gamma}'+\big(\mathbf{e}_i\cdot\gamma'\big)\sum_j\kappa_j\mathbf{v}_j+\big(\mathbf{e}_i\cdot\gamma'\big)\kappa_n\mathbf{n}-\big(\mathbf{e}_i\cdot\gamma'\big)\sum_j\kappa_j\mathbf{v}_j\\
&\quad+\sum_j\sum_\ell\big(\mathbf{e}_i\cdot\kappa_{j_\ell}\mathbf{N}_\ell\big)\mathbf{v}_j-\sum_j\big(\mathbf{e}_i\cdot\kappa_j\mathbf{N}_j\big)\tilde{\gamma}'+\sum_\ell\sum_j\big(\mathbf{e}_i\cdot\mathbf{N}_\ell\big)\kappa_{\ell_j}\mathbf{v}_j=(\mathbf{e}_i\cdot\gamma')\kappa_n\mathbf{n}
\end{aligned}$$

where we used the fact that $\kappa_{i_j}=-\kappa_{j_i}$. By Lemma 4.9, $1-\sum_{j=1}^{n-1}s_j\kappa_j(t)>0$ for all $(t,s)\in\Sigma^\gamma$. Therefore,

$$\left(\nabla\frac{\partial}{\partial x_i}u\right)(\Phi(t,s))\gamma'(t)=\frac{(\mathbf{e}_i\cdot\gamma'(t))\kappa_{\mathbf{n}}(t)\mathbf{n}(t)}{1-\sum_{j=1}^{n-1}s_j\kappa_j(t)}. \tag{4.33}$$

Since Φ is Lipschitz with $J_\Phi(t,s) = 1 - \sum_{j=1}^{n-1} s_j\kappa_j(t) > 0$, the change of variable $x = \Phi(t,s)$ with (4.23) and (4.33) yields

$$\int_{\Omega(\gamma)} |u(x)|^2 dx = \int_0^\ell \int_{\Sigma^\gamma(t)} \left|\tilde{\gamma}(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_i(t)\right|^2 \cdot \left(1 - \sum_{j=1}^{n-1} s_j\kappa_j(t)\right) d\mathcal{H}^{n-1}(s)dt, \tag{4.34}$$

$$\int_{\Omega(\gamma)} |\nabla u(x)|^2 dx = n|\Omega(\gamma)|, \tag{4.35}$$

$$\begin{aligned}\int_{\Omega(\gamma)} |\nabla^2 u(x)|^2 dx &= \int_0^\ell \int_{\Sigma^\gamma(t)} \frac{\sum_i \left(\mathbf{e}_i \cdot \gamma'(t)\right)^2 \kappa_{\mathbf{n}}^2(t)}{\left(1 - \sum_{j=1}^{n-1} s_j\kappa_j(t)\right)} d\mathcal{H}^{n-1}(s)dt \\ &= \int_0^\ell \int_{\Sigma^\gamma(t)} \frac{\kappa_{\mathbf{n}}^2(t)}{\left(1 - \sum_{j=1}^{n-1} s_j\kappa_j(t)\right)} d\mathcal{H}^{n-1}(s)dt.\end{aligned} \tag{4.36}$$

4.5. Approximation process for $u|_{\Omega(\gamma)}$

Recall $L_s^\gamma(t)$ and $S_s^\gamma(t)$ defined in (4.13) and (4.8) respectively. Since all leading fronts meet outside Ω, we must have $L_s^\gamma(t) \geq S_s^\gamma(t)$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0,\ell]$.

Lemma 4.10. *There exists a sequence of isometries $u_m \in W^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ converging strongly to u with the property that each u_m has a suitable leading curve $\gamma_m : [0,\ell_m] \to \mathbb{R}^n$ for which $L_s^{\gamma_m}(t) - S_s^{\gamma_m}(t) > \rho_m > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0,\ell_m]$.*

Proof. The proof is exactly the same as the 2-dimensional case, [22, Proposition 3.2]. For this reason, it is omitted. □

Remark 4.11. By the above lemma, we can just assume u has a suitable leading curve γ that satisfies $L_s^\gamma(t) - S_s^\gamma(t) > \rho > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0,\ell]$.

Lemma 4.12. *Suppose $L_s^\gamma(t) - S_s^\gamma(t) > \rho > 0$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0,\ell]$. Then there is a sequence of smooth maps in $I^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ converging strongly to u.*

Proof. The idea is to construct a smooth curve γ_m approximating γ. We do not know yet this curve is a leading curve of u_m or not, so we cannot call the $(n-2)$-dimensional hyperplane orthogonal to γ_m at t leading fronts. Instead we call them *orthogonal fronts* and denote them by $F_{\gamma_m}(t)$. If we manage to show all such orthogonal fronts meet outside $\Omega(\gamma_m)$, γ_m becomes a leading curve for u_m and $F_{\gamma_m}(t)$ are actually the leading fronts. We then define u_m to be isometric affine mapping along each leading front $F_{\gamma_m}(t)$. Since all the leading fronts intersect outside Ω, u_m is well-defined. □

We first need the following:

Lemma 4.13. *There exists smooth curve γ_m such that $\gamma_m(t) \to \gamma(t)$ strongly in $W^{2,p}([0,\ell],\mathbb{R}^n)$ for all $1 \le p < \infty$ and satisfies $F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \overline{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0,\ell]$.*

Proof. The construction is long and technical so we postpone the proof to Appendix A. □

We also need to define the curves $\tilde{\gamma}_m$ in the target space $u(\Omega(\gamma))$ corresponding to γ_m. Recall that the normal curvature $\kappa_{\mathbf{n}}$ defined in (4.29) is bounded. We choose a sequence of uniformly bounded smooth function $\tilde{\kappa}_{\mathbf{n},m}$ such that $\tilde{\kappa}_{\mathbf{n},m} \to \kappa_{\mathbf{n}}$ a.e. in $[0,\ell]$, (and hence in L^p for all $1 \le p < \infty$).

