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Multiple positive or sign-changing solutions for a type of nonlinear Schrödinger equation

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Abstract. This paper is concerned with the existence of multiple non-radial positive solutions for

$$\begin{cases} -\Delta u + (1+\beta V(y))u = |u|^{p-2}u & y \in \mathbb{R}^N \\ u(y) \to 0 & \text{as } |y| \to +\infty \end{cases}$$

where $2 < p < 2^*$, $2^* = \frac{2N}{N-2}$ for $N > 2$ and $2^* = +\infty$ for $N = 2$, β can be regarded as a parameter and $V(|y|) > 0$ decays exponentially to zero at infinity. We prove that, for any positive integer $k > 1$, there exists a suitable range of β such that the above problem has a non-radial positive solution with exactly k maximum points which tending to infinity as $\beta \to +\infty$ (or 0^+).

Mathematics Subject Classification (2010): 35J20 (primary); 35J60 (secondary).

1. Introduction

In this paper we consider the following nonlinear Schrödinger problem

$$\begin{cases} -\Delta u + (1+\beta V(y))u = |u|^{p-2}u & y \in \mathbb{R}^N \\ u(y) \to 0 & \text{as } |y| \to +\infty \end{cases} \tag{1.1}$$

where $2 < p < 2^*$, $2^* = \frac{2N}{N-2}$ for $N > 2$ and $2^* = +\infty$ for $N = 2$, β can be regarded as a parameter and $V(y) > 0$ decays to zero exponentially at infinity.

Problem (1.1) arises from looking for standing waves $\Psi(t, y) = \exp(iEt)u(y)$ for the following nonlinear Schrödinger equation in $\mathbb{R}^N$,

$$i\frac{\partial \Psi}{\partial t} = -\Delta\Psi + \hat{V}\Psi - |\Psi|^{p-2}\Psi, \tag{1.2}$$

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where i is the imaginary unit, $E \in \mathbb{R}$ and $\hat{V}(y) : \mathbb{R}^N \to \mathbb{R}^+$ is a continuous function. Problem (1.2) arises in many applications (see, *e.g.*, [2,7,8]). For example, in some problems arising in nonlinear optics, in plasma physics and in condensed matter physics, the presence of many particles leads one to considering nonlinear terms which simulate the interaction effect among them. The function $\hat{V}(y)$ represents the potential acting on the particle, which avoids the spreading of the wave packets in the time-dependent version of the above equation.

Assuming that the amplitude $u(y)$ is positive and vanishes at infinity, we see that $\Psi(t, y)$ satisfies (1.2) if and only if u solves the nonlinear elliptic problem (1.1) with $1 + \beta V(y) = \hat{V}(y) - E$.

The study of the existence of ground states and higher energy solutions for (1.1) has attracted considerable attention in recent years, and there are a lot of results in the literature: one can refer to [2–4,6,12,16–18,20,21,24,25] and the references therein. If $\beta > 0$ and

$$\inf_{y\in\mathbb{R}^N} V(y) < \lim_{|y|\to+\infty} V(y), \tag{1.3}$$

then, using the concentration compactness principle [20,21], one can show that (1.1) has a least energy solution. See also for example [13,20–22]. But if (1.3) does not hold, (1.1) may not have least energy solutions. For example, if $V(y)$ satisfies

$$V(y) > \inf_{y\in\mathbb{R}^N} V(y) = \lim_{|y|\to+\infty} V(y),$$

then it is easy to see that problem (1.1) has no least energy solutions. So, in this case, one needs to find solutions with higher energy. For results on this aspect, the readers can refer to [3,4,24,25] and the references therein.

Here we want to mention some results in [5], where Bartsch and Wang considered (1.1) with $\Omega = int V^{-1}(0)$ non-empty and $\mu\{y \in \mathbb{R}^n : \ V(y) \le M_0\} < \infty$ for some $M_0 > 0$. They considered the existence of the least energy solution, multiplicity of solutions and certain concentration behavior of the solutions for large $\beta > 0$. Very recently, in [17], Lin, Liu and Chen showed that if $V(y)$ satisfies

$$\lim_{|y|\to\infty} V(y) = 0, \text{ and } \lim_{|y|\to\infty} \frac{\ln|V(y)|}{|y|} = 0, \tag{1.4}$$

then as $\beta \to 0$, equation (1.1) has multiple positive solutions. In [25], Wei and Yan used a construction argument and obtained a very interesting result, which says that if

$$V(y) = V(|y|) \sim \frac{1}{r^m} + O\left(\frac{1}{r^{m+\theta}}\right), \ (m > 1, \ \theta > 0), \quad \text{as } r = |y| \to +\infty, \tag{1.5}$$

(this is a special case of condition (1.4)), then for any β fixed, problem (1.1) has infinitely many non-radial positive solutions.

Now an interesting problem left is the following case:

$$\lim_{|y|\to\infty} \frac{\ln|V(y)|}{|y|} < 0. \tag{1.6}$$

In this paper we will consider (1.1) under the assumption (1.6). Our aim is to prove the existence of multiple positive solutions to equation (1.1).

For simplicity, we suppose that $V(y) = V(|y|)$ is a positive continuous function in $\mathbb{R}^N$ and satisfies the following decay assumption at infinity, which is a special case of (1.6):

(V) There are $a \in \mathbb{R}$, $\alpha \in (0, 1]$, such that

$$V(r) \sim r^a e^{-\alpha r}, \text{ as } r \to +\infty.$$

Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that* (V) *holds and* $k > 1$ *is an integer. Then problem* (1.1) *has a non-radial positive solution with exactly* k *maximum points provided that* k *and* β *satisfy one of the following conditions:*

(i) *If* $2 \sin \frac{\pi}{k} < \alpha$, *then* $\beta > \beta_1^*$ *for suitably large* $\beta_1^* > 0$*;*
(ii) *If* $2 \sin \frac{\pi}{k} > \alpha$, *then* $0 < \beta < \bar{\beta}_1^*$ *for suitably small* $\bar{\beta}_1^* > 0$.

Here β_1^* *and* $\bar{\beta}_1^*$ *depend on* α, k *and* N.

Remark 1.2. In fact, our result is true for more general problems with more general $V(y)$

$$\begin{cases} -\Delta u + (1 + \beta V(y))u = f(u) & y \in \mathbb{R}^N \\ u(y) \to 0 & \text{as } |y| \to +\infty \end{cases}$$

where $f(t)$ satisfies the following assumptions (see, for example [11]):

(f_1) $f \in C^{1,\gamma}(\mathbb{R})$, $f(0) = f'(0) = 0$, $f(-t) = -f(t)$ for all $t \in \mathbb{R}$;
(f_2) the following problem

$$\begin{cases} -\Delta w + w = f(w),\ u > 0 & y \in \mathbb{R}^N, \\ w(0) = \max\limits_{\mathbb{R}^N} w(y) & w \in H^1\left(\mathbb{R}^N\right) \end{cases}$$

has a unique solution w, which is nondegenerate, *i.e.*, denoting by L the linearized operator

$$L:\ H^2\left(\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N\right),\ L(u) = -\Delta u + u - f'(w)u,$$

then

$$\text{Kernel}(L) = \text{span}\left\{\frac{\partial w}{\partial x_i}:\ i = 1, \cdots, N\right\}.$$

$V(y) = V(y', y'') = V(|y'|, |y_3|, \cdots, |y_N|)$ $((y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2})$ satisfies

(V') There is $\alpha \in (0, 1]$, such that

$$V(y) \sim k(|y|)e^{-\alpha|y|}, \text{ as } |y| \to +\infty,$$

where $k(|y|) > 0$ satisfies $a_1|y|^a \le k(|y|) \le b_1|y|^b$ for some constants $a_1 > 0$, $b_1 > 0$ and $a, b \in \mathbb{R}$, $a \le b$.

