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Optimal Liouville theorems for supersolutions of elliptic equations with the Laplacian

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Abstract. In this paper we consider the question of nonexistence of positive supersolutions of the equation $-\Delta u = f(u)$ in exterior domains of $\mathbb{R}^N$, where f is continuous and positive in $(0, +\infty)$. When $N \geq 3$, we find that positive supersolutions exist if and only if

$$\int_0^\delta \frac{f(t)}{t^{\frac{2(N-1)}{N-2}}} dt < +\infty$$

for some $\delta > 0$. A similar condition is found for $N = 2$: positive supersolutions exist if and only if

$$\int_M^\infty e^{at} f(t) dt < +\infty$$

for some $a, M > 0$. The proofs are extended to consider some more general operators, which include the Laplacian with gradient terms, the p-Laplacian or uniformly elliptic fully nonlinear operators with radial symmetry, like the Pucci's extremal operators $\mathcal{M}^{\pm}_{\lambda,\Lambda}$, with $\Lambda > \lambda > 0$.

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1. Introduction and results

It is well-known that nonlinear Liouville theorems play an important role in the study of some nonlinear partial differential equations of elliptic type. They are usually employed to obtain a priori bounds for all possible (positive) solutions, which in turn give existence of such solutions by means of topological arguments.

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One of the most famous theorems of this type is concerned with the model equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N \tag{1.1}$$

and was proved in the celebrated work [23] (see also [15] for a simplified proof). It was shown there that (1.1) does not have positive solutions provided that $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$ (observe that nonexistence for $N = 2$ follows irrespective of p because nonconstant positive superharmonic functions cannot exist in $\mathbb{R}^2$). This result has been generalized to deal with some more general nonlinearities than the power: for instance, in [29] the problem

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N$$

was analyzed under the assumption that $f(t)$ is a continuous nonlinearity such that $f(t)t^{-\frac{N+2}{N-2}}$ is nonincreasing in $(0, +\infty)$, and some nonexistence theorems were obtained. We refer also to [35] for related theorems with the Laplacian replaced with the p-Laplacian.

On the other hand, it is worth remarking that the critical exponent $\frac{N+2}{N-2}$ plays a role only if the equation is posed in the whole $\mathbb{R}^N$. If it is considered in exterior domains of $\mathbb{R}^N$ (which amounts to saying in $\mathbb{R}^N \setminus B_{R_0}$ for some $R_0 > 0$, where B_{R_0} stands for the ball with radius R_0 centered at the origin), it is known that positive solutions of (1.1) cannot exist in the range $1 < p \leq \frac{N}{N-2}$ (*cf.* [10] for a proof, in the context of the p-Laplacian operator). This condition on p is optimal since for $p > \frac{N}{N-2}$ a singular solution of the form $u = A|x|^{-\frac{2}{p-1}}$ can be constructed. It was also realized that actually the exponent $\frac{N}{N-2}$ is critical with regard to existence of positive supersolutions, both in $\mathbb{R}^N$ or in exterior domains (see [22] for the first proof and [32] for a simplified one).

Numerous works dealt with the question of nonexistence of supersolutions with some more general nonlinearities and operators. Without being exhaustive with the references, we mention [1, 2, 4, 5, 7–9, 11–14, 18, 24, 26] and [30] (and references therein). We refer to the survey [25] for a list of references.

A qualitative step further was given in [4], where the problem

$$-\Delta u \geq f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{1.2}$$

was considered for a general continuous, positive nonlinearity f (although it has to be remarked that the emphasis there was to deal with a general class of fully nonlinear operators, instead of the Laplacian). Among other things, it was shown that if f verifies

$$\liminf_{t\to 0+} \frac{f(t)}{t^{\frac{N}{N-2}}} > 0, \tag{1.3}$$

then no positive solutions of (1.2) can exist. When $N = 2$, the problem was also analyzed in [4], and they showed that if f verifies

$$\liminf_{t\to +\infty} e^{at} f(t) > 0 \qquad \text{for every } a > 0, \tag{1.4}$$

then no solutions of (1.2) can exist. Conditions (1.3) and (1.4) are, at the best of our knowledge, the best ones obtained so far to ensure the nonexistence of positive solutions of problem (1.2).

Conversely, it can be easily checked that if $N \geq 3$ and f verifies

$$\limsup_{t \to 0+} \frac{f(t)}{t^p} < +\infty \tag{1.5}$$

for some $p > \frac{N}{N-2}$, then positive supersolutions in $\mathbb{R}^N \setminus B_{R_0}$ can be constructed. When $N = 2$, if f verifies

$$\limsup_{t \to +\infty} e^{at} f(t) < +\infty \tag{1.6}$$

for some $a > 0$, then a positive supersolution in $\mathbb{R}^2 \setminus B_{R_0}$ can also be constructed. It is clear that conditions (1.3) and (1.5) on one side and (1.4) and (1.6) on the other are not exhaustive, so that the question still remains: is there an optimal condition ensuring the nonexistence of positive solutions of (1.2), both when $N \geq 3$ and when $N = 2$? This is precisely the question we are addressing in this paper.

Although a more general type of supersolutions can be dealt with, we will be mainly dealing for simplicity with weak supersolutions $u \in C^1(\mathbb{R}^N \setminus B_{R_0})$, that is, functions verifying

$$\int_{\mathbb{R}^N \setminus B_{R_0}} \nabla u \nabla \phi \geq \int_{\mathbb{R}^N \setminus B_{R_0}} f(u)\phi$$

for every non-negative $\phi \in C_0^\infty(\mathbb{R}^N \setminus B_{R_0})$.

We begin by stating our results for the case $N \geq 3$.

Theorem 1.1. *Let $f : (0, \infty) \to \mathbb{R}$ be continuous and positive and assume $N \geq 3$. Then problem* (1.2) *admits a positive supersolution if and only if*

$$\int_0^\delta \frac{f(t)}{t^{\frac{2(N-1)}{N-2}}} dt < +\infty. \tag{1.7}$$

for some $\delta > 0$.

As a byproduct of our proofs it follows that, when (1.7) holds, infinitely many radially symmetric solutions can be constructed in $\mathbb{R}^N \setminus B_R$ for adequately large $R > 0$. These solutions verify

$$\lim_{x \to \infty} |x|^{N-2} u(x) = \lambda, \qquad \lim_{x \to \infty} |x|^{N-1} |\nabla u(x)| = (N-2)\lambda$$

for small $\lambda > 0$.

It is worthy of mention that Theorem 1.1 can be applied for the nonlinearity $f(t) = t^{\frac{N}{N-2}}(|\log t| + 1)^\beta$ to obtain a nonexistence result when $\beta \geq -1$. Previous results in the literature could not deal with this nonlinearity when $\beta < 0$.

Also, let us remark that, if f is nondecreasing in $(0, \delta)$ and (1.7) holds, then the equation $-\Delta u = f(u)$ admits supersolutions in $\mathbb{R}^N$ (*cf.* Remark 4.1 in Section 4). Thus, roughly speaking, existence of supersolutions in $\mathbb{R}^N$ is equivalent to existence of radially symmetric solutions in exterior domains.

Next, let us turn to the case $N = 2$.

Theorem 1.2. *Let $f : (0, \infty) \to \mathbb{R}$ be continuous and positive and assume $N = 2$. Then problem* (1.2) *admits a positive supersolution if and only if there exist $M, a > 0$ such that*

$$\int_M^\infty e^{at} f(t)dt < +\infty. \tag{1.8}$$

By means of a counterexample, it can be shown that the nonexistence condition in Theorem 1.2,

$$\int_M^\infty e^{at} f(t)dt = +\infty$$

for every $a, M > 0$, is weaker than (1.4) (*cf.* Remark 4.2 in Section 4).

Let us make a brief comment on the methods we use. Our proofs are based on a reduction of problem (1.2) to a radial setting: assuming a positive supersolution of (1.2) exists, we show that there exists a radially symmetric positive solution in an exterior domain. With a change of variables which involves the fundamental solution of the Laplacian, the problem is transformed into a sort of initial value problem for a one-dimensional equation which has a singularity. Then the important point is to show existence and nonexistence of solutions for this problem under suitable conditions. We stress that the one-dimensional problem so obtained is not integrable, since it is nonautonomous. Moreover, it has a singularity, hence to obtain existence and nonexistence of solutions is not a trivial task.

This method of proof has the advantage to be easily adapted to more general elliptic, radially symmetric operators. We consider some possible generalizations, which include the Laplacian with a gradient term, the p-Laplacian and fully nonlinear uniformly elliptic operators, in Section 5. Just to give a flavor of what is to be expected there, consider the inequality

$$-F\big(D^2u\big) \geq f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{1.9}$$

where F is a fully nonlinear operator which is positively homogeneous, radially symmetric and uniformly elliptic (*cf.* Section 5 for precise assumptions). This operator has two radially symmetric fundamental solutions $\phi^\pm(x)$, with either $\phi^\pm(x) = \pm|x|^{-\alpha}$ if $\alpha > 0$ and $\phi^\pm(x) = \mp|x|^{-\alpha}$ if $\alpha < 0$ for some $\alpha = \alpha^\pm(F)$ or $\phi^\pm(x) = \mp \log|x|$ (if this is the case then we set $\alpha^\pm(F) = 0$). Notice that for a given operator F there are two unique numbers $\alpha^+(F)$ and $\alpha^-(F)$, and therefore only two fundamental solutions ϕ^+ and ϕ^-. For simplicity of the presentation, we will assume for the moment that $\alpha^+(F) > 0$.

Then, problem (1.9) has a positive solution if and only if

$$\int_0^\delta \frac{f(t)}{t^{\frac{2(\alpha^+(F)+1)}{\alpha^+(F)}}} dt < +\infty$$

for some $\delta > 0$ when $\alpha^-(F) > 0$,

$$\int_M^\infty e^{at} f(t) dt < +\infty$$

for some $M, a > 0$ when $\alpha^-(F) = 0$, and

$$\int_M^\infty \frac{f(t)}{t^{\frac{2(\alpha^-(F)+1)}{\alpha^-(F)}}} dt < +\infty$$

for some $M > 0$, when $\alpha^-(F) < 0$. These conditions improve in the case of radially symmetric operators those in [4].