We need to flatten $\tilde{\kappa}_{\mathbf{n},m}$ around the end points 0 and ℓ for two reasons: first, it might happen that $\Omega(\gamma) \nsubseteq \Omega(\gamma_m)$ so we need to extend the isometric immersion defined on $\Omega(\gamma_m)$ smoothly to the region of $\Omega(\gamma)$ outside $\Omega(\gamma_m)$. Second, so far all the construction is on one covered domain $\Omega(\gamma)$ and our final goal is to glue all the different covered domains together smoothly. By flattening $\tilde{\kappa}_{\mathbf{n},m}$ around the end point 0 and ℓ, u_m constructed later is affine near the leading planes $P_\gamma(0)$ and $P_\gamma(\ell)$ (for definition of leading planes see Definition 4.5) so that we can join all the pieces smoothly. The modification goes as follows: by (4.33), the second derivative of u vanishes whenever $\kappa_{\mathbf{n}} = 0$. Put

$$\ell_m^* = \begin{cases} \ell & \text{if} \quad \Omega(\gamma) \subset \Omega(\gamma_m) \text{ and,} \\ \sup\left\{t \in [0,\ell],\, F_{\gamma_m}(t) \cap F_\gamma(\ell) \cap \overline{\Omega}(\gamma) = \emptyset\right\} & \text{otherwise.} \end{cases}$$

By step 1 of Lemma 4.13 in the Appendix, $F_{\gamma_m}(t) \to F_\gamma(t)$ uniformly, hence $\ell_m^* \to \ell$ as $m \to \infty$.

Let ψ_1 be any smooth non-negative function which is 0 on $[-1,\infty)$ and 1 on $(-\infty,-2)$. Let ψ_2 be any smooth positive function which is 0 on $(-\infty,1]$ and 1 on $(2,\infty)$. We put,

$$\kappa_{\mathbf{n},m}(t) := \psi_1(m(t-\ell_m^*))\psi_2(mt)\tilde{\kappa}_{\mathbf{n},m}(t), \quad t \in [0,\ell]$$

and we solve the following linear system for initial values $\tilde{\gamma}_m'(0) = \tilde{\gamma}'(0)$, $\mathbf{v}_{i,m}(0) = \mathbf{v}_i(0)$, and $\mathbf{n}_m(0) = \mathbf{n}(0)$:

$$\begin{pmatrix} \tilde{\gamma}_m' \\ \mathbf{v}_{1,m} \\ \mathbf{v}_{2,m} \\ \vdots \\ \mathbf{v}_{n-1,m} \\ \mathbf{n}_m \end{pmatrix}' = \mathcal{K}_m \begin{pmatrix} \tilde{\gamma}_m' \\ \mathbf{v}_{1,m} \\ \mathbf{v}_{2,m} \\ \vdots \\ \mathbf{v}_{n-1,m} \\ \mathbf{n}_m \end{pmatrix}.$$

Here the matrix $\mathcal{K}_m$ is given by

$$\mathcal{K}_m = \begin{pmatrix} 0 & \kappa_{1,m} & \kappa_{2,m} & \cdots & \kappa_{n-1,m} & \kappa_{\mathbf{n},m} \\ -\kappa_{1,m} & 0 & \kappa_{1_2,m} & \cdots & \kappa_{1_{n-1},m} & 0 \\ -\kappa_{2,m} & -\kappa_{1_2,m} & 0 & \cdots & \kappa_{2_{n-1},m} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\kappa_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \cdots & 0 & 0 \\ -\kappa_{\mathbf{n},m} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We define

$$\tilde{\gamma}_m(t) = \tilde{\gamma}(0) + \int_0^t \tilde{\gamma}_m'(\tau)d\tau.$$

By the same argument as in Step 1 in the proof of Lemma 4.13, $\tilde{\gamma}_m \to \tilde{\gamma}$ in $W^{2,p}([0,\ell],\mathbb{R}^{n+1})$ and the moving frame $(\tilde{\gamma}_m', \mathbf{v}_{1,m}, \cdots, \mathbf{v}_{n-1,m}, \mathbf{n}_m)$ converges to $(\tilde{\gamma}', \mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{n})$ uniformly.

Eventually, we define our approximating sequence u_m on $\Omega(\gamma_m)$

$$u_m\left(\gamma_m(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t)\right) = \tilde{\gamma}_m(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_{i,m}(t), \tag{4.37}$$

where γ_m is defined in Lemma 4.13. Such γ_m assures that all its leading fronts intersect outside $\overline{\Omega}$, hence u_m is well-defined and smooth over $\Omega(\gamma) \cap \Omega(\gamma_m)$.

As before, let $\Phi_m : [0,\ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ be defined as

$$\Phi_m(t,s) = \gamma_m(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t),$$

and let $\Delta^{\gamma_m} = \{(t,s) : \Phi_m(t,s) \in \Omega(\gamma)\}$. The same argument as in Step 6 in Lemma 4.13 gives that $\Phi_m(t,s)$ is a bi-Lipschitz mapping of Δ^{γ_m} onto $\Omega(\gamma) \cap \Omega(\gamma_m)$. By differentiating with respect to $t, s_1, \cdots, s_{n-1}$, as in (4.27) and (4.25), we see that at each point of x, $\nabla u_m(x)$ maps an orthonormal frame to an orthonormal frame. Hence $\nabla u_m(x)^T \nabla u_m(x) = \mathrm{I}$. Moreover, u_m is affine near $P_{\gamma_m}(\ell)$ and can be extended by an affine isometry over $\Omega(\gamma)$. Therefore, $u_m \in I^{2,2}(\Omega(\gamma), \mathbb{R}^n)$. Everything we have proved for isometric immersions of course applies, in particular, by (4.30), (4.33), and (4.31) we have for all $1 \le i \le n-1$

$$\frac{\partial}{\partial x_i} u_m \circ \Phi_m(t,s) = \big(\mathbf{e}_i \cdot \gamma_m'(t)\big)\tilde{\gamma}_m'(t) + \sum_{j=1}^{n-1} \big(\mathbf{e}_i \cdot \mathbf{N}_{j,m}(t)\big)\mathbf{v}_{j,m}(t), \tag{4.38}$$

