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Comparing $\mathbb{A}^1$-h-cobordism and $\mathbb{A}^1$-weak equivalence

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Abstract. We study the problem of classifying projectivizations of rank-two vector bundles over $\mathbb{P}^2$ up to two notions of equivalence that arise naturally in $\mathbb{A}^1$-homotopy theory, namely $\mathbb{A}^1$-weak equivalence and $\mathbb{A}^1$-h-cobordism.

First, we classify such varieties up to $\mathbb{A}^1$-weak equivalence: over algebraically closed fields having characteristic unequal to two the classification can be given in terms of characteristic classes of the underlying vector bundle. When the base field is $\mathbb{C}$, this classification result can be compared to a corresponding topological result and we find that the algebraic and topological homotopy classifications agree.

Second, we study the problem of classifying such varieties up to $\mathbb{A}^1$-h-cobordism using techniques of deformation theory. To this end, we establish a deformation rigidity result for $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ which links $\mathbb{A}^1$-h-cobordisms to deformations of the underlying vector bundles. Using results from the deformation theory of vector bundles we show that if X is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ and Y is the projectivization of a direct sum of line bundles on $\mathbb{P}^2$, then if X is $\mathbb{A}^1$-weakly equivalent to Y, X is also $\mathbb{A}^1$-h-cobordant to Y.

Finally, we discuss some subtleties inherent in the definition of $\mathbb{A}^1$-h- cobordism. We show, for instance, that direct $\mathbb{A}^1$-h-cobordism fails to be an equivalence relation.

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1. Introduction

In this note, we study the relation of two classification problems in the topology of algebraic varieties. On the one hand, there is the problem of classifying smooth

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proper varieties over a fixed field k up to $\mathbb{A}^1$-weak equivalence. We refer to this as the $\mathbb{A}^1$-homotopy classification problem. On the other hand, there is the problem of classifying smooth proper varieties having a fixed $\mathbb{A}^1$-homotopy type. This is an analogue of the surgery problem in differential topology. These problems were initially studied in [4] for varieties of dimension at most two. For this the notion of $\mathbb{A}^1$-h-cobordism of smooth proper varieties was introduced as an algebraic replacement of h-cobordism of smooth manifolds. By definition, varieties that are $\mathbb{A}^1$-h-cobordant are $\mathbb{A}^1$-weakly equivalent and in [4] examples are produced to show that $\mathbb{A}^1$-h-cobordant varieties need not be isomorphic.

The present work takes the next step, studying these classification problems in dimension three. The varieties we consider are projectivizations of rank-two vector bundles on the projective plane $\mathbb{P}^2$ over a fixed base field, which will be suppressed from the notation.

1.1. Classification up to $\mathbb{A}^1$-weak equivalence

As a first result in this direction, we can provide a complete classification of such varieties up to $\mathbb{A}^1$-weak equivalence, at least for certain base fields.

Theorem 1.1 (see Theorem 4.5). *Assume k is an algebraically closed field having characteristic unequal to two. If $\mathscr{E}$ and $\mathscr{F}$ are two vector bundles over the projective plane over $\mathbb{P}^2$, each of rank two, then the following are equivalent:*

(1.1.1) *The pairs of Chern classes $\big(c_1(\mathscr{E}), c_2(\mathscr{E})\big)$ and $\big(c_1(\mathscr{F}), c_2(\mathscr{F})\big)$ are in the same orbit for the action of* $\mathrm{Pic}(\mathbb{P}^2)$ *on* $\mathrm{Pic}(\mathbb{P}^2) \times CH^2(\mathbb{P}^2)$ *induced from twisting by line bundles,* cf. *Theorem* 3.10 *and Corollary* 3.11*;*
(1.1.2) *There is an $\mathbb{A}^1$-weak equivalence $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \simeq_{\mathbb{A}^1} \mathbb{P}_{\mathbb{P}^2}(\mathscr{F})$.*

To establish this result, we first provide an $\mathbb{A}^1$-homotopy classification of PGL_2-torsors over $\mathbb{P}^2$. This classification is obtained by appeal to techniques of obstruction theory, *cf.* Theorem 3.10 and Corollary 3.11. Results from the theory of fiber sequences then show that homotopies of classifying maps of Zariski locally trivial $\mathbb{P}^1$-bundles yield $\mathbb{A}^1$-weak equivalences of total spaces, *cf.* Corollary 4.2. Conversely, using the cubic form on the Picard group and some results from classical invariant theory, we show that if two $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ are $\mathbb{A}^1$-weakly equivalent, then the underlying vector bundles have the same classifying maps. The weak equivalence between total spaces is then induced from a homotopy between those classifying maps; roughly speaking "every $\mathbb{A}^1$-weak equivalence between the total spaces of Zariski locally trivial $\mathbb{P}^1$-bundles is induced by a fiber homotopy equivalence".

1.2. Classification up to cobordism

The second part of the paper is devoted to understanding $\mathbb{A}^1$-h-cobordism classes within a given $\mathbb{A}^1$-homotopy type in the special case of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$. We obtain the following partial classification result, which exhibits some interesting subtleties of the notion of $\mathbb{A}^1$-h-cobordism.

Theorem 1.2 (see Proposition 6.5 and Theorem 6.6). *Let k be an algebraically closed field having characteristic unequal to two, and let $c_1, c_2 \in \mathbb{Z}$ be integers. The following results concerning $\mathbb{A}^1$-h-cobordism classes of rank-two vector bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2 hold:*

(1.2.1) *If there exists an integer d such that $d^2 - dc_1 + c_2 = 0$, then for any two rank-two vector bundles $\mathscr{E}$ and $\mathscr{F}$ on $\mathbb{P}^2$ with Chern classes c_1 and c_2, the corresponding projective bundles $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$ and $\mathbb{P}_{\mathbb{P}^2}(\mathscr{F})$ are $\mathbb{A}^1$-h-cobordant. In particular, any $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ which is $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}^1 \times \mathbb{P}^2$ is also $\mathbb{A}^1$-h-cobordant to $\mathbb{P}^1 \times \mathbb{P}^2$;*

(1.2.2) *There are infinitely many rank-two vector bundles $(\mathscr{E}_i)_{i \in \mathbb{N}}$ on $\mathbb{P}^2$ with Chern classes c_1 and c_2 such that no two of the varieties $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_i)$ are directly $\mathbb{A}^1$-h-cobordant. In particular, direct $\mathbb{A}^1$-h-cobordism is rather far from being an equivalence relation.*

These results rely on a certain deformation rigidity result, which provides a close relation between $\mathbb{A}^1$-h-cobordisms and $\mathbb{A}^1$-deformations of the underlying vector bundles: given an $\mathbb{A}^1$-h-cobordism with $f^{-1}(0)$ a given $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, there is a Zariski open neighborhood of $0 \in \mathbb{A}^1$ over which the resulting deformation is induced from a deformation of rank-two vector bundles over $\mathbb{P}^2$; see Theorem 2.11 for a precise statement. This result allows us to import some results of Strømme, [30], to help investigate the $\mathbb{A}^1$-h-cobordism classification of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$. Assuming some condition on the Chern classes, like those imposed in Part (1.2.1) of Theorem 1.2, we observe that there are "enough" deformations of rank-two vector bundles over $\mathbb{P}^2$ to guarantee that the $\mathbb{A}^1$-h-cobordism classification is not finer than the $\mathbb{A}^1$-homotopy classification, *cf.* Proposition 6.5. On the other hand, the non-deformability results of Strømme imply the existence of infinitely many varieties in each of the above $\mathbb{A}^1$-weak homotopy types which cannot be connected by direct $\mathbb{A}^1$-h-cobordisms. These observations lead to the results spelled out in Part (1.2.2) of Theorem 1.2.

Our $\mathbb{A}^1$-h-cobordism classification result is incomplete because we impose restrictions on the Chern classes of the vector bundles under consideration. The main reason for these restrictions stems from the difficulties inherent in providing an isomorphism or deformation classification of vector bundles on $\mathbb{P}^2$. While the explicit families of vector bundles produced in Strømme's work are enough to prove connectedness of the "moduli space of rank two bundles on $\mathbb{P}^2$", they do not allow us to establish the $\mathbb{A}^1$-chain connectedness of that space. At the moment, we are unable to decide if Part (1.2.1) of Theorem 1.2 can be extended to all projective bundles or if there exist projective bundles which are $\mathbb{A}^1$-weakly equivalent but not $\mathbb{A}^1$-h-cobordant.

Finally, we take a moment to indicate more abstractly the main difficulties involved in the study of $\mathbb{A}^1$-h-cobordism. Our problem, phrased a bit more broadly, is to understand all the smooth proper varieties having a fixed $\mathbb{A}^1$-homotopy type, say modulo various notions of equivalence. Varieties in a fixed $\mathbb{A}^1$-homotopy type can appear in families. Thus, it is natural to try to construct a "moduli space of scheme structures in a fixed $\mathbb{A}^1$-homotopy type." In order to analyze $\mathbb{A}^1$-h-cobordisms, we would ideally like this moduli problem to be representable by a smooth scheme: if that was true, then we could try to construct $\mathbb{A}^1$-h-cobordisms by producing maps from $\mathbb{A}^1$ to the moduli space. However, difficulties arise involving both of these ideas. Indeed, the moduli problem need not be representable by a smooth scheme, and it turns out to be hard to construct $\mathbb{A}^1$-h-cobordisms.

Already in the case of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ the moduli problem is not representable by a smooth scheme. Nevertheless, after fixing an additional invariant, the generic splitting type, one can construct suitable moduli schemes within the $\mathbb{A}^1$-homotopy type, though we make no claim that these moduli schemes actually exhaust the $\mathbb{A}^1$-homotopy type. Indeed, it seems likely that there are smooth projective varieties that are $\mathbb{A}^1$-h-cobordant to $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ but that are not themselves of this form. In this case, deformations of a vector bundle parameterized by the affine line give rise to $\mathbb{A}^1$-h-cobordisms, so there is a close connection between affine lines on the moduli space and $\mathbb{A}^1$-h-cobordisms. The results of Strømme establish that the moduli problem as a whole is connected (in the usual topology) for each $\mathbb{A}^1$-homotopy type of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$, but has infinitely many irreducible components. These observations lead to the failure of $\mathbb{A}^1$-h-cobordism to be an equivalence relation.

We summarize these observations as a slogan: $\mathbb{A}^1$-h-cobordism is sensitive to the geometry of moduli of scheme structures. In fact, it seems likely that $\mathbb{A}^1$-h-cobordism is only well-behaved if the "moduli space of scheme structures" is well-behaved, say, locally $\mathbb{A}^1$-contractible. In view of Murphy's law for moduli spaces, [31], this "local $\mathbb{A}^1$-contractibility of the moduli space of scheme structures" is likely to hold, if ever, only in very special cases.

1.3. Structure of the paper

Section 2 gathers a number of results concerning rank-two vector bundles on $\mathbb{P}^2$ and their associated projective bundles. To the best of our knowledge, some of these results are new and might be of independent interest. In Section 3 we discuss the $\mathbb{A}^1$-homotopy classification of PGL_2-torsors over $\mathbb{P}^2$, from which we deduce the $\mathbb{A}^1$-homotopy classification of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ in Section 4. Then, we turn to the more geometric equivalence relations. We define $\mathbb{A}^1$-concordance and discuss the classification of rank-two bundles over $\mathbb{P}^2$ up to $\mathbb{A}^1$-concordance in Section 5; consequences of these results for the $\mathbb{A}^1$-h-cobordism classification of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ are contained in Section 6. Finally, in Section 7, we compare the algebraic classification results with corresponding topological classification results in the setting of complex manifolds.

1.4. Notation and global conventions

Throughout this paper, we work with schemes that are separated and have finite type over an algebraically closed field k. With the exception of Section 3, the characteristic of k will always be unequal to two. Following notation from Hartshorne's book, an *abstract variety* is an integral, separated scheme of finite type over k. We use the word "sheaf" to mean "coherent sheaf", unless noted otherwise.

Throughout this paper, we fix a hyperplane class H on $\mathbb{P}^2$ and use this to identify $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z} \cdot H$ and $CH^2(\mathbb{P}^2) \cong \mathbb{Z} \cdot H^2$. If $\mathscr{E}$ is a rank-two vector bundle on $\mathbb{P}^2$, we use these identifications to view the Chern classes $c_i(\mathscr{E})$ as integers.

2. Vector bundles over $\mathbb{P}^2$ and associated projective bundles

For the reader's convenience, we briefly recall in this section notation and results pertaining to vector bundles on $\mathbb{P}^2$, to families of vector bundles, and to their moduli spaces. Section 2.1 begins by recalling some results about the Hartshorne-Serre correspondence relating vector bundles to codimension-two local complete intersections. We are particularly interested in the relative setting. Section 2.2 recalls a Bertini-type theorem, which appears in the work of Kleiman. Section 2.3 contains a uniqueness result about projective bundle structures, see Theorem 2.10. Section 2.4 contains the deformation rigidity result mentioned in the introduction, Theorem 2.11. Finally, Section 2.5 recalls some results about the deformation theory of vector bundles on $\mathbb{P}^2$. With the exception of Section 2.5, which requires the notion of *type* of a vector bundle, see Definition 2.9, these sections are written to be independent of each other.

2.1. The Hartshorne-Serre correspondence

We will briefly recall the well-known correspondence between rank-two vector bundles on a given smooth variety X and codimension-two local complete intersections $Y \subset X$. The following simplified version suffices for our purposes.

