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Asymptotic estimates and compactness of expanding gradient Ricci solitons

ALIX DERUELLE

Abstract. We first investigate the asymptotics of conical expanding gradient Ricci solitons by proving sharp decay rates to the asymptotic cone both in the generic and in the asymptotically Ricci flat case. We then establish a compactness theorem concerning nonnegatively curved expanding gradient Ricci solitons.

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1. Introduction

The Ricci flow, introduced by Hamilton in the early eighties, is a nonlinear heat equation on the space of metrics of a given manifold, modulo the action of diffeomorphisms and homotheties. Therefore, one expects the Ricci flow to smooth out singular geometric structures. We consider here the smoothing of metric cones over smooth compact manifolds by expanding self-similarities of the Ricci flow. More precisely, we recall that an expanding gradient Ricci soliton is a triplet $(M, g, \nabla f)$ where $f : M \to \mathbb{R}$ is a smooth function such that

$$\nabla^2 f = \mathrm{Ric}(g) + \frac{g}{2}.$$

At least formally, one can associate a Ricci flow solution by defining $g(\tau) := (1+\tau)\phi_\tau^* g$ where $(\phi_\tau)_{\tau>-1}$ is the one-parameter family of diffeomorphisms generated by the vector field $-\nabla^g f/(1+\tau)$.

Let us describe a first class of nontrivial examples, *i.e.* non-Einstein, of Ricci expanders discovered by Bryant [6, Chapter 1]: it is a one-parameter family of rotationally symmetric metrics $(g_c)_{c>0}$ on $\mathbb{R}^n$ whose asymptotic cones (in the Gromov sense) are the metric cones $(C(\mathbb{S}^{n-1}), dr^2 + (cr)^2 g_{\mathbb{S}^{n-1}})$, with $c \in \mathbb{R}_+^*$, where

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$(\mathbb{S}^{n-1}, g_{\mathbb{S}^{n-1}})$ is the Euclidean sphere of curvature 1. The metrics g_c have positive curvature operator if $c < 1$ and are negatively curved if $c > 1$. The metric cone $(C(\mathbb{S}^{n-1}), dr^2 + (cr)^2 g_{\mathbb{S}^{n-1}})$ is both the asymptotic cone of (M, g_c) and the singular initial condition of the associated Ricci flow at time $\tau = -1$. Such a phenomenon has been generalized at least in two directions. In the positively curved case, Schulze and Simon [24] have shown that most of the asymptotic cones of non collapsed Riemannian manifolds are smoothed out by Ricci gradient expanders. On the other hand, Chen and the author [7] proved that any Ricci gradient expander with quadratic curvature decay has a unique asymptotic cone $(C(X), dr^2 + r^2 g_X)$, where X is a smooth compact manifold and g_X is a $C^{1,\alpha}$ metric for any $\alpha \in (0, 1)$.

The main purpose of this article consists in understanding how the geometry of the link (X, g_X) is reflected globally or asymptotically on the Ricci soliton. We describe first the asymptotics of Ricci expanders. We recall the definition of a gradient Ricci expander being asymptotically conical.

Definition 1.1. An expanding gradient Ricci soliton $(M, g, \nabla f)$ is *asymptotically conical* to $(C(X), g_{C(X)} := dr^2 + r^2 g_X, r\partial r/2)$ if there exists a compact $K \subset M$, a positive radius R and a diffeomorphism $\phi : M \setminus K \to C(X) \setminus B(o, R)$ such that

$$\sup_{\partial B(o,r)} |\nabla^k(\phi_* g - g_{C(X)})|_{g_{C(X)}} = O(f_k(r)), \quad \forall k \in \mathbb{N}, \tag{1.1}$$

$$f(\phi^{-1}(r, x)) = \frac{r^2}{4}, \quad \forall (r, x) \in C(X) \setminus B(o, R), \tag{1.2}$$

where $f_k(r) = o(1)$ as $r \to +\infty$.

Condition (1.1) is one of the classical definitions of asymptotically conical metrics (*e.g.*, in [23] with Schauder norms instead) and condition (1.2) taken from [8] reflects the compatibility with the expanding structure.

We are mainly interested in the following situations: either the convergence is polynomial at rate τ, *i.e.* $f_k(r) = r^{-\tau-k}$ for some positive τ and any nonnegative integer k, or the convergence is exponential at rate τ, *i.e.* $f_k(r) = r^{-\tau+k} e^{-r^2/4}$ for any nonnegative integer k.

The first main result of this paper is the following:

Theorem 1.2. *Let $(M, g, \nabla f)$ be an expanding gradient Ricci soliton such that, for some $p \in M$,*

$$\limsup_{x \to +\infty} r_p(x)^{2+i} |\nabla^i \operatorname{Ric}(g)|(x) < +\infty, \quad \forall k \in \mathbb{N}.$$

Then there exists a unique metric cone over a smooth compact manifold $(C(X), dr^2 + r^2 g_X)$ with a smooth metric such that $(M, g, \nabla f)$ is asymptotic to $(C(X), dr^2 + r^2 g_X, r\partial_r/2)$ at polynomial rate $\tau = 2$. In particular, it shows that

$$\limsup_{x \to +\infty} r_p(x)^{2+i} |\nabla^i \operatorname{Rm}(g)|(x) < +\infty, \quad \forall i \in \mathbb{N}.$$

See [7, Theorem 1.2] and [23, Theorem 4.3.1] for earlier and related results. This rate is sharp regarding the existing examples: the Bryant examples or the Ricci expanders built by Feldman-Ilmanen-Knopf [14] on the negative line bundle $(L^{-k}, g_{k,p})_{k,p}$ with $k,\ p\in\mathbb{N}^*$ coming out of the cone $(C(\mathbb{S}^{2n-1}/\mathbb{Z}_k), i\partial\bar{\partial}(|z|^{2p}/p))$, where $\mathbb{Z}_k$ acts on $\mathbb{C}^n$ by rotations.

Theorem 1.2 shows the existence and uniqueness of an asymptotic cone. This comes essentially from the use of the Morse flow associated to the potential function f: see the introduction of [7] for references of counterexamples in case one drops the expanding structure.

Finally, Theorem 1.2 holds for Ricci shrinkers, *i.e.* triplets $(M, g, \nabla f)$ such that $\mathrm{Ric}(g)+\nabla^2 f = g/2$. As shrinkers are ancient solutions to the Ricci flow, local Shi's estimates [9, Chapter 6] imply immediately the corresponding decay of higher covariant derivatives of the curvature tensor in case the curvature tensor decays quadratically at infinity. This does not hold for Ricci expanders as the recent paper [13] shows: there exist asymptotically conical Ricci expanders with low regularity $(C^{2,\alpha})$ at infinity. That is why we insist in Theorem 1.2 and in the remaining part of this paper on appropriate covariant derivatives decay of the curvature tensor.

Now, if the curvature tensor of the link X satisfies some elliptic constraints, we expect to get faster convergence to the asymptotic cone or at least a faster convergence of some tensor made out of the curvature tensor of the Ricci expander. Indeed, for Ricci expanders, there is a one-to-one correspondence between the asymptotic cone (in the Gromov-Hausdorff sense) and the singular initial condition of the corresponding Ricci flow: see the introduction of [13] for an explanation. Therefore, the smoother the initial condition (or the asymptotic cone) is, the faster the convergence rate to the initial condition will be.

The second main result of this paper is:

Theorem 1.3. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton. Assume*

$$\lim_{+\infty} r_p^2 \,\mathrm{Ric}(g) = 0.$$

(1) *Then, $\nabla^k \mathrm{Ric}(g) = O(f^{1+\frac{k-n}{2}}e^{-f})$, for any $k \geq 0$;*
(2) *$(M, g, \nabla f)$ is asymptotic to a smooth Ricci flat metric cone $(C(X), dr^2+r^2 g_X)$ at exponential rate $\tau = n$;*
(3) *If the scalar curvature is positive then*

$$\inf_M f^{n/2-1}e^f\,\mathrm{R}_g$$
$$\geq C(n, \sup_{M^n}\mathrm{R}_g, \sup_{M^n} f\,\mathrm{R}_g)\min\left\{\liminf_{+\infty} f^{n/2-1}e^f\,\mathrm{R}_g;\ \min_{f\leq C(\mu(g),n,\sup_{M^n}\mathrm{R}_g)}\mathrm{R}_g\right\}.$$

Remark 1.4.

- The assumption on the Ricci curvature is satisfied if, for instance, the convergence to its asymptotic cone is at least C^k, for $k \geq 2$ and if the asymptotic cone is Ricci flat;

- The Ricci curvature decay obtained in Theorem 1.3 is sharp. Indeed, according to [23, Example 3.3.3], the Ricci curvature decay of the Ricci expanders $(L^{-k}, g_{k,1})_k$ with $k \in \mathbb{N}^*$ coming out of the cone $(C(\mathbb{S}^{2n-1}/\mathbb{Z}_k), i\partial\bar{\partial}(|z|^2))$ built in [14] is exactly $O(f^{1-n/2}e^{-f})$;
- The rate $\tau = n$ is given as in Theorem 1.2 by the decay of the Ricci curvature at infinity.

If (X, g_X) is Einstein with constant scalar curvature different from $(n-1)(n-2)$, then

$$T := \mathrm{Ric}(g) - \frac{\mathrm{R}_g}{n-1}\left(g - \frac{\nabla f}{|\nabla f|} \otimes \frac{\nabla f}{|\nabla f|}\right) = O(r^{-4}).$$

This tensor T reflects the conical geometry at infinity and has already been proved useful in other contexts: [3]. This is proved in Theorems 3.6 and 3.8 under weak assumptions on the convergence to the asymptotic cone (less than C^2).

Finally, we also focus on how bounds on asymptotic covariant derivatives of the Ricci curvature imply corresponding bounds on asymptotic covariant derivatives of the full curvature: see the proof of Theorem 1.2.

Now, once this said, we must say a few words about the proof of Theorem 1.3 which is almost more important because of the robustness of its method based on the maximum principle with the help of new barriers. The main contribution of this paper is to derive a priori new decay at infinity of (sub)solutions of the following eigenvalue problem:

$$\begin{cases} \Delta_f u = -\lambda u \quad \lambda \in \mathbb{R} \\ u = o(f^{-\lambda}) \end{cases} \tag{1.3}$$

where $\Delta_f := \Delta + \nabla_{\nabla f}$ denotes the weighted Laplacian associated to a Ricci expander $(M^n, g, \nabla f)$. Notice that they are two approximate solutions to this eigenvalue problem: $f^{-\lambda}$ and $f^{\lambda - n/2}e^{-f}$. To give a flavor of the main technical Lemma 2.9, we state a light but striking version of one of its corollary:

Corollary 1.5. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton such that f is an exhausting function. Assume there exists a C^2 tensor T such that*

$$\Delta_f T = -\lambda T + \mathrm{Rm}(g) * T,$$

for some real number λ. Assume $R_0 := \sup_{M^n}(f|\mathrm{Rm}(g)|)$ is finite.

(1) *Then,*

$$\sup_{M^n} f^\lambda |T| \le C\left(\limsup_{+\infty} f^\lambda |T| + \sup_{f \le f_0} |T|\right),$$
$$C = C(n, \lambda, R_0, \min_{M^n} f),$$
$$f_0 = f_0(n, \lambda, R_0, \min_{M^n} f);$$

(2) *Moreover, if* $\limsup_{+\infty} f^{\lambda}|T| = 0$ *then*

$$\sup_{M^n} f^{\frac{n}{2}-\lambda} e^f |T| \leq C\left(n, \lambda, R_0, \min_{M^n} f, \sup_{f \leq f_0} |T|\right),$$
$$f_0 = f_0\left(n, \lambda, R_0, \min_{M^n} f\right).$$

This method is even useful in the flat case, *i.e.* for the Gaussian soliton ($\mathbb{R}^n$, eucl, $r\partial_r/2$). Indeed, the $L^2\left(e^{|\cdot|^2/4} d\mu(\text{eucl})\right)$-eigenvalues of the weighted Laplacian $\Delta_{r\partial_r/2}$ are conjugate to the Hermite functions since the weighted Laplacian is unitarily conjugate to the harmonic oscillator $\Delta - c_1 r^2 - c_2$ where c_1 and c_2 are positive constants. Therefore, Corollary 1.5 gives an exact decay for the Hermite functions.

We emphasize the fact that all these estimates are quantitative, *i.e.* they hold outside a compact set that might depend on the expander, that is why they are not so useful regarding compactness questions which is the second aspect of this article. Indeed, all the previous results are asymptotic estimates, they do not control all the topology and the geometry of the Ricci expander. Besides, Corollary 1.5 applied to the curvature tensor is of no help.

The first result about (pre)compactness of Riemannian metrics goes back to Cheeger and Gromov. As we are mainly interested in the C^∞ topology, we state Hamilton's compactness theorem with bounds on covariant derivatives of the curvature tensor and bounded diameter [15]:

Theorem 1.6 (Hamilton).

$$\mathfrak{M}(n, D, v, (\Lambda_k)_{k\geq 0}) := \Big\{(N^n, h) \textit{ closed} \mid \operatorname{diam}(h) \leq D;\ \operatorname{Vol}(N, h) \geq v; \quad (1.4)$$
$$|\nabla^k \operatorname{Rm}(h)| \leq \Lambda_k, \forall k \in \mathbb{N}\Big\}, \quad (1.5)$$

where D, v *and* $(\Lambda_k)_{k\in\mathbb{N}}$ *are positive real numbers, is compact in the* C^∞ *topology.*

If one drops the diameter bound, a similar statement holds in the C^∞ *pointed* convergence for complete Riemannian manifolds: then, the lower volume bound concerns the volume of geodesic balls of a fixed radius, say 1. The conditions on the covariant derivatives of the curvature tensor can be actually replaced by bounds on the Ricci curvature and are given for free when an elliptic or parabolic equation is assumed to hold: [15] in the Ricci flow case and [16] for Ricci shrinkers and the references therein. In our setting, as $(C(X), g_{C(X)})$ is the initial condition of a nonlinear heat equation, we expect heuristically at least that if (X, g_X) belongs to a compact set of metrics $\mathfrak{M}$ then so will be the set of expanders with asymptotic cone $(C(X), g_{C(X)})_{(X,g_X)\in\mathfrak{M}}$. The notion of compactness in this setting is that of the C^∞ conical topology whose Definition 4.4, rather technical, is given in Section 4. In particular, converging in this topology for asymptotically conical Ricci expanders implies the convergence of the corresponding asymptotic cones.

We are able to prove compactness of Ricci expanders with non negative curvature operator. In this case, the potential function is a strictly convex function with quadratic growth, in particular, the topology is that of $\mathbb{R}^n$ and the level sets of the potential function are diffeomorphic to $\mathbb{S}^{n-1}$. More precisely, if $\mathrm{Crit}(f)$ denotes the set of critical points of a function f, we have the following:

Theorem 1.7. *The class*

$$\mathfrak{M}^{\mathrm{Vol}}_{\mathrm{Exp}}(n, (\Lambda_k)_{k\geq 0}, V_0) := \left\{ \begin{array}{l|l} (M^n, g, \nabla f, p) \text{ normalized} & \mathrm{Rm}(g) \geq 0; \\ \text{expanding gradient Ricci soliton} & \mathrm{Crit}(f) = \{p\}; \\ \text{smoothly converging to its asymptotic} & \mathrm{AVR}(g) \geq V_0; \\ \text{cone } (C(X), dr^2 + r^2 g_X, r\partial_r/2, o) & (X, g_X) \in \mathfrak{M}(n-1, \pi, nV_0, (\Lambda_k)_{k\geq 0}) \end{array} \right\}$$

is compact in the C^∞ conical topology.

Remark 1.8.

- We prove a more general compactness theorem for conical expanders in case one assumes the non negativity of the Ricci curvature only: see Theorem 4.6.
- It would be interesting to allow different underlying topologies in Theorem 4.6 as in the Feldman-Ilmanen-Knopf examples [14].

The main difficulty in proving Theorems 1.7 and 4.6 consists in inverting limits, *i.e.* formally speaking,

$$\lim_{t\to+\infty} \lim_{i\to+\infty} (M_i, t^{-2} g_i, p_i) = \lim_{i\to+\infty} \lim_{t\to+\infty} (M_i, t^{-2} g_i, p_i), \tag{1.6}$$

where $(M_i, g_i, p_i)_i$ is a sequence of expanding gradient Ricci solitons as in Theorem 1.7. More precisely, we need to bound independently of the sequence the following rescaled covariant derivatives:

$$\sup_{x\in M_i} (1 + r_{p_i}(x))^{2+k} |\nabla^k \mathrm{Rm}(g_i)|(x), \quad k \geq 0.$$

The main step for proving this inversion is to derive new a priori estimates of differential inequalities with quadratic nonlinearities. Indeed, the norm of the curvature tensor $|\mathrm{Rm}(g)| =: u$ satisfies

$$\Delta_f u \geq -u - c(n)u^2,$$

as soon as $|\mathrm{Rm}(g)|$ does not vanish. Therefore, the second main (technical) contribution of this paper is to bound globally the rescaled norm of such a solution u by its value at infinity:

Proposition 1.9. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton with bounded scalar curvature and such that the potential function f is an exhausting function. Assume there exists a C^2 bounded nonnegative function $u : M \to \mathbb{R}$ such that*

$$\Delta_f u \geq -u - cu^2, \quad \limsup_{+\infty} fu < +\infty,$$

for some positive constant c. Then

$$\sup_{M^n} fu \leq C(c, \sup_{M^n} u, \limsup_{+\infty} fu).$$

Again, the only tool used massively all along the proofs of asymptotic and a priori global estimates is the maximum principle for functions. This contrasts with the shrinking case where integral estimates appear to be more useful: see [16]. Finally, on the one hand, Theorem 1.7 can be interpreted as the extension of Hamilton's compactness theorem for non negatively curved expanding gradient Ricci solitons with conical initial condition, on the other hand, the motivation of Theorem 1.7 comes equally from an analogous statement (but which is much harder to prove) in the setting of conformally compact Einstein manifolds where the notion of asymptotic cone $(C(X), g_{C(X)})$ is replaced by the one of conformal infinity $(X, [g_X])$: see [1] for details. Last but not least, Theorem 1.7 was highly motivated to show the connectedness of the moduli space of conical Ricci expanders with positive curvature operator [13].