$$\left(\nabla \frac{\partial}{\partial x_i} u_m\right)\big(\Phi_m(t,s)\big)\gamma_m'(t) = \frac{\big(\mathbf{e}_i \cdot \gamma_m'(t)\big)\kappa_{\mathbf{n},m}(t)\mathbf{n}(t)}{1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)}, \quad \text{and} \tag{4.39}$$

$$\left(\nabla \frac{\partial}{\partial x_i} u_m\right)\big(\Phi_m(t,s)\big)\mathbf{N}_{j,m}(t) = 0, \quad \text{for all } 1 \le j \le n-1 \tag{4.40}$$

for all $t \in [0, \ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \Delta^{\gamma_m}(t)$. Moreover, by (4.34), (4.35), and (4.36) we compute

$$\begin{aligned}
&\int_{\Omega(\gamma)} \left|u_m(x)\right|^2 dx \\
&= \int_{\Omega(\gamma)\cap\Omega(\gamma_m)} \left|u_m(x)\right|^2 dx + \int_{\Omega(\gamma)\setminus\Omega(\gamma_m)} \left|u_m(x)\right|^2 dx \\
&= \int_0^\ell \int_{\Delta^{\gamma_m}(t)} \left|\tilde{\gamma}_m(t) + \sum_{i=1}^{n-1} s_i \mathbf{v}_{i,m}(t)\right|^2 \cdot \left(1 - \sum_{i=1}^{n-1} s_j \kappa_{j,m}(t)\right) d\mathcal{H}^{n-1}(s) dt \\
&+ \int_{\Omega(\gamma)\setminus\Omega(\gamma_m)} \left|u_m(\ell) + \nabla u_m(\ell)(x - \gamma_m(\ell))\right|^2 dx,
\end{aligned} \tag{4.41}$$

$$\int_{\Omega(\gamma)} \left|\nabla u_m(x)\right|^2 dx = n|\Omega(\gamma)|, \tag{4.42}$$

$$\begin{aligned}
&\int_{\Omega(\gamma)} \left|\nabla^2 u_m(x)\right|^2 dx \\
&= \int_{\Omega(\gamma)\cap\Omega(\gamma_m)} \left|\nabla^2 u_m(x)\right|^2 dx + \int_{\Omega(\gamma)\setminus\Omega(\gamma_m)} \left|\nabla^2 u_m(x)\right|^2 dx \\
&= \int_0^\ell \int_{\Delta^{\gamma_m}(t)} \frac{\kappa^2_{\mathbf{n},m}(t)}{\left(1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)\right)} d\mathcal{H}^{n-1}(s) dt + 0.
\end{aligned} \tag{4.43}$$

It is easy to see that $u_m \to u$ in $W^{2,2}(\Omega(\gamma), \mathbb{R}^{n+1})$ because $(\gamma_m', \mathbf{N}_{1,m}, \cdots \mathbf{N}_{n-1,m})$ converges to $(\gamma', \mathbf{N}_1, \cdots \mathbf{N}_{n-1})$ uniformly, $(\tilde{\gamma}_m', \mathbf{v}_{1,m}, \cdots, \mathbf{v}_{n-1,m}, \mathbf{n}_m)$ converges to $(\tilde{\gamma}', \mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{n})$ uniformly, $\kappa_{\mathbf{n},m} \to \kappa_{\mathbf{n}}$, $\kappa_{i,m} \to \kappa_i$, $1 \le i \le n-1$ in $L^p([0,\ell])$ for all $1 \le p < \infty$, $1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) \ge \min\{\rho/16d, 1/2\}$, $\Delta^{\gamma_m}(t) \to \Sigma^\gamma(t)$ for all $t \in [0,\ell]$ and $|\Omega(\gamma) \setminus \Omega(\gamma_m)| \to 0$. The proof is complete. □

Combining Lemmas 4.10 and 4.12 we get a smooth approximation sequence for any isometry u in $\Omega(\gamma)$.

4.6. Approximation for u in Ω

The proof is exactly the same as the proof in [22, Section 3.3]. Since it is the final part of the argument, we briefly review it for the convenience of the reader.

Recall that we defined a maximal region on which u is affine a *body* if its boundary contains more than *two* different $(n-1)$-planes in Ω (recall Definition 3.9 for the definition of $(n-1)$-planes in Ω) and we have shown that we can assume Ω has only a finite number of bodies and is partitioned into bodies and covered domains. We call the maximal subdomain covered by some leading curve γ an *arm*. Similarly to Lemma 4.1 we also have,

Lemma 4.14. *It is sufficient to prove Theorem* 1.5 *for a function in* $I^{2,2}(\Omega, \mathbb{R}^{n+1})$ *with a finite number of arms.*

Proof. The proof is the same as the two-dimensional case in [22, Lemma 3.9] and is omitted for brevity. □

Now since Ω is convex and simply-connected, we claim that two bodies are connected through one chain of bodies and arms: it suffices to consider the graph obtained by retracting bodies to vertices and arms to edges. This graph is simply connected because it is a deformation retract of Ω. Therefore every two vertices are connected through only one chain of edges, which proves the claim (Figure 4.5).

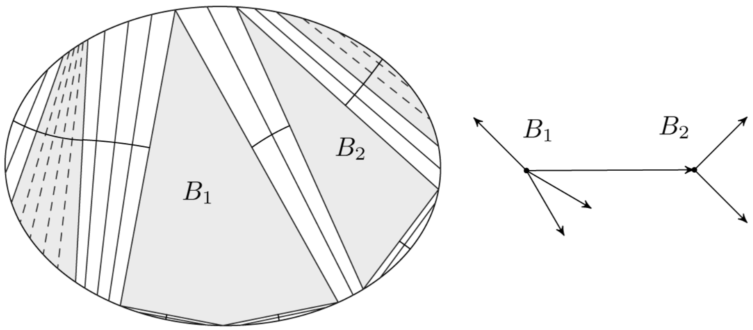

Figure 4.5. Graph of retraction of Ω.