Fact 2.1 (Hartshorne-Serre correspondence [5, Theorem 1.1]). Let X be any smooth variety of dimension $\dim X \geq 2$, and $Y \subseteq X$ be a local complete intersection of codimension two, with ideal sheaf $\mathscr{I}_Y \subset \mathscr{O}_X$. Let $\mathscr{L} \in \mathrm{Pic}(X)$ be any line bundle such that the dualizing sheaf ω_Y is isomorphic to $(\mathscr{L} \otimes \omega_X)|_Y$. Then, there exists a canonically defined, functorial exact sequence

$$\begin{aligned} H^1\big(X,\, \mathscr{L}^*\big) &\to \mathrm{Ext}^1\big(\mathscr{I}_Y \otimes \mathscr{L},\, \mathscr{O}_X\big) \\ &\to H^0\big(Y,\, \wedge^2 \mathscr{N}_{X/Y} \otimes \mathscr{L}^*|_Y\big) \to H^2\big(X,\, \mathscr{L}^*\big). \end{aligned} \tag{2.1}$$

Remark 2.2. In the setting of Fact 2.1, if we assume in addition that the cohomology groups $H^1\big(X,\, \mathscr{L}^*\big)$ and $H^2\big(X,\, \mathscr{L}^*\big)$ vanish, then Sequence (2.1) yields a canonical isomorphism $\mathrm{Ext}^1\big(\mathscr{I}_Y \otimes \mathscr{L},\, \mathscr{O}_X\big) \cong H^0\big(Y,\, \wedge^2 \mathscr{N}_{X/Y} \otimes \mathscr{L}^*|_Y\big)$.

Theorem 2.3 (Characterisation of locally frees, I). *In the setting of Fact 2.1, given an extension of the form*

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\ s\ } \mathscr{E} \longrightarrow \mathscr{I}_Y \otimes \mathscr{L} \longrightarrow 0, \tag{2.2}$$

then $\mathscr{E}$ is locally free if any only if the section of $\wedge^2 \mathscr{N}_{X/Y} \otimes \mathscr{L}^*$ that is associated with the extension class of (2.2) *generates that sheaf.*

Proof. This result is established in [5, Theorem 1.1] under the additional assumption that the cohomology groups $H^1(X, \mathscr{L}^*)$ and $H^2(X, \mathscr{L}^*)$ vanish. Note that Sequence (2.1) is functorial with respect to restriction maps. By picking an open affine cover of X, we can always guarantee the necessary cohomology vanishing, and local freeness can be checked by restriction to each open affine in the cover. □

Remark 2.4 (Characterisation of locally frees, II). Let X be any smooth variety of dimension $\dim X \geq 2$, and $Y \subseteq X$ be a local complete intersection of codimension two, with ideal sheaf $\mathscr{I}_Y \subset \mathscr{O}_X$. Given an extension of the form (2.2), observe that the sheaf $\mathscr{E}$ is locally free if and only if the section s vanishes precisely on Y.

Remark 2.5 (Hartshorne-Serre correspondence for bundles on $\mathbb{P}^2$). Consider the case where $X = \mathbb{P}^2$, where $Y \subset X$ is any finite, reduced subscheme and where $\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^2}(d)$ with $d < 3$. It will then follow directly from Serre duality that $H^2(X, \mathscr{L}^*) = 0$. The assumption that ω_Y be isomorphic to $(\mathscr{L} \otimes \omega_X)|_Y$ is vacuous in this case. The bundle $\bigwedge^2 \mathscr{N}_{X/Y} \otimes \mathscr{L}^*|_Y$ is the trivial line bundle on Y. Since $H^1(\mathbb{P}^2, \mathscr{L}^*) = 0$, each nowhere-vanishing section σ in $H^0(Y, \mathscr{O}_Y)$ gives rise to a (unique up to isomorphism) rank-two vector bundle on $\mathbb{P}^2$.

The following corollary applies this result. It will later be used to construct deformations of the bundle $\mathscr{E}$ by moving points in Y within $\mathbb{P}^2$. The following notation will be useful.

Notation 2.6. Using that $\mathrm{Pic}(\mathbb{P}^2) \cong \mathrm{Pic}(\mathbb{P}^2 \times \mathbb{A}^1)$, identify $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{A}^1) \cong \mathbb{Z}$. Given any integer n, write $\mathscr{O}_{\mathbb{P}^2 \times \mathbb{A}^1}(n)$ for the corresponding line bundle. In a similar vein, identify $CH^2(\mathbb{P}^2 \times \mathbb{A}^1) \cong \mathbb{Z}$. Given any rank-two vector bundle $\mathscr{E}$ on $\mathbb{P}^2 \times \mathbb{A}^1$, we can thus identify the Chern classes $c_i(\mathscr{E})$ with integers.

Corollary 2.7 (Extension of vector bundles). *Consider the quasi-projective variety $X := \mathbb{P}^2 \times \mathbb{A}^1$ and the projection onto the second factor $\pi : X \to \mathbb{A}^1$. Let $Y \subset X$ be the union of m pairwise disjoint sections of π and $d \leq 2$ be any integer. Write $X_0 := \mathbb{P}^2 \times \{0\}$ and $Y_0 := X_0 \cap Y$. Assume we are given a rank-two bundle $\mathscr{E}_0$ on X_0, defined by an extension,*

$$0 \longrightarrow \mathscr{O}_{X_0} \xrightarrow{s_0} \mathscr{E}_0 \longrightarrow \mathscr{I}_{Y_0} \otimes \mathscr{O}_{X_0}(d) \longrightarrow 0. \tag{2.3}$$

Then, there exists a rank-two bundle $\mathscr{E}$ on X, defined by an extension

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{s} \mathscr{E} \longrightarrow \mathscr{I}_Y \otimes \mathscr{O}_X(d) \longrightarrow 0,$$

such that $\mathscr{E}|_{X_0} \cong \mathscr{E}_0$ and $s|_{X_0} = s_0$.

Proof of Corollary 2.7

For the reader's convenience, the proof is subdivided into two steps.

Step 1*: Establishing prerequisites for the Hartshorne-Serre correspondence*. In order to construct the bundle $\mathscr{E}$, we aim to apply the results of Fact 2.1 and Theorem 2.3 to X, with $\mathscr{L} = \mathscr{O}_X(d)$. Observe that Y is a local complete intersection, being a smooth, closed subscheme of X. Since Y is isomorphic to a disjoint union of m copies of $\mathbb{A}^1$, it follows that all locally free sheaves on Y are free. The assumption that ω_Y be isomorphic to $(\mathscr{L} \otimes \omega_X)|_Y$ is therefore vacuous.

In order to verify vanishing of $H^1\big(X, \mathscr{L}^*\big)$ and $H^2\big(X, \mathscr{L}^*\big)$, consider the Leray spectral sequence associated with π, [12, Chapter II.4.17], which takes the form

$$E^2_{ij} = H^i\big(\mathbb{A}^1, \mathbb{R}^j\pi_*\mathscr{L}^*\big) \Longrightarrow H^{i+j}\big(X, \mathscr{L}^*\big).$$

Since $\mathbb{A}^1$ is affine, the cohomology groups $H^i\big(\mathbb{A}^1, \mathscr{F}\big)$ vanish for any quasi-coherent sheaf $\mathscr{F}$ and any number $i > 0$. In particular, the spectral sequence collapses at the E^2-page, [22, Chapter I.1, Ex. 1.B], and yields isomorphisms

$$H^0\big(\mathbb{A}^1, \mathbb{R}^j\pi_*\mathscr{L}^*\big) \cong H^j\big(X, \mathscr{L}^*\big) \quad \text{for all } j \geq 0. \tag{2.4}$$

Identify $X = \mathbb{P}^2 \times \mathbb{A}^1$ with the projectivization of the trivial rank-three bundle on $\mathbb{A}^1$. With this identification, it follows from the special case of relative duality discussed in [17, Chapter III, Exc. 8.4c] that there is a canonical isomorphism

$$\mathbb{R}^2\pi_*\mathscr{L}^* \cong \big(\pi_*(\mathscr{L} \otimes \omega_X)\big)^* \cong \big(\pi_*\mathscr{O}_X(d-3)\big)^* = 0 \quad \text{since } d \leq 2. \tag{2.5}$$

Combining (2.4) and (2.5), we see that $H^2\big(X, \mathscr{L}^*\big) = 0$. A somewhat simpler argument, left to the reader, shows that $H^1\big(X, \mathscr{L}^*\big)$ vanishes as well.

Step 2*: Application of the Hartshorne-Serre correspondence*. All prerequisites satisfied, Fact 2.1 identifies

$$\operatorname{Ext}^1\big(\mathscr{I}_Y \otimes \mathscr{O}_X(d), \mathscr{O}_X\big) \cong H^0\big(Y, \wedge^2\mathscr{N}_{X/Y} \otimes \mathscr{L}^*|_Y\big).$$

Likewise, the extension class of (2.3) is identified with an element

$$\begin{aligned} \sigma_0 \in \operatorname{Ext}^1\big(\mathscr{I}_{Y_0} \otimes \mathscr{O}_{X_0}(d), \mathscr{O}_{X_0}\big) &\cong H^0\big(Y_0, \wedge^2\mathscr{N}_{X_0/Y_0} \otimes \mathscr{L}^*|_{Y_0}\big) \\ &= H^0\big(Y_0, (\wedge^2\mathscr{N}_{X/Y} \otimes \mathscr{L}^*)|_{Y_0}\big) \end{aligned}$$

that, by Theorem 2.3, generates $\wedge^2\mathscr{N}_{X_0/Y_0} \otimes \mathscr{L}^*|_{Y_0}$. To conclude, it will therefore suffice to find a section $\sigma \in H^0\big(Y, \wedge^2\mathscr{N}_{X/Y} \otimes \mathscr{L}^*|_Y\big)$, which generates $\wedge^2\mathscr{N}_{X/Y} \otimes \mathscr{L}^*|_Y$ and restricts to σ_0. Since $\wedge^2\mathscr{N}_{X/Y} \otimes \mathscr{L}^*|_Y$ is isomorphic to the trivial sheaf $\mathscr{O}_Y$, this is easily possible. □

2.2. A Bertini-type theorem

Generalizing the classical Bertini theorem, Kleiman gave conditions guaranteeing that the zero locus of a sufficiently general section of a vector bundle is non-singular. We state a version of Kleiman's Bertini theorem here; our formulation is quoted from a paper of Hartshorne [18, Proposition 1.4].

Fact 2.8 (Bertini-type theorem for sections in vector bundles [19,Corollary 3.6]). Let $\mathscr{E}$ by any rank-two vector bundle on $\mathbb{P}^n$, for $n \geq 2$. If $\mathscr{E}(-1)$ is generated by global sections, then for all sufficiently general $s \in H^0\big(\mathbb{P}^n, \mathscr{E}\big)$, the associated scheme of zeros is non-singular.

2.3. Uniqueness of the bundle structure

The goal of this subsection is to establish Theorem 2.10, which shows that the $\mathbb{P}^1$-bundle structure on the projectivization of a rank-two bundle on $\mathbb{P}^2$ is often unique. In order to state the result, we need to recall the following definition. This notion was studied by Strømme, [30], and will reappear in later sections.

Definition 2.9 (Type of a bundle on $\mathbb{P}^2$ [30, Section 1.1]). If $\mathscr{E}$ is a vector bundle on $\mathbb{P}^2$, set

$$d(\mathscr{E}) := \begin{cases} -1 & \text{if } \mathscr{E} \text{ is slope-stable} \\ \max\big\{d \mid H^0\big(\mathbb{P}^2, \mathscr{E}(-d)\big) > 0\big\} & \text{otherwise.} \end{cases}$$

The number $d(\mathscr{E})$ is called the *generic splitting type* of $\mathscr{E}$, and $\mathscr{E}$ will be said to be "of *type d*".

Theorem 2.10 (Uniqueness of bundle structure). *Fix two numbers $c_1 \in \{0, -1\}$ and $c_2 \in \mathbb{Z}$ and let d be any number such that $d > 3 + c_1$. Let $\mathscr{E}$ be any rank-two vector bundle on $\mathbb{P}^2$ with Chern classes c_1 and c_2 and type d, and let $\pi : \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \to \mathbb{P}^2$ be the obvious bundle map. Given any other morphism $\phi : \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \to \mathbb{P}^2$ that has the structure of a Zariski locally trivial $\mathbb{P}^1$-bundle, there exists an automorphism $\psi : \mathbb{P}^2 \to \mathbb{P}^2$ fitting into a commutative diagram of the form:*

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) & = & \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^2 & \xrightarrow[\psi]{} & \mathbb{P}^2. \end{array}$$

Proof of Theorem 2.10

We prove Theorem 2.10 in the remaining part of Section 2.3. For the reader's convenience, the proof is subdivided into three relatively independent steps.

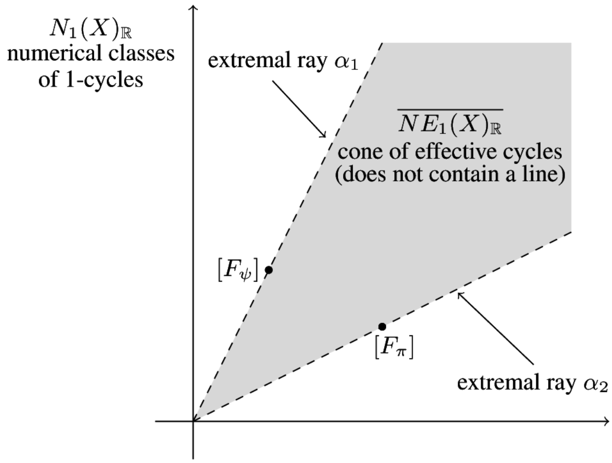

Figure 2.1. The figure illustrates the vector space $N_1(X)_{\mathbb{R}}$ of numerical curve classes that appears in the proof of Theorem 2.10. The closed cone $\overline{NE_1(X)_{\mathbb{R}}}$, which is spanned by effective cycles, does not contain a line and therefore has exactly two extremal rays, α_1 and α_2. Under the assumptions made in the proof, it will turn out that these rays are generated by numerical classes of fibers of the bundles π and ψ, respectively.