In the course of the proof of the a priori estimates, we are able to partially answer questions (9) and (11) asked in [9, Section 7.2, Chapter 9]: see Proposition 3.10 concerning a lower bound on the rescaled scalar curvature and Proposition 3.5 for Ricci pinched metrics.

We give a brief outline of the organization of the paper.

In Section 2, we first recall and (re)-prove curvature identities on expanding gradient Ricci solitons, then we focus on precise estimates of the potential function (Proposition 2.4). Secondly, we (re)-prove local estimates in the spirit of Shi both for the curvature tensor (Lemma 2.6) and general solutions of weighted elliptic equations (Lemma 2.8). Finally, we prove a priori estimates for subsolutions of weighted elliptic equations: Lemma 2.9, though abstract, is at the core of most results of this paper.

Section 3 is devoted to the proof of asymptotic estimates on conical expanding gradient Ricci solitons: we treat separately the generic (Theorem 1.2) and the asymptotically Ricci flat case (Theorem 1.3). Then we prove Proposition 3.5 concerning the rigidity of Ricci pinched metrics. Then we focus on Einstein constraints at infinity: Theorems 3.6 and 3.8. Subsection 3.4 focuses on a priori lower bounds on the curvature operator in case the asymptotic cone is positively curved.

Finally, in Section 4, we first prove general global a priori estimates: Propositions 1.9 and 4.3, Corollaries 4.1 and 4.2. This leads us directly to the proof of the two compactness theorems: Theorems 4.6 and 1.7.

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2. Soliton equations and rough estimates

2.1. Algebraic identities on Ricci expanders

We start with a couple of definitions:

- Let (M, g) be a Riemannian manifold and let $w : M \to \mathbb{R}$ be a smooth function on M. The *weighted Laplacian* (with respect to w) is
$$\Delta_w T := \Delta T + \nabla_{\nabla w} T,$$
where T is a tensor on M. This weighted Laplacian is symmetric with respect to the weighted measure $e^w d\mu(g)$. We emphasize the fact that this is not the usual convention for the weighted Laplacian: such a choice is made to keep the potential function f of an expander positive;
- An expanding gradient Ricci soliton $(M,g,\nabla f)$ is said *normalized* if $\int_M e^{-f} d\mu_g = (4\pi)^{n/2}$ (whenever it makes sense).

The next lemma gathers well-known Ricci soliton identities together with the (static) evolution equations satisfied by the curvature tensor.

Lemma 2.1. *Let $(M^n, g, \nabla f)$ be a normalized expanding gradient Ricci soliton. Then the trace and first order soliton identities are:*

$$\Delta f = \mathrm{R}_g + \frac{n}{2}, \tag{2.1}$$
$$\nabla \mathrm{R}_g + 2\,\mathrm{Ric}(g)(\nabla f) = 0, \tag{2.2}$$
$$|\nabla f|^2 + \mathrm{R}_g = f + \mu(g), \tag{2.3}$$
$$\mathrm{div}\,\mathrm{Rm}(g)(Y, Z, T) = \mathrm{Rm}(g)(Y, Z, \nabla f, T), \tag{2.4}$$

for any vector fields Y, Z, T and where $\mu(g)$ is a constant called the entropy.

The evolution equations for the curvature operator, the Ricci tensor and the scalar curvature are:

$$\Delta_f \mathrm{Rm}(g) + \mathrm{Rm}(g) + \mathrm{Rm}(g) * \mathrm{Rm}(g) = 0, \tag{2.5}$$
$$\Delta_f \mathrm{Ric}(g) + \mathrm{Ric}(g) + 2\,\mathrm{Rm}(g) * \mathrm{Ric}(g) = 0, \tag{2.6}$$
$$\Delta_f \mathrm{R}_g + \mathrm{R}_g + 2|\mathrm{Ric}(g)|^2 = 0, \tag{2.7}$$

where, if A and B are two tensors, $A * B$ denotes any linear combination of contractions of the tensorial product of A and B.

Proof. See [6, Chapter1] for instance. □

The next lemma is valid for any gradient Ricci soliton and only uses the second Bianchi identity:

Lemma 2.2. *Let* $(M^n, g, \nabla f)$ *be a gradient Ricci soliton. Then*

$$\begin{aligned}\nabla_{\nabla f} \operatorname{Rm}(g)(W, X, Y, Z) = {} & \nabla_W \operatorname{div} \operatorname{Rm}(g)(Y, Z, X) - \nabla_X \operatorname{div} \operatorname{Rm}(g)(Y, Z, W) \\ & + \operatorname{Rm}(g)(X, \nabla^2 f(W), Y, Z) \\ & - \operatorname{Rm}(g)(W, \nabla^2 f(X), Y, Z).\end{aligned}$$

Proof. By the second Bianchi identity,

$$\nabla_{\nabla f} \operatorname{Rm}(g)(W,X,Y,Z) = -\nabla_W \operatorname{Rm}(g)(X, \nabla f, Y, Z) - \nabla_X \operatorname{Rm}(g)(\nabla f, W, Y, Z).$$

Now,

$$\begin{aligned}\nabla_W \operatorname{Rm}(g)(X, \nabla f, Y, Z) = {} & W \cdot \operatorname{Rm}(g)(X, \nabla f, Y, Z) - \operatorname{Rm}(g)(\nabla_W X, \nabla f, Y, Z) \\ & - \operatorname{Rm}(g)(X, \nabla^2 f(W), Y, Z) \\ & - \operatorname{Rm}(g)(X, \nabla f, \nabla_W Y, Z) \\ & - \operatorname{Rm}(g)(X, \nabla f, Y, \nabla_W Z) \\ = {} & -\nabla_W \operatorname{div} \operatorname{Rm}(g)(Y, Z, X) \\ & - \operatorname{Rm}(g)(X, \nabla^2 f(W), Y, Z). \qquad \square\end{aligned}$$

We pursue this section by computing the evolution of the norm of the covariant derivatives of the curvature along the Morse flow generated by the potential function in terms of the covariant derivatives of the Ricci curvature.

Proposition 2.3. *Let* $(M^n, g, \nabla f)$ *be an expanding gradient Ricci soliton.*

(1) *Then,*

$$\begin{aligned}& \nabla_{\nabla f} |\operatorname{Rm}(g)|^2 + 4\nabla^2 f(\operatorname{Rm}(g), \operatorname{Rm}(g)) \\ & = -4 \operatorname{tr}\left(\star \to \langle \nabla_\star \operatorname{div} \operatorname{Rm}(g)(\cdot, \cdot, \cdot), \operatorname{Rm}(g)(\cdot, \cdot, \cdot, \star)\rangle\right) \\ & = -4\nabla_i \operatorname{div} \operatorname{Rm}(g)_{abc} \operatorname{Rm}(g)_{abci},\end{aligned}$$

where $\nabla^2 f(\operatorname{Rm}(g), \operatorname{Rm}(g)) := \langle \operatorname{Rm}(g)(\cdot, \nabla^2 f(\cdot), \cdot, \cdot), \operatorname{Rm}(g)(\cdot, \cdot, \cdot, \cdot)\rangle$*;*

(2) *(Commutation identities) For any tensor* T*, and any positive integer* k*,*

$$\left[\nabla_{\nabla f}, \nabla^k\right] T = -\frac{k}{2} \nabla^k T + \sum_{i=0}^{k} \nabla^{k-i} \operatorname{Ric}(g) * \nabla^i T, \tag{2.8}$$

$$\left[\Delta, \nabla^k\right] T = \nabla^k T * \operatorname{Ric}(g) + \sum_{i=0}^{k-1} \nabla^{k-i} \operatorname{Rm}(g) * \nabla^i T. \tag{2.9}$$

(3) *For any integer $k \geq 1$,*

$$\begin{aligned}
&\nabla_{\nabla f}|\nabla^k \operatorname{Rm}(g)|^2 + (2+k)|\nabla^k \operatorname{Rm}(g)|^2\\
&= \nabla^{k+2}\operatorname{Ric}(g) * \nabla^k \operatorname{Rm}(g) + \operatorname{Ric}(g) * \nabla^k \operatorname{Rm}(g)^{*2}\\
&\quad + \sum_{i=1}^{k} \nabla^i \operatorname{Ric}(g) * \nabla^{k-i}\operatorname{Rm}(g) * \nabla^k \operatorname{Rm}(g).
\end{aligned}$$

Proof. By Lemma 2.2,

$$\begin{aligned}
\nabla_{\nabla f}|\operatorname{Rm}(g)|^2 &= 2\nabla_{\nabla f}\operatorname{Rm}(g)_{ijkl}\operatorname{Rm}(g)_{ijkl}\\
&= 2\big(\nabla_i \operatorname{div}\operatorname{Rm}(g)_{klj} - \nabla_j \operatorname{div}\operatorname{Rm}(g)_{kli}\\
&\quad + \operatorname{Rm}(g)_{j\nabla^2 f(i)kl} - \operatorname{Rm}(g)_{i\nabla^2 f(j)kl}\big)\operatorname{Rm}(g)_{ijkl}\\
&= -4\nabla_i \operatorname{div}\operatorname{Rm}(g)_{jkl}\operatorname{Rm}(g)_{jkli} - 4\operatorname{Rm}(g)_{i\nabla^2 f(j)kl}\operatorname{Rm}(g)_{ijkl}.
\end{aligned}$$

The commutation identities can be proved by induction on k.

Concerning the third identity, according to (2.8) and Proposition 2.3,

$$\begin{aligned}
\nabla_{\nabla f}|\nabla^k \operatorname{Rm}(g)|^2 &= 2\Big\langle \nabla^k \nabla_{\nabla f}\operatorname{Rm}(g), \nabla^k \operatorname{Rm}(g)\Big\rangle\\
&\quad +2\Big\langle \big[\nabla_{\nabla f}, \nabla^k\big]\operatorname{Rm}(g), \nabla^k \operatorname{Rm}(g)\Big\rangle\\
&= -(2+k)|\nabla^k \operatorname{Rm}(g)|^2 + \nabla^{k+2}\operatorname{Ric}(g) * \nabla^k \operatorname{Rm}(g)\\
&\quad + \operatorname{Ric}(g) * \nabla^k \operatorname{Rm}(g)^{*2}\\
&\quad + \sum_{i=1}^{k} \nabla^{k-i}\operatorname{Ric}(g) * \nabla^i \operatorname{Rm}(g) * \nabla^k \operatorname{Rm}(g).
\end{aligned}$$ □

For an expanding gradient Ricci soliton $(M^n, g, \nabla f)$, we define $v : M^n \to \mathbb{R}$ by

$$v(x) := f(x) + \mu(g) + n/2,$$

for $x \in M^n$. Then, v enjoys the following properties.

Proposition 2.4. *Let $(M^n, g, \nabla f)$ be a non-Einstein expanding gradient Ricci soliton. Then,*

$$\Delta_f v = v, \tag{2.10}$$

$$v > |\nabla v|^2. \tag{2.11}$$

Assume $\mathrm{Ric}(g) \geq 0$ *and assume* $(M^n, g, \nabla f)$ *is normalized. Then* M^n *is diffeomorphic to* $\mathbb{R}^n$ *and*

$$v \geq \frac{n}{2} > 0. \tag{2.12}$$

$$\frac{1}{4} r_p(x)^2 + \min_{M^n} v \leq v(x) \leq \left(\frac{1}{2} r_p(x) + \sqrt{\min_{M^n} v} \right)^2, \quad \forall x \in M^n, \tag{2.13}$$

$$\mathrm{AVR}(g) := \lim_{r \to +\infty} \frac{\mathrm{Vol}\, B(q,r)}{r^n} > 0, \quad \forall q \in M^n, \tag{2.14}$$

$$-C(n, V_0, R_0) \leq \min_{M^n} f \leq 0 \quad ; \quad \mu(g) \geq \max_{M^n} \mathrm{R}_g \geq 0, \tag{2.15}$$

where V_0 *is a positive number such that* $\mathrm{AVR}(g) \geq V_0$*,* R_0 *is such that* $\sup_{M^n} \mathrm{R}_g \leq R_0$ *and* $p \in M^n$ *is the unique critical point of* v*.*

Remark 2.5.

- (2.14) is due to Hamilton: [9, Chapter 9].
- In [10] it is proved that $\mu(g) \geq 0$ with equality if and only if it is isometric to the Euclidean space by studying the linear entropy along the heat kernel. Their argument works for Riemannian manifolds with nonnegative Ricci curvature. We give here a simpler but maybe less enlightening proof in the setting of gradient Ricci expanders.

Proof. (2.10) comes from adding equations (2.1) and (2.3). (2.11) comes from the maximum principle at infinity established in [19] ensuring that for a nontrivial expanding gradient Ricci soliton, one has $\Delta v = \Delta f > 0$.

Now, if $\mathrm{Ric}(g) \geq 0$, the soliton equation ensures that the potential function f (or v) satisfies $\nabla^2 f \geq (g/2)$ which implies that v is a proper strictly convex function. By Morse-type arguments, one can show that M is diffeomorphic to $\mathbb{R}^n$ and the lower bound of (2.13) is achieved by integrating the previous differential inequality involving the Hessian of v. The upper bound of (2.13) holds more generally since it comes from (2.11).

Now, for the convenience of the reader, we reprove briefly (2.14) with the help of the sublevel set of the potential function. Indeed, with the help of the coarea formula, the rescaled volume of the level sets $t \to \mathrm{Vol}\{f = t\} t^{-(n-1)/2}$ is non decreasing for t larger than $\min_{M^n} f$. By the coarea formula again, it implies directly that the rescaled volume of the sublevel sets $t \to \mathrm{Vol}\{f \leq t\} t^{-n/2}$ is bounded from below by a positive constant. As the potential function and the distance function to the square are comparable by (2.13), the volume ratio $r^{-n}\, \mathrm{Vol}\, B(p, r)$ is bounded from below by a positive constant for large radii r hence for any radius by the Bishop-Gromov theorem.

Finally, we show first that the scalar curvature is bounded. Indeed, if $(\phi_t)_t$ is the Morse flow associated to f, then

$$\partial_t \mathrm{R}_g = -2\, \mathrm{Ric}(g) \left(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|} \right) \leq 0.$$

In particular, it means that $t \to \max_{f=t} \mathrm{R}_g$ is non increasing and therefore, by continuity of the scalar curvature, we get

$$\sup_{M^n} \mathrm{R}_g \leq \max_{f=\min_{M^n} f} \mathrm{R}_g = \mathrm{R}_g(p),$$

where $\mathrm{Crit}(f) = \{p\}$.

Now, as $\nabla^2 f \geq g/2$,

$$\begin{aligned} f(x) &\leq \frac{r_p(x)^2}{4} + \min_{M^n} f + \sqrt{\min_{M^n} f + \mu(g)}\frac{r_p(x)}{2} \\ &\leq \frac{r_p(x)^2}{4} + \min_{M^n} f + \sqrt{\max_{M^n} \mathrm{R}_g}\frac{r_p(x)}{2}, \end{aligned}$$

for any $x \in M^n$ since $\min_{M^n} f + \mu(g) = \max_{M^n} \mathrm{R}_g$. As f is normalized, we have

$$\begin{aligned} (4\pi)^{n/2} &= \int_{M^n} e^{-f} d\mu(g) \geq \left(nV_0 e^{-\min_{M^n} f}\right) \int_0^{+\infty} e^{-r^2/4 - \sqrt{\max_{M^n} \mathrm{R}_g} r} r^{n-1} dr \\ &\geq e^{-\min_{M^n} f} C(n, V_0, R_0), \end{aligned}$$

for some positive constant $C(n, V_0, R_0)$. Hence a lower bound on $\min_{M^n} f$. By using a lower bound on f of the form

$$\min_{M^n} f + \frac{r_p(x)^2}{4} \leq f(x),$$

for any $x \in M^n$, one gets by a similar argument using the Bishop-Gromov theorem that $\min_{M^n} f \leq 0$ since

$$\int_{\mathbb{R}^n} e^{-\|x\|^2/4} d\mu(\mathrm{eucl})(x) = (4\pi)^{n/2}.$$

As a consequence of the previous estimates, we get for free that $\mu(g) \geq \max_{M^n} \mathrm{R}_g \geq 0$ without using [10] and also an upper bound for $\mu(g)$ as expected: $\mu(g) \leq C(n, V_0, R_0)$. Therefore, if $\mu(g) = 0$, one has $\mathrm{R}_g = 0$ which implies that $\mathrm{Ric}(g) = 0$ by equation (2.7). [21] shows then that $(M^n, g, \nabla f)$ is isometric to the Gaussian soliton $(\mathbb{R}^n, \mathrm{eucl}, r\partial_r/2)$. Indeed, the potential function satisfies $\nabla^2 f = g/2$ which means that $f(x) = \min_{M^n} f + r_p^2(x)/4$ for any $x \in M^n$, where p is the only critical point of f. In particular, it means that r_p^2 is smooth on M^n and that $\Delta r_p^2 = 2n$ on a Riemannian manifold with nonnegative Ricci curvature: the (infinitesimal) Bishop-Gromov theorem then implies that (M^n, g) is Euclidean. □

2.2. Local Shi's estimates

We derive now some baby Shi's estimates for expanding gradient Ricci solitons for large radii. The main thing here is to estimate the influence of the supremum of the curvature operator.

Lemma 2.6. *If $(M^n, g, \nabla f)$ is an expanding gradient Ricci soliton such that $\mathrm{Ric}(g) \geq -(n-1)K$ on $B(p,r)$ with $K \geq 0$, then, for any $k \geq 0$ and $r \geq 1$,*

$$\sup_{B(p,r/2)} |\nabla^k \mathrm{Rm}(g)| \leq C\left(n, \sup_{A(p,r/2,r)} \frac{|\nabla f|}{r}, k, Kr^2\right) \sup_{B(p,r)} |\mathrm{Rm}(g)| \left(1 + \sup_{B(p,r)} |\mathrm{Rm}(g)|^{k/2}\right).$$

Remark 2.7. Lemmata 2.6 and 2.8 below do not use the expanding structure in an essential way, therefore, these results still hold for shrinkers and steady solitons as well.

Proof. Along the proof, c is a constant which only depends on n but can vary from a line to another. We only give the proof for $k = 1$.

Using the commutation identities of Proposition 2.3, one can prove that

$$\Delta_f |\nabla \mathrm{Rm}(g)|^2 \geq 2|\nabla^2 \mathrm{Rm}(g)|^2 - \left(\frac{3}{2} + c(n)|\mathrm{Rm}(g)|\right) |\nabla \mathrm{Rm}(g)|^2.$$

The coefficient $3/2$ has no importance in this proof.