We begin by a central body B_1 and define our approximating sequence on each arm as in Subsection 4.5. Note that for this final purpose, we have constructed our approximating smooth isometric immersion to be affine near both ends, this allows us to apply an affine transformation to the target space of each arm so that the affine regions near its ends join together smoothly all the way till we reach B_2. Meanwhile, we also apply an affine transformation to $u(B_2)$ so that it joins the last arm smoothly. It is easy to see from the uniform convergence of each term in representation (4.38) that such affine transformation converges to identity as $m \to 0$. Now we continue our construction using B_2 as a new starting point. Note that we will never come back to B_1 because they are connected through only one chain of arms. The construction of the approximating sequence on the entire domain Ω is complete. □

Appendix

A. Proof of Lemma 4.13

Step 1. Recall from the matrix of moving frame defined in Subsection 4.2. that $\gamma''(t) = \sum_{i=1}^{n-1} \kappa_i(t)\mathbf{N}_i(t))$, with κ_i bounded. We can choose uniformly bounded smooth functions $\tilde{\kappa}_{i,m} \to \kappa_i$ a.e. on $[0, \ell]$, and hence in measure due to the fact that $[0, \ell]$ is bounded. Since the sequence $\tilde{\kappa}_{i,m}$ are uniformly bounded, it follows $\tilde{\kappa}_{i,m} \to \kappa_i$ in L^p for all $1 \leq p < \infty$. Similarly we can find uniformly bounded smooth functions $\kappa_{i_j,m} \to \kappa_{i_j}$ a.e. on $[0, \ell]$ (hence in L^p for all $1 \leq p < \infty$) for κ_{i_j}, $1 \leq i, j \leq n-1$. By solving the system of ODEs

$$\begin{pmatrix} \Gamma'_m \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}' = \widetilde{\mathcal{K}}_m \begin{pmatrix} \Gamma'_m \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix},$$

where the matrix $\widetilde{\mathcal{K}}_m$ is given by

$$\widetilde{\mathcal{K}}_m = \begin{pmatrix} 0 & \tilde{\kappa}_{1,m} & \tilde{\kappa}_{2,m} & \cdots & \tilde{\kappa}_{n-1,m} \\ -\tilde{\kappa}_{1,m} & 0 & \kappa_{1_2,m} & \cdots & \kappa_{1_{n-1},m} \\ -\tilde{\kappa}_{2,m} & -\kappa_{1_2,m} & 0 & \cdots & \kappa_{2_{n-1},m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\tilde{\kappa}_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \cdots & 0 \end{pmatrix},$$

we obtain a unique orthogonal frame $(\Gamma'_m(t), \mathbf{N}_{1,m}(t), \cdots, \mathbf{N}_{n-1,m}(t))$ with initial condition $\Gamma'_m(0) = \gamma'(0)$, and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$. We can then define

$$\Gamma_m(t) = \Gamma(0) + \int_0^t \Gamma'_m(\tau)d\tau.$$

We want to show that $(\Gamma'_m, \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m}) \to (\gamma', \mathbf{N}_1, \cdots, \mathbf{N}_{n-1})$ uniformly. This result is given by the following theorem due to Opial [21, Theorem 1]:

Lemma A.1 (Opial). *Suppose the linear system of differential equations*

$$x'(t) = A_k(t)x(t), \quad x(0) = a_k, \quad k = 0, 1, 2, \cdots \tag{A.1}$$

admits a solution $x_k(t)$ in $[0, \ell]$ for all k. Suppose $a_k \to a_0$,

$$\int_0^t A_k(s)\,ds \to \int_0^t A_0(s)\,ds$$

uniformly for all $t \in [0, \ell]$ and A_k is a bounded sequence in L^1, i.e.

$$\sup_k \|A_k\|_{L^1([0,\ell])} < \infty.$$

Then the solutions $x_k(t)$ converge to $x_0(t)$ uniformly.

Since $\tilde{\kappa}_{i,m} \to \kappa_i$ and $\kappa_{i_j,m} \to \kappa_{i_j}$ in L^p for all $1 \le p < \infty$, in particular for $p = 1$, the conditions in Lemma A.1 are satisfied, hence $(\Gamma'_m, \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m})$ converges to $(\gamma', \mathbf{N}_1, \cdots, \mathbf{N}_{n-1})$ uniformly. Since $\Gamma''_m = \sum_{i=1}^{n-1} \tilde{\kappa}_{i,m} \mathbf{N}_{i,m}$, Γ''_m are uniformly bounded, and $\Gamma''_m \to \gamma''$ a.e. (and hence in L^p for all $1 \le p < \infty$), Poincaré inequality for intervals implies that $\Gamma_m \to \gamma$ in $W^{2,p}([0,\ell], \mathbb{R}^n)$ for all $1 \le p < \infty$.

However Γ_m is not our desired curve since we cannot guarantee that all its leading fronts intersect outside $\overline{\Omega}$. This happens if Γ_m is too "curvy". We need to "flatten" its curvature continuously. This needs to be done in several steps:

Step 2. We construct $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ continuous on $t \in [0,\ell]$ and for each $t \in [0,\ell]$ and $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$,

$$\left(S_s^{\Gamma_m}(t) + \frac{\rho}{2}\right)\left(\sum_{i=1}^{n-1} s_i \tilde{\tilde{\kappa}}_{i,m}(t)\right) \le 1 \tag{A.2}$$

where

$$S_s^{\Gamma_m}(t) = \sup\left\{S \ge 0 : \Gamma_m(t) + S\left(\sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t)\right) \in \Omega\right\}.$$

We first need the following lemma using the implicit function theorem for C^1 functions:

Lemma A.2. *$S_s^{\Gamma_m}(t)$ is uniformly continuous on $(s,t) \in \mathbb{S}^{n-2} \times [0,\ell]$ and $S_s^{\Gamma_m}(t)$ converges to $S_s^{\gamma}(t)$ uniformly on $(s,t) \in \mathbb{S}^{n-2} \times [0,\ell]$.*

Proof. Let $t_0 \in [0,\ell]$ and $s^0 = (s_1^0, \cdots, s_{n-1}^0) \in \mathbb{S}^{n-1}$ be arbitrary. We parametrize locally $\mathbb{S}^{n-2}$ by the polar coordinates: $s_i = s_i(\theta)$ where $\theta = (\theta_1, \cdots, \theta_{n-2}) \in U_1 \subset [0,\pi)^{n-3} \times [0, 2\pi)$. Let $\theta^0 \in U_1$ be such that $s_i^0 = s_i(\theta^0)$.

