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PGL_2-equivariant strata of point configurations in $\mathbb{P}^1$

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Abstract. We compute the integral Chow ring of the quotient stack $[(\mathbb{P}^1)^n/PGL_2]$, which contains $\mathcal{M}_{0,n}$ as a dense open, and determine a natural $\mathbb{Z}$-basis for the Chow ring in terms of certain ordered incidence strata. We further show that all $\mathbb{Z}$-linear relations between the classes of ordered incidence strata arise from an analogue of the WDVV relations in $A^\bullet(\overline{\mathcal{M}}_{0,n})$. Next we compute the classes of unordered incidence strata in the integral Chow ring of the quotient stack $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$ and classify all $\mathbb{Z}$-linear relations between the strata via these analogues of WDVV relations. Finally, we compute the rational Chow rings of the complement of a union of unordered incidence strata.

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1. Introduction

The objective of this paper is to gain a comprehensive understanding of equivariant Chow classes of strata of points on the projective line. In particular, we will compute formulas, find minimal sets of generators, and classify relations for these classes. We will consider two main settings:

(1) In the setting of n ordered points in $\mathbb{P}^1$, we consider for each k-part partition P of $\{1,\dots,n\}$ the strata $\Delta_P \subset (\mathbb{P}^1)^n$, where $x_i = x_j$ if i, j are in the same part of P. We are interested in the equivariant classes $[\Delta_P] \in A^{n-k}_{PGL_2}((\mathbb{P}^1)^n) \cong A^{n-k}([(\mathbb{P}^1)^n/PGL_2])$;

(2) In the setting of n unordered points in $\mathbb{P}^1$, we consider for each partition $\lambda = \{\lambda_1,\dots,\lambda_k\} \vdash n$ the strata $Z_\lambda \subset \mathrm{Sym}^n\mathbb{P}^1 = \mathbb{P}^n$ of degree n cycles on $\mathbb{P}^1$ of the form $\sum \lambda_i p_i$ for not necessarily distinct points $p_1,\dots,p_k \in \mathbb{P}^1$. We are interested in the equivariant classes $[Z_\lambda] \in A^{n-k}_{PGL_2}(\mathbb{P}^n) \cong A^{n-k}([\mathbb{P}^n/PGL_2])$.

By working in the PGL_2-equivariant setting, our results specialize to describe universal relations between the relative strata in $A^\bullet(\mathcal{P}^n)$ and $A^\bullet(\mathrm{Sym}^n\mathcal{P})$ for arbitrary $\mathbb{P}^1$-bundles $\mathcal{P} \to B$. Our results imply analogous results for GL_2-equivariant classes, which govern the universal relations in the special case that $\mathcal{P}$ is the projectivization of a rank 2 vector bundle. Working in the PGL_2-equivariant setting

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makes our task more difficult, as the PGL_2-equivariant Chow rings we work with can have 2-torsion, and torus localization is no longer injective as it is with GL_2. We now briefly describe some of our results.

1.1. Ordered point configurations on $\mathbb{P}^1$

In the context of n ordered points on $\mathbb{P}^1$, our viewpoint is that, like the moduli space $\overline{\mathcal{M}_{0,n}}$ of n-pointed genus zero curves, the quotient stack $[(\mathbb{P}^1)^n/PGL_2]$ is a compactification of $\mathcal{M}_{0,n} = ((\mathbb{P}^1)^n \setminus \bigcup_{i<j} \Delta_{i,j})/PGL_2$, where $\Delta_{i,j} \subset (\mathbb{P}^1)^n$ is where the i'th and j'th factors coincide.

Recall that the $WDVV$ relation in $A^\bullet(\overline{\mathcal{M}_{0,4}})$ says two points in $\overline{\mathcal{M}_{0,4}} \cong \mathbb{P}^1$ corresponding to reducible curves have the same class [24, Section 0.1]. It was shown by Keel [22] that $A^\bullet(\overline{\mathcal{M}_{0,n}})$ is generated as a ring by its boundary divisors, and the only nontrivial relations come from pulling back the $WDVV$ relation under forgetful maps $\overline{\mathcal{M}_{0,n}} \to \overline{\mathcal{M}_{0,4}}$.

We show that $A^\bullet([(\mathbb{P}^1)^n/PGL_2]) = A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ has a similar nice presentation in terms of strata.

Theorem 1.1 (Theorem 3.1). *For $n \geq 3$, the ring $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ is given by $\frac{\mathbb{Z}[\{[\Delta_{i,j}]\}_{1\leq i<j\leq n}]}{relations}$, where the relations are (notating $[\Delta_{j,i}] := [\Delta_{i,j}]$ for $j > i$)*

(1) $[\Delta_{i,j}] + [\Delta_{k,l}] = [\Delta_{i,k}] + [\Delta_{j,l}]$ *for distinct i, j, k, l* *(square relations);*
(2) $[\Delta_{i,j}][\Delta_{i,k}] = [\Delta_{i,j}][\Delta_{j,k}]$ *for distinct i, j, k* *(diagonal relations).*

The square relations relate to the $WDVV$ relations as follows. Consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}_{0,4}}(\mathbb{P}^1, 1) & \xrightarrow{\text{ev}} & (\mathbb{P}^1)^4 \\ \downarrow{\scriptstyle\pi} & & \\ \overline{\mathcal{M}_{0,4}} & & \end{array}$$

where ev is the (PGL_2-equivariant) total evaluation map from the Kontsevich mapping space [16, Section 1] and π remembers only the source of the stable map and stabilizes. The square relation is $\text{ev}_* \pi^*$ applied to the $WDVV$ relation. Concretely, for any closed point $a \in \mathbb{P}^1 \cong \overline{\mathcal{M}_{0,4}}$ we can consider the locus $A_a \subset (\mathbb{P}^1)^n$ consisting of the quadruples of points with cross ratio a, and the square relation comes from equating the classes of A_0 and A_∞.

We also show that all additive relations between classes of strata $[\Delta_P] \in A^{n-k}_{PGL_2}((\mathbb{P}^1)^n)$ for partitions P of size k arise from pushing forward square relations under inclusions $(\mathbb{P}^1)^{k+1} \cong \Delta_{P'} \hookrightarrow (\mathbb{P}^1)^n$ for partitions P' into $k+1$ parts (see Theorem 3.6), and find an explicit subset of the $[\Delta_P]$ which form an additive basis for $A^{n-k}_{PGL_2}((\mathbb{P}^1)^n)$ (see Theorem 3.4).

1.2. Unordered point configurations on $\mathbb{P}^1$

In the context of n unordered points on $\mathbb{P}^1$, we similarly view the quotient stack $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2] = [\mathbb{P}^n/PGL_2]$ as compactifying $\mathcal{M}_{0,n}/S_n$. The equivariant Chow ring $A^\bullet_{PGL_2}(\mathbb{P}^n)$ is no longer generated by classes of strata, however the computation of the classes of strata and the classification of their additive relations becomes more interesting.

We generalize the computation of GL_2-equivariant classes of strata given in [12] to PGL_2-equivariant classes (Theorem 3.7), the primary challenge being the 2-torsion part of the class, which lies in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$. In addition, we give the following classification of the relations between classes of strata, showing in particular that any relation holding rationally also holds integrally.

Theorem 1.2. *(Theorem 3.8) Fix k, n and choose $a_\lambda \in \mathbb{Z}$ for each partition $\lambda \vdash n$ into k parts. Then the following are equivalent:*

(1) $\sum a_\lambda[Z_\lambda] = 0$ *in* $A^{n-k}_{PGL_2}(\mathbb{P}^n)$*;*
(2) $\sum a_\lambda[Z_\lambda] = 0$ *in* $A^{n-k}_{GL_2}(\mathbb{P}^n)$*;*
(3) *The following identity holds in* $\mathbb{Q}[z]$*:*

$$\sum_{\lambda=a_1^{e_1}\dots a_k^{e_k}} \frac{a_\lambda}{\prod_{i=1}^k e_i!}\prod_{i=1}^k (z^{a_i}-1)^{e_i} = 0.$$

In addition to the explicit formula in part (3) of Theorem 1.2, there is a simple recipe to generate all the relations. Namely, we can pushforward the additive relations in $A^{n-k}_{PGL_2}((\mathbb{P}^1)^n)$ as given in Section 1.1 under the symmetrization map $(\mathbb{P}^1)^n \to \mathbb{P}^n$. Since the symmetrization map is degree $n!$, it is easy to see that this generates all the additive relations in $A^{n-k}_{PGL_2}(\mathbb{P}^n)$ up to a factor of $n!$. In fact, we will show how to write arbitrary $[Z_\lambda] \in A^{n-k}_{PGL_2}(\mathbb{P}^n)$ as $\mathbb{Q}$-linear combinations of classes of the form $[Z_{\{a,b,1^{k-2}\}}]$, which we show form a $\mathbb{Q}$-basis for $A^{n-k}_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ (see Theorem 3.10). As Theorem 1.2 implies any relation between the Z_λ holding rationally actually holds integrally, this gives a recipe for generating all the relations in $A^{n-k}_{PGL_2}(\mathbb{P}^n)$ between the $[Z_\lambda]$.

Example 1.3. When $n = 6$, Theorem 1.2 implies

$$[Z_{\{4,1,1\}}] + 3[Z_{\{2,2,2\}}] = [Z_{\{3,2,1\}}] \tag{1.1}$$

in $A^\bullet_{PGL_2}(\mathbb{P}^n)$. This relation is also obtained by pushing forward the square relation $[\Delta_{1,2}]-[\Delta_{2,3}]+[\Delta_{3,4}]-[\Delta_{4,1}]$ via the sequence of maps $(\mathbb{P}^1)^4 \cong \Delta_{\{\{1,5\},\{2,6\},\{3\},\{4\}\}} \hookrightarrow (\mathbb{P}^1)^6 \to \mathbb{P}^6$ to obtain $2[Z_{4,1,1}] + 6[Z_{2,2,2}] = 2[Z_{3,2,1}]$ in $A^\bullet_{PGL_2}(\mathbb{P}^6)$. By Theorem 1.2 the relation still holds after dividing by 2, giving (1.1).

Each relation between classes $[Z_\lambda]$ in the equivariant Chow ring $A^\bullet_{PGL_2}(\mathbb{P}^n)$ gives relations between enumerative problems. For example, (1.1) implies:

(1) Let $C_t \subset \mathbb{P}^2$ be a general pencil of degree 6 plane curves. Then, as we vary C_t over $t \in \mathbb{P}^1$, the number of hyperflex lines plus thrice the number of tritangent lines is equal to the number of lines that are both flex and bitangent.
(2) Let $X \subset \mathbb{P}^3$ be a general degree 6 surface. Then in $A^\bullet(\mathbb{G}(1,3))$, the class of the curve of lines that meet X to order 4 at a point plus three times the class of the curve of tritangent lines to X is equal to the class of the curve of lines that meet X at three points with multiplicities $1, 2, 3$.

In the absence of a transversality argument, the equalities need to be taken with appropriate multiplicities.

Remark 1.4. Lines with prescribed orders of contact with a hypersurface were also studied in [30, Section 5]. Counts of these lines are also related to counting line sections of a hypersurface with fixed moduli [5,23]. For the surface $X \subset \mathbb{P}^3$ in Theorem 1.3, the points $p \in X$ for which a line meets X at p to order 4 is the *flecnode curve*, which is always of expected dimension 1 if X is not ruled by lines by the Cayley-Salmon theorem [21, Theorem 6], which is a primary tool for bounding the number of lines on a smooth surface in $\mathbb{P}^3$ (see [29] and [4, Appendix]). Also, there is no reason not to consider a general variety $X \subset \mathbb{P}^N$ other than the difficulty of finding a projective variety of higher codimension that has at least a 3-dimensional family of 6-secant lines.

1.3. Excision

As an application of our understanding of incidence strata, we compute PGL_2-equivariant Chow rings obtained by excising strata, which often arise in the studies of GIT quotients.

Recall that for an inclusion $\iota : Y \hookrightarrow X$ of a closed subvariety Y into a variety X and a group G which acts on X and Y, the excision exact sequence implies that $A^\bullet_G(X \setminus Y) = A^\bullet_G(X)/I_Y$ where I_Y is the ideal in $A^\bullet_G(X)$ generated by $\iota_* A^\bullet_G(Y)$. In particular, for a sequence of subvarieties $Y_1, \dots, Y_\ell$, we have $I_{Y_1 \cup \dots \cup Y_\ell} = I_{Y_1} + \dots + I_{Y_\ell}$. Hence determining the PGL_2-equivariant Chow rings of $(\mathbb{P}^1)^n \setminus \bigcup \Delta_{P_i}$ and $\mathbb{P}^n \setminus \bigcup Z_{\lambda_i}$ is equivalent to determining the excision ideals for $\Delta_P \subset (\mathbb{P}^1)^n$ and $Z_\lambda \subset \mathbb{P}^n$.

1.3.1. *Ordered points on* $\mathbb{P}^1$

For a k part partition P of $\{1, \dots, n\}$, we have $\Delta_P \cong (\mathbb{P}^1)^k$, so applying Theorem 1.1 (see Theorem 3.2 when P has ≤ 2 parts), $A^\bullet_{PGL_2}(\Delta_P)$ is generated by $A^1_{PGL_2}(\Delta_P)$ and the restriction $A^1_{PGL_2}((\mathbb{P}^1)^n) \to A^1_{PGL_2}(\Delta_P)$ is surjective. Hence the restriction map $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to A^\bullet_{PGL_2}(\Delta_P)$ is surjective, so by the push-pull formula we have the following.

Corollary 1.5 (Theorem 3.11). *For a partition P of $\{1, \dots, n\}$ we have $I_{\Delta_P} = \langle [\Delta_P] \rangle \subset A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$.*

If we pick a linearization of the PGL_2-action on $(\mathbb{P}^1)^n$ and there are no strictly semistable points, then excising the unstable locus (a union of Δ_P strata) and applying [8, Theorem 3] gives the rational Chow ring of the GIT quotient. See [13] for an approach via quiver representations. These GIT quotients are Hassett spaces with total weight $2+\epsilon$ [18, Section 8] and receive maps from $\overline{\mathcal{M}}_{0,n}$ via reduction morphisms [18, Theorem 4.1], as induced maps between GIT quotients [19, Theorem 3.4], or by viewing $\overline{\mathcal{M}}_{0,n}$ as a Chow quotient [20].

1.3.2. *Unordered points on* $\mathbb{P}^1$

The main difficulty with computing the ideals of excision I_{Z_λ} is that the varieties Z_λ are singular and the structure of subvarieties on Z_λ is difficult to determine. However, we can say much more if we work rationally.

Theorem 1.6 (Theorem 3.12). *For a partition $\lambda \vdash n$, $I_{Z_\lambda} \otimes \mathbb{Q} \subset A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ is generated by all $[Z_{\lambda'}]$ with λ' obtained by merging various parts of λ.*

Not all $Z_{\lambda'}$ are always necessary for generation however, and in particular we generalize [7] and [12], showing for $\lambda = \{a, 1^{n-a}\}$ that $I_{Z_\lambda} \otimes \mathbb{Q} = \langle Z_\lambda, Z_{\lambda'} \rangle$ for a particular λ' (see Theorem 3.14).

A number of situations were considered by previous works. In the special case $\lambda = \{2, 1^{n-2}\}$, computing I_λ is the technical heart of the computation of Edidin and Fulghesu of the Chow ring of the stack of hyperelliptic curves of even genus [7]. For n odd and Z_λ the unstable locus (so $\lambda = \{\frac{n+1}{2}, 1^{\frac{n-1}{2}}\}$), the rational Chow ring $A^\bullet_{GL_2}(\mathbb{P}^n \setminus Z_\lambda) \otimes \mathbb{Q}$ equals $A^\bullet(\mathbb{P}^n // GL_2) \otimes \mathbb{Q}$, the rational Chow ring of the GIT quotient [8, Theorem 3]. For all n and $Z_\lambda \subset \mathbb{P}^n$ the locus of unstable and strictly semistable points, Fehér, Némethi, and Rimányi computed $A^\bullet_{GL_2}(\mathbb{P}^n \setminus Z_\lambda) \otimes \mathbb{Q}$ using a spectral sequence and used the result to compute the rational Chow ring of the GIT quotient [12, Theorems 4.3 and 4.10]. They actually work with the affine space $\operatorname{Sym}^n K^2$ instead of $\mathbb{P}^n$, but the two settings are essentially the same (see Theorem 10.5).

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2. Background and conventions

Conventions:

(1) The base field K is algebraically closed of arbitrary characteristic
(2) GL_2 acts linearly on $\mathbb{P}^1$ and hence on all products $(\mathbb{P}^1)^n$, symmetric powers $\mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$, and their duals
(3) $T \subset GL_2$ is the standard maximal torus with standard characters u and v

(4) $[n]$ denotes the set $\{1, \dots, n\}$
(5) $\Phi : (\mathbb{P}^1)^n \to \mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$ denotes the multiplication map, where n will be clear from context.

2.1. Universal relations and equivariant intersection theory

Equivariant intersection theory was formalized in [8] as the correct notion of integral Chow groups and rings for quotient stacks. See also [1] for an exposition.

For the purposes of computation and application to enumerative problems (see Theorem 1.3), we would like to know that equivariant Chow classes of equivariant subvarieties specialize to nonequivariant classes of similar subvarieties in a relative setting. Therefore, we have chosen to introduce equivariant intersection theory in terms of the construction.

Suppose we have a group G (typically $G = T, GL_2, PGL_2$) acting on a variety X (typically $(\mathbb{P}^1)^n, \mathrm{Sym}^n\mathbb{P}^1 = \mathbb{P}^n$), and G-invariant subvarieties Y_i (typically incidence strata in $(\mathbb{P}^1)^n$ or $\mathbb{P}^n$). Given a principal G-bundle $\mathcal{P} \to B$, we have the X-bundle $X_{\mathcal{P}} \to B$, where $X_{\mathcal{P}} := \mathcal{P} \times^G X$. Inside $X_{\mathcal{P}}$, we have the cycles

$$(Y_i)_{X_{\mathcal{P}}} := (Y_i)_{\mathcal{P}} \subset X_{\mathcal{P}}$$

restricting to Y_i in each fiber X, inducing classes $[Y_i]_{X_{\mathcal{P}}} \in A_\bullet(X_{\mathcal{P}})$. We would like to understand what "universal" linear relations exist between these classes (*i.e.*, which don't depend on B or $\mathcal{P}$).

For example, if we take $G = PGL_2$, then we are seeking universal relations between classes $[Z_P]_{\mathcal{F}^n}$ and between classes $[Z_\lambda]_{\mathrm{Sym}^n\mathcal{F}}$ for $\mathcal{F} \to B$ a $\mathbb{P}^1$-bundle. If we use $G = GL_2$ instead the relations hold a priori only for $\mathcal{F}$ the projectivization of a rank 2 vector bundle on B.

As we will see in Section 2.2, there is a universal group $A_\bullet^G(X)$ approximated by certain $A_\bullet(X_{\mathcal{P}'})$ which is equipped with maps $A_\bullet^G(X) \to A_\bullet(X_{\mathcal{P}})$ for all $\mathcal{P}$ and there are classes $[Y_i] \in A_\bullet^G(X)$ such that $[Y_i] \mapsto [Y_i]_{\mathcal{P}}$, so any relations in $A_\bullet^G(X)$ between the $[Y_i]$ descend to relations between the $[Y_i]_{\mathcal{P}}$. Conversely, we will see by construction that any relation between the $[Y_i]_{\mathcal{P}}$ for all $\mathcal{P}$ induces a relation between the $[Y_i]$.

2.2. Equivariant intersection theory

The equivariant Chow group $A_\bullet^G(X)$ is defined as follows. Suppose G acts linearly on a vector space V with an open subset U of codimension c on which it acts freely. Then for any $k < c$, we define $A^G_{\dim(X)-k}(X) := A_{\dim(X \times^G U)-k}(X \times^G U)$. Note that $X \times^G U = X_{\mathcal{P}}$ where $\mathcal{P}$ is the principal G-bundle $U \to U/G$. This does not depend on the choice of V [8, Definition-Proposition 1].

For $\mathcal{P} \to B$ a principal G-bundle over an equidimensional base B, we have a map

$$A_\bullet^G(X) \to A_{\dim(B)+\bullet}(P \times^G X)$$

via the composition

$$\begin{aligned} A^G_{>\dim(X)-c}(X) &\cong A_{>\dim(X\times^G U)-c}(X \times^G U) \\ &\to A_{>\dim((P\times X)\times^G U)-c}((P \times X) \times^G U) \\ &\cong A_{>\dim((P\times X)\times^G U)-c}((P \times X) \times^G V) \\ &\cong A_{>\dim(P\times^G X)-c}(P \times^G X), \end{aligned}$$

where the second map is induced by flat pullback from the projection, the third map follows from excising $(P \times X) \times^G (V \setminus U)$, and the last map follows from the Chow groups of a vector bundle [17, Theorem 3.3(a)].

Now, we define $A^\bullet_G(X)$ to be the ring of operational G-equivariant Chow classes on X, *i.e.*, $A^i_G(X)$ is all assignments

$$(Y \to X) \mapsto (A^G_\bullet(Y) \to A^G_{\bullet-i}(Y))$$

for every G-equivariant map $Y \to X$, compatible with the standard operations on Chow groups [8, Section 2.6]. In our case X is always smooth, and we have the Poincaré duality isomorphism $A^\bullet_G(X) = A^G_{\dim(X)-\bullet}(X)$ [8, Proposition 4], and the identification

$$A^\bullet([X/G]) \cong A^\bullet_G(X),$$

where $[X/G]$ is the quotient stack [8, Section 5.3].