Step 1. *Setup*. Since $\mathbb{P}^2$ is normal, the claim of Theorem 2.10 follows from Zariski's main theorem as soon as we show that any ϕ-fiber F is also a fiber of π. Since fibers of π are characterized as those curves that intersect $c_1\big(\pi^*\mathscr{O}_{\mathbb{P}^2}(1)\big)$ trivially, it suffices to show that the numerical classes of π-fibers and ϕ-fibers agree up to multiplication with a positive constant. We argue by contradiction and assume that this is not the case. Using standard arguments of minimal model theory, we will see in Step 2 that this assumption implies that $X := \mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$ is Fano, that is, that the anti-canonical divisor $-K_X$ is ample. Step 3 then shows that the numerical assumptions made in Theorem 2.10 are incompatible with the Fano property.

Step 2. The Picard-number of X being two, it follows from the Theorem of the base of Néron-Severi, [20, II Theorem 4.5] and references there in, that the vector space of numerical curve classes, $N_1(X)_{\mathbb{R}}$, is likewise two-dimensional. Given any ample divisor D on X, recall from Kleiman's ampleness criterion, [20, IV Theorem 2.19], that any numerical class α contained in the closure of the cone of effective cycles, $\overline{NE_1(X)_{\mathbb{R}}}$, intersects D positively, $D.\alpha > 0$. In particular, the cone $\overline{NE_1(X)_{\mathbb{R}}}$ does not contain any lines. As it is convex by definition, $\overline{NE_1(X)_{\mathbb{R}}}$ is spanned by two extremal classes, say α_1 and α_2.

Intersection with $c_1(\pi^*\mathscr{O}_{\mathbb{P}^2}(1))$ defines a non-trivial form on $N_1(X)_{\mathbb{R}}$, which is non-negative on $\overline{NE_1(X)_{\mathbb{R}}}$ and trivial on the ray $\mathbb{R}^+ \cdot [F_\pi]$ spanned by the numerical class of any π-fiber F_π. It follows that this ray must be one of the two extremal rays of $\overline{NE_1(X)_{\mathbb{R}}}$. The same holds for the numerical class of any ϕ-fiber F_ϕ. Using the assumption that the numerical classes $[F_\pi]$ and $[F_\phi]$ are no positive multiples of each other, we have thus identified $\overline{NE_1(X)_{\mathbb{R}}}$ as the cone spanned by these two classes,

$$\overline{NE_1(X)_{\mathbb{R}}} = \mathbb{R}^{\geq 0} \cdot [F_\pi] + \mathbb{R}^{\geq 0} \cdot [F_\phi].$$

This observation has further consequences. Using the $\mathbb{P}^1$-bundle structure of π and ϕ, it follows from the adjunction formula that

$$-K_X.F_\pi = -K_X.F_\phi = 2.$$

It follows that $-K_X \cdot C > 0$ for any class $C \in \overline{NE_1(X)_{\mathbb{R}}} \setminus \{0\}$ and thus by Kleiman's ampleness criterion, we conclude that $-K_X$ is ample. In other words, X is Fano.

Step 3. In order to derive a contradiction, we will now construct a curve $C \subset X$ which intersects $-K_X$ negatively. To this end, we choose a general line $\ell \subset \mathbb{P}^2$. A classical result of Dedekind and Weber [10], often attributed to Grothendieck, allows us to write $\mathscr{E}|_\ell$ as a sum of line bundles,

$$\mathscr{E}|_\ell \cong \begin{cases} \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(-a) & \text{if } c_1(\mathscr{E}) = 0 \\ \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(-a-1) & \text{if } c_1(\mathscr{E}) = -1, \end{cases}$$

where a is a non-negative integer. Since ℓ is general, it follows immediately from the definition of generic splitting type that $d \leq a$. In particular, $3 + c_1 < a$. In either case, $a > 2$. We obtain that the preimage of ℓ is a Hirzebruch surface of type

$$\pi^{-1}(\ell) \cong \mathbb{F}_b \quad \text{where } b > 4.$$

Let $C \subset \mathbb{F}_b$, $C \cong \mathbb{P}^1$ denote the unique section whose self-intersection equals $-b$. A two-fold application of the adjunction formula then shows the following

$$-K_X.C = \underbrace{c_1(N_{\mathbb{F}_b/X}).C}_{=1} + (-K_{\mathbb{F}_b}.C) = 1 + \underbrace{c_1(N_{C/\mathbb{F}_b}).C}_{=-b} + \underbrace{\deg T_C}_{=-2} = -b-1 < 0.$$

This contradicts the result obtained in Step 2 and therefore ends the proof of Theorem 2.10. □

2.4. Deformation rigidity

Assume we are given a proper, surjective morphism of varieties, $X \to \mathbb{A}^1$, and assume that the fiber X_0 over the origin is of the form $X_0 \cong \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_0)$, for a suitable rank-two vector bundle $\mathscr{E}_0$ on $\mathbb{P}^2$. Under favorable conditions, the following Theorem 2.11 guarantees that nearby fibers are also of this form, $X_t \cong \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_t)$, and that the bundles $\mathscr{E}_t$ vary smoothly over $\mathbb{A}^1$.

Theorem 2.11 (Deformation rigidity of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$). *Let $f : X \to \mathbb{A}^1$ be a proper, surjective morphism of abstract varieties defined over k. Write $X_0 := f^{-1}(0)$ for the scheme theoretic fiber over 0 of f. Assume that there exists a locally* *free sheaf $\mathscr{E}_0$ of rank two on $\mathbb{P}^2$ and an isomorphism $\phi_0 : X_0 \to \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_0)$. Write X_{reg} for the regular locus of X and assume further that the natural restriction map* $\text{Pic}(X_{\text{reg}}) \to \text{Pic}(X_0)$ *is surjective. Then, there exists a Zariski-open neighborhood $U = U(0) \subseteq \mathbb{A}^1$ such that all fibers $(X_t)_{t\in U}$ are of the form $X_t \cong \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_t)$.*

More precisely, there exist a rank-three, locally free sheaf $\mathscr{F}_U$ on U, a rank-two locally free sheaf $\mathscr{E}_U$ on $Y_U := \mathbb{P}_U(\mathscr{F}_U)$ and a commutative diagram of the form

$$\begin{array}{ccccccccc}
X_0 & \xrightarrow[\cong]{\phi_0} & \mathbb{P}(\mathscr{E}_0) & \xrightarrow{\text{closed immersion}} & \mathbb{P}_{Y_U}(\mathscr{E}_U) & \xleftarrow[\cong]{\phi_U} & X_U & \xrightarrow{\text{open immersion}} & X \\
 & & \downarrow {\scriptstyle \mathbb{P}^1\text{-bundle}} & & {\scriptstyle \mathbb{P}^1\text{-bundle}} \downarrow {\scriptstyle \alpha_U} & & & & \\
{\scriptstyle f|_{X_0}} \downarrow & & \mathbb{P}^2 & \xrightarrow[\text{closed immersion}]{} & Y_U & & \downarrow {\scriptstyle f|_{X_U}} & & \downarrow {\scriptstyle f} \\
 & & \downarrow & & {\scriptstyle \mathbb{P}^2\text{-bundle}} \downarrow {\scriptstyle \beta_U} & & & & \\
\{0\} & \xleftrightarrow[=]{} & \{0\} & \xrightarrow[\text{closed immersion}]{} & U & \xleftrightarrow[=]{} & U & \xrightarrow[\text{open immersion}]{} & \mathbb{A}^1,
\end{array}$$

where $X_U := f^{-1}(U)$.

Remark 2.12 (Smoothness of X near X_0). Since $\mathbb{A}^1$ is one-dimensional and smooth, it follows that the morphism f of Theorem 2.11 is flat, [17, III Proposition 9.7]. The assumption that $X_0 \cong \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_0)$ therefore implies the existence of an open neighborhood $V = V(0) \subseteq \mathbb{A}^1$ such that $X_V := f^{-1}(V)$ and $f|_{X_V}$ are smooth, [17, III Ex. 10.2]. The restriction map $\text{Pic}(X_{\text{reg}}) \to \text{Pic}(X_0)$ used in Theorem 2.11 is therefore well-defined.

Proof of Theorem 2.11

As before, the proof of Theorem 2.11 spans the rest of the present Section 2.4.

Step 1. *Choices and identifications*. Choose a rank-two locally free sheaf $\mathscr{E}_0$ on $\mathbb{P}^2$ and one identification $\phi_0 : X_0 \to \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_0)$. With these choices made, consider the natural projection morphism $\eta_0 : X_0 \to \mathbb{P}^2$ and the invertible sheaves $\mathscr{A}_0 := \mathscr{O}_{\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_0)}(1)$ and $\mathscr{B}_0 := \eta_0^*\big(\mathscr{O}_{\mathbb{P}^2}(1)\big)$. Using the assumption that the natural restriction map $\text{Pic}(X_{\text{reg}}) \to \text{Pic}(X_0)$ is surjective, choose invertible sheaves $\mathscr{A}$, $\mathscr{B}$ on X_{reg} whose restrictions to X_0 agree with $\mathscr{A}_0$ and $\mathscr{B}_0$, respectively. Finally, choose an open neighborhood $U = U(0) \subseteq \mathbb{A}^1$ of the point $0 \in \mathbb{A}^1$ such that f is smooth over U.

With the exception of U, maintain the choices made in this section throughout the proof. For simplicity, we will abuse notation slightly and shrink the neighborhood U several times in the proof, whenever it becomes clear that there exists a sub-neighborhood $U' \subseteq U$ where some desirable property holds.

Step 2. *Notation*. If $V \subseteq \mathbb{A}^1$ is any open set, denote the f-preimage of V by $X_V := f^{-1}(V) \subseteq X$. If X_V is smooth, denote the restriction of $\mathscr{A}$ by $\mathscr{A}_V := \mathscr{A}|_{X_V}$, similarly for $\mathscr{B}$. The restriction of f to V is written as $f_V : X_V \to V$. In a similar vein, if $t \in \mathbb{A}^1$ is any closed point, write $X_t := f^{-1}(t)$ and $\mathscr{A}_t := \mathscr{A}|_{X_t}$, etc.

To avoid an awkward notation, write $Y_0 := \mathbb{P}^2$ when thinking of $\mathbb{P}^2$ as the base of the $\mathbb{P}^1$-bundle η_0. Fibers of η_0 will always be denoted by ℓ.

Step 3. *Observations*. Semicontinuity of the flat, proper morphism f, [24, Corollary on page 50], guarantees that there exists an open neighborhood $V = V(0) \subseteq U$ such that $(f_V)_*(\mathscr{O}_{X_V}) = \mathscr{O}_V$. In particular, fibers of f_V will be connected. Shrinking U, if necessary, we assume that this holds true on all of U.

Assumption w.l.o.g. 2.13. All fibers of the morphism $f_U : X_U \to U$ are connected and $(f_U)_*(\mathscr{O}_{X_U}) = \mathscr{O}_U$.

Step 4. *Construction of* Y_U. We will show in this step that the push-forward of the sheaf $\mathscr{B}_U$ is locally free. The space Y_U will be constructed as the projectivization of this sheaf.

Claim 2.14. The cohomology groups $H^i\big(X_0,\ \mathscr{B}_0\big)$ vanish, for all $i \in \mathbb{N}^+$.

Proof of Claim 2.14. Let $\ell \subset X_0$ be any fiber of η_0. Then $\ell \cong \mathbb{P}^1$, the sheaf $\mathscr{B}_0|_\ell$ is isomorphic to $\mathscr{O}_{\mathbb{P}^1}$, and $h^i\big(\ell,\ \mathscr{B}_0|_\ell\big) = 0$ for all $i \in \mathbb{N}^+$. In particular, $\mathbb{R}^i(\eta_0)_*\mathscr{B}_0 = 0$ for all $i \in \mathbb{N}^+$, [24, Corollary 2 on page 50]. Given any specific number $i \in \mathbb{N}^+$, the cohomology group in question is thus computed as follows,

$$\begin{aligned} H^i\big(X_0,\ \mathscr{B}_0\big) &= H^i\big(Y_0,\ (\eta_0)_*\mathscr{B}_0\big) && \text{(Leray spectral sequence, [17, III Ex. 8.1])} \\ &= H^i\big(\mathbb{P}^2,\ \mathscr{O}_{\mathbb{P}^2}(1)\big) && \text{(definition of } \mathscr{B}_0\text{)} \\ &= 0. && \text{(cohomology of } \mathbb{P}^n\text{, [17, III Theorem 5.1]).} \end{aligned}$$

This finishes the proof of Claim 2.14. □

Claim 2.15. There exists an open, affine neighborhood $V = V(0) \subseteq U$ with the following properties:

(2.15.1) The sheaf $(f_V)_*\mathscr{B}_V$ is locally free of rank three;
(2.15.2) Given any closed point $t \in V$, let $k(t)$ denote the associated residue field. With this notation, the natural maps $(f_*\mathscr{B}) \otimes_{\mathscr{O}_V} k(t) \to H^0\big(X_t,\ \mathscr{B}_t\big)$ are isomorphisms, for all closed points $t \in V$;
(2.15.3) The natural restriction map $r_t : H^0\big(X_V,\ \mathscr{B}_V\big) \to H^0\big(X_t,\ \mathscr{B}_t\big)$ is surjective, for all closed points $t \in V$.

Proof of Claim 2.15. Recall the following standard continuity and semicontinuity properties of the flat, proper morphism f [24, Corollary on page 50]:

(2.15.4) The functions $\phi_i : U \to \mathbb{N}, t \mapsto h^i\big(X_t,\ \mathscr{B}_t\big)$ are upper semicontinuous for all $i \in \mathbb{N}$;
(2.15.5) The function $\chi : U \to \mathbb{N}, t \mapsto \sum_{i\in\mathbb{N}}(-1)^i\phi_i(t)$ is constant.