Remark 1.3. In assumption (V) (or (V')), if $\alpha \in (1,2)$, then part (ii) of Theorem 1.1 is still true, which will be clarified in Remark 3.5 in Section 3 later.

There is a lot of literature concerning time-independent semilinear Schrödinger equations. In particular, the singular perturbed equation

$$-\varepsilon^2 \Delta u + V(y)u = |u|^{p-2}u, \ \ u \in H^1\left(\mathbb{R}^N\right) \tag{1.7}$$

has been extensively studied. Solutions of (1.7) as $\varepsilon \to 0^+$ are called semi-classical states, which usually exhibit a concentration phenomenon, that is, the solutions may concentrate at some points such at the critical points of $V(y)$, see for instance [8–11,14,15,18,19,22,23]. The solutions we obtain in Theorem 1.1 do not concentrate near any fixed point, and they have multiple bumps separated far apart with each bump resembling the shape of the solution of (1.8). Similar phenomenon was also observed in [16,17,25].

Now, let us outline the main idea in the proof of Theorem 1.1.

We will use the unique ground state U of

$$\begin{cases} -\Delta u + u = u^{p-1}, \ u > 0 & y \in \mathbb{R}^N \\ u(0) = \max\limits_{\mathbb{R}^N} u(y) & u \in H^1\left(\mathbb{R}^N\right), \end{cases} \tag{1.8}$$

to build up the approximate solutions for (1.1). It is well-known that $U(y) = U(|y|)$ is nondegenerate (see the definition (f_2) in Remark 1.3) and satisfies

$$U'(r) < 0, \ \lim_{r\to\infty} r^{\frac{N-1}{2}} e^r U(r) = C > 0, \ \lim_{r\to\infty} \frac{U'(r)}{U(r)} = -1.$$

Let

$$x^j = \left(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0, \cdots, 0\right) \in \mathbb{R}^N, \ j = 1, 2, \cdots, k,$$

where $r > 0$ will be determined later.

Set $y = (y', y''), \ y' \in \mathbb{R}^2, \ y'' \in \mathbb{R}^{N-2}$. Define

$$\begin{aligned} H_s = \Bigg\{ & u : u \in H^1\left(\mathbb{R}^N\right), u \text{ is even in } y_i, i = 2, \cdots, N, \\ & u(r\cos\theta, r\sin\theta, y'') = u\left(r\cos\left(\theta + \frac{2j\pi}{k}\right), r\left(\theta + \sin\frac{2j\pi}{k}\right), y''\right)\Bigg\}. \end{aligned}$$

Denote

$$U_r(y) = \sum_{j=1}^{k} U_{x^j}(y),$$

where $U_{x^j}(y) = U(y - x^j)$.

To prove Theorem 1.1, it suffices to verify the following result:

Theorem 1.4. *If* $2\sin\frac{\pi}{k} \neq \alpha$ *then* (1.1) *has a solution* $u = U_r + \omega_r$ *provided that one of the following conditions holds:*

(i) *If* $2\sin\frac{\pi}{k} < \alpha$, *then* $\beta > \beta_1^*$ *for suitably large* $\beta_1^* > 0$;
(ii) *If* $2\sin\frac{\pi}{k} > \alpha$, *then* $0 < \beta < \bar{\beta}_1^*$ *for suitably small* $\bar{\beta}_1^* > 0$,

Here β_1^* *and* $\bar{\beta}_1^*$ *depend on* α, k *and* N.

Moreover, $r \in \left[\frac{(1-\delta)\ln\beta}{\alpha-2\sin\frac{\pi}{k}}, \frac{(1+\delta)\ln\beta}{\alpha-2\sin\frac{\pi}{k}}\right]$ *for small* $\delta > 0$, *and* $\omega_r \in H_s$ *satisfies*

$$\int_{\mathbb{R}^N} |\nabla\omega_r|^2 + |\omega_r|^2 \to 0, \ \ as\ \beta \to +\infty \ \big(or\ 0^+\big).$$

The idea of our proof is essentially inspired by [25]. The key part of the idea is to establish a balance between two main terms in the expansion of the energy functional, one is $\int_{\mathbb{R}^N} V(y)U_r(y)^2$ arising from the effect of $V(y)$, the other is $\int_{\mathbb{R}^N} U_{x^i}^{p-1}U_{x^j}$ $(i \neq j)$ which is from interaction among the peaks. In [25], the potential $V(y)$ satisfies assumption (1.5), which means that $V(y)$ decays to zero algebraically at infinity. Hence the term $\int_{\mathbb{R}^N} V(y)U_r(y)^2$ decays also to zero algebraically as $r \to +\infty$. The other main term $\int_{\mathbb{R}^N} U_{x^i}^{p-1}U_{x^j}$ $(i \neq j)$ approaches to zero exponentially as $r \to +\infty$. A balance between these two main terms can be obtained for any fixed parameter $\beta > 0$. However, in this paper, since the potential $V(y)$ decays to zero exponentially at infinity, both of the main terms $\int_{\mathbb{R}^N} V(y)U_r(y)^2$ and $\int_{\mathbb{R}^N} U_{x^i}^{p-1}U_{x^j}$ $(i \neq j)$ are exponentially small as $r \to +\infty$, which implies that one term can control the other. In this case, we need to adjust the parameter β to keep a balance between these two terms. More precisely, if $2\sin\frac{\pi}{k} < \alpha$, the term $\int_{\mathbb{R}^N} V(y)U_r(y)^2$ can dominate $\int_{\mathbb{R}^N} V(y)U_r(y)^2$. Hence we need the parameter β to be large. Otherwise, when $2\sin\frac{\pi}{k} > \alpha$, the term $\int_{\mathbb{R}^N} V(y)U_r(y)^2$ is overwhelming, so the parameter β should be small.

This paper is organized as follows. In Section 2, we will make an energy expansion for the functional corresponding to problem (1.1). In Section 3, we will carry out a reduction procedure and study the reduced finite dimensional problem to prove Theorem 1.4.

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2. Energy expansion

The variational functional corresponding to (1.1) can be defined as follows

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla u|^2 + (1+\beta V(|y|))u^2\Big) - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p, \quad \forall\, u \in H^1\left(\mathbb{R}^N\right).$$

In this section, we will give an energy expansion related to the approximate solutions. In the sequel, we always assume

$$r \in S =: \left[\frac{(1-\delta)\ln\beta}{\alpha - 2\sin\frac{\pi}{k}}, \frac{(1+\delta)\ln\beta}{\alpha - 2\sin\frac{\pi}{k}}\right], \tag{2.1}$$

where α is the constant in expansion for V, and $\delta \in (0,1)$ is a small constant.

The following result can be found in [1, Lemma 3.7].