Just as an example, for the case of the Pucci's minimal operator $\mathcal{M}^-_{\lambda,\Lambda}$ (to be defined in Section 5) with $\Lambda > \lambda$, we have $\alpha^+(\mathcal{M}^-_{\lambda,\Lambda}) = \frac{\Lambda}{\lambda}(N-1) - 1$ and $\alpha^-(\mathcal{M}^-_{\lambda,\Lambda}) = \frac{\lambda}{\Lambda}(N-1) - 1$ and if $N \geq 2$ then $\alpha^+(\mathcal{M}^-_{\lambda,\Lambda}) > 0$, so the above mentioned results hold depending on whether $\frac{\lambda}{\Lambda}(N-1) - 1$ is positive, negative or zero. The fundamental solutions were introduced in the case of extremal Pucci's operators in [18,27,28]. For other more general radially symmetric operators see [20] and for the general case see [6].

The rest of the paper is organized as follows: in Section 2, we will collect some preliminary properties on problem (1.2) and on its radial solutions. Section 3 is devoted to analyze some existence and nonexistence results for some special nonautonomous, singular one-dimensional problems, and in Section 4 the proofs of Theorems 1.1 and 1.2 are performed. In Section 5 we consider some more general operators, and finally the Appendix deals with some remarks on the classical method of sub- and supersolutions.

2. Preliminaries

In this section, we consider some preliminary properties which deal mainly with the reduction of problem (1.2) to a radial one and with properties of radial solutions. For a non-negative weak superharmonic function $u \in C^1(\mathbb{R}^N \setminus B_{R_0})$ (that is, verifying $-\Delta u \geq 0$ in $\mathbb{R}^N \setminus B_{R_0}$ in the weak sense), we begin by considering the function

$$m(R) = \inf_{|x|=R} u(x) \tag{2.1}$$

for $R > R_0$. When u is nontrivial, it follows from the maximum principle that m is strictly positive. The main property we need with regard to this function concerns its monotonicity. Next lemma is a slight variant of [1, Lemma 1]. We include a short sketch of the proof for the reader's convenience.

Lemma 2.1. *Let $u \in C^1(\mathbb{R}^N \setminus B_{R_0})$ be a weak superharmonic function in $\mathbb{R}^N \setminus B_{R_0}$. Then there exists $\overline{R} > R_0$ such that $m(R)$ is monotone for $R > \overline{R}$.*

Sketch of the proof. Choose $R_2 > R_1$ and apply the maximum principle in the annulus $A(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$. We obtain that $\inf_{A(R_1,R_2)} u = \min\{m(R_1), m(R_2)\}$, thus the function $\min\{m(R_1), m(R_2)\}$ is increasing in R_1 and decreasing in R_2. It follows that $m(R)$ cannot have a local minimum, hence the lemma. □

The next step is to show that problem (1.2) can be reduced to a radially symmetric one. This is the key in our analysis of problem (1.2).

Lemma 2.2. *Assume f is continuous and positive in $(0, +\infty)$ and there exists a positive weak solution of*

$$-\Delta u \geq f(u) \qquad \text{in } \mathbb{R}^N \setminus B_{R_0}.$$

Then for some $R_1 > R_0$ there exists a C^1, positive, radially symmetric function $\bar{u}$ verifying

$$-\Delta \bar{u} = f(\bar{u}) \qquad \text{in } \mathbb{R}^N \setminus B_{R_1}$$

in the weak sense. Moreover, if $\bar{u}(x) = v(|x|)$, then $v \in C^2(R_1, +\infty)$, and there exists $\overline{R} \geq R_1$ such that v is monotone for $r > \overline{R}$.

Proof. Let $m(R)$ be given by (2.1). According to Lemma 2.1, there exists $R_1 > R_0$ such that $m(R)$ is monotone for $R > R_1$.

For $R_2 > R_1$ denote $A(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ and consider the problem

$$\begin{cases} -\Delta v = f(v) & \text{in } A(R_1, R_2) \\ v = m(R_1) & \text{on } |x| = R_1 \\ v = m(R_2) & \text{on } |x| = R_2. \end{cases} \tag{2.2}$$

Now let $\underline{v}(x) = m(R_2)$ if $m(R)$ is nonincreasing, and $\underline{v} = m(R_1)$ if $m(R)$ is nondecreasing. In either case, $\underline{v}$ is a subsolution and $u \geq \underline{v}$ in $A(R_1, R_2)$. Thus by Corollary A.2 in the Appendix we obtain a radially symmetric solution v_{R_2} of (2.2) verifying $\underline{v} \leq v_{R_2} \leq u$ in $A(R_1, R_2)$.

Next, observe that the inequality $v_{R_2} \leq u$ gives local bounds for the family $\{v_{R_2}\}_{R_2 > R_1}$. Thus it is standard to obtain local $C^{1,\alpha}$ bounds in $\mathbb{R}^N \setminus B_{R_1}$ for every $\alpha \in (0, 1)$ (notice that these bounds hold up to the boundary of ∂B_{R_1}), hence by a diagonal argument we get a sequence $R_{2,n} \to \infty$ such that $v_{R_{2,n}} \to \bar{u}$ in $C^1(\mathbb{R}^N \setminus B_{R_1})$ for some $\bar{u} \in C^1(\mathbb{R}^N \setminus B_{R_1})$. Passing to the limit in (2.2) we see that $\bar{u}$ is a non-negative radial weak solution to the equation $-\Delta \bar{u} = f(\bar{u})$ in $\mathbb{R}^N \setminus B_{R_1}$ with $\bar{u} \leq u$. Since the convergence is uniform up to the boundary of ∂B_{R_1}, $\bar{u}$ verifies $\bar{u} = m(R_1)$ on $|x| = R_1$, hence $\bar{u}$ is nontrivial and according to the strong maximum principle, $\bar{u}$ is strictly positive.

Finally, if $\bar{u}(x) = v(|x|)$, then $v \in C^1(R_1, +\infty)$, and it is a weak solution of $-v'' - \frac{N-1}{r}v' = f(v)$ for $r > R_1$. It follows that $v \in C^2(R_1, +\infty)$, and the existence of $\overline{R}$ such that v is monotone for $r > \overline{R}$ is a consequence of Lemma 2.1. The proof is concluded. □

Since most of our work from now on will be based on the radial solution v, we finally consider some of its more relevant properties. It turns out that these properties depend on whether $N = 2$ or $N \geq 3$ (see also a generalization in Lemma 5.5 of Section 5).

Lemma 2.3. *Assume f is continuous and positive in* $(0, +\infty)$. *Let $v \in C^2(\overline{R}, +\infty)$ be a positive monotone solution of*

$$-v'' - \frac{N-1}{r}v' = f(v) \qquad \text{in } r > \overline{R}. \tag{2.3}$$

We have:

(a) *If $N = 2$, then $v'(r) > 0$ and* $\lim_{r\to+\infty} v(r) = +\infty$.
(b) *When $N \geq 3$, $v'(r) < 0$ and* $\lim_{r\to+\infty} v(r) = 0$.

Proof. First of all, let us prove that v verifies $v' > 0$ and $\lim_{r\to+\infty} v(r) = +\infty$ or $v' < 0$ and $\lim_{r\to+\infty} v(r) = 0$. To see this, observe that, when v is unbounded, we have by monotonicity $\lim_{r\to+\infty} v(r) = +\infty$ and v has to be nondecreasing. Thus assume in what follows that v is bounded and let $l = \lim_{r\to+\infty} v(r)$. Our intention is to show that $l = 0$, hence we will assume $l > 0$ and we will reach a contradiction. Take $\delta < l$ and for large r introduce the function

$$H(r) = \frac{v'(r)^2}{2} + F(v(r)),$$

where $F(t) = \int_\delta^t f(s)ds$. Since $v(r) > \delta$ for large r, $F(v(r))$ is well-defined and positive. Moreover, it is well known that H is nonincreasing. Thus H is positive for large r and has a non-negative limit. Since $F(v(r)) \to F(l)$, it follows that $v'(r)$ also has a limit, which has to be zero. Passing to the limit in (2.3) we obtain that $\lim_{r\to+\infty} v''(r) = -f(l)$, this implying that $f(l) = 0$, contradicting the facts that $l > 0$ and f is positive in $(0, +\infty)$. Thus $l = 0$, as we wanted to show.

Moreover, by monotonicity, either $v' \geq 0$ or $v' \leq 0$ in $(\overline{R}, +\infty)$, and if we had $v' = 0$ at some point, then v' would attain either a minimum or a maximum, hence $v'' = 0$ at that point, which is not possible due to $f(v) > 0$. Hence $v' > 0$ in the first case and $v' < 0$ in the second.

To summarize, either $v' > 0$ and v tends to infinity or $v' < 0$ and v goes to zero. Let us see that the first case arises precisely when $N = 2$ and the second one when $N \geq 3$.

Consider first $N = 2$ and assume $v' < 0$ in $(\overline{R}, +\infty)$. Then since $rv'(r)$ is nonincreasing, we have for an arbitrary $r_0 > \overline{R}$, $v'(r) \leq \frac{r_0 v'(r_0)}{r}$ if $r > r_0$, so that

$v(r) \le v(r_0) + r_0 v'(r_0)\log(r/r_0)$ in $(r_0, +\infty)$, and $v(r) \to -\infty$ as $r \to +\infty$ follows, which is not possible. Hence $v' > 0$ in $(\overline{R}, +\infty)$ and $\lim_{r\to+\infty} v(r) = +\infty$. This shows part (a).

