Tamed symplectic structures on compact solvmanifolds of completely solvable type

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Abstract. A compact solvmanifold of completely solvable type, *i.e.* a compact quotient of a completely solvable Lie group by a lattice, has a Kähler structure if and only if it is a complex torus. We show that a compact solvmanifold M of completely solvable type endowed with an invariant complex structure J admits a symplectic form taming J if and only if M is a complex torus. This result generalizes the one obtained in [7] for nilmanifolds.

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1. Introduction

We will say that a symplectic form Ω on a complex manifold (M, J) tames the complex structure J if $\Omega(X, JX) > 0$ for every non-zero vector field X on M or, equivalently, if the $(1, 1)$-part of the 2-form Ω is positive. By [14, 18] a compact complex surface admitting a symplectic structure taming a complex structure is necessarily Kähler. Moreover, by [13] non-Kähler Moishezon complex structures on compact manifolds cannot be tamed by a symplectic form (see also [19]). However, it is still an open problem to find out an example of a compact manifold having a symplectic structure taming a complex structure but not admitting any Kähler structure. Some negative results have been obtained for compact nilmanifolds in [7] and for special classes of compact solvmanifolds in [8], where by a compact nilmanifold (respectively solvmanifold) we mean a compact quotient of a nilpotent (resp. solvable) Lie group G by a discrete subgroup Γ.

We recall that a compact nilmanifold is Kähler if and only if it is diffeomorphic to a torus ([3, 11]). Benson and Gordon in [3] conjectured that if a compact

solvmanifold $\Gamma \backslash G$ of completely solvable type admits a Kähler metric then M is diffeomorphic to a standard $2n$-torus. The conjecture was proved by Hasegawa [12] and more generally it was showed that a compact solvmanifold is Kähler if and only if it is a finite quotient of a complex torus which has the structure of a complex torus bundle over a complex torus. A similar result was proved by Baues and Cortes in [1] for Kähler infra-solvmanifolds, showing the relation of the Benson-Gordon conjecture to the more general problem of aspherical Kähler manifolds with solvable fundamental group. In [7] it has been shown that a compact nilmanifold $\Gamma \backslash G$ endowed with an invariant complex structure J, *i.e.* a complex structure which comes from a left invariant complex structure on G, admits a symplectic form taming J if and only if $\Gamma \backslash G$ is a torus. The result was obtained by using a characterization of compact nilmanifolds admitting pluriclosed metrics, *i.e.* Hermitian metrics such that its fundamental form ω satisfies the condition $\partial \bar{\partial} \omega = 0$. For compact solvmanifolds by [8] if J is invariant under the action of a nilpotent complement of the nilradical of G and J is abelian or G is almost abelian (not of type (I)), then the compact solvmanifold $\Gamma \backslash G$ cannot admit any symplectic form taming the complex structure J, unless $\Gamma \backslash G$ is Kähler. For a compact solvmanifold $\Gamma \backslash G$ of completely solvable type, no general result about the existence of symplectic forms taming complex structures is known. In the present paper we will show that if such symplectic structures exist then the compact solvmanifold has to be a torus and therefore a Kähler manifold. We note that at the level of Lie algebras, a unimodular completely solvable Kähler Lie algebra is necessarily abelian. As remarked in [1] this follows from Hano's result [10] that a unimodular Kähler Lie group has to be flat and from the classification of flat Lie groups obtained in [16]. Kähler Lie algebras have been also studied in [4], where the so-called Kähler double extension has been introduced. The Kähler double extension realizes a Kähler Lie algebra as the Kähler reduction of another one. In [2] it is shown that every symplectic Lie group admits a sequence of symplectic reductions to a unique irreducible symplectic Lie group. In particular, every symplectic completely solvable Lie group is always symplectically reduced. By using this symplectic reduction we show that if $\mathfrak{g}$ is unimodular completely solvable Lie algebra admitting a symplectic form Ω taming a complex structure J, then $\mathfrak{g}$ has to be abelian. As a consequence we prove the following:

Theorem. *A compact solvmanifold M of completely solvable type endowed with an invariant complex structure J admits a symplectic form taming J if and only if M is a complex torus.*

2. Preliminaries

A Lie algebra $\mathfrak{g}$ is called completely solvable if $\text{ad}_X : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for every $X \in \mathfrak{g}$. Equivalently, $\mathfrak{g}$ is isomorphic to a subalgebra of the (real)

upper triangular matrices in $\mathfrak{gl}(m, \mathbb{R})$ for some m. In particular, nilpotent Lie groups are completely solvable.