Lemma 2.1. *Suppose that $u, u' : \mathbb{R}^N \to \mathbb{R}$ two positive continuous radial functions satisfy*

$$u(x) \sim |x|^a e^{-b|x|}, u'(x) \sim |x|^{a'} e^{-b'|x|}, \quad (|x| \to \infty),$$

where $a, a' \in \mathbb{R}$, $b' > b > 0$. Let $\xi \in \mathbb{R}^N$ tend to infinity. Then,

$$\int_{\mathbb{R}^N} u_\xi u' \sim e^{-b|\xi|}|\xi|^a,$$

where $u_\xi = u(y+\xi)$.

Lemma 2.2. *For $\beta > 0$ large (or small) enough, there holds*

$$\int_{\mathbb{R}^N} V(y)U_r^2 = B_\beta r^a e^{-\alpha r} + O\left(e^{-(1+\tau)\alpha r}\right),$$

where $B_\beta \in [C_1, C_2]$, C_1, C_2 and τ are positive constants independent of β.

Proof. Set

$$\Omega_j = \left\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \left\langle \frac{y'}{|y'|}, \frac{x^j}{|x^j|} \right\rangle \geq \cos\frac{\pi}{k} \right\}, \; j = 1, 2, \cdots, k.$$

For any $y \in \Omega_i$ and $j \neq i$, we have

$$\left|y - x^j\right| \geq \left|y - x^i\right|, \quad \forall\, y \in \Omega_i,$$

which gives $|y - x^j| \geq \frac{1}{2}|x^j - x^i|$ if $|y - x^i| \geq \frac{1}{2}|x^j - x^i|$. On the other hand, if $|y - x^i| \leq \frac{1}{2}|x^j - x^i|$, then

$$\left|y - x^j\right| \geq \left|x^j - x^i\right| - \left|y - x^i\right| \geq \frac{1}{2}\left|x^j - x^i\right|.$$

So, we find

$$\left|y - x^j\right| \geq \frac{1}{2}\left|x^j - x^i\right|, \quad \forall\, y \in \Omega_i.$$

Thus, for any $\eta \in (0, 1)$,

$$\begin{aligned} U_{x^j} &\le Ce^{-\eta|y-x^j|}e^{-(1-\eta)|y-x^j|} \le Ce^{-(1-\eta)|y-x^i|}e^{-\frac{\eta}{2}|x^j-x^i|} \\ &\le Ce^{-\eta r \sin\frac{\pi}{k}}e^{-(1-\eta)|y-x^i|}, \quad \forall\ y \in \Omega_i. \end{aligned} \tag{2.2}$$

Hence using (2.2), we know

$$\begin{aligned} \int_{\mathbb{R}^N} V(y)U_{x^i}U_{x^j} &\le Ck\int_{\Omega_i} V(y)U_{x^i}U_{x^j} \\ &\le Ck\int_{\Omega_i} V(y)U_{x^i}e^{-\eta r \sin\frac{\pi}{k}}e^{-(1-\eta)|y-x^i|} \\ &\le Cke^{-\eta r \sin\frac{\pi}{k}}\int_{\mathbb{R}^N} V(y-x_i)Ue^{-(1-\eta)|y|} \\ &= Cke^{-\eta r \sin\frac{\pi}{k}}\int_{B_{(1-\sigma)r}(0)} V(y-x_i)Ue^{-(1-\eta)|y|} \\ &\quad + Cke^{-\eta r \sin\frac{\pi}{k}}\int_{\mathbb{R}^N\setminus B_{(1-\sigma)r}(0)} V(y-x_i)Ue^{-(1-\eta)|y|}. \end{aligned}$$

On the other hand, using Lemma 2.1, we find

$$\begin{aligned} \int_{B_{(1-\sigma)r}(0)} V(y-x_i)Ue^{-(1-\eta)|y|} &\le C\int_{B_{(1-\sigma)r}(0)} \left|y-x^i\right|^a e^{-\alpha|y-x^i|}e^{-(1-\eta)|y|}U \\ &\le C\int_{\mathbb{R}^N} \left|y-x^i\right|^a e^{-\alpha|y-x^i|}e^{-(1-\eta)|y|}U \\ &\le Cr^a e^{-\alpha r} \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N\setminus B_{(1-\sigma)r}(0)} V(y-x_i)Ue^{-(1-\eta)|y|} &\le C\int_{\mathbb{R}^N\setminus B_{(1-\sigma)r}(0)} V(y-x_i)e^{-(2-\eta)|y|} \\ &\le Ce^{-(2-\eta)(1-\sigma)r}\int_{\mathbb{R}^N} V(y-x_i) \\ &\le Ce^{-\alpha r}, \end{aligned}$$

provided $(2-\eta)(1-\sigma) > \alpha$.

Thus, we see

$$\begin{aligned} \int_{\mathbb{R}^N} V(y)U_{x^i}U_{x^j} &\le Cke^{-\eta r \sin\frac{\pi}{k}}\int_{\mathbb{R}^N} V(y-x_i)Ue^{-(1-\eta)|y|} \\ &\le Cke^{-\eta r \sin\frac{\pi}{k}}\left(r^a e^{-\alpha r} + e^{-\alpha r}\right) \\ &\le Ce^{-(1+\tau)\alpha r}, \end{aligned} \tag{2.3}$$

where C, τ are constants independent of β.

Similarly, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} V(y)U_{x^i}^2 &= \int_{\mathbb{R}^N} V\left(y-x^i\right)U^2 \\
&=(1+o_\beta(1))\int_{B_{(1-\sigma)r}(0)} \left|y-x^i\right|^a e^{-\alpha|y-x^i|}U^2 \\
&\quad+\int_{\mathbb{R}^N\setminus B_{(1-\sigma)r}(0)} V\left(y-x^i\right)U^2 \\
&=B_{\beta,1}r^a e^{-\alpha r}+O\left(e^{-(1+\tau)\alpha r}\right),
\end{aligned} \tag{2.4}
$$

where $B_{\beta,1}\in[C_1', C_2']$, C_1', C_2' are positive constants independent of β and $o_\beta(1)$ denotes a small data satisfying $o_\beta(1)\to 0$ as $\beta\to+\infty$ (or 0^+). Combining (2.3) and (2.4), we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} V(|y|)U_r^2 &= \int_{\mathbb{R}^N} V(y)\left(\sum_{i=1}^k U_{x^i}\right)^2 \\
&= k\int_{\mathbb{R}^N} V(y)U_{x^i}^2+\sum_{i\neq j}^k\int_{\mathbb{R}^N}|y|^a e^{-\alpha|y|}U_{x_i}U_{x_j} \\
&= B_\beta r^a e^{-\alpha r}+O\left(e^{-(1+\tau)\alpha r}\right).
\end{aligned}
$$

□

Proposition 2.3. *There is a small constant $\tau>0$, such that*

$$
\begin{aligned}
I(U_r) &= A+\beta B_\beta r^a e^{-\alpha r}-B_\beta' r^{\frac{1-N}{2}}e^{-2r\sin\frac{\pi}{k}} \\
&\quad+O\left(\beta e^{-(1+\tau)\alpha r}+e^{-(1+\tau)2r\sin\frac{\pi}{k}}\right),
\end{aligned}
$$

where $A=\left(\frac{1}{2}-\frac{1}{p}\right)k\int_{\mathbb{R}^N}U^p$, and $B_\beta, B_\beta'\in[C_1,C_2]$, C_1, C_2 are positive constants independent of β.