Following [22] and [5], define $U := |\mathrm{Rm}(g)|^2$, $V := |\nabla \mathrm{Rm}(g)|^2$ and $W := |\nabla^2 \mathrm{Rm}(g)|^2$. Consider the function $F := (U + a)V$ where a is a positive constant to be defined later. F satisfies,

$$\begin{aligned}
\Delta_f F &= V(\Delta_f U) + (U + a)(\Delta_f V) + 2\langle \nabla U, \nabla V\rangle \\
&\geq V(2V - (c|\mathrm{Rm}(g)| + 2)U) + (U + a)(2W - (c|\mathrm{Rm}(g)| + 2)W) \\
&\quad - 8U^{1/2} V W^{1/2} \\
&\geq 2V^2 + 2(U + a)W - 16UW - V^2 - (c|\mathrm{Rm}(g)| + 4)F \\
&= V^2 + 2W(a - 7U) - (c|\mathrm{Rm}(g)| + 4)F,
\end{aligned}$$

where, in the third inequality, we use $2ab \leq a^2/\alpha + \alpha b^2$, for positive numbers a and b with $\alpha = 4$. If $a := 7 \sup_{B(p,r)} U$, one gets on $B(p,r)$,

$$\Delta_f F \geq V^2 - (c|\mathrm{Rm}(g)| + 4)F. \tag{2.16}$$

Let $\phi : M^n \to [0,1]$ be a smooth positive function with compact support defined by $\phi(x) := \psi(r_p(x)/r)$ where $r > 0$ and $\psi : [0, +\infty[\to [0,1]$ is a smooth positive function satisfying

$$\psi|_{[0,1/2]} \equiv 1, \quad \psi|_{[1,+\infty[} \equiv 0, \quad \psi' \leq 0, \quad \frac{\psi'^2}{\psi} \leq c, \quad \psi'' \geq -c.$$

Let $G := \phi F$. Then G satisfies

$$\Delta_f G = \phi(\Delta_f F) + 2\langle \nabla\phi, \nabla F\rangle + F(\Delta_f \phi).$$

At a point where G attains its maximum,

$$\nabla G = F\nabla\phi + \phi\nabla F = 0, \quad \Delta G \le 0.$$

Multiplying (2.16) by ϕ, we get at a point where G attains its maximum,

$$0 \ge (\phi V)^2 - \phi G(c|\operatorname{Rm}(g)| + 4) - 2G\frac{|\nabla\phi|^2}{\phi} + G(\Delta\phi + \langle\nabla f, \nabla\phi\rangle).$$

Now, in any case, $a\phi V \le G \le 2a\phi V$. Therefore, if $a > 0$,

$$0 \ge \left(\frac{G}{8a}\right)^2 + G\left[-\phi(c|\operatorname{Rm}(g)| + 4) - 2\frac{|\nabla\phi|^2}{\phi} + \Delta_f\phi\right].$$

So that if G does not vanish identically on $B(p, r)$,

$$\phi V \le ca\left[\phi(c|\operatorname{Rm}(g)| + 4) + 2\frac{|\nabla\phi|^2}{\phi} - \Delta_f\phi\right].$$

Now,

$$\nabla\phi = \frac{\psi'}{r}\nabla r_p, \quad \Delta\phi = \frac{\psi''}{r^2} + \frac{\psi'}{r}\Delta r_p.$$

Hence,

$$2\frac{|\nabla\phi|^2}{\phi} - \Delta_f\phi = \frac{1}{r^2}\left[\frac{2\psi'^2}{\psi} - \psi''\right] - \frac{\psi'}{r}(\langle\nabla f, \nabla r_p\rangle + \Delta r_p).$$

On the other hand, $\operatorname{Ric}(g) \ge -(n-1)K$ on $B(p, r)$, with $K \ge 0$, by the comparison theorem,

$$\begin{aligned}\Delta r_p &\le (n-1)K^{1/2}\coth(K^{1/2}r_p)\\ &\le \frac{n-1}{r_p}(1 + K^{1/2}r_p) \le \frac{2(n-1)}{r}(1 + K^{1/2}r) \quad \text{on } B(p, r)\backslash B(p, r/2).\end{aligned}$$

Therefore, by the very definition of ψ,

$$\begin{aligned}&\sup_{B(p,r/2)} |\nabla\operatorname{Rm}(g)|\\ &\le c\sup_{B(p,r)}|\operatorname{Rm}(g)|\left[\sup_{B(p,r)}|\operatorname{Rm}(g)| + 1 + \frac{\sup_{A(p,r/2,r)}|\nabla f|}{r} + \frac{1}{r^2}(1 + K^{1/2}r)\right]^{1/2}.\end{aligned}$$

□

Finally, we consider local Shi's estimates for solutions of the weighted Laplacian with a potential depending on the curvature.

Lemma 2.8 (Local covariant derivatives estimates). *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton such that* $\mathrm{Ric}(g) \geq -(n-1)K$ *on* $B(p, r)$ *with* $K \geq 0$*. Let T be a C^2 tensor satisfying*

$$\Delta_f T = -\lambda T + \mathrm{Rm}(g) * T, \tag{2.17}$$

for some $\lambda \in \mathbb{R}$. Then, for any $k \geq 0$ and $r \geq 1$,

$$\sup_{B(p,r/2)} |\nabla^k T| \leq C\left(n, \lambda, \sup_{A(p,r/2,r)} \frac{|\nabla f|}{r}, k, Kr^2\right) \sup_{B(p,r)} |T|(1 + \sup_{B(p,r)} |\mathrm{Rm}(g)|^{k/2})$$

Proof. The proof is essentially the same as for Lemma 2.6. We only give the major steps for $k = 1$ and $k = 2$.

By the evolution equation for T given by (2.17), we estimate,

$$\begin{aligned}
\Delta_f |T|^2 &\geq 2|\nabla T|^2 - (c(n)|\mathrm{Rm}(g)| + 2\lambda)|T|^2. \\
\Delta_f |\nabla T|^2 &\geq 2|\nabla^2 T|^2 - (c(n)(|\mathrm{Rm}(g)| + |\nabla \mathrm{Rm}(g)|^{2/3}) + 2\lambda + 1)|\nabla T|^2 \\
&\quad - c|\nabla \mathrm{Rm}(g)|^{4/3}|T|^2.
\end{aligned}$$

Adapting the proof of Lemma 2.6, with the same notation, one gets

$$\Delta_f F \geq V^2 - c(\lambda, n)(1 + (|\mathrm{Rm}(g)| + |\nabla \mathrm{Rm}(g)|^{2/3}))F - c(n)|\nabla \mathrm{Rm}(g)|^{4/3} a^2.$$

We conclude following the same procedure and by invoking the results of Lemma 2.6.

Concerning the estimates of $\nabla^2 T$:

$$\begin{aligned}
\Delta_f |\nabla^2 T|^2 &\geq 2|\nabla^3 T|^2 - c(n, \lambda)|\nabla^2 T|^2(1 + |\mathrm{Rm}(g)|) \\
&\quad - c(n, \lambda)|\nabla^2 T|\left(|\nabla T||\nabla \mathrm{Rm}(g)| + |\nabla^2 \mathrm{Rm}(g)||T|\right) \\
&\geq 2|\nabla^3 T|^2 - \psi_1 |\nabla^2 T|^2 - \psi_2,
\end{aligned}$$

where

$$\begin{aligned}
\psi_1 &:= c(n, \lambda)(1 + |\mathrm{Rm}(g)| + |\nabla \mathrm{Rm}(g)|^{2/3} + |\nabla^2 \mathrm{Rm}(g)|^{1/2}) \\
\psi_2 &:= c(n, \lambda)(|\nabla T|^2 |\nabla \mathrm{Rm}(g)|^{4/3} + |\nabla^2 \mathrm{Rm}(g)|^{3/2}|T|^2).
\end{aligned}$$

Now, define $U := |\nabla T|^2$, $V := |\nabla^2 T|^2$ and $W := |\nabla^3 T|^2$. Consider the function $F := (U + a)V$ where $a := c \sup_{B(p,r)} U$. F satisfies,

$$\Delta_f F \geq V^2 - \left(2\psi_1 + \frac{|\nabla \mathrm{Rm}(g)|^{4/3}|T|^2}{a}\right) F - 2a\psi_2.$$

Therefore, as in the proof of Lemma 2.6, we obtain:

$$\sup_{B(p,r/2)} V \leq c \sup_{B(p,r)} \left[a\psi_1 + |\nabla \mathrm{Rm}(g)|^{4/3}|T|^2 + (a\psi_2)^{1/2}\right]. \qquad \square$$

2.3. A priori estimates for subsolutions of the weighted Laplacian

The purpose of this section is to prove the following lemma that is crucial for the rest of the paper.

Lemma 2.9. *Let $(M^n, g, \nabla f)$ be a normalized expanding gradient Ricci soliton such that f is an exhausting function. Assume $u : M^n \to \mathbb{R}_+$ is a C^2 function that is a subsolution of the following weighted elliptic equation:*

$$\Delta_f u \geq -\lambda u - c_1 v^{-\alpha} u - Q, \tag{2.18}$$

where $\lambda \in \mathbb{R}$, $c_1 \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+^*$ and $Q : M^n \to \mathbb{R}$ is a nonneg ative function. Define $\tilde{\alpha} := \min\{1, \alpha\}$. Assume $\sup_{M^n} f^{\tilde{\alpha}} |\mathrm{R}_g|$ is finite.

(1) *If $Q \equiv 0$, then there exists some positive height $t_0 = t_0(n, \lambda, \alpha, c_1, \min_{M^n} f, \sup_{M^n} f^{\tilde{\alpha}} |\mathrm{R}_g|)$ and some positive constants $C_i = C_i(n, \lambda, \alpha, c_1, \min_{M^n} f, \sup_{M^n} f^{\tilde{\alpha}} |\mathrm{R}_g|)$ for $i = 0, 1$ such that for $t \geq t_0$, the function*

$$v^{\lambda} e^{-C_0 v^{-\tilde{\alpha}}} u - A_1 v^{2\lambda - \frac{n}{2}} e^{-v - C_1 v^{-\tilde{\alpha}}} : \{t_0 \leq f \leq t\} \to \mathbb{R},$$

attains its maximum on the boundary $\{f = t_0\} \cup \{f = t\}$ for any nonnegative constant A_1;

(2) *If $Q = O(v^{-\beta})$, for some real number β such that $\lambda < \beta$ then there exists some positive height $t_0 = t_0(n, \lambda, \alpha, c_1, \beta, \sup_{M^n} f^{\tilde{\alpha}} |\mathrm{R}_g|, \min_{M^n} f)$ and some positive constants $C_i = C_i(n, \lambda, \alpha, c_1, \min_{M^n} f, \sup_{M^n} f^{\tilde{\alpha}} |\mathrm{R}_g|)$ for $i = 0, 1$ such that for $t \geq t_0$, the function*

$$v^{\lambda} e^{-C_0 v^{-\tilde{\alpha}}} u - A_0 v^{\lambda - \beta} - A_1 v^{2\lambda - \frac{n}{2}} e^{-v - C_1 v^{-\tilde{\alpha}}} : \{t_0 \leq f \leq t\} \to \mathbb{R},$$

attains its maximum on the boundary $\{f = t_0\} \cup \{f = t\}$ for any nonnegative constant A_1, and $A_0 \geq A_0(n, \lambda, c_1, \alpha, \beta, \sup_{M^n}(v^{\beta} Q), \min_{M^n} f)$;

(3) *If $Q = O(v^{\beta} e^{-v})$, for some real number β such that $\lambda > \beta + n/2$, then there exists some positive height $t_0 = t_0(n, \lambda, \alpha, c_1, \beta, \sup_{M^n} f^{\tilde{\alpha}} |\mathrm{R}_g|, \min_{M^n} f)$ and some positive constants $C_i = C_i(n, \lambda, \alpha, c_1, \min_{M^n} f, \sup_{M^n} f^{\tilde{\alpha}} |\mathrm{R}_g|)$ for $i = 0, 1$ such that for $t \geq t_0$, the function*

$$v^{\lambda} e^{-C_0 v^{-\tilde{\alpha}}} u + A_0 v^{\beta + \lambda} e^{-v} - A_1 v^{2\lambda - \frac{n}{2}} e^{-v - C_1 v^{-\tilde{\alpha}}} : \{t_0 \leq f \leq t\} \to \mathbb{R},$$

attains its maximum on the boundary $\{f = t_0\} \cup \{f = t\}$ for any nonnegative constant A_1 and $A_0 \geq A_0(n, \lambda, c_1, \alpha, \beta, \sup_{M^n}(v^{-\beta} e^{v} Q), \min_{M^n} f)$.

Proof. We first absorb the linear term on the right-hand side of (2.18):

$$\Delta_f(v^{\lambda} u) \geq -\left(\lambda(\lambda+1)|\nabla \ln v|^2 + c_1 v^{-\alpha}\right)(v^{\lambda} u) - v^{\lambda} Q + 2\lambda \langle \nabla \ln v, \nabla(v^{\lambda} u)\rangle,$$

i.e.

$$\begin{aligned}\Delta_{v-2\lambda\ln v}(v^\lambda u) &\geq -\left(\lambda(\lambda+1)|\nabla \ln v|^2 + c_1 v^{-\alpha}\right)(v^\lambda u) - v^\lambda Q\\ &\geq -\frac{C(\lambda, c_1, \min_{M^n} v)}{v^{\min\{\alpha,1\}}}(v^\lambda u) - v^\lambda Q.\end{aligned}$$

Then, if $\tilde{\alpha} := \min\{\alpha, 1\}$, multiply the previous differential inequality by a function of the form $e^{-Cv^{-\tilde{\alpha}}}$ to get,

$$\begin{aligned}&\Delta_{v-2\lambda\ln v+2Cv^{-\tilde{\alpha}}}\left(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u\right)\\ &\geq \Delta_{v-2\lambda\ln v}\left(e^{-Cv^{-\tilde{\alpha}}}\right)\cdot(v^\lambda u) - 2C^2|\nabla v^{-\tilde{\alpha}}|^2(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u)\\ &\quad -\frac{C(\lambda, c_1, \min_{M^n} v)}{v^{\tilde{\alpha}}}\left(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u\right) - v^\lambda e^{-Cv^{-\tilde{\alpha}}}Q.\end{aligned}$$

Now,

$$\begin{aligned}\Delta_{v-2\lambda\ln v}\left(e^{-Cv^{-\tilde{\alpha}}}\right) &= \left(-C\Delta_{v-2\lambda\ln v}v^{-\tilde{\alpha}} + C^2|\nabla v^{-\tilde{\alpha}}|^2\right)e^{-Cv^{-\tilde{\alpha}}}\\ &= \frac{C}{v^{\tilde{\alpha}}}\left(\tilde{\alpha}(1+[\tilde{\alpha}+1-2\lambda+C\tilde{\alpha}v^{-\tilde{\alpha}}]|\nabla\ln v|^2)\right)e^{-Cv^{-\tilde{\alpha}}}\\ &\geq \frac{C\tilde{\alpha}}{2v^{\tilde{\alpha}}}e^{-Cv^{-\tilde{\alpha}}},\end{aligned}$$

for $v \geq v_0(C, \min_{M^n} v, \alpha)$. Therefore,

$$\Delta_{v-2\lambda\ln v+2Cv^{-\tilde{\alpha}}}\left(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u\right) \geq \left(\frac{C\tilde{\alpha}}{2} - C(\lambda, c_1, \min_{M^n} v)\right)v^{-\tilde{\alpha}}\left(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u\right) \tag{2.19}$$

$$-2C^2|\nabla v^{-\tilde{\alpha}}|^2(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u) - v^\lambda e^{-Cv^{-\tilde{\alpha}}}Q, \tag{2.20}$$

$$\Delta_{v-2\lambda\ln v+2Cv^{-\tilde{\alpha}}}\left(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u\right) > -v^\lambda e^{-Cv^{-\tilde{\alpha}}}Q, \tag{2.21}$$

for $C = C(\lambda, c_1, \alpha, \min_{M^n} v)$ and $v \geq v_0(\lambda, c_1, \min_{M^n} v, \alpha)$.