To prove part (b), let $N \ge 3$. Taking into account that $r^{N-1}v'(r)$ is decreasing, we have

$$v'(r) \le \left(\frac{\overline{R}}{r}\right)^{N-1} v'(\overline{R})$$

for $r > \overline{R}$, and an integration gives

$$v(r) \le v(\overline{R}) + \frac{\overline{R}^{N-1} v'(\overline{R})}{N-2}\left(\frac{1}{\overline{R}^{N-2}} - \frac{1}{r^{N-2}}\right)$$

for $r > \overline{R}$. Thus v is bounded, so that $v' < 0$ and $\lim_{r\to+\infty} v(r) = 0$. This concludes the proof. $\square$

3. One-dimensional results

In this section, we deal with existence and nonexistence of positive solutions of some one-dimensional problems, which are the core of our proofs of Theorems 1.1 and 1.2. In this generality, we believe our results are new and they can be of independent interest. Let us consider first the problem

$$\begin{cases} -w'' = s^{-\gamma} f(w) & \text{in } (0, s_0) \\ w(0) = 0 \end{cases} \tag{3.1}$$

where $\gamma > 2$ and f is a given continuous function. We will be dealing with solutions $w \in C^2(0, s_0) \cap C[0, s_0)$.

Theorem 3.1. *Let $f : (0, \infty) \to \mathbb{R}$ be continuous and positive. Assume $\gamma > 2$ and*

$$\int_0^\delta \frac{f(t)}{t^\gamma} dt = +\infty \tag{3.2}$$

for some small $\delta > 0$. Then problem (3.1) *does not admit positive solutions. Conversely, if* (3.2) *does not hold, then for every $\lambda > 0$ small enough,* (3.1) *admits a positive solution $w \in C^1[0, s_0) \cap C^2(0, s_0)$ which verifies*

$$w'(0) = \lambda.$$

Proof. Assume (3.2) holds and (3.1) admits a positive solution w. First of all, notice that since $w(0) = 0$, $w'' < 0$ and $w > 0$ in $(0, s_0)$, we have $w' \ge 0$ in some small interval of the form $(0, s_1)$. Diminishing s_0 a little bit we can always assume that $w' \ge 0$ in $(0, s_0)$. Observe that if we had $w' = 0$ at some point, then w' would achieve a minimum, so that $w'' = 0$ at that point, which is not possible. Thus $w' > 0$ in $(0, s_0)$. On the other hand, the mean value theorem gives $w(s) = w'(\xi)s \geq w'(s)s$ where ξ is some point in the interval $(0, s)$. Hence

$$0 < w'(s) \leq \frac{w(s)}{s} \quad \text{in } (0, s_0). \tag{3.3}$$

Since w' is decreasing, it follows that $w(s) \geq C_0 s$ for some $C_0 > 0$ and every $s \in (0, s_0)$. Multiplying the equation in (3.1) by w' and integrating in (s, s_0) we obtain

$$w'(s)^2 \geq 2 \int_s^{s_0} \frac{f(w(t))}{w(t)^\gamma} \left(\frac{w(t)}{t}\right)^\gamma w'(t)dt \tag{3.4}$$

for every $s \in (0, s_0)$. Hence

$$w'(s)^2 \geq 2C_0^\gamma \int_s^{s_0} \frac{f(w(t))}{w(t)^\gamma} w'(t)dt = 2C_0^\gamma \int_{w(s)}^{w(s_0)} \frac{f(\tau)}{\tau^\gamma} d\tau.$$

In particular, since $w(0) = 0$, (3.2) implies $\lim_{s \to 0} w'(s) = +\infty$. Diminishing s_0 again if necessary, we can always assume that $w'(s) \geq 1$ in $(0, s_0)$.

Next, let us iterate the use of (3.4). Since $w(s)/s \geq 1$ in $(0, s_0)$, we obtain

$$w'(s)^2 \geq 2 \int_{w(s)}^{w(s_0)} \frac{f(\tau)}{\tau^\gamma} d\tau =: H(w(s)) \quad \text{in } (0, s_0).$$

Taking this inequality again in (3.4) with the aid of (3.3) we have

$$\begin{aligned} w'(s)^2 &\geq 2 \int_s^{s_0} \frac{f(w(t))}{w(t)^\gamma} H(w(t))^{\frac{\gamma}{2}} w'(t)dt = 2 \int_{w(s)}^{w(s_0)} \frac{f(\tau)}{\tau^\gamma} H(\tau)^{\frac{\gamma}{2}} d\tau \\ &= -\int_{w(s)}^{w(s_0)} H'(\tau) H(\tau)^{\frac{\gamma}{2}} d\tau = \frac{H(w(s))^{\frac{\gamma}{2}+1}}{\frac{\gamma}{2}+1}, \end{aligned}$$

where we have used $H(w(s_0)) = 0$. Iterating this procedure we obtain two sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ given by $a_1 = 1$, $a_k = \left(\frac{\gamma}{2}\right) a_{k-1} + 1$, $b_1 = 1$, $b_k = b_{k-1}^{\frac{\gamma}{2}} a_k$ such that

$$w'(s)^2 \geq \frac{H(w(s))^{a_k}}{b_k} \quad \text{in } (0, s_0) \text{ for every } k. \tag{3.5}$$

It is not hard to see that

$$a_k = \sum_{j=0}^{k-1} \left(\frac{\gamma}{2}\right)^j = \frac{\left(\frac{\gamma}{2}\right)^k - 1}{\frac{\gamma}{2} - 1},$$

and we directly obtain

$$\left(\frac{\gamma}{2}\right)^{k-1} \le a_k \le C_1 \left(\frac{\gamma}{2}\right)^{k-1} \tag{3.6}$$

for some positive constant C_1. It follows in particular from (3.6) that $b_k \le C_1 b_{k-1}^{\frac{\gamma}{2}} \left(\frac{\gamma}{2}\right)^{k-1}$ for $k \ge 1$. Iterating this inequality from $k = 1$ we obtain

$$b_k \le C_1^{\sum_{j=0}^{k-1} \left(\frac{\gamma}{2}\right)^j} \left(\frac{\gamma}{2}\right)^{\sum_{j=0}^{k-1}(k-j)\left(\frac{\gamma}{2}\right)^j}, \tag{3.7}$$

for $k \ge 1$. To calculate the sum in the last exponent, we just notice that it is an arithmetic-geometric sum, so that

$$\sum_{j=0}^{k-1}(k-j)\left(\frac{\gamma}{2}\right)^j = \frac{\left(\frac{\gamma}{2}\right)^k - 1}{\frac{\gamma}{2}-1}\left(1 + \frac{1}{\frac{\gamma}{2}-1}\right) - \frac{k}{\frac{\gamma}{2}-1}. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$b_k \le C_2^{\left(\frac{\gamma}{2}\right)^{k-1}} \tag{3.9}$$

for some $C_2 > 1$.

Now take $\delta \in (0, s_0)$ such that $H(w(s)) > 2C_2$ in $(0, \delta)$. Taking (3.6) and (3.9) into (3.5) we see that

$$w'(s)^2 \ge 2^{\left(\frac{\gamma}{2}\right)^{k-1}} \quad \text{in } (0, \delta) \text{ for every } k.$$

Letting $k \to +\infty$, we arrive at a contradiction, which shows that there are no positive solutions of (3.1) when condition (3.2) holds.

Next, assume that (3.2) does not hold. Fix $\lambda > 0$ and denote $z_\lambda(s) = \lambda s$. In the Banach space $X = \{z \in C^1[0, s_0] : z(0) = 0\}$ equipped with the standard C^1 norm $|z|_{C^1} = \max\{|z|_\infty, |z'|_\infty\}$, consider the set $B = \{z \in X : |z - z_\lambda|_{C^1} \le \frac{\lambda}{2}\}$, which is closed and convex. Define the operator

$$Tz(s) = \lambda s - \int_0^s \int_0^t \tau^{-\gamma} f(z(\tau))\, d\tau\, dt, \quad s \in [0, s_0].$$

We claim that T is well-defined, maps B in B and is compact. To show the first two assertions, notice that $z \le \frac{3\lambda}{2}s$, $z' \ge \frac{\lambda}{2}$ in $[0, s_0]$ for every $z \in B$, so that

$$\begin{aligned}
|(Tz)'(s) - \lambda| &\le \int_0^s \tau^{-\gamma} f(z(\tau))d\tau = \int_0^s \frac{f(z(\tau))}{z(\tau)^\gamma}\left(\frac{z(\tau)}{\tau}\right)^\gamma \frac{z'(\tau)}{z'(\tau)}d\tau \\
&\le 3^\gamma 2^{1-\gamma}\lambda^{\gamma-1}\int_0^s \frac{f(z(\tau))}{z(\tau)^\gamma}z'(\tau)d\tau \\
&= 3^\gamma 2^{1-\gamma}\lambda^{\gamma-1}\int_0^{z(s)} \frac{f(t)}{t^\gamma}dt \\
&\le 3^\gamma 2^{1-\gamma}\lambda^{\gamma-1}\int_0^{\frac{3\lambda}{2}s} \frac{f(t)}{t^\gamma}dt \le \frac{\lambda}{2}
\end{aligned} \tag{3.10}$$

in $[0, s_0]$, taking λ small enough. It also follows if λ is small that $|Tz(s) - z_\lambda| \leq \frac{\lambda}{2}$. Hence T is well defined and $T(B) \subset B$.

In addition, T is a compact operator. Indeed, let $\{w_n\}_{n=1}^{\infty}$ be an arbitrary sequence and denote $z_n = Tw_n$. By (3.10), $\{z_n'\}_{n=1}^{\infty}$ is uniformly bounded in $[0, s_0]$, so that $\{z_n\}$ is equicontinuous and uniformly bounded, hence precompact. Passing to a subsequence, which we still denote $\{z_n\}_{n=1}^{\infty}$, we have $z_n \to z$ uniformly in $[0, s_0]$, for some $z \in C[0, s_0]$. Let us prove that $z \in C^1[0, s_0]$ and $z_n' \to z'$ uniformly in $[0, s_0]$.