Proof. Using the symmetry, we see

$$
\begin{aligned}
\int_{\mathbb{R}^N}\left(|\nabla U_r|^2+U_r^2\right) &= \sum_{j=1}^k\sum_{i=1}^k\int_{\mathbb{R}^N}U_{x^j}^{p-1}U_{x^i} \\
&= k\int_{\mathbb{R}^N}U^p+k\sum_{i=2}^k\int_{\mathbb{R}^N}U_{x^1}^{p-1}U_{x^i}.
\end{aligned} \tag{2.5}
$$

By Lemma 2.1, we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} U_r^p &= k\int_{\mathbb{R}^N} U_{x^1}^p + kp\int_{\mathbb{R}^N}\sum_{i=2}^k U_{x^1}^{p-1}U_{x^i} \\
&\quad + \begin{cases} O\left(\displaystyle\int_{\mathbb{R}^N}\sum_{i\neq j}^k U_{x^i}^{\frac{p}{2}}U_{x^j}^{\frac{p}{2}}\right) & (2<p\le 3) \\ O\left(\displaystyle\int_{\mathbb{R}^N}\sum_{i\neq j}^k U_{x^i}^{p-2}U_{x^j}^{2}\right) & (p>3) \end{cases} \\
&= k\int_{\mathbb{R}^N}\left(U^p + pU_{x^1}^{p-1}\sum_{i=2}^k U_{x^i}\right) + kO\left(e^{-(1+\tau)|x^2-x^1|}\right).
\end{aligned}
\tag{2.6}
$$

Combining Lemma 2.2, (2.5) and (2.6), we have

$$
\begin{aligned}
I(U_r) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla U_r|^2 + (1+\beta V(|y|))U_r^2\right) - \frac{1}{p}\int_{\mathbb{R}^N} U_r^p \\
&= \frac{1}{2}k\left(\int_{\mathbb{R}^N} U_{x^1}^p + \sum_{i=2}^k\int_{\mathbb{R}^N} U_{x^1}^{p-1}U_{x^i}\right) - \frac{1}{p}k\left(\int_{\mathbb{R}^N} U_{x^1}^p + p\sum_{i=2}^k\int_{\mathbb{R}^3} U_{x^1}^{p-1}U_{x^i}\right) \\
&\quad + \frac{\beta}{2}\int_{\mathbb{R}^N} V(|y|)U_r^2 + kO\left(e^{-(1+\tau)|x^2-x^1|}\right) \\
&= \left(\frac{1}{2}-\frac{1}{p}\right)k\int_{\mathbb{R}^N} U^p - \frac{k}{2}\int_{\mathbb{R}^N}\left(U_{x^1}^{p-1}U_{x^2} + U_{x^1}^{p-1}U_{x^k}\right) + \frac{\beta}{2}\int_{\mathbb{R}^N} V(|y|)U_r^2 \\
&\quad + kO\left(e^{-(1+\tau)|x^2-x^1|}\right) \\
&= A + \beta B_\beta r^a e^{-\alpha r} - C_\beta |x^2-x^1|^{-\frac{N-1}{2}} e^{-|x^2-x^1|} \\
&\quad + O\left(\beta e^{-(1+\tau)\alpha r} + e^{-(1+\tau)|x^2-x^1|}\right) \\
&= A + \beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r\sin\frac{\pi}{k}} + O\left(\beta e^{-(1+\tau)\alpha r} + e^{-(1+\tau)2r\sin\frac{\pi}{k}}\right). \qquad \square
\end{aligned}
$$

3. Proof of the main result

Let

$$
Z_j = \frac{\partial U_{x^j}}{\partial r},\ j = 1, 2, \cdots, k.
$$

Define

$$
E := \left\{ u : u \in H_s, \sum_{j=1}^k \int_{\mathbb{R}^N} U_{x^j}^{p-2} Z_j u = 0 \right\}.
$$

The norm of $H^1(\mathbb{R}^N)$ is defined as

$$\|u\|^2 = \langle u, u\rangle, \quad u \in H^1\left(\mathbb{R}^N\right),$$

where $\langle u_1, u_2\rangle = \int_{\mathbb{R}^N} \nabla u_1 \cdot \nabla u_2 + (1+\beta V(|y|))u_1 u_2$.

We define the following linear operator L on E, satisfying

$$\langle Lu_1, u_2\rangle = \int_{\mathbb{R}^N} \nabla u_1 \cdot \nabla u_2 + (1+\beta V(|y|))u_1 u_2 - (p-1)U_r^{p-2} u_1 u_2. \quad (3.1)$$

Lemma 3.1. *There is a constant $\rho > 0$ such that*

$$\|Lu\| \geq \rho\|u\|, \quad \forall\, u \in E$$

provided that one of the following conditions holds:

(i) *If $2\sin\frac{\pi}{k} < \alpha$, then $\beta > \beta_1^*$ for suitably large $\beta_1^* > 0$;*
(ii) *If $2\sin\frac{\pi}{k} > \alpha$, then $0 < \beta < \bar{\beta}_1^*$ for suitably small $\bar{\beta}_1^* > 0$.*

Proof. We only prove the lemma for the case $2\sin\frac{\pi}{k} < \alpha$ since the other one is similar.

We argue by contradiction. Suppose that there are $u_n \in E$, $\beta_n \to +\infty$, such that

$$\|Lu_n\| = o(1), \quad \|u_n\| = 1.$$

For simplicity, we use β to denote β_n and x^j to denote

$$x^{j,n} = \left(r_n \cos\frac{2(j-1)\pi}{k}, r_n \sin\frac{2(j-1)\pi}{k}, \mathbf{0}\right).$$

We have

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla\varphi + (1+\beta V(|y|))u_n\varphi - (p-1)U_r^{p-2} u_n\varphi \\ &= \langle Lu_n, \varphi\rangle = o(1)\|\varphi\|, \quad \varphi \in E.\end{aligned} \quad (3.2)$$

In particular,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + (1+\beta V(|y|))u_n^2 - (p-1)U_r^{p-2} u_n^2 = o(1)$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + (1+\beta V(|y|))u_n^2 = 1. \quad (3.3)$$

Set $\widetilde{u}_n(y) = u_n(y + x^1)$. Then for any $R > 0$,

$$\int_{B_R(x^1)} |\nabla u_n|^2 + (1+\beta V(|y|))u_n^2 \leq 1,$$

which implies

$$\int_{B_R(0)} |\nabla \widetilde{u}_n|^2 + (1+\beta V(|y|))\widetilde{u}_n^2 \leq 1.$$

So we can suppose that there is a $u \in H^1(\mathbb{R}^N)$, such that as $n \to +\infty$,

$$\widetilde{u}_n \rightharpoonup u, \quad in \quad H^1\left(\mathbb{R}^N\right),$$

and

$$\widetilde{u}_n \to u, \quad in \quad L^2_{\text{loc}}\left(\mathbb{R}^N\right).$$