Claim 2.14 and Item (2.15.4) imply the existence of an open, affine neighborhood $V = V(0) \subseteq U$ such that $\phi_i(t) = 0$ for all closed points $t \in V$ and all indices $i \in \mathbb{N}^+$. Together with Item (2.15.5), we see that $\chi = \phi_0$ is constant on V. By [24, Corollary 2 on page 50], this already implies that $f_*\mathscr{B}|_V$ is locally free and that (2.15.2) holds. As for (2.15.1), the rank of $(f_V)_*\mathscr{B}_V$ is computed as follows,

$$\begin{aligned} \operatorname{rank}\big((f_V)_*\mathscr{B}_V\big) &= h^0\big(X_0,\ \mathscr{B}_0\big) && \text{(isomorphism (2.15.2) in case } t=0) \\ &= h^0\big(X_0,\ (\eta_0)^*\mathscr{O}_{\mathbb{P}^2}(1)\big) = 3 && \text{(definition of } \mathscr{B}_0). \end{aligned}$$

Surjectivity of r_t, as asserted in (2.15.3), follows because V was taken to be affine. This finishes the proof of Claim 2.15. □

To simplify the notation, we shrink U if necessary, and assume the following:

Assumption w.l.o.g. 2.16. Items (2.15.1)–(2.15.3) of Claim 2.15 hold on U.

Construct Y_U as a $\mathbb{P}^2$-bundle over U by setting $\mathscr{F}_U := (f_U)_*\mathscr{B}_U$ and $Y_U := \mathbb{P}_U(\mathscr{F}_U)$. Maintain these choices for the remainder of the proof.

Step 5. *Factorization of f* In this step, it will be shown that the morphism f_U factorizes via Y_U. The following claim will be important.

Claim 2.17. There exists an open neighborhood $V = V(0) \subseteq U$ such that f is smooth over V and such that the natural evaluation morphism,

$$e : (f_U)^*(f_U)_*\mathscr{B}_U \to \mathscr{B}_U,$$

is surjective on X_V.

Proof of Claim 2.17. Let $\operatorname{Bs}(\mathscr{B}_U) \subset X_U$ be the base point locus of the sheaf $\mathscr{B}$ on X_U. More precisely, let $\operatorname{Bs}(\mathscr{B}_U)$ be the support of $\operatorname{coker}(e)$, with its natural structure as a proper closed, reduced subscheme of X_U. We claim that $\operatorname{Bs}(\mathscr{B}_U)$ does not intersect the fiber X_0, that is, $\operatorname{Bs}(\mathscr{B}_U) \cap X_0 = \emptyset$. Once this is shown, set

$$V := U \setminus f_U\big(\operatorname{Bs}(\mathscr{B}_U)\big).$$

Since f_U is proper, this will be an open neighborhood of $0 \in U$ with all desired properties.

In order to prove the claim, it suffices to show that the natural restriction

$$r_x : H^0\big(X_U,\ (f_U)^*(f_U)_*\mathscr{B}_U\big) \to H^0\big(\{x\},\ \mathscr{B}_U|_{\{x\}}\big)$$

is surjective, for any closed point $x \in X_0$. However, given any such x, observe that the morphism r_x factors as follows,

$$\begin{array}{c} H^0\big(X_U, (f_U)^*(f_U)_*\mathscr{B}_U\big) \xrightarrow{\qquad\qquad r_x \qquad\qquad} H^0\big(\{x\},\ \mathscr{B}_U|_{\{x\}}\big) \\ \phi \big\downarrow \text{isomorphism} \\ H^0\big(X_U,\ \mathscr{B}_U\big) \xrightarrow[\text{restr. to } X_0]{r_1} H^0\big(X_0,\ \mathscr{B}_U|_{X_0}\big) \xrightarrow[\text{restr. to } \{x\}]{r_2} H^0\big(\{x\},\ \mathscr{B}_U|_{\{x\}}\big). \end{array}$$

In the diagram above, the morphism ϕ is the inverse of the natural map $H^0\big(X_U,\, \mathscr{B}_U\big) \to H^0\big(X_U,\, (f_U)^*(f_U)_*\mathscr{B}_U\big)$, which is isomorphic because the fibers of f_U are connected by Assumption 2.13. Surjectivity of r_1 holds by Assumption 2.16. Surjectivity of r_2 holds by choice of $\mathscr{B}_U|_{X_0} = \mathscr{B}_0$. It follows that r_x is surjective. This finishes the proof of Claim 2.17. □

As before, we shrink U if necessary, and assume the following.

Assumption w.l.o.g. 2.18. The evaluation morphism e is surjective on X_U.

Recall from [17, II Proposition 7.12] that to give a morphism $X_U \to Y_U = \mathbb{P}_U(\mathscr{F}_U)$ over U, it is equivalent to give an invertible sheaf $\mathscr{L}$ on X_U and a surjective map of sheaves $(f_U)^*(\mathscr{F}_U) = (f_U)^*(f_U)_*\mathscr{B}_U \to \mathscr{L}$. Setting $\mathscr{L} := \mathscr{B}_U$, the evaluation map e considered above therefore gives rise to a factorization of f_U,

$$X_U \xrightarrow{\eta_U} Y_U \xrightarrow[\beta_U,\ \mathbb{P}^2\text{-bundle}]{} U, \qquad f_U = \beta_U \circ \eta_U.$$

Step 6. *The central fiber of* β_U. We claim that the fiber $F := \beta_U^{-1}(0)$ is canonically isomorphic to $Y_0 \cong \mathbb{P}^2$, and that this isomorphism identifies the restricted map $\eta_U|_{X_0} : X_0 \to F$ with the projection map $\eta_0 : X_0 \to Y_0$. With these identifications, our choice of notation is consistent: $\eta_0 = \eta_U|_{X_0}$ and $Y_0 = \beta_U^{-1}(0)$.

Both claims follow from compatibility of **Proj** and base change, [14, Proposition 3.5.3]. More precisely,

$$\begin{aligned} F = \beta_U^{-1}(0) &= \mathbf{Proj}\,\mathrm{Sym}\big(\mathscr{F}_U \otimes_{\mathscr{O}_U} k(0)\big) && \text{(base change)}\\ &= \mathrm{Proj}\,\mathrm{Sym}\,H^0\big(X_0,\, \mathscr{B}_0\big) && \text{(Claim 2.15, Item (2.15.2))}\\ &= \mathrm{Proj}\,\mathrm{Sym}\,H^0\big(X_0,\, (\eta_0)^*\mathscr{O}_{Y_0}(1)\big) && \text{(definition of } \mathscr{B}_0).\end{aligned}$$

Step 7. *Fibers of the morphism* η_U.

Claim 2.19 (Smoothness of η). There exists an open neighborhood $V = V(0) \subseteq U$ such that η_V is smooth over Y_V.

Proof of Claim 2.19. Let $B \subset X_U$ be the closed set where the morphism η_U is *not* smooth. We claim that B does not intersect the fiber X_0, that is, $B \cap X_0 = \emptyset$. Once this is shown, set

$$V := U \setminus f_U(B).$$

Since f_U is proper, this will be an open neighborhood of $0 \in U$ with all desired properties.

In order to establish the claim, let $x \in X_0$ be any closed point. We will show that η_U is smooth at x by using the criterion [16, II Corollary 2.2]: the morphism η_U is smooth at x if $f_U = \beta_U \circ \eta_U$ is smooth at x, and if the restriction of η_U to the fibers, $\eta_U|_{X_0} : X_0 \to \eta_U^{-1}(0)$ is smooth. Smoothness of f_U at x holds by Remark 2.12. Smoothness of $\eta_U|_{X_0}$ has been established in Step 6 above. This finishes the proof of Claim 2.19. □

Claim 2.19 and the same reasoning as in Step 3 allow us to make the following additional assumptions:

Assumption w.l.o.g. 2.20. The morphism η_U is smooth. Its fibers are connected.

Claim 2.21. If $y \in Y_U$ is any closed point with associated fiber $X_y := \eta_U^{-1}(y)$, then $X_y \cong \mathbb{P}^1$.

Proof of Claim 2.21. Assumption 2.20 implies that the fibers of η_U are complete, smooth, connected curves. As before, [24, Corollary on page 50], guarantees that the function

$$\chi : Y_U \to \mathbb{N}, \quad y \mapsto \sum_{i \in \mathbb{N}} (-1)^i h^i\big(X_y, \mathscr{O}_{X_y}\big)$$

is constant on Y_U. Since $\chi(y) = 1 - g(X_y)$ for any closed point $y \in Y_U$ and since $X_y \cong \mathbb{P}^1$ if $y \in X_0$, it follows that all fibers of η_U are isomorphic to $\mathbb{P}^1$. This finishes the proof of Claim 2.21. □

Step 8. *End of the proof.* To conclude, we need to show that the smooth morphism η_U has the structure of a $\mathbb{P}^1$-bundle. Since all its fibers are isomorphic to $\mathbb{P}^1$ and since the invertible sheaf $\mathscr{A}$ has degree one on each fiber, this follows quickly from arguments that are quite similar to those used in Steps 4 and 5. For projective morphisms between complex varieties, everything has been shown in [11, Lemma 2.12].

We aim to construct an explicit $\mathbb{P}^1$-bundle which will then turn out to be isomorphic to X_U. To this end, set $\mathscr{E}_U := (\eta_U)_*(\mathscr{A})$ and observe that $\mathscr{E}_U|_{Y_0} \cong \mathscr{E}_0$ by choice of $\mathscr{A}$. Since all fibers X_y are isomorphic to $\mathbb{P}^1$ and since the invertible sheaf $\mathscr{A}$ has degree one on these fibers, it follows that the function

$$\phi : Y_U \to \mathbb{N}, \quad y \mapsto h^0\big(X_y, \mathscr{A}_U|_{X_y}\big)$$

is constant of value two. As before, invoke [24, Corollary 2 on page 50] to conclude that $\mathscr{E}_U$ is locally free of rank two. Using that $\mathscr{A}_U|_{X_y}$ is identified with $\mathscr{O}_{\mathbb{P}^1}(1)$ and is hence basepoint-free for any closed point $y \in Y_U$, a minor variant of the argumentation used in the proof of Claim 2.17 reveals that the evaluation map

$$(\eta_U)^*(\eta_U)_*\mathscr{A}_U \to \mathscr{A}_U$$

is surjective. As before, we have thus constructed a refined factorization of f_U,

$$X_U \xrightarrow{\phi_U} \mathbb{P}_{Y_U}(\mathscr{E}_U) \xrightarrow[\alpha_U,\, \mathbb{P}^1\text{-bundle}]{} Y_U \xrightarrow[\beta_U,\, \mathbb{P}^2\text{-bundle}]{} U, \qquad f_U : X_U \to U.$$

By construction, the restriction of the ϕ_U to any fiber X_y is identified with the morphism induced by the very ample invertible sheaf $\mathscr{A}_U|_{X_y} \cong \mathscr{O}_{\mathbb{P}^1}(1)$, that is,

$$\mathbb{P}^1 \to \mathbb{P}\Big(H^0\big(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)\big)\Big).$$

This has two consequences. First, the smoothness criterion [16, II Corollary 2.2] applies to show that ϕ_U is smooth. In particular, ϕ_U is separable, [8, Chapter AG, Theorem 17.3]. Second, it follows that the morphism ϕ_U is bijective. By [13, Sect. 2] or [8, Theorem on page 43], the induced morphism between functions fields has separable degree equal to one. It follows that ϕ_U is birational. Since all spaces in question are smooth, hence normal, Zariski's Main Theorem, [15, Lemma 8.12.10.1], therefore guarantees that ϕ_U is isomorphic. This finishes the proof of Theorem 2.11.

2.5. Deformations and moduli

We recall Strømme's results on moduli of vector bundles and draw first conclusions concerning deformability and non-deformability of vector bundles.

2.5.1. *Notation and known facts*

Projectivizations of rank-two vector bundles are the main objects of interest in this paper. In the discussion, we will often be free to twist any given vector bundle with a suitable line bundle, allowing to assume that the bundle's first Chern class is either zero or minus one.

Setting 2.22 (Choice of Chern classes). Fix two numbers $c_1 \in \{0, -1\}$ and $c_2 \in \mathbb{Z}$.

Definition 2.23 (Families of bundles). Let T be a k-scheme. Given numbers c_1 and c_2, a *family* $\mathfrak{E}/T$ of rank-two vector bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2 is a rank-two bundle $\mathscr{E}$ on $T \times \mathbb{P}^2$ such that for any k-valued point $t \in T$, the fiber $\mathscr{E}_t$ is a rank-two bundle on $\mathbb{P}^2$, with Chern classes $c_1(\mathscr{E}_t) = c_1$ and $c_2(\mathscr{E}_t) = c_2$.

Definition 2.24 (Pure type [30, Section 2.3]). In the setting of Definition 2.23, given any integer $d \geq 0$, the family $\mathfrak{E}/T$ is said to be of *pure type* d, if $\mathbb{R}^2\pi_*\big(\mathscr{E}^*(d-3)\big)$ is invertible, where $\pi : T \times \mathbb{P}^2 \to T$ is the natural projection map.

Fact 2.25 (Type and pure type [30, Remark 2.4]). In the setting of Definition 2.24, if $\mathfrak{E}/T$ is of pure type d, then all bundles $\mathscr{E}_t$ have generic splitting type d.

Fact 2.26 (Semicontinuity [30, Section 2.2]). In the setting of Definition 2.23, the generic splitting type is upper semicontinuous as a function on the closed points of T. Given any $d \geq 0$, there exists a maximal, locally closed subscheme $T(d) \subseteq T$ over which the bundle is of pure type d.