Observe that $|z_n''(s)| = s^{-\gamma} f(w_n(s))$ is uniformly bounded in compacts of $(0, s_0]$. Hence by means of Arzelá-Ascoli's theorem and a diagonal argument we may assume that also $z_n' \to \bar{z}$ uniformly in compacts of $(0, s_0]$ for some $\bar{z} \in C(0, s_0]$. It follows that $z \in C^1(0, s_0]$ and $\bar{z} = z'$. But the convergence is actually uniform in $[0, s_0]$ (where we define $z'(0) = \lambda$). To see this, let $\varepsilon > 0$. By (3.10):

$$\left|z_n'(s) - \lambda\right| \leq 3^\gamma 2^{1-\gamma} \lambda^{\gamma-1} \int_0^{\frac{3\lambda}{2}s} \frac{f(t)}{t^\gamma} dt,$$

and the same is true for $z(s)$ by passing to the limit. Hence

$$\left|z_n'(s) - z(s)\right| \leq 3^\gamma 2^{2-\gamma} \lambda^{\gamma-1} \int_0^{\frac{3\lambda}{2}\delta} \frac{f(t)}{t^\gamma} dt \leq \varepsilon$$

provided $s \in [0, \delta]$ for $\delta < s_0$ small enough. Since $z_n' \to z'$ uniformly in $[\delta, s_0]$, we also have $|z_n'(s) - z'(s)| \leq \varepsilon$ if n is large enough. Thus $z_n' \to z'$ uniformly in $[0, s_0]$, that is, $z_n \to z$ in X and T is a compact operator.

We finally remark that the continuity of T follows by using essentially the same argument, so that we may apply Schauder's fixed point theorem, to obtain that T has a fixed point $w \in B$, which is a solution of (3.1) verifying $w'(0) = \lambda$, $w > 0$ in $(0, s_0]$. This concludes the proof. □

Remark 3.2. It is important to stress that, for the slightly more general problem

$$\begin{cases} -w'' = a(s)f(w) & \text{in } (0, s_0) \\ w(0) = 0 \end{cases}$$

where $a(s)$ is continuous in $(0, s_0)$, solutions can be constructed as long as

$$\int_0^\delta sa(s)ds < +\infty \tag{3.11}$$

and $f(0+) := \limsup_{t \to 0} f(t) < +\infty$. This can be easily seen since for every $\lambda > 0$ the function $\underline{w}(s) = \lambda s$ is a subsolution, while

$$\overline{w}(s) = \lambda + A \int_0^s \int_t^s a(\tau) d\tau dt$$

(which is well-defined by (3.11) and Fubini's theorem) is a supersolution provided $A > f(0+)$ and s_0 is chosen small enough. In particular, problem (3.1) with $\gamma < 2$ always has solutions, irrespective of the growth of f at zero (as long as $f(0+)$ is finite). The case $\gamma = 2$ is subtler and things are slightly different (see the proof of Theorem 5.1 in Section 5).

We turn now to consider our second problem, which arises in the analysis of problem (1.2) in the case $N = 2$ (Theorem 1.2). Namely,

$$\begin{cases} -w'' = e^{2s} f(w) \text{in } (s_0, +\infty) \\ \lim\limits_{s\to+\infty} w(s) = +\infty. \end{cases} \tag{3.12}$$

The hypotheses in Theorem 3.3 below do not seem so straightforward to check as in Theorem 3.1, but they are also necessary and sufficient.

Theorem 3.3. *Let $f : (0, +\infty) \to \mathbb{R}$ be continuous and positive. Assume there exists $M > 0$ such that*

$$\int_M^\infty e^{at} f(t)dt = +\infty \quad \textit{for every } a > 0. \tag{3.13}$$

Then problem (3.1) *does not admit positive solutions. Conversely, if* (3.13) *does not hold, then for every sufficiently large positive λ,* (3.12) *admits a positive solution verifying*

$$\frac{\lambda}{2} s \le w(s) \le \frac{3\lambda}{2} s, \quad s > s_0.$$

Proof. Assume (3.12) has a positive solution w, defined in $[s_0, +\infty)$. Since $w'' < 0$, the function w' is decreasing. If we had $w'(s_1) \le 0$ for some $s_1 > s_0$, then $w' < 0$ in $(s_1, +\infty)$. Taking $s_2 > s_1$ we would have $w(s) \le w(s_2) + w'(s_2)(s - s_2)$ for $s > s_2$, thus implying $w(s) \to -\infty$ as $s \to +\infty$, which is against the assumption that w is positive in $[s_0, +\infty)$. Thus $w' > 0$ in $(s_0, +\infty)$. It also follows that $w(s) \le Cs$ in $[s_0, +\infty)$ for some positive constant C. Multiplying the equation by w' and integrating in (s_0, s), for some $s > s_0$, we arrive at

$$\begin{aligned} w'(s_0)^2 &\ge 2\int_{s_0}^{s} e^{2t} f(w(t))w'(t)dt \ge 2\int_{s_0}^{s} e^{\frac{2}{C}w(t)} f(w(t))w'(t)dt \\ &\ge 2\int_{w(s_0)}^{w(s)} e^{\frac{2}{C}\tau} f(\tau)d\tau, \end{aligned}$$

and letting $s \to +\infty$ we obtain a contradiction with hypothesis (3.13). Thus no positive solutions of (3.12) can exist.

The second part of the proof is similar to that of Theorem 3.1.

Assume $\int_M^\infty e^{at} f(t)dt < \infty$ for some $M, a > 0$. Consider the Banach space $\widetilde{X} = \{z \in C^1[s_0, +\infty) : \|z\| < +\infty\}$, where

$$\|z\| = \max\left\{\sup_{[s_0,+\infty)} \frac{|z(s)|}{s}, \sup_{[s_0,+\infty)} \left|z'(s)\right|\right\},$$

and the set $\widetilde{B} = \{z \in \widetilde{X} : \|z - z_\lambda\| \le \frac{\lambda}{2}\}$, where $z_\lambda(s) = \lambda s$ and $\lambda > 0$ is fixed. Define the operator

$$Tz(s) = \lambda s - \int_{s_0}^s \int_{s_0}^t e^{2\tau} f(z(\tau))d\tau dt, \quad s \in [s_0, +\infty).$$

Let us prove that T is well-defined, maps $\widetilde{B}$ into $\widetilde{B}$ and is compact, provided we choose λ large enough. Observe that $z(s) \ge \frac{\lambda}{2}s$, $z'(s) \ge \frac{\lambda}{2}$ in $[s_0, +\infty)$ for every $z \in \widetilde{B}$. Then if $\lambda \ge \frac{4}{a}$:

$$\begin{aligned} |(Tz)'(s) - \lambda| &= \int_{s_0}^s e^{2\tau} f(z(\tau))\frac{z'(\tau)}{z'(\tau)}d\tau \le \frac{2}{\lambda}\int_{s_0}^s e^{\frac{4}{\lambda}z(\tau)} f(z(\tau))z'(\tau)d\tau \\ &= \frac{2}{\lambda}\int_{z(s_0)}^{z(s)} e^{\frac{4}{\lambda}t} f(t)dt \le \frac{2}{\lambda}\int_{z(s_0)}^{z(s)} e^{at} f(t)dt \\ &\le \frac{2}{\lambda}\int_{\frac{\lambda}{2}s_0}^{\infty} e^{at} f(t)dt \le \frac{\lambda}{2}, \end{aligned} \tag{3.14}$$

if λ is chosen large enough. Moreover:

$$\left|\frac{Tz(s)}{s} - \lambda\right| \le \frac{1}{s}\int_{s_0}^s \int_{s_0}^t e^{2\tau} f(z(\tau))d\tau dt \le \frac{\lambda}{2}\frac{s - s_0}{s} \le \frac{\lambda}{2},$$

for every $s > s_0$, hence T is well-defined and $T(\widetilde{B}) \subset \widetilde{B}$.

To see that T is compact, take a sequence $\{z_n\}_{n=1}^\infty \subset \widetilde{B}$ and let $w_n = Tz_n$. Notice that, by Arzelá-Ascoli's theorem and a diagonal argument, we may assume that $z_n \to z$ uniformly in compacts of $[s_0, +\infty)$, where $z \in C[s_0, \infty)$. It follows easily that $w_n' \to \bar{w}$ uniformly in compacts of $[s_0, +\infty)$, where

$$\bar{w}(s) = \lambda - \int_{s_0}^s e^{2\tau} f(z(\tau))d\tau, \quad s \ge s_0.$$

But this convergence is also uniform in $[s_0, +\infty)$. Indeed, observe that, if we fix $s_1 > s_0$ take $s > s_1$, and argue as in (3.14) we arrive at

$$|w_n'(s) - w_n'(s_1)| = \int_{s_1}^s e^{2\tau} f(z_n(\tau))d\tau \le \frac{2}{\lambda}\int_{\frac{\lambda}{2}s_1}^{z_n(s)} e^{at} f(t)dt,$$

and a similar equality holds with w_n and z_n replaced by $\bar{w}$ and z, by passing to the limit. Hence

$$\left|w_n'(s) - \bar{w}(s)\right| \le \left|w_n'(s_1) - \bar{w}(s_1)\right| + \frac{4}{\lambda}\int_{\frac{\lambda}{2}s_1}^{\infty} e^{at} f(t)dt$$

for every $s > s_1$. Next take $\varepsilon > 0$. Choosing s_1 large enough we have the last integral less than $\frac{\varepsilon}{2}$. Taking n large enough we also have $|w_n'(s_1) - \bar{w}(s_1)| \le \frac{\varepsilon}{2}$, hence $|w_n'(s) - \bar{w}(s)| \le \varepsilon$ if $s > s_1$. Since this inequality also holds in $[s_0, s_1]$ for large enough n, we obtain that $w_n' \to \bar{w}$ uniformly in $[s_0, +\infty)$.

Let $w(s) = \lambda + \int_{s_0}^{s} \bar{w}(t)dt$. Then $w_n' \to w'$ uniformly in $[s_0, +\infty)$. Hence

$$\frac{|w_n(s) - w(s)|}{s} \le \frac{1}{s}\int_{s_0}^{s} |w_n'(t) - w(t)|dt \to 0$$

uniformly in $[s_0, +\infty)$, as $n \to +\infty$. This shows that T is compact in $\widetilde{B}$. The continuity of T is shown in a similar manner.