Since $\widetilde{u}_n$ is even in $y_i, i = 2, \ldots, N$, it is easy to see that u is even in $y_i, i = 2, \ldots, N$. It follows from $\int_{\mathbb{R}^N} U_{x^1}^{p-2} Z_1 u_n = 0$ that $\int_{\mathbb{R}^N} U^{p-2} \frac{\partial U}{\partial x_1} \widetilde{u}_n = 0$. So, u satisfies

$$\int_{\mathbb{R}^N} U^{p-2} \frac{\partial U}{\partial x_1} u = 0. \tag{3.4}$$

Now, we claim that u satisfies

$$-\Delta u + u - (p-1)U^{p-2}u = 0 \quad in \ \mathbb{R}^N. \tag{3.5}$$

Indeed, we set

$$\widetilde{E} = \left\{\varphi : \varphi \in H^1\left(\mathbb{R}^N\right), \ \int_{\mathbb{R}^N} U^{p-2} \frac{\partial U}{\partial x_1} \varphi = 0\right\}.$$

For any $R > 0$, let $\varphi \in C_0^\infty(B_R(0) \cap \widetilde{E}$ be any function, satisfying that φ is even in $y_i, i = 2, \ldots, N$. Then $\varphi_1(y) = \varphi(y - x^1) \in C_0^\infty(B_R(x^1))$. Using (3.2), we see

$$\int_{B_R(x^1)} \nabla u_n \cdot \nabla \varphi_1 + (1+\beta V(|y|))u_n\varphi_1 - pU_r^{p-1}u_n\varphi_1 = o(1),$$

i.e.,

$$\begin{aligned}
&\int_{B_R(0)} \nabla \widetilde{u}_n \nabla \varphi + (1+\beta V(|y+x^1|))\widetilde{u}_n\varphi - pU^{p-1}\widetilde{u}_n\varphi \\
&+O\left(\int_{B_R(0)} \sum_{j=2}^{k} U^{p-2}\left(y+x^1-x^j\right) U\widetilde{u}_n\varphi\right) = o(1).
\end{aligned}$$

Since

$$\left|\int_{B_R(0)} \beta V(|y+x^1|))\widetilde{u}_n\varphi\right| \leq C\beta r^a e^{-\alpha r}\|\tilde{u}_n\| = C\beta^{\frac{-2\sin\frac{\pi}{k}+\alpha\delta}{\alpha-2\sin\frac{\pi}{k}}} = o(1)$$

and

$$\left|\int_{B_R(0)} \sum_{j=2}^{k} U^{p-2}\big(y + x^1 - x^j\big) U \widetilde{u}_n \varphi\right| \le C e^R e^{-\min\{p-2,1\}|x^2-x^1|} \|\tilde{u}_n\| = o(1),$$

we find

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi + u\varphi - pU^{p-1} u\varphi = 0. \tag{3.6}$$

On the other hand, since u is even in y_i, $i = 2, \dots, N$, (3.6) holds for any $\varphi \in C_0^\infty(B_R(0) \cap \widetilde{E})$. We know that $\varphi = \frac{\partial U}{\partial x_1}$ is a solution of (3.4), and thus (3.6) is true for any $\varphi \in H^1(\mathbb{R}^N)$. So we have proved (3.5).

Since U is nondegenerate, we see that $u = c\frac{\partial U}{\partial x_1}$ because u is even in y_i, $i = 2, \dots, N$. From $\int_{\mathbb{R}^N} U^{p-2} \frac{\partial U}{\partial x_1} u = 0$, we find

$$u = 0.$$

As a result,

$$\int_{B_R(x^1)} u_n^2 = o(1), \quad \forall R > 0.$$

Thus,

$$\begin{aligned} o(1)\|u_n\| &= \langle Lu_n, u_n\rangle = \int_{\mathbb{R}^N} |\nabla u_n|^2 + (1 + \beta V(y)) u_n^2 - (p-1)U^{p-2} u_n^2 \\ &\ge c\|u_n\|^2 + o_R(1)\|u_n\|^2 + o(1) \\ &> c' > 0 \end{aligned}$$

provided that R and n are large enough.

As a result, we get a contradiction. □

Let

$$\begin{aligned} J(\omega) &= I(U_r + \omega) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla(U_r + \omega)|^2 + (1 + \beta V(|y|))(U_r + \omega)^2\Big) - \frac{1}{p}\int_{\mathbb{R}^N} |U_r + \omega|^p. \end{aligned}$$

By a direct calculation, we have

$$J(\omega) = J(0) + l(\omega) + \frac{1}{2}\langle L\omega, \omega\rangle + R(\omega),$$

where

$$l(\omega) = \int_{\mathbb{R}^N} \sum_{j=1}^{k} \beta V(y) U_{x^j} \omega - \int_{\mathbb{R}^N} \left(U_r^{p-1} - \sum_{j=1}^{k} U_{x^j}^{p-1}\right)\omega$$

and

$$R(\omega) = -\frac{1}{p}\int_{\mathbb{R}^N}\left(|U_r+\omega|^p - U_r^p - pU_r^{p-1}\omega - \frac{p(p-1)}{2}U_r^{p-2}\omega^2\right).$$

We have the following estimates.

Lemma 3.2. *There is a constant $C > 0$ independent of β such that*

$$\|R'(\omega)\| = O\left(\|\omega\|^{\min\{2,p-1\}}\right)$$

and

$$\|R''(\omega)\| = O\left(\|\omega\|^{\min\{1,p-2\}}\right).$$

Proof. By direct calculation, we have

$$\left\langle R'(\omega), \psi\right\rangle = -\int_{\mathbb{R}^N}\left((U_r+\omega)^{p-1} - U_r^{p-1} - (p-1)U_r^{p-2}\omega\right)\psi$$

and

$$\left\langle R''(\omega)(\psi,\xi)\right\rangle = -(p-1)\int_{\mathbb{R}^N}\left((U_r+\omega)^{p-2} - U_r^{p-2}\right)\psi\xi.$$

Here we only deal with the case $p > 3$, since the situation $2 < p < 3$ is similar. Noting

$$\left|\left\langle R'(\omega), \psi\right\rangle\right| \le C\int_{\mathbb{R}^N} U_r^{p-3}|\omega|^2|\psi| \le C\left(\int_{\mathbb{R}^N}\left(U_r^{p-3}|\omega|^2\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\|\psi\|_s,$$

we find

$$\left\|R'(\varphi)\right\| \le C\left(\int_{\mathbb{R}^N}\left(U_r^{p-3}|\varphi|^2\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.$$

Considering that U_r is bounded and $2 < \frac{2p}{p-1} < p$, we obtain

$$\left\|R'(\varphi)\right\| \le C\left(\int_{\mathbb{R}^N}|\varphi|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}} \le C\|\varphi\|_s^2.$$

For the estimate of $\|R''(\varphi)\|$, we have

$$\begin{aligned}\left|R''(\varphi)(\psi,\xi)\right| &\le C\int_{\mathbb{R}^N} U_r^{p-3}|\varphi||\psi||\xi| \\ &\le C\int_{\mathbb{R}^N}|\varphi||\psi||\xi| \le C\left(\int_{\mathbb{R}^N}|\varphi|^3\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^N}|\psi|^3\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^N}|\xi|^3\right)^{\frac{1}{3}} \\ &\le C\|\varphi\|_s\|\psi\|_s\|\xi\|_s.\end{aligned}$$