2.3. GL_2 and T-equivariant Chow rings of $(\mathbb{P}^1)^n$ and $\mathbb{P}^n$

We will postpone discussing PGL_2-equivariant intersection rings to Section 4. The equivariant Chow rings $A^\bullet_T((\mathbb{P}^1)^n)$ and $A^\bullet_T(\mathbb{P}^n)$ can be approximated by the ordinary Chow rings of $(\mathbb{P}^1)^n$ and $\mathbb{P}^n$ bundles over $\mathbb{P}^N \times \mathbb{P}^N$. Similarly, $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ and $A^\bullet_{GL_2}(\mathbb{P}^n)$ can be approximated by the ordinary Chow rings of the Grassmannian of lines $\mathbb{G}(1, N)$ for $N >> 0$.

Let u, v be the standard characters of T. If pt is a point with trivial GL_2 action, then

$$A^\bullet_T(\mathrm{pt}) = \mathbb{Z}[u, v], \qquad A^\bullet_{GL_2}(\mathrm{pt}) = \mathbb{Z}[u, v]^{S_2},$$

where S_2 acts on $\mathbb{Z}[u, v]$ by swapping u, v. By the Chow ring of a vector bundle [17, Theorem 3.3(a)], the T (respectively GL_2) equivariant Chow ring of an affine space is isomorphic to the equivariant Chow ring of a point. By the projective bundle theorem [10, Theorem 9.6], we have

$$A^\bullet_T((\mathbb{P}^1)^n) = \mathbb{Z}[u, v][H_1, \dots, H_n]/(F(H_i)), \quad A^\bullet_T(\mathbb{P}^n) = \mathbb{Z}[u,v][H]/(G(H)),$$
$$A^\bullet_{GL_2}((\mathbb{P}^1)^n) = \mathbb{Z}[u,v]^{S_2}[H_1, \dots, H_n]/(F(H_i)), \quad A^\bullet_{GL_2}(\mathbb{P}^n) = \mathbb{Z}[u,v]^{S_2}[H]/(G(H)),$$

where H_i is $c_1(\mathscr{O}_{\mathbb{P}^1}(1))$ pulled back to $(\mathbb{P}^1)^n$ under the ith projection and H is $c_1(\mathscr{O}_{\mathbb{P}^n}(1))$, and we define

$$F(z) = (z+u)(z+v), \qquad G(z) = \prod_{k=0}^{n}(z+ku+(n-k)v)$$

for the rest of the document. Even though one might want to use GL_2-equivariant Chow rings for applications, GL_2-equivariant Chow rings inject into T-equivariant Chow rings, so it suffices to only consider T-equivariant Chow rings.

The formula for the class of the projectivization of a subbundle [10, Proposition 9.13] shows the ith coordinate hyperplane in $\mathbb{P}^n$ has class $H+iu+(n-i)v$. This gives the formula for any torus fixed linear space (for example the torus-fixed points) in $(\mathbb{P}^1)^n$ or $\mathbb{P}^n$ by multiplying a subset of these classes.

2.4. Ordered and unordered strata of n points on $\mathbb{P}^1$

Definition 2.1. Given a collection $P = \{A_1, \dots, A_d\}$ of disjoint subsets of $[n]$, let $\Delta_P \subset (\mathbb{P}^1)^n$ denote the d-dimensional locus of points $(p_1, \dots, p_n)$ where $p_i = p_j$ whenever i, j are in the same set A_k of P.

Example 2.2. If $P = \{\{1,2,4\},\{3,6\}\}$ and $A = [6]$, then $Z_P \subset (\mathbb{P}^1)^6$ consists of points $(p_1, \dots, p_6)$ such that $p_1 = p_2 = p_4$ and $p_3 = p_6$.

Definition 2.3. Given a partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of a positive integer n, we define the d-dimensional subvariety $Z_\lambda \subset \mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$ to be the image of Δ_P under the multiplication map $\Phi : (\mathbb{P}^1)^n \to \mathbb{P}^n$, where $P = \{A_1, \dots, A_d\}$ is any partition of $[n]$ with $|A_i| = \lambda_i$.

Remark 2.4. If we view $\mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$ as binary degree n forms on the dual of $\mathbb{P}^1$, then Z_λ is the closure of the degree n forms with multiplicity sequence given by λ, whose equivariant Chow classes were studied by Fehér, Némethi, and Rimányi [12].

In order to compactify notation, we make the following definitions.

Definition 2.5. Given P a partition of $[n]$ and λ a partition of n, we let

$$[\Delta_P] \in A^\bullet_G((\mathbb{P}^1)^n)$$
$$[\lambda] := \left(\prod_{i=1}^{n} e_i^\lambda!\right)[Z_\lambda] \in A^\bullet_G(\mathbb{P}^n),$$

where G is T, GL_2 or PGL_2, depending on the context and $e_i^\lambda = \#\{j \mid \lambda_j = i\}$. For $\lambda = \{a_1, \dots, a_d\}$, we will often write $[a_1, \dots, a_d]$ or $[1^{e_1^\lambda}, \dots, n^{e_n^\lambda}]$ for $[\lambda]$.

Remark 2.6. For any such partition P and λ as in Theorem 2.3, then Φ maps Δ_P onto Z_λ with degree $\prod_{i=1}^n e_i^\lambda!$, so $\Phi_*[\Delta_P] = [\lambda]$.

2.5. Affine and projective Thom polynomials

Definition 2.7. Given a T-invariant subvariety $V \subset \mathbb{P}^n$, let $\widetilde{V} \subset \mathbb{A}(\mathrm{Sym}^n K^2)$ denote the cone of $V \subset \mathbb{P}^n$ in $\mathbb{A}(\mathrm{Sym}^n K^2) \cong \mathbb{A}^{n+1}$.

Given a T-invariant subvariety $V \subset \mathbb{P}^n$, its class $[V] \in A^\bullet_T(\mathbb{P}^n)$ is a polynomial $p(H, u, v)$ of degree at most n. The degree 0 term in H, $p_0(u, v)$, is $[\widetilde{V}] \in A^\bullet_T(\mathbb{A}^{n+1}) \cong \mathbb{Z}[u, v]$. This is seen by considering the diagram

$$A^\bullet_T(\mathbb{P}^n) \xleftarrow{\sim} A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1}\backslash\{0\}) \to A^\bullet_T(\mathbb{A}^{n+1}\backslash\{0\})$$

and noting that $A^k_T(\mathbb{A}^{n+1}\backslash\{0\}) \cong A^k_T(\mathbb{A}^{n+1})$ for $k \le n$.

It turns out $p_0(u, v)$ determines all of p.

Lemma 2.8 ([11, Theorem 6.1]). *We have* $p(u, v) = p_0(u + \frac{H}{n}, v + \frac{H}{n})$.

This argument is written down in its natural generality in [11, Theorem 6.1], but we specialize their argument here.

Proof. Consider the maps of groups

$$T \times \mathbb{G}_m \xleftarrow{(t_0,t_1),t^n \leftarrow\!\shortmid (t_0,t_1),t} T \times \mathbb{G}_m \xrightarrow{(t_0,t_1),t\mapsto(t_0t,t_1t)} T.$$

This induces the maps of rings

$$A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1}) \xrightarrow[(u,v),H\mapsto(u,v),nH]{\alpha} A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1}) \xleftarrow[(u+H,v+H)\leftarrow\!\shortmid(u,v)]{\beta} A^\bullet_T(\mathbb{A}^{n+1}), \tag{2.1}$$

where we use u, v as the torus characters on T dual to the coordinates t_0, t_1 and H as the torus character on $\mathbb{G}_m$ dual to the coordinate t. As an abuse of notation, we should note that the $T \times \mathbb{G}_m$ actions on $\mathbb{A}^{n+1}$ for the two equivariant Chow rings in (2.1) are not the same: the first Chow ring has the $\mathbb{G}_m$ factor acting on $\mathbb{A}^{n+1}$ by multiplication by λ and the second has the $\mathbb{G}_m$ factor acting by multiplication by λ^n.

Let $\widetilde{p}(u, v, H) \in A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1})$ be the class of $[\widetilde{V}]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^{n+1}$ by multiplication by λ. Since $p(u, v)$ is the class of $[\widetilde{V}]$ in $A^\bullet_T(\mathbb{A}^{n+1})$, we must have that

$$\alpha(\widetilde{p}) = \beta(p) \Rightarrow \widetilde{p}(u, v, nH) = p_0(u + H, v + H).$$

Therefore, $\widetilde{p}(u, v, H) = p(u + \frac{H}{n}, v + \frac{H}{n})$. Finally, as $(\mathbb{A}^{n+1}\backslash\{0\})/\mathbb{G}_m \cong \mathbb{P}^n$, the class $[V] \in A^\bullet_T(\mathbb{P}^n)$ can be computed from $[\widetilde{V}] \in A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1})$ by mapping to $A^\bullet_T(\mathbb{P}^n)$ via

$$A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1}) \to A^\bullet_{T\times\mathbb{G}_m}(\mathbb{A}^{n+1}\backslash\{0\}) \cong A^\bullet_T(\mathbb{P}^n).$$

Since $\widetilde{p}(u, v, H)$ maps to $p(u, v, H)$, we are done. □

3. Statement of results

Now that we have formally introduced all the necessary definitions, we will collect in this section a complete list of our results for the reader's convenience.

3.1. Ordered strata in $[(\mathbb{P}^1)^n/PGL_2]$

We compute a ring presentation in Theorem 3.1 for $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ similar to that of $A^\bullet(\overline{\mathcal{M}}_{0,n})$ computed by Keel [22]. The incidence strata Δ_P play a fundamental role in the equivariant Chow ring. Additionally, in Theorem 3.4 we compute a $\mathbb{Z}$-basis for $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$, which consists in degree $\le n-2$ of certain incidence strata, and in Theorem 3.6 we show all relations between incidence strata arise from an analogue of the WDVV relation in $A^\bullet(\overline{\mathcal{M}}_{0,4})$ (see Section 1.1).

Theorem 3.1. *The following are true*

(1) (Theorem 6.16) *For $n \ge 3$, the ring $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) = \frac{\mathbb{Z}[\{[\Delta_{i,j}]\}_{1\le i<j\le n}]}{relations}$, where the relations are (notating $[\Delta_{j,i}] := [\Delta_{i,j}]$ for $j > i$)*
   (a) $[\Delta_{i,j}] + [\Delta_{k,l}] = [\Delta_{i,k}] + [\Delta_{j,l}]$ *for distinct i, j, k, l (square relations);*
   (b) $[\Delta_{i,j}][\Delta_{i,k}] = [\Delta_{i,j}][\Delta_{j,k}]$ *for distinct i, j, k. (diagonal relations);*

(2) (Theorem 6.4) *The group $A^k_{PGL_2}((\mathbb{P}^1)^n)$ is a free $\mathbb{Z}$-module of rank*

$$\sum_{\substack{i\le k \\ i\equiv k \bmod 2}} \binom{n}{i};$$

(3) (Theorem 4.3) *For $n \ge 3$, $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ is the subring of $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ where $[\Delta_{i,j}]$ is identified with the class $H_i + H_j + u + v$;*

(4) (Theorem 4.5) *If the base field is $\mathbb{C}$, then the map $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to H^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ to equivariant cohomology is an isomorphism.*

Remark 3.2. Parts (1) and (3) Theorem 3.1 extend to the cases when $n = 1, 2$ if we include the classes $\psi_i = \pi_i^* c_1(T^\vee\mathbb{P}^1) \in A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ pulled back from the ith projection π_i. They correspond to $-(2H_i + u + v)$ under inclusion from (3) (see Theorem 5.4) and their definition is analogous to the ψ-classes on $\overline{M}_{0,n}$ [26, Section 2]. The additional relations for part (1) are $\psi_i[\Delta_{i,j}] = -[\Delta_{i,j}]^2$, $\psi_i + \psi_j = -2[\Delta_{i,j}]$ and $\psi_i = [\Delta_{j,k}] - [\Delta_{i,j}] - [\Delta_{i,k}]$ (though we remark the last of these implies the first two for $n \ge 3$).

Definition 3.3. Call a partition P of $\{1, \dots, n\}$ *good* if it can be written as $P = \{A_1, \dots, A_d\}$ with $A_1 \sqcup A_2$ an initial segment of $\{1, \dots, n\}$, and $A_3, \dots, A_d$ intervals.

Theorem 3.4 (Theorem 6.16). *For $n \ge 3$, the additive group $A^k_{PGL_2}((\mathbb{P}^1)^n)$ has a $\mathbb{Z}$-basis consisting of the following.*

1. *If $k \leq n-2$, the classes $[\Delta_P]$ for P a good partition into $n-k$ parts.*
2. *If $k > n-2$, the classes $[\Delta_{i_P, j_P}]^{k-n+2}[\Delta_P]$ for P a partition of $\{1, \dots, n\}$ into two parts and $[\Delta_{i_{\{[n]\}}, j_{\{[n]\}}}]^{k-n+1}[\Delta_{\{[n]\}}]$, where for each P the pair i_P, j_P are chosen to lie in the same part of P.*

In Section 6.1 we describe a simple algorithm to write arbitrary classes in this $\mathbb{Z}$-basis, along with a worked example.

Definition 3.5. Denote by $\mathrm{Part}(d, n)$ the set of partitions of $[n]$ into d parts. Let $\mathrm{Sq}(d, n)$ be the subgroup of the free Abelian group $\mathbb{Z}^{\mathrm{Part}(d,n)}$ generated by formal square relations $P_{i,j} - P_{j,k} + P_{k,l} - P_{l,i}$ for $P \in \mathrm{Part}(d+1, n)$ and $i, j, k, l \in \{1, \dots, n\}$ indices in different parts of P, where $P_{x,y}$ denotes the partition formed by merging the parts of P containing x and y.

We have the relation $[\Delta_{P_{i,j}}] - [\Delta_{P_{j,k}}] + [\Delta_{P_{k,l}}] - [\Delta_{P_{l,i}}] = [\Delta_P(\Delta_{i,j}] - [\Delta_{j,k}] + [\Delta_{k,l}] - [\Delta_{l,i})] = 0$ in $A^{\bullet}_{n-d}((\mathbb{P}^1)^n)$ by pushing forward the square relation for i, j, k, l from Theorem 3.1 (1) in $\Delta_P \cong \mathbb{P}^{d+1}$ to $\mathbb{P}^n$ under the inclusion $\Delta_P \subset \mathbb{P}^n$. These relations in fact generate all linear relations for $n \geq d \geq 2$.

Theorem 3.6 (Theorem 6.14). *For $n \geq d \geq 2$, the map*

$$\mathbb{Z}^{\mathrm{Part}(d,n)} / \mathrm{Sq}(d, n) \to A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$$

sending $P \mapsto \Delta_P$ is an isomorphism.

In particular, since every square relation between the $[\Delta_P]$ classes comes from an explicit PGL_2-invariant degeneration in $(\mathbb{P}^1)^n$ (see Section 1.1), Theorem 3.6 implies that all linear relations between the $[\Delta_P]$ classes can be realized by a sequence of PGL_2-invariant degenerations in $(\mathbb{P}^1)^n$.

Non-equivariantly, there are relations between the classes $[\Delta_P] \in A^{\bullet}((\mathbb{P}^1)^n)$ not generated by these square relations. For example, if $n = 4$ we have

$$\begin{aligned}[\Delta_{\{\{1,2,3\},\{4\}\}}] + [\Delta_{\{\{1,2,4\},\{3\}\}}] + [\Delta_{\{\{1,3,4\},\{2\}\}}] + [\Delta_{\{\{2,3,4\},\{1\}\}}] \\ = [\Delta_{\{\{1,2\},\{3,4\}\}}] + [\Delta_{\{\{1,3\},\{2,4\}\}}] + [\Delta_{\{\{1,4\},\{2,3\}\}}]\end{aligned}$$

in $A^2((\mathbb{P}^1)^4)$.

3.2. Unordered strata in $[\mathrm{Sym}^n \mathbb{P}^1 / PGL_2]$

The PGL_2-action on $\mathbb{P}^1$ induces an action on the symmetric power $\mathrm{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$, which parameterizes degree n divisors on $\mathbb{P}^1$. For each partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of n, we have the PGL_2-invariant subvariety $Z_\lambda \subset \mathbb{P}^n$ consisting of divisors that can be written in the form $\sum_{i=1}^d \lambda_i p_i$ where $p_i \in \mathbb{P}^1$. For convenience we often write $\lambda = a_1^{e_1} \dots a_k^{e_k}$ to be the partition of n where a_i appears e_i times.

3.2.1. *Integral classes of strata*

We compute the class of $[Z_\lambda]$ in $A^\bullet_{PGL_2}(\mathbb{P}^n)$ generalizing the GL_2-equivariant computation of Fehér, Némethi, and Rimányi [12]. The class of $[Z_\lambda]$ in $A^\bullet_{GL_2}(\mathbb{P}^n)$ was given in [12], and we will give a quick independent proof and more compact form in Theorem 5.5. If n is odd, the map $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ induced by the projection $GL_2 \to PGL_2$ is injective (see Theorem 4.8). Therefore, all of the difficulty lies in computing $[Z_\lambda]$ in $A^\bullet_{PGL_2}(\mathbb{P}^n)$ for n even. It turns out (see Section 8) that it suffices to compute the class in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$, which takes on a particularly simple form.

Theorem 3.7. *Let n be even and $\lambda = a_1^{e_1} \dots a_k^{e_k}$ be a partition of n into $d = e_1 + \dots + e_k$ parts. The class of $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2[c_2, c_3, H]/(q_n(H))$ where*

$$q_n(t) = \begin{cases} t^{(n+4)/4}(t^3 + c_2 t + c_3)^{n/4} & n \equiv 0 \bmod 4 \\ t^{(n-2)/2}(t^3 + c_2 t + c_3)^{(n+2)/4} & n \equiv 2 \bmod 4 \end{cases}$$

is non-zero precisely when all a_i and $\frac{d!}{e_1! \dots e_k!}$ are odd and all e_i are even, in which case it is equal to $(\frac{q_n}{q_d})(H)$.

3.2.2. *Relations between strata*

If $\lambda = \{\lambda_1, \dots, \lambda_d\} = \{a_1^{e_1}, \dots, a_k^{e_k}\} = a_1^{e_1} \dots a_k^{e_k}$ is a partition of n, then taking $\Phi : (\mathbb{P}^1)^n \to \mathrm{Sym}^n \mathbb{P}^1$ to be the multiplication map, if $P = \{A_1, \dots, A_d\}$ is any partition of $[n]$ with $|A_i| = \lambda_i$, we have

$$\Phi_* \Delta_P = \left(\prod e_i!\right) [Z_\lambda].$$

In particular, every square relation between the classes of ordered strata induces a relation between $[Z_\lambda]$ classes by pushing forward along Φ.

Theorem 3.8 (Section 8). *Fix n and choose $a_\lambda \in \mathbb{Z}$ for each partition of n. The following are equivalent:*

(1) $\sum a_\lambda [Z_\lambda] = 0$ *in* $A^\bullet_{PGL_2}(\mathbb{P}^n)$;
(2) $\sum a_\lambda [Z_\lambda] = 0$ *in* $A^\bullet_{GL_2}(\mathbb{P}^n)$;
(3) $\sum a_\lambda [Z_\lambda]$ *is formally a rational linear combination of pushforwards of square relations from* $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$;
(4) *The following identity holds in* $\mathbb{Q}[z]$:

$$\sum_{\lambda = a_1^{e_1} \dots a_k^{e_k}} \frac{a_\lambda}{\prod_{i=1}^k e_i!} \prod_{i=1}^k (z^{a_i} - 1)^{e_i} = 0.$$

Corollary 3.9. *Every $\mathbb{Z}$-linear relation that holds between Chow classes of relative Z_λ-cycles in $A^\bullet(\mathrm{Sym}^n \mathbb{P}(V))$ for every rank 2 vector bundle $V \to B$ and base B holds in $A^\bullet(\mathrm{Sym}^n \mathcal{P})$ for every $\mathbb{P}^1$-bundle $\mathcal{P} \to B$ and base B.*

We remark that there is 2-torsion in $A^\bullet_{PGL_2}(\mathbb{P}^n)$ for n even, but Theorem 3.8 implies that if each a_λ is even and $\sum a_\lambda[Z_\lambda]$ is zero in $A^\bullet_{PGL_2}(\mathbb{P}^n)$, then in fact the same is true for $\sum \frac{a_\lambda}{2}[Z_\lambda]$.

Rather than search for linear relations between $[Z_\lambda]$ classes using Theorem 3.8 (4), the following corollary identifies certain partitions whose corresponding strata are a $\mathbb{Q}$-linear basis for $A^{\leq n-2}_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$, and gives an explicit formula for writing every such class in this basis. Every part of Theorem 3.10 can be deduced from Theorem 3.8 except that the strata that we choose span $A^{\leq n-2}_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$.

For $\lambda = a_1^{e_1} \dots a_k^{e_k}$, denote by $[\lambda]$ the normalization

$$[\lambda] = \left(\prod e_i!\right)[Z_\lambda].$$

Corollary 3.10 (Theorem 7.4 and Theorem 7.2). *For fixed $d \geq 2$, the classes $[\{a, b, 1^{d-2}\}]$ form a $\mathbb{Q}$-basis for $A^{n-d}_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q} \subset A^{n-d}_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$. Writing the polynomial*

$$-\frac{1}{(z-1)^{d-2}}\prod_{i=1}^{d}(z^{a_i}-1) = \sum_{\substack{0\leq k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}(z^{k_1}+z^{k_2}),$$

we have $\alpha_i \in \mathbb{Z}$ and

$$[\{a_1,\dots,a_d\}] = \sum_{\substack{1\leq k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}[\{k_1,k_2,1^{d-2}\}].$$

3.3. Excision

As an application of our results, we deduce results on equivariant Chow rings of complements of strata in $(\mathbb{P}^1)^n$ and $\mathbb{P}^n$.