(1) If $Q \equiv 0$, then

$$\Delta_{v-2\lambda\ln v+2Cv^{-\tilde{\alpha}}}\left(v^\lambda e^{-Cv^{-\tilde{\alpha}}}u\right) > 0, \tag{2.22}$$

for $C = C(\lambda, c_1, \alpha, \min_{M^n} v)$ and $v \geq v_0(\lambda, c_1, \min_{M^n} v, \alpha)$. Now, we compute the weighted Laplacian of a function of the form $v^\gamma e^{-v}$ with $\gamma \in \mathbb{R}$:

$$\begin{aligned}
\Delta_{v-2\lambda \ln v} v^\gamma &= \left(\gamma v^{\gamma-1}\Delta_{v-2\lambda\ln v} v + \gamma(\gamma-1)|\nabla \ln v|^2 v^\gamma\right) \\
&= \gamma v^\gamma \left(1 + (\gamma - 1 - 2\lambda)|\nabla \ln v|^2\right), \\
\Delta_{v-2\lambda \ln v} e^{-v} &= (-\Delta_{v-2\lambda\ln v} v + |\nabla v|^2)e^{-v} = \left(-\Delta v + 2\lambda \frac{|\nabla v|^2}{v}\right) e^{-v} \\
&= \left(-\mathrm{R}_g - \lambda \frac{2\,\mathrm{R}_g + n}{v} + 2\lambda - \frac{n}{2}\right) e^{-v}, \\
\Delta_{v-2\lambda \ln v}(v^\gamma e^{-v}) &= \gamma v^\gamma e^{-v}\left(1 + (\gamma - 1 - 2\lambda)|\nabla \ln v|^2\right) - 2\gamma v^\gamma e^{-v}\frac{|\nabla v|^2}{v} \\
&\quad + \left(-\mathrm{R}_g - \lambda \frac{2\,\mathrm{R}_g + n}{v} + 2\lambda - \frac{n}{2}\right) v^\gamma e^{-v} \\
&= \left(2\lambda - \frac{n}{2} - \gamma - \mathrm{R}_g + (\gamma - \lambda)\frac{2\,\mathrm{R}_g + n}{v}\right. \\
&\quad \left. + \gamma(\gamma - 1 - 2\lambda)|\nabla \ln v|^2\right) v^\gamma e^{-v}, \\
\langle \nabla v^{-\tilde{\alpha}}, \nabla v^\gamma e^{-v}\rangle &\leq \frac{C(\gamma, \min_{M^n} v)}{v^{\tilde{\alpha}}} v^\gamma e^{-v}.
\end{aligned}$$

Therefore, if $\gamma := 2\lambda - n/2$, then,

$$\Delta_{v-2\lambda\ln v + 2Cv^{-\tilde{\alpha}}}(v^{2\lambda - n/2}e^{-v}) \leq \frac{C_1}{v^{\tilde{\alpha}}} v^{2\lambda - n/2} e^{-v}, \tag{2.23}$$

for some positive constant $C_1 = C_1(C, \sup_{M^n} v^{\tilde{\alpha}}|\,\mathrm{R}_g\,|, n, \min_{M^n} f)$. Now,

$$\Delta_{v-2\lambda\ln v + 2Cv^{-\tilde{\alpha}}}(v^{2\lambda - n/2}e^{-v}e^{-2C_2 v^{-\tilde{\alpha}}}) \leq \left(\frac{C_1 - C_2}{v^{\tilde{\alpha}}}\right) v^{2\lambda - n/2}e^{-v}e^{-2C_2 v^{-\tilde{\alpha}}},$$

for any positive constant C_2, for $v \geq v_0(C_2, C, n, \min_{M^n} v)$ since,

$$\begin{aligned}
\Delta_{v-2\lambda\ln v + 2Cv^{-\tilde{\alpha}}}\left(e^{-2C_2 v^{-\tilde{\alpha}}}\right) &\leq \frac{C_2\tilde{\alpha}}{v^{\tilde{\alpha}}} e^{-2C_2 v^{-\tilde{\alpha}}}, \\
2\left\langle \nabla(v^{2\lambda - n/2}e^{-v}), \nabla e^{-2C_2 v^{-\tilde{\alpha}}}\right\rangle &\leq -\frac{2C_2\tilde{\alpha}}{v^{\tilde{\alpha}}} v^{2\lambda - n/2}e^{-v}e^{-2C_2 v^{-\tilde{\alpha}}}.
\end{aligned}$$

Choose $C_2 = C_1/\tilde{\alpha}$ such that

$$\Delta_{v-2\lambda\ln v + 2Cv^{-\tilde{\alpha}}}(v^{2\lambda - n/2}e^{-v}e^{-2C_1 v^{-\tilde{\alpha}}}) \leq 0, \tag{2.24}$$

for $v \geq v_0(C, \sup_{M^n} v^{\tilde{\alpha}}|\mathrm{R}_g|)$. Combining the two differential inequalities (2.22) and (2.24), one gets for any nonnegative constant A_1, for $v \geq v_0(n, \lambda, c_1, \min_{M^n} v, \alpha, \sup_{M^n} v^{\tilde{\alpha}}|\mathrm{R}_g|)$,

$$\Delta_{v-2\lambda \ln v+2Cv^{-\tilde{\alpha}}}\left(v^{\lambda}e^{-Cv^{-\tilde{\alpha}}}u - A_1 v^{2\lambda-n/2}e^{-v}e^{-C_1 v^{-\tilde{\alpha}}}\right) > 0, \quad (2.25)$$

where $C = C(n, \lambda, c_1, \alpha, \min_{M^n} v)$ and $C_1 = C_1(n, \lambda, c_1, \alpha, \min_{M^n} v, \sup_{M^n} v^{\tilde{\alpha}}|\mathrm{R}_g|)$. Hence the result by applying the maximum principle on a slice $\{v_0 \leq v \leq t\}$.

(2) If $Q = O(v^{-\beta})$ with $\lambda < \beta$, then, thanks to (2.21) and (2.25),

$$\Delta_{v-2\lambda \ln v+2Cv^{-\tilde{\alpha}}}\left(v^{\lambda}e^{-Cv^{-\tilde{\alpha}}}u - A_1 v^{2-n/2}e^{-v}e^{-C_1 v^{-\tilde{\alpha}}}\right) \geq -\tilde{Q}v^{\lambda-\beta}e^{-Cv^{-\tilde{\alpha}}},$$

where $\tilde{Q} := \sup_{M^n} v^{\beta}Q$, for $v \geq v_0(n, \lambda, c_1, \alpha, \sup_{M^n}(v^{\tilde{\alpha}}|\mathrm{R}_g|), \min_{M^n} v)$. We notice that, for $\gamma > 0$,

$$\Delta_{v-2\lambda \ln v+2Cv^{-\tilde{\alpha}}}\left(v^{-\gamma}\right) \leq -\frac{\gamma}{2}v^{-\gamma},$$

for $v \geq v_0(n, \lambda, C, \alpha, \gamma, \min_{M^n} v)$. Therefore, if $\gamma := \beta - \lambda$,

$$\Delta_{v-2\lambda \ln v+2Cv^{-\tilde{\alpha}}}\left(v^{\lambda}e^{-Cv^{-\tilde{\alpha}}}u - A_1 v^{2\lambda-n/2}e^{-v}e^{-C_1 v^{-\tilde{\alpha}}} - A_0 v^{\lambda-\beta}\right) > 0,$$

for

$$v \geq v_0(n, \lambda, c_1, \alpha, \beta, \sup_{M^n}(v^{\tilde{\alpha}}|\mathrm{R}_g|), \min_{M^n} v),$$
$$A_0 \geq A_0(n, \lambda, c_1, \alpha, \beta, \sup_{M^n}(v^{\beta}Q), \min_{M^n} v),$$
$$A_1 \geq 0.$$

Again, applying the maximum principle on a slice $\{v_0 \leq v \leq t\}$ gives the result.

(3) If $Q = O(v^{\beta}e^{-v})$ with $\beta + n/2 < \lambda$, then,

$$\Delta_{v-2\lambda \ln v+2Cv^{-\tilde{\alpha}}}\left(v^{\lambda}e^{-Cv^{-\tilde{\alpha}}}u - A_1 v^{2\lambda-n/2}e^{-v}e^{-C_1 v^{-\tilde{\alpha}}}\right) \geq -\tilde{Q}v^{\lambda+\beta}e^{-v-Cv^{-\tilde{\alpha}}},$$

where $\tilde{Q} := \sup_{M^n} v^{-\beta}e^{v}Q$, for $v \geq v_0(n, \lambda, c_1, \alpha, \sup_{M^n}(v^{\tilde{\alpha}}|\mathrm{R}_g|), \min_{M^n} v)$. A similar computation leading to the estimate (2.23) gives

$$\Delta_{v-2\lambda \ln v+2Cv^{-\tilde{\alpha}}}\left(v^{\lambda+\beta}e^{-v}\right) \geq \frac{\lambda - \beta - n/2}{2}v^{\lambda+\beta}e^{-v},$$

for $v \geq v_0(n, \lambda, C, \alpha, \lambda, \beta, \sup_{M^n}(v^{\tilde{\alpha}}|\mathrm{R}_g|), \min_{M^n} v)$. Therefore,

$$\Delta_{v-2\lambda \ln v+2Cv^{-\tilde{\alpha}}}\left(v^{\lambda}e^{-Cv^{-\tilde{\alpha}}}u - A_1 v^{2\lambda-n/2}e^{-v}e^{-C_1 v^{-\tilde{\alpha}}} + A_0 v^{\lambda+\beta}e^{-v}\right) \geq 0,$$

for

$$\begin{aligned}
&v \geq v_0\big(n, \lambda, c_1, \alpha, \beta, \sup_{M^n}(v^{\tilde{\alpha}} |\operatorname{R}_g|), \min_{M^n} v\big),\\
&A_0 \geq A_0\big(n, \lambda, c_1, \alpha, \beta, \sup_{M^n}(v^{-\beta} e^v Q), \min_{M^n} v\big),\\
&A_1 \geq 0.
\end{aligned}$$

□

The main application of Lemma 2.9 is the following corollary dealing with tensors:

Corollary 2.10. *Let $(M^n, g, \nabla f)$ be a normalized expanding gradient Ricci soliton such that f is an exhausting function. Assume there exists a C^2 tensor T such that*

$$\Delta_f T = -\lambda T + \operatorname{Rm}(g) * T,$$

for some real number λ. Assume $R_0 := \sup_{M^n} \left(v^{\min\{\alpha,1\}} |\operatorname{Rm}(g)|\right)$ is finite, for some $\alpha > 0$.

(1) *Then,*

$$\begin{aligned}
&\sup_{M^n} v^{\lambda} |T| \leq C \left(\limsup_{+\infty} v^{\lambda} |T| + \sup_{v \leq v_0} |T| \right),\\
&C = C(n, \lambda, \alpha, R_0, \min_{M^n} f),\\
&v_0 = v_0(n, \lambda, \alpha, R_0, \min_{M^n} f).
\end{aligned}$$

Moreover, if $\limsup_{+\infty} v^{\lambda} |T| = 0$ *then*

$$\begin{aligned}
&\sup_{M^n} v^{\frac{n}{2}-\lambda} e^v |T| \leq C \left(n, \lambda, \alpha, R_0, \min_{M^n} f, \sup_{v \leq v_0} |T| \right),\\
&v_0 = v_0(n, \lambda, \alpha, R_0, \min_{M^n} f).
\end{aligned}$$

(2) *If* $\sup_{M^n} v^{\min\{\alpha,1\}+i/2} |\nabla^i \operatorname{Rm}(g)| =: R_i^{\alpha}$ *and* $\limsup_{+\infty} v^{\lambda+i/2} |\nabla^i T| =: T_i$ *are finite for $i = 0, \dots, k$ then*

$$\sup_{M^n} v^{\lambda+k/2} |\nabla^k T| \leq C \left(n, k, \lambda, \alpha, (R_i^{\alpha})_{0 \leq i \leq k}, \sup_{v \leq v_0} |T|, (T_i)_{0 \leq i \leq k}, \min_{M^n} f \right),$$

where

$$v_0 = v_0 \left(n, k, \lambda, \alpha, (R_i^{\alpha})_{0 \leq i \leq k}, \min_{M^n} f \right).$$

(3) *If $T_0 = 0$ and if R_0^α is finite then*

$$\sup_{M^n} v^{-\lambda-\frac{k}{2}+\frac{n}{2}} e^v |\nabla^k T| \leq C\left(n, k, \alpha, \lambda, R_0^\alpha, \sup_{v\leq v_0} |T|, \min_{M^n} f\right),$$
$$v_0 = v_0\left(n, k, \lambda, \alpha, R_0^\alpha, \min_{M^n} f\right).$$

Proof.

(1) We want to apply Lemma 2.9 to the function $u_\epsilon := \sqrt{|T|^2+\epsilon^2}$. Indeed, u_ϵ satisfies

$$\Delta_f u_\epsilon \geq -\lambda u_\epsilon - c_1 v^{-\alpha} u_\epsilon.$$

Therefore, there exists some positive height $v_0 = v_0(n, \lambda, \alpha, R_0^\alpha, \min_{M^n} f)$ and some positive constants $C_i = C_i(n, \lambda, \alpha, R_0^\alpha, \min_{M^n} f)$ for $i = 0, 1$ such that for $t \geq v_0$, the function

$$v^\lambda e^{-C_0 v^{-\tilde{\alpha}}} u_\epsilon - A_1 v^{2\lambda-\frac{n}{2}} e^{-v-C_1 v^{-\tilde{\alpha}}} : \{v_0 \leq f \leq t\} \to \mathbb{R},$$

attains its maximum on the boundary $\{v = v_0\} \cup \{v = t\}$ for any nonnegative constant A_1. If we choose A_1 large enough (independent of $\epsilon \leq 1$) such that

$$\sup_{v\leq v_0}\left(v^\lambda e^{-C_0 v^{-\tilde{\alpha}}} u_\epsilon - A_1 v^{2\lambda-\frac{n}{2}} e^{-v-C_1 v^{-\tilde{\alpha}}}\right) \leq 0,$$
$$A_1 = A_1(n, \lambda, \alpha, R_0^\alpha, \min_{M^n} f, \sup_{v\leq v_0} u).$$

Hence the result by letting ϵ go to 0 and then letting $v = t$ go to $+\infty$.

(2) We are now in a position to prove the second part of Corollary 2.10. Indeed, by using (2.8) and (2.9) of Proposition 2.3, for $k \geq 1$,

$$\begin{aligned}\Delta_f(\nabla^k T) &= \nabla^k \Delta_f T + [\Delta_f, \nabla^k] T \\ &= -\left(\lambda + \frac{k}{2}\right)\nabla^k T + \mathrm{Rm}(g) * \nabla^k T + \sum_{i=0}^{k-1} \nabla^{k-i}\,\mathrm{Rm}(g) * \nabla^i T \\ &= -\left(\lambda + \frac{k}{2}\right)\nabla^k T + \mathrm{Rm}(g) * \nabla^k T + Q_k,\end{aligned}$$

where $Q_k := \sum_{i=0}^{k-1} \nabla^{k-i}\,\mathrm{Rm}(g) * \nabla^i T$. By induction on k, $k \geq 2$ (the proof of case $k = 1$ is straightforward)

$$\sup_{M^n} v^{\lambda+\frac{k}{2}+\tilde{\alpha}} |Q_k| = C\left(n, k, \lambda, \alpha, (R_i^\alpha)_{0\leq i\leq k}, \sup_{v\leq v_{k-1}} |T|, (T_i)_{0\leq i\leq k-1}, \min_{M^n} f\right),$$
$$v_{k-1} = v_{k-1}\left(n, k, \lambda, \alpha, (R_i^\alpha)_{0\leq i\leq k-1}, \min_{M^n} f\right).$$

By Lemma 2.9, as $\lambda + k/2 < \lambda + k/2 + \tilde{\alpha}$, there exists some positive height

$$v_k = v_k(n, k, \lambda, \alpha, R_0^\alpha), \min_{M^n} f)$$

and some positive constants $C_i = C_i(n, \lambda, \alpha, R_0^\alpha, \min_{M^n} f)$ for $i = 0, 1$ such that for $t \geq v_k$, the function

$$v^{\lambda+\frac{k}{2}} e^{-C_0 v^{-\tilde{\alpha}}} |\nabla^k T| - A_0 v^{-\frac{k}{2}-\tilde{\alpha}} - A_1 v^{2\lambda+k-\frac{n}{2}} e^{-v-C_1 v^{-\tilde{\alpha}}} : \{v_k \leq v \leq t\} \to \mathbb{R},$$

attains its maximum on the boundary $\{v = v_k\} \cup \{v = t\}$ for any nonnegative constant A_1, and $A_0 \geq A_0(n, k, \lambda, R_0^\alpha, \alpha, \sup_{M^n}(v^{\lambda+\frac{k}{2}+\tilde{\alpha}}|Q_k|), \min_{M^n} f)$. For A_0 large enough one can ensure that

$$\sup_{v=v_k} \left(v^{\lambda+\frac{k}{2}} e^{-C_0 v^{-\tilde{\alpha}}} |\nabla^k T| - A_0 v^{-\frac{k}{2}-\tilde{\alpha}} \right) \leq 0,$$
$$A_0 \geq C(n, k, \lambda, R_0^\alpha, \alpha, \sup_{M^n}(v^{\lambda+\frac{k}{2}+\tilde{\alpha}}|Q_k|), \min_{M^n} f) \sup_{v \leq v_k} |\nabla^k T|.$$

Therefore, if t goes to $+\infty$,

$$\sup_{M^n} v^{\lambda+\frac{k}{2}+\tilde{\alpha}} |\nabla^k T| \leq C(\limsup_{+\infty} v^{\lambda+\frac{k}{2}+\tilde{\alpha}} |\nabla^k T| + \sup_{v \leq v_k} |\nabla^k T|),$$
$$C = C\left(n, k, \lambda, \alpha, (R_i^\alpha)_{0 \leq i \leq k}, \sup_{v \leq v_{k-1}} |T|, (T_i)_{0 \leq i \leq k}, \sup_{v \leq v_k} |\nabla^k T|, \min_{M^n} f\right),$$
$$v_k = v_k\left(n, k, \lambda, \alpha, (R_i^\alpha)_{0 \leq i \leq k}, \min_{M^n} f\right).$$

Now, by the local estimates given by Lemma 2.8,

$$\begin{aligned} \sup_{v \leq v_k} |\nabla^k T| &\leq C(n, k, \lambda, \alpha, \sup_{v \leq 2v_k} |\mathrm{Rm}(g)|, \min_{M^n} f) \sup_{v \leq 2v_k} |T|. \\ &\leq C(n, k, \lambda, \alpha, R_0^\alpha, \min_{M^n} f) \sup_{v \leq 2v_k} |T| \end{aligned}$$

which concludes the proof.