Therefore,

$$\left\|R''(\varphi)\right\| \le C\|\varphi\|_s. \qquad \square$$

Lemma 3.3. *For any $\tau \in (0,1)$, we can find $C > 0$ independent of β such that*

$$\|l\| \le C\left(\beta e^{-(1-\tau)\alpha r} + e^{-\min\{p-1-\tau,\,2-\tau\} r \sin\frac{\pi}{k}}\right).$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned}
|\langle l,\omega\rangle| &= \left|\int_{\mathbb{R}^N}\sum_{j=1}^{k}\beta V(y)U_{x^j}\omega - \left(U_r^{p-1}-\sum_{j=1}^{k}U_{x^j}^{p-1}\right)\omega\right| \\
&\le k\beta\int_{\mathbb{R}^N} V(y)U_{x^1}|\omega| + \int_{\mathbb{R}^N}\left|U_r^{p-1}-\sum_{j=1}^{k}U_{x^j}^{p-1}\right||\omega| \\
&\le k\beta\left[\int_{\mathbb{R}^N} V^2(|y+x^1|)U^2\right]^{\frac12}\left(\int_{\mathbb{R}^N}\omega^2\right)^{\frac12} \\
&\quad + \begin{cases} C\displaystyle\int_{\mathbb{R}^N}\sum_{i\ne j}U_{x^i}^{p-2}U_{x^j}|\omega| & (p>3) \\ C\displaystyle\int_{\mathbb{R}^N}\sum_{i\ne j}U_{x^i}^{\frac{p-1}{2}}U_{x^j}^{\frac{p-1}{2}}|\omega| & (2<p\le 3)\end{cases} \\
&\le C\beta e^{-(1-\tau)\alpha r}\|\omega\| + Ce^{-\min\{p-1-\tau,\,2-\tau\} r \sin\frac{\pi}{k}}\|\omega\|.
\end{aligned}$$

Hence

$$\|l\| \le C\left(\beta e^{-(1-\tau)\alpha r} + e^{-\min\{p-1-\tau,\,2-\tau\} r \sin\frac{\pi}{k}}\right). \qquad \square$$

Proposition 3.4. *Under the conditions of Lemma* 3.1*, for any $r \in S$, there is a unique $\omega \in E$ satisfying*

$$J'(\omega)|_E = 0. \tag{3.7}$$

Moreover,

$$\|\omega\| \le C\left(\beta e^{-(1-\tau)\alpha r} + e^{-\min\{p-1-\tau,\,2-\tau\} r \sin\frac{\pi}{k}}\right),$$

where τ is the same as that of Lemma 3.3.

Proof. By Lemma 3.3, $l(\omega)$ is a bounded linear functional on E. We know by Riesz representation theorem that there is an $l \in E$, such that

$$l(\omega) = \langle l,\omega\rangle.$$

So, finding a critical point for $J(\omega)$ is equivalent to solving

$$l + L\omega + R'(\omega) = 0. \tag{3.8}$$

By Lemma 3.1, L is invertible. Thus, (3.8) is equivalent to

$$\omega = A(\omega) =: -L^{-1}\big(l + R'(\omega)\big).$$

Set

$$S_r := \left\{ \omega \in E : \ \ \|\omega\| \le \beta e^{-(1-\tau_1)\alpha r} + e^{-\min\{p-1-\tau_1,\, 2-\tau_1\} r \sin\frac{\pi}{k}} \right\},$$

where $\tau < \tau_1 < 1$.

Now we verify that A is a contraction from S_r to S_r. Indeed, we see

$$\begin{aligned}
\|A(\omega)\| &\le \left(\|l\| + \|R'(\omega)\|\right) \\
&\le \|l\| + C\|\omega\|^{\min\{2,p-1\}} \\
&\le C\left(\beta e^{-(1-\tau)\alpha r} + e^{-\min\{p-1-\tau,\, 2-\tau\} r \sin\frac{\pi}{k}}\right) \\
&\quad + C\left(\beta e^{-(1-\tau_1)\alpha r} + e^{-\min\{p-1-\tau_1,\, 2-\tau_1\} r \sin\frac{\pi}{k}}\right)^{\min\{2,p-1\}} \\
&\le \beta e^{-(1-\tau_1)\alpha r} + e^{-\min\{p-1-\tau_1,\, 2-\tau_1\} r \sin\frac{\pi}{k}},
\end{aligned}$$

which implies that A maps S_r to S_r. On the other hand, for any $\omega_1,\ \omega_2 \in S_r$,

$$\begin{aligned}
\|A(\omega_1) - A(\omega_2)\| &= \left\|L^{-1}R(\omega_1) - L^{-1}R(\omega_2)\right\| \\
&\le C\|R(\omega_1) - R(\omega_2)\| \\
&\le C\left\|R'(\theta\omega_1 - (1-\theta)\omega_2)\right\| \|\omega_1 - \omega_2\| \\
&\le C\|\omega_1 + \omega_2\| \|\omega_1 - \omega_2\| \\
&\le \frac{1}{2}\|\omega_1 - \omega_2\|.
\end{aligned}$$

Hence, A is a contraction map in S_r and the result follows from the contraction mapping theorem. $\square$

Now we are ready to prove Theorem 1.4. Let $\omega = \omega_r$ be obtained in Proposition 3.4, and define

$$F(r) = I(U_r + \omega), \quad \forall r \in S.$$

It is well known that if r is a critical point of $F(r)$, then $U_r + \omega_r$ is a solution of (1.1) (see [15,18]).

Proof of Theorem 1.4. Since

$$\begin{aligned}
F(r) &= I(U_r + \omega) \\
&= I(U_r) + l(\omega) + \frac{1}{2}\langle L\omega, \omega\rangle + R(\omega) \\
&= I(U_r) + O\left(\|l\|\|\omega\| + \|\omega\|^2\right),
\end{aligned}$$

it follows from Propositions 2.3 and 3.4 that

$$\begin{aligned} F(r) &= A + \beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r \sin\frac{\pi}{k}} \\ &\quad + O\big(e^{-2(1+\tau)r\sin\frac{\pi}{k}} + \beta^2 e^{-2(1-\tau)\alpha r}\big) \\ &= A + \big(\beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r\sin\frac{\pi}{k}}\big) \\ &\quad + O\big(e^{-2(1+\tau)r\sin\frac{\pi}{k}} + \beta e^{-(1+\tau)\alpha r}\big). \end{aligned} \tag{3.9}$$

We first consider the case $2\sin\frac{\pi}{k} < \alpha$.

Define

$$F_1(r) = \beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r\sin\frac{\pi}{k}} + O\big(\beta e^{-(1+\tau)\alpha r} + e^{-2(1+\tau)r\sin\frac{\pi}{k}}\big).$$

We consider the following minimization problem:

$$\min_{r\in S} F_1(r).$$

Supposing that $\hat{r}$ is a minimizer, we will prove that $\hat{r}$ is an interior point of S.