Lattices in completely solvable Lie groups satisfy strong rigidity properties, which are similar to the Malcev rigidity [15] of lattices in nilpotent Lie groups. These properties were first shown by Saito in [17].

A unimodular Kähler Lie group has to be flat, as proved by Hano in [10]. Therefore a completely solvable unimodular Kähler Lie group is abelian by virtue of Hano's result and the classification of flat Lie groups obtained in [16]. For non-unimodular Kähler Lie algebras general results have been obtained by Gindikin, Vinberg, Pyatetskii-Shapiro in [9] (see also [5]).

In [2] it is shown that every symplectic Lie group admits a sequence of subsequent symplectic reductions to a unique irreducible symplectic Lie group.

Let $(\mathfrak{g}, \Omega)$ be a symplectic Lie algebra. An ideal $\mathfrak{h}$ of $\mathfrak{g}$ is called an isotropic ideal of $(\mathfrak{g}, \Omega)$ if $\mathfrak{h}$ is an isotropic subspace for Ω, *i.e.* $\Omega|_{\mathfrak{h} \times \mathfrak{h}} = 0$. Note that $\mathfrak{h}$ is isotropic if and only if $\mathfrak{h} \subseteq \mathfrak{h}^{\perp\Omega}$, where $\mathfrak{h}^{\perp\Omega}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to Ω.

By [2, Lemma 2.1] one has the following properties:

(1) If $\mathfrak{h}$ is an isotropic ideal of a symplectic Lie algebra $(\mathfrak{g}, \Omega)$ then $\mathfrak{h}$ is abelian.
(2) If $\mathfrak{h}$ is an ideal of a symplectic Lie algebra $(\mathfrak{g}, \Omega)$, then $\mathfrak{h}^{\perp\Omega}$ is a Lie subalgebra of $\mathfrak{g}$.
(3) If $\mathfrak{h}$ is an ideal of a symplectic Lie algebra $(\mathfrak{g}, \Omega)$, then $\mathfrak{h}^{\perp\Omega}$ is an ideal in $\mathfrak{g}$ if and only if $[\mathfrak{h}^{\perp\Omega}, \mathfrak{h}] = 0$.

We will now briefly review the symplectic reduction. If $(\mathfrak{g}, \Omega)$ is a symplectic Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ is an isotropic ideal, then $\mathfrak{h}^{\perp\Omega}$ with respect to Ω is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ and therefore Ω descends to a symplectic form $\tilde{\Omega}$ on the quotient Lie algebra $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$. The symplectic Lie algebra $(\mathfrak{h}^{\perp\Omega}/\mathfrak{h}, \tilde{\Omega})$ is called the symplectic reduction of $(\mathfrak{g}, \Omega)$ with respect to the isotropic ideal $\mathfrak{h}$.

If $\mathfrak{g}$ is completely solvable then by [2, Example 2.4], $\mathfrak{g}$ contains a nontrivial ideal $\mathfrak{h}$ which is isotropic. In the case of symplectic forms taming complex structures we now show the following:

Lemma 2.1. *Let $\mathfrak{g}$ be a completely solvable Lie algebra endowed with a symplectic form Ω taming a complex structure J. Then one has the following decomposition (in terms of vector spaces):*

$$\mathfrak{g} = J\mathfrak{h} \oplus \mathfrak{h}^{\perp\Omega},$$

with $\mathfrak{h}$ an abelian isotropic ideal of $\mathfrak{g}$ and $\mathfrak{h}^{\perp\Omega}$ a Lie subalgebra of $\mathfrak{g}$.

Proof. Since $\dim J\mathfrak{h} = \dim \mathfrak{h} = \dim \mathfrak{g} - \dim \mathfrak{h}^{\perp\Omega}$, it is sufficient to show that $J\mathfrak{h} \cap \mathfrak{h}^{\perp\Omega} = \{0\}$. Let Y be a non zero element belonging to the intersection $J\mathfrak{h} \cap \mathfrak{h}^{\perp\Omega}$. Then one has that $Y = JX$, with $X \in \mathfrak{h}$ and $X \neq 0$. But then since $Y \in \mathfrak{h}^{\perp\Omega}$, one should have in particular

$$\Omega(JX, X) = 0,$$

which is not possible since Ω tames J. $\qquad\square$

The ideal $\mathfrak{h}$ is at least 1-dimensional, so we can suppose that dim $\mathfrak{h} = 1$.