(3) The proof of (3) is similar. Here, by induction on k,

$$\sup_{M^n} \left(v^{\frac{n}{2}-\lambda-\frac{k-1}{2}} e^v |Q_k| \right) \leq C\left(n, k, \lambda, \alpha, \sup_{v \leq v_{k-1}} |T|, R_0^\alpha, \min_{M^n} f\right),$$
$$v_{k-1} = v_{k-1}\left(n, k, \lambda, \alpha, R_0^\alpha, \min_{M^n} f\right).$$

According to Lemma 2.9, since trivially,

$$\lambda + \frac{k}{2} > \lambda + \frac{k-1}{2},$$

there exists some positive height

$$v_k = v_k(n, k, \lambda, \alpha, R_0^\alpha, \min_{M^n} f)$$

and some positive constants $C_i = C_i(n, \lambda, \alpha, R_0^\alpha, \min_{M^n} f)$ for $i = 0, 1$ such that for $t \geq v_k$, the function defined on $\{v_k \leq v \leq t\}$,

$$v^{\lambda+\frac{k}{2}} e^{-C_0 v^{-\tilde{\alpha}}} |\nabla^k T| + A_0 v^{2\lambda+k-\frac{n+1}{2}} e^{-v} - A_1 v^{2\lambda+k-\frac{n}{2}} e^{-v-C_1 v^{-\tilde{\alpha}}},$$

attains its maximum on the boundary $\{v = v_k\} \cup \{v = t\}$ for any nonnegative constant A_1 and $A_0 \geq A_0(n, \lambda, R_0^\alpha, \alpha, \sup_{M^n}(v^{\frac{n}{2}-\lambda-\frac{k-1}{2}} e^v |Q_k|), \min_{M^n} f)$. For A_1 large enough, one can ensure that

$$\sup_{v=v_k} \left(v^{\lambda+\frac{k}{2}} e^{-C_0 v^{-\tilde{\alpha}}} |\nabla^k T| + A_0 v^{2\lambda+\frac{k-1}{2}} e^{-v} - A_1 v^{2\lambda+k-\frac{n}{2}} e^{-v-C_1 v^{-\tilde{\alpha}}} \right) \leq 0,$$

$$A_1 \geq C(n, k, \lambda, R_0^\alpha, \alpha, \sup_{M^n}(v^{\frac{n}{2}-\lambda-\frac{k-1}{2}} e^v |Q_k|), \min_{M^n} f) \sup_{v \leq v_k} |\nabla^k T|.$$

Now, by the local estimates given by Lemma 2.8, as $T = O(v^{2\lambda - n/2} e^{-v})$, $T_k = 0$ for any $k \geq 0$.
Therefore, if t goes to $+\infty$ together with Lemma 2.8,

$$\sup_{M^n} \left(v^{\frac{n}{2}-\lambda-\frac{k}{2}} e^v |\nabla^k T| \right) \leq C(n, k, \alpha, \lambda, \min_{M^n} f, R_0^\alpha, \sup_{v \leq v_k} |T|),$$

$$v_k = v_k\left(n, k, \lambda, \alpha, R_0^\alpha, \min_{M^n} f\right). \qquad \square$$

3. Asymptotic estimates

3.1. Rate of convergence to the asymptotic cone: generic case

By the very definition of the asymptotic cone, the asymptotic geometry is close to the cone $(C(X), dr^2 + r^2 g_X)$ in the sense of smooth Cheeger-Gromov convergence. The aim of this section is to quantify this convergence. This approach is inspired by [2] and [11] which are the cornerstones of the study of asymptotically locally euclidean manifolds (ALE).

Definition 3.1. $\mathbb{A}_g^k(T) := \limsup_{x \to +\infty} r_p(x)^{2+k} |\nabla^k T|_g(x)$ for a tensor T and a nonnegative integer k.

The purpose of this section is to give a proof of Theorem 1.2 that we restate in terms of the invariants defined above.

Theorem 3.2. *Let $(M, g, \nabla f)$ be an expanding gradient Ricci soliton such that*

$$\mathbb{A}_g^k(\mathrm{Ric}(g)) < +\infty, \quad \forall k \in \mathbb{N}.$$

Then there exists a metric cone over a smooth compact manifold $(C(X), dr^2 + r^2 g_X)$ *with a smooth metric such that* $(M, g, \nabla f)$ *is asymptotic to* $(C(X), dr^2 + r^2 g_X, r\partial_r/2)$ *at polynomial rate* $\tau = 2$. *In particular, the invariants* $(\mathbb{A}^k_g(\mathrm{Rm}(g))_{k\geq 0}$ *are finite.*

Remark 3.3. Instead of assuming the finiteness of all the invariants $(\mathbb{A}^k_g(\mathrm{Ric}(g)))_{k\geq 0}$, we could have stated Theorem 3.2 with a finite number of finite such invariants: if $(\mathbb{A}^i_g(\mathrm{Ric}(g))_{0\leq i\leq k}$ are finite then the proof of Theorem 3.2 shows that the asymptotic cone is of regularity C^k.

Proof. The proof is essentially the same as in [7]. Nonetheless, we clarify the situation by pointing out the crucial estimates. Indeed, as the Ricci curvature has quadratic curvature decay, one can show that the potential function is equivalent to $r_p^2/4$ for any $p \in M$, hence proper and by (2.4), the level sets of $f + \mu(g)$ (this choice will be made clearer below), denoted by $M_t := \{f + \mu(g) = t\}$, are well-defined compact diffeomorphic hypersurfaces, for t large enough. Let us define the following diffeomorphism

$$\begin{aligned}\phi : [t_0, +\infty) \times M_{t_0^2/4} =: [t_0, +\infty) \times X &\to M_{\geq t_0^2/4}\\ (x, t) &\to \phi_{\frac{t^2}{4}-\frac{t_0^2}{4}}(x),\end{aligned}$$

for t_0 large enough, where $(\phi_t)_t$ is the Morse flow associated to the potential function $f + \mu(g)$, *i.e.*

$$\partial_t \phi_t = \frac{\nabla f}{|\nabla f|^2}.$$

Then the pull-back metric $\phi^* g$ can be written as

$$\begin{aligned}\phi^* g &= \frac{t^2}{4|\nabla f|^2} dt^2 + t^2 \bar{g}_t\\ &= \left(1 + \frac{4\phi^* \mathrm{R}_g}{t^2 - 4\phi^* \mathrm{R}_g}\right) dt^2 + t^2 \bar{g}_t,\end{aligned}$$

where, by definition, $\bar{g}_t := t^{-2} \phi^*_{t^2/4 - t_0^2/4}(g|_{M_{t^2/4}})$. Therefore, showing the expected asymptotic estimates amounts to study the convergence of the one-parameter family of metrics $\bar{g}_t$ as t goes to $+\infty$. Moreover, by the very definition of ϕ, one checks immediately that it is compatible with the soliton structure:

$$(f + \mu(g))(\phi(t, x)) = \frac{t^2}{4} \quad ; \quad \partial_t \phi = \sqrt{f + \mu(g)} \frac{\nabla f}{|\nabla f|^2}.$$

Under the sole assumption $\mathbb{A}^0_g(\mathrm{Rm}(g)) < +\infty$, Chen and the author show in [7] that $(\bar{g}_t)_{t\geq t_0}$ $C^{1,\alpha}$ converges to a $C^{1,\alpha}$ metric $\bar{g}_\infty$ without giving the rate of the convergence. Moreover, if one imposes bounds on the rescaled covariant derivatives of the Ricci tensor, the family $(\bar{g}_t)_t$ converges to $\bar{g}_\infty$ in the C^∞ topology.

Indeed, we are even able to precise the convergence of this family of metrics to its limit $\bar{g}_\infty$:

$$\partial_t \bar{g}_t = -2t^{-1}\bar{g}_t + t^{-2} \cdot \frac{t}{2}\frac{2\phi^*\nabla^2 f}{|\nabla f|^2} \tag{3.1}$$

$$= \left(-2t^{-1} + \frac{t}{2|\nabla f|^2}\right)\bar{g}_t + \frac{\phi^* \operatorname{Ric}(g)}{t|\nabla f|^2} \tag{3.2}$$

$$= \frac{8\phi^* \mathrm{R}_g}{t(t^2 - \phi^* \mathrm{R}_g)}\bar{g}_t + \frac{4\phi^* \operatorname{Ric}(g)}{t(t^2 - 4\phi^* \mathrm{R}_g)}. \tag{3.3}$$

Hence, if the Ricci curvature decays quadratically,

$$\bar{g}_t - \bar{g}_\infty = O(t^{-2}). \tag{3.4}$$

With appropriate bounds on the rescaled covariant derivatives of the Ricci curvature, one gets more precise estimates concerning the covariant derivatives of the difference $\bar{g}_t - \bar{g}_\infty$: firstly, one derivates equation (3.3) and then, by using the assumption on the decay of the covariant derivatives of the Ricci curvature, one integrates as we did previously to get $\nabla^{\bar{g}_\infty, i}\bar{g}_t = O(t^{-2})$ for $i \in \mathbb{N}$.

We finally prove that the full curvature tensor decays quadratically if a sufficient number (two exactly) of covariant derivatives of the Ricci curvature are bounded in a rescaled sense.

By a previous observation, if $\mathbb{A}^0_g(\operatorname{Ric}(g)) < +\infty$, f is quadratic in the distance to a fixed point and its Morse flow is well-defined outside a compact set. Therefore, by Proposition 2.3, one obtains on M_t:

$$\partial_t |\operatorname{Rm}(g)|^2 + \frac{4\nabla^2 f(\operatorname{Rm}(g), \operatorname{Rm}(g))}{|\nabla f|^2} = \frac{1}{|\nabla f|^2}\nabla^2 \operatorname{Ric}(g) * \operatorname{Rm}(g),$$

which means in particular that

$$\partial_t (t^2 |\operatorname{Rm}(g)|^2) \leq O(t^{-2})(t^2 |\operatorname{Rm}(g)|^2) + c(n)|\nabla^2 \operatorname{Ric}(g)|(t|\operatorname{Rm}(g)|). \tag{3.5}$$

By integrating this differential inequality, we get for $t_0 \leq t$,

$$t \sup_{M_t} |\operatorname{Rm}(g)| \leq t_0 \sup_{M_{t_0}} |\operatorname{Rm}(g)| + O\left(\int_{t_0}^t \sup_{M_s} |\nabla^2 \operatorname{Ric}(g)| ds\right),$$

giving the finiteness of $\mathbb{A}^0_g(\operatorname{Rm}(g)$ if $\mathbb{A}^2_g(\operatorname{Ric}(g)) < +\infty$. The finiteness of the invariants $\mathbb{A}^k_g(\operatorname{Rm}(g))$ follows by induction by mimicking the previous argument together with the second part of Proposition 2.3. □

Remark 3.4. Theorem 1.2 does not assume a priori bounds on the full curvature tensor: the sole assumption on the rescaled covariant derivatives of the Ricci curvature implies the corresponding bounds for the curvature tensor. Nonetheless, the proof of Theorem 1.2 shows a loss of two derivatives when estimating the convergence and the finiteness of the invariants $(\mathbb{A}^k_g(\mathrm{Rm}(g))_{k\geq 0}$ which seems to be a usual phenomenon when dealing with geodesic normal coordinates: [11] and [2]. Can one prove the existence of a harmonic diffeomorphism at infinity which preserves the soliton structure ?

3.2. Ricci flat asymptotic cones and rigidity of pinched metrics

The first part of this section consists in proving Theorem 1.3:

Proof of Theorem 1.3.

(1) By Lemma 2.1, equation (2.6), the Ricci curvature satisfies

$$\Delta_f \mathrm{Ric}(g) = -\mathrm{Ric}(g) - 2\,\mathrm{Rm}(g) * \mathrm{Ric}(g).$$

Moreover, as $\lim_{+\infty} r_p^2 \mathrm{Ric}(g) = 0$, it implies $\lim_{+\infty} f\,\mathrm{Ric}(g) = 0$ since f is equivalent to $r_p^2/4$. In order to apply Lemma 2.10, we have to prove that the curvature tensor decays appropriately at infinity. A priori, we have no information on the curvature growth at infinity, nonetheless, one can adapt the work of Munteanu and Wang [18] to the expanding case by proving that the curvature tensor grows at most polynomially providing the Ricci curvature is bounded and f is an exhausting function which is the case here. As $\mathrm{Ric}(g)$ decays quadratically, by Lemma 2.8, $\nabla^2 \mathrm{Ric}(g)$ grows at most like $f^{\alpha-1}$ if $\mathrm{Rm}(g) = O(f^\alpha)$ for some positive α. Then, by using Lemma 2.3 as we did in the proof of Theorem 1.2, one can show that $\mathrm{Rm}(g) = O(f^{\alpha-1})$. Hence, by induction, one shows actually that $\mathrm{Rm}(g) = O(f^{-1+\alpha'})$ for any $\alpha' \in (0,1)$.
Now, we apply Lemma 2.10 with $T = \mathrm{Ric}(g)$, $\lambda = 1$ and $\alpha' \in (0,1)$ to get

$$\sup_{M^n} v^{\frac{n-k}{2}-1} e^v |\nabla^k \mathrm{Ric}(g)| < +\infty, \quad \forall k \geq 0.$$

(2) Clearly, by the previous step, the invariants $(\mathbb{A}^k_g(\mathrm{Ric}(g)))_{k\geq 0}$ are finite. Therefore, we can apply Theorem 1.2 to claim that $(M^n, g, \nabla f))$ is asymptotically conical to a metric cone $(C(X), dr^2 + r^2 g_X)$ with (X, g_X) a smooth Riemannian manifold. Moreover, as $\lim_{+\infty} f\,\mathrm{Ric}(g) = 0$, the metric cone is Ricci flat.
The convergence is exponential since, according to equation (3.3), the rate is given by

$$O\left(\int_t^{+\infty} s^2 \sup_{M_{s^2/4}} |\mathrm{Ric}(g)| \cdot s^{-3} ds\right) = O\left(t^{-n} e^{-\frac{t^2}{4}}\right).$$

(3) Concerning the lower bound for the scalar curvature, we employ the same method, that is, we compute the evolution of the quantity $v^{n/2-1}e^v \mathrm{R}_g$ hoping that some weighted Laplacian of it is positive outside a compact set.
We first consider $e^v \mathrm{R}_g$:

$$\Delta_v(e^v \mathrm{R}_g) = \left(\Delta_v e^v\right) \mathrm{R}_g + 2\left\langle \nabla v, \nabla\left(\mathrm{R}_g e^v\right)\right\rangle - 2|\nabla v|^2 \mathrm{R}_g e^v + e^v \Delta_v \mathrm{R}_g, \quad (3.6)$$

$$\Delta_{-v}(e^v \mathrm{R}_g) = (v - |\nabla v|^2 - 1)\mathrm{R}_g e^v - 2|\mathrm{Ric}(g)|^2 e^v \quad (3.7)$$

$$= \left(\mathrm{R}_g + \frac{n}{2} - 1\right) e^v \mathrm{R}_g - 2|\mathrm{Ric}(g)|^2 e^v. \quad (3.8)$$

Then, for some real α,

$$\begin{aligned}\Delta_{-v}(v^\alpha e^v \mathrm{R}_g) &= \Delta_{-v}\left(v^\alpha\right)(e^v \mathrm{R}_g) + 2\alpha\langle \nabla \ln v, \nabla(v^\alpha e^v \mathrm{R}_g)\rangle \\ &\quad - 2\alpha^2 |\nabla \ln v|^2 v^\alpha e^v \mathrm{R}_g + \left(\mathrm{R}_g + \frac{n}{2} - 1\right) v^\alpha e^v \mathrm{R}_g \\ &\quad - 2|\mathrm{Ric}(g)|^2 v^\alpha e^v, \\ \Delta_{-v-2\alpha \ln v}(v^\alpha e^v \mathrm{R}_g) &= \left(\frac{n}{2} - 1 - \alpha + \mathrm{R}_g + 2\alpha \frac{\Delta v}{v} - \alpha(\alpha+1)|\nabla \ln v|^2\right) v^\alpha e^v \mathrm{R}_g \\ &\quad - 2|\mathrm{Ric}(g)|^2 v^\alpha e^v \\ &= \left(\frac{n}{2} - 1 - \alpha + \frac{2\alpha \Delta v - \alpha(\alpha+1)}{v} + \alpha(\alpha+1)\frac{\Delta v}{v^2}\right) v^\alpha e^v \mathrm{R}_g \\ &\quad + (\mathrm{R}_g^2 - 2|\mathrm{Ric}(g)|^2) v^\alpha e^v.\end{aligned}$$

If $\alpha := n/2 - 1$, we get

$$\begin{aligned}\Delta_{-v-(n-2)\ln v}(v^{n/2-1} e^v \mathrm{R}_g) &= \left(\frac{(n/2-1)n/2 + (n-2)\mathrm{R}_g}{v}\right) e^v \mathrm{R}_g \\ &\quad + \left(n/2(n/2-1)\frac{\mathrm{R}_g + n/2}{v^2}\right) v^{n/2-1} e^v \mathrm{R}_g \\ &\quad + (\mathrm{R}_g^2 - 2|\mathrm{Ric}(g)|^2) v^{n/2-1} e^v.\end{aligned}$$

Therefore, one has

$$\Delta_{-v-(n-2)\ln v}(v^{n/2-1} e^v \mathrm{R}_g) \leq \left(\frac{C(n, \sup_{M^n} \mathrm{R}_g)}{v} + \mathrm{R}_g\right) v^{n/2-1} e^v \mathrm{R}_g. \quad (3.9)$$

According to the differential inequality (3.9), one has

$$\Delta_{-v-(n-2)\ln v}(v^{n/2-1} e^v \mathrm{R}_g) \leq \left(\frac{C(n, \sup_{M^n} \mathrm{R}_g, \sup_{M^n} v\,\mathrm{R}_g)}{v}\right) v^{n/2-1} e^v \mathrm{R}_g.$$

By multiplying by a function of the form $e^{Cv^{-1}}$ where C is a constant depending on n, $\sup_{M^n} \mathrm{R}_g$ and $\sup_{M^n} v\,\mathrm{R}_g$ as we did before for the Ricci curvature, one has

$$\Delta_{-v-(n-2)\ln v - 2Cv^{-1}}(v^{n/2-1} e^v e^{-Cv^{-1}} \mathrm{R}_g) < 0,$$

outside a sublevel set of the form $\{v \leq C(n, \sup_{M^n} \mathrm{R}_g)\}$. Hence the result. □

We decide to prove a proposition on Ricci pinched metrics as a natural follow-up to the proof of the previous theorem.

Recall that a Riemannian manifold (M^n, g) is *Ricci pinched* if there exists some $\epsilon \in (0, 1]$ such that

$$\mathrm{Ric}(g) \geq \frac{\epsilon}{n} \mathrm{R}_g \, g.$$

We are able to prove the following:

Proposition 3.5. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton with non negative scalar curvature which is Ricci pinched. Then:*

(1) *$(M^n, g, \nabla f)$ is asymptotically conical to a Ricci flat metric cone $\big(C(\mathbb{S}^{n-1}), g_{C(\mathbb{S}^{n-1})}, r\partial_r/2\big)$ at exponential rate;*
(2) *If* $\dim M = 4$*, $(M^4, g, \nabla f)$ is isometric to the Gaussian soliton* $(\mathbb{R}^4, \mathrm{eucl}, |\cdot|^2/4)$*;*
(3) *If g is sufficiently pinched,* i.e. *if $\epsilon \in [\epsilon(n), 1]$ where* $\lim_{n\to+\infty} \epsilon(n) = 1 - 2^{-1/2}$*, then $(M^n, g, \nabla f)$ is isometric to the Gaussian soliton.*

Note that we do not assume a bound on the full curvature tensor: we derive such bounds by the sole assumption of being Ricci pinched.