As discussed in Section 1.3, by Theorem 3.1 and Theorem 3.2 we can easily deduce the following corollary

Corollary 3.11. *For a partition P of $\{1,\dots,n\}$ we have $I_{\Delta_P} = \langle[\Delta_P]\rangle \subset A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$.*

We also show that $I_\lambda \otimes \mathbb{Q}$ is generated by the classes of strata contained in Z_λ.

Theorem 3.12 (Theorem 9.8). *Given a partition λ of n, $I_\lambda \otimes \mathbb{Q}$ is generated by $[Z_{\lambda'}]$ for all λ' that can be obtained from λ by merging parts.*

Remark 3.13. Theorem 3.12 is false if we replace $I_\lambda \otimes \mathbb{Q}$ with I_λ. This already fails nonequivariantly in the case $n = 4$ and $\lambda = \{2, 1, 1\}$. Indeed, $\Phi : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^4$ maps birationally onto Z_λ. Let H_1 and H_2 be the hyperplane classes in the factors of $\mathbb{P}^1 \times \mathbb{P}^2$ and H be the hyperplane class of $\mathbb{P}^4$. Then $\Phi_* H_1 = H^2$, while $[Z_{\{2,2\}}] = 8H^2$, $[Z_{\{3,1\}}] = 6H^2$, and $[Z_{\{2,1,1\}}] = 6H$.

We typically don't need to use every merged partition λ' for dimension reasons by Theorem 3.10. When $\lambda = \{a, 1^{n-a}\}$ is a partition with only one part of size greater than 1, we in fact show that $I_\lambda \otimes \mathbb{Q}$ is generated by just two generators.

Theorem 3.14 (Theorem 9.2). *Given the partition $\lambda = \{a, 1^{n-a}\}$ of n, $I_\lambda \otimes \mathbb{Q}$ is generated by $[Z_\lambda]$ and $[Z_{\lambda'}]$, where*

$$\lambda' = \begin{cases} \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\ \{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}. \end{cases}$$

In fact we will also show the analogous results with $A^\bullet_{GL_2}(\mathbb{P}^n \setminus \cup_\lambda Z_\lambda) \otimes \mathbb{Q}$, and if we further replace $\mathbb{P}^n \setminus \cup_\lambda Z_\lambda$ with its affine cone $\mathbb{A}^{n+1} \setminus \cup_\lambda \widetilde{Z_\lambda}$ and consider $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \cup_\lambda \widetilde{Z_\lambda})$ (see Theorem 10.2).

Remark 3.15. The affine analogue of Theorem 3.14 as given in Theorem 10.2 in the special case $a = \lceil \frac{n}{2} \rceil$ recovers the GL_2-equivariant Chow rings of the stable locus computed in [12, Theorems 4.3 and 4.10] as described above. The Chow ring of the semistable locus required a separate argument.

We conclude in Appendix A by describing a combinatorial branching rule for multiplying the affine analogue of the class of a strata $[\widetilde{Z_\lambda}] \in A^\bullet_{GL_2}(\mathrm{Sym}^n K^2) \cong \mathbb{Z}[u, v]^{S_2}$ by a generator $u + v$ or uv. This generalizes [12, Remark 3.9 (1)].

4. PGL_2 and GL_2-equivariant Chow rings

In this section we compare certain PGL_2-equivariant Chow rings to their GL_2-equivariant counterparts, which are easier to work with because GL_2 is special, so restricting to the maximal torus is an injection on equivariant Chow rings [8, Proposition 6].

In particular, we show in Theorem 4.3 that $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ is injective and identify its image. For the unordered case, we show in Theorem 4.8 that $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ is injective for n odd and injective up to 2-torsion when n is even.

4.1. Ordered case

To start, we recall a lemma.

Lemma 4.1 ([25, Lemma 2.1]). *Given a linear algebraic group G acting on a smooth variety X, let H be a normal subgroup of G that acts freely on X with quotient X/H. Then, there is a canonical isomorphism of graded rings*

$$A^\bullet_G(X) \cong A^\bullet_{G/H}(X/H).$$

Remark 4.2. Theorem 4.1 was proven in [25, Lemma 2.1] directly from the definitions, but it can also be seen as a consequence of the fact that the ring $A^\bullet_G(X)$ depends only on the quotient stack $[X/G]$ [8, Proposition 16] and $[[X/H]/(G/H)] \cong [X/G]$ (see [28, Remark 2.4] or [2, Lemma 4.3]).

Proposition 4.3. *For $n \geq 1$, the ring homomorphism*

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to A^\bullet_{GL_2}((\mathbb{P}^1)^n)$$

induced by the quotient map $GL_2 \to PGL_2$ is an injection, and the image is generated by the classes $-(2H_i + u + v)$ and $[\Delta_{i,j}] = H_i + H_j + u + v$.

Remark 4.4. We will show in Theorem 5.4 that $\psi_i := \pi_i^* c_1(T^\vee \mathbb{P}^1) = -(2H_i + u + v)$, as mentioned in Theorem 3.2. For $n \geq 3$ this class is redundant as

$$-(2H_i + u + v) = [\Delta_{j,k}] - [\Delta_{i,j}] - [\Delta_{i,k}].$$

Proof. We show the injectivity of $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \xrightarrow{\iota} A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ using the commutativity of the diagram

$$\begin{array}{ccc} A^\bullet_{PGL_2}((\mathbb{P}^1)^n) & \xrightarrow[q_1]{\sim} & A^\bullet_{GL_2}((\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle f} \\ A^\bullet_{GL_2}((\mathbb{P}^1)^n) & \xrightarrow[q_2]{\sim} & A^\bullet_{GL_2 \times \mathbb{G}_m}((\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}) \end{array}$$

with f induced by the multiplication map $GL_2 \times \mathbb{G}_m \to GL_2$.

We have the isomorphisms q_1 and q_2 by Theorem 4.1.

To prove commutativity of the diagram, we can identify each of the rings $A^\bullet_G(X)$ with $A^\bullet([X/G])$ as in Section 2.2, so it suffices to show the following diagram of stacks is commutative.

$$\begin{array}{ccc} \left[(\mathbb{P}^1)^n/PGL_2\right] & \xleftarrow{\sim} & \left[(\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}/GL_2\right] \\ \uparrow & & \uparrow \\ \left[(\mathbb{P}^1)^n/GL_2\right] & \xleftarrow{\sim} & \left[(\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}/(GL_2 \times \mathbb{G}_m)\right]. \end{array}$$

Suppose we start with a principal $GL_2 \times \mathbb{G}_m$-bundle $P \to S$ together with a $GL_2 \times \mathbb{G}_m$-equivariant map $P \to (\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}$, giving a map $S \to [(\mathbb{A}^2\backslash\{0\}) \times (\mathbb{P}^1)^{n-1}/(GL_2 \times \mathbb{G}_m)]$. Following the diagram around clockwise or counterclockwise, we get a map $S \to [(\mathbb{P}^1)^n/PGL_2]$ given by a PGL_2-equivariant morphism

$$P \times^{GL_2 \times \mathbb{G}_m} GL_2 \times^{GL_2} PGL_2 \cong P \times^{GL_2 \times \mathbb{G}_m} PGL_2 \to (\mathbb{P}^1)^n.$$

When going counterclockwise, the product $P \times^{GL_2 \times \mathbb{G}_m} GL_2$ is taken with respect to the multiplication map $GL_2 \times \mathbb{G}_m \to GL_2$, while when going clockwise, the product is taken with respect to the projection map $GL_2 \times \mathbb{G}_m \to GL_2$. However, the resulting principal PGL_2-bundle is the same as the compositions with the quotient $GL_2 \to PGL_2$ are identical.

Now, we will find the induced map

$$A^\bullet_{GL_2}((\mathbb{A}^2\backslash 0) \times (\mathbb{P}^1)^{n-1}) \to A^\bullet_{GL_2}((\mathbb{P}^1)^n)$$

in terms of generators and show it is injective. Consider the diagram

$$\begin{array}{ccccc}
A^\bullet_{GL_2}((\mathbb{A}^2\backslash 0)\times(\mathbb{P}^1)^{n-1}) & \xrightarrow{f} & A^\bullet_{GL_2\times\mathbb{G}_m}((\mathbb{A}^2\backslash\{0\})\times(\mathbb{P}^1)^{n-1}) & \xrightarrow[\sim]{q_2} & A^\bullet_{GL_2}((\mathbb{P}^1)^n) \\
\downarrow\sim & & \downarrow\sim & & \downarrow\sim \\
A^\bullet_{GL_2\times(\mathbb{G}_m)^{n-1}}((\mathbb{A}^2\backslash 0)^n) & \xrightarrow{f'} & A^\bullet_{GL_2\times(\mathbb{G}_m)^n}((\mathbb{A}^2\backslash\{0\})^n) & \xrightarrow[\sim]{q_2'} & A^\bullet_{GL_2\times(\mathbb{G}_m)^n}((\mathbb{A}^2\backslash\{0\})^n),
\end{array}$$

where GL_2 acts in the standard way in all cases. In the middle term of the top row, $\mathbb{G}_m$ acts by scaling $\mathbb{A}^2 \setminus \{0\}$. In the last term of the second row, $(\mathbb{G}_m)^n$ acts by having the ith copy of $\mathbb{G}_m$ scale the ith copy of $\mathbb{A}^2 \setminus \{0\}$. In the middle term of the second row, $(\mathbb{G}_m)^n$ acts by having the first copy of $\mathbb{G}_m$ act by scaling all copies of $\mathbb{A}^2 \setminus \{0\}$ and the ith copy of $\mathbb{G}_m$ with $2 \le i \le n$ acting by scaling the ith copy of $\mathbb{A}^2 \setminus \{0\}$. In the first term of the second row, the ith copy of $\mathbb{G}_m^{n-1}$ scales the $i+1$st copy of $\mathbb{A}^2 \setminus \{0\}$.

To compute f', we let H_1 be the standard character on the first factor of $\mathbb{G}_m$ in $GL_2 \times (\mathbb{G}_m)^n$ and let $H_2, \dots, H_n$ be the standard characters on the remaining $n-1$ factors and the $n-1$ factors of $\mathbb{G}_m$ in $GL_2 \times (\mathbb{G}_m)^{n-1}$. The induced map $T \times (\mathbb{G}_m)^n \to T \times (\mathbb{G}_m)^{n-1}$ of tori induces $u \mapsto u + H_1$ and $v \mapsto v + H_1$. Therefore,

$$f' : \frac{\mathbb{Z}[u,v]^{S_2}[H_2,\dots,H_n]}{(uv, F(H_2),\dots,F(H_n))} \to \frac{\mathbb{Z}[u,v]^{S_2}[H_1][H_2,\dots,H_n]}{(uv, F(H_2+H_1),\dots,F(H_n+H_1))},$$

where $u \mapsto u + H_1$, $v \mapsto v + H_1$, and $H_i \mapsto H_i$.

For q_2', the induced map $T \times (\mathbb{G}_m)^n \to T \times (\mathbb{G}_m)^n$ of tori induces $H_1 \mapsto H_1$, $H_i \mapsto H_i - H_1$ for $2 \le i \le n$ and $u \mapsto u$, $v \mapsto v$, and gives the map

$$q_2' : \frac{\mathbb{Z}[u,v]^{S_2}[H_1][H_2,\dots,H_n]}{(uv, F(H_2+H_1),\dots,F(H_n+H_1))} \to \frac{\mathbb{Z}[u,v]^{S_2}[H_1,\dots,H_n]}{(F(H_1),\dots,F(H_n))}.$$

The composite

$$q_2' \circ f' : \frac{\mathbb{Z}[u,v]^{S_2}[H_2,\dots,H_n]}{(uv, F(H_2),\dots,F(H_n))} \to \frac{\mathbb{Z}[u,v]^{S_2}[H_1,\dots,H_n]}{(F(H_1),\dots,F(H_n))}$$

is given by $u \mapsto u + H_1$, $v \mapsto v + H_1$, $H_i \mapsto H_i - H_1$ for $2 \le i \le n$. The image is therefore generated by $2H_1 + u + v$ and $H_i - H_1$ for $2 \le i \le n$. If $n \ge 3$, then this is generated by the collection

$$\{H_i + H_j + u + v \mid 1 \le i < j \le n\} = \{[\Delta_{i,j}] \mid 1 \le i < j \le n\}$$

(see Theorem 5.4). □

Remark 4.5. Suppose our base field is $\mathbb{C}$. We have a commutative diagram

$$\begin{array}{ccc} A^\bullet_{PGL_2}((\mathbb{P}^1)^n) & \xhookrightarrow{q_1} & A^\bullet_{GL_2}((\mathbb{P}^1)^n) \\ \downarrow & & \downarrow \\ H^\bullet_{PGL_2}((\mathbb{P}^1)^n) & \xrightarrow[q_1^H]{} & H^\bullet_{GL_2}((\mathbb{P}^1)^n). \end{array}$$

The map $A^\bullet_{GL_2}((\mathbb{P}^1)^n) \to H^\bullet_{GL_2}((\mathbb{P}^1)^n)$ is an isomorphism by the Leray-Hirsch theorem applied to $\mathbb{P}^1_{\mathbb{C}}$-bundles. Running the proof of Theorem 4.3 for the map q_1^H shows q_1^H is injective. Here we replace the projective bundle theorem in algebraic geometry by the Leray-Hirsch theorem applied to $\mathbb{P}^1_{\mathbb{C}}$-bundles and the application of Theorem 4.1 with the fact that if G acts on X and H is a normal subgroup which acts freely, then $(X \times EG)/G \cong ((X \times EG)/H)/(G/H)$, and $(X \times EG)/H$ is homotopy equivalent to X/H and has a free action by G/H.

This implies $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \to H^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism.

4.2. Unordered case

To start, we will recall the computation of $A^\bullet_{PGL_2}(\text{pt})$. This was first done in [27, Theorem 1] over $\mathbb{C}$, where we feel the characteristic assumption was not crucial for the argument. We will follow [6, Lemma 4.4] for this paper, which contains a similar argument adapted to an algebraic setting.

Following the notation in [6], let $\mathbb{A}(2,2)$ be the affine space of homogenous degree 2 forms in three variables. We let the group GL_3 act on $\mathbb{A}(2,2)$ where $A \in GL_3$ sends $\phi \in \mathbb{A}(2,2)$ to $\det(A)\phi \circ A^{-1}$. Let $\mathcal{S} \subset \mathbb{A}(2,2)$ denote the open locus of homogenous forms ϕ that cut out a smooth conic in $\mathbb{P}^2$.

By [6, Proposition 2.5], there is a natural isomorphism $[\text{pt}/PGL_2] \cong [\mathcal{S}/GL_3]$, so their Chow rings agree. Excision gives the $A^\bullet_{PGL_2}$ as a quotient of $A^\bullet_{GL_3}$ and [6, Lemma 4.4] computes the kernel, yielding the following presentation.

Lemma 4.6 ([6, Lemma 4.4]). *We have* $A^\bullet_{PGL_2}(\text{pt}) \cong \mathbb{Z}[c_1, c_2, c_3]/(c_1, 2c_3)$, *where* c_1, c_2, c_3 *are the image of the generators of* $A^\bullet_{GL_3}(\text{pt})$ *after pullback under* $[\text{pt}/PGL_2] \cong [\mathcal{S}/GL_3] \to [\text{pt}/GL_3]$.

In this paper, we allow for our base field K to be characteristic 2. The argument in [6, Lemma 4.4] needs to be modified slightly at the end to work in characteristic 2, so we repeat the argument here for clarity. The idea is unchanged from before.

Proof of Theorem 4.6. Let $\mathbb{G}_m$ act on $\mathbb{A}(2,2)$ by scaling all the coordinates. Since $\mathcal{S}$ is an open subset of $\mathbb{A}(2,2)$, $A^\bullet_{GL_3}(\mathcal{S})$ is generated by the generators c_1, c_2, c_3 of $A^\bullet_{GL_3}(\text{pt})$. Similarly, $A^\bullet_{GL_3 \times \mathbb{G}_m}(\mathcal{S})$ is generated by c_1, c_2, c_3 and the standard character of $\mathbb{G}_m$, which we will call h.[1] The inclusion $GL_3 \to GL_3 \times \mathbb{G}_m$ sending

[1] Our h is $h - c_1$ in the context of [6, Proof of Lemma 4.4].

A to $(A, 1)$ induces a surjection of Chow rings $A^\bullet_{GL_3\times\mathbb{G}_m}(\mathcal{S}) \to A^\bullet_{GL_3}(\mathcal{S})$, sending c_i to c_i for $i = 1, 2, 3$ and h to 0. Therefore, it suffices to compute $A^\bullet_{GL_3\times\mathbb{G}_m}(\mathcal{S}) \cong A^\bullet_{GL_3}(\mathbb{P}(\mathcal{S})$.

The complement of $\mathbb{P}(\mathcal{S}) \subset \mathbb{P}(\mathbb{A}(2,2))$ consists of singular conics. Let $\mathcal{W} \subset \mathbb{P}(\mathbb{A}(2,2)) \times \mathbb{P}^2$ consists of pairs (C, p) where C is a conic singular at p. Since $\mathcal{W} \to \mathbb{P}(\mathbb{A}(2,2))$ has image the complement of $\mathcal{S}$, we have the following exact sequence

$$A^\bullet_{GL_3}(\mathcal{W}) \to A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2))) \to A^\bullet_{GL_3}(\mathcal{S}) \to 0.$$

Therefore, it suffices to compute the image of $A^\bullet_{GL_3}(\mathcal{W}) \to A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)))$ under pushforward. To this end, we will compute the class of $[\mathcal{W}]\in A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2))\times\mathbb{P}^2)$ and conclude using the structure of $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)) \times \mathbb{P}^2)$. The ring $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2))\times\mathbb{P}^2)$ is generated over $\mathbb{Z}$ by the generators c_1, c_2, c_3 of $A^\bullet_{GL_3}(\mathrm{pt})$ and the two hyperplane classes $t \in A^\bullet_{GL_3}(\mathbb{P}^2)$ and $h \in A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)))$, pulled back to $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)) \times \mathbb{P}^2)$.

The computation of $[\mathcal{W}]$ is the only place where this proof diverges from [6, Proof of Lemma 4.4]. Viewing $\mathbb{A}(2,2) \times \mathbb{P}^2$ as an equivariant vector bundle over $\mathbb{P}^2$, we can form the exact sequence of vector bundles

$$0 \to W \to \mathbb{A}(2,2) \times \mathbb{P}^2 \to P^1(\mathscr{O}_{\mathbb{P}^2}(2t)) \otimes D \to 0,$$

where D is the 1-dimensional representation of GL_3 given by the determinant and $P^1(\mathscr{O}_{\mathbb{P}^2}(2t))$ is the sheaf of principal parts parameterizing Taylor expansions up to order 1 at each point of $\mathbb{P}^2$. The map $\mathbb{A}(2,2) \times \mathbb{P}^2 \to P^1(\mathscr{O}_{\mathbb{P}^2}(2)) \otimes D$ is given by twisting the global section map $H^0(\mathscr{O}_{\mathbb{P}^2}(2t)) \to P^1(\mathscr{O}_{\mathbb{P}^2}(2t))$ by D.

In particular, $\mathcal{W}$ is the projectivation $\mathbb{P}(W)$ of a vector bundle $W \to \mathbb{P}^2$. Therefore, $\mathcal{W} \subset \mathbb{P}(\mathbb{A}(2,2))\times\mathbb{P}^2$ is the projectivation of a subbundle, which by [10, Proposition 9.13] has class

$$[\mathcal{W}] = c_3((\mathbb{A}(2,2)/W) \otimes \mathscr{O}(h)) = c_3(P^1(\mathscr{O}(2t)) \otimes D \otimes \mathscr{O}(h)).$$

The exact sequence for the sheaf of principal parts and the Euler exact sequence are given by

$$0 \to \Omega_{\mathbb{P}^2}(2t) \to P^1(\mathscr{O}_{\mathbb{P}^2}(2t)) \to \mathscr{O}_{\mathbb{P}^2}(2t) \to 0$$
$$0 \to \Omega_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(-t) \otimes (K^3)^* \to \mathscr{O}_{\mathbb{P}^2} \to 0.$$

These two exact sequences imply

$$\begin{aligned} c(P^1(\mathscr{O}(2t)) \otimes D \otimes \mathscr{O}(h)) &= c(\Omega_{\mathbb{P}^2}(t + D + h))c(\mathscr{O}_{\mathbb{P}^2}(2t + D + h)) \\ &= \frac{c((K^3)^* \otimes \mathscr{O}(t + D + h))}{c(\mathscr{O}_{\mathbb{P}^2}(2t + D + h))} c(\mathscr{O}_{\mathbb{P}^2}(2t + D + h)) \\ &= c((K^3)^* \otimes \mathscr{O}(t + D + h)). \end{aligned}$$

Letting u, v, w be the chern roots of c_1, c_2, c_3, we find

$$\begin{aligned}[\mathcal{W}] &= (1-u+t+D+h)(1-v+t+D+h)(1-w+t+D+h)\\ &= (t+c_1+h)^3 - c_1(t+c_1+h)^2 + c_2(t+c_1+H) - c_3\\ &= t^2(c_1+3h) + t(4c_1h + c_1^2 + 3h^2) + 2c_1h^2 + c_1^2h + c_2h\\ &\quad + c_1c_2 - 2c_3 + h^3,\end{aligned} \tag{4.1}$$

using the relation $t^3 + c_1t^2 + c_2t + c_3 = 0$. Our final goal is to compute the image of $A^\bullet_{GL_3}(\mathcal{W}) \to A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)))$ under pushforward. Since $\mathcal{W}$ is a projective subbundle of $\mathbb{P}(\mathbb{A}(2,2)) \times \mathbb{P}^2$, viewed as bundles over $\mathbb{P}^2$, the restriction map $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)) \times \mathbb{P}^2) \to A^\bullet_{GL_3}(\mathcal{W})$ is surjective. This means the image of $A^\bullet_{GL_3}(\mathcal{W}) \to A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)))$ is the image of the ideal generated by $[\mathcal{W}]$ in $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)) \times \mathbb{P}^2)$ pushed forward to $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)))$.