Fact 2.27 (Existence of moduli spaces [30, Propositions 1.2 and 2.7]). Given numbers c_1 and c_2 as in Setting 2.22, let $d \geq 0$ be any number. Then, there exists a coarse moduli scheme $M(d)$ for families of rank-two vector bundles on $\mathbb{P}^2$ of pure type d, modulo isomorphism and twists by line bundles coming from the base. The dimension of $M(d)$ is computed as follows:

(2.27.1) If $d^2 - dc_1 + c_2 < 0$, then $M(d)$ is empty;
(2.27.2) If $d^2 - dc_1 + c_2 = 0$, then $M(d)$ is a point;
(2.27.3) If $d^2 - dc_1 + c_2 > 0$, then $\dim M(d) = 3(d^2 - dc_1 + c_2) - 1$.

The scheme $M(d)$ is either empty, or irreducible, nonsingular, quasiprojective and rational.

Fact 2.28. (Existence of maximal families, [30, Section 3.1–3.6 and Theorem 3.9]). Given numbers c_1 and c_2 as in Setting 2.22 and $d \geq -1$, then there exists a smooth, irreducible scheme $Q(d)$ and a family $\mathfrak{E}/Q(d)$ of rank-two vector bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2, such that $\mathfrak{E}/Q(d)$ is pure type d and such that the induced moduli map $Q(d) \to M(d)$ is surjective.

Definition 2.29. (Deformability to given type over irreducible base, [30, Section 2.12 and Theorem 3.13]). Given numbers c_1 and c_2 as in Setting 2.22, numbers $d > e \geq 0$, and a rank-two vector bundle $\mathscr{E}$ on $\mathbb{P}^2$ of type d. We say that $\mathscr{E}$ is *deformable to type e over an irreducible base* if there exists an irreducible k-scheme T and a family of bundles $\mathfrak{E}/T$ with Chern classes c_1 and c_2 that is generically of type e and contains $\mathscr{E}$ as a fiber.

Fact 2.30. (Locus of deformable bundles, [30, Sect. 3.12 and Theorems 3.13, 4.7]). Given numbers c_1 and c_2 as in Setting 2.22, and $d > e \geq -1$. Then, there exists a closed subset $M(d; e) \subseteq M(d)$ whose k-rational points are exactly those isomorphism classes of bundles that are deformable to type e over an irreducible base. If $M \subseteq M(d; e)$ is any irreducible component, then $\operatorname{codim}_{M(d)} M \geq \gamma(d; e)$, where

$$\gamma(d; e) := \begin{cases} P(d) & \text{if } e = -1 \text{ or } e = c_1 = c_2 = 0 \\ P(d) - P(e) + 1 & \text{otherwise} \end{cases} \quad \text{and}$$
$$P(x) := (x-1)(x-2-c_1) - c_2.$$

If $\binom{d-e-1}{2} \geq e^2 - e \cdot c_1 + c_2$, then $M(d; e)$ contains an irreducible component for which equality holds.

Observation 2.31 (Numerology). In the setting of Fact 2.30, elementary computations show that if $d \gg 0$ is sufficiently large, then $\gamma(d; e) > 0$ for all numbers e satisfying $d > e \geq -1$. In particular, for any such e, the locus $M(d; e)$ of bundles that are deformable to type e over an irreducible base is either empty, or a proper closed subset, $M(d; e) \subsetneq M(d)$.

Corollary 2.32 (Non-emptyness of $M(d; e)$). *Given numbers c_1 and c_2 as in Setting 2.22 and $e \geq -1$ such that $M(e)$ is not empty. If $d \gg 0$ is any sufficiently large number, then $M(d; e) \subsetneq M(d)$ is a proper, non-empty subvariety.*

Proof. Given c_1, c_2 and e, consider the polynomials $P(\cdot)$ and $\gamma(\cdot; e)$ as in Fact 2.30. If $d \gg 0$ is sufficiently large, then any of the following polynomials in d, which all have positive leading coefficients, takes strictly positive values.

$$Q_1(d) := d^2 - d \cdot c_1 + c_2 \qquad Q_2(d) := 3(d^2 - d \cdot c_1 + c_2) - 1$$
$$Q_3(d) := \binom{d-e-1}{2} - e^2 - e \cdot c_1 + c_2 \qquad Q_4(d) := Q_2(d) - \gamma(d; e)$$
$$Q_5(d) := \gamma(d; e).$$

Fact 2.27 asserts that $M(d)$ is non-empty as soon as $Q_1(d) \geq 0$. The dimension of $M(d)$ is then given as $Q_2(d)$. Fact 2.30 claims that once $Q_3(d)$ is positive, the space $M(d;e) \subseteq M(d)$ contains a component M whose dimension $\dim M$ is equal to $Q_4(d)$, and therefore again positive. The minimal codimension in $M(d)$ of components of $M(d;e)$ is given by $Q_5(d)$, showing that $M(d;e) \neq M(d)$. □

2.5.2. *Deformability and non-deformability*

As a consequence of Observation 2.31 we will see in Proposition 2.33 that most vector bundles cannot be deformed over an irreducible base to bundles of smaller type. In striking contrast, we will see in Proposition 2.34 that any two vector bundles whose Chern classes are equal are deformable into each other, over a base that is not necessarily irreducible.

Proposition 2.33 (Non-deformability over irreducible base). *Given numbers c_1 and c_2 as in Setting* 2.22. *If $d \gg 0$ is sufficiently large, then there exists a rank-two vector bundle $\mathscr{E}$ of type d on $\mathbb{P}^2$ with Chern classes c_1 and c_2 that is not deformable to type e over an irreducible base, for any $d > e \geq -1$, in the sense of Definition* 2.29.

Proof. Recall from Observation 2.31 that the open complement $M(d)\setminus \bigcup_{d>e\geq -1} M(d;e)$ is not empty. Choose a k-rational point in there and let $\mathscr{E}$ be the corresponding bundle. □

Proposition 2.34 (Deformability over reducible base). *Given numbers c_1, c_2 as in Setting* 2.22 *and vector bundles $\mathscr{A}$ and $\mathscr{B}$ with Chern classes c_1, c_2. Then $\mathscr{A}$ and $\mathscr{B}$ are deformable into each other, over a base scheme that need not necessarily be irreducible.*

Proof. Denote the splitting types of $\mathscr{A}$ and $\mathscr{B}$ by $e_{\mathscr{A}}$ and $e_{\mathscr{B}}$, respectively. Corollary 2.32 then gives a number $d \gg 0$ such that $M(d;e_{\mathscr{A}})$ and $M(d;e_{\mathscr{B}})$ are both non-empty. Choose a bundle $\mathscr{C} \in M(d)$. By Fact 2.28, there exists a deformation family that connects the bundle $\mathscr{C}$ to one in $M(d;e_{\mathscr{A}})$. By definition of $M(d;e_{\mathscr{A}})$, this bundle can be deformed into one in $M(e_{\mathscr{A}})$, which, by Fact 2.28 again, can be deformed into $\mathscr{A}$. We have thus found a deformation over a reducible base that has $\mathscr{C}$ and $\mathscr{A}$ as fibers. In a similar manner, find a deformation that connects $\mathscr{C}$ and $\mathscr{B}$. Connect these deformation families to conclude. □

3. Homotopy classification of PGL_2-torsors over $\mathbb{P}^2$

In this section we discuss the $\mathbb{A}^1$-homotopy classification of (Nisnevich locally trivial) PGL_2-torsors on $\mathbb{P}^2$. In other words, we describe the pointed set $[\mathbb{P}^2, \mathrm{BPGL}_2]_{\mathbb{A}^1}$. To formulate a useful description of this set, we observe that Nisnevich locally trivial

PGL_2-torsors are always obtained from GL_2-torsors by change of structure group. We then investigate the induced map

$$[\mathbb{P}^2, \mathrm{BGL}_2]_{\mathbb{A}^1} \longrightarrow [\mathbb{P}^2, \mathrm{BPGL}_2]_{\mathbb{A}^1},$$

show that this map is surjective, and describe the right hand side as a quotient of the left hand side by the natural action of $\mathrm{Pic}(\mathbb{P}^2)$ coming from "tensoring by line bundles". Using an explicit description of $[\mathbb{P}^2, \mathrm{BGL}_2]_{\mathbb{A}^1}$ that stems from techniques of obstruction theory, we then obtain a description of $[\mathbb{P}^2, \mathrm{BPGL}_2]_{\mathbb{A}^1}$. The main results of this section are Theorem 3.10 and Corollary 3.11.

Conventions 3.1. In Sections 3 and 4, we deviate from our global conventions. Fix an algebraically closed field k. Contrary to our global assumptions fixed in Section 1.4, these sections use different assumptions on the characteristic of k; we will always be explicit about the primes we want to exclude.

We write $\mathcal{S}\Updownarrow_k$ for the category of schemes that are separated, finite type and smooth over $\mathrm{Spec}\, k$. We write $\mathcal{S}\sqrt{\rfloor}_k$ for the category of simplicial Nisnevich sheaves of sets on $\mathcal{S}\Updownarrow_k$, equipped with the $\mathbb{A}^1$-local model structure of [25]. In the rest of this section, the word "sheaf" will by synonymous with "Nisnevich sheaf of groups on $\mathcal{S}\Updownarrow_k$."

A presheaf $\mathbf{F}$ on $\mathcal{S}\Updownarrow_k$ is called $\mathbb{A}^1$-invariant if $\mathbf{F}(U) \to \mathbf{F}(U \times \mathbb{A}^1)$ is a bijection for any $U \in \mathcal{S}\Updownarrow_k$. A sheaf of groups $\mathbf{G}$ is strongly $\mathbb{A}^1$-invariant if its cohomology presheaves $H^i(\cdot, \mathbf{G})$ are $\mathbb{A}^1$-invariant, for $i \in \{0, 1\}$. A sheaf of abelian groups $\mathbf{A}$ is called strictly $\mathbb{A}^1$-invariant if all its cohomology presheaves are $\mathbb{A}^1$-invariant. By [23, Theorem 1.9], if $(\mathcal{X}, x)$ is a pointed space, then $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathcal{X}, x)$ is strongly $\mathbb{A}^1$-invariant for $i = 1$, and strictly $\mathbb{A}^1$-invariant for $i > 1$.

Remark 3.2. The classification results of Sections 3 and 4 hold in greater generality: statements and proofs apply verbatim to the case where the base field k is quadratically closed.

3.1. Torsors and classifying spaces in $\mathbb{A}^1$-homotopy theory

If $\mathcal{G}$ is a sheaf of groups, we write $B\mathcal{G}$ for the simplicial bar construction of the sheaf of groups $\mathcal{G}$, [25, Section 4.1]. By [25, Proposition 4.1.15], we know that (free) simplicial homotopy classes of maps from a smooth scheme X to $B\mathcal{G}$ are in bijection with Nisnevich locally trivial $\mathcal{G}$-torsors on X. Thus, if $B\mathcal{G}^f$ is a simplicially fibrant model of $B\mathcal{G}$, then, given a Nisnevich locally trivial $\mathcal{G}$-torsor $\pi : P \to X$, we can pick a morphism $f_\pi : X \to B\mathcal{G}^f$ such that π is the pullback of the universal $\mathcal{G}$-torsor along f_π.

The space $B\mathcal{G}$ is a reduced simplicial sheaf (i.e., the sheaf of 0-simplices is reduced to a point) and is therefore simplicially 0-connected. It follows from [25, Corollary 2.3.22] that $B\mathcal{G}$ is $\mathbb{A}^1$-connected. We write $*$ for the canonical base-point of $B\mathcal{G}$. If we write X_+ for X with a disjoint base-point attached, then "forgetting the base-point" induces a bijection between the set of pointed morphisms from X_+

to $B\mathcal{G}$ and the set of morphisms from X to $B\mathcal{G}$. In particular, we can always assume that f_π is represented by a pointed morphism from X_+.

If G is a linear algebraic group, then G can be viewed as an étale sheaf of groups, and we can consider the étale classifying space $B_{\text{ét}}G$; see [25, Section 4.2] for the construction. There is a canonical adjunction morphism $BG \to B_{\text{ét}}G$ that is a simplicial weak equivalence if and only if étale locally trivial G-torsors are Nisnevich locally trivial.

If G is a finite étale group scheme of order coprime to the characteristic of k, then $B_{\text{ét}}G$ is $\mathbb{A}^1$-local by [25, Proposition 4.3.5]. As a consequence, if X is a smooth scheme, then $[X, B_{\text{ét}}G]_{\mathbb{A}^1}$ is in natural bijection with the set of étale locally trivial G-torsors on X. We define $\mathcal{H}^1_{\text{ét}}(G)$ to be the Nisnevich sheafification of the presheaf $U \mapsto [U, B_{\text{ét}}G]_{\mathbb{A}^1}$. If G is abelian, the sheaf $\mathcal{H}^1_{\text{ét}}(G)$ is a sheaf of abelian groups, and under the hypothesis on k, is also strictly $\mathbb{A}^1$-invariant. The important fact, used below without explicit reference, is that morphisms of strictly $\mathbb{A}^1$-invariant sheaves are determined by their sections over extensions of the base field. This follows because such sheaves are unramified in the sense of [23, Definition 2.1], *cf.* [23, Corollary 6.9 and Rem. 6.10].

3.2. Some $\mathbb{A}^1$-homotopy theory of $\mathbf{PGL_2}$

In this section we produce some $\mathbb{A}^1$-fiber sequences related to PGL_n and BPGL_n. We refer to [34] for discussion of the general theory of $\mathbb{A}^1$-fiber sequences.

Lemma 3.3. *There is an $\mathbb{A}^1$-fiber sequence of the form* $\mathrm{PGL}_n \to B\mathbf{G}_m \to \mathrm{BGL}_n$.