Let $\gamma^0 = \gamma(t_0)$ and $\mathbf{N}_i^0 = \mathbf{N}_i(t_0)$. Let x_0 be the intersection of the line segment $L = \{\gamma^0 + S\big(\sum_{i=1}^{n-1} s_i^0 \mathbf{N}_i^0\big), 0 \le S\}$ and $\partial\Omega$. Then $x_0 = \gamma^0 + S_0\big(\sum_{i=1}^{n-1} s_i^0 \mathbf{N}_i^0\big)$ for some $S_0 > 0$.

Since Ω is a C^1 domain, there exits an open subset of $U_2 \subset \mathbb{R}^{n-1}$ and a C^1 function $\alpha : U_2 \to \partial\Omega$ and $\alpha(\eta_1^0, \cdots, \eta_{n-1}^0) = x_0$ for some $(\eta_1^0, \cdots, \eta_{n-1}^0) \in U_2$.

Consider $F : \mathbb{R}^n \times \mathbb{R}^{n\times(n-1)} \times U_1 \times \mathbb{R} \times U_2 \to \mathbb{R}^n$

$$F\big(\gamma, \mathbf{N}_1, \cdots \mathbf{N}_{n-1}, \theta, S, \eta_1, \cdots, \eta_{n-1}\big) = \gamma + S\left(\sum_{i=1}^{n-1} s_i(\theta)\mathbf{N}_i\right) - \alpha(\eta_1, \cdots, \eta_{n-1}).$$

Since $x_0 \in \partial\Omega \cap L$,

$$F\big(\gamma^0, \mathbf{N}_1^0, \cdots, \mathbf{N}_{n-1}^0, \theta^0, S^0, \eta_1^0, \cdots, \eta_{n-1}^0\big) = 0.$$

Let

$$\mathbf{x} = \big(\gamma, \mathbf{N}_1, \cdots, \mathbf{N}_{n-1}, \theta\big), \quad \mathbf{y} = \big(S, \eta_1, \cdots, \eta_{n-1}\big)$$

and

$$\alpha_k := \frac{\partial \alpha}{\partial \eta_k}, \quad 1 \le k \le n-1.$$

Then

$$\det\left[\left(\frac{\partial F}{\partial \mathbf{y}}\big(\gamma^0, \mathbf{N}_1^0, \cdots, \mathbf{N}_{n-1}^0, \theta^0, S^0, \eta_1^0, \cdots, \eta_{n-1}^0\big)\right)\right]$$
$$= \det\left[\sum_{i=1}^{n-1} s_i\big(\theta^0\big)\mathbf{N}_i^0, \alpha_1\big(\eta_1^0, \cdots, \eta_{n-1}^0\big), \cdots, \alpha_{n-1}\big(\eta_1^0, \cdots, \eta_{n-1}^0\big)\right] \ne 0.$$

Otherwise, the line segment L would be parallel to the tangent plane of $\partial\Omega$ at x_0, which is not possible since Ω is convex.

By the implicit function theorem, there is an open neighborhood

$$V_1 \subset \mathbb{R}^n \times \mathbb{R}^{n\times(n-1)} \times U_1$$

of $\mathbf{x}_0 = (\gamma^0, \mathbf{N}_1^0, \cdots, \mathbf{N}_{n-1}^0, \theta^0)$, $V_2 \subset \mathbb{R} \times U_2$ of $\mathbf{y}_0 = (S^0, \eta_1^0, \cdots, \eta_{n-1}^0)$, and a C^1 map $\mathbf{y} : V_1 \to V_2$ such that

$$F(\mathbf{x}, \mathbf{y}(\mathbf{x})) = F(\mathbf{x}, S(\mathbf{x}), \eta_1(\mathbf{x}), \cdots, \eta_{n-1}(\mathbf{x})) = 0,$$

for all $\mathbf{x} \in V_1$.

Since $\gamma, \mathbf{N}_i$, $1 \le i \le n-1$ are Lipschitz on $[0, \ell]$ and $\Gamma_m \to \gamma$ uniformly and $\mathbf{N}_{i,m} \to \mathbf{N}_i$ uniformly on $[0, \ell]$ for all $1 \le i \le n-1$, there exists an open interval $O \subset \mathbb{R}$ containing t_0, an open subset $\Delta \subset U_1$ containing θ_0 and an integer M such that for all $t \in [0, \ell] \cap O, \theta \in \Delta$ and $m \ge M$

$$\mathbf{x}(t, \theta) = (\gamma(t), \mathbf{N}_1(t), \cdots, \mathbf{N}_{n-1}(t), \theta) \in V_1 \quad \text{and}$$
$$\mathbf{x}_m(t, \theta) = (\Gamma_m(t), \mathbf{N}_{1,m}(t), \cdots, \mathbf{N}_{n-1,m}(t), \theta) \in V_1.$$

Evidently $\mathbf{x}_m(t, \theta) \to \mathbf{x}(t, \theta)$ uniformly for all $t \in [0, \ell] \cap O$ and $\theta \in \Delta$. Since S is C^1 on $\mathbf{x} \in V_1$,

$$S\big(\mathbf{x}_m(t, \theta)\big) \to S(\mathbf{x}(t, \theta)) \text{ uniformly on } t \in [0, \ell] \cap O \text{ and } \theta \in \Delta. \tag{A.3}$$

Moreover, since S is C^1 on $\mathbf{x} \in V_1$ and $\mathbf{x}$ is uniformly continuous on $t \in [0, \ell] \cap O$ and $s \in s(\Delta)$, S is uniformly continuous on $t \in [0, \ell] \cap O$ and $s \in s(\Delta)$.