Thus we may apply Schauder's fixed point theorem to obtain that T has a fixed point w in $\widetilde{B}$, which is a solution of (3.12) verifying $\frac{\lambda}{2}s \le w(s) \le \frac{3\lambda}{2}s$, hence it is positive in $[s_0, +\infty)$. This concludes the proof. □

Remark 3.4. It can be shown by entirely similar methods that the problem

$$\begin{cases} -w'' = s^{\gamma} f(w) \text{ in } (s_0, +\infty) \\ \lim\limits_{s \to +\infty} w(s) = +\infty, \end{cases} \tag{3.15}$$

with $\gamma > 0$, admits a positive solution if and only if

$$\int_M^{\infty} t^{\gamma} f(t)dt < \infty$$

for some $M > 0$.

4. Proof of Theorems 1.1 and 1.2

In the present section, we are performing the proof of our main theorems. As we have already remarked in the Introduction, the equation (1.2) is first reduced to a radial one, which is turned into a one-dimensional problem, so that the results in Section 3 can be used.

Proof of Theorem 1.1. Assume there exists a supersolution of $-\Delta u = f(u)$ in $\mathbb{R}^N \setminus B_{R_0}$ and condition (1.7) does not hold. According to Lemma 2.2, there exists a radial solution $\bar{u}$ of the same equation. Setting $\bar{u}(x) = v(r)$, with $r = |x|$, we obtain that v is monotone for r greater than some $\overline{R} > 0$ and

$$-v'' - \frac{N-1}{r} v' = f(v) \quad \text{in } (\overline{R}, +\infty) \tag{4.1}$$

Lemma 2.3 also gives $v' < 0$, $\lim_{r\to+\infty} v(r) = 0$.

We next introduce the change of variables $s = \frac{1}{(N-2)r^{N-2}}$, $w(s) = v(r)$. Then it is not hard to see that w is a positive solution of:

$$\begin{cases} -w'' = as^{-\gamma} f(w) & \text{in } (0, s_0) \\ w(0) = 0, \end{cases} \tag{4.2}$$

for some $a > 0$ and some small positive s_0, where $\gamma = \frac{2(N-1)}{N-2} > 2$. We obtain an immediate contradiction with Theorem 3.1. Thus no positive supersolutions can exist in this case.

Reciprocally, assume condition (1.7) holds. By Theorem 3.1, for every positive, small enough λ, there exists a positive solution w of (4.2) with $w'(0) = \lambda$. This solution gives rise to a solution v of (4.1) verifying $r^{N-1}v'(r) \to -\lambda$ as $r \to +\infty$, thus $(N-2)r^{N-2}v(r) \to \lambda$ as $r \to +\infty$. Hence there exist infinitely many positive radially symmetric solutions of the equation $-\Delta u = f(u)$ in $\mathbb{R}^N \setminus B_{R_0}$, as was to be seen. This concludes the proof. $\square$

Remark 4.1. When f is nondecreasing in $(0, \delta)$ for some small δ, it can also be shown that condition (1.7) implies the existence of a positive supersolution in $\mathbb{R}^N$. Indeed, if f is nondecreasing for small values then (1.7) implies $\lim_{t\to0+} t^{1-\gamma} f(t) \to 0$, where $\gamma = \frac{2(N-1)}{N-2}$. In particular, the solution of (4.2) verifies $s|w''(s)| = s^{1-\gamma} f(w(s)) \le Cw(s)^{1-\gamma} f(w(s)) \to 0$. Thus $(N-2)sw'' + (N-1)w' > 0$ for small s (since $w'(0) = \lambda > 0$). This implies in turn that $v''(r) \ge 0$ for $r \ge R_0$ and R_0 suitable large.

Now let $z(r) = v\left(\sqrt{r^2 + R_0^2}\right)$, $r \ge 0$. It is easily checked that $z'(0) = 0$, while

$$\begin{aligned} -z'' - \frac{N-1}{r} z' &= -v'' \frac{r^2}{r^2 + R_0^2} - v' \frac{R_0^2}{\left(r^2 + R_0^2\right)^{\frac{3}{2}}} - (N-1) \frac{v'}{\sqrt{r^2 + R_0^2}} \\ &\ge -v'' - (N-1) \frac{v'}{\sqrt{r^2 + R_0^2}} \ge f(v) = f(z) \end{aligned}$$

(observe that v is evaluated at $\sqrt{r^2 + R_0^2}$). Then $z(|x|)$ is a supersolution of the equation in $\mathbb{R}^N$.

The proof of our second theorem is much the same as that of Theorem 1.1. The only major difference is that Theorem 3.3 is used instead of Theorem 3.1.

Proof of Theorem 1.2. Assume $N = 2$, condition (1.8) does not hold, and there exists a positive supersolution of (1.2). According to Lemmas 2.2 and 2.3, there exists a solution v of the equation

$$-v'' - \frac{1}{r}v' = f(v) \quad \text{in } (\overline{R}, +\infty)$$

verifying $v' > 0$, $\lim_{r\to+\infty} v(r) = +\infty$. The change of variables $s = \log r$, $v(r) = w(s)$ leads to the problem

$$\begin{cases} -w'' = e^{2s} f(w) & \text{in } (s_0, +\infty) \\ w(+\infty) = +\infty, \end{cases} \tag{4.3}$$

and we arrive at a contradiction with Theorem 3.3.

Conversely, if condition (1.8) holds, then by Theorem 3.3 problem (4.3) admits infinitely many positive solutions. Thus problem (1.2) admits infinitely many positive radial solutions. It is worth mentioning that these radial solutions can never be extended to be supersolutions in $\mathbb{R}^2$, due to the Liouville property. □

Remark 4.2. It is interesting to remark that the contrary of condition (1.8) is weaker than the condition found in [4], namely $\liminf_{s\to+\infty} e^{as} f(s) > 0$ for every $a > 0$. Let us see it by means of an example.

Choose $\varphi \in C_0(-1, 1)$ with $0 < \varphi \leq 1$, $\varphi(0) = 1$. Define the sequence $\{b_n\}_{n=1}^{\infty}$ by $b_n = \frac{1}{n}$ if n is even and $b_n = 1$ if n is odd, and let

$$h(s) = \sum_{n=2}^{\infty} b_n \varphi(2(s-n)),$$

and $f(s) = \exp(-sh(s))$, $s > 1$. Observe that $h(n) = b_n$ for $n \in \mathbb{N}$, so that, if $a \in (0, 1)$, we have

$$\lim_{k\to+\infty} e^{a(2k+1)} f(2k+1) = \lim_{k\to+\infty} e^{(a-1)(2k+1)} = 0,$$

so that the condition in [4] does not hold. Let us show that $\int_1^{\infty} e^{as} f(s)ds = +\infty$ for every $a > 0$. Indeed, take an arbitrary $a > 0$. Then

$$\begin{aligned} \int_1^{\infty} e^{as} f(s)ds &= \int_1^{\infty} e^{\left(a - \sum_{n=2}^{\infty} b_n \varphi(2(s-n))\right)s} ds \\ &\geq \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} e^{(a - b_n \varphi(2(s-n)))s} ds \geq \sum_{n=2}^{\infty} \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} e^{(a-b_n)s} ds \\ &\geq \sum_{k=k_0}^{\infty} \int_{2k-\frac{1}{2}}^{2k+\frac{1}{2}} e^{(a-b_{2k})s} ds \end{aligned}$$

for every integer $k_0 > 1$. Choosing k_0 large enough so that $b_{2k} < \frac{a}{2}$ if $k \geq k_0$, we obtain

$$\int_1^{\infty} e^{as} f(s)ds \geq \sum_{k=k_0}^{\infty} e^{\frac{a}{2}\left(2k-\frac{1}{2}\right)} = +\infty$$

as was to be shown.

5. Some more general problems

Our main results can be extended to some more general equations with similar proofs. We consider just a few possibilities and give only the main points in the proofs.

Problems with weights

Problem (1.2) can also be dealt with when a weight function of power type appears in front of the nonlinearity, namely:

$$-\Delta u \geq \lambda |x|^{\alpha} f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}, \tag{5.1}$$

where $\alpha \in \mathbb{R}$ and f is, as before, continuous and positive in $(0, +\infty)$. The appearance of the parameter $\lambda \in \mathbb{R}$ in (5.1) will become clear in the statement corresponding to the "critical" case $\alpha = -2$. For simplicity, we only state the case $N \geq 3$, but the case $N = 2$ could be easily treated as well.

Theorem 5.1. *Assume $N \geq 3$. Let $f : (0, +\infty) \to \mathbb{R}$ be continuous and positive. Then*

(a) *If $\alpha < -2$ and $\limsup_{t\to 0+} f(t) < +\infty$, problem (5.1) has a positive supersolution.*
(b) *If $\alpha = -2$ and f is nondecreasing in $(0, \delta]$ for some small δ, we have:*

- *When $\limsup_{t\to 0+} \frac{f(t)}{t} = +\infty$, problem (5.1) does not have positive supersolutions for any $\lambda > 0$.*
- *If $0 < \limsup_{t\to 0+} \frac{f(t)}{t} < +\infty$, there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ problem (5.1) admits a positive supersolution, while no positive supersolutions exist for $\lambda > \lambda_0$.*
- *If $\lim_{t\to 0+} \frac{f(t)}{t} = 0$, then there exist supersolutions of (5.1) for every $\lambda > 0$.*

(c) *If $\alpha > -2$, problem (5.1) has a positive supersolution if and only if*

$$\int_0^{\delta} \frac{f(t)}{t^{\frac{\alpha+2(N-1)}{N-2}}} dt < \infty \tag{5.2}$$

for some $\delta > 0$.