Proof. Write EGL_n for the Čech simplicial object associated with the structure map $\mathrm{GL}_n \to \operatorname{Spec} k$. The inclusion of the center $\mathbf{G}_m \hookrightarrow \mathrm{GL}_n$ yields an isomorphism $\mathrm{GL}_n/\mathbf{G}_m \xrightarrow{\sim} \mathrm{PGL}_n$ and there is a natural left translation action of GL_n on $\mathrm{GL}_n/\mathbf{G}_m$. Consider the associated fiber space $\mathrm{EGL}_n \times^{\mathrm{GL}_n} \mathrm{GL}_n/\mathbf{G}_m$. Projection onto the first factor gives a morphism $\mathrm{EGL}_n \times^{\mathrm{GL}_n} \mathrm{GL}_n/\mathbf{G}_m \to \mathrm{BGL}_n$ that, as the associated fiber space of a GL_n-torsor, is automatically an $\mathbb{A}^1$-fiber sequence by [34, Proposition 5.1 and Theorem 5.3]. On the other hand it is straightforward to show that $\mathrm{EGL}_n \times^{\mathrm{GL}_n} \mathrm{GL}_n/\mathbf{G}_m$ is simplicially weakly equivalent to $B\mathbf{G}_m$. This is established in exactly the same fashion as the proof of [6, Lemma 3.8]. □

The map $\mathrm{EGL}_n \times^{\mathrm{GL}_n} \mathrm{GL}_n/\mathbf{G}_m \to \mathrm{BGL}_n$ in the proof of Lemma 3.3 is, as the associated fiber space of a GL_n-torsor, Nisnevich locally trivial; under the identification $\mathrm{GL}_n/\mathbf{G}_m \cong \mathrm{PGL}_n$ this map is furthermore a PGL_n-torsor. As a consequence, there exists a classifying morphism $\mathrm{BGL}_n \to \mathrm{BPGL}_n$ for this map. The next result then follows from [34, Proposition 5.1 and Theorem 5.3].

Lemma 3.4. *There is an $\mathbb{A}^1$-fiber sequence of the form* $B\mathbf{G}_m \to \mathrm{BGL}_n \to \mathrm{BPGL}_n$.

The following result is essentially contained in [6, Corollary 3.17] and [33, Propositions 5.11, 5.12], though the formulation and proof below are somewhat different.

Proposition 3.5. *Let $n \geq 2$ be a natural number, and assume that the base field k has characteristic that does not divide n. Then there is a canonical isomorphism*

$$\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathrm{BPGL}_n, *) \xrightarrow{\sim} \mathcal{H}^1_{\text{ét}}(\mu_n)$$

and a short exact sequence of strictly $\mathbb{A}^1$-invariant sheaves of the form

$$0 \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(\mathrm{BGL}_n) \longrightarrow \boldsymbol{\pi}_2^{\mathbb{A}^1}(\mathrm{BPGL}_n) \longrightarrow \mu_n \longrightarrow 0$$

with $\boldsymbol{\pi}_2^{\mathbb{A}^1}(\mathrm{BGL}_2) \cong \mathbf{K}_2^{MW}$ and $\boldsymbol{\pi}_2^{\mathbb{A}^1}(\mathrm{BGL}_n) \cong \mathbf{K}_2^{M}$ for $n \geq 3$.

Proof. The $\mathbb{A}^1$-fiber sequence

$$B\mathbf{G}_m \to \mathrm{BGL}_n \to \mathrm{BPGL}_n \tag{3.1}$$

induces a long exact sequence in $\mathbb{A}^1$-homotopy sheaves [6, Lemma 2.10]. There is a canonical isomorphism $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathrm{BGL}_n) \xrightarrow{\sim} \mathbf{G}_m$ induced by the determinant homomorphism. As described in the proof of Lemma 3.3, the map $B\mathbf{G}_m \to \mathrm{BGL}_n$ in the above $\mathbb{A}^1$-fiber sequence is induced by the inclusion of the center $\mathbf{G}_m \to \mathrm{GL}_n$. If t is a coordinate on $\mathbf{G}_m$, then the composite map $\mathbf{G}_m \to \mathrm{GL}_n \to \mathbf{G}_m$, where the second homomorphism is induced by the determinant, is given by $t \mapsto t^n$. In particular, the map $\mathbf{G}_m \cong \boldsymbol{\pi}_1^{\mathbb{A}^1}(B\mathbf{G}_m) \to \boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathrm{BGL}_n) \cong \mathbf{G}_m$ in the long exact sequence is precisely the map $t \mapsto t^n$. It follows that $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathrm{BPGL}_n)$ is isomorphic to the Nisnevich sheaf quotient $\mathbf{G}_m/\mathbf{G}_m^n$.

The Kummer sequence of étale sheaves $\mu_n \to \mathbf{G}_m \to \mathbf{G}_m$ yields an exact sequence of cohomology presheaves

$$\mathbf{G}_m(\cdot) \xrightarrow{\times n} \mathbf{G}_m(\cdot) \longrightarrow H^1_{\text{ét}}(\cdot, \mu_n) \longrightarrow H^1_{\text{ét}}(\cdot, \mathbf{G}_m).$$

Sheafifying this sequence of presheaves for the Nisnevich topology on $\mathcal{S}m_k$, and observing that the Nisnevich sheafification of $H^1_{\text{ét}}(\cdot, \mathbf{G}_m)$ is trivial, yields a canonical isomorphism of Nisnevich sheaves $\mathbf{G}_m/\mathbf{G}_m^n \to \mathcal{H}^1_{\text{ét}}(\mu_n)$. Combining these facts and observing that $B\mathbf{G}_m$ is $\mathbb{A}^1$-connected yields the first isomorphism.

The kernel of the map $\boldsymbol{\pi}_1^{\mathbb{A}^1}(B\mathbf{G}_m) \to \boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathrm{BGL}_n)$ is μ_n. Since $\boldsymbol{\pi}_2^{\mathbb{A}^1}(\mathrm{BGL}_2) \cong \mathbf{K}_2^{MW}$ and $\boldsymbol{\pi}_i^{\mathbb{A}^1}(B\mathbf{G}_m) = 0$ for $i \geq 2$, the second result also follows from the long exact sequence associated with (3.1) above. □

Remark 3.6. One can show that $\boldsymbol{\pi}_2^{\mathbb{A}^1}(\mathrm{BPGL}_2)$ is the pullback of the diagram $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbf{G}_m \hookleftarrow \mu_2$, in particular a subgroup sheaf of $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathbb{P}^1)$. The group structure on the above extension is inherited from this inclusion.

3.3. PGL_n-torsors vs. GL_n-torsors

If X is a smooth scheme, then there is a function

$$[X, \mathrm{BPGL}_n]_s \longrightarrow [X, \mathrm{BPGL}_n]_{\mathbb{A}^1}$$

induced by the map sending BPGL_n to its $\mathbb{A}^1$-localization. In general, there is no reason for this function to be surjective, as $\mathbb{A}^1$-homotopy classes of maps with source X and target BPGL_n need not come from an actual PGL_n-torsor on X. The next result is a partial replacement for this deficiency.

Proposition 3.7. *Let $n \geq 2$ be a natural number, and assume that the base field k has characteristic that does not divide n. For X a smooth k-scheme, the canonical map*

$$[X, \mathrm{BGL}_n]_{\mathbb{A}^1} \to [X, \mathrm{BPGL}_n]_{\mathbb{A}^1}$$

is surjective. Moreover, given any element $\zeta \in [X, \mathrm{BPGL}_n]_{\mathbb{A}^1}$ and any smooth affine scheme Y that is $\mathbb{A}^1$-weakly equivalent to X, then there exists a vector bundle $\mathscr{E}$ on Y such that the map ζ is $\mathbb{A}^1$-homotopic to the classifying map of the PGL_n-torsor associated with $\mathscr{E}$.

Proof. We consider the Moore-Postnikov factorization of the map $BGL_n \to BPGL_n$. For details regarding the Moore-Postnikov factorization in $\mathbb{A}^1$-homotopy theory, we refer the reader to [2, Theorem 6.1.1]. Roughly speaking, this factorization corresponds to looking at the Postnikov tower of the $\mathbb{A}^1$-homotopy fiber of $\mathrm{BGL}_n \to \mathrm{BPGL}_n$, which we identified above with $B\mathbf{G}_m$. There is a canonical action of $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathrm{BPGL}_n)$ on the $\mathbb{A}^1$-homotopy fiber of $\mathrm{BGL}_n \to \mathrm{BPGL}_n$ induced by change of base-point. This yields an action of $\mathcal{H}^1_{\text{ét}}(\mu_n)$ on $\boldsymbol{\pi}_i^{\mathbb{A}^1}(B\mathbf{G}_m)$ and the latter is only non-trivial if $i = 1$, in which case it is isomorphic to $\mathbf{G}_m$.

The sheaf of automorphisms of $\mathbf{G}_m$ is isomorphic to the constant sheaf $\mathbb{Z}/2$, which is, in particular, strictly $\mathbb{A}^1$-invariant. The action of $\mathcal{H}^1_{\text{ét}}(\mu_n)$ on the homotopy sheaves of $B\mathbf{G}_m$ is determined by a homomorphism of sheaves $\mathcal{H}^1_{\text{ét}}(\mu_n) \to \mathbb{Z}/2$. The source and target sheaves here are strictly $\mathbb{A}^1$-invariant and consequently such a homomorphism is uniquely determined by its behavior on sections over finitely generated extensions of the base-field. Since $\mathbb{Z}/2$ is a constant sheaf, to determine the value of such a homomorphism over a finitely generated extension L of the base field, we can pass to an algebraic closure $\bar{L}$ of L. In that case, the sections of $\mathcal{H}^1_{\text{ét}}(\mu_n)$ are necessarily trivial, so we conclude that any morphism of sheaves $\mathcal{H}^1_{\text{ét}}(\mu_n) \to \mathbb{Z}/2$ is trivial.

It follows that there is precisely one obstruction to lifting an $\mathbb{A}^1$-homotopy class of maps $X \to \mathrm{BPGL}_n$ to an $\mathbb{A}^1$-homotopy class of maps $X \to \mathrm{BGL}_n$, and that obstruction lies in the group $H^2_{\mathrm{Nis}}(X, \mathbf{G}_m)$. Note that we have an untwisted obstruction in this case because the action of $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathrm{BPGL}_n)$ on $\mathbf{G}_m$ is trivial. We refer the reader to [2, Section 6.1] for more details on these twisted obstructions.

We now claim that the group $H^2_{\mathrm{Nis}}(X, \mathbf{G}_m)$ vanishes for any smooth scheme. Indeed, since the sheaf $\mathbf{G}_m$ is strictly $\mathbb{A}^1$-invariant, we know that $H^2_{\mathrm{Nis}}(X, \mathbf{G}_m) \cong$

$H^2_{\mathrm{Zar}}(X, \mathbf{G}_m)$ and the latter cohomology can be computed by means of the Cousin resolution for $\mathbf{G}_m = \mathbf{K}^M_1$. This fact is standard, but it is difficult to find an explicit reference. In lieu of a reference of this precise fact, we refer the reader to [28, Propositions 5.6–5.8] where much more general results are established.

Since X is smooth, the Jouanolou-Thomason homotopy lemma asserts that there exists a smooth affine scheme Y that is $\mathbb{A}^1$-weakly equivalent to X, [32, Proposition 4.4]. Thus, there is a bijection between isomorphism classes of rank-n vector bundles on Y and (free) $\mathbb{A}^1$-homotopy classes of maps $[X, \mathrm{BGL}_n]$ by [3, Theorem 5.2.3]. In particular, the lift constructed in the previous paragraph is represented by a vector bundle on Y. It is straightforward to check that the PGL_n-torsor associated with this vector bundle has the properties mentioned in the statement. □

3.4. $\mathbb{A}^1$-homotopy classification of PGL_2-torsors on $\mathbb{P}^2$

If X is any smooth variety, then mapping X_+ into the $\mathbb{A}^1$-fiber sequence $B\mathbf{G}_m \to \mathrm{BGL}_n \to \mathrm{BPGL}_n$ and using [25, Proposition 4.3.8] to identify $[X, B\mathbf{G}_m] \cong \mathrm{Pic}(X)$ yields an exact sequence of groups and pointed sets of the form

$$[X, \mathrm{PGL}_n]_{\mathbb{A}^1} \longrightarrow \mathrm{Pic}(X) \longrightarrow [X, \mathrm{BGL}_n]_{\mathbb{A}^1} \longrightarrow [X, \mathrm{BPGL}_n]_{\mathbb{A}^1}.$$

The action of $\mathrm{Pic}(X)$ on $[X, \mathrm{BGL}_n]_{\mathbb{A}^1}$ admits the following description. While $[X, \mathrm{BGL}_n]_{\mathbb{A}^1}$ need not be in bijection with the set of isomorphism classes of vector bundles on X if X is not affine, we can always find a smooth affine scheme X' and an $\mathbb{A}^1$-weak equivalence $X' \to X$. In that case, for any space $\mathcal{Y}$, the induced map $[X, \mathcal{Y}]_{\mathbb{A}^1} \to [X', \mathcal{Y}]_{\mathbb{A}^1}$ is a bijection. Thus, we obtain an exact sequence as above with X replaced by X' throughout.

In that case, we identify $\mathrm{Pic}(X')$ with the set of isomorphism classes of line bundles on X, $[X', \mathrm{BGL}_n]_{\mathbb{A}^1}$ with the set of isomorphism classes of rank n vector bundles on X' and describe the action of $\mathrm{Pic}(X')$ on $[X', \mathrm{BGL}_n]_{\mathbb{A}^1}$ as follows,

$$\mathrm{Pic}(X') \times [X', \mathrm{BGL}_n]_{\mathbb{A}^1} \to [X', \mathrm{BGL}_n]_{\mathbb{A}^1} \qquad (\mathscr{L}, \mathscr{E}) \mapsto \mathscr{L} \otimes \mathscr{E}.$$

With these identifications, the next result follows from Proposition 3.7.