Since $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)) \times \mathbb{P}^2)$ is a free module over $A^\bullet_{GL_3}(\mathbb{P}(\mathbb{A}(2,2)))$ generated by $1, t, t^2$, and the pushforward map is known explicitly to be extracting the coefficient of the t^2 term after reducing modulo $t^3 + c_1t^2 + c_2t + c_3 = 0$, one can argue as in [14, Theorem 5.5] to see that the image of the ideal generated by $[\mathcal{W}]$ has image generated by the coefficients

$$c_1 + 3h, \quad 4c_1h + c_1^2 + 3h^2, \quad 2c_1h^2 + c_1^2h + c_2h + c_1c_2 - 2c_3 + h^3$$

of t^2, t and 1 in the expression for $[\mathcal{W}]$ (4.1). Finally, to get the relations in $A^\bullet_{GL_3}(S)$, we also add the relation $h = 0$, yielding

$$A^\bullet_{GL_3}(S) = \mathbb{Z}[c_1, c_2, c_3]/(c_1, c_1^2, c_1c_2 - 2c_3) = \mathbb{Z}[c_1, c_2, c_3]/(c_1, 2c_3),$$

as desired. □

In addition to knowing the presentation of $A^\bullet_{PGL_2}(\mathrm{pt})$, we also want to know the map $A^\bullet_{PGL_2}(\mathrm{pt}) \to A^\bullet_{GL_2}(\mathrm{pt})$ induced by the quotient $GL_2 \to PGL_2$.

Lemma 4.7. *Let T be the standard torus inside of GL_2. The composition $T \hookrightarrow GL_2 \to PGL_2$ induces the map*

$$\mathbb{Z}[c_1, c_2, c_3]/(c_1, 2c_3) \cong A^\bullet_{PGL_2}(\mathrm{pt}) \to A^\bullet_T(\mathrm{pt}) \cong \mathbb{Z}[u, v],$$

sending $c_1 \mapsto 0$, $c_2 \mapsto -(u-v)^2$, $c_3 \mapsto 0$, where u and v are the standard characters of T.

Proof. Theorem 4.7 amounts to following the composition of maps

$$[\mathrm{pt}/T] \to [\mathrm{pt}/GL_2] \to [\mathrm{pt}/PGL_2] \to [\mathcal{S}/GL_3] \to [\mathrm{pt}/GL_3].$$

To this end, let us start with a map $S \to [\mathrm{pt}/GL_2]$ from a scheme, which is equivalent to the data of a vector bundle $V \to S$. For the purposes of computing the map on Chow rings, it suffices by definition to consider the case where S is the Grassmannian $G(2,N)$ and V is the tautological subbundle. This yields a map $S \to [\mathrm{pt}/PGL_2]$, which is given by the data of the $\mathbb{P}^1$ bundle $\pi : \mathbb{P}(V) \to S$. For the key step, this $\mathbb{P}^1$ bundle yields the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(V) & \hookrightarrow & \mathbb{P}((\pi_* \omega^{-1}_{\mathbb{P}(V)/B})^*) \\ \downarrow \pi & \swarrow & \\ B & & \end{array}$$

where each $\mathbb{P}^1$ fiber of π is embedded into each $\mathbb{P}^2$ fiber of $\mathbb{P}((\pi_* \omega^{-1}_{\mathbb{P}(V)/B})^*) \to B$ as a conic by the anticanonical map. This diagram yields the data of a map $S \to [\mathcal{S}/GL_3]$. Finally, to get the data of $S \to [\mathrm{pt}/GL_3]$, we only remember the vector bundle $(\pi_* \omega^{-1}_{\mathbb{P}(V)/B})^* \to B$.

By the Euler exact sequence, we have

$$0 \to \mathscr{O}_{\mathbb{P}(V)} \to V \otimes \mathscr{O}_{\mathbb{P}(V)}(1) \to \omega^{-1}_{\mathbb{P}(V)/B} \to 0,$$

which yields

$$0 \to \mathscr{O}_S \to V \otimes V^* \to \pi_* \omega^{-1}_{\mathbb{P}(V)/B} \to 0.$$

If we assume that the vector bundle V has chern roots u and v, then we see that the quotient of $V \otimes V^*$ by the trivial bundle has chern roots $0, u-v, v-u$. Taking the dual yields again the same chern roots. Therefore, the chern classes $c_1, c_2, c_3 \in A^\bullet_{GL_3}(\mathrm{pt})$ pull back to the elementary symmetric functions in $0, u-v, v-u$, finishing the proof. □

Proposition 4.8. *We have*

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \cong \begin{cases} \mathbb{Z}[u,v]^{S_2}/(\prod_{i=0}^n((\frac{n+1}{2}-i)u + (\frac{-n+1}{2}+i)v)) & \text{if } n \text{ is odd} \\ \mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H)) & \text{if } n \text{ is even} \end{cases}$$

where $p_n(t) \in A^\bullet_{PGL_2}(\mathrm{pt})[t]$ *is defined as*

$$p_n(t) = \begin{cases} t\prod_{k=1}^{\frac{n}{2}}(t^2+k^2c_2) + t^{\frac{n}{4}+1}\sum_{k=1}^{\frac{n}{4}}\binom{\frac{n}{4}}{k}(t^3+c_2t)^{\frac{n}{4}-k}c_3^k & n \equiv 0 \pmod 4 \\ t\prod_{k=1}^{\frac{n}{2}}(t^2+k^2c_2) + t^{\frac{n-2}{4}}\prod_{k=1}^{\frac{n+2}{4}}\binom{\frac{n+2}{4}}{k}(t^3+c_2t)^{\frac{n+2}{4}-k}c_3^k & n \equiv 2 \pmod 4. \end{cases}$$

The map

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$$

induced by $GL_2 \to PGL_2$ *is given by*

$$u \mapsto H + \frac{n+1}{2}u + \frac{n-1}{2}v \quad v \mapsto H + \frac{n-1}{2}u + \frac{n+1}{2}v \qquad \text{if } n \text{ is odd}$$

$$c_2 \mapsto -(u-v)^2 \quad c_3 \mapsto 0 \quad H \mapsto H + \frac{n}{2}(u+v) \qquad \text{if } n \text{ is even.}$$

Finally, $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ *is injective for* n *odd and injective up to 2-torsion when* n *is even.*

Proof. The injectivity statements immediately follow from the explicit descriptions of all of the rings maps in the statement of Theorem 4.8. Indeed, when n is odd, we have

$$\prod_{i=0}^{n}\left(\left(\frac{n+1}{2}-i\right)u + \left(\frac{-n+1}{2}+i\right)v\right) \mapsto \prod_{i=0}^{n}(H+(n-i)u+iv),$$

which is a relation in $A^\bullet_{GL_2}(\mathbb{P}^n)$, and similarly for n even we have

$$\begin{aligned} p_n(H)|_{c_3=0} \mapsto \left(H+\frac{n}{2}(u+v)\right)&\prod_{k=1}^{\frac{n}{2}} \\ &\left(H+\frac{n}{2}(u+v)-k(u-v)\right)\left(H+\frac{n}{2}(u+v)+k(u-v)\right) \\ = \prod_{i=0}^{n}&(H+(n-i)u+iv). \end{aligned}$$

We do the cases n is odd and even separately. First suppose n is odd. Consider the commutative diagram

$$\begin{array}{ccc} A^\bullet_{PGL_2}(\mathbb{P}^n) & \longrightarrow & A^\bullet_{GL_2}(\mathbb{P}^n) \\ \downarrow\sim & & \downarrow\sim \\ A^\bullet_{GL_2}(\mathbb{A}^{n+1}\setminus\{0\}) & \xrightarrow[\phi]{} & A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1}\setminus\{0\}) \end{array}$$

For the Chow ring $A^\bullet_{GL_2}(\mathbb{A}^{n+1}\setminus\{0\})$ in the bottom left corner, GL_2 acts on $\mathbb{A}^{n+1}$ by sending a degree n binary form $f \in K[x,y]_d$ to $\det(A)^{\frac{n-1}{2}} f(A^{-1}\begin{pmatrix} x \\ y\end{pmatrix})$ for $A \in GL_2$. For the Chow ring $A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1}\setminus\{0\})$ in the bottom right corner, GL_2 acts as normal, sending f as before to $f(A^{-1}\begin{pmatrix} x \\ y\end{pmatrix})$ and $\mathbb{G}_m$ acts by scaling. The first vertical arrow is an isomorphism because the matrices $\begin{pmatrix} t & 0 \\ 0 & t\end{pmatrix}$ act on $\mathbb{A}^2$ by scaling by t.

To determine $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \{0\})$ it suffices to check the maximal torus $T \subset GL_2$ acts on $\mathbb{A}^{n+1}$ with characters $\{(\frac{n+1}{2}-i)u + (\frac{-n+1}{2}+i)v \mid 0 \le i \le n\}$. This shows

$$A^\bullet_{PGL_2}(\mathbb{P}^n) = \mathbb{Z}[u,v]^{S_2}/(\prod_{i=0}^{n}((\frac{n+1}{2}-i)u + (\frac{-n+1}{2}+i)v))$$

in this case.

To find the map $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \{0\}) \to A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1} \setminus \{0\})$, we consider the map $GL_2 \times \mathbb{G}_m \to GL_2$ and find it maps the pair $\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}, \lambda$ to $\begin{pmatrix}\lambda\lambda_1^{\frac{n+1}{2}}\lambda_2^{\frac{n-1}{2}} & \\ & \lambda\lambda_1^{\frac{n-1}{2}}\lambda_2^{\frac{n+1}{2}}\end{pmatrix}$ in GL_2. This shows the map

$$\begin{aligned}&\mathbb{Z}[u,v]^{S_2}\Big/\left(\prod_{i=0}^{n}\left(\left(\frac{n+1}{2}-i\right)u + \left(\frac{-n+1}{2}+i\right)v\right)\right)\\ &\quad\to \mathbb{Z}[u,v]^{S_2}[H]\Big/\left(\prod_{i=0}^{n}(H+iu+(n-i)v)\right)\end{aligned}$$

giving $A^\bullet_{GL_2}(\mathbb{A}^{n+1} \setminus \{0\}) \to A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1} \setminus \{0\})$ is given by

$$u \mapsto H + \frac{n+1}{2}u + \frac{n-1}{2}v \quad v \mapsto H + \frac{n-1}{2}u + \frac{n+1}{2}v.$$

Now, we do the case n is even. Let $V \cong K^2$ be the standard representation of GL_2, and $D = \det(V) \cong K$. Then $(\operatorname{Sym}^n V) \otimes (D^\vee)^{\otimes n/2}$ is a GL_2 representation that descends to a PGL_2 representation.

To determine

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \cong A^\bullet_{PGL_2}(\mathbb{P}((\operatorname{Sym}^n V) \otimes (D^\vee)^{\otimes\frac{n}{2}}))$$

it suffices to find the chern classes of the PGL_2 representation $(\operatorname{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}}$ regarded as a PGL_2-equivariant vector bundle over a point. These chern classes are given in [15, Corollary 6.3]. The reader should also note that [15] contains mistakes elsewhere in the document (see [6, Introduction]). As a result, we have $A^\bullet_{PGL_2}(\mathbb{P}^n)$ is $\mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H))$, where $p_n(t) \in A_{PGL_2}(\mathrm{pt})[t]$ is given as in the statement of the proposition.

Therefore, we have

$$A^\bullet_{PGL_2}\left(\mathbb{P}\left((\operatorname{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}}\right)\right) \to A^\bullet_{GL_2}\left(\mathbb{P}\left((\operatorname{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}}\right)\right)$$

given by $c_2 \mapsto -(u-v)^2$ and $c_3 \mapsto 0$ by Theorem 4.7.

Also, the $\mathscr{O}_{\mathbb{P}((\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}})}(1)$ class in $A^\bullet_{PGL_2}(\mathbb{P}((\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}})$ maps to the $\mathscr{O}_{\mathbb{P}((\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes n})}(1)$ class in $A^\bullet_{GL_2}(\mathbb{P}((\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}})$ by the projective bundle formula.

Finally, since $(\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}}$ is a twist of $\mathrm{Sym}^n V$ by a GL_2-equivariant line bundle, the $\mathscr{O}_{\mathbb{P}((\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}})}(1)$ class in $A^\bullet_{GL_2}(\mathbb{P}((\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}}))$ maps to $\mathscr{O}_{\mathbb{P}(\mathrm{Sym}^n V)}(1) - c_1^{GL_2}((D^\vee)^{\otimes\frac{n}{2}})$ in $A^\bullet_{GL_2}(\mathbb{P}(\mathrm{Sym}^n V)^{\otimes\frac{n}{2}})$.

Since $-c_1^{GL_2}((D^\vee)^{\otimes\frac{n}{2}}) = c_1^{GL_2}(D^{\otimes n/2}) = \frac{n}{2}(u+v)$, we find the composite map

$$A^\bullet_{PGL_2}\left(\mathbb{P}\left((\mathrm{Sym}^n V)\otimes(D^\vee)^{\otimes\frac{n}{2}}\right)\right) \to A^\bullet_{GL_2}\left(\mathbb{P}(\mathrm{Sym}^n V)\right)$$

is given by

$$c_2 \mapsto -(u-v)^2 \quad c_3 \mapsto 0 \quad H \mapsto H + \frac{n}{2}(u+v). \qquad \square$$

5. Formulas and initial reductions

In this section we express the $[\Delta_P]$ and $[\lambda]$ classes in terms of our equivariant Chow ring presentations. We also compute formulas for $[\Delta_P] \in A^\bullet_T((\mathbb{P}^1)^n)$ and give a quick, alternative computation of the classes $[Z_\lambda] \in A^\bullet_T(\mathbb{P}^n)$ given in [12, Theorem 3.4]. The simple presentation for the class of the diagonal in $(\mathbb{P}^1)^n$ works especially well with the formula for the pushforward $\Phi_* : A^\bullet_T((\mathbb{P}^1)^n) \to A^\bullet_T(\mathbb{P}^n)$ via the classes of torus fixed points, and appears not to have been previously exploited in this fashion.

5.1. Class of the diagonal in $(\mathbb{P}^1)^n$

We now compute the T-equivariant class of the diagonal $\Delta_{\{[n]\}} \subset (\mathbb{P}^1)^n$. This formula would also follow from localization to the torus fixed points, but the derivation below is simpler.

Proposition 5.1. *The class of $\Delta_{\{[n]\}}$ in $A^\bullet_T((\mathbb{P}^1)^n)$ is given by*

$$[\Delta_{\{[n]\}}] = \frac{1}{u-v}\left(\prod_{i=1}^n (H_i+u) - \prod_{i=1}^n (H_i+v)\right).$$

Proof. This follows from the fact that $\Delta_{\{[n]\}}$ intersected with $\{[0:1]\}\times(\mathbb{P}^1)^{n-1}$ and $\{[1:0]\}\times(\mathbb{P}^1)^{n-1}$ are the torus-fixed points $[0:1]^n$ and $[1:0]^n$ respectively, so

$$((H_1+u)-(H_1+v))[\Delta_{\{[n]\}}] = \prod_{i=1}^n (H_i+u) - \prod_{i=1}^n (H_i+v). \qquad \square$$

5.2. Formula for $[\Delta_P]$

When two strata $[\Delta_P]$ and $[\Delta_{P'}]$ intersect transversely in $(\mathbb{P}^1)^n$, it is easy to describe their intersection as another stratum.

Proposition 5.2. *The class $[\Delta_P] \in A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$ for P a partition of $[n]$ into d parts is given by the product $\prod_{\{i,j\}\in Edge(\mathcal{F})}[\Delta_{i,j}]$, where $\mathcal{F}$ is any forest with vertex set $[n]$ consisting of one spanning tree for each part of P. In particular,*

(1) *If i, j are in distinct parts of P, then if P_{ij} is the partition merging the parts containing i and j, we have $[\Delta_{i,j}][\Delta_P] = [\Delta_{P_{ij}}]$;*
(2) *If i, j, i', j' are in the same part of P, we have $\Delta_{i,j}\Delta_P = \Delta_{i',j'}\Delta_P$.*

Proof. Writing $P = \{P_1, \ldots, P_k\}$, we have $[\Delta_P] = \prod[\Delta_{P_i}]$ so it suffices to show the result for $P = \{[n]\}$ into a single part. Consider a tree $\mathcal{T}$ spanning the vertices $[n]$, and take a leaf i with associated edge ij. Then $\mathcal{T} \setminus ij$ spans $[n] \setminus i$, and so by induction it suffices to show $[\Delta_{i,j}][\Delta_{\{[n]\setminus j\}}] = [\Delta_{\{[n]\}}]$ which follows from the transversality of the intersection $\Delta_{i,j} \cap \Delta_{[n]\setminus j} = \Delta_{[n]}$. Item (1) now follows immediately. For item (2), it suffices to show that $[\Delta_{i,j}][\Delta_P] = [\Delta_{k,j}][\Delta_P]$ whenever i, j, k are distinct and in the same part of P, which follows by taking the forest $\mathcal{T}$ to contain the edge ik and using the diagonal relation $[\Delta_{i,j}][\Delta_{i,k}] = [\Delta_{k,j}][\Delta_{i,k}]$. □

Proposition 5.3. *Let $P = \{V_1, \ldots, V_d\}$ be a partition of $[n]$, then*

$$[\Delta_P] = \frac{1}{(u-v)^d}\prod_{i=1}^{d}\left(\prod_{j\in V_i}(H_j+u) - \prod_{j\in V_i}(H_j+v)\right).$$

Proof. From Theorem 5.2, $[\Delta_P] = \prod_{i=1}^d \Delta_{\{V_i\}}$. Now apply Theorem 5.1. □

5.3. The ψ_i and $[\Delta_{i,j}]$ classes

At this point, we can prove the formula for $[\Delta_{i,j}]$ in item (3) of Theorem 3.1 and for ψ_i as mentioned in Theorem 3.2.

Proposition 5.4. *We have*

$$[\Delta_{i,j}] = H_i + H_j + u + v$$
$$\psi_i = -(2H_i + u + v).$$

Proof. The formula for $[\Delta_{i,j}]$ is an immediate consequence of Theorem 5.3.

To compute ψ_i, it suffices to show that $c_1(T_{\mathbb{P}^1}) \in A^\bullet_{GL_2}(\mathbb{P}^1)$ is $2H + u + v$, where $H = c_1(\mathcal{O}(1))$. We note that $c_{\text{top}}(T_X)$ for any smooth X is the pullback of the diagonal under the diagonal map $X \to X \times X$. The pullback $A^\bullet_{GL_2}((\mathbb{P}^1)^2) \to A^\bullet_{GL_2}(\mathbb{P}^1)$ under the inclusion $\mathbb{P}^1 \cong \Delta_{1,2} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is given by $H_1, H_2 \mapsto H$. Under this map, $[\Delta_{1,2}]$ pulls back to $2H + u + v$ as desired. □

5.4. Pullback and Pushforward under Φ

The pullback map $\Phi^* : A_T^\bullet(\mathbb{P}^n) \to A_T^\bullet((\mathbb{P}^1)^n)$ is induced by

$$\Phi^*(H) = \sum_{i=1}^{n} H_i.$$

We now consider Φ_*. By considering the classes of the torus-fixed loci, we have for any $A \subset [n]$,

$$\Phi_*\left(\prod_A (H_i + u) \prod_{[n]\setminus A} (H_j + v)\right) = \prod_{([n]\cup\{0\})\setminus|A|} (H + kv + (n-k)u).$$

This in fact uniquely characterizes Φ_*, which can be seen either from localization [9, Theorem 2] or because

$$\frac{\prod_A (H_i + u) \prod_{[n]\setminus A} (H_j + v)}{\prod_A (-v + u) \prod_{[n]\setminus A} (-u + v)}$$

is a Lagrange interpolation basis for polynomials in $H_1, \ldots, H_n$ modulo $F(H_i)$ for each i.

5.5. Formula for $[\lambda]$

Fehér, Némethi, and Rimányi computed the class of $[\lambda]$ for λ a partition of n [12, Theorem 3.4]. We can give a quick self-contained computation from Section 5.4 and theorem 5.1 as follows.

Theorem 5.5 ([12, Theorem 3.4]). *The class $[a_1, \ldots, a_d]$ is the result of first expanding the polynomial*

$$\prod_{i=1}^{d} (z^{a_i} - 1) = \sum_{k \geq 0} c_k z^k \qquad (c_k \in \mathbb{Z}),$$

and then replacing each monomial

$$z^k \mapsto \frac{1}{(u-v)^d} \prod_{([n]\cup\{0\})\setminus k} (H + jv + (n-j)u).$$

Proof. Let $P = \{V_1, \ldots, V_d\}$ be a partition of $[n]$ with $|V_i| = a_i$. We expand the formula from Theorem 5.3

$$\Delta_P = \frac{1}{(u-v)^d} \prod_{i=1}^{d} \left(\prod_{j \in V_i} (H_i + u) - \prod_{j \in V_i} (H_i + v) \right)$$

to a sum of terms of the form $\prod_{i\in A}(H_i + u) \prod_{j\in[n]\setminus A}(H_j + v)$. Then, Section 5.4 implies that each such term pushes forward to $\prod_{([n]\cup\{0\})\setminus|A|}(H + jv + (n-j)u)$. The result follows immediately. □

6. Strata in $[(\mathbb{P}^1)^n/PGL_2]$

In this section we prove all of our results on ordered point configurations in $\mathbb{P}^1$. Up to Section 6.1, the only result that we use is Theorem 4.3, and in particular the identification of $[\Delta_{i,j}]$ in $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ as $H_i + H_j + u + v$.