Now note that since $F\big(\mathbf{x}(t, \theta), \mathbf{y}(\mathbf{x}(t, \theta))\big) = 0$ and $F\big(\mathbf{x}_m(t, \theta), \mathbf{y}(\mathbf{x}_m(t, \theta))\big) = 0$, for each $s = s(\theta) \in s(\Delta) \subset \mathbb{S}^{n-2}$, we have $S_s^{\gamma}(t) = S(\mathbf{x}(t, \theta))$ and $S_s^{\Gamma_m}(t) = S(\mathbf{x}_m(t, \theta))$. Thus by (A.3),

$$S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t) \text{ uniformly on } t \in [0, \ell] \cap O \text{ and } s \in s(\Delta),$$

and $S_s^{\Gamma_m}(t)$ is uniformly continuous on $t \in [0, \ell] \cap O$ and $s \in s(\Delta)$.

It remains to observe that since $[0, \ell]$ and $\mathbb{S}^{n-2}$ are both compact they can be covered by a finite union of neighborhoods on which (A.3) holds. The proof is complete. □

Define

$$\lambda_m(t) := \min \left\{ 1, 1/ \left(\sup_{|s|=1} \left\{ \left(S_s^{\Gamma_m}(t) + \frac{\rho}{2} \right) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \right) \right\} \right) \right\}$$

where $\tilde{\kappa}_{i,m}$, $1 \leq i \leq n-1$ are those found in Step 1. A first observation is that $0 < \lambda_m \leq 1$. Indeed, there must exist $s \in \mathbb{S}^{n-2}$ such that $\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \geq 0$ so the supreme over all $s \in \mathbb{S}^{n-2}$ must be nonnegative. On the other hand, $S_s^{\Gamma_m}$ as well as all $\tilde{\kappa}_{i,m}$ are bounded so λ_m is bounded below by a positive number.

Second, we observe that λ_m is continuous. Indeed, by Lemma A.2,

$$\left(S_s^{\Gamma_m}(t) + \rho/2 \right) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \right)$$

is uniformly continuous on $(s, t) \in \mathbb{S}^{n-2} \times [0, \ell]$. Hence the supremum over $\mathbb{S}^{n-2}$ is attained and a simple argument gives

$$h(t) := \sup_{|s|=1} \left\{ \left(S_s^{\Gamma_m}(t) + \frac{\rho}{2} \right) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \right) \right\}$$

is continuous.

We then define a vector-valued function $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ as

$$\left(\tilde{\tilde{\kappa}}_{1,m}(t), \cdots, \tilde{\tilde{\kappa}}_{n-1,m}(t) \right) := \lambda_m(t) \big(\tilde{\kappa}_{1,m}(t), \cdots, \tilde{\kappa}_{n-1,m}(t) \big)$$

$\tilde{\tilde{\kappa}}_m$ is obviously continuous. It remains to show that $\tilde{\tilde{\kappa}}_m$ satisfies (A.2). Indeed, for any $s = (s_1, \cdots, s_{n-1}) \in \mathbb{S}^{n-2}$,

$$\left(S_s^{\Gamma_m}(t) + \frac{\rho}{2} \right) \left(\sum_{i=1}^{n-1} s_i \tilde{\tilde{\kappa}}_{i,m}(t) \right) = \lambda_m \left(S_s^{\Gamma_m}(t) + \frac{\rho}{2} \right) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \right).$$

If $\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \geq 0$, then by the definition of λ_m,

$$\lambda_m(t) \Big(S_s^{\Gamma_m}(t) + \frac{\rho}{2} \Big) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \right) \leq \min \left\{ \left(S_s^{\Gamma_m}(t) + \frac{\rho}{2} \right) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \right), 1 \right\} \leq 1.$$

If $\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) < 0$, then

$$\lambda_m(t) \left(S_s^{\Gamma_m}(t) + \frac{\rho}{2} \right) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) \right) < 0 \leq 1.$$

Thus (A.2) is satisfied.

Step 3. We want to show that $(\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m}) \to (\kappa_1, \cdots, \kappa_{n-1})$ a.e. Indeed, we know that $\tilde{\kappa}_m = (\tilde{\kappa}_{1,m}, \cdots, \tilde{\kappa}_{n-1,m}) \to (\kappa_1, \cdots, \kappa_{n-1})$ a.e.. Therefore, all we need to show is $\lambda_m \to 1$ a.e.

By possibly replacing λ_m by a subsequence, it suffices to prove $\lambda_m \to 1$ in measure. From the definition of λ_m, it is enough to show that the Lebesgue measure of the set

$$E_m = \left\{ t \in [0, l], \exists s \in \mathbb{S}^{n-2}, \left(S_s^{\Gamma_m}(t) + \frac{\rho}{2}\right) \left(\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t)\right) > 1 \right\}$$

goes to zero. First by assumption, $L_s^{\gamma}(t) - S_s^{\gamma}(t) > \rho > 0$ and by Lemma 4.7, $L_s^{\gamma}(t)(\sum_{i=1}^{n-1} \kappa_i(t) s_i) \leq 1$, thus

$$\big(S_s^{\gamma}(t) + \rho\big) \left(\sum_{i=1}^{n-1} s_i \kappa_i(t)\right) \leq 1 \tag{A.4}$$

for all $t \in [0, \ell]$ and $s \in \mathbb{S}^{n-2}$. Indeed, if $\sum_{i=1}^{n-1} s_i \kappa_i(t)) \geq 0$,

$$\big(S_s^{\gamma}(t) + \rho\big) \left(\sum_{i=1}^{n-1} s_i \kappa_i(t)\right) \leq L_s^{\gamma}(t) \left(\sum_{i=1}^{n-1} \kappa_i(t) s_i\right) \leq 1$$

which gives (A.4).