Proposition 3.8. *Let $n \geq 2$ be a natural number, and assume that the base field k has characteristic that does not divide n. For X a smooth k-scheme, there is a canonical bijection*

$$[X, \mathrm{BGL}_n]_{\mathbb{A}^1} / \mathrm{Pic}(X) \xrightarrow{\sim} [X, \mathrm{BPGL}_n]_{\mathbb{A}^1},$$

where the action of $\mathrm{Pic}(X)$ *on* $[X, \mathrm{BGL}_n]_{\mathbb{A}^1}$ *is, up to $\mathbb{A}^1$-weak equivalence described in the preceding paragraph.*

The Chow ring of BGL_2 is isomorphic to a formal power series ring over $\mathbb{Z}$ in two variables c_1 and c_2, the first and second Chern class. It follows from this observation that c_1 and c_2 yield well-defined (pointed) functions

$$c_i : [X, \mathrm{BGL}_2]_{\mathbb{A}^1} \longrightarrow CH^i(X).$$

These functions are useful in describing the set $[X, \mathrm{BGL}_2]_{\mathbb{A}^1}$ for X a smooth surface. More precisely, we have the following result.

Lemma 3.9. *Assume that the base field k has characteristic unequal to two, and that X is a (connected) smooth k-scheme which is $\mathbb{A}^1$-weakly equivalent to a smooth scheme of dimension ≤ 2. Then the map $(c_1, c_2) : [X, \mathrm{BGL}_2]_{\mathbb{A}^1} \to \mathrm{Pic}(X) \times CH^2(X)$ is a bijection.*

Proof. We compute $[X, \mathrm{BGL}_2]_{\mathbb{A}^1}$ using obstruction theory. By the same argument as [1, Proposition 6.2], under the hypothesis on X, the canonical map

$$[X, \mathrm{BGL}_2^{(2)}]_{\mathbb{A}^1} \longrightarrow [X, \mathrm{BGL}_2]_{\mathbb{A}^1}$$

is a bijection.

The second stage of the Postnikov tower of BGL_2 is described in [1, Section 6]. In particular, if X is as in the statement, then by [1, Proposition 6.3] a map $X \to \mathrm{BGL}_2^{(2)}$ consists of a pair $(\mathscr{L}, \alpha)$ where $\mathscr{L}$ is a line bundle on X, and $\alpha \in H^2\big(X, \mathbf{K}_2^{MW}(\mathscr{L})\big)$. If $\mathscr{L}'$ is another line bundle on X, then there are canonical isomorphisms $H^2\big(X, \mathbf{K}_2^{MW}(\mathscr{L})\big) \cong H^2\big(X, \mathbf{K}_2^{MW}(\mathscr{L} \otimes \mathscr{L}'^{\otimes 2})\big)$. Since the base field k is assumed algebraically closed, the canonical map $H^2\big(X, \mathbf{K}_2^{MW}(\mathscr{L})\big) \to H^2\big(X, \mathbf{K}_2^{M}\big)$ is a bijection, *cf.* the proof of [1, Corollary 5.3]. In that case, the identification of $\mathscr{L}$ with c_1 is clear, and the identification of the class in $H^2\big(\mathbb{P}^2, \mathbf{K}_2^{MW}(\mathscr{L})\big)$ with c_2 is contained in the proof of [1, Theorem 6.6]. □

Theorem 3.10. *Assume that the base field k has characteristic unequal to two, and that X is a (connected) smooth k-scheme which is $\mathbb{A}^1$-weakly equivalent to a smooth scheme of dimension ≤ 2. Then there is a bijection*

$$\mathrm{Pic}(X) \times CH^2(X) \big/ \mathrm{Pic}(X) \xrightarrow{\sim} [X, \mathrm{BPGL}_2]_{\mathbb{A}^1},$$

where the action of $l \in \mathrm{Pic}(X)$ on $(c_1, c_2) \in \mathrm{Pic}(X) \times CH^2(X)$ is given by the formula

$$l \cdot (c_1, c_2) = (c_1 + 2l, c_2 + lc_1 + l^2).$$

Proof. This follows from Proposition 3.8, Lemma 3.9 and the following observation. If $\mathscr{L}$ is a line bundle on X whose class in $\mathrm{Pic}(X)$ is l, and if $\mathscr{E}$ is a rank-two vector bundle with Chern classes $c_1 = c_1(\mathscr{E})$ and $c_2 = c_2(\mathscr{E})$, then $c_1(\mathscr{L} \otimes \mathscr{E}) = c_1(\mathscr{E}) + 2c_1(\mathscr{L})$ while $c_2(\mathscr{L} \otimes \mathscr{E}) = c_2(\mathscr{E}) + c_1(\mathscr{E})c_1(\mathscr{L}) + c_1(\mathscr{L})^2$. □

If H is a hyperplane class on $\mathbb{P}^2$, then $\mathrm{Pic}(\mathbb{P}^2) \times CH^2(\mathbb{P}^2) \cong \mathbb{Z} \cdot H \times \mathbb{Z} \cdot H^2$. In this special case, Theorem 3.10 simplifies to the following result.

Corollary 3.11. *Assume that the base field k has characteristic unequal to two. Then there is an identification*

$$(\mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot H^2)/\mathbb{Z} \xrightarrow{\sim} [\mathbb{P}^2, \mathrm{BPGL}_2]_{\mathbb{A}^1},$$

where $\mathbb{Z}$ acts on $\mathbb{Z}^{\oplus 2}$ by the formula

$$n \cdot (a, b) = (a + 2n, b + an + n^2). \qquad \square$$

Remark 3.12. Schwarzenberger showed that for arbitrary pairs of integers (a, b), there exists a vector bundle on $\mathbb{P}^2$ with first Chern class a and second Chern class b, [29, Theorem 8]. In particular, his construction yields an alternative verification of the surjectivity of

$$H^1_{\mathrm{Nis}}(\mathbb{P}^2,\ \mathrm{PGL}_2) \longrightarrow [\mathbb{P}^2, \mathrm{BPGL}_2]_{\mathbb{A}^1}$$

that is independent of Proposition 3.7.

4. $\mathbb{A}^1$-homotopy classification of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$

In this section we classify Nisnevich locally trivial $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ up to $\mathbb{A}^1$-weak equivalence of total spaces, using Corollary 3.11. Each Nisnevich locally trivial PGL_2-torsor on a smooth scheme X yields a Nisnevich locally trivial $\mathbb{P}^1$-bundle on X by "passing to the associated fiber space". Conversely, any Nisnevich locally trivial $\mathbb{P}^1$-bundle on X yields a PGL_2-torsor on X by "forming the scheme of automorphisms". A Nisnevich locally trivial $\mathbb{P}^1$-bundle on a smooth scheme is automatically Zariski locally trivial and is therefore the projectivization of a rank-two vector bundle on X. Thus, this section aims to classify projectivizations of rank-two vector bundles on $\mathbb{P}^2$ up to $\mathbb{A}^1$-weak equivalence.

We first show in Corollary 4.2 that, given a pair of Nisnevich locally trivial PGL_2-torsors on a smooth scheme X whose classifying maps coincide in $[X, \mathrm{BPGL}_2]_{\mathbb{A}^1}$, the total spaces of the associated $\mathbb{P}^1$-bundles on X are $\mathbb{A}^1$-weakly equivalent. Specializing to the case where $X = \mathbb{P}^2$, we then observe in Theorem 4.5, by means of Chow ring computations, that the $\mathbb{P}^1$-bundles corresponding to distinct elements of $[\mathbb{P}^2, \mathrm{BPGL}_2]_{\mathbb{A}^1}$ can be distinguished.

4.1. $\mathbb{A}^1$-classification of projective bundles

Let $n \geq 2$ be a natural number, and assume that the characteristic of k does not divide n. Write $\mathcal{G}\nabla_n$ for the infinite Grassmannian parametrizing n-dimensional subspaces of the free k-vector space generated by $\mathbb{Z}$. Suppose X is a (connected) smooth k-scheme, and $\mathscr{E}$ is a rank-n vector bundle on X. Since $\mathcal{G}\nabla_n$ is $\mathbb{A}^1$-weakly equivalent to the space BGL_n described in the previous Section 3, the classifying map $f_{\mathscr{E}}$ of $\mathscr{E}$, as discussed in Section 3.1, determines an element in $[X, \mathcal{G}\nabla_n]_{\mathbb{A}^1}$.

Write γ_n for the universal rank-n vector bundle on $\mathcal{G}\nabla_n$. The class of the map $f_{\mathscr{E}}$ in $[X, \mathcal{G}\nabla_n]_{\mathbb{A}^1}$ need not be represented by an actual morphism from X to $\mathcal{G}\nabla_n$. Since X is smooth, the Jouanolou-Thomason homotopy lemma, [32, Proposition 4.4], guarantees that there always exists a smooth affine scheme X' and a morphism $\pi : X' \to X$ that is a torsor under a vector bundle on X. In particular,

π is an $\mathbb{A}^1$-weak equivalence. In general, the pair (X', π) is not unique, and we refer to a choice of such a pair as *a Jouanolou device*. If we write $\mathscr{E}'$ for $\pi^*\mathscr{E}$, then by [3, Theorem 5.2.3] the classifying map $f_{\mathscr{E}'}$ is represented by a morphism $X \to \mathcal{G}\nabla_n$ that, by abuse of terminology, we will also denote $f_{\mathscr{E}'} : X' \to \mathcal{G}\nabla_n$.

It follows that the morphism $\mathbb{P}_{X'}(\mathscr{E}') \to X'$ is the pullback of $\mathbb{P}_{\mathcal{G}\nabla_n}(\gamma_n) \to \mathcal{G}\nabla_n$ along the morphism $f_{\mathscr{E}'}$ of the previous paragraph. On the other hand, there is a pullback square of the form

$$\begin{array}{ccc} \mathbb{P}_{X'}(\mathscr{E}') & \longrightarrow & X' \\ \downarrow & & \downarrow \\ \mathbb{P}_X(\mathscr{E}) & \longrightarrow & X \end{array}$$

since $\mathbb{P}_X(\mathscr{E}) \times_X X' \cong \mathbb{P}_{X'}(\mathscr{E}')$ by [14, Proposition 3.5.3]. Since the right hand vertical morphism is a torsor under a vector bundle, the left hand vertical map is a torsor under a vector bundle as well[1] and is, in particular, an $\mathbb{A}^1$-weak equivalence.

Proposition 4.1. *Let $n \geq 2$ be a natural number, and assume k has characteristic that does not divide n. Suppose X is a smooth k-scheme and $\mathscr{E}_0$, $\mathscr{E}_1$ are a pair of rank-n vector bundles on X with classifying maps f_0 and f_1. If the classes of f_0 and f_1 are equal in $[X, \mathcal{G}\nabla_n(k)]_{\mathbb{A}^1}$, then the projective bundles $\mathbb{P}_X(\mathscr{E}_0)$ and $\mathbb{P}_X(\mathscr{E}_1)$ are $\mathbb{A}^1$-weakly equivalent.*

Proof. We produce an explicit chain of three $\mathbb{A}^1$-weak equivalences between the two projective bundles. First, fix a Jouanolou device $\pi : X' \to X$. Write $\mathscr{E}'_i := \pi^*\mathscr{E}_i$. By the discussion just prior to the statement, the maps $\mathbb{P}_{X'}(\mathscr{E}'_i) \to \mathbb{P}_X(\mathscr{E}_i)$ are $\mathbb{A}^1$-weak equivalences. If the classifying maps f_i of the vector bundles $\mathscr{E}_i$ lie in the same class in $[X, \mathcal{G}\nabla_n]_{\mathbb{A}^1}$, then, since the map $[X, \mathcal{G}\nabla_n]_{\mathbb{A}^1} \to [X', \mathcal{G}\nabla_n]_{\mathbb{A}^1}$ induced by pullback is a bijection, it follows from [3, Theorem 5.2.3] that the bundles $\mathscr{E}'_i$ are actually isomorphic as vector bundles on X'. A choice of such an isomorphism induces an isomorphism of the total spaces of the associated projective bundles $\mathbb{P}_{X'}(\mathscr{E}'_0) \cong \mathbb{P}_{X'}(\mathscr{E}'_1)$. Thus we have constructed a diagram

$$\mathbb{P}_X(\mathscr{E}_0) \longleftarrow \mathbb{P}_{X'}(\mathscr{E}'_0) \longrightarrow \mathbb{P}_{X'}(\mathscr{E}'_1) \longrightarrow \mathbb{P}_X(\mathscr{E}_1)$$

where each morphism is an $\mathbb{A}^1$-weak equivalence. □

Corollary 4.2. *Assume that k has characteristic unequal to two and that X is a smooth k-scheme that is $\mathbb{A}^1$-weakly equivalent to a smooth k-scheme of dimension ≤ 2. Suppose $\mathscr{E}_0$, $\mathscr{E}_1$ are two rank-two vector bundles on X. If f_i is the classifying morphism of the Zariski locally trivial* PGL_2*-torsor associated with $\mathbb{P}_X(\mathscr{E}_i)$, and if the classes of f_i in $[X, \mathrm{BPGL}_2]_{\mathbb{A}^1}$ coincide, then $\mathbb{P}_X(\mathscr{E}_0)$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{P}_X(\mathscr{E}_1)$.*

[1] In fact, it is a torsor under the pull-back along π of the vector bundle on X under which π is a torsor.