Previous works on this question were done in the following cases:

- If (M, g) has a sufficiently pinched curvature tensor (implying the nonnegativity of the sectional curvature): [12];
- If $(M, g) \times \mathbb{R}^2$ has pinched isotropic curvature (implying the nonnegativity of the sectional curvature): [4] extending [12], proving the non existence of non flat non compact such pinched metrics without any expanding structure;
- If $\dim M = 3$: [17] by essentially using the same argument of the proof of Proposition 3.5 in dimension 4.

Proof of Proposition 3.5.

(1) As the Ricci curvature is pinched and the scalar curvature is nonnegative, the Ricci curvature is trivially nonnegative, therefore, the potential function is quadratic in the distance function by (2.13) in Proposition 2.4. Moreover, the Ricci curvature decays exponentially at infinity: this has already been shown in previous works on Ricci pinched expanders. We reprove it for the convenience of the reader. Consider the Morse flow $(\phi_t)_t$ associated to the potential function. Then, by (2.2) of Lemma 2.1,

$$\partial_t \mathrm{R}_g = \left\langle \nabla \mathrm{R}_g, \frac{\nabla f}{|\nabla f|^2} \right\rangle = -2 \,\mathrm{Ric}(g)(\mathbf{n}, \mathbf{n}).$$

Since the Ricci curvature is pinched,

$$\partial_t \mathrm{R}_g \leq -\frac{2\epsilon}{n} \mathrm{R}_g,$$

which implies that $\mathrm{R}_g(\phi_t(x)) \leq e^{-2\epsilon t/n}\,\mathrm{R}_g(x)$, for $t \geq 0$, *i.e.* the Ricci curvature decays as $e^{-2\epsilon f/n}$. In particular, one has $\lim_{+\infty} r_p^2 \,\mathrm{Ric}(g) = 0$ and Theorem 1.3 applies. Moreover, as it is explained in Lemma 2.1, the levels sets of f are diffeomorphic to $\mathbb{S}^{n-1}$: the link of the asymptotic cone of $(M^n, g, \nabla f)$ is therefore a smooth standard sphere endowed with an Einstein metric with constant $n-2$;

(2) According to the previous step of the proof of Proposition 3.5, the asymptotic cone is $(C(X), dr^2 + r^2 g_X)$ where X is diffeomorphic to $\mathbb{S}^{n-1}$ and g_X satisfies $\mathrm{Ric}(g_X) = (n-2)g_X$, which, in (global) dimension 3 or 4, implies that (X, g_X) is isometric to the Euclidean sphere of curvature 1. In particular, the asymptotic volume ratio is

$$\mathrm{AVR}(g) = \lim_{r\to+\infty} \frac{\mathrm{Vol}\, B(p,r)}{r^n} = \omega_n,$$

where ω_n is the volume of the unit ball of the n-dimensional Euclidean space. By the Bishop-Gromov theorem, (M^n, g) is isometric to the Euclidean space when $n \in \{3, 4\}$;

(3) On the one hand, if the Ricci curvature is pinched, that is, there exists some positive number ϵ less than or equal to 1 such that

$$\mathrm{Ric}(g) \geq \frac{\epsilon\,\mathrm{R}_g}{n} g,$$

then we estimate the norm of the Ricci curvature as follows:

$$\begin{aligned} |\mathrm{Ric}(g)|^2 &= \left|\mathrm{Ric}(g) - \frac{\epsilon\,\mathrm{R}_g}{n} g + \frac{\epsilon\,\mathrm{R}_g}{n} g\right|^2 \\ &\leq (1-\epsilon)^2\,\mathrm{R}_g^2 + \frac{2\epsilon(1-\epsilon)}{n}\,\mathrm{R}_g^2 + \frac{\epsilon^2}{n}\,\mathrm{R}_g^2 \\ &\leq \left(c(n)\epsilon^2 - 2c(n)\epsilon + 1\right)\mathrm{R}_g^2, \end{aligned}$$

where $c(n) = 1 - 1/n$. Consequently, $2|\mathrm{Ric}(g)|^2 \leq \mathrm{R}_g^2$ pointwisely if and only if $\epsilon \in [\epsilon_-, 1]$ where

$$\epsilon_- := 1 - \sqrt{1 - \frac{1}{2\left(1 - \frac{1}{n}\right)}},$$

which means that $e^v\,\mathrm{R}_g$ satisfies on M^n according to equation (3.8),

$$\Delta_{-v}(e^v\,\mathrm{R}_g) \geq \left(\frac{n}{2} - 1\right) e^v\,\mathrm{R}_g\,.$$

By applying the maximum principle, we get that (M^n, g) is scalar flat since

$$\lim_{+\infty} e^v\,\mathrm{R}_g = 0,$$

as soon as $n \geq 3$ by the Ricci curvature decay obtained in Theorem 1.3. By the evolution equation (2.7), (M^n, g) is Ricci flat hence isometric to the Euclidean case according to [21]. □

3.3. Estimates for non Ricci flat asymptotic cones with an Einstein link

In this section, we investigate the asymptotic geometry of Ricci expanders whose asymptotic cones have an Einstein link.

Theorem 3.6. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton. Assume the following weak decays on the curvature tensor:*

$$\mathrm{Rm}(g) = O(r_p^{-2}), \quad \operatorname{div}\mathrm{Rm}(g) = O(r_p^{-3}).$$

If $(M, g, \nabla f)$ is weakly Einstein at infinity, i.e. $\mathrm{Ric}(g) - \frac{\mathrm{R}_g}{(n-1)}(g - \mathbf{n}\otimes\mathbf{n}) = o(r_p^{-2})$, *then*

$$\mathrm{Ric}(g) - \frac{\mathrm{R}_g}{(n-1)}(g - \mathbf{n}\otimes\mathbf{n}) = O(r_p^{-3})$$

Remark 3.7. Theorem 3.6 is interesting only if the convergence to the asymptotic cone is less than C^2 a priori. Recall from [7] that if the curvature tensor decays quadratically, the asymptotic cone is only $C^{1,\alpha}$, for any $\alpha \in (0, 1)$.

Proof. Define

$$T := \mathrm{Ric}(g) - \frac{R_g}{(n-1)}(g - \mathbf{n}\otimes\mathbf{n}).$$

T is well-defined outside a compact set (or even a point if $\mathrm{Ric}(g) \geq -(\delta/2)g$ for $\delta \in [0, 1)$). Then, $\operatorname{tr} T = 0$ and

$$\begin{aligned}\Delta_f T + T &= -2\,\mathrm{Rm}(g) * \mathrm{Ric}(g) + \frac{2}{n-1}|\mathrm{Ric}(g)|^2(g - \mathbf{n}\otimes\mathbf{n})\\ &\quad + \frac{\mathrm{R}_g}{n-1}\Delta_f(\mathbf{n}\otimes\mathbf{n}) + \frac{2}{n-1}\nabla\mathrm{R}_g \cdot \nabla(\mathbf{n}\otimes\mathbf{n}).\end{aligned}$$

Now, recall that $\mathrm{Rm}(g) * \mathrm{Ric}(g)_{ij} := \mathrm{Rm}(g)_{iklj}\,\mathrm{Ric}(g)_{kl}$. Therefore,

$$\begin{aligned}\langle \mathrm{Rm}(g) * \mathrm{Ric}(g), T\rangle &= \langle \mathrm{Rm}(g) * T, T\rangle + \frac{\mathrm{R}_g}{n-1}\langle \mathrm{Rm}(g) * (g - \mathbf{n}\otimes\mathbf{n}), T\rangle\\ &= \langle \mathrm{Rm}(g) * T, T\rangle + \frac{\mathrm{R}_g}{n-1}\langle \mathrm{Ric}(g) - \mathrm{Rm}(g)(\mathbf{n}, \cdot, \cdot, \mathbf{n}), T\rangle\\ &= \langle \mathrm{Rm}(g) * T, T\rangle + \frac{\mathrm{R}_g}{n-1}|T|^2 - \frac{\mathrm{R}_g^2}{(n-1)^2}T(\mathbf{n}, \mathbf{n})\\ &\quad - \frac{\mathrm{R}_g}{n-1}\langle \mathrm{Rm}(g)(\mathbf{n}, \cdot, \cdot, \mathbf{n}), T\rangle.\end{aligned}$$

Now, according to Lemma 2.1, we know that

$$\begin{aligned}\langle \mathrm{Rm}(g)(\mathbf{n}, \cdot, \cdot, \mathbf{n}), T\rangle &= O(v^{-2})|T|\\ T(\mathbf{n}, \mathbf{n}) &= \mathrm{Ric}(\mathbf{n}, \mathbf{n}) = O(v^{-2})\\ |T|^2 &= |\mathrm{Ric}(g)|^2 - \frac{\mathrm{R}_g^2}{n-1} + \frac{2\,\mathrm{R}_g}{n-1}\mathrm{Ric}(\mathbf{n}, \mathbf{n}),\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\left(\Delta_f|T|^2+2|T|^2-2|\nabla T|^2\right) = &-2\langle \mathrm{Rm}(g)*T,T\rangle\\ &-\frac{2}{n-1}(\mathrm{R}_g+\mathrm{Ric}(g)(\mathbf{n},\mathbf{n}))|T|^2\\ &+\frac{4\,\mathrm{R}_g}{(n-1)^2}\mathrm{Ric}(\mathbf{n},\mathbf{n})^2+\frac{2\,\mathrm{R}_g}{n-1}\mathrm{Rm}(g)_{\mathbf{n}..\mathbf{n}}*T\\ &+\frac{\mathrm{R}_g}{n-1}\langle\Delta_f(\mathbf{n}\otimes\mathbf{n}),T\rangle\\ &+\frac{2}{n-1}\langle\nabla\mathrm{R}_g\cdot\nabla(\mathbf{n}\otimes\mathbf{n}),T\rangle.\end{aligned}$$

Now, since,

$$\nabla\mathbf{n}=\frac{\nabla^2 f-\nabla^2 f(\mathbf{n})\otimes\mathbf{n}}{|\nabla f|},$$

$$\begin{aligned}\langle\nabla\mathrm{R}_g\cdot\nabla(\mathbf{n}\otimes\mathbf{n}),T\rangle &= 2\langle\nabla_{\nabla\mathrm{R}_g}\mathbf{n},\mathrm{Ric}(g)(\mathbf{n})\rangle\\ &= O(v^{-4}),\\ \langle\nabla_{\nabla f}(\mathbf{n}\otimes\mathbf{n}),T\rangle &= 2\langle\nabla_{\nabla f}\mathbf{n},\mathrm{Ric}(g)(\mathbf{n})\rangle\\ &= 2\langle\mathrm{Ric}(g)(\mathbf{n})-\mathrm{Ric}(g)(\mathbf{n},\mathbf{n})\mathbf{n},\mathrm{Ric}(g)(\mathbf{n})\rangle = O(v^{-4}),\\ \langle\Delta(\mathbf{n}\otimes\mathbf{n}),T\rangle &= 2\langle\Delta(\mathbf{n}),\mathrm{Ric}(g)(\mathbf{n})\rangle+2\langle\nabla\mathbf{n}*\nabla\mathbf{n},T\rangle\\ &= O(v^{-1})|T|.\end{aligned}$$

Finally, we get:

$$\begin{aligned}\frac{1}{2}\left(\Delta_v|T|^2+2|T|^2-2|\nabla T|^2\right) &= O(v^{-1})|T|^2+O(v^{-2})|T|+O(v^{-4})\\ &= O(v^{-1})|T|^2+O(v^{-3}),\end{aligned}$$

where we used the Young inequality in the last line. Define now $u:=|T|^2$. Then, there are positive constants c_0, c_1 such that

$$\Delta_v u+2u\geq -c_0v^{-1}u-c_1v^{-3}.$$

We apply Lemma 2.9 to u, $\lambda=-2$, $\alpha=1$ and $Q=c_1v^{-3}$ to get that $\sup_{M^n} v^3u<+\infty$ since $\lim_{+\infty} v^2u=0$ by assumption on T, which is excatly the expected decay on T. □

The decay of the tensor T considered in Theorem 3.6 can be sharpened as soon as the Laplacian of the curvature tensor decays appropriately:

Theorem 3.8. *Let $(M^n,g,\nabla f)$ be an expanding gradient Ricci soliton. Assume the curvature tensor decays quadratically at infinity,* i.e. $\mathrm{Rm}(g)=O(r_p^{-2})$.

(1) *(Weak asymptotically constant scalar curvature) If* $\lim_{+\infty} f\,\mathrm{R}_g = \mathrm{R}_\infty = cst$ *and if* $\Delta_g \mathrm{R}_g = O(f^{-2})$*, then*

$$\mathrm{R}_g = \frac{\mathrm{R}_\infty}{f} + O(f^{-2});$$

(2) *(Weak asymptotically constant Ricci curvature) If*

$$T := \mathrm{Ric}(g) - \frac{\mathrm{R}_g}{(n-1)}(g - \mathbf{n}\otimes\mathbf{n}) = o(f^{-1}),$$
$$\Delta_g \mathrm{Ric}(g) = O(f^{-2}),$$

then $T = O(f^{-2})$*;*

(3) *(Weak asymptotically constant curvature operator) If*

$$\tilde{T} := \mathrm{Rm}(g) - \frac{R_g}{(n-1)(n-2)}(g - \mathbf{n}\otimes\mathbf{n}) \odot (g - \mathbf{n}\otimes\mathbf{n}) = o(f^{-1}),$$
$$\Delta_g \mathrm{Rm}(g) = O(f^{-2}),$$

then $\tilde{T} = O(f^{-2})$.

Remark 3.9. Again, Theorem 3.8 is only interesting if the convergence to the asymptotic cone is less than C^2.

Proof. First of all, denote by R_∞ the limit of the rescaled curvature, *i.e.* $\mathrm{R}_\infty := \lim_{+\infty} v\,\mathrm{R}_g$. Then, by the soliton identities given by Lemma 2.1,

$$\begin{aligned}\nabla_{\nabla f}(v\,\mathrm{R}_g - \mathrm{R}_\infty) &= |\nabla v|^2\,\mathrm{R}_g + v\nabla_{\nabla f}\mathrm{R}_g\\ &= -\Delta v\,\mathrm{R}_g - v\left(\Delta\,\mathrm{R}_g + 2|\mathrm{Ric}(g)|^2\right)\\ &= O(v^{-1}).\end{aligned}$$

In particular, if we integrate the previous estimate along the flow generated by $\nabla f/|\nabla f|^2$, then we get,

$$v\,\mathrm{R}_g - \mathrm{R}_\infty = O(v^{-1}).$$

The same procedure applies to the Ricci curvature if the metric of the link of the asymptotic cone has constant Ricci curvature. Indeed, by the previous estimate on the scalar curvature,

$$vT = v\,\mathrm{Ric}(g) - \frac{\mathrm{R}_\infty}{n-1}(g - \mathbf{n}\otimes\mathbf{n}) + O(v^{-1}).$$

Therefore, it suffices to prove that

$$v\,\mathrm{Ric}(g) - \frac{\mathrm{R}_\infty}{n-1}(g - \mathbf{n}\otimes\mathbf{n}) = O(v^{-1}).$$

Again, by Lemma 2.1,

$$\nabla_{\nabla f}\left(v\,\mathrm{Ric}(g)-\frac{\mathrm{R}_\infty}{n-1}(g-\mathbf{n}\otimes\mathbf{n})\right)=-v(\Delta\,\mathrm{Ric}(g)+2\,\mathrm{Rm}(g)*\mathrm{Ric}(g))-\Delta v\,\mathrm{Ric}(g)$$
$$+\frac{\mathrm{R}_\infty}{n-1}\left[(\nabla_{\nabla f}\mathbf{n})\otimes\mathbf{n}+\mathbf{n}\otimes(\nabla_{\nabla f}\mathbf{n})\right]$$
$$=O(v^{-1}).$$

The result follows by integrating along the Morse flow of the potential function f. The same procedure applies to the curvature tensor if the metric of the link of the asymptotic cone has constant curvature. □

3.4. Positively curved expanding gradient Ricci solitons

Even if the following proposition belongs to the global estimates, we find it more convenient to give its proof now:

Proposition 3.10. *Let $(M, g, \nabla f)$ be a normalized expanding gradient Ricci soliton with non negative Ricci curvature. Then,*

$$\left(\sqrt{\mu(g)}+\frac{r_p(x)}{2}\right)^2\mathrm{R}_g(x)\geq\liminf_{y\to+\infty}\frac{r_p(y)^2}{4}\,\mathrm{R}_g(y),\quad\forall x\in M,$$

where p is the unique critical point of the potential function f and $\mu(g)$ is the entropy.

The definition of the entropy of a Ricci expander can be found in Lemma 2.1.

Note that if the convergence to the asymptotic cone $(C(X), g_{C(X)})$ of such a Ricci expander is at least C^2 and if the scalar curvature R_{g_X} is positive then according to Proposition 3.10, the rescaled scalar curvature has a definite positive lower bound.

Proof of Proposition 3.10. Recall first that $\Delta_f v = v$. Then, by equation (2.7), one has

$$\Delta_f(v\,\mathrm{R}_g)=-2|\,\mathrm{Ric}(g)|^2v+2<\nabla(v\,\mathrm{R}_g),\nabla\ln v>-2(v\,\mathrm{R}_g)|\nabla\ln v|^2\leq 0.$$

One gets the expected estimate by applying the maximum principle to sub level set $\{v\leq t\}$ with t tending to $+\infty$. □

Remark 3.11.

- The proof of Proposition 3.10 does not require the Ricci curvature to be nonnegative, one only needs nonnegative scalar curvature plus the fact that v is an exhaustion function;

- In particular, Proposition 3.10 gives a positive lower bound for the rescaled scalar curvature as soon as the metric g_X of the link of the asymptotic cone $C(X)$ satisfies $\mathrm{R}_{g_X} > (n-1)(n-2)$ on X.