For any $k > 0$ satisfying

$$2\sin\frac{\pi}{k} < \alpha,$$

we can check that the function

$$G(r) = \beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r\sin\frac{\pi}{k}}$$

has a minimum point

$$\widetilde{r} = \left(\frac{1}{\alpha - 2\sin\frac{\pi}{k}} + o(1)\right)\ln\beta$$

and

$$e^{-(\alpha - 2\sin\frac{\pi}{k})\widetilde{r}} = \frac{1}{\beta}\frac{B'_\beta}{B_\beta}\widetilde{r}^{\frac{1-N}{2}-a}\frac{\frac{N-1}{2} + 2\widetilde{r}\sin\frac{\pi}{k}}{\alpha\widetilde{r} - a}.$$

By direct computation, we deduce that

$$\begin{aligned} F_1(\widetilde{r}) &= e^{-2\widetilde{r}\sin\frac{\pi}{k}}\Big(B_\beta\beta\widetilde{r}^a e^{-(\alpha-2\sin\frac{\pi}{k})\widetilde{r}} - B'_\beta\widetilde{r}^{\frac{1-N}{2}}\Big) \\ &\quad + O\left(\beta e^{-(1+\tau)\alpha\widetilde{r}} + e^{-(1+\tau)2\widetilde{r}\sin\frac{\pi}{k}}\right) \\ &= e^{-2\widetilde{r}\sin\frac{\pi}{k}} B'_\beta \widetilde{r}^{\frac{1-N}{2}}\left(\frac{2\sin\frac{\pi}{k}\widetilde{r} + \frac{N-1}{2}}{\alpha\widetilde{r} - 1} - 1\right) + O\left(\beta^{\frac{-2\sin\frac{\pi}{k} - \delta_0}{\alpha - 2\sin\frac{\pi}{k}}}\right) \\ &= \beta^{\frac{-2\sin\frac{\pi}{k}}{\alpha-2\sin\frac{\pi}{k}}} B'_\beta\widetilde{r}^{\frac{1-N}{2}}\left(\frac{2\sin\frac{\pi}{k}}{\alpha} - 1 + o\left(\frac{1}{\widetilde{r}}\right)\right) < 0. \end{aligned}$$

On the other hand, we find

$$\begin{aligned}
F_1\left(\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right) &= B_\beta\beta r^a e^{-\alpha\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} - B'_\beta r^{\frac{1-N}{2}} e^{-2\sin\frac{\pi}{k}\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} \\
&\quad + O\left(\beta e^{-(1+\tau)\alpha r} + e^{-(1+\tau)2r\sin\frac{\pi}{k}}\right) \\
&= B_\beta\left(\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^a \beta^{\frac{-2\sin\frac{\pi}{k}+\alpha\delta}{\alpha-2\sin\frac{\pi}{k}}} \\
&\quad - B'_\beta\left(\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^{\frac{1-N}{2}} \beta^{\frac{-2\sin\frac{\pi}{k}+2\sin\frac{\pi}{k}\delta}{\alpha-2\sin\frac{\pi}{k}}} \\
&\quad + O\left(\beta^{\frac{-2\sin\frac{\pi}{k}+\alpha\delta-\delta_0}{\alpha-2\sin\frac{\pi}{k}}}\right) \\
&> 0 > F_1(\widetilde{r})
\end{aligned}$$

and

$$\begin{aligned}
F_1\left(\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right) &= B_\beta\beta r^a e^{-\alpha\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} - B'_\beta r^{\frac{1-N}{2}} e^{-2\sin\frac{\pi}{k}\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} \\
&\quad + O\left(\beta e^{-(1+\tau)\alpha r} + e^{-(1+\tau)2r\sin\frac{\pi}{k}}\right) \\
&= B_\beta\left(\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^a \beta^{\frac{-2\sin\frac{\pi}{k}-\alpha\delta}{\alpha-2\sin\frac{\pi}{k}}} \\
&\quad - B'_\beta\left(\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^{\frac{1-N}{2}} \beta^{\frac{-2\sin\frac{\pi}{k}-2\sin\frac{\pi}{k}\delta}{\alpha-2\sin\frac{\pi}{k}}} \\
&\quad + O\left(\beta^{\frac{-2\sin\frac{\pi}{k}-2\sin\frac{\pi}{k}\delta-\delta_0}{\alpha-2\sin\frac{\pi}{k}}}\right) \\
&\geq -C(\ln\beta)^{\frac{1-N}{2}}\beta^{\frac{-2\sin\frac{\pi}{k}-2\sin\frac{\pi}{k}\delta}{\alpha-2\sin\frac{\pi}{k}}} > F_1(\widetilde{r}),
\end{aligned}$$

for some $\delta_0 > 0$.

The above estimates imply that $\hat{r}$ is indeed an interior point of S. Thus

$$U_{\hat{r}} + \omega_{\hat{r}}$$

is a solution of (1.1).

Now we investigate the case $2\sin\frac{\pi}{k} > \alpha$. In this case, we should solve (1.1) for $\beta > 0$ sufficiently small.

We consider the following maximization problem:

$$\max_{r\in S} F_1(r).$$

We will prove that the maximizer $\check{r}$ is an interior point of S.

For any $k > 0$ satisfying
$$2\sin\frac{\pi}{k} > \alpha,$$
we find that
$$G(r) = \beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r\sin\frac{\pi}{k}}$$
has a maximum point
$$\widetilde{r} = \left(\frac{1}{\alpha - 2\sin\frac{\pi}{k}} + o(1)\right)\ln\beta$$
and
$$e^{-(\alpha-2\sin\frac{\pi}{k})\widetilde{r}} = \frac{1}{\beta}\frac{B'_\beta}{B_\beta}\widetilde{r}^{\frac{1-N}{2}-a}\frac{\frac{N-1}{2}+2\widetilde{r}\sin\frac{\pi}{k}}{\alpha\widetilde{r}-a}.$$
By direct computation, we deduce that
$$\begin{aligned}
F_1(\widetilde{r}) &= e^{-2\widetilde{r}\sin\frac{\pi}{k}}\left(B_\beta\beta\widetilde{r}^a e^{-(\alpha-2\sin\frac{\pi}{k})\widetilde{r}} - B'_\beta\widetilde{r}^{\frac{1-N}{2}}\right) \\
&\quad + O\left(\beta e^{-(1+\tau)\alpha\widetilde{r}} + e^{-(1+\tau)2\widetilde{r}\sin\frac{\pi}{k}}\right) \\
&= e^{-2\widetilde{r}\sin\frac{\pi}{k}} B'_\beta\widetilde{r}^{\frac{1-N}{2}}\left(\frac{2\sin\frac{\pi}{k}\widetilde{r}+\frac{N-1}{2}}{\alpha\widetilde{r}-1} - 1\right) + O\left(\beta^{\frac{2\sin\frac{\pi}{k}+\delta_0}{2\sin\frac{\pi}{k}-\alpha}}\right) \\
&= \beta^{\frac{2\sin\frac{\pi}{k}}{2\sin\frac{\pi}{k}-\alpha}} B'_\beta\widetilde{r}^{\frac{1-N}{2}}\left(\frac{2\sin\frac{\pi}{k}}{\alpha} - 1 + o\left(\frac{1}{\widetilde{r}}\right)\right) > 0.
\end{aligned}$$