Sketch of proof. Assume there exists a positive supersolution u. Then there exists a radial solution v, which is decreasing and vanishes at infinity. With the change of variables $v(r) = w(s)$, where $s = \frac{1}{(N-2)r^{N-2}}$, we obtain a solution of

$$\begin{cases} -w'' = s^{-\gamma} f(w) & \text{in } (0, s_0) \\ w(0) = 0 \end{cases} \tag{5.3}$$

with $\gamma = \frac{\alpha+2(N-1)}{N-2}$ and s_0 sufficiently small. When $\alpha > -2$, it follows that $\gamma > 2$, so that solutions of (5.3) exist if and only if (5.2) holds, by Theorem 3.1. If $\alpha < -2$, we obtain $\gamma < 2$, and problem (5.3) always has a positive solution, irrespective of f, provided $\limsup_{t\to 0+} f(t) < +\infty$ (*cf.* Remark 3.2 in Section 3).

Thus only the case $\alpha = -2$, that is, $\gamma = 2$, remains to be proved. With regard to existence, assume $\limsup_{t\to 0+} \frac{f(t)}{t} < +\infty$. Then there exists $K > 0$ such that $f(t) \le Kt$ if t is small enough. The function $\bar{w}(s) = s^\alpha$ is a supersolution of (5.3) when $\lambda \le 1/4K$, provided α is chosen to satisfy $\alpha(1-\alpha) = \lambda K$, since

$$-\bar{w}''(s) = \alpha(1-\alpha)s^{-2}\bar{w}(s) = \lambda K s^{-2}\bar{w}(s) \ge s^{-2} f(\bar{w}(s)).$$

If $\lim_{t\to 0+} \frac{f(t)}{t} = 0$, for every $\lambda > 0$ we can take δ such that $\lambda\delta \le 1/4$, and since $f(t) \le \delta t$ if t is small enough, the function $\bar{w}$ will be again a supersolution.

Now, for the nonexistence results, observe that (*cf.* (3.3) in the proof of Theorem 3.1) $w(s)/s$ is decreasing, hence $r^{N-2}v(r)$ is increasing. The rest of the proof does not take into account any more problem (5.3) and deals only with the radial solution $v(r)$. It is essentially different from the previous ones, and relies in an argument borrowed from [18] (see also [21]).

Choose a test function $\phi \in C_0^\infty(1, 4)$ with $0 \le \phi \le 1$ and $\phi = 1$ in $(2, 3)$. For $R > R_0$ introduce the function

$$z(r) = v(r) - v(2R)\phi\left(\frac{r}{R}\right), \quad r > R_0.$$

Observe that $z = v > 0$ if $R_0 < r < R$ or $r > 4R$, and $z(2R) = 0$. In particular, z achieves a nonpositive minimum at some point $R < r_0 < 4R$. Hence $z'(r_0) = 0$, $z''(r_0) \ge 0$. This implies

$$\begin{aligned} \lambda r_0^{-2} f(v(r_0)) &= -v''(r_0) - \frac{N-1}{r_0} v'(r_0) \\ &\le \frac{v(2R)}{R^2}\left(-\phi''\left(\frac{r_0}{R}\right) - \frac{(N-1)R}{r_0}\phi'\left(\frac{r_0}{R}\right)\right) \\ &\le C\frac{v(2R)}{R^2}, \end{aligned}$$

where C is a constant which does not depend on R nor on v. Since v is decreasing for large r, $R < r_0 < 4R$ and f is nondecreasing for small values, we obtain

$\lambda f(v(4R)) \leq Cv(2R)$ for large R. By the monotonicity of $R^{N-2}v(R)$, we also have $v(2R) \leq 2^{N-1}v(4R)$, hence, after replacing $2R$ by r, we obtain for all sufficiently large r:

$$\lambda \frac{f(v(r))}{v(r)} \leq C,$$

where C does not depend on r nor on v. Letting $r \to \infty$, we arrive at a contradiction with existence if $\limsup_{t\to 0+} \frac{f(t)}{t} = +\infty$. Otherwise we obtain

$$\lambda \limsup_{t\to 0+} \frac{f(t)}{t} \leq C,$$

and when the lim sup is positive, there are no positive solutions when λ is large enough. Since there are supersolutions for small λ we can define λ_0 to be the supremum of the values of λ for which there exists a positive supersolution. It clearly follows that there exist positive supersolutions for $\lambda < \lambda_0$, while there are no supersolutions for $\lambda > \lambda_0$. □

Gradient terms

When the Laplacian is perturbed with a gradient term – but keeping the homogeneity – a similar analysis can be performed. As an example, consider the problem:

$$-\Delta u + |\nabla u| \geq \lambda f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{5.4}$$

where f is as before. This problem was analyzed in [2] in the special case $f(t) = t^p$, $p > 0$. Our results for problem (5.4) are as follows:

Theorem 5.2. *Assume $N \geq 2$. Let $f : [0, +\infty) \to \mathbb{R}$ be continuous and positive in $(0, +\infty)$, and there exists $\delta > 0$ such that f is nondecreasing in $(0, \delta)$. Then*

(a) *When $\limsup_{t\to 0+} \frac{f(t)}{t} = +\infty$, problem (5.4) does not have positive bounded supersolutions for any $\lambda > 0$.*
(b) *If $0 < \limsup_{t\to 0+} \frac{f(t)}{t} < +\infty$, there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ problem (5.4) admits a positive bounded supersolution, while no positive bounded supersolutions exist for $\lambda > \lambda_0$.*
(c) *If $\lim_{t\to 0+} \frac{f(t)}{t} = 0$, then there exist positive bounded supersolutions of (5.4) for every $\lambda > 0$.*

Sketch of proof. Assuming a positive bounded supersolution exists, and arguing as in Lemma 2.2, we obtain a radial solution v verifying $-(r^{N-1}v')' + r^{N-1}|v'| = r^{N-1}f(v)$, $r > \overline{R}$ and $v(+\infty) = 0$. Performing the change of variables $v(r) = w(s)$, where $s = \phi(r)$ and

$$\phi(r) = \int_r^\infty \frac{1}{s^{N-1}} e^{-s} ds$$

the radial problem reduces to

$$\begin{cases} -w'' = \dfrac{1}{\phi'(\phi^{-1}(s))^2} f(w) & \text{in } (0, s_0) \\ w(0) = 0 \end{cases} \tag{5.5}$$

for some small positive s_0. Now observe that $\phi(r) \sim r^{1-N}e^{-r}$ as $r \to \infty$ (l'Hôpital's rule), this implying that $(\phi^{-1}(s))^{1-N}e^{-\phi^{-1}(s)} \sim s$ as $s \to 0$. Thus

$$-\phi'\big(\phi^{-1}(s)\big) = \big(\phi^{-1}(s)\big)^{1-N} e^{-\phi^{-1}(s)} \sim s \quad \text{as } s \to 0.$$

Hence solutions of (5.5) also verify

$$\begin{cases} -w'' \geq \dfrac{C}{s^2} f(w) & \text{in } (0, \varepsilon) \\ w(0) = 0 \end{cases}$$

for some $C > 0$ (as close to one as we desire). The further change of variables $w(s) = z(t)$, where $t = s^{\frac{1}{2-N}}$, shows that $\bar{z}(x) = z(|t|)$ verifies

$$-\Delta \bar{z} \geq C|x|^{-2} f(\bar{z}) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

for some $R_0 > 0$ large enough. Thus Theorem 5.1 can be used to give the nonexistence results in parts (a) and (b). The existence results are exactly as in the proof of Theorem 5.1. □

Remarks 5.3. (a) Let us mention in passing that the Liouville property is not valid in $\mathbb{R}^2$ for this operator. That is, there exist positive functions verifying $-\Delta u + |\nabla u| \geq 0$ in $\mathbb{R}^2$.

(b) Solutions which are not bounded can also be dealt with, following a slightly different approach. In this case, the radial solution v verifies $v(+\infty) = +\infty$, and the change of variables $v(r) = w(s)$ with $s = \widetilde{\phi}(r)$ and

$$\widetilde{\phi}(r) = \int_1^r \frac{e^s}{s^{N-1}} ds$$

followed by $w(s) = z(t)$, with $t = s^{\frac{1}{2-N}}$, leads to the problem

$$-\Delta \bar{z} \geq C|x|^{-2} f(\bar{z}) \quad \text{in } B_{R_0} \setminus \{0\},$$

where $\bar{z}(x) = z(|x|)$. The proof concludes essentially as that of Theorem 5.1.

(c) The related problem with a minus sign in front of the gradient can also be analyzed in a similar way.

Uniformly elliptic fully nonlinear operators

The linearity of the Laplacian is not an essential point in any of our proofs. Therefore, it is to be expected that they can be extended to cover problems containing nonlinear operators. Let us consider the fully nonlinear equation

$$-F\left(D^2u\right) \geq f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0} \tag{5.6}$$

where $F : \mathcal{S}_N \to \mathbb{R}$ is continuous and positively homogeneous of degree one. Here, $\mathcal{S}_N$ denotes the set of all symmetric $N \times N$ real matrices. On one hand, we assume that F satisfies the uniform ellipticity condition

$$\mathcal{M}^-_{\lambda,\Lambda}(P) \leq F(M+P) - F(M) \leq \mathcal{M}^+_{\lambda,\Lambda}(P) \tag{5.7}$$

for all $M, P \in \mathcal{S}_N$, and some $\Lambda \geq \lambda > 0$, where $\mathcal{M}^\pm_{\lambda,\Lambda}$ are the extremal Pucci's operators, given by

$$\mathcal{M}^+_{\lambda,\Lambda}(P) = \lambda \sum_{\substack{\mu\in\sigma(P)\\ \mu<0}} \mu + \Lambda \sum_{\substack{\mu\in\sigma(P)\\ \mu>0}} \mu$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(P) = \Lambda \sum_{\substack{\mu\in\sigma(P)\\ \mu<0}} \mu + \lambda \sum_{\substack{\mu\in\sigma(P)\\ \mu>0}} \mu$$

respectively, and $\sigma(P)$ denotes the spectrum of the matrix P. On the other hand, we also assume that F has radial symmetry as in [19], that is,

$$\begin{cases} F\left(\dfrac{p}{r}I + \left(m - \dfrac{p}{r}\right)\dfrac{x\otimes x}{r^2}\right) \text{ depends on } x \\ \text{only through } |x| \text{ for } m, p \in \mathbb{R}, \end{cases} \tag{5.8}$$

where, as usual, $r = |x|$ and $\otimes$ denotes tensorial product. It is known, by the results of [6], that F has two radially symmetric fundamental solutions $\phi^\pm(x)$, with either $\phi^\pm(x) = \pm|x|^{-\alpha}$ if $\alpha > 0$ and $\phi^\pm(x) = \mp|x|^{-\alpha}$ if $\alpha < 0$ for some $\alpha = \alpha^\pm(F)$ or $\phi^\pm(x) = \mp\log|x|$ (if this is the case then we set $\alpha^\pm(F) = 0$). Notice that for a given operator F there are two unique numbers $a^+(F)$ and $\alpha^-(F)$, and therefore only two fundamental solutions ϕ^+ and ϕ^-.