If $t \in E_m$, there is $s \in \mathbb{S}^{n-2}$ such that

$$\sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) > \frac{1}{S_s^{\Gamma_m}(t) + \rho/2}.$$

Therefore all $t \in E_m$ and our choice of $s = s(t)$ as above, we have

$$\begin{aligned} \big|\tilde{\kappa}_m(t) - \kappa(t)\big| &\geq \sum_{i=1}^{n-1} s_i \tilde{\kappa}_{i,m}(t) - \left(\sum_{i=1}^{n-1} s_i \kappa_i(t)\right) \\ &> \frac{\rho/2 + S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)}{\big(S_s^{\Gamma_m}(t) + \rho/2\big)\big(S_s^{\gamma}(t) + \rho\big)} \geq \frac{\rho/2 - \big|S_s^{\gamma}(t) - S_s^{\Gamma_m}(t)\big|}{\rho^2/2}. \end{aligned}$$

By Lemma A.2, we have

$$S_s^{\Gamma_m}(t) \to S_s^{\gamma}(t) \text{ uniformly on } s \in \mathbb{S}^{n-2} \text{ and } t \in [0, \ell],$$

hence we can find m sufficiently large so that $|S_s^\gamma(t) - S_s^{\Gamma_m}(t)| < \rho/4$ for all $s \in \mathbb{S}^{n-2}$ and $t \in [0, \ell]$. Since $\tilde{\kappa}_m \to \kappa$ a.e.

$$\lim_{m\to\infty} \left|E_m\right| \leq \lim_{m\to\infty} \left|\left\{t : \left|\tilde{\kappa}_m(t) - \kappa(t)\right| \geq \frac{1}{2\rho}\right\}\right| = 0$$

which is what we wanted to show.

Step 4. Since $\tilde{\tilde{\kappa}}_m = (\tilde{\tilde{\kappa}}_{1,m}, \cdots, \tilde{\tilde{\kappa}}_{n-1,m})$ are continuous, for each m we can find κ_m smooth and $|\tilde{\tilde{\kappa}}_m - \kappa_m| \to 0$ uniformly on $t \in [0, \ell]$. Hence for m sufficiently large

$$\left(S_s^{\Gamma_m}(t) + \frac{\rho}{4}\right)\left(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)\right) \leq 1. \tag{A.5}$$

Step 5. We now define our desired curve γ_m. Given $\kappa_m = (\kappa_{1,m}, \cdots \kappa_{n-1,m})$ smooth as found in Step 4, and $\kappa_{i_j,m} \to \kappa_{i_j}$ found in step 1, we again solve the system of ODEs

$$\begin{pmatrix} \gamma_m' \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix}' = \mathcal{K}_m^{n\times n} \begin{pmatrix} \gamma_m' \\ \mathbf{N}_{1,m} \\ \mathbf{N}_{2,m} \\ \vdots \\ \mathbf{N}_{n-1,m} \end{pmatrix},$$

where

$$\mathcal{K}_m^{n\times n} = \begin{pmatrix} 0 & \kappa_{1,m} & \kappa_{2,m} & \cdots & \kappa_{n-1,m} \\ -\kappa_{1,m} & 0 & \kappa_{1_2,m} & \cdots & \kappa_{1_{n-1},m} \\ -\kappa_{2,m} & -\kappa_{1_2,m} & 0 & \cdots & \kappa_{2_{n-1},m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\kappa_{n-1,m} & -\kappa_{1_{n-1},m} & -\kappa_{2_{n-1},m} & \cdots & 0 \end{pmatrix},$$

and denote by the orthogonal frame $(\gamma_m'(t), \mathbf{N}_{1,m}(t), \cdots \mathbf{N}_{n-1,m}(t))$ the unique solution with initial conditions $\gamma_m'(0) = \gamma'(0)$ and $\mathbf{N}_{i,m}(0) = \mathbf{N}_i(0)$. Moreover, by Lemma A.1, $(\gamma_m'(t), \mathbf{N}_{1,m}(t), \cdots \mathbf{N}_{n-1,m}(t)) \to (\gamma'(t), \mathbf{N}_1(t), \cdots \mathbf{N}_{n-1}(t))$ uniformly. Let

$$\gamma_m(t) = \gamma(0) + \int_0^t \gamma_m'(\tau)d\tau.$$

We claim γ_m satisfies for m sufficiently large

$$\left(S_s^{\gamma_m}(t) + \frac{\rho}{8}\right)\left(\sum_{i=1}^{n-1} s_i \kappa_{i,m}(t)\right) \leq 1. \tag{A.6}$$

Indeed, by the same argument of Lemma A.2 using the implicit function theorem, $S_s^{\gamma_m}$ also converges to S_s^{γ} uniformly. Together with Lemma A.2 we obtain that $|S_s^{\gamma_m} - S_s^{\Gamma_m}|$ converges to 0 uniformly. Thus the claim follows from (A.5).

Step 6. Finally, we claim that orthogonal fronts satisfy $F_{\gamma_m}(t) \cap F_{\gamma_m}(\tilde{t}) \cap \overline{\Omega} = \emptyset$ for all $t, \tilde{t} \in [0, \ell]$.

For γ_m and its moving frame $(\gamma_m', \mathbf{N}_{1,m}, \cdots, \mathbf{N}_{n-1,m})$ found in Step 5, let

$$\Phi_m : [0, \ell] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$$

be defined as

$$\Phi_m(t, s) = \gamma_m(t) + \sum_{i=1}^{n-1} s_i \mathbf{N}_{i,m}(t).$$

Let $\Sigma^{\gamma_m} = \{(t, s) : \Phi_m(t, s) \in \overline{\Omega}\}$. By the same argument as Lemma 4.6, Φ_m maps Σ^{γ_m} onto $\overline{\Omega(\gamma_m)}$ where $\Omega(\gamma_m)$ is the subset of Ω covered by all orthogonal fronts $F_{\gamma_m}(t)$, $t \in [0, \ell]$. By the same computation as in (4.22)

$$J_{\Phi_m}(t, s) = 1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t).$$

Let $d := \operatorname{diam}(\Omega)$; we claim that

$$1 - \sum_{i=1}^{n-1} s_i \kappa_{i,m}(t) \geq \min\{\rho/16d, 1/2\}$$

for all $(t, s) \in \Sigma^{\gamma_m}$. Indeed, if $\sum_{i=1}^{n-1} (s_i/|s|)\kappa_{i,m}(t) \geq 1/2d$, then by (A.6),

$$\begin{aligned} 1 - |s| \left(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) &\geq 1 - S_s^{\gamma_m}(t) \left(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) \\ &\geq \frac{\rho}{8} \left(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) \geq \frac{\rho}{8} \cdot \frac{1}{2d}. \end{aligned}$$