The next proposition gives an asymptotic lower bound on the Ricci curvature in case the asymptotic cone is sufficiently positively curved:

Proposition 3.12. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton with positive Ricci curvature that converges (in the C^4 sense at least) to its asymptotic cone $(C(X), dr^2 + r^2 g_X)$ where* $\mathrm{Ric}(g_X) > (n-2)g_X$. *Then, for some $p \in M^n$,*

$$\liminf_{r\to+\infty} r^4 \inf_{\partial B(p,r)} \mathrm{Ric}(g) > 0.$$

Proof. By assumption on the positivity of the Ricci curvature of the metric of the link of the cone, it is sufficient to prove it for the radial direction. By the soliton identity (2.2),

$$\begin{aligned} 2v^2 \mathrm{Ric}(g)(\mathbf{n}, \mathbf{n}) &= -\frac{v^2}{|\nabla f|^2} \langle \nabla \mathrm{R}_g, \nabla f \rangle \\ &= \frac{v^2}{|\nabla f|^2} \left(\Delta \mathrm{R}_g + 2|\mathrm{Ric}(g)|^2 + \mathrm{R}_g \right). \end{aligned}$$

In particular, using the decay of $\Delta \mathrm{R}_g$ and the fact that $\liminf_{+\infty} v \,\mathrm{R}_g > 0$, one gets $\inf_{M^n} v^2 \mathrm{Ric}(g)(\mathbf{n}, \mathbf{n}) > 0$.

Hence the result. □

We end this subsection by proving the analogue of Proposition 3.12 for the curvature operator.

Proposition 3.13. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton with positive curvature operator that converges (in the C^4 sense) to its asymptotic cone $(C(X), dr^2 + r^2 g_X)$ where* $\mathrm{Rm}(g_X) > \mathrm{Id}_{\Lambda^2 TX}$. *Then, for $p \in M^n$,*

$$\liminf_{r\to+\infty} r^4 \inf_{\partial B(p,r)} \mathrm{Rm}(g) > 0.$$

Proof. Any bivector $\alpha \in \Lambda^2 TM$ can be written as $\alpha = \alpha' + \beta \wedge \mathbf{n}$, where α' is a linear combination of bivectors spanning 2-planes orthogonal to $\mathbf{n}$ and β is a vector orthogonal to $\mathbf{n}$. Then, by the soliton identity (2.4)

$$\begin{aligned} \mathrm{Rm}(g)(\alpha, \alpha) = {} & \mathrm{Rm}(g)(\alpha', \alpha') + 2\,\mathrm{Rm}(g)(\alpha', \beta \wedge \mathbf{n}) + \mathrm{Rm}(g)(\beta \wedge \mathbf{n}, \beta \wedge \mathbf{n}) \\ & \geq \mathrm{Rm}(g)(\alpha', \alpha') - c(n) v^{-1/2} |\operatorname{div} \mathrm{Rm}(g)| |\alpha'| |\beta| + \mathrm{Rm}(g)(\beta, \mathbf{n}, \mathbf{n}, \beta). \end{aligned}$$

Now,

$$\begin{aligned}\mathrm{Rm}(g)(\nabla f, U, V, \nabla f) &= -\operatorname{div}\mathrm{Rm}(g)(\nabla f, U, V)\\ &= -\nabla_{\nabla f}\mathrm{Ric}(g)(U, V) + \nabla_U \mathrm{Ric}(g)(\nabla f, V),\end{aligned}$$

for any vector U and V. On the other hand, by the soliton identities given by Lemma 2.1,

$$\begin{aligned}\nabla_U \mathrm{Ric}(g)(\nabla f, V) &= U \cdot \mathrm{Ric}(g)(\nabla f, V) - \mathrm{Ric}(g)(\nabla_U \nabla f, V) - \mathrm{Ric}(g)(\nabla f, \nabla_U V)\\ &= \langle \nabla_U(\mathrm{Ric}(g)(\nabla f)), V\rangle - \mathrm{Ric}(g)\left(\frac{U}{2} + \mathrm{Ric}(g)(U), V\right)\\ &= -\frac{\nabla^2 \mathrm{R}_g}{2}(U, V) - \mathrm{Ric}(g)\left(\frac{U}{2} + \mathrm{Ric}(g)(U), V\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Rm}(g)(\mathbf{n}, U, V, \mathbf{n}) = {} & \frac{1}{2|\nabla f|^2}\left(2\Delta\,\mathrm{Ric}(g) + \mathrm{Ric}(g) + 4\,\mathrm{Rm}(g) * \mathrm{Ric}(g)\right)(U, V)\\ & - \frac{1}{2|\nabla f|^2}\left(\nabla^2 \mathrm{R}_g(U, V) + 2\,\mathrm{Ric}(g) \otimes \mathrm{Ric}(g)\right)(U, V).\end{aligned}$$

Hence, by the decay of the curvature and the assumption on the positivity of the curvature operator of the metric of the link of the cone,

$$\inf_{x \in M^n} v^2(x) \min_{\beta \perp \mathbf{n}} \mathrm{Rm}(g)(x)(\beta \wedge \mathbf{n}, \beta \wedge \mathbf{n}) > 0.$$

Finally, we get, by the Young inequality,

$$\begin{aligned}\mathrm{Rm}(g)(\alpha, \alpha) &\geq \frac{C}{v}|\alpha'|^2 - O(v^{-2})|\alpha'||\beta| + \frac{C}{v^2}|\beta|^2\\ &\geq \frac{C}{v}\left(1 - O(v^{-1})\right)|\alpha'|^2 + \frac{C}{v^2}|\beta|^2,\end{aligned}$$

where C is a positive constant space independent which can vary from line to line. Hence the expected lower bound for the curvature operator. □

4. A priori curvature estimates and compactness theorems

We start by establishing Proposition 1.9: it is an a priori estimate for general subsolutions of weighted elliptic equations with quadratic nonlinearities on a Ricci expander.

Proof of Proposition 1.9. First of all, as the potential function is exhausting, the level sets $\{f = t\}$ of the potential function are smooth compact diffeomorphic hypersurfaces as soon as $t > \sup_{M^n} \mathrm{R}_g$ by Lemma 2.1. Without loss of generality, one can assume $c = 1$ by considering $\tilde{u} := cu$.

As $\Delta_f v = v$,

$$\begin{aligned}\Delta_f(vu) &\geq -\frac{(vu)^2}{v} + 2\langle \nabla(vu), \nabla \ln v\rangle - 2(vu)|\nabla \ln v|^2\\ &\geq 2\langle \nabla(vu), \nabla \ln v\rangle - \frac{(vu+1)^2}{v} + \frac{1}{v},\end{aligned}$$

by Proposition 2.4 Now, define $w := (vu+1)^{-1}$. Then,

$$\begin{aligned}\Delta_{v-2\ln v} w &= -\frac{\Delta_{v-2\ln v}(vu)}{(vu+1)^2} + 2\frac{|\nabla(vu)|^2}{(vu+1)^3} \leq \frac{1}{v} + 2|\nabla w|^2(uv+1)\\ &\leq \frac{1}{v} + 2|\nabla w|^2 w^{-1}.\end{aligned}$$

Finally, consider $W := w + 2v^{-1}$. Then, as

$$\begin{aligned}\Delta_{v-2\ln v} v^{-1} &= -\frac{\Delta_{v-2\ln v} v}{v^2} + 2|\nabla v|^2 v^{-3}\\ &= -\frac{1}{v} + 4|\nabla v|^2 v^{-3},\end{aligned}$$

we get,

$$\begin{aligned}\Delta_{v-2\ln v} W &\leq -v^{-1} + 8|\nabla v|^2 v^{-3} + 2|\nabla w|^2 w^{-1}\\ &\leq -v^{-1} + 8v^{-2} + 2|\nabla w|^2 w^{-1}.\end{aligned}$$

Assume that W attains its minimum at an interior point in the slice $\{t_1 \leq f \leq t_2\}$ with t_1 to be chosen later. Then, at this point, one has by the previous differential inequality:

$$0 \leq -v^{-1} + 8v^{-2} + 2|\nabla 2v^{-1}|^2 w^{-1}.$$

That is,

$$1 - 8v^{-1} \leq 8(\sup_{M^n} u + v^{-1})v^{-1},$$

which is impossible if $t_1 > C(\sup_{M^n} u)$. Therefore, W attains its minimum on the boundary of $\{t_1 \leq f \leq t_2\}$ as soon as t_1 is larger than $C(\sup_{M^n} u) > 0$. By letting t_2 go to $+\infty$, one has

$$\min_M W \geq \min\left(C(\sup_{M^n} u), \liminf_{+\infty} W\right),$$

which turns out to be exactly the expected estimate. □

It turns out that Proposition 1.9 can be applied to the full curvature tensor and the scalar curvature when the Ricci curvature is nonnegative as the next two lemmata show.

Corollary 4.1. *Let* $(M^n, g, \nabla f)$ *be an expanding gradient Ricci soliton with finite asymptotic curvature ratio,* i.e. $\mathbb{A}^0_g(\mathrm{Rm}(g)) < +\infty$. *Then,*

$$\sup_{M^n} v|\mathrm{Rm}(g)| \leq C\left(n, \sup_{M^n}|\mathrm{Rm}(g)|, \mathbb{A}^0_g\big(\mathrm{Rm}(g)\big)\right).$$

Proof. First of all, as $\mathbb{A}^0_g(\mathrm{Rm}(g)) < +\infty$, it can be shown that f is equivalent to $r_p^2/4$, in particular f is proper.

By equation (1.9), the curvature operator $\mathrm{Rm}(g)$ of an expanding gradient Ricci soliton satisfies

$$\Delta_f \mathrm{Rm}(g) = -\mathrm{Rm}(g) + \mathrm{Rm}(g) * \mathrm{Rm}(g).$$

Therefore,

$$\Delta_f|\mathrm{Rm}(g)|^2 \geq 2|\nabla \mathrm{Rm}(g)|^2 - 2|\mathrm{Rm}(g)|^2 - c(n)|\mathrm{Rm}(g)|^3.$$

Now, since the curvature may vanish, we have to deal with $u_\epsilon := \sqrt{|\mathrm{Rm}(g)|^2 + \epsilon^2}$ first, where $\epsilon > 0$. By routine computations, one has

$$\Delta_f u_\epsilon \geq -u_\epsilon - c(n)u_\epsilon^2.$$

Using the proof of Proposition 1.9, one gets, with the same notations, $W_\epsilon := (vu_\epsilon + 1)^{-1}$,

$$\min_{t_1 \leq f \leq t_2} W_\epsilon = \min\left\{\min_{f=t_1} W_\epsilon, \min_{f=t_2} W_\epsilon\right\}, \tag{4.1}$$

for t_1 larger than a positive constant depending on $\sup_{M^n} u_\epsilon$ which does not depend on ϵ if ϵ is less than 1. Therefore, if ϵ goes to 0 in (4.1), and by letting t_2 go to $+\infty$, we get the expected estimate. □

Corollary 4.2. *Let* $(M^n, g, \nabla f)$ *be an expanding gradient Ricci soliton with nonnegative Ricci curvature such that* $\limsup_{+\infty} v\,\mathrm{R}_g < +\infty$. *Then,*

$$\sup_{M^n} v\,\mathrm{R}_g \leq C\left(\sup_{M^n}\mathrm{R}_g, \mathbb{A}^0_g(\mathrm{R}_g)\right).$$

Proof. By Proposition 2.4, f is an exhausting function and the scalar curvature is bounded. Moreover, the scalar curvature satisfies the following differential inequality since the Ricci curvature is nonnegative:

$$\Delta_f \mathrm{R}_g = -\mathrm{R}_g - 2|\mathrm{Ric}(g)|^2 \geq -\mathrm{R}_g - 2\mathrm{R}_g^2.$$

Hence the result by applying Proposition 1.9 to the scalar curvature R_g. □

As we are dealing with smooth convergence, we need to control globally the rescaled covariant derivatives of the curvature tensor in terms of the asymptotic curvature ratios $(\mathbb{A}^k_g(\mathrm{Rm}(g)))_{k\geq 0}$ and a bound on the curvature tensor. We state and prove the following proposition dealing only with the full curvature tensor.

Proposition 4.3. *Let $(M^n, g, \nabla f)$ be an expanding gradient Ricci soliton such that*

$$\mathbb{A}^i_g(\mathrm{Rm}(g)) < +\infty,$$

for $i \in \{0, ..., k\}$. Then,

$$\begin{aligned}&\sup_{M^n}\left(v^{1+k/2}|\nabla^k \mathrm{Rm}(g)|\right)\\ &\leq C\left(n, k, \min_{M^n} v, \sup_{M^n} |\mathrm{Rm}(g)|, \mathbb{A}^0_g(\mathrm{Rm}(g)), ..., \mathbb{A}^k_g(\mathrm{Rm}(g))\right).\end{aligned}$$

Proof of Proposition 4.3. We cannot apply directly Corollary 2.10, since it assumes a priori bounds on the full rescaled covariant derivatives of the curvature.

Nonetheless, we prove these estimates in a similar way by induction on k.

We begin by deriving a nice differential inequality satisfied by $\nabla^k \mathrm{Rm}(g)$:

$$\begin{aligned}\Delta_f|\nabla^k \mathrm{Rm}(g)|^2 = {}& 2|\nabla^{k+1}\mathrm{Rm}(g)|^2 + 2\langle[\Delta, \nabla^k]\mathrm{Rm}(g), \nabla^k \mathrm{Rm}(g)\rangle\\ &+2\langle[\nabla_{\nabla f}, \nabla^k]\mathrm{Rm}(g), \nabla^k \mathrm{Rm}(g)\rangle\\ &+2\langle\nabla^k(-\mathrm{Rm}(g) + \mathrm{Rm}(g) * \mathrm{Rm}(g)), \nabla^k \mathrm{Rm}(g)\rangle.\end{aligned}$$

According to (2.8) and (2.9) , we get for $k \geq 2$, (the case $k = 1$ can be proved in a more straightforward way)

$$\begin{aligned}\Delta_f|\nabla^k \mathrm{Rm}(g)|^2 \geq {}& 2|\nabla^{k+1}\mathrm{Rm}(g)|^2\\ &-2\left(1 + \frac{k}{2} + c(n, k)|\mathrm{Rm}(g)|\right)|\nabla^k \mathrm{Rm}(g)|^2 \qquad (4.2)\\ &-c(n, k)\sum_{i=1}^{k-1}|\nabla^i \mathrm{Rm}(g)||\nabla^{k-i}\mathrm{Rm}(g)||\nabla^k \mathrm{Rm}(g)|. \qquad (4.3)\end{aligned}$$

By applying the induction assumption, we get

$$\begin{aligned}&\Delta_f|\nabla^k \mathrm{Rm}(g)|^2 \geq 2|\nabla^{k+1}\mathrm{Rm}(g)|^2 - 2\left(1+\frac{k}{2}+c(n, k)|\mathrm{Rm}(g)|\right)|\nabla^k \mathrm{Rm}(g)|^2\\ &-C\left(n,k,\min_{M^n} v,\sup_{M^n}|\mathrm{Rm}(g)|,\mathbb{A}^0_g(\mathrm{Rm}(g)), \dots ,\mathbb{A}^{k-1}_g(\mathrm{Rm}(g))\right)v^{-(k/2+2)}|\nabla^k \mathrm{Rm}(g)|.\end{aligned}$$

By considering $u^k_\epsilon := \sqrt{|\nabla^k \mathrm{Rm}(g)|^2 + \epsilon^2}$, one has

$$\begin{aligned}\Delta_f u^k_\epsilon \geq {}& -\left(1 + \frac{k}{2} + c(n, k)|\mathrm{Rm}(g)|\right)u^k_\epsilon\\ &-C\left(n,k,\min_{M^n} v,\sup_{M^n}|\mathrm{Rm}(g)|, \mathbb{A}^0_g(\mathrm{Rm}(g)), \dots ,\mathbb{A}^{k-1}_g(\mathrm{Rm}(g))\right)v^{-(k/2+2)}.\end{aligned}$$

Since

$$\Delta_f(v^{k/2+1}) = \left(1+\frac{k}{2}\right) v^{k/2+1} + \frac{k}{2}\left(1+\frac{k}{2}\right) |\nabla \ln v|^2 v^{k/2+1},$$

we get, by Corollary 4.1,

$$\begin{aligned}\Delta_f\left(u_\epsilon^k v^{k/2+1}\right) \geq 2\left(1+\frac{k}{2}\right)&\left\langle \nabla(u_\epsilon^k v^{k/2+1}), \nabla \ln v\right\rangle\\ &-C(n,k,\sup_{M^n}|\operatorname{Rm}(g)|, \mathbb{A}_g^0(\operatorname{Rm}(g)))v^{-1}\left(u_\epsilon^k v^{k/2+1}\right)\\ &-C\left(n,k,\min_{M^n} v, \sup_{M^n}|\operatorname{Rm}(g)|, \mathbb{A}_g^0(\operatorname{Rm}(g)),\dots,\mathbb{A}_g^{k-1}(\operatorname{Rm}(g))\right)v^{-1}.\end{aligned}$$

Now,

$$\begin{aligned}&\Delta_{v-(2+k)\ln v} e^{-C_0 v^{-1}} = \frac{C_0}{v^2}\left(v-(2+k)\frac{|\nabla v|^2}{v}\right) + C_0^2|\nabla v^{-1}|^2,\\ &\Delta_{v-(2+k)\ln v}\left(u_\epsilon^k v^{k/2+1} e^{-C_0 v^{-1}}\right)\\ \geq\ & -2C_0\left\langle \nabla\left(u_\epsilon^k v^{k/2+1} e^{-C_0 v^{-1}}\right), \nabla v^{-1}\right\rangle\\ &+\left(\frac{C_0}{2} - C(n,k,\sup_{M^n}|\operatorname{Rm}(g)|, \mathbb{A}_g^0(\operatorname{Rm}(g))) - C_0^2 v^{-2}\right)\\ &\cdot v^{-1}\cdot\left(u_\epsilon^k v^{k/2+1} e^{-C_0 v^{-1}}\right)\\ &-C\left(n,k,\min_{M^n} v, \sup_{M^n}|\operatorname{Rm}(g)|, \mathbb{A}_g^0(\operatorname{Rm}(g)), \dots, \mathbb{A}_g^{k-1}(\operatorname{Rm}(g))\right) e^{-C_0 v^{-1}} v^{-1}\\ \geq\ & -2C_0\left\langle \nabla\left(u_\epsilon^k v^{k/2+1} e^{-C_0 v^{-1}}\right), \nabla v^{-1}\right\rangle\\ &-C\left(n,k,\min_{M^n} v, \sup_{M^n}|\operatorname{Rm}(g)|, \mathbb{A}_g^0(\operatorname{Rm}(g)), \dots, \mathbb{A}_g^{k-1}(\operatorname{Rm}(g))\right) e^{-C_0 v^{-1}} v^{-1},\end{aligned}$$

for

$$\begin{aligned}C_0 &= C\left(n,k,\sup_{M^n}|\operatorname{Rm}(g)|, \mathbb{A}_g^0(\operatorname{Rm}(g))\right),\\ v \geq C &:= C\left(n,k,\sup_{M^n}|\operatorname{Rm}(g)|, \mathbb{A}_g^0(\operatorname{Rm}(g))\right).\end{aligned}$$