It is to be noted that $\alpha^\pm(F) > -1$ always holds (see [6]). Then a similar procedure as before gives us the next result:

Theorem 5.4. *Assume $F : \mathcal{S}_{\mathcal{N}} \to \mathbb{R}$ is continuous, positively homogenous of degree one, and verifies* (5.7) *and* (5.8). *Then*

I. *Assume $\alpha^+(F) > 0$.*

(a) *If $\alpha^-(F) > 0$, problem* (5.6) *admits a positive supersolution if and only if*

$$\int_0^\delta \frac{f(t)}{t^{\frac{2(\alpha^+(F)+1)}{\alpha(F)}}} dt < +\infty$$

for some $\delta > 0$.

(b) *If $\alpha^-(F) = 0$, problem* (5.6) *admits a positive supersolution if and only if*

$$\int_M^\infty e^{at} f(t) dt < +\infty$$

for some $M, a > 0$.

(c) *If $\alpha^-(F) < 0$, problem* (5.6) *admits a positive supersolution if and only if*

$$\int_M^\infty t^{-\frac{2(\alpha^-(F)+1)}{\alpha^-(F)}} f(t) dt < +\infty$$

for some $M > 0$.

II. *Assume $\alpha^+(F) \leq 0$.*

(a) *If $\alpha^-(F) > 0$, problem* (5.6) *does not admit positive supersolutions. The other possibilities* (b) *and* (c) *are exactly as in Case* I.

Proof. We remark that all solutions and supersolutions have to be considered in the viscosity sense, but we are not giving precise details here. If there exists a positive supersolution u, by arguing as in the proof of Lemma 2.2, there exists a radial solution v verifying:

$$-F\left(\left(\frac{v''(r)}{r^2} - \frac{v'(r)}{r^3}\right) x \otimes x + \frac{v'(r)}{r} I\right) = f(v),$$

where I is the $N \times N$ identity matrix.

Moreover, notice that Lemma 2.1 holds since only comparison is needed so either $v' \geq 0$ or $v' \leq 0$ in $(\overline{R}, +\infty)$, and if we had $v' = 0$ at some point, then v' would attain either a minimum or a maximum, hence $v'' = 0$ at that point, which is not possible due to $f(v) > 0$. Hence $v' > 0$ in the first case and $v' < 0$ in the second. Now we will see that if $v' > 0$ then $\lim_{r\to+\infty} v(r) = +\infty$. In fact if v is bounded then we can argue as in Theorem 5.1: choose a test function $\phi \in C_0^\infty(1, 4)$ with $0 \leq \phi \leq 1$ and $\phi = 1$ in $(2, 3)$. For $R > R_0$ introduce the function

$$z(r) = v(r) - v(2R)\phi\left(\frac{r}{R}\right), \quad r > R_0.$$

Since $z = v > 0$ if $R_0 < r < R$ or $r > 4R$ and $z(2R) = 0$, z achieves a nonpositive minimum at some point $r_0(R) \in (R, 4R)$. This implies

$$f(v(r_0(R))) \leq C\frac{v(2R)}{R^2},$$

where C is a constant which does not depend on R nor on v. Since v is bounded $f(v(r_0(R))) \to 0$ as $R \to \infty$ thus $v(r_0(R)) \to 0$ as $R \to \infty$ which contradicts $v' > 0$. Observe that in the case $v' < 0$ with the same argument we get $\lim_{r\to+\infty} v(r) = 0$.

The next result is a generalization of Lemma 2.3.

Lemma 5.5.

(a) *If* $\alpha^+(F) \leq 0$, *then* $\lim_{r\to+\infty} v(r) = +\infty$.
(b) *If* $\alpha^-(F) \leq 0$, *then* $\lim_{r\to+\infty} v(r) = +\infty$.
(c) *If* $\alpha^-(F) > 0$, *then* $\lim_{r\to+\infty} v(r) = 0$.

Proof. (a) Suppose by contradiction that $\lim_{r\to+\infty} v(r) = 0$, hence $v' < 0$. We define $w(r) = c\phi^+(r) + d$ where c and d are taken to have $w(\bar{R}) = v(\bar{R})$ and $w'(\bar{R}) > v'(\bar{R})$. Observe that since $v'(\bar{R}) < 0$, it is possible to choose $c > 0$. Then, since $\lim_{r\to+\infty} w(r) = -\infty$, there exists R^* such that $v(R^*) = w(R^*)$. This contradicts the comparison principle.
(b) Take $R_1 > \bar{R}$ and for $r \in (R_1, \bar{R})$ define

$$w(r) = \frac{v(\bar{R})(\phi^-(r) - \phi^-(R_1)) + v(R_1)(\phi^-(\bar{R}) - \phi^-(r))}{\phi^-(\bar{R}) - \phi^-(R_1)}.$$

It is to be noted that $-F(D^2w) = 0$, while $w(R_1) = v(R_1)$, $w(\bar{R}) = v(\bar{R})$. Then by comparison $v \geq w$ in $(\bar{R}, R_1)$. Now we can let $R_1 \to +\infty$ noticing that $\phi^-(R_1) \to \infty$ to get $v \geq v(\bar{R})$ in $(\bar{R}, \infty)$. This shows that $\lim_{r\to+\infty} v(r) = +\infty$.
(c) The proof is similar to that of part (a) and therefore will be omitted. $\square$

Observe that parts (a) and (c) of the lemma immediately prove case II (a). To deal with the other cases in the proof, we make the change of variables $s = \phi(r)$ (where in case (a) we take $\phi = \phi^+$ and in cases (b) and (c) $\phi = \phi^-$), $v(r) = w(s)$ to arrive at

$$-F\left(\frac{\phi'(r)^2}{r^2}w''x\otimes x + w'\left[\left(\frac{\phi''(r)}{r^2} - \frac{\phi'(r)}{r^3}\right)x\otimes x + \frac{\phi'(r)}{r}I\right]\right) = f(v).$$

Now notice that by (5.7) we have

$$\begin{aligned}
&F\left(\frac{\phi'(r)^2}{r^2}w''x\otimes x + w'\left[\left(\frac{\phi''(r)}{r^2} - \frac{\phi'(r)}{r^3}\right)x\otimes x + \frac{\phi'(r)}{r}I\right]\right)\\
&\geq F\left(w'\left[\left(\frac{\phi''(r)}{r^2} - \frac{\phi'(r)}{r^3}\right)x\otimes x + \frac{\phi'(r)}{r}I\right]\right) + \mathcal{M}^-_{\lambda,\Lambda}\left(\frac{\phi'(r)^2}{r^2}w''x\otimes x\right).
\end{aligned}$$

Next observe that $w' = v'\phi'$, so that by Lemma 5.5 and the definition of ϕ we get $w' > 0$. Hence

$$F\left(\frac{\phi'(r)^2}{r^2}w'' x\otimes x + w'\left[\left(\frac{\phi''(r)}{r^2}-\frac{\phi'(r)}{r^3}\right)x\otimes x+\frac{\phi'(r)}{r}I\right]\right)$$
$$\geq \mathcal{M}^-_{\lambda,\Lambda}\left(\frac{\phi'(r)^2}{r^2}w''x\otimes x\right) = \frac{\phi'(r)^2}{r^2}\mathcal{M}^-_{\lambda,\Lambda}\left(w''x\otimes x\right),$$

since F is positively homogeneous. It follows that w verifies the inequality $-\frac{\phi'(r)^2}{r^2}\mathcal{M}^-_{\lambda,\Lambda}\left(w''x\otimes x\right) \geq f(w)$ in an interval of the form $(0, s_0)$ in case (a) or $(s_0, +\infty)$ in cases (b) and (c). If we had $w'' \geq 0$ at some point then $\mathcal{M}^-_{\lambda,\Lambda}\left(w''x\otimes x\right) = \lambda r^2 w''$, which leads to $-\lambda\phi'(r)^2 w'' \geq f(w) > 0$, impossible. Then $w'' < 0$ at every point and $\mathcal{M}^-_{\lambda,\Lambda}\left(w''x\otimes x\right) = \Lambda r^2 w''$. Thus w verifies

$$-w'' \geq \frac{1}{\Lambda\phi'\left(\phi^{-1}(s)\right)^2}f(w),$$

in an interval of the form $(0, s_0)$ (with $w(0) = 0$) or $(s_0, +\infty)$ (with $w(+\infty) = +\infty$). Taking into account that $\phi'(\phi^{-1}(s)) = s^{\frac{\alpha^+(F)+1}{\alpha^+(F)}}$ in case (a), $\phi'(\phi^{-1}(s)) = e^{-s}$ in case (b) and $\phi'(\phi^{-1}(s)) = s^{\frac{\alpha^-(F)+1}{\alpha^-(F)}}$ in case (c), it follows that w is a (viscosity) supersolution of the equation

$$-w'' = a(s)f(w), \tag{5.9}$$

where $a(s) = \Lambda^{-1}s^{-\frac{2(\alpha^+(F)+1)}{\alpha^+(F)}}$, $s \in (0, s_0)$ in case (a), $a(s) = \Lambda^{-1}e^{2s}$, $s \in (s_0, +\infty)$ in case (b) and $a(s) = \Lambda^{-1}s^{-\frac{2(\alpha^-(F)+1)}{\alpha^-(F)}}$, $s \in (s_0, +\infty)$ in case (c).