If $\sum_{i=1}^{n-1} (s_i/|s|)\kappa_{i,m}(t) < 1/2d$, then

$$1 - |s| \left(\sum_{i=1}^{n-1} \frac{s_i}{|s|} \kappa_{i,m}(t) \right) > 1 - \frac{|s|}{2d} \geq \frac{1}{2}.$$

Hence, the claim follows. By the inverse function theorem due to Clarke [5], Φ admits a local Lipschitz inverse, actually a *global* Lipschitz inverse $\Phi_m^{-1} : \overline{\Omega(\gamma_m)} \to \Sigma^{\gamma_m}$ since the Jacobian is everywhere bounded below by a positive constant in Σ^{γ_m}. In particular, Φ_m is one-to-one on Σ^{γ_m}. This implies that all orthogonal fronts $F_{\gamma_m}(t)$, $t \in [0, \ell]$ meet outside $\overline{\Omega}$. The proof of Lemma 4.13 is complete. □

References

[1] YU. F. BORISOV, *The parallel translation on a smooth surface, III* (Russian), Vestnik Leningrad. Univ. **14** (1959), 34–50.

[2] YU. F. BORISOV, *Irregular surfaces of the class $C^{1,\beta}$ with an analytic metric* (Russian) Sibirsk. Mat. Zh. **45** (2004), 25–61; translation in Siberian Math. J. **45** (2004), 19–52.

[3] V. BORRELLI, S. JABRANE, F. LAZARUS and B. THIBERT, *Flat tori in three-dimensional space and convex integration*, Proc. Natl. Acad. Sci. USA **109** (2012), 7218–7223.

[4] F. CAJORI, *Generalizations in geometry as seen in the history of developable surfaces*, Amer. Math. Monthly **36** (1929), 431–437.

[5] F. H. CLARKE, *On the inverse function theorem*, Pacific J. Math. **64** (1976), 97–102.

[6] S. CONTI, C. DE LELLIS and L. JR. SZÉKELYHIDI, *h-principle and rigidity for $C^{1,\alpha}$ isometric embeddings*, In: "Nonlinear Partial Differential Equations", Abel Symp., Vol. 7, Springer, Heidelberg, 2012, 83–116.

[7] L. C. EVANS and R. F. GARIEPY "Measure Theory and Fine Properties of Functions", Studies in Advanced Mathematics, CRC Press, 1992.

[8] M. GROMOV, "Partial Differential Relations", Ergebn. der Math., Vol. 9, Springer-Verlag, Berlin, 1986.

[9] P. HARTMAN and L. NIRENBERG, *On spherical image maps whose Jacobians do not change sign*, Amer. J. Math. **81** (1959), 901–920.

[10] P. HORNUNG, *Approximating $W^{2,2}$ isometric immersions*, C. R. Math. Acad. Sci. Paris **346** (2008), 189–192.

[11] P. HORNUNG, *Fine level set structure of flat isometric immersions*, Arch. Ration. Mech. Anal. **199** (2011), 943–1014.

[12] P. HORNUNG, *Approximation of flat $W^{2,2}$ isometric immersions by smooth ones*, Arch. Ration. Mech. Anal. **199** (2011), 1015–1067.

[13] R. L. JERRARD, *Some remarks on Monge-Ampère functions*, In: "Singularities in PDE and the Calculus of Variations", CRM Proc. Lecture Notes, Vol. 44, Amer. Math. Soc., Providence, RI, 2008, 89–112.

[14] R. L. JERRARD, *Some rigidity results related to Monge-Ampère functions*, Canad. J. Math. **62** (2010), 320–354.

[15] R. L. JERRARD and M. R. PAKZAD, *Sobolev spaces of isometric immersions of arbitrary dimension and codimension*, in preparation.

[16] B. KIRCHHEIM, "Geometry and Rigidity of Microstructures", Habilitation Thesis, Leipzig, 2001.

[17] N. H. KUIPER, *On C^1-isometric imbeddings I, II*, Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955) 545–556, 683–689.

[18] S. MÜLLER and M. R. PAKZAD, *Regularity properties of isometric immersions*, Math. Z. **251** (2005), 313–331.

[19] J. NASH, *The imbedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20–63.

[20] J. NASH, *C^1 isometric imbeddings*, Ann. of Math. **60** (1954), 383–396.

[21] Z. OPIAL, *Continuous parameter dependence in linear systems of differential equations*, J. Differential Equations **3** (1967), 571–579.

[22] M. R. PAKZAD, *On the Sobolev space of isometric immersions*, J. Differential Geom. **66** (2004), 47–69.

[23] A. V. POGORELOV, "Surfaces with Bounded Extrinsic Curvature" (Russian), Kharhov, 1956.

[24] A. V. POGORELOV, "Extrinsic Geometry of Convex Surfaces", Translation of mathematical monographs, Vol. 35, Amer. Math. Soc., 1973.

[25] M. SPIVAK, "A Comprehensive Introduction to Differential Geometry", Vol. V, 2nd edition, Publish or Perish Inc., 1979.

[26] S. C. VENKATARAMANI, T. A. WITTEN, E. M. KRAMER and R. P. GEROCH, *Limitations on the smooth confinement of an unstretchable manifold*, J. Math. Phys. **41** (2000), 5107–5128.

Department of Mathematical Analysis
Charles University
Czech Republic
liuzhuomin@hotmail.com

Department of Mathematics
University of Pittsburgh
301, Thackeray Hall
Pittsburgh, PA 15260, USA
pakzad@pitt.edu