Finally, for any constant C_1 larger than C, one has

$$\Delta_{v-(2+k)\ln v+2C_0 v^{-1}}\left(u_\epsilon^k v^{k/2+1} e^{-C_0 v^{-1}} - C_1 v^{-1}\right) \geq 0,$$

for $v \geq C\left(n, k, \sup_{M^n} |\mathrm{Rm}(g)|, \mathbb{A}^0_g(\mathrm{Rm}(g))\right)$. Therefore, by the maximum principle and the fact that these estimates does not depend on $\epsilon \leq 1$, one gets the expected estimates since one can assume that

$$u_\epsilon^k v^{k/2+1} e^{-C_0 v^{-1}} - C_1 v^{-1} \leq 0$$

on $v \leq C(n, k, \sup_{M^n} |\mathrm{Rm}(g)|, \mathbb{A}^0_g(\mathrm{Rm}(g)))$. □

Before proving Theorem 1.7, we need to introduce the notion of (C^∞) conical convergence for asymptotically conical Ricci expanders:

Definition 4.4. A sequence of pointed asymptotically conical Ricci expanders $(M_i^n, g_i, \nabla^{g_i} f_i, p_i)_i$ with corresponding asymptotic cones $(C(X_i), dr^2 + r^2 g_{X_i}, r\partial_r/2, o_i)_i$ converges to a pointed asymptotically conical Ricci expander $(M^n_\infty, g_\infty, \nabla^{g_\infty} f_\infty, p_\infty)$ with corresponding asymptotic cone $(C(X_\infty), dr^2 + r^2 g_{X_\infty}, r\partial_r/2, o_\infty)$ in the C^∞ *conical* topology if the following holds:

- $(M_i, g_i, p_i)_i$ converges to $(M_\infty, g_\infty, p_\infty)$ in the sense of the Cheeger-Gromov pointed smooth convergence;
- There exists a sequence of diffeomorphisms $\psi_i : X_\infty \to X_i$ such that for any positive ϵ and any nonnegative integer k, there exist a radius R and an index i_0 such that

$$\sup_{r \geq R} \sup_{\partial B(o_\infty, r)} r^l \left| \nabla^{\phi_\infty^* g_\infty, l} \left[(\phi_i^{-1} \circ (Id_r, \psi_i))^* g_i - \phi_\infty^* g_\infty \right] \right| \leq \epsilon, \quad i \geq i_0, \quad l \leq k,$$

  where $(\phi_i)_i$ and ϕ_∞ are the diffeomorphisms at infinity given by Definition 1.1 and where Id_r denotes the identity on the radial coordinates of the cone $C(X_\infty)$.

Remark 4.5. Notice that the convergence in the smooth conical topology implies the convergence of the corresponding asymptotic cones.

We are now in a position to state and prove the first compactness theorem:

Theorem 4.6. *The class*

$$\mathfrak{M}^0_{\mathrm{Exp}}(n, \lambda_0, (\Lambda_k)_{k \geq 0}) := \Big\{ (M^n, g, \nabla f, p) \text{ normalized expanding gradient Ricci soliton such that } \mathrm{Ric}(g) \geq 0; \quad \mathrm{Crit}(f) = \{p\}; \quad \sup_{M^n} |\mathrm{Rm}(g)| \leq \lambda_0; \quad \limsup_{x \to +\infty} r_p(x)^{2+k} |\nabla^k \mathrm{Rm}(g)|(x) \leq \Lambda_k, \quad \forall k \geq 0 \Big\}$$

is compact in the C^∞ conical topology.

Remark 4.7.

- Theorem 4.6 does not assume a uniform lower bound for the rescaled volume: this is actually a consequence of the proof of Theorem 4.6;
- We could also have assumed a bound on the Ricci curvature of the form $\mathrm{Ric}(g) \geq -\delta g$ with $\delta \in [0, 1/2)$ giving again the same properties for the potential function, the statement of Theorem 4.6 should be modified slightly then and would have been less concise;
- There is no a priori lower bound on the volume in the statement of Theorem 4.6. Actually, the nonnegativity of the Ricci curvature implies that X is diffeomorphic to a codimension one sphere endowed with a metric g_X with a positive lower bound on the Ricci curvature depending only on the dimension and a curvature bound Λ_0: one can then invoke a non-collapsing theorem due to Petrunin and Tuschmann [20] to bound from below the volume of g_X.

Proof of Theorem 4.6.

- Firstly, we remark that if $(M^n, g, \nabla f, p) \in \mathfrak{M}^0_{\mathrm{Exp}}(n, \lambda_0, (\Lambda_k)_{k\geq 0})$ then $(M^n, g, \nabla f)$ is asymptotically conical to $(C(X), g_{C(X)}, r\partial_r/2)$ by Theorem 1.2. Moreover, as $\mathrm{Ric}(g) \geq 0$, X is diffeomorphic to $\mathbb{S}^{n-1}$. We claim that

$$\mathrm{AVR}(g) \geq C(n, \mathbb{A}^0_g(\mathrm{Rm}(g))) \geq C(n, \Lambda_0) > 0.$$

Indeed, (X, g_X) is a smooth Riemannian manifold such that

$$\mathrm{Ric}(g_X) \geq (n-2)g_X,$$
$$\sup_X |\mathrm{Rm}(g_X)| \leq C(n, \mathbb{A}^0_g(\mathrm{Rm}(g))) \leq C(n, \Lambda_0).$$

On the one hand, if n is odd, then the Gauss-Bonnet applied to (X, g_X) gives

$$2 = \chi(X) = \frac{2}{\mathrm{Vol}(\mathbb{S}^{n-1})}\int_X \mathbf{K},$$

where

$$\mathbf{K} = \frac{1}{(n-1)!}\sum_{i_1<\ldots<i_{n-1}} \epsilon_{i_1,\ldots,i_{n-1}} \mathrm{Rm}(g_X)_{i_1,i_2} \wedge \ldots \wedge \mathrm{Rm}(g_X)_{i_{n-2},i_{n-1}},$$

where $\epsilon_{i_1,\ldots i_{n-1}}$ is the signature of the permutation $(i_1, \ldots, i_{n-1})$. Therefore,

$$0 < C(n) \leq \mathrm{Vol}_{g_X}(X)(C(n) + \mathbb{A}^0_g(\mathrm{Rm}(g))^{\frac{n-1}{2}} \leq C(n, \Lambda_0)\,\mathrm{AVR}(g),$$

which proves the claim if n is odd.

On the other hand, if n is even, we apply the π_2-theorem of [20] asserting in our setting that there exists a positive constant $V_0 = V_0(n, \Lambda_0)$ such that the

volume of any simply connected compact Riemannian manifold (X, g_X) with finite second homotopy group and such that $\mathrm{Ric}(g_X) \geq c(n)g_X > 0, \mathrm{Rm}(g_X) \leq C(n, \Lambda_0)$ is bounded from below by V_0.
As a consequence, any link of the asymptotic cones of such Ricci expanders belongs to $\mathfrak{M}(n-1, \pi, C(n, \Lambda_0), (\widetilde{\Lambda_k})_{k\geq 0})$ where $\widetilde{\Lambda_k}$ depends on a finite number of Λ_i. Indeed, the diameter estimate follows by the Myers theorem and the curvature estimates follow by Gauss equations. Hence, these metrics lie in a compact set for the C^∞ topology according to Theorem 1.6;

- Thanks to Proposition 2.4, we notice that

$$|\min_{M^n} f| + |\mu(g)| \leq C(n, \lambda_0, \Lambda_0). \tag{4.4}$$

Therefore, if $(M_i^n, g_i, \nabla f_i, p_i)_i \in \mathfrak{M}^0_{\mathrm{Exp}}(n, \lambda_0, (\Lambda_k)_{k\geq 0})$, then $(M_i^n, g_i, p_i)_i$ belongs to

$$\mathfrak{M}_{Pt}(n, v, (C(n,k,\lambda_0))_{k\geq 0}) := \Big\{(M^n, g, p) \text{ complete} \mid \mathrm{Vol}_g B_g(p,1) \geq v; \\ |\nabla^k \mathrm{Rm}(g)| \leq C(n,k,\lambda_0), \forall k \in \mathbb{N}\Big\},$$

where $v = v(n, \Lambda_0)$, which is compact for the pointed C^∞ topology. Indeed, the bounds on the covariant derivatives of the curvature tensor come from Lemma 2.6 and the volume estimate is due to the Bishop-Gromov theorem together with the estimate $\mathrm{AVR}(g) \geq C(n, \Lambda_0)$. In particular, $(M_i, g_i, p_i)_i$ (sub)converges to a smooth Riemannian manifold $(M_\infty, g_\infty, p_\infty) \in \mathfrak{M}_{Pt}(n, v, (C(n,k,\lambda_0))_{k\geq 0})$ with non negative Ricci curvature. Moreover, by the Ricci soliton equation together with the bounds (4.4), $(f_i)_i$ subconverges to a smooth function f_∞ satisfying the Ricci soliton as well: f_∞ is a smooth strictly convex function with $\mathrm{Crit}(f_\infty) = \{p_\infty\}$. We also check that f_∞ is normalized. Indeed, for a positive radius $R > 0$ and some index i, we have

$$\begin{aligned}\left|(4\pi)^{n/2} - \int_{B(p_i,R)} e^{-f_i} d\mu(g_i)\right| &\leq \int_{r_{p_i}\geq R} e^{-f_i} d\mu(g_i) \\ &\leq e^{-\min_{M_i^n} f_i} \int_{r_{p_i}\geq R} e^{-r_{p_i}^2/4} d\mu(g_i) \\ &\leq C(n, \lambda_0, \Lambda_0) \int_R^{+\infty} e^{-r^2/4} r^{n-1} dr.\end{aligned}$$

If R is fixed and if i tends to $+\infty$,

$$\left|(4\pi)^{n/2} - \int_{B(p_\infty,R)} e^{-f_\infty} d\mu(g_\infty)\right| \leq C(n, \lambda_0, \Lambda_0) \int_R^{+\infty} e^{-r^2/4} r^{n-1} dr,$$

which implies in particular that f_∞ is normalized;

- Now, we justify the inversion of limits.
  Let $(M_i^n, g_i, \nabla f_i, p_i)_i \in \mathfrak{M}^0_{\mathrm{Exp}}(n, \lambda_0, (\Lambda_k)_{k\geq 0})$ be converging to a normalized expanding gradient Ricci soliton $(M_\infty, g_\infty, \nabla f_\infty, p_\infty)$ with nonnegative Ricci curvature. Let $(C(X_i), dr^2+r^2 g_{X_i})_i$ be the sequence of asymptotic cones corresponding to $(M_i, g_i, p_i)_i$. As noticed previously, the sequence of compact Riemannian manifolds $(X_i, g_{X_i})_i$ has a subsequence converging to a smooth Riemannian manifold (X_∞, g_{X_∞}). On the other hand, according to Proposition 4.3 together with the crucial estimate (3.3) given in the proof of Theorem 1.2, one has,

$$\partial_t \nabla^{g_{X_i},k} \bar{g}_{i\,t} = O(t^{-3}), \quad \forall k \geq 0, \tag{4.5}$$

  where O depends on k and is uniform in the indices i. In particular,

$$\nabla^{g_{X_i},k}((\bar{g}_i)_t - g_{X_i}) = O(t^{-2}),$$

  which proves that the asymptotic cone of $(M_\infty, g_\infty, p_\infty)$ is isometric to $(C(X_\infty), dr^2 + r^2 g_{X_\infty})$.
  The last step is to make sure that the invariants $(\mathbb{A}^k_{g_\infty}(\mathrm{Rm}(g_\infty))_{k\geq 0}$ have not improved. Indeed,

$$\mathbb{A}^k_{g_\infty}(\mathrm{Rm}(g_\infty)) = \lim_{i\to+\infty} \limsup_{x\to+\infty} d^{2+k}_{g_i}(p_i, x)|\nabla^k \mathrm{Rm}(g_i)| \leq \Lambda_k. \qquad \square$$

Remark 4.8. What if one only assumes bounds on the asymptotic covariant derivatives of the Ricci curvature $(\mathbb{A}^k_g(\mathrm{Ric}(g)))_{k\geq 0}$? This question is motivated by Theorem 1.2 where it is proved that the full curvature tensor is actually asymptotically controlled by the Ricci curvature.

We are now in a position to prove Theorem 1.7 for Ricci expander with nonnegative curvature operator: notice that the bound on the curvature is replaced by a lower bound on the asymptotic volume ratio.

Remark 4.9.

- If one is only interested in $C^{1,\alpha}$ convergence in the preceding theorem, assuming $\mathbb{A}^0(\mathrm{R}_g) \leq \Lambda_0$ is enough to ensure such convergence since the scalar curvature controls pointwise the curvature operator;
- Again, compared to Theorem 4.6, Theorem 1.7 does not assume any a priori bound on the curvature tensor: this is actually a consequence as shown in the proof below;
- The proof of Theorem 1.7 is more self-contained than the one of Theorem 1.7 since it does not use the involved non collapsing theory developed in [20].

Proof of Theorem 1.7. By Theorem 4.6, it suffices to prove that if $(M^n, g, \nabla f, p)$ belongs to $\mathfrak{M}^{\mathrm{Vol}}_{\mathrm{Exp}}(n, (\Lambda_k)_{k\geq 0}, V_0)$ then

$$\sup_{M^n} |\mathrm{Rm}(g)| \leq C(n, V_0).$$

This estimate has already been proved in greater generality by Schulze and Simon [24]. Nonetheless we give here a self-contained proof. This goes by contradiction. Assume there exists a sequence of expanding gradient Ricci solitons $(M_i^n, g_i, \nabla f_i, p_i)_i$ such that

$$\mathrm{Rm}(g_i) \geq 0; \quad \mathrm{AVR}(g_i) \geq V_0; \quad \mathrm{Crit}(f_i) = \{p_i\}; \quad \sup_i \sup_{M_i^n} \mathrm{R}_{g_i} = +\infty.$$

As the Ricci curvatures of the metrics g_i are nonnegative, the scalar curvatures R_{g_i} attain their maximum at the (unique) point $p_i \in M_i^n$. Moreover, we have the following estimates of the potential functions by Proposition 2.4:

$$\frac{1}{4} r_{p_i}(x)^2 + \min_{M_i^n} v_i \leq v_i(x) \leq \left(\frac{1}{2} r_{p_i}(x) + \sqrt{\min_{M_i^n} v_i} \right)^2, \tag{4.6}$$

for any $x \in M_i^n$. By (2.2), we have $\min_{M_i^n} v_i = \max_{M_i^n} \mathrm{R}_{g_i} + n/2 = \mathrm{R}_{g_i}(p_i) + n/2$. Define the rescaled metrics $\tilde{g_i}(\tau) := Q_i g_i(Q_i^{-1}\tau)$ where $Q_i := \mathrm{R}_{g_i}(p_i) + n/2$ and $g_i(\cdot)$ is the associated Ricci flow to the expanding gradient Ricci soliton $(M_i^n, g_i, \nabla f_i)$. Then the sequence $(M_i^n, \tilde{g_i}(\tau))_{\tau \in (-Q_i, +\infty)}$ of Ricci flows satisfies: $\sup_{M^n} |\mathrm{Rm}(\tilde{g_i}(\tau))|$ is uniformly bounded in the indices i for τ in a compact interval of $(-Q_i, +\infty)$, $\mathrm{R}_{\tilde{g_i}(0)}(p_i) = 1$, $\mathrm{Rm}(\tilde{g_i}(\tau)) \geq 0$ and $\mathrm{AVR}(\tilde{g_i}(\tau)) \geq V_0$ for $\tau \in (-Q_i, +\infty)$ which means in particular that $\mathrm{inj}_{\tilde{g_i}(0)}(p_i) \geq C(n, V_0) > 0$. Therefore, by Hamilton's compactness theorem [15], there is a subsequence, still denoted by $(M_i^n, \tilde{g_i}(\tau), p_i)_i$ converging in the pointed Cheeger-Gromov topology to a non flat eternal solution of the Ricci flow $(M_\infty^n, g_\infty(\tau), p_\infty)$ with nonnegative curvature operator and Euclidean volume growth, *i.e.* such that $\mathrm{AVR}(g_\infty) > 0$. Moreover, $\tilde{v_i} := v_i - \min_{M_i^n} v_i$ satisfy by the estimates (4.6):

$$0 \leq \frac{r^2_{\tilde{g_i}(0), p_i}}{4Q_i} \leq \tilde{v_i} \leq \frac{r^2_{\tilde{g_i}(0), p_i}}{4Q_i} + r_{\tilde{g_i}(0), p_i}; \quad \tilde{v_i}(p_i) = 0.$$

Moreover, by (2.2),

$$|\nabla^{\tilde{g_i}(0)} \tilde{v_i}|_{\tilde{g_i}(0)} \leq Q_i^{-1} \left(\frac{r^2_{\tilde{g_i}(0), p_i}}{4Q_i} + r_{\tilde{g_i}(0), p_i} + Q_i \right),$$

which means in particular that the sequence of functions $(\tilde{v_i})_i$ is equicontinuous. Finally, the soliton equation can be rewritten as

$$\nabla^{2, \tilde{g_i}(0)} \tilde{v_i} = \mathrm{Ric}(\tilde{g_i}(0)) + \frac{\tilde{g_i}(0)}{2Q_i}.$$

Therefore, the higher covariant derivatives of $\tilde{v_i}$ are uniformly bounded and $\tilde{v_i}$ converges smoothly to a smooth function v_∞ which satisfies

$$\nabla^{2, g_\infty} v_\infty = \mathrm{Ric}(g_\infty),$$

i.e. $(M_\infty, g_\infty, \nabla v_\infty)$ is a non flat steady gradient Ricci soliton with nonnegative curvature operator and positive asymptotic volume ratio: a contradiction to Hamilton's argument [9, Chapter 9, Section 3] which can be proved by induction on the dimension. □

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Institut de Mathématiques de Jussieu
Paris Rive Gauche (IMJ-PRG)
4, place Jussieu
75252 Paris Cedex 05, France
alix.deruelle@imj-prg.fr