We observe that $\underline{w} = \lambda s$ is a subsolution of (A) in either case and $w \geq \underline{w}$ for sufficiently small λ. By means of the method of sub- and supersolutions, we obtain a (viscosity) solution z of (A), which by bootstrapping verifies $z \in C^2(0, s_0) \cap C[0, s_0]$ or $z \in C^2(s_0, +\infty) \cap C[s_0, +\infty)$. Thus all nonexistence results follow from Theorems 3.1 and 3.3 and Remark 3.4.

The sufficiency of the three conditions in the different cases can be seen by taking into account that the problem

$$-w'' = \frac{1}{\lambda\phi'\left(\phi^{-1}(s)\right)^2}f(w)$$

has a positive solution with either of the conditions $w(0) = 0$ or $w(+\infty) = +\infty$, verifying $w' > 0$, thanks to the results in Section 3.

Taking again ϕ as above in the three different cases we have

$$F\left(\frac{\phi'(r)^2}{r^2}w''x\otimes x+w'\left[\left(\frac{\phi''(r)}{r^2}-\frac{\phi'(r)}{r^3}\right)x\otimes x+\frac{\phi'(r)}{r}I\right]\right)$$
$$\le \mathcal{M}^+_{\lambda,\Lambda}\left(\frac{\phi'(r)^2}{r^2}w''x\otimes x\right)=\lambda\phi'\big(\phi^{-1}(s)\big)^2w''=-f(w)$$

and $u(x)=v(|x|)$ is a radially symmetric supersolution of the equation $-F(D^2u)=f(u)$ in $\mathbb{R}^N\setminus B_{R_0}$ for suitably large R_0. □

p-Laplacian operator

Elliptic problems which are not uniformly elliptic can also be considered with our approach. As a prototype, for the p-Laplacian version of problem (1.2), namely

$$-\Delta_p u\ge f(u)\quad \text{in } \mathbb{R}^N\setminus B_{R_0}, \tag{5.10}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p>1$, we have the following results, depending on whether $p<N$, $p=N$ or $p>N$.

Theorem 5.6. *Let $f:[0,\infty)\to\mathbb{R}$, be continuous and positive in $(0,\infty)$ and $p<N$. Then problem (5.10) admits a positive weak supersolution if and only if*

$$\int_0^\delta \frac{f(t)}{t^{\frac{p(N-1)}{N-p}}}dt<+\infty$$

for some $\delta>0$.

Theorem 5.7. *Let $f:[0,\infty)\to\mathbb{R}$, be continuous and positive in $(0,\infty)$ and $p=N$. Then problem (5.10) admits a positive weak supersolution if and only if there exist $M,a>0$ such that*

$$\int_M^\infty e^{at}f(t)dt<\infty.$$

Theorem 5.8. *Let $f:[0,\infty)\to\mathbb{R}$, be continuous and positive in $(0,\infty)$ and $p>N$. Then problem (5.10) admits a positive weak supersolution if and only if there exists $M>0$ such that*

$$\int_M^\infty t^{\frac{p(N-1)}{p-N}}f(t)dt<\infty.$$

As in the previous proofs, the basic ingredient is the existence of a fundamental solution, which takes the form $\phi(x)=|x|^{\frac{p-N}{p-1}}$ if $p\neq N$ and $\phi(x)=\log|x|$ when $p=N$. Then the relevant point is an adaptation of Theorems 3.1 and 3.3 and

Remark 3.4 for the corresponding one-dimensional problems obtained after setting $s = \phi(r)$ in the radial version of (5.10):

$$\begin{cases} -\left(|w'|^{p-2}w'\right)' = s^{-\gamma} f(w) & \text{in } (0, s_0) \\ w(0) = 0 \end{cases}$$

where $\gamma > p$,

$$\begin{cases} -\left(\left|w'\right|^{p-2}w'\right)' = e^{ps} f(w) & \text{in } (s_0, \infty) \\ \lim\limits_{s \to +\infty} w(s) = +\infty \end{cases}$$

and

$$\begin{cases} -\left(\left|w'\right|^{p-2}w'\right)' = s^{\gamma} f(w) & \text{in } (s_0, +\infty) \\ \lim\limits_{s \to +\infty} w(s) = +\infty, \end{cases}$$

with $\gamma > 0$.

A. Appendix

In this appendix, we collect some results on the method of sub- and supersolutions which are instrumental in our proofs, since we could not find a pertinent reference for some of them (especially for those concerning radial symmetry). We only consider the model problem:

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where Ω is a smooth bounded domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$ are continuous. But with similar techniques, some more general problems can also be dealt with. For instance, the fully nonlinear equation

$$\begin{cases} -F\left(D^2 u\right) = f(x, u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Most proofs are essentially the same (the existence part can be obtained from Theorem 1.1 of [17] or from the use of Schauder's fixed point theorem together with a truncation argument as in [31]). For the p-Laplacian version of (A.1), namely

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

similar results hold (*cf.* [34] for the standard method of sub- and supersolutions).

We say that $\underline{u} \in H^1(\Omega) \cap C(\overline{\Omega})$ is a (weak) subsolution of (A.1) if $\underline{u} \le g$ on $\partial\Omega$ and

$$\int_\Omega \nabla \underline{u} \nabla \phi \le \int_\Omega f(x, \underline{u})\phi$$

for every $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$. Supersolutions are defined by reversing the above inequalities. It is well-known that the maximum of two subsolutions is again a subsolution, while the minimum of two supersolutions is also a supersolution.

When problem (A.1) admits a subsolution $\underline{u}$ and a supersolution $\bar{u}$ with $\underline{u} \le \bar{u}$ in Ω and the function f is locally Lipschitz continuous or locally Hölder continuous with respect to the second variable, a weak solution u of (A.1) can be obtained by means of a monotone iteration (*cf.* [33] or [3]). This method gives further information, since it provides in addition with a minimal and a maximal solution in the order interval $[\underline{u}, \bar{u}]$.

A remarkable fact is that the existence of a minimal and a maximal solution can be always ensured even if the solutions are not obtained by means of a monotone iteration. This is the case in our present situation since f is assumed to be merely continuous.

Theorem A.1. *Assume $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are continuous, and that there exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of* (A.1) *with $\underline{u} \le \overline{u}$ in Ω. Then there exist a minimal and a maximal weak solution of* (A.1) *in the order interval $[\underline{u}, \overline{u}]$.*

Proof. The existence of a weak solution in the interval $[\underline{u}, \overline{u}]$ for continuous nonlinearities (and even more general operators) is well-known (see for instance [16,34] or [36] for a variational proof). Thus we are only proving that there exists a maximal weak solution (of course the existence of a minimal weak solution follows similarly). This proof is taken from [31].

Define

$$A = \sup \left\{ \int_\Omega u \,:\, u \in [\underline{u}, \overline{u}] \text{ is a solution of (A.1)} \right\}.$$

Then A is clearly well defined. By its definition, there exists a sequence of weak solutions $\{u_n\}_{n=1}^\infty \subset [\underline{u}, \overline{u}]$ such that $\int_\Omega u_n \to A$. Since $\{u_n\}$ is a uniformly bounded sequence, we obtain in a standard way that it is bounded in $H^1(\Omega)$, thus by passing to a subsequence, we may assume $u_n \to u$ weakly in $H^1(\Omega)$ and strongly in $L^1(\Omega)$, where $u \in [\underline{u}, \overline{u}]$ is a weak solution of (A.1) with of course $\int_\Omega u = A$. We claim that u is the maximal weak solution.

For this aim, let $v \in [\underline{u}, \overline{u}]$ be an arbitrary weak solution. Since $w = \max\{v, u\}$ is a subsolution with $w \le \overline{u}$, there exists a weak solution $z \in [w, \overline{u}]$. But then $u \le w \le z$ and

$$A = \int_\Omega u \le \int_\Omega z \le A,$$

by the definition of A. It follows that $u = z \ge w \ge v$, as was to be proved. □

An important consequence of Theorem A.1 is that, when Ω is rotationally invariant, $f = f(|x|, u)$, $g = g(|x|)$ and both $\underline{u}$ and $\bar{u}$ are radially symmetric, then the minimal and the maximal weak solutions $u_{\min}$ and $u_{\max}$ are also radially symmetric. This holds because if R is a rotation the $u_{\min}(Rx)$ is again a solution in the order interval $[\underline{u}, \bar{u}]$, hence $u_{\min}(Rx) \geq u_{\min}(x)$, which shows the radial symmetry of $u_{\min}$. When f is locally Hölder continuous, thanks to the monotone iteration, it suffices that $\underline{u}$ is radially symmetric to obtain that $u_{\min}$ also is. But this property is valid in general without assuming the extra amount of regularity on f.

Corollary A.2. *Under the same hypotheses as in Theorem* A.1*, assume in addition that $f = f(|x|, u)$, $g = g(|x|)$ and Ω is rotationally invariant. If $\underline{u}$* (*respectively $\overline{u}$*) *is radially symmetric then so is the minimal* (*respectively maximal*) *weak solution.*

Proof. Assume $\underline{u}$ is radially symmetric, and let $u_{\min}$ denote the minimal weak solution. If R denotes a rotation, then $u_{\min}(Rx)$ is also a weak solution of (A.1). It follows that

$$\overline{\overline{u}}(x) = \min\{u_{\min}(Rx), \overline{u}(x)\}$$

is a supersolution, which verifies, by the radial symmetry of $\underline{u}$, $\overline{\overline{u}} \geq \underline{u}$. By Theorem A.1 there exists a weak solution w in the order interval $[\underline{u}, \overline{\overline{u}}] \subset [\underline{u}, \overline{u}]$. Thus $w \geq u_{\min}$. But on the other hand $w(x) \leq u_{\min}(Rx)$, so that $u_{\min}(x) \leq u_{\min}(Rx)$. Since R is an arbitrary rotation, it follows that $u_{\min}$ is radially symmetric. □

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