Lemma 2.2. *Let $\mathfrak{g}$ be a completely solvable Lie algebra endowed with a symplectic form Ω taming a complex structure J and let $\mathfrak{h}$ be a 1-dimensional isotropic ideal. Then the Lie algebra $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$ admits a symplectic form $\tilde{\Omega}$ taming a complex structure $\tilde{J}$.*

If $\mathfrak{g}$ is unimodular and $\mathfrak{h}^{\perp\Omega}$ is an abelian ideal of $\mathfrak{g}$, then $\mathfrak{g}$ is abelian.

Proof. Suppose that $\mathfrak{h} = \operatorname{span}\langle X \rangle$. So $J\mathfrak{h} = \operatorname{span}\langle JX \rangle$ and we know that, since $\mathfrak{h}$ is an ideal then $[X, \mathfrak{g}] \in \operatorname{span}\langle X \rangle$. We have that

$$J \left(Y - \frac{\Omega(JY, X)}{\Omega(JX, X)} X \right) \in \mathfrak{h}^{\perp\Omega},$$

for every $Y \in \mathfrak{h}^{\perp\Omega}$. Moreover, since $\mathfrak{h}$ is an ideal, the complex structure J induces a complex structure $\tilde{J}$ on $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$, defined by

$$\tilde{J}(Y + \mathfrak{h}) = JY + \mathfrak{h}.$$

Indeed, if JY does not belong to $\mathfrak{h}^{\perp\Omega}$, one changes $Y + \mathfrak{h}$ to

$$Y - \frac{\Omega(JY, X)}{\Omega(JX, X)} X + \mathfrak{h}.$$

Moreover, $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$ has a symplectic structure $\tilde{\Omega}$ induced by Ω. Moreover, the complex structure on $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$ is tamed by $\tilde{\Omega}$. Suppose that $\mathfrak{h}^{\perp\Omega}$ is an abelian ideal of $\mathfrak{g}$. Then $\mathfrak{g} = J\mathfrak{h} \ltimes \mathfrak{h}^{\perp\Omega}$ and it is almost abelian. Since $\mathfrak{g}$ is not of type (I), by [8, Proposition 7.1] we have that $\mathfrak{g}$ has to be abelian. $\square$

3. Main result

By [6] a 4-dimensional completely solvable Lie algebra endowed with a symplectic form Ω taming a complex structure J is necessarily Kähler and, if it is unimodular, then it is abelian. We now show that in every dimension a unimodular completely solvable Lie algebra endowed with a symplectic form Ω taming a complex structure J is abelian.

Theorem 3.1. *Let $\mathfrak{g}$ be a $2n$-dimensional unimodular completely solvable Lie algebra endowed with a symplectic form Ω taming a complex structure J, then $\mathfrak{g}$ is abelian.*

Proof. For $n = 2$ we know by [6] that the theorem holds. We will prove the theorem by induction.

Let $\mathfrak{h} = \operatorname{span}\langle X \rangle$ be a 1-dimensional isotropic ideal $\mathfrak{h}$ of $\mathfrak{g}$. Since dim $\mathfrak{g} = 2n$, by the previous lemma we can choose a basis $\{X, JX, Y_1, JY_1, \ldots, Y_{n-1}, JY_{n-1}\}$ of $\mathfrak{g}$

with $Y_l, JY_l \in \mathfrak{h}^{\perp\Omega}, l = 1, \ldots, n-1$. By Lemma 2.2 the $(2n-2)$-dimensional Lie algebra $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$, which can be identified with $\mathfrak{v} = \operatorname{span}\langle Y_1, JY_1, \ldots, Y_{n-1}, JY_{n-1}\rangle$, has a tamed complex structure. For every $Y \in \mathfrak{v}$ we have $\Omega(Y, X) = 0 = \Omega(JY, X)$ and

$$[X, Y] = a_1 X, \quad [X, JY] = a_2 X, \quad [JX, Y] = b_1 X + b_2 JX + Z_1,$$
$$[JX, JY] = c_1 X + c_2 JX + Z_2,$$

with $\Omega(Z_i, X) = \Omega(JZ_i, X) = 0$, for $i = 1, 2$. By the integrability of J