On the other hand, we see
$$\begin{aligned}
F_1\left(\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right) &= B_\beta\beta r^a e^{-\alpha\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} - B'_\beta r^{\frac{1-N}{2}} e^{-2\sin\frac{\pi}{k}\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} \\
&\quad + O\left(\beta e^{-(1+\tau)\alpha r} + e^{-(1+\tau)2r\sin\frac{\pi}{k}}\right) \\
&= B_\beta\left(\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^a \beta^{\frac{2\sin\frac{\pi}{k}-\alpha\delta}{2\sin\frac{\pi}{k}-\alpha}} \\
&\quad - B'_\beta\left(\frac{1-\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^{\frac{1-N}{2}} \beta^{\frac{2\sin\frac{\pi}{k}-2\sin\frac{\pi}{k}\delta}{2\sin\frac{\pi}{k}-\alpha}} \\
&\quad + O\left(\beta^{\frac{2\sin\frac{\pi}{k}-\alpha\delta+\delta_0}{2\sin\frac{\pi}{k}-\alpha}}\right) \\
&< 0 < F_1(\widetilde{r})
\end{aligned}$$

and

$$\begin{aligned}F_1\left(\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right) &= B_\beta\beta r^a e^{-\alpha\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} - B'_\beta r^{\frac{1-N}{2}} e^{-2\sin\frac{\pi}{k}\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta} \\ &\quad + O\left(\beta e^{-(1+\tau)\alpha r} + e^{-(1+\tau)2r\sin\frac{\pi}{k}}\right) \\ &= B_\beta\left(\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^a \beta^{\frac{2\sin\frac{\pi}{k}+\alpha\delta}{2\sin\frac{\pi}{k}-\alpha}} \\ &\quad - B'_\beta\left(\frac{1+\delta}{\alpha-2\sin\frac{\pi}{k}}\ln\beta\right)^{\frac{1-N}{2}} \beta^{\frac{2\sin\frac{\pi}{k}+2\sin\frac{\pi}{k}\delta}{2\sin\frac{\pi}{k}-\alpha}} \\ &\quad + O\left(\beta^{\frac{2\sin\frac{\pi}{k}+2\sin\frac{\pi}{k}\delta+\delta_0}{2\sin\frac{\pi}{k}-\alpha}}\right) \\ &\le C(|\ln\beta|)^a \beta^{\frac{2\sin\frac{\pi}{k}+\alpha\delta}{2\sin\frac{\pi}{k}-\alpha}} < F_1(\widetilde{r})\end{aligned}$$

for some $\delta_0 > 0$.

The above estimate implies that $\check{r}$ is actually an interior point of S. Thus

$$U_{\check{r}} + \omega_{\check{r}}$$

is a solution of (1.1). □

Remark 3.5. In assumption (V) (or (V')), if $\alpha \in (1,2)$, then part (ii) of Theorem 1.4 is still true. Indeed, in this case, we should modify the proof of Lemma 3.3 and obtain the following estimate on l

$$\|l\| \le C\left(\beta e^{-(1-\tau)r} + e^{-\min\{p-1-\tau,\,2-\tau\}r\sin\frac{\pi}{k}}\right).$$

Hence, we have the following energy expansion

$$\begin{aligned}F(r) &= A + \beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r\sin\frac{\pi}{k}} + O\left(e^{-2(1+\tau)r\sin\frac{\pi}{k}} + \beta^2 e^{-2(1-\tau)r}\right) \\ &= A + \beta B_\beta r^a e^{-\alpha r} - B'_\beta r^{\frac{1-N}{2}} e^{-2r\sin\frac{\pi}{k}} + O\left(e^{-2(1+\tau)r\sin\frac{\pi}{k}} + \beta e^{-(1+\tau)\alpha r}\right).\end{aligned}$$

Now proceeding as we have done to prove Theorem 1.4, we can complete the proof.

References

[1] A. AMBROSETTI, E. COLORADO and D. RUIZ, *Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations*, Calc. Var. Partial Differential Equations **30** (2007), 85–112.

[2] A. AMBROSETTI, A. MALCHIODI and S. SECCHI, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Ration. Mech. Anal. **159** (2001), 253–271.

[3] A. BAHRI and Y. Y. LI, *On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$*, Rev. Mat. Iberoamericana **6** (1990), 1–15.

[4] A. BAHRI and P. L. LIONS, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 365–413.

[5] T. BARTSCH and Z. Q. WANG, *Multiple positive solutions for a nonlinear Schrödinger equation*, Z. Angew. Math. Phys. **51** (2000), 336–384.

[6] T. BARTSCH, A. PANKOV and Z. Q. WANG, *Nonlinear Schrödinger equations with steep potential well*, Commun. Contemp. Math. **3** (2001), 549–569.

[7] I. M. BESIERIS, "Solitons in Randomly Inhomogeneous Media", Nonlinear Electromagenetics, Academic Press, New York, 1980, 87–116.

[8] J. BYEON and Y. OSHITA, *Existence of multi-bump stading waves with a critical frequency for nonlinear Schrödinger equations*, Comm. Partial Differential Equations **29** (2004), 1877–1904.

[9] D. CAO and S. PENG, *Multi-bump bound states of Schrödinger equations with a critical frequency*, Math. Ann. **336** (2006), 925–948.

[10] M. CLAPP and Y. DING, *Positive solutions of a Schrödinger equations with critical nonlinearity*, Z. Angew. Math. Phys. **55** (2004), 592–605.

[11] T. D'APRILE and A. PISTOIA, *Existence, multiplicity and profile of sign-changing clustered solutions of a semiclassical nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), 1423–1451.

[12] Y. DING and K. TANAKA, *Multiplicity of positive solutions of a nonlinear Schrödinger equation*, Manuscripta Math. **112** (2003), 109–135.

[13] W. Y. DING and W. M. NI, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Ration. Mech. Anal. **91** (1986), 283–308.

[14] A. FLOER and A. WEINSTEIN, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. **69** (1986), 397–408.

[15] X. KANG and J. WEI, *On interacting bumps of semi-classical states of nonlinear Schrödinger equations*, Adv. Differential Equations **5** (2000), 899–928.

[16] L. LIN and Z. LIU, *Multi-bump solutions and multi-tower solutions for equations on $\mathbb{R}^N$*, J. Funct. Anal. **257** (2009), 485–505.

[17] L. LIN, Z. LIU and S. CHEN, *Multi-bump solutions for a semilinear Schrödinger equation*, Indiana Univ. Math. J. **58** (2009), 1659–1689.

[18] F. LIN, W. M. NI and J. WEI, *On the number of interior peak solutions for a singularly perturbed Neumann problem*, Comm. Pure Appl. Math. **60** (2007), 252–281.

[19] Y. Y. LI, *On a singularly perturbed elliptic equation*, Adv. Differential Equations, **2** (1997), 955–980.

[20] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The locally compact case I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145.

[21] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The locally compact case II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 223–283.

[22] P. H. RABINOWITZ, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.

[23] Y. SATO, *Multi-peak positive solutions for nonlinear Schrödinger equations with critical frequency*, Calc. Var. Partial Differential Equations **29** (2007), 365–395.
[24] J. WEI and S. YAN, *New solutions for nonlinear Schrödinger equations with critical nonlinearity*, J. Differential Equations **237** (2007), 446–472.
[25] J. WEI and S. YAN, *Infinite many positive solutions for the nonlinear Schrödinger equation in* $\mathbb{R}^n$, Calc. Var. Partial Differential Equations **37** (2010), 423–439.

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