Remark 6.1. Whenever we write $\Delta_{i,j}$ in any context, we will always treat $\{i, j\}$ as an unordered tuple, so that implicitly

$$\Delta_{i,j} := \Delta_{j,i}$$

for $i > j$.

Recall from Theorem 4.3 and Section 2.3, we have the inclusions

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \subset A^\bullet_{GL_2}((\mathbb{P}^1)^n) \subset A^\bullet_T((\mathbb{P}^1)^n).$$

We first consider the square relation in $(\mathbb{P}^1)^4$.

Proposition 6.2. *In $A^\bullet_{PGL_2}((\mathbb{P}^1)^4)$, we have the square relation*

$$[\Delta_{1,2}] + [\Delta_{3,4}] = [\Delta_{2,3}] + [\Delta_{4,1}].$$

Proof. Both sides are equal to $H_1 + H_2 + H_3 + H_4 + 2(u + v)$ by Theorem 5.4. This can also be shown using the fact that the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ has a torus-equivariant deformation to $\{0\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{\infty\}$. It also holds by the discussion in Section 1.1. □

Definition 6.3. Let $R(n)$ be the ring

$$R(n) = \mathbb{Z}[\{\Delta_{i,j} \mid 1 \le i < j \le n\}]/\text{relations},$$

generated by the symbols $\Delta_{i,j} = \Delta_{j,i}$ together with the relations

(1) $\Delta_{i,j} + \Delta_{k,l} = \Delta_{i,k} + \Delta_{j,l}$ for distinct i, j, k, l (square relations);
(2) $\Delta_{i,j}\Delta_{i,k} = \Delta_{i,j}\Delta_{j,k}$ for distinct i, j, k (diagonal relations)

given in Theorem 3.1 (1). If n is clear from context or irrelevant, we will let $R := R(n)$. If we let each $\Delta_{i,j}$ have degree 1, then the ideal of relations is homogenous, so R is a graded ring, and we will denote by R_k the kth graded part of R.

By Theorem 4.3, we can identify $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ as a subring of $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$, where the image

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \hookrightarrow A^\bullet_{GL_2}((\mathbb{P}^1)^n) = \mathbb{Z}[u, v]^{S_2}[H_1, \ldots, H_n]/(F(H_1), \ldots, F(H_n))$$

is generated by $\Delta_{i,j} = H_i + H_j + u + v$ for $n \geq 3$. If $n \leq 2$, we also have to add the classes $\psi_i = -(2H_i + u + v)$ (see Theorem 5.4). Therefore for $n \geq 3$ by Theorem 6.2, we have a surjective map

$$R \twoheadrightarrow A^\bullet_{PGL_2}((\mathbb{P}^1)^n), \tag{6.1}$$

sending each symbol $\Delta_{i,j} \in R$ to $\Delta_{i,j} \in A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$. To show Theorem 3.1 (1), we need to show this surjection is an isomorphism for $n \geq 3$.

As $A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ is free as an Abelian group, $A^k_{PGL_2}((\mathbb{P}^1)^n)$ is a free Abelian group for each k. We first compute the rank of these groups for varying k.

Lemma 6.4. *For every $n \geq 1$, the free Abelian group $A^k_{PGL_2}((\mathbb{P}^1)^n)$ has rank*

$$\sum_{\substack{0 \leq i \leq k \\ i \equiv k \bmod 2}} \binom{n}{i}.$$

Proof. We compute the rank of $A^k_{PGL_2}((\mathbb{P}^1)^n)$ by working instead with the rational subring

$$A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q} \subset A^\bullet_{GL_2}((\mathbb{P}^1)^n)) \otimes \mathbb{Q},$$

which is generated by the elements $H'_i := H_i + \frac{1}{2}(u+v)$ by Theorem 4.3. Noting that $H'^2_i = \frac{1}{4}(u-v)^2$, we see the $\mathbb{Q}$-vector space $A^k_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}$ is spanned by the elements

$$\mathcal{B} = \left\{ \left(\frac{u-v}{2}\right)^{k-|B|} \prod_{i \in B} H'_i \;\middle|\; B \subset [n],\ |B| \leq k,\ |B| \equiv k \pmod 2 \right\},$$

which has size

$$|\mathcal{B}| = \sum_{\substack{0 \leq i \leq k \\ i \equiv k \bmod 2}} \binom{n}{i}.$$

To finish, it suffices to show that the elements of $\mathcal{B}$ are linearly independent. Indeed, the elements of $\mathcal{B}$ become distinct monomials in the H'_i after setting $u = 1$ and $v = -1$ (after which the defining relations $F(H_i) = 0$ become $H'^2_i = 1$ for each i). □

Definition 6.5. Let Part(d, n) denote the set of partitions of $[n]$ into d parts. For $P \in$ Part(d, n), for any forest $\mathcal{F}$ with vertex set $[n]$ consisting of one spanning tree for each part of P, we define

$$\Delta_P = \prod_{\{i,j\} \in \mathrm{Edge}(\mathcal{F})} \Delta_{i,j} \in R.$$

Note that by the diagonal relations this is independent of the choice of $\mathcal{F}$, and $\Delta_P \mapsto \Delta_P$ under the map $R \to A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ by Theorem 5.2.

Remark 6.6. The two items (1), (2) in Theorem 5.2 are also true for the elements $\Delta_P \in R$ as the proof only uses the diagonal relations in $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$.

Lemma 6.7. *For $k \leq n-2$, R_k is generated by $\{\Delta_P \mid P \in \mathrm{Part}(n-k,n)\}$.*

Proof. Given a product $\prod_{\ell=1}^{k} \Delta_{i_\ell, j_\ell}$, we will produce an algorithm for rewriting this product in terms of Δ_P with P a partition of $[n]$ into $n-k$ parts.

By induction, we can write $\prod_{\ell=1}^{k-1} \Delta_{i_\ell, j_\ell}$ as $\sum_{P' \in \mathrm{Part}(n-k+1,n)} a_{P'} \Delta_{P'}$, so it suffices to show that $\Delta_{i_k, j_k} \Delta_{P'}$ for $P' \in \mathrm{Part}(n-k+1,n)$ can be written as a $\mathbb{Z}$-linear combination $\sum_{P \in \mathrm{Part}(n-k,n)} a_P \Delta_P$.

If i_k, j_k are in different parts of P', then $\Delta_{i_k, j_k} \Delta_{P'} = \Delta_P$ where P merges the parts containing i_k and j_k, and we are done. Otherwise, if they are in the same part A_1, let A_2, A_3 be two parts of P' distinct from A_1 (which exist as $n-k+1 \geq 3$), with elements $x_2 \in A_2$ and $x_3 \in A_3$. By applying a square relation, we have

$$\Delta_{i_k, j_k} \Delta_{P'} = (\Delta_{i_k, x_2} - \Delta_{x_2, x_3} + \Delta_{x_3, j_k}) \Delta_{P'},$$

and each of the three terms on the right is some Δ_P with $P \in \mathrm{Part}(n-k,n)$. $\square$

Definition 6.8. Given a partition P of $[n]$ and $i, j \in [n]$ in distinct parts of P, let $P_{i,j}$ be the partition of $[n]$ obtained by merging the parts in P containing i and j.

From Theorem 6.6, the following relations hold in $R(n)$ (and hence also in $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$).

Definition 6.9. For i_1, i_2, i_3, i_4 in distinct parts of a partition P of $[n]$, define the *square relation* for P associated to i_1, i_2, i_3, i_4 to be the relation

$$\Delta_{P_{i_1,i_2}} - \Delta_{P_{i_2,i_3}} + \Delta_{P_{i_3,i_4}} - \Delta_{P_{i_4,i_1}} = 0.$$

Definition 6.10. Inside the free Abelian group $\mathbb{Z}^{\mathrm{Part}(d,n)}$, denote by $\mathrm{Sq}(d,n)$ the subgroup generated by formal square relations $P_{i,j} - P_{j,k} + P_{k,l} - P_{l,i}$ for $P \in \mathrm{Part}(d+1,n)$ and i, j, k, l in distinct parts of P. Then we define

$$\mathcal{A}(d,n) := \mathbb{Z}^{\mathrm{Part}(d,n)} / \mathrm{Sq}(d,n).$$

Theorem 6.7 shows for $d \geq 2$ we have a surjection

$$\mathcal{A}(d,n) \twoheadrightarrow R_{n-d}$$

that sends $P \mapsto \Delta_P$. We will in fact show this is an isomorphism.

Definition 6.11. Say a partition $P \in \mathrm{Part}(d,n)$ for $d \geq 2$ is *good* if P can be written as $P = \{A_1, \ldots, A_d\}$ with $A_1 \sqcup A_2$ a partition of an initial segment of $[n]$, and $A_3, \ldots, A_d$ all contiguous intervals. Denote

$$\mathrm{Good}(d,n) := \{P \in \mathrm{Part}(d,n) \mid P \text{ good}\}.$$

Lemma 6.12. *For $2 \le d \le n$, $\mathcal{A}(d,n)$ is generated by the set of $P \in \operatorname{Good}(d,n)$.*

Proof. We use induction on n and d. For $d = 2$ every partition is good, and for $d = n$ the result is trivial. Suppose now we have $2 < d < n$. Take $Q \in \operatorname{Part}(d,n)$.

If $n-1$ and n are in the same part, then $Q' := Q \setminus n \in \operatorname{Part}(d,n-1)$, and by the induction hypothesis applied to $\mathcal{A}(d,n-1)$ we can write $Q' = \sum_{P' \in \operatorname{Good}(d,n-1)} a_{P'} P'$. There is a map

$$\mathcal{A}(d,n-1) \to \mathcal{A}(d,n)$$

mapping each P' for $P' \in \operatorname{Part}(d,n-1)$ to P, where P is obtained by adding n to the same part as $n-1$ in P'. Furthermore, under this map $P \in \operatorname{Good}(d,n)$ if $P' \in \operatorname{Good}(d,n-1)$, so we get Q as a $\mathbb{Z}$-linear combination of P for $P \in \operatorname{Good}(d,n)$.

If n is isolated in Q, then let $Q' = Q \setminus n \in \operatorname{Part}(d-1,n-1)$. By the induction hypothesis applied to $\mathcal{A}(d-1,n-1)$, we can write $Q = \sum_{P' \in \operatorname{Good}(d-1,n-1)} a_{P'} P'$. There is a map

$$\mathcal{A}(d-1,n-1) \to \mathcal{A}(d,n)$$

mapping each P' for $P' \in \operatorname{Part}(d-1,n-1)$ to P, where P is obtained by adding n as an isolated part. Furthermore, under this map $P \in \operatorname{Good}(d,n)$ if $P' \in \operatorname{Good}(d-1,n-1)$, so we get Q as a $\mathbb{Z}$-linear combination of P for $P \in \operatorname{Good}(d,n)$.

If neither of the above two cases hold, then $n-1$ and n are not in the same part and n is not isolated in Q. Let $x \in [n]$ be another element in the same part as n, and let $y \in [n]$ be in a different part as $n-1$ and n (which exists as $d > 2$). Then if we let $\widetilde{Q} \in \operatorname{Part}(d+1,n)$ be the result of taking Q and isolating n into its own part, the square relation for $\widetilde{Q}$ associated to $n-1, n, x, y$ yields Q as a combination of 3 terms, each of which either has n isolated or $n-1, n$ in the same group. □

Lemma 6.13. *For $2 \le d \le n$,*

$$\#\operatorname{Good}(d,n) = \sum_{\substack{0 \le i \le n-d \\ i \equiv n-d \bmod 2}} \binom{n}{i}.$$

Proof. From the definition of $\operatorname{Good}(d,n)$,

$$\#\operatorname{Good}(d,n) = \sum_{k=1}^{n-d+2} (2^{k-1}-1)\binom{n-k-1}{n-k-d+2}.$$

To see this, suppose $A_1 \cup A_2$ is the initial segment $\{1,\dots,k\}$. Then, there are $2^k - 2$ partitions of a size k set into two nonempty subsets, where the subsets have an ordering. Since we do not want our subsets to have an ordering, this yields $2^{k-1}-1$ ways to choose A_1 and A_2. The factor of $\binom{n-k-1}{n-k-d+2} = \binom{n-k-1}{d-3}$ appearing in the summation is the number of ways to partition $\{k+1,\dots,n\}$ into $d-2$ nonempty contiguous blocks. In particular, when $k=n$ or $d=2$, it is understood that this factor is 1.

Let

$$G_{d,n} = \sum_{k=1}^{n-d+2} (2^{k-1}-1)\binom{n-k-1}{n-k-d+2}$$

$$G'_{d,n} = \sum_{\substack{0\le i\le n-d\\ i\equiv n-d \bmod 2}} \binom{n}{i}.$$

We will show $G_{d,n}=G'_{d,n}$ for all $n\ge 2$ and $d\ge 2$ by induction on n. For the base case if $n=2$ and $d\ge 2$ arbitrary, we have two cases: if $d=2$, $\#\mathrm{Good}(2,2)=G_{2,2}=1$ and if $d>2$, $\#\mathrm{Good}(d,2)=G_{d,2}=0$. If $d=2$ and $n\ge 2$ arbitrary, then $G'_{d,n}=2^{n-1}-1$ by the binomial theorem, and $G_{d,n}=2^{n-1}-1$ because only the $k=n$ term $(2^{n-1}-1)\binom{-1}{0}$ contributes.

Now, assume we know $G_{d,n}=G'_{d,n}$ for some n and all $d\ge 2$. For the induction step,

$$\begin{aligned} G_{d,n}+G_{d+1,n} &= \sum_{k=1}^{n-d+2}(2^{k-1}-1)\left(\binom{n-k-1}{n-k-d+2}+\binom{n-k-1}{n-k-d+1}\right)\\ &= \sum_{k=1}^{n-d+2}(2^{k-1}-1)\binom{n-k}{n-k-d+2} = G_{d+1,n+1}, \end{aligned}$$

and similarly applying Pascal's identity, $G'_{d,n}+G'_{d+1,n}=G'_{d+1,n+1}$. □

Corollary 6.14. *For* $2\le d\le n$ *and* $n\ge 3$ *we have the isomorphisms*

$$\mathbb{Z}^{\mathrm{Good}(d,n)} \xrightarrow{\sim} \mathcal{A}(d,n) \xrightarrow{\sim} R_{n-d} \xrightarrow{\sim} A^{n-d}_{PGL_2}((\mathbb{P}^1)^n).$$

Proof. By Theorems 6.7 and 6.12 and (6.1), we have

$$\mathbb{Z}^{\mathrm{Good}(d,n)} \twoheadrightarrow \mathcal{A}(d,n) \twoheadrightarrow R_{n-d} \twoheadrightarrow A^{n-d}_{PGL_2}((\mathbb{P}^1)^n).$$

Since $A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$ is a finitely generated, free $\mathbb{Z}$-module of rank equal to the rank of $\mathbb{Z}^{\mathrm{Good}(d,n)}$ by Theorems 6.4 and 6.13, the composite $\mathbb{Z}^{\mathrm{Good}(d,n)}\to A^{n-d}_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism. □

We now find an explicit basis for R_k for $k>n-2$ of size 2^{n-1}.

Lemma 6.15. *For each partition $P \in \text{Part}(d,n)$ for $d \leq 2$, arbitrarily choose i_P, j_P that lie in the same part. Then for $k > n-2$, R_k is generated by the 2^{n-1} elements*

$$S_k := \left\{\Delta_{\{[n]\}}\Delta^{k-n+1}_{i_{\{[n]\}},j_{\{[n]\}}}\right\} \cup \left\{\Delta_P \Delta^{k-n+2}_{i_P,j_P} \mid P \in \text{Part}(2,n)\right\}.$$

Proof. Let $P = \{A, B\} \in \text{Part}(2,n)$. By Theorem 6.7, it suffices to show $\Delta_P \prod_{a=1}^{k-n+2} \Delta_{i_a,j_a}$ is generated by S_k for any choices of $i_a \neq j_a$. We proceed by induction on $k > n-2$. For the base case $k = n-1$, it suffices to show $\Delta_{i,j}\Delta_P$ is generated by S_k for any $i \neq j$. If $k > n-1$, then by the induction hypothesis, it suffices to show $\Delta_{i,j}\Delta_P\Delta^{k-n+1}_{i_P,j_P}$ and $\Delta_{i,j}\Delta_{\{[n]\}}\Delta^{k-n}_{i_{\{[n]\}},j_{\{[n]\}}}$ are generated by S_k. Both the base case and the induction step will work in the same way.

First, $\Delta_{i,j}\Delta_{\{[n]\}}\Delta^{k-n}_{i_{\{[n]\}},j_{\{[n]\}}} = \Delta_{\{[n]\}}\Delta^{k-n+1}_{i_{\{[n]\}},j_{\{[n]\}}}$ by Theorem 6.6 (2). To deal with $\Delta_{i,j}\Delta_P\Delta^{k-n+1}_{i_P,j_P}$, we have two cases.

(1) If $\{i, j\}$ is not contained in A or B, then $\Delta_{i,j}\Delta_P$ is the diagonal $\Delta_{\{[n]\}}$ by Theorem 6.6 (1). Then, by Theorem 6.6 (2), $\Delta_{i,j}\Delta_P\Delta^{k-n+1}_{i_P,j_P} = \Delta^{k-n+1}_{i_{\{[n]\}},j_{\{[n]\}}}\Delta_{\{[n]\}}$;
(2) Suppose now each $\{i, j\}$ is in A or B, and that without loss of generality, $i_P, j_P \in A$. If $i, j \in A$, then using Theorem 6.6 (2) we may replace $\Delta_{i,j}$ with Δ_{i_P,j_P}. If $i, j \in B$, we can use a square relation to replace it with $\Delta_{i,i_P} - \Delta_{i_P,j_P} + \Delta_{j_P,i}$. We then have $\Delta_{i,i_P}\Delta_P = \Delta_{\{[n]\}} = \Delta_{j_P,i}\Delta_P$, so

$$\Delta_{i,j}\Delta_P\Delta^{k-n+1}_{i_P,j_P} = 2\Delta^{k-n+1}_{i_{\{[n]\}},j_{\{[n]\}}}\Delta_{\{[n]\}} - \Delta_P\Delta^{k-n+2}_{i_P,j_P}$$

by Theorem 6.6 (2). □

Theorem 6.16. *For $n \geq 3$, the natural surjection $R \twoheadrightarrow A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism. Furthermore, R_k has $\mathbb{Z}$-basis given by*

(1) $\{\Delta_P \mid P \in \text{Good}(n-k,n)\}$ *for $k \leq n-2$;*
(2) $S_k = \{\Delta_{\{[n]\}}\Delta^{k-n+1}_{i_{\{[n]\}},j_{\{[n]\}}}\} \cup \{\Delta_P\Delta^{k-n+2}_{i_P,j_P} \mid P \in \text{Part}(2,n)\}$, *where for each partition $P \in \text{Part}(d,n)$ for $d \leq 2$, arbitrarily choose i_P, j_P that lie in the same part.*

Proof. If $k \leq n-2$, we have $R_k \twoheadrightarrow A^k_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism with $\mathbb{Z}$-basis given by $\{\Delta_P \mid P \in \text{Good}(n-k,n)\}$ by Theorem 6.14. Now, we consider the case $k > n-2$.

The S_k span R_k by Theorem 6.15, so applying (6.1) yields

$$\mathbb{Z}^{S_k} \twoheadrightarrow R_k \twoheadrightarrow A^k_{PGL_2}((\mathbb{P}^1)^n),$$

whose composite is a surjection of free $\mathbb{Z}$-modules of the same rank 2^{n-1} by Theorems 6.4 and 6.15, so it is an isomorphism. This proves $R_k \twoheadrightarrow A^k_{PGL_2}((\mathbb{P}^1)^n)$ is an isomorphism and identifies S_k as a basis. □

6.1. Algorithm and example

We can describe an algorithm for writing arbitrary classes in $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$ in terms of our $\mathbb{Z}$-basis. The key fact is that if $\mathrm{pr}^n : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^{n-1}$ is projection by forgetting the last factor, then by definition of the pushforward of a cycle

$$\mathrm{pr}^n_* \Delta_P = \begin{cases} \Delta_{P\setminus n} & \text{if } n \text{ is not isolated} \\ 0 & \text{if } n \text{ is isolated.} \end{cases}$$

At the level of formulae, if we write our class as a polynomial in the H_i, u, v with each H_i appearing to degree at most 1, then pr^n_* extracts the H_n-coefficient. Also, if we have a Δ_P and we know that either n is isolated or $n-1, n$ are in the same part, then as $[\Delta_{n-1,n}](H_n - H_{n-1}) = 0$ we also have

$$\mathrm{pr}^n_*([\Delta_P](H_n - H_{n-1})) = \begin{cases} 0 & \text{if } n-1, n \text{ are in the same group} \\ [\Delta_{P\setminus n}] & \text{if } n \text{ is isolated.} \end{cases}$$

Suppose we have a class

$$\alpha = \sum_{P\in \mathrm{Good}(d,n)} a_P[\Delta_P] = \sum_{\substack{P\in \mathrm{Good}(d,n)\\ n \text{ isolated}}} a_P[\Delta_P] + \sum_{\substack{P\in \mathrm{Good}(d,n)\\ n-1,n \text{ together}}} a_P[\Delta_P]$$

and we want to find the coefficients a_P.