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY]$$

we obtain

$$(c_1 + b_2 - a_1)X + (c_2 - b_1 - a_2)JX + Z_2 - JZ_1 = 0$$

and therefore

$$(c_2 - b_1 - a_2)\Omega(JX, X) = 0.$$

Since $\Omega(JX, X) \neq 0$, we get

$$c_2 = b_1 + a_2, \quad Z_2 = JZ_1 + (a_1 - b_2 - c_1)X.$$

As a consequence

$$[JX, JY] = (a_1 - b_2)X + (b_1 + a_2)JX + JZ_1.$$

Since $d\Omega = 0$, we have

$$\Omega([JX, Y], X) = -\Omega([Y, X], JX) - \Omega([X, JX], Y)$$

and thus

$$b_2\Omega(JX, X) = -a_1\Omega(JX, X)$$

which implies that $b_2 = -a_1$. By the condition

$$\Omega([JX, JY], X) = -\Omega([JY, X], JX) - \Omega([X, JX], JY)$$

we get

$$-(b_1 + a_2)\Omega(X, JX) = a_2\Omega(X, JX)$$

and therefore $b_1 = -2a_2$. Then, for every $Y \in \mathfrak{v}$, we have

$$[X, Y] = aX, \quad [X, JY] = bX, \quad [JX, Y] = -2bX - aJX + Z_1,$$
$$[JX, JY] = 2aX - bJX + JZ_1,$$

(3.1)

with

$$a = \frac{\Omega([X, Y], JX)}{\Omega(X, JX)}, \quad b = \frac{\Omega([X, JY], JX)}{\Omega(X, JX)}.$$

Moreover by using that $\Omega(Z_1, X) = 0 = \Omega(JZ_1, X)$ we obtain that

$$Z_1 \in \text{span}\langle Y_1, JY_1, \ldots, Y_{n-1}, JY_{n-1}\rangle = \mathfrak{v}.$$

By (3.1) and using the basis $\{X, JX, Y_1, JY_1, \ldots, Y_{n-1}, JY_{n-1}\}$ of $\mathfrak{g}$ we have that for every $Y \in \mathfrak{v}$

$$\text{trace}(\text{ad}_Y) = (-a) + (a) + \text{trace}(\text{ad}_Y \mid_\mathfrak{v}) = 0,$$

since $\mathfrak{g}$ is unimodular, where $\text{ad}_Y \mid_\mathfrak{v}$ denotes the $(2n - 2) \times (2n - 2)$ block matrix of $\text{ad}_Y \mid_\mathfrak{v}$ with respect to the basis $Y_1, JY_1, \ldots, Y_{n-1}, JY_{n-1}$. This implies that $\text{trace}(\text{ad}_Y \mid_\mathfrak{v}) = 0$, for every $Y \in \mathfrak{v}$. Note that by using the identification $\mathfrak{v} \cong \mathfrak{h}^{\perp\Omega}/\mathfrak{h}$ as vector spaces we have that

$$[Y, Z] + \mathfrak{h} = [Y, Z]_\mathfrak{v} + \mathfrak{h}, \quad \forall Y, Z \in \mathfrak{v}$$

where by $[Y, Z]_\mathfrak{v}$ we denote the component of $[Y, Z]$ on $\mathfrak{v}$. This implies that $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$ has to be unimodular.

Therefore by induction we can suppose that the symplectic reduction $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$ has a Kähler structure. Since $\mathfrak{h}^{\perp\Omega}/\mathfrak{h}$ is completely solvable and unimodular, it has to be abelian. So as a vector space:

$$\mathfrak{g} = \text{span}\langle X, JX\rangle \oplus \mathfrak{v},$$

with $\text{span}\langle X\rangle \oplus \mathfrak{v}$ a Lie subalgebra of $\mathfrak{g}$ and

$$\text{span}\langle Y_1, JY_1, \ldots, Y_{n-1}, JY_{n-1}\rangle = \mathfrak{v}$$

J-invariant and such that $[\mathfrak{v}, \mathfrak{v}] \subseteq \text{span}\langle X\rangle$. Suppose that $\mathfrak{h}^{\perp\Omega}$ is not an ideal of $\mathfrak{g}$, then there exists a non-zero $Y \in \mathfrak{v}$, such that $[JX, Y] \notin \mathfrak{h}^{\perp\Omega}$. Thus

$$\begin{aligned} &[X, Y] = aX, \quad [X, JY] = bX, \\ &[JX, Y] = -2bX - aJX + Z_1, \quad [JX, JY] = 2aX - bJX + JZ_1 \end{aligned} \tag{3.2}$$