We first show how to reduce down to the case $d = 2$. By the above, we have

$$\mathrm{pr}^n_* \alpha = \sum_{\substack{P\in \mathrm{Good}(d,n)\\ n-1,n \text{ together}}} a_P[\Delta_{P\setminus n}], \quad \mathrm{pr}^n_*(\alpha(H_n - H_{n-1})) = \sum_{\substack{P\in \mathrm{Good}(d,n)\\ n \text{ isolated}}} a_P[\Delta_{P\setminus n}].$$

In the first case each $P \setminus n \in \mathrm{Good}(d, n-1)$, and in the second case each $P \setminus n \in \mathrm{Good}(d-1, n-1)$ so we can apply induction to determine all of these coefficients.

Once we have reduced down to the case $d = 2$, we can now identify each a_P separately for $P = \{A, B\}$ a partition of $[n]$ into two parts by evaluating at $H_i = -u$ for $i \in A$ and $H_i = -v$ for $i \in B$ (which is localization at a torus-fixed point). By Theorem 5.3, this evaluates to $a_{\{A,B\}}(u-v)^{n-2}(-1)^{|A|-1}$.

The same method for $d = 2$ works for elements $\alpha \in A^k((\mathbb{P}^1)^n)$ with $k > n-2$. Applying the same substitution to

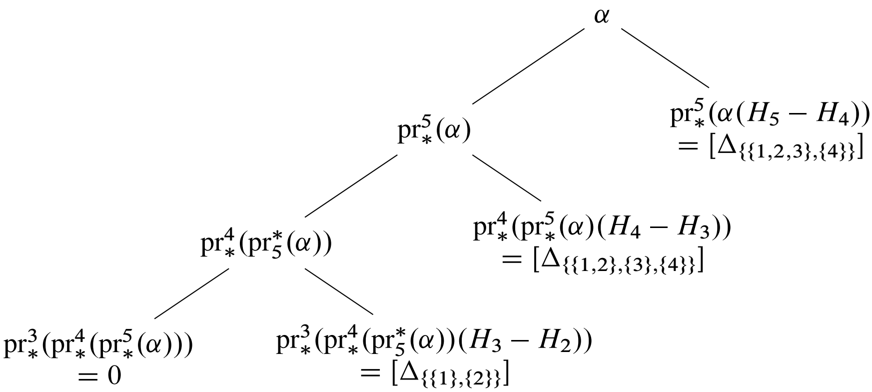
$$\alpha = \sum a_P[\Delta_{i_P,j_P}]^{k-n+2}[\Delta_P] + a_{\{[n]\}}[\Delta_{i_{\{[n]\}},j_{\{[n]\}}}]^{n-k+1}[\Delta_{\{[n]\}}]$$

extracts the a_P-coefficient for $P = \{A, B\}$ a partition of $[n]$ into two parts as this is the only term that does not vanish under this substitution. Then, we subtract off all of these terms to recover $a_{[n]}$.

Example 6.17. As a simple example, consider the PGL_2-orbit closure of a generic point in $(\mathbb{P}^1)^5$. The formula computed in [23, Corollary 4.8] shows that the class of this orbit is

$$\alpha = e_2(H_1, H_2, H_3, H_4, H_5) + 2(u+v)(H_1 + H_2 + H_3 + H_4 + H_5) + (3u^2 + 4uv + 3v^2),$$

where e_2 is the second elementary symmetric polynomial. We have



$$\begin{aligned}
\mathrm{pr}^5_*\,\alpha &= (H_1 + H_2 + H_3 + H_4) + 2(u+v)\\
\mathrm{pr}^5_*(\alpha(H_5 - H_4)) &= e_2(H_1, H_2, H_3) + (u+v)(H_1 + H_2 + H_3)\\
&\quad + (u^2 + uv + v^2)\\
\mathrm{pr}^4_*(\mathrm{pr}^5_*\,\alpha) &= 1\\
\mathrm{pr}^4_*(\mathrm{pr}^5_*\,\alpha(H_4 - H_3)) &= H_1 + H_2 + u + v\\
\mathrm{pr}^3_*(\mathrm{pr}^4_*(\mathrm{pr}^5_*\,\alpha)) &= 0\\
\mathrm{pr}^3_*(\mathrm{pr}^4_*(\mathrm{pr}^5_*\,\alpha)(H_3 - H_2)) &= 1.
\end{aligned}$$

The only non-trivial identification was $\mathrm{pr}^5_*(\alpha(H_5 - H_4)) = \Delta_{\{\{1,2,3\},\{4\}\}}$, which we can identify as follows. Substitute $-u$'s and $-v$'s for the H_i corresponding to all nontrivial partitions $\{A, B\}$ of [4] into two parts. We find the only choice that gives a nonzero result is $A = \{1,2,3\}$, $B = \{4\}$, yielding $(u-v)^2$, which is the same as for $\Delta_{\{\{1,2,3\},\{4\}\}}$ by Theorem 5.3. Putting this together yields

$$\alpha = \Delta_{\{\{1\},\{2\},\{3,4,5\}\}} + \Delta_{\{\{1,2\},\{3\},\{4,5\}\}} + \Delta_{\{\{1,2,3\},\{4\},\{5\}\}}.$$

We remark that the PGL_2-orbit closure $X_n \subset (\mathbb{P}^1)^n$ of a general point in $(\mathbb{P}^1)^n$ decomposes into good incidence strata as

$$[X_n] = \sum_{a=1}^{n-2} \Delta_{\{\{1,\ldots,a\},\{a+1\},\{a+2,\ldots,n\}\}} \tag{6.2}$$

which can be geometrically explained as follows. Consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1,1) & \xrightarrow{\text{ev}} & (\mathbb{P}^1)^n \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{0,n} & & \end{array}$$

(see Section 1.1 for notation). The left and right-hand side of (6.2) can both be described as $\text{ev}_*\pi^*(\text{pt})$ for $\text{pt} \in \overline{\mathcal{M}}_{0,n}$ being a general point and the point corresponding to a chain of $n-2$ rational curves (respectively), and the result follows from the flatness of π. See [23, Section 4] for a generalization of this degeneration to PGL_{r+1} orbits closures of general points in $(\mathbb{P}^r)^n$.

7. GL_2-equivariant classes of strata in $\mathrm{Sym}^n\mathbb{P}^1$

Recall from Theorem 2.5 that $[\lambda] \in A^\bullet_{GL_2}(\mathbb{P}^n)$ for λ a partition of n is the push-forward of $[\Delta_P] \in A^\bullet_{GL_2}((\mathbb{P}^1)^n)$ under the multiplication map $(\mathbb{P}^1)^n \to \mathbb{P}^n$ for P a partition of $[n]$ into subsets with cardinalities given by λ. Up to a constant factor given in Theorem 2.5, this is the class of the closure Z_λ given in Theorem 2.3 of degree n forms on $(\mathbb{P}^1)^\vee$ whose roots have multiplicities given by λ as studied by Fehér, Némethi, and Rimányi [12].

Definition 7.1. Denote by $[a, b, 1^c] := [\{a, b, 1, 1, \ldots, 1\}]$ where there are c 1's.

From writing the expressions for $[\lambda]$ in Theorem 5.5 using generating functions, we find the following new Corollary.

Corollary 7.2. *For $d \geq 2$, consider the polynomial*

$$-\frac{1}{(z-1)^{d-2}}\prod_{i=1}^{d}(z^{a_i}-1) = \sum_{\substack{0\leq k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}(z^{k_1}+z^{k_2}). \tag{7.1}$$

Then $\alpha_i \in \mathbb{Z}$ and

$$[a_1, \ldots, a_d] = \sum_{\substack{1\leq k_1\leq k_2\\ k_1+k_2=n-d+2}} \alpha_{k_1}[k_1, k_2, 1^{d-2}]$$

Proof. We first note that the left side of (7.1) is a polynomial because every term in the product is divisible by $z-1$. In particular, both sides of (7.1) is a polynomial divisible by $(z-1)^2$.

To show integrality of the α_i, it is clear that all $\alpha_i \in \mathbb{Z}$ except possibly $\alpha_{\frac{n-d+2}{2}}$, which a priori only lies in $\mathbb{Z}[\frac{1}{2}]$. But plugging in $z = 1$ to both sides shows the integrality.

By Theorem 5.5, it suffices to show

$$\prod_{i=1}^{d}(z^{a_i}-1)=\sum_{\substack{1\le k_1\le k_2\\ k_1+k_2=n-d+2}}\alpha_{k_1}(z^{k_1}-1)(z^{k_2}-1)(z-1)^{d-2}.$$

or equivalently

$$\frac{1}{(z-1)^{d-2}}\prod_{i=1}^{d}(z^{a_i}-1)=\sum_{\substack{k_1\le k_2\\ k_1+k_2=n-d+2}}\alpha_{k_1}(z^{k_1}-1)(z^{k_2}-1).$$

By definition of α_k, the coefficients of both sides agree except possibly the z^0 and $z^{n-(d-2)}$-coefficient. Also, the coefficients of z^0 and $z^{n-(d-2)}$ are equal to each other on the left-hand side, and the same is true on the right side. To see they agree between the left and right sides, we note both sides are 0 after substituting $z=1$. □

Lemma 7.3. *The rational T-equivariant classes in $\mathbb{P}^n$ of the torus fixed points*

$$\prod_{j\in([n]\cup\{0\})\setminus\{k\}}(H+jv+(n-j)u)\in A_T^{\bullet}(\mathbb{P}^n)\otimes\mathbb{Q}$$

are linearly independent as k ranges in $\{0,\dots,n\}$.

Proof. For fixed k, $H\mapsto -ku-(n-k)v$ maps $\prod_{j\in[n]\setminus\{k'\}}(H+jv+(n-j)u)$ to 0 if and only if $k'\neq k$ □

Theorem 7.4. *For fixed $c\ge 0$, the classes $[a,b,1^c]$ with $a+b=n-c$ and $a\ge b$ form a $\mathbb{Q}$-basis for $A_{PGL_2}^{n-c-2}(\mathbb{P}^n)\otimes\mathbb{Q}\subset A_{GL_2}^{n-c-2}(\mathbb{P}^n)\otimes\mathbb{Q}$.*

Proof. To show the linear independence, by Theorem 7.3 and Theorem 5.5 it suffices to show for fixed c that the polynomials $(z^a-1)(z^b-1)(z-1)^c$ with $a\ge b$ and $a+b=n-c$ are linearly independent. Indeed, dividing out by $(z-1)^c$, we note that the polynomials $(z^a-1)(z^b-1)$ for $a+b=n-c$ and $a\ge b\ge 1$ are linearly independent. This can be seen for example by noting that the monomial z^i appears only in $(z^a-1)(z^b-1)$ for $b=i$ and $a=n-c-i$, as for all integral i between 1 and $\frac{n-c}{2}$ inclusive.

To see that the $\mathbb{Q}$-linear span of the classes $[a,b,1^c]$ is precisely $A_{PGL_2}^{n-c-2}(\mathbb{P}^n)\otimes\mathbb{Q}$, we note that we have just shown that the dimension of the $\mathbb{Q}$-linear span of the $[a,b,1^c]$ is precisely $\lfloor\frac{n-c}{2}\rfloor$ by linear independence, which we can check is the same as the dimension of $A_{PGL_2}^{n-c-2}(\mathbb{P}^n)\otimes\mathbb{Q}$ by Theorem 4.8. □

8. Integral classes of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$

In this section, we compute the integral classes of $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n)$. By Theorem 4.8, if n is odd, then $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ is injective and we know the image of the $[Z_\lambda]$ in $A^\bullet_{GL_2}(\mathbb{P}^n)$ by Theorem 5.5, so it suffices to consider the case n is even, which we assume for the remainder of this section.

Recall the polynomials $p_n(t) \in A^\bullet_{PGL_2}(\mathrm{pt})[t]$ defined in Theorem 4.8 for even n and let q_n be the image of p_n in $A^\bullet_{PGL_2}(\mathrm{pt})/(2)[t] \cong \mathbb{F}_2[c_2, c_3, t]$. It is easy to see by the binomial theorem or directly from [15, Lemma 6.1] that

$$q_n(t) = \begin{cases} t^{(n+4)/4}(t^3 + c_2 t + c_3)^{n/4} & \text{if } n \equiv 0 \bmod 4 \\ t^{(n-2)/4}(t^3 + c_2 t + c_3)^{(n+2)/4} & \text{if } n \equiv 2 \bmod 4, \end{cases}$$

and $q_n(t) \mid q_{n+k}(t)$ for $k = 0$ or $k \geq 4$ for any even n.

By Theorem 4.8, for n even,

$$A^\bullet_{PGL_2}(\mathbb{P}^n) \cong \mathbb{Z}[c_2, c_3, H]/(2c_3, p_n(H)),$$

which is isomorphic to

$$\left(\bigoplus_{i=0}^{n} \mathbb{Z}[c_2]H^i\right) \oplus \left(\bigoplus_{i=0}^{n} c_3\mathbb{F}_2[c_2, c_3]H^i\right) \tag{8.1}$$

as Abelian groups. So to determine the class $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n)$, it suffices to find its image in $\bigoplus_{i=0}^{n} \mathbb{Z}[c_2]H^i$ and $\bigoplus_{i=0}^{n} c_3\mathbb{F}_2[c_2, c_3]H^i$. Equivalently, if we write the class of $[Z_\lambda]$ as a polynomial in c_2, c_3, and H with degree at most n in H, then it suffices to consider the terms not containing c_3 and the terms containing c_3 separately. Under the map $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$, Theorem 4.8 shows that the first factor maps injectively and the second factor maps to zero.

We can determine the image of $[Z_\lambda]$ in the first factor using Theorem 5.5, so it suffices to determine the image of $[Z_\lambda]$ in the second factor to identify its class. To do this, we will work modulo 2 and determine $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$. Discarding those monomials not containing c_3 then yields the image of $[Z_\lambda]$ in the second factor.

Definition 8.1. We say a partition $\lambda = a_1^{e_1} \ldots a_k^{e_k}$ of n into $d = \sum_{i=1}^{k} e_i$ parts is *special* if all a_i and $\frac{d!}{e_1! \cdots e_k!}$ are odd, and all e_i are even.

Theorem 8.2. *Let d and n be integers with n even. The class of $[Z_\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ for λ a partition of n into d parts is given by*

$$\begin{cases} \frac{q_n}{q_d}(H) & \text{if } \lambda \text{ is special} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 8.3. *Given a ring $R[H]/(P(H))$ for P a monic polynomial of degree $n+1$, define the R-linear map $\int : R[H]/(P(H)) \to R$ given by taking a polynomial $f(H)$, and outputting the H^n-coefficient of the reduction $\widetilde{f}(H)$ of $f(H)$ $\pmod{P(H)}$ to a polynomial of degree at most n with respect to H. Then letting t be an indeterminate, we have*

$$\int \frac{P(H)-P(t)}{H-t} f(H) = \widetilde{f}(t).$$

Proof. We have

$$\begin{aligned}
&\int \frac{P(H)-P(t)}{H-t} f(H) \\
&= \int \frac{P(H)-P(t)}{H-t} \widetilde{f}(H) \\
&= \int \frac{P(H)\widetilde{f}(H) - P(H)\widetilde{f}(t) - P(t)\widetilde{f}(H) + P(t)\widetilde{f}(t) + P(H)\widetilde{f}(t) - P(t)\widetilde{f}(t)}{H-t},
\end{aligned}$$

which by linearity is

$$\begin{aligned}
&\int P(H)\frac{\widetilde{f}(H)-\widetilde{f}(t)}{H-t} - \int P(t)\frac{\widetilde{f}(H)-\widetilde{f}(t)}{H-t} + \int \frac{P(H)-P(t)}{H-t}\widetilde{f}(t) \\
&= 0 + 0 + \widetilde{f}(t) = \widetilde{f}(t).
\end{aligned}$$

The first term is zero because the integrand is a multiple of $P(H)$, the second term is zero because $\frac{\widetilde{f}(H)-\widetilde{f}(t)}{H-t}$ is a polynomial of degree at most $n-1$, and the last term is $\widetilde{f}(t)$ because $\frac{P(H)-P(t)}{H-t}$ is monic of degree n. □

Remark 8.4. Let G be a linear algebraic group and V be a representation. Then,

$$\begin{aligned}
A_G^\bullet(\mathbb{P}(V)) &\cong A_G^\bullet(\mathrm{pt})[H]/(P(H)) \\
A_G^\bullet(\mathbb{P}(V)\times\mathbb{P}(V)) &\cong A_G^\bullet(\mathrm{pt})[H_1,H_2]/(P(H_1),P(H_2)),
\end{aligned}$$

where $P \in A_G^\bullet[T]$ is $T^{\dim(V)} + c_1^G(V)T^{\dim(V)-1} + \cdots + c_{\dim(V)}^G(V)$ by the projective bundle theorem and the class of the diagonal in $\mathbb{P}(V)\times\mathbb{P}(V)$ is $(P(H_1) - P(H_2))/(H_1 - H_2)$, giving a geometric interpretation of Theorem 8.3. This can be proven, for example, by first noting that it suffices to consider the case $G = GL(V)$. Then, we can restrict to a maximal torus [8, Proposition 6] and use the fact that the diagonal in $\mathbb{P}(V)\times\mathbb{P}(V)$ admits a torus-equivariant deformation into a union of products of coordinate linear spaces [3, Theorem 3.1.2].

Proof of Theorem 8.2. Note that when all a_i are odd and all e_i are even then $n = \sum a_i e_i$ is either equal to $\sum e_i$, or exceeds it by at least 4, so $q_{e_1+\ldots+e_k} \mid q_n$ and the claimed expression for $[Z_\lambda]$ is well-defined.

We resolve Z_λ birationally with the map

$$\Psi : \prod_{i=1}^{k} \mathbb{P}^{e_i} \to \mathbb{P}^n$$

taking $(D_1, \dots, D_k) \mapsto a_1 D_1 + \dots + a_k D_k$ (treating $P^r = \mathrm{Sym}^r \mathbb{P}^1$ for all r).

If at least one e_i is odd, then we claim $c_3[Z_\lambda] = 0$. Indeed,

$$c_3[Z_\lambda] = \Psi_* c_3,$$

and $c_3 \in A^\bullet_{PGL_2}(\mathrm{pt})$ maps to 0 in $A^\bullet_{PGL_2}(\prod_{i=1}^k \mathbb{P}^{e_i})$ as the projection $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathrm{pt}$ can be factored as the composite $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathbb{P}^{e_i} \to \mathrm{pt}$, and if e_i is odd then c_3 pulls back to zero in $A^\bullet_{PGL_2}(\mathbb{P}^{e_i})$ by Theorem 4.8.

Hence, as $c_3[Z_\lambda] = 0$, we must have $[Z_\lambda]$ is zero in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$.

Now, suppose that all e_i are even. This means each $\mathbb{P}^{e_i}$ is the projectivization of a PGL_2-representation with Chern classes given as the coefficients of $p_{e_i}(t)$, so we have the Chow ring

$$A^\bullet_{PGL_2}(\prod_{i=1}^{k} \mathbb{P}^{e_i}) \cong A^\bullet_{PGL_2}(\mathrm{pt})[H_1, \dots, H_k]/(p_{e_1}(H_1), \dots, p_{e_k}(H_k))$$

by repeatedly applying the projective bundle formula.

For the remainder of the proof all integrals are in Chow rings after tensoring with $\mathbb{Z}/2\mathbb{Z}$, so each $p_r(t)$ gets replaced with $q_r(t)$. By Theorem 8.3, it suffices to show

$$\int_{\mathbb{P}^n} \frac{q_n(t) - q_n(H)}{t - H} \cap \Psi_* 1 = \begin{cases} \frac{q_n}{q_d}(t) & \text{if all } a_i \text{ and } \frac{d!}{e_1! \dots e_k!} \text{ are odd} \\ 0 & \text{otherwise.} \end{cases}$$

By the projection formula applied to Ψ, we have

$$\int_{\mathbb{P}^n} \frac{q_n(t) - q_n(H)}{t - H} \cap \Psi_* 1 = \int_{\prod_{i=1}^k \mathbb{P}^{e_i}} \frac{q_n(t) - q_n(\sum a_i H_i)}{t - \sum a_i H_i}.$$

Now, if any a_i is even, then as we are working modulo 2, $\frac{q_n(t) - q_n(\sum a_i H_i)}{t - \sum a_i H_i}$ will not contain H_i, so the integral is clearly zero. Hence we may assume from now on that all a_i are odd, so that $\sum a_i H_i = \sum H_i \bmod 2$.

We claim that $q_d(\sum H_i) = 0$ and that

$$\int_{\prod_{i=1}^k \mathbb{P}^{e_i}} \frac{q_d(t) - q_d(\sum H_i)}{t - \sum H_i} = \frac{d!}{e_1! \cdots e_k!}.$$

The first of these follows from pulling back $q_d(H)$ under the multiplication map $\prod_{i=1}^k \mathbb{P}^{e_i} \to \mathbb{P}^d$, and the second of these follows from applying Theorem 8.3 to

$1 \in A^\bullet_{PGL_2}(\mathbb{P}^d)$ together with the projection formula as the multiplication map has degree $\frac{d!}{e_1!\cdots e_k!}$.