with $a \neq 0$ and $Z_1 \in \mathfrak{v}$. By the Jacobi identity we obtain

$$[[JX, Y], JY] + [[Y, JY], JX] + [[JY, JX], Y] = 0$$

and therefore in particular that the component in $\mathfrak{v}$ has to vanish

$$-aJZ_1 + bZ_1 = 0$$

and so $Z_1 = 0$. If we put $[X, JX] = hX$, by the Jacobi identity

$$[[X, JX], Y] + [[Y, X], JX] + [[JX, Y], X] = ahX = 0$$

we get $h = 0$ and so $[X, JX] = 0$. By using again the Jacobi identity

$$[[JX, Y], JY] + [[JY, JX], Y] + [[Y, JY], JX] = (-4b^2 - 4a^2)X = 0$$

we get a contradiction. So $\mathfrak{h}^{\perp \Omega}$ has to be an ideal of $\mathfrak{g}$. Moreover, with the same argument as before we can show that $[X, Y] = 0$, for every $Y \in \mathfrak{v}$. Indeed, if there exists a non-zero Y such that $[X, Y] \neq 0$, then $[JX, Y]$ has a non zero component along JX. Therefore, for every $Y \in \mathfrak{v}$, we have $[JX, Y] \in \mathfrak{v}$. Thus, we have the relations (3.2) with $a = b = 0$, *i.e.*

$$[X, Y] = 0, \ [X, JY] = 0, \ [JX, Y] = Z_1, \ [JX, JY] = JZ_1,$$

for every $Y \in \mathfrak{v}$ and so in particular $[JX, JY] = J[JX, Y]$. Therefore

$$\mathrm{ad}_{JX} \circ J(Y) = J \circ \mathrm{ad}_{JX}(Y), \quad \forall Y \in \mathfrak{v}. \tag{3.3}$$

Since $\mathfrak{h}^{\perp \Omega} = \mathrm{span}\langle X \rangle \oplus \mathfrak{v}$ is nilpotent, $\mathfrak{h}^{\perp \Omega}$ coincides with the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. Indeed $[\mathfrak{h}^{\perp \Omega}, \mathfrak{h}^{\perp \Omega}] \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}^{\perp \Omega}] = \{0\}$. Note that if $[X, JX] = 0$, then we have that $\mathrm{span}\langle JX \rangle$ is a nilpotent complement of $\mathfrak{n}$ such that $J \, \mathrm{ad}_{JX} = \mathrm{ad}_{JX} J$ and we can apply [8, Theorem 1.2] to conclude that $\mathfrak{g}$ has to be abelian.

Now in order to complete the proof we claim that if $h \neq 0$ then we get a contradiction. Suppose that $[JX, X] = hX$, with $h \neq 0$. If $\mathfrak{h}^{\perp \Omega}$ is abelian, then $\mathfrak{g} = \mathbb{R}JX \ltimes \mathbb{R}^{2n-1}$ and by [8] an almost abelian completely solvable Lie algebra admitting a symplectic form taming a complex structure has to be abelian.

Since ad_{JX} is unimodular on $\mathfrak{h}^{\perp \Omega}$, ad_{JX} has an eigenvalue k such that $\mathrm{sign}(h) \neq \mathrm{sign}(k)$. Take the generalized eigenspace (*i.e.* the eigenspace of the semi-simple part $(\mathrm{ad}_{JX})_s$ of ad_{JX}) $V \subset \mathfrak{h}^{\perp \Omega}$ for the eigenvalue k. Since $\mathrm{ad}_{JX}(\mathfrak{v}) \subset \mathfrak{v}$, $\mathrm{ad}_{JX}(X) = hX$ and $h \neq k$, we have $V \subset \mathfrak{v}$. Then we get $[JX, V] \subset V$ and by (3.3) $\mathrm{ad}_{JX} J(Y) = J \, \mathrm{ad}_{JX}(Y)$, for every $Y \in V$. For $Y_1, Y_2 \in V$, we have $[Y_1, Y_2] = cX$ for some c. Since the semi-simple part $(\mathrm{ad}_{JX})_s$ of ad_{JX} is a derivation on $\mathfrak{h}^{\perp \Omega}$, we get

$$2kcX = hcX.$$

By $\mathrm{sign}(h) \neq \mathrm{sign}(k)$, we obtain $c = 0$ and hence $[V, V] = 0$. Take a subspace $W \subset \mathfrak{h}^{\perp \Omega}$ such that

$$\mathfrak{h}^{\perp \Omega} = V \oplus W \oplus \mathrm{span}\langle X \rangle$$

and

$$[JX, W \oplus \mathrm{span}\langle X \rangle] \subset W \oplus \mathrm{span}\langle X \rangle$$

by using the generalized eigenspace decomposition. We have $[V, W] \subset \mathrm{span}\langle X \rangle$ and $[W, W] \subset \mathrm{span}\langle X \rangle$. Consider

$$\bigwedge \mathfrak{g}^* = \bigwedge (\mathrm{span}\langle Jx \rangle \oplus V^* \oplus W^* \oplus \mathrm{span}\langle x \rangle).$$

Then we have $dV^* \subset Jx \wedge V^*, dW^* \subset Jx \wedge W^*$ and

$$dx \in V^* \wedge W^* \oplus Jx \wedge W^* \oplus W^* \wedge W^* \oplus \mathrm{span}\langle Jx \wedge x \rangle$$

by $[JX, V] \subset V$ and $[V, V] = 0$. We obtain

$$d(Jx \wedge V^*) = 0,$$

$$d(Jx \wedge W^*) = 0,$$

$$d(Jx \wedge x) \subset jx \wedge V^* \wedge W^* \oplus Jx \wedge W^* \wedge W^*,$$

$$d(V^* \wedge V^*) \subset Jx \wedge V^* \wedge V^*,$$

$$d(V^* \wedge W^*) \subset Jx \wedge V^* \wedge W^*,$$

$$d(V^* \wedge x) \subset Jx \wedge V^* \wedge x \oplus V^* \wedge V^* \wedge W^* \oplus Jx \wedge V^* \wedge W^* \oplus V^* \wedge V^* \wedge W^*,$$

$$d(W^* \wedge W^*) \subset Jx \wedge W^* \wedge W^*$$

and

$$d(W^* \wedge x) \subset Jx \wedge W^* \wedge x \oplus V^* \wedge W^* \wedge W^* \oplus W^* \wedge W^* \wedge W^*$$
$$\oplus x \wedge W^* \wedge W^* \oplus Jx \wedge W^* \wedge W^*.$$

Consider a generic 2-form $\omega \in \bigwedge^2 \mathfrak{g}^*$. We can write ω as

$$\omega = \omega_1 + \omega_2$$

with $\omega_1 \in V^* \wedge V^*$ and ω_2 belonging to the complement of $V^* \wedge V^*$. If we impose $d\omega = 0$, then $d\omega_1 = 0$. Since $\mathrm{ad}_{JX}(V) \subset V$ and $\mathrm{ad}_{JX} \circ J = J \circ \mathrm{ad}_{JX}$, considering the triangulation of ad_{JX} on V, we can take a basis $(Y_1, Y_2, \ldots, Y_{2s-1}, Y_{2s})$ of V such that

$$[JX, Y_i] = kY_i \qquad \mathrm{mod} \ \mathrm{span}\langle Y_1, \ldots, Y_{i-1}\rangle$$

and $JY_{2i} = Y_{2i-1}$. Consider the dual basis $(y_1, y_2, \ldots, y_{2s-1}, y_{2s})$ of $(Y_1, Y_2, \ldots, Y_{2s-1}, Y_{2s})$. Then

$$d(y_{2i} \wedge y_{2i-1}) = -2kJx \wedge y_{2i} \wedge y_{2i-1} \qquad \mathrm{mod} \ Jx \wedge \mathrm{span}_{j_1+j_2<4i-1}\langle y_{j_1} \wedge y_{j_2}\rangle.$$

If ω_1 has a non-trivial $y_{2s} \wedge Jy_{2s}$-component, then we have $d\omega_1 \neq 0$. Hence we get

$$\omega(Y_{2s}, JY_{2s}) = \omega_1(Y_{2s}, Y_{2s-1}) = 0$$

for every closed 2-form $\omega \in \bigwedge^2 \mathfrak{g}^*$. This implies that any closed 2-form $\omega \in \bigwedge^2 \mathfrak{g}^*$ cannot be a symplectic form taming J. Thus $h \neq 0$ is impossible. $\square$

As a consequence of previous theorem we can prove the following:

Theorem 3.2. *A compact solvmanifold $M = \Gamma \backslash G$ of completely solvable type endowed with an invariant complex structure J admits a symplectic form taming J if and only if M is a complex torus.*

Proof. Since G admits a compact quotient by a lattice, by Milnor's result [16] the Lie group G has to be unimodular. Suppose that M admits a symplectic structure Ω taming an invariant complex structure J on M, by using a symmetrization process we can suppose that Ω is invariant (see [7]). So the Lie algebra of G has to be a unimodular completely solvable Lie algebra admitting a symplectic form taming a complex structure. Therefore, by previous theorem $\mathfrak{g}$ has to be abelian. $\square$

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