From the vanishing of $q_d(\sum H_i)$, we have

$$\begin{aligned}\frac{q_n(t)-q_n(\sum H_i)}{t-\sum H_i} &= \frac{q_n}{q_d}(t)\frac{q_d(t)-q_d(\sum H_i)}{t-\sum H_i} + q_d\left(\sum H_i\right)\frac{\frac{q_n(t)}{q_d(t)}-\frac{q_n\left(\sum H_i\right)}{q_d\left(\sum H_i\right)}}{t-\sum H_i}\\ &= \frac{q_n}{q_d}(t)\frac{q_d(t)-q_d\left(\sum H_i\right)}{t-\sum H_i},\end{aligned}$$

and the result now follows from the second claim after applying $\int_{\prod_{i=1}^k \mathbb{P}^{e_i}}$ to both sides. □

We now prove surprisingly that despite the presence of occasional 2-torsion, integral relations between $[Z_\lambda]$ classes in $A^\bullet_{GL_2}(\mathbb{P}^n)$ are equivalent to integral relations between $[Z_\lambda]$-classes in $A^\bullet_{PGL_2}(\mathbb{P}^n)$.

Theorem 8.5. *Let n, d be integers. A linear combination $\sum a_\lambda[Z_\lambda]$ with $a_\lambda \in \mathbb{Z}$ and each λ a partition of n into d parts is zero in $A^\bullet_{PGL_2}(\mathbb{P}^n)$ if and only if it is zero in $A^\bullet_{GL_2}(\mathbb{P}^n)$. In particular, $\sum a_\lambda[Z_\lambda] = 0$ if and only if*

$$\sum_{\lambda=a_1^{e_1}\ldots a_n^{e_k}} a_\lambda \prod_{i=1}^{k} \frac{(z^{a_i}-1)^{e_i}}{e_i!} = 0.$$

Proof. One direction is trivial, as we have the map $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ induced by $GL_2 \to PGL_2$, so if a linear relation holds in $A^\bullet_{PGL_2}(\mathbb{P}^n)$, then it also holds in $A^\bullet_{GL_2}(\mathbb{P}^n)$. Conversely, suppose that we have $\sum a_\lambda[Z_\lambda] = 0$ in $A^\bullet_{GL_2}(\mathbb{P}^n)$. We only have to care about the case that n is even, because when n is odd, $A^\bullet_{PGL_2}(\mathbb{P}^n) \hookrightarrow A^\bullet_{GL_2}(\mathbb{P}^n)$ is an injection by Theorem 4.8.

For n even, suppose we have a sum $\sum a_\lambda[Z_\lambda]$, which is 0 in $A^\bullet_{GL_2}(\mathbb{P}^n)$. Then since the kernel of $A^\bullet_{PGL_2}(\mathbb{P}^n) \to A^\bullet_{GL_2}(\mathbb{P}^n)$ is 2-torsion by Theorem 4.8, we know $\sum a_\lambda[Z_\lambda]$ is 2-torsion in $A^\bullet_{PGL_2}(\mathbb{P}^n)$. By Theorem 8.2, the class $[Z_\lambda]$ in $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Z}/2\mathbb{Z}$ is either 0 or $\frac{q_n}{q_d}(H)$, and the second possibility occurs precisely when λ is special. Hence to prove Theorem 8.5, by Theorem 5.5 and Theorem 7.3 it suffices to show that if

$$\sum_{\lambda=a_1^{e_1}\ldots a_n^{e_k}} a_\lambda \prod_{i=1}^{k} \frac{(z^{a_i}-1)^{e_i}}{e_i!} = 0, \tag{8.2}$$

then

$$\sum_{\lambda \text{ special}} a_\lambda \equiv 0 \pmod 2.$$

Note first that if no special λ appears we are done, so we may assume that at least one special λ appears. As $d = \sum_{i=1}^{k} e_i$ for any partition $\lambda = a_1^{e_1} \dots a_n^{e_k}$ appearing, we must have d is even if a special λ appears. Multiplying (8.2) by $\frac{d!}{(z-1)^d}$ and plugging in $z = 1$, we have

$$\sum_{\lambda=a_1^{e_1}\dots a_n^{e_n}} a_\lambda \frac{d!}{e_1!\cdots e_k!} \prod_{i=1}^{k} a_i{}^{e_i} = 0.$$

Now we claim that $\frac{d!}{e_1!\cdots e_k!}$ is even if any e_i is odd. Indeed, as d is even, if not all e_i are even, then at least two of the e_i are odd. If e_i, e_j are both odd, then replacing $e_i!e_j!$ in $\frac{d!}{e_1!\cdots e_k!}$ with $(e_i - 1)!(e_j + 1)!$ yields an integer with a smaller power of 2 dividing it.

Hence, $\frac{d!}{e_1!\cdots e_k!} \prod_{i=1}^{k} a_i{}^{e_i}$ is odd precisely when λ is special. Taking the equality $\pmod 2$ then yields the desired result. □

We complete the proof of Theorem 3.8.

Proof of Theorem 3.8. We have (1), (2) and (4) are equivalent by Theorem 8.5. Also (3) implies (2) is clear as $A^\bullet_{GL_2}(\mathbb{P}^n)$ is free as an Abelian group, so $A^\bullet_{GL_2}(\mathbb{P}^n) \hookrightarrow A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$.

To finish, it suffices to show (2) implies (3). Let $\lambda = (\lambda_1, \dots, \lambda_d)$ for $\lambda_1 \geq \dots \geq \lambda_d$.

Claim 8.6. Suppose $\lambda_3 > 1$. Then using pushforwards of square relations in $A^\bullet_{PGL_2}((\mathbb{P}^1)^n)$, we can express $[\lambda] \in A^\bullet_{PGL_2}(\mathbb{P}^n)$ in terms of classes $[\lambda']$ where $\lambda' = (\lambda'_1, \dots, \lambda'_d)$ where $\lambda'_1 + \lambda'_2 > \lambda_1 + \lambda_2$.

Proof of Claim. Pick a partition $P = \{A_1, \dots, A_d\}$ of $[n]$ with $|A_i| = \lambda_i$. Since $|A_3| > 1$, we can partition it as $A_3 = A'_3 \sqcup A''_3$ into nonempty parts. Now, applying the square relation associated to $P' = \{A_1, A_2, A'_3, A''_3, \dots, A_d\}$ of $[n]$ into $d + 1$ parts and the parts A_1, A_2, A_3, A''_3 shows

$$[\lambda] = [\lambda_1] + [\lambda_2] - [\lambda_3],$$

where $\lambda'_3 = |A'_3|$ and $\lambda''_3 = |A''_3|$ and

$$\begin{aligned}\lambda_1 &= \{\lambda_1 + \lambda'_3, \lambda_2, \lambda''_3, \dots, \lambda_d\}\\ \lambda_2 &= \{\lambda_1, \lambda_2 + \lambda''_3, \lambda'_3, \dots, \lambda_d\}\\ \lambda_3 &= \{\lambda_1 + \lambda_2, \lambda'_3, \lambda''_3, \dots, \lambda_d\}.\end{aligned}$$ □

Returning to the proof of Theorem 3.8, iterating the claim shows that the pushforward of square relations allow us to rewrite any $[\lambda]$ in terms of the $\mathbb{Q}$-basis found in Theorem 7.4, which shows (2) implies (3). □

9. Excision of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$

As an application of our results in the ordered case, we will prove the following result on the PGL_2-equivariant Chow ring of $\mathbb{P}^n$ with strata excised, which we will adapt in the next section to the case of GL_2-equivariant Chow rings with strata in both $\mathbb{P}^n$ and in $\mathbb{A}^{n+1}$.

Theorem 9.1. *Given a partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of n,*

$$A^\bullet_{PGL_2}(\mathbb{P}^n \setminus Z_\lambda) = A^\bullet_{PGL_2}(\mathbb{P}^n)/I,$$

where the ideal $I \otimes \mathbb{Q} \subset A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ is generated by all $[\lambda']$ for λ' a partition formed by merging some of the parts of λ.

Even though Theorem 9.1 requires many generators for I, in some cases fewer generators suffice.

Theorem 9.2. *Given the partition $\lambda = \{a, 1^{n-a}\}$ of n, the ideal $I \otimes \mathbb{Q}$ in Theorem* 9.1 *is generated by* $[\lambda]$ *and* $[\lambda']$, *where*

$$\lambda' = \begin{cases} \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\ \{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}. \end{cases}$$

See Theorem 10.3 for the connection to similar results proved in [12].

By the excision exact sequence [17, Proposition 1.8], the ideal I is the same as the pushforward ideal I_λ which we define in Theorem 9.3.

Definition 9.3. Given a partition λ of n and for $G = PGL_2$ or GL_2, let I^G_λ be the ideal of $A^\bullet_G(\mathbb{P}^n)$ given by the pushforward via the inclusion $\iota_\lambda : Z_\lambda \hookrightarrow \mathbb{P}^n$

$$I^G_\lambda = (\iota_\lambda)_* A^G_\bullet(Z_\lambda) \subset A^G_\bullet(\mathbb{P}^n)$$

and the identification $A^G_\bullet(\mathbb{P}^n) \cong A^{n-\bullet}_G(\mathbb{P}^n)$ via Poincaré duality [8, Proposition 4]. When G is clear from context we will simply write I_λ.

Since Z_λ is possibly singular, we will want to instead work with a desingularization (as was done in [12]).

Definition 9.4. Given a partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of n, let $e^\lambda_i = \#\{j \mid \lambda_j = i\}$ and $Y_\lambda = \prod_{i=1}^n \mathbb{P}^{e^\lambda_i}$. We have a map

$$\hat{\iota}_\lambda : Y_\lambda \to \mathbb{P}^n$$

that is birational onto its image Z_λ given by the composition

$$Y_\lambda \hookrightarrow \prod_{i=1}^n \mathbb{P}^{ie^\lambda_i} \to \mathbb{P}^n$$

of the ith power map on each factor $\mathbb{P}^{e_i}$ together with the multiplication map. Equivalently, if we view projective space $\mathbb{P}^n$ as parameterizing degree n divisors on $\mathbb{P}^1$, then the map is given by $(D_1, \dots, D_n) \mapsto \sum_{i=1}^n iD_i$.

Since $\hat{\iota}_\lambda$ is birational onto its image, I_λ is also given by the image of $(\hat{\iota}_\lambda)_*$. Since we will work rationally, we can take a finite cover of Y_λ.

Definition 9.5. Given a partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of n, define the finite map $\Phi_\lambda : (\mathbb{P}^1)^d \to Y_\lambda$ to be

$$\Phi_\lambda : (\mathbb{P}^1)^d = \prod_{i=1}^{n} (\mathbb{P}^1)^{e_i^\lambda} \to \prod_{i=1}^{n} \mathbb{P}^{e_i^\lambda} = Y_\lambda$$

given by the multiplication map $(\mathbb{P}^1)^{e_i^\lambda} \to \mathbb{P}^{e_i^\lambda}$ on each factor.

Since Φ_λ is finite,

$$(\Phi_\lambda)_* : A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q} \to A^\bullet_{PGL_2}(Y_\lambda) \otimes \mathbb{Q}$$

is surjective, so $I_\lambda \otimes \mathbb{Q}$ is the image of

$$(\hat{\iota}_\lambda \circ \Phi_\lambda)_* : A^\bullet_{PGL_2}((\mathbb{P}^1)^d) \otimes \mathbb{Q} \to A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}.$$

The map Φ_λ has the nice property that given a partition P of $[d]$, the pushforward of the strata $(\hat{\iota}_\lambda \circ \Phi_\lambda)_*[\Delta_P]$ is $[\lambda']$, where λ' is the partition of n given by merging the parts of λ according to the partition P. From this, we will be able to deduce certain symmetrized strata generate $I_\lambda \otimes \mathbb{Q}$ based on the generation properties of strata in $(\mathbb{P}^1)^d$.

Definition 9.6. Given a set of partitions $\mathcal{P}$ of $[d]$ and $G = PGL_2$ or GL_2, let $\Lambda^G_{\mathcal{P}} \subset A^\bullet_G((\mathbb{P}^1)^d) \otimes \mathbb{Q}$ be the submodule over $A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q}$ generated by the classes $[\Delta_P]$. Explicitly,

$$\Lambda^G_{\mathcal{P}} = \sum_{P \in \mathcal{P}} [\Delta_P] \Phi^*_\lambda \hat{\iota}^*_\lambda (A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q}).$$

When G is clear from context we will notate $\Lambda^G_{\mathcal{P}}$ simply by $\Lambda_{\mathcal{P}}$.

Lemma 9.7. *Let $\lambda = \{\lambda_1, \dots, \lambda_d\}$ be a partition of n, and let $G = PGL_2$ or GL_2. Suppose we have a collection of partitions $\mathcal{P}$ of $[d]$ such that in $A^\bullet_G((\mathbb{P}^1)^d) \otimes \mathbb{Q}$*

$$A^\bullet_G((\mathbb{P}^1)^d)^{\prod_{i=1}^n S_{e_i^\lambda}} \otimes \mathbb{Q} \subset \Lambda^G_{\mathcal{P}}.$$

Then $\{(\hat{\iota}_\lambda \circ \Phi_\lambda)_* \Delta_P \mid P \in \mathcal{P}\}$ generates $I^G_\lambda \otimes \mathbb{Q} \subset A^\bullet_G(\mathbb{P}^n) \otimes \mathbb{Q}$.

Proof. Since

$$\Phi^*_\lambda(A^\bullet_G(Y_\lambda) \otimes \mathbb{Q}) \subset A^\bullet_G((\mathbb{P}^1)^d)^{\prod_{i=1}^n S_{e_i^\lambda}} \otimes \mathbb{Q} \subset \Lambda^G_{\mathcal{P}},$$

we have

$$(\Phi_\lambda)_*\Lambda_{\mathcal{P}}^G \supset (\Phi_\lambda)_*(\Phi_\lambda^*(A_G^\bullet(Y_\lambda)) \otimes \mathbb{Q}) = A_G^\bullet(Y_\lambda) \otimes \mathbb{Q}$$

and by the projection formula, $(\Phi_\lambda)_*\Lambda_{\mathcal{P}}^G$ is

$$(\Phi_\lambda)_* \sum_{P\in\mathcal{P}} [\Delta_P]\Phi_\lambda^*\hat{\iota}_\lambda^*(A_G^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}) = \sum_{P\in\mathcal{P}} (\Phi_\lambda)_*[\Delta_P] \cap \hat{\iota}_\lambda^*(A_G^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}).$$

By the projection formula again, we thus have

$$I_\lambda^G \otimes \mathbb{Q} = (\hat{\iota}_\lambda)_*(A_G^\bullet(Y_\lambda) \otimes \mathbb{Q}) = \sum_{P\in\mathcal{P}} (\hat{\iota}_\lambda \circ \Phi_\lambda)_*[\Delta_P] \cap A_G^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}$$

as desired. □

Lemma 9.8. *Let $\lambda = \{\lambda_1, \dots, \lambda_d\}$ be a partition of $[n]$ and $\mathcal{P}$ be all partitions of $[d]$. Then*

$$\Lambda_{\mathcal{P}}^{PGL_2} = \begin{cases} A_{PGL_2}^\bullet((\mathbb{P}^1)^2)^{S_2} \otimes \mathbb{Q} & \text{if } d = 2 \text{ and } \lambda_1 = \lambda_2 \\ A_{PGL_2}^\bullet((\mathbb{P}^1)^d) \otimes \mathbb{Q} & \text{otherwise.} \end{cases}$$

In particular, given a partition $\lambda = \{\lambda_1, \dots, \lambda_d\}$ of n, $I_\lambda^{PGL_2} \otimes \mathbb{Q}$ is generated by all $[\lambda']$ with λ' formed by merging parts of λ.

Proof. Given the description of $\Lambda_{\mathcal{P}}^{PGL_2}$, the result about $I_\lambda^{PGL_2} \otimes \mathbb{Q}$ follows directly from Theorem 9.7. We will now show the description of $\Lambda_{\mathcal{P}}^{PGL_2}$.

We may identify $A_{PGL_2}^\bullet(\mathbb{P}^n) \otimes \mathbb{Q} \subset A_{GL_2}^\bullet(\mathbb{P}^n) \otimes \mathbb{Q}$ as the subring generated by $H + \frac{n}{2}(u+v)$ and $(u-v)^2$ by Theorem 4.8. Define

$$H_i' = H_i + \frac{1}{2}(u+v) \quad \text{and} \quad H' = H + \frac{n}{2}(u+v).$$

Note that with these definitions, we have

$$\Phi_\lambda^*\hat{\iota}_\lambda^*(H') = \sum \lambda_i H_i', \qquad H_i'^2 = \frac{1}{4}(u-v)^2.$$

We have the $\mathbb{Q}$-linear span

$$\Lambda_{\mathcal{P}} = \mathrm{Span}_{\mathbb{Q}}\left(\{\Delta_P(u-v)^{2k}\left(\sum \lambda_i H_i'\right)^\ell \mid k, \ell \geq 0, P \in \mathcal{P}\}\right).$$

The trivial partition is in $\mathcal{P}$, so 1 is automatically in $\Lambda_{\mathcal{P}}$.

Recall by Theorem 5.4 that

$$\Delta_{i,j} = H_i' + H_j',$$

and that $A^\bullet_{PGL_2}((\mathbb{P}^1)^d)\otimes\mathbb{Q}$ is generated by the H'_i and $(u-v)^2$. As $H_i'^2=\frac{1}{4}(u-v)^2$, to show $\Lambda_{\mathcal{P}}=A^\bullet_{PGL_2}((\mathbb{P}^1)^d)\otimes\mathbb{Q}$ it suffices to show that every monomial $\prod_{i\in C}H'_i$ is in $\Lambda_{\mathcal{P}}$ for $C\subset[d]$.

For $d=1$, $\lambda=\{[n]\}$, we are done as $H'_1=\frac{1}{\lambda_1}\Phi^*_\lambda\iota^*_\lambda H'$.

For $d=2$ and $\lambda_1\neq\lambda_2$,

$$\begin{aligned}H'_1&=\frac{1}{\lambda_1-\lambda_2}(\Phi^*_\lambda\hat{\iota}^*_\lambda(H')-\lambda_2\Delta_{1,2})\\ H'_2&=\frac{1}{\lambda_2-\lambda_1}(\Phi^*_\lambda\hat{\iota}^*_\lambda(H')-\lambda_1\Delta_{1,2})\\ H'_1H'_2&=\frac{1}{2\lambda_1\lambda_2}\left(\Phi^*_\lambda\hat{\iota}^*_\lambda(H')^2-\frac{1}{4}(\lambda_1^2+\lambda_2^2)(u-v)^2\right).\end{aligned}$$

For $d=2$ and $\lambda_1=\lambda_2=a$, we have to show $\Lambda_{\mathcal{P}}=A^\bullet_{PGL_2}((\mathbb{P}^1)^2)^{S_2}\otimes\mathbb{Q}$. As $H_i'^2=\frac{1}{4}(u-v)^2$, it suffices to show 1, $H'_1+H'_2$ and $H'_1H'_2$ are in $\Lambda_{\mathcal{P}}$. We already know that $1\in\Lambda_{\mathcal{P}}$, and

$$\begin{aligned}H'_1+H'_2&=\frac{1}{a}\Phi^*_\lambda\hat{\iota}^*_\lambda H',\\ H'_1H'_2&=\frac{1}{2a^2}\Big(\Phi^*_\lambda\hat{\iota}^*_\lambda(H')^2-\frac{1}{2}a^2(u-v)^2\Big).\end{aligned}$$

We will now show that $\Lambda_{\mathcal{P}}=A^\bullet_{PGL_2}((\mathbb{P}^1)^d)\otimes\mathbb{Q}$ when $d\geq 3$.

Up to degree $d-2$, we can take $k,\ell=0$ as the classes Δ_P for $P\in\mathcal{P}$ generate $A^{\leq d-2}_{PGL_2}((\mathbb{P}^1)^d)$ by Theorem 6.12. Hence to conclude the proof of Theorem 9.8, it suffices to show that $\prod_{k\neq i}H'_k$ for all i and $\prod H'_k$ are in $\Lambda_{\mathcal{P}}$.

For $\prod_{k\neq i}H_k$, without loss of generality suppose $i=1$. We have each of

$$\begin{aligned}&\frac{1}{a_1a_2}\Big(\prod_{k\neq 1,2}H'_k\Big)\cap\Phi^*_\lambda\hat{\iota}^*_\lambda H'\\ &=\frac{1}{a_1}\prod_{k\neq 1}H'_k+\frac{1}{a_2}\prod_{k\neq 2}H'_k+\frac{1}{4a_1a_2}(u-v)^2\sum_{j\neq 1,2}a_j\prod_{k\neq 1,2,j}H'_k,\\ &\frac{1}{a_1a_3}\Big(\prod_{k\neq 1,3}H'_k\Big)\cap\Phi^*_\lambda\hat{\iota}^*_\lambda H\\ &=\frac{1}{a_1}\prod_{k\neq 1}H_k+\frac{1}{a_3}\prod_{k\neq 3}H_k+\frac{1}{4a_2a_3}(u-v)^2\sum_{j\neq 2,3}a_j\prod_{k\neq 2,3,j}H'_k,\\ &\frac{1}{a_2a_3}\Big(\prod_{k\neq 2,3}H'_k\Big)\cap\Phi^*_\lambda\hat{\iota}^*_\lambda H\\ &=\frac{1}{a_2}\prod_{k\neq 2}H_k+\frac{1}{a_3}\prod_{k\neq 3}H_k+\frac{1}{4a_1a_3}(u-v)^2\sum_{j\neq 1,3}a_j\prod_{k\neq 1,3,j}H'_k\end{aligned}$$

lie in $\Lambda_{\mathcal{P}}$ as we have already shown each $\prod_{k\neq i,j} H_k$ lies in Λ. Also, the last term on each right-hand side lies in $\Lambda_{\mathcal{P}}$ as the number of terms in the H'_k monomial is $d-3$. Hence taking a linear combination we get $\prod_{k\neq 1} H'_k \in \Lambda_{\mathcal{P}}$.

To show $\prod_{i=1}^n H'_i \in \Lambda_{\mathcal{P}}$, we can proceed similarly to above, or expand

$$\frac{1}{a_1\ldots a_n}\Phi_\lambda^*\hat{\iota}_\lambda^*(H')^d = \prod H'_i + (u-v)^2\left(\text{lower order terms in the } H'_i\right),$$

using $H_i'^2 = \frac{1}{4}(u-v)^2$. □

Proof of Theorem 9.1. This follows from the excision exact sequence [17, Proposition 1.8] and Theorem 9.8. □

Lemma 9.9. *Let $\lambda = \{a, 1^b\}$ be a partition of n. Define $\mathcal{P}_\lambda$ to be the set of partitions*

$$\mathcal{P}_\lambda = \{T\} \sqcup \begin{cases} \{T_{1,i}\}_{i\geq 2} & a\neq b \\ \{T_{i,j}\}_{2\leq i<j\leq n} & a=b, \end{cases}$$

where T is the trivial partition and $T_{i,j}$ is the partition with $n-1$ parts and i, j in the same part. Then

$$\Lambda_{\mathcal{P}_\geq}^{PGL_2} = A^\bullet_{PGL_2}((\mathbb{P}^1)^{b+1})^{S_1\times S_b}\otimes\mathbb{Q}.$$

Proof. Define

$$H' = H + \frac{n}{2}(u+v) \quad\text{and}\quad H'_i = H_i + \frac{1}{2}(u+v).$$

Then in particular,

$$\begin{aligned}\Delta_{i,j} &= H'_i + H'_j \\ \Phi_\lambda^*\hat{\iota}_\lambda^*(H') &= aH'_1 + H'_2 + \ldots + H'_{b+1},\end{aligned}$$

so $\Lambda_{\mathcal{P}_\lambda}$ is the $\mathbb{Q}$-linear span

$$\Lambda_{\mathcal{P}_\lambda} = \mathrm{Span}_{\mathbb{Q}}\left\{\Delta_P(u-v)^{2k}(aH'_1 + H'_2 + \ldots + H'_{b+1})^\ell \mid k,\ell\geq 0,\ P\in\mathcal{P}_\lambda\right\}.$$

We first show that $H'_1 \in \Lambda_{\mathcal{P}_\lambda}$. Consider the case $b\neq a$. Then

$$H'_1 = \frac{1}{a-b}\left(\Phi_\lambda^*\hat{\iota}_\lambda^*(H') - \sum_{i\geq 2}\Delta_{1,i}\right)\in\Lambda_{\mathcal{P}_\lambda}.$$

Now consider the case $b = a$. Then

$$H_1' = \frac{1}{a}\left(\Phi_\lambda^* \hat{\iota}_\lambda^*(H') - \frac{1}{a-1}\sum_{2\le i<j\le a+1}\Delta_{i,j}\right) \in \Lambda_{\mathcal{P}_\lambda}.$$

Now that we have shown that $H_1' \in \Lambda$, it therefore suffices to show that the invariant subring $A^\bullet_{PGL_2}((\mathbb{P}^1)^{b+1})^{S_1\times S_b}$ is given by

$$\mathrm{Span}_{\mathbb{Q}}\Big\{(u-v)^{2k}(aH_1'+H_2'+\ldots+H_{b+1}')^\ell,\ H_1'(u-v)^{2k}(aH_1'+H_2'+\ldots+H_{b+1}')^\ell \,|\, k,\ell\ge 0\Big\}.$$

Note that by using the relation $H_1'^2 = \frac{1}{4}(u-v)^2$, we see this is the same as

$$\begin{aligned}
&\mathrm{Span}_{\mathbb{Q}}\{H_1'^k(aH_1' + H_2' + \ldots + H_{b+1}')^\ell \mid k,\ell \ge 0\}\\
&= \mathrm{Span}_{\mathbb{Q}}\{H_1'^k(H_2' + \ldots + H_{b+1}')^\ell \mid k,\ell \ge 0\}\\
&= \mathrm{Span}_{\mathbb{Q}}\{(u-v)^{2k}(H_1' + H_2' + \ldots + H_{b+1}')^\ell,\\
&\quad H_1'(u-v)^{2k}(H_1' + H_2' + \ldots + H_{b+1}')^\ell \mid k,\ell \ge 0\}.
\end{aligned}$$

By using the relations $H_i'^2 = \frac{1}{4}(u-v)^2$ whenever possible, we see that an element of the invariant subring is a sum of terms of the form $(u-v)^{2k}e_j(H_2',\ldots,H_{b+1}')$ and $(u-v)^{2k}H_1'e_j(H_2',\ldots,H_{b+1}')$ where e_j is the jth elementary symmetric polynomial, hence it suffices to show that

$$\begin{aligned}
&\mathrm{Span}_{\mathbb{Q}}\{(u-v)^{2k}e_j(H_2',\ldots,H_{b+1}') \mid j,k \ge 0\}\\
&\subset \mathrm{Span}_{\mathbb{Q}}\{(u-v)^{2k}(H_2' + \ldots + H_{b+1}')^\ell \mid k,\ell \ge 0\}.
\end{aligned}$$

This follows by induction on j and the relation

$$\begin{aligned}
&e_j(H_2',\ldots,H_{b+1}')(H_2' + \ldots + H_{b+1}')\\
&= (j+1)e_{j+1}(H_2'+\ldots+H_{b+1}') + \frac{1}{4}(u-v)^2(n-j+1)e_{j-1}(H_2',\ldots,H_{b+1}').
\end{aligned}$$

□

Proof of Theorem 9.2. This follows from the excision exact sequence [17, Proposition 1.8], Theorem 9.7, and Theorem 9.9. □

10. Excision of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/GL_2]$ and $[\mathrm{Sym}^n K^2/GL_2]$

In this section, we show how our results about excision of unordered strata in $[\mathrm{Sym}^n\mathbb{P}^1/PGL_2]$ imply similar results in $[\mathrm{Sym}^n\mathbb{P}^1/GL_2]$ and $[\mathrm{Sym}^n K^2/GL_2]$, recovering and extending some results of [12] (see Theorem 10.3).

Definition 10.1. Given a partition λ of n, let $\widetilde{I_\lambda}$ be the ideal of $A^\bullet_{GL_2}(\mathbb{A}^{n+1})$ given by the image of the pushforward $A^{GL_2}_\bullet(\widetilde{Z_\lambda}) \hookrightarrow A^{GL_2}_\bullet(\mathbb{A}^{n+1})$ and the identification $A^{GL_2}_\bullet(\mathbb{A}^{n+1}) \cong A^{n+1-\bullet}_{GL_2}(\mathbb{A}^{n+1})$ via Poincaré duality [8, Proposition 4].

Theorem 10.2. *$I^{GL_2}_\lambda \otimes \mathbb{Q}$ (respectively $\widetilde{I_\lambda} \otimes \mathbb{Q}$) is generated by all $[Z_{\lambda'}]$ (respectively $[\widetilde{Z}_{\lambda'}]$) with λ' formed by merging parts of λ. For $\lambda = \{a, 1^{n-a}\}$ only two generators are required, namely $[Z_\lambda]$ (respectively $[\widetilde{Z}_\lambda]$) and $[Z_{\lambda'}]$ (respectively $[\widetilde{Z}_{\lambda'}]$) where*

$$\lambda' = \begin{cases} \{a+1, 1^{n-a-1}\} & \text{if } a \neq \frac{n}{2} \\ \{a, 2, 1^{n-a-2}\} & \text{if } a = \frac{n}{2}. \end{cases}$$

Remark 10.3. In the affine case, when n is odd and $a = \lceil \frac{n}{2} \rceil$ this recovers [12, Theorem 4.3], and when n is even and $a = \frac{n}{2}$ this recovers the rational Chow ring of the stable locus in [12, Theorem 4.10].

Lemma 10.4. *We have*

$$\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}} = \left(A^\bullet_{GL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}} \text{ and}$$
$$\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}\right) = A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}.$$

In particular, if a set of partitions $\mathcal{P}$ satisfies the hypotheses of Theorem 9.7 for $G = PGL_2$, then they also satisfy the hypotheses of Theorem 9.7 for $G = GL_2$.

Proof. We identify $A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}$ as the subring of $A^\bullet_{GL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}$ via Theorem 4.8 generated by $H' := H + \frac{n}{2}(u+v)$ and $(u-v)^2$. Since $A^\bullet_{GL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}$ is generated by H' over $\mathbb{Q}[u,v]^{S_2}$, and $(u-v)^2$ and $u+v$ generate $\mathbb{Q}[u,v]^{S_2}$, $\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}\right) = A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$.

For the other equality, we use Theorem 4.3 to identify $A^\bullet_{PGL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ as the subring of $A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$ generated by $H_i' := H_i + \frac{u+v}{2}$. Then,

$$\left(A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}}$$

is generated $\mathbb{Z}$-linearly by all $p(H_1', \ldots, H_n')$, where p is a polynomial invariant under the action of $S_{e_1} \times \cdots \times S_{e_k}$. Similarly, $\left(A^\bullet_{GL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}}$ is generated by all such $p(H_1', \ldots, H_n')$, together with $u+v$ and uv. Therefore,

$$\mathbb{Q}[u,v]^{S_2}\left(A^\bullet_{PGL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}} = \left(A^\bullet_{GL_2}((\mathbb{P}^1)^n) \otimes \mathbb{Q}\right)^{S_{e_1}\times\ldots\times S_{e_k}}.$$

□

As we will now see, the cones over generators of $I^{GL_2}_\lambda \otimes \mathbb{Q}$ also generate $\widetilde{I_\lambda} \otimes \mathbb{Q}$. We will use a certain property about the classes of unordered strata to prove this, which as we will see is that Z_λ contains a cycle whose class divides the class of the origin in $A^\bullet_{GL_2}(\mathbb{A}^{n+1}) \otimes \mathbb{Q}$.

Lemma 10.5. *Given a partition λ of n and a set of generators S of $I_\lambda^{GL_2} \otimes \mathbb{Q}$ of degree at most n, $\widetilde{I_\lambda} \otimes \mathbb{Q}$ is generated by*

$$\{\alpha_0 \mid \alpha \in S\},$$

where α_0 is the constant term of $\alpha \in A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$, after writing α as a polynomial in H, u, v that is degree at most n in H using the relation $G(H) = 0$ (see Section 2.3*).*

Proof. Let $\widetilde{I'_\lambda} \subset A^\bullet_{GL_2}(\mathbb{A}^{n+1}) \otimes \mathbb{Q}$ be the ideal generated by $\{\alpha_0 \mid \alpha \in S\}$, so we want to show $\widetilde{I'_\lambda} = \widetilde{I_\lambda} \otimes \mathbb{Q}$. Consider the diagram of rational Chow rings (we omit $\otimes\mathbb{Q}$ for brevity)

$$\begin{array}{ccccccc}
A^\bullet_{GL_2}(\mathbb{P}^n) & \xleftarrow{\sim} & A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1}\backslash\{0\}) & \twoheadrightarrow & A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\}) & \twoheadleftarrow & A^\bullet_{GL_2}(\mathbb{A}^{n+1}) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_4} \\
A^\bullet_{GL_2}(\mathbb{P}^n\backslash Z_\lambda) & \xleftarrow{\sim} & A^\bullet_{GL_2\times\mathbb{G}_m}(\mathbb{A}^{n+1}\backslash\widetilde{Z_\lambda}) & \twoheadrightarrow & A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\widetilde{Z_\lambda}) & \xleftarrow{\sim} & A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\widetilde{Z_\lambda})
\end{array}$$

where $\mathbb{G}_m$ acts by scaling on $\mathbb{A}^{n+1}$. We know $I_\lambda \otimes \mathbb{Q}$ is the kernel of π_1, so it maps surjectively to the kernel of π_3 in $A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\})$. Each generator $\alpha \in S$ maps to the image of α_0 in $A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\}) \otimes \mathbb{Q}$. Since the kernel of $A^\bullet_{GL_2}(\mathbb{A}^{n+1}) \otimes \mathbb{Q} \to A^\bullet_{GL_2}(\mathbb{A}^{n+1}\backslash\{0\}) \otimes \mathbb{Q}$ is generated by $\prod_{i=0}^n (iu + (n-i)v)$, we have $\widetilde{I'_\lambda} + \langle \prod_{i=0}^n (iu + (n-i)v) \rangle = \widetilde{I_\lambda} \otimes \mathbb{Q}$. To finish, it suffices to see $\prod_{i=0}^n (iu + (n-i)v) \in \widetilde{I'_\lambda}$.

As $Z_{\{n\}}$ is a cycle in Z_λ, $[\{n\}]$ can be expressed as an $A^\bullet_{GL_2}(\mathbb{P}^n) \otimes \mathbb{Q}$-linear combination of the elements of S, and taking the constant terms yields

$$[\{n\}]_0 = n \prod_{i=1}^{n-1} (iu + (n-i)v) \in \widetilde{I'_\lambda}$$

by Theorem 5.1 and Section 5.4, which divides $\prod_{i=0}^n (iu + (n-i)v)$. □

Proof of Theorem 10.2. Apply Theorem 10.4 to Theorems 9.8 and 9.9 to get the statements on $I_\lambda^{GL_2} \otimes \mathbb{Q}$. Then, apply Theorem 10.5 to get the statements on $\widetilde{I_\lambda}$. □

Appendix

A. Multiplicative relations between symmetrized strata

In this section, we investigate certain multiplicative relations between the classes $[\widetilde{Z_\lambda}] \in A^\bullet_{GL_2}(\mathrm{Sym}^n K^2)$. These are equivalent to certain relations between the degree 0 terms of the expressions for $[\lambda] \in A^{\bullet}_{GL_2}(\mathbb{P}^n)$ by Section 2.5. For this, it suffices to restrict ourselves to the $\mathbb{Q}$-basis given by the $[a, b, 1^c]$-classes from Theorem 7.4.

Definition A.1. Denote by $[a, b, 1^c]_0 \in \mathbb{Z}[u, v]^{S_2}$ be the term of $[a, b, 1^c] \in A^{\bullet}_{GL_2}(\mathbb{P}^n)$ that is degree zero in H.

We show how to write $(u+v)[a, b, 1^c]_0$ and $uv[a, b, 1^c]_0$ as a $\mathbb{Q}$-linear combination of strata. A few of these multiplicative relations have been explicitly written down [12, Remark 3.9] and shown to exist abstractly [12, Theorems 4.3 and 4.10] using the degeneration of a spectral sequence of a filtered CW-complex. We give a combinatorial method to do this in general in Theorems A.2 and A.4.

Theorem A.2. *For $c \geq 1$ and $a + b + c = n$,*

$$\begin{aligned} n(u + v)[a, b, 1^c]_0 = {} & (c + a - b)[a + 1, b, 1^{c-1}]_0 \\ & + (b + c - a)[a, b + 1, 1^{c-1}]_0 \\ & + (a + b - c)[a + b, 1, 1^{c-1}]_0. \end{aligned}$$

Proof. We will prove Theorem A.2 by pulling back to $(\mathbb{P}^1)^n$. By Theorem 2.8, we want to show

$$(2H + nu + nv)[a, b, 1^c] = (c + a - b)[a + 1, b, 1^{c-1}] \tag{A.1}$$

$$+ (b + c - a)[a, b + 1, 1^{c-1}] \tag{A.2}$$

$$+ (a + b - c)[a + b, 1, 1^{c-1}]. \tag{A.3}$$

Let $A = \{1, \ldots, a\}$, $B = \{a + 1, \ldots, a + b\}$. As in Section 2, let $\Phi : (\mathbb{P}^1)^n \to \mathrm{Sym}^n\mathbb{P}^1 \cong \mathbb{P}^n$ be the degree $n!$ multiplication map. As $\Phi_*[\Delta_{\{A,B\}}] = [a, b, 1^c]$, the left-hand side of (A.1) is

$$\Phi_*([\Delta_{\{A,B\}}] \cap \Phi^*(2H + nu + nv)).$$

The pullback of $2H + nu + nv$ along Φ is

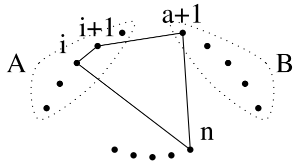
$$\begin{aligned} & (H_1 + H_2 + u + v) + (H_2 + H_3 + u + v) + \ldots + (H_n + H_1 + u + v) \\ & = [\Delta_{1,2}] + [\Delta_{2,3}] + \ldots + [\Delta_{n,1}] \end{aligned}$$

by Theorem 5.4. In this way, we now only have to intersect strata using Theorem 5.2 and the square relation as in Theorem 6.2.

There are 6 cases: $1 \leq i \leq a - 1$, $i = a$, $a + 1 \leq i \leq a + b - 1$, $i = a + b$, $a + b + 1 \leq i \leq n - 1$, and $i = n$. We will deal with each of these cases in the same way outlined above.

To calculate $\Phi_*([\Delta_{i,i+1}][\Delta_{\{A,B\}}])$ for $1 \leq i \leq a - 1$, we use the square relation to replace $[\Delta_{i,i+1}]$ with $[\Delta_{i,n}] - [\Delta_{n,a+1}] + [\Delta_{a+1,i+1}]$. Using Theorem 5.2, each of the products is itself a strata, and the pushforward is

$$[a + 1, b, 1^{c-1}] - [a, b + 1, 1^{c-1}] + [a + b, 1, 1^{c-1}].$$



For $i = a$, Theorem 5.2 implies $[\Delta_{a,a+1}][\Delta_{A,B}] = [\Delta_{\{A \sqcup B\}}]$, which pushes forward to

$$[a+b, 1, 1^{c-1}].$$

Similarly to before, for $a+1 \leq i \leq a+b-1$, the pushforward is

$$[a, b+1, 1^{c-1}] - [a+1, b, 1^{c-1}] + [a+b, 1, 1^{c-1}].$$

For $i = a+b$, Theorem 5.2 implies $[\Delta_{a+b,a+b+1}][\Delta_{\{A,B\}}] = [\Delta_{\{A, B \sqcup \{a+b+1\}\}}]$, which pushes forward to

$$[a, b+1, 1^{c-1}].$$

For $a+b+1 \leq i \leq n-1$, replace $[\Delta_{i,i+1}]$ with $[\Delta_{i,a}] - [\Delta_{a,a+1}] + [\Delta_{a+1,i+1}]$, and similarly to before we get the pushforward is

$$[a+1, b, 1^{c-1}] - [a+b, 1, 1^{c-1}] + [a, b+1, 1^{c-1}].$$

Finally, for $i = n$, using Theorem 5.2, $\Delta_{n,1}\Delta_{\{A,B\}} = \Delta_{\{A \sqcup \{n\}, B\}}$, so this will pushforward to

$$[a+1, b, 1^{c-1}].$$

Combining these yields the desired result. □

Remark A.3. Given a partition λ of n with at least three nontrivial parts, the argument of Theorem A.2 is a combinatorial algorithm that can non-canonically express $n(u+v)[\lambda]$ in terms of other classes $[\lambda']$ with one fewer part. The number of square relations can be drastically reduced in practice by an appropriate choice of the partition pushing forward to $[a_1, \ldots, a_d]$.

Theorem A.4. *For $c \geq 2$, and $a+b+c = n$*

$$\begin{aligned} n^2 uv[a, b, 1^c]_0 = (2ab + ac + bc + c(c-1))&[a+1, b+1, 1^{c-2}]_0 \\ +(-ab - bc)&[a+2, b, 1^{c-2}]_0 \\ +(-ab - ac)&[a, b+2, 1^{c-2}]_0 \\ +(-ac - bc - c(c-1))&[a+b+1, 1, 1^{c-2}]_0 \\ +(ac + bc)&[a+b, 2, 1^{c-2}]_0. \end{aligned}$$

Proof. As in the previous theorem letting $A = \{1, \dots, a\}$, $B = \{a+1, \dots, a+b\}$ the statement is equivalent to

$$\begin{aligned}\Phi_*\Big(\Big(\sum H_i + nu\Big)\Big(\sum H_i + nv\Big)[\Delta_{\{A,B\}}]\Big)&\\ = (2ab + ac + bc + c(c-1))[a+1, b+1, 1^{c-2}]&\\ +(-ab - bc)[a+2, b, 1^{c-2}]&\\ +(-ab - ac)[a, b+2, 1^{c-2}]&\\ +(-ac - bc - c(c-1))[a+b+1, 1, 1^{c-2}]&\\ +(ac + bc)[a+b, 2, 1^{c-2}].&\end{aligned}$$

We have $(H_i + u)(H_i + v) = 0$, so

$$\begin{aligned}\Big(\sum H_i + nu\Big)\Big(\sum H_i + nv\Big) &= \sum_{1\le i<j\le n} (H_i + u)(H_j + v) + (H_j + u)(H_i + v)\\ &= \sum_{1\le i<j\le n} -(H_i - H_j)^2\\ &= \sum_{1\le i<j\le n} -([\Delta_{i,k_{i,j}}] - [\Delta_{j,k_{i,j}}])^2\end{aligned}$$

where $k_{i,j} \in [n] \setminus \{i, j\}$ is arbitrary. There are 6 cases depending on which of $A, B, [n] \setminus \{A, B\}$ each of i, j lie in, and for each of these cases an appropriate choice of $k_{i,j}$ can be made so that the strata combine via Theorem 5.2 as in the proof of Theorem A.2 and push forward to $[a', b', 1^{c-2}]$-classes. □

Remark A.5. Similarly to Theorem A.2, the argument of Theorem A.4 is a combinatorial algorithm that can express $n^2uv[\lambda]$ in terms of other classes $[\lambda']$ with two fewer parts for any partition λ of n with at least four parts.

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