Proof. The classifying maps f_i lie in the same class in $[X, \mathrm{BPGL}_2]_{\mathbb{A}^1}$ by assumption. By Theorem 3.10 it follows that the Chern classes of $\mathscr{E}_0$ and those of $\mathscr{E}_1$ lie in the same orbit for the action of $\mathrm{Pic}(X)$ on $\mathrm{Pic}(X) \times CH^2(X)$ coming from tensoring by a line bundle.

In other words, there exists $\mathscr{L} \in \mathrm{Pic}(X)$ such that the Chern classes of the twist $\mathscr{E}_0 \otimes \mathscr{L}$ coincide with those of $\mathscr{E}_1$. Therefore, by Lemma 3.9, the classifying maps of the vector bundles $\mathscr{E}_0 \otimes \mathscr{L}$ and $\mathscr{E}_1$ lie in the same class in $[X, \mathcal{G}\nabla_n]_{\mathbb{A}^1}$. Applying Proposition 4.1 in this situation allows us to complete the proof. □

4.2. Chow rings and Chern classes

Computation 4.3. If X is a smooth scheme, if $\mathscr{E}$ is a rank-n vector bundle on X and $\mathbb{P}_X(\mathscr{E})$ is the associated projective space bundle, then the Chow ring $CH^*\big(\mathbb{P}_X(\mathscr{E})\big)$ is described by the projective bundle formula. More precisely,

$$CH^*\big(\mathbb{P}_X(\mathscr{E})\big) \cong {CH^*(X)[\tau]}\big/{\big\langle P_{\mathscr{E}}(\tau)\big\rangle}, \quad \text{where} \quad P_{\mathscr{E}}(\tau) := \sum_{i=0}^{n} c_i(\mathscr{E})\tau^{n-i}. \quad (4.1)$$

If $X = \mathbb{P}^2$, if H is a hyperplane class, and $\mathscr{E}$ a rank-two vector bundle on $\mathbb{P}^2$ with Chern classes c_1, c_2, then (4.1) simplifies to

$$CH^*\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})\big) \cong {\mathbb{Z}[H,\tau]}\big/{\langle H^3, \tau^2 + c_1 H\tau + c_2 H^2\rangle}.$$

The ring structure equips $\mathrm{Pic}\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})\big)$ with an integral cubic form which is computed to be the following,

$$\Phi : \mathrm{Pic}\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})\big) \to CH^3\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})\big) \cong \mathbb{Z}, \quad aH + b\tau \mapsto 3a^2b - 3c_1ab^2 + (c_1^2 - c_2)b^3.$$

The discriminant of Φ is $c_1^2 - 4c_2$.

Now assume that we are given two rank-two bundles on $\mathbb{P}^2$, say $\mathscr{E}_1$ and $\mathscr{E}_2$, with arbitrary Chern classes. Any isomorphism of graded rings, $CH^*\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_1)\big) \to CH^*\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_2)\big)$, induces an invertible linear map of Picard groups, $\mathrm{Pic}\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_1)\big) \to \mathrm{Pic}\big(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_2)\big)$, which, in terms of the bases above can be identified with an element of $\mathrm{GL}_2(\mathbb{Z})$. From [26, Ex. 5, Proposition 18], it follows that the $\mathrm{GL}_2(\mathbb{Z})$-orbits are distinguished by the discriminant. We formulate this as a lemma.

Lemma 4.4. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be any two rank-two vector bundles on $\mathbb{P}^2$. Then, the Chow rings of $\mathbb{P}(\mathscr{E}_1)$ and $\mathbb{P}(\mathscr{E}_2)$ are isomorphic if and only if the discriminants of the associated cubic forms on Picard groups are equal.* □

Theorem 4.5. *Let k be an algebraically closed field having characteristic unequal to two. Suppose $\mathscr{E}$ and $\mathscr{E}'$ are rank-two vector bundles on $\mathbb{P}^2$ with Chern classes (c_1, c_2) and (c_1', c_2'), respectively. Then, an $\mathbb{A}^1$-weak equivalence $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \simeq_{\mathbb{A}^1} \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}')$ exists if and only if (c_1, c_2) and (c_1', c_2') lie in the same orbit for the $\mathbb{Z}$-action on $\mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot H^2$ described in Corollary* 3.11.

Proof. If (c_1, c_2) lies in the same $\mathbb{Z}$-orbit as (c_1', c_2'), then the associated projective bundles are $\mathbb{A}^1$-weakly equivalent: by Corollary 3.11 the $\mathbb{A}^1$-homotopy class of $[\mathbb{P}^2, \mathrm{BPGL}_2]$ is equivalent to specifying the $\mathbb{Z}$-orbit of the pair (c_1, c_2). It follows that the $\mathbb{A}^1$-homotopy classes corresponding to the classifying maps of $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{E}')$ agree. Given this fact, Corollary 4.2 implies that the projective bundles associated with these vector bundles are $\mathbb{A}^1$-weakly equivalent.

Conversely, suppose that we have an $\mathbb{A}^1$-weak equivalence $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \simeq_{\mathbb{A}^1} \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}')$. In this case, there is a ring isomorphism $CH^*(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})) \cong CH^*(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}'))$ and in particular, the cubic forms on Picard groups are isomorphic and therefore have equal discriminants by Lemma 4.4. We need to show that (c_1, c_2) and (c_1', c_2') lie in the same $\mathbb{Z}$-orbit. Note that $c_1(\mathscr{E})^2 - 4c_2(\mathscr{E}) \equiv c_1(\mathscr{E}) \mod 2$. By definition, tensoring $\mathscr{E}$ by a line bundle preserves the $\mathbb{Z}$-orbit of (c_1, c_2). After replacing $\mathscr{E}$ and $\mathscr{E}'$ by $\mathscr{E} \otimes \mathscr{L}$ and $\mathscr{E}' \otimes \mathscr{L}'$ if necessary, we can assume that $c_1(\mathscr{E})$ and $c_1(\mathscr{E}')$ are either both equal to zero or both equal to one. Now, the equality of discriminants implies that $4 \cdot c_2(\mathscr{E}) = 4 \cdot c_2(\mathscr{E}')$, and therefore the second Chern classes of the bundles must be equal as well. It follows that the vector bundles $\mathscr{E}$ and $\mathscr{E}'$ can be assumed to have $(c_1, c_2) = (c_1', c_2')$. But then (c_1, c_2) and (c_1', c_2') obviously lie in the same $\mathbb{Z}$-orbit. □

5. Concordance classification of rank-two vector bundles over $\mathbb{P}^2$

This section discusses the $\mathbb{A}^1$-concordance classification of rank-two vector bundles on $\mathbb{P}^2$. Our main result, Theorem 5.4, asserts that among all bundles with vanishing second Chern class, the first Chern class is the only concordance-invariant. To be more precise, we show that any rank-two vector bundle on $\mathbb{P}^2$ with arbitrary first Chern class c_1 and second Chern class $c_2 = 0$ is $\mathbb{A}^1$-concordant to the split bundle $\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(c_1)$. This result allows us, in the subsequent Section 6, to obtain $\mathbb{A}^1$-h-cobordism classification results for projectivizations of "topologically split" bundles.

Definition 5.1 (Concordance and direct concordance). Given a k-scheme X and two vector bundles $\mathscr{E}_0, \mathscr{E}_1$ on X, we say that $\mathscr{E}_0$ and $\mathscr{E}_1$ are *directly* $\mathbb{A}^1$*-concordant* if there exists a vector bundle $\mathscr{E}$ over $X \times \mathbb{A}^1$ such that $\mathscr{E}_0 \cong \iota_0^*\mathscr{E}$ and $\mathscr{E}_1 \cong \iota_1^*\mathscr{E}$, where $\iota_i : X \to X \times \{i\} \subset X \times \mathbb{A}^1$ are the obvious inclusions. The vector bundles $\mathscr{E}_0$ and $\mathscr{E}_1$ on X are said to be $\mathbb{A}^1$*-concordant* if they are equivalent under the equivalence relation generated by direct $\mathbb{A}^1$-concordance.

Remark 5.2. A direct $\mathbb{A}^1$-concordance is a deformation of vector bundles. If two vector bundles are $\mathbb{A}^1$-concordant, then they can be deformed into each other, over a base space that need not be irreducible. On the other hand, if two vector bundles are deformation equivalent, then they need not be $\mathbb{A}^1$-concordant since the parameter space of the deformation need not contain any affine lines.

Remark 5.3. The homotopy invariance results of Quillen-Suslin [28] and Lindel [21] imply that for smooth, affine X, the notion of $\mathbb{A}^1$-concordance agrees with vector bundle isomorphism. However, over non-smooth or non-affine base schemes, there are non-trivial deformations and $\mathbb{A}^1$-concordances of vector bundles.

Theorem 5.4 (Concordance classification of vector bundles with $c_2 = 0$). *If $\mathscr{E}$ is a rank-two vector bundle on $\mathbb{P}^2$ with arbitrary first Chern class c_1 and with vanishing second Chern class $c_2 = 0$, then $\mathscr{E}$ is $\mathbb{A}^1$-concordant to $\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(c_1)$. Thus, any two rank-two vector bundles on $\mathbb{P}^2$ with first Chern class c_1 and second Chern class 0 are $\mathbb{A}^1$-concordant.*

A proof of Theorem 5.4 is given in Section 5.2.

5.1. Explicit construction of $\mathbb{A}^1$-concordances

To prepare for the proof Theorem 5.4, we aim to refine the results of Proposition 2.34 to statements about $\mathbb{A}^1$-concordance. The proof of Proposition 2.34 made use of the irreducibility of the moduli spaces $M(d)$, as well as deformations that connect bundles in $M(d; e) \subseteq M(d)$ to bundles in $M(e)$. The deformations that go from $M(d; e)$ to $M(e)$ are explicitly described in Strømme's paper, [30, Sect. 4], and are easily seen to be $\mathbb{A}^1$-concordances. For the reader's convenience, we briefly recall their construction.

Construction 5.5. Fix the following data:

(5.5.1) A rank-two vector bundle $\mathscr{F}$ on $\mathbb{P}^2$ with Chern classes $c_1, c_2 \in \mathbb{Z}$ and splitting type $e \geq -1$;
(5.5.2) An integer $d > e$ and a global section $\tau \in H^0\big(\mathbb{P}^2, \mathscr{F}(d-c_1)\big)$ that vanishes in a codimension-two subscheme $Y \subseteq \mathbb{P}^2$;
(5.5.3) A section $F \in H^0\big(\mathbb{P}^2, \mathscr{O}(2d - c_1)\big)$ whose zero locus is disjoint from Y.

We obtain a sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \xrightarrow{\tau} \mathscr{F}(d - c_1) \to \mathscr{I}_Y \otimes \mathscr{O}_{\mathbb{P}^2}(2d - c_1) \to 0$$

and an associated extension class $\xi \in \operatorname{Ext}^1\big(\mathscr{I}_Y, \mathscr{O}_{\mathbb{P}^2}(c_1 - 2d)\big)$.

To continue with the construction, consider the standard projection $\pi : \mathbb{P}^2 \times \mathbb{A}^1 \to \mathbb{P}^2$, consider the map

$$\tau \wedge (-) : \mathscr{F} \to \underbrace{(\wedge^2 \mathscr{F})(d - c_1)}_{\cong \mathscr{O}_{\mathbb{P}^2}(d)} \qquad \sigma \mapsto \tau \wedge \sigma,$$

and choose a coordinate T on $\mathbb{A}^1$. We obtain the following monad of vector bundles on $\mathbb{P}^2 \times \mathbb{A}^1$,

$$\pi^*\mathscr{O}(c_1-d) \xrightarrow{b:=(T\cdot\mathrm{Id},\tau,-F)^t} \pi^*\mathscr{O}(c_1-d) \oplus \pi^*\mathscr{F} \oplus \pi^*\mathscr{O}(d) \xrightarrow{a:=(F,\tau\wedge(-),T\cdot\mathrm{Id})} \pi^*\mathscr{O}(d).$$

The vanishing loci of τ and $-F$ are disjoint by assumption. This implies that b is injective and that a is surjective. It also implies that both maps have constant rank, so that the cohomology sheaf $\mathscr{C}$ is locally free. Given $i \in \{0, 1\}$, let $\iota_i : \mathbb{P}^2 \times \{i\} \to \mathbb{P}^2 \times \mathbb{A}^1$ be the corresponding inclusion. The bundle $\mathscr{C}$ provides an $\mathbb{A}^1$-concordance from the bundle $\mathscr{F}_0 \cong \iota_0^*(\mathscr{C})$ to the bundle $\mathscr{F}_1 := \iota_1^*(\mathscr{C})$. Strømme proves in [30, Proposition 4.3] that the following properties hold:

(5.5.4) The bundle $\mathscr{F}_1$ is isomorphic to $\mathscr{F}$;

(5.5.5) The bundle $\mathscr{F}_0$ has splitting type d. The Chern classes of $\mathscr{F}_0$ and $\mathscr{F}_1$ agree;

(5.5.6) The bundle $\mathscr{F}_0$ appears in a sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{F}_0(-d) \to \mathscr{I}_Y \otimes \mathscr{O}_{\mathbb{P}^2}(c_1 - 2d) \to 0.$$

whose extension class is the image of ξ under the map

$$F^2 \cdot (-) : \operatorname{Ext}^1\big(\mathscr{I}_Y,\ \mathscr{O}_{\mathbb{P}^2}(c_1 - 2d)\big) \to \operatorname{Ext}^1\big(\mathscr{I}_Y,\ \mathscr{O}_{\mathbb{P}^2}(2d - c_1)\big).$$

Remark 5.6 (Explanation of $F^2 \cdot (-)$). Using the Hartshorne-Serre correspondence, we obtain the following diagram, which gives an elementary description of the map $F^2 \cdot (-)$ that appears in Item (5.5.6) above,

$$\begin{array}{ccc}
\operatorname{Ext}^1\big(\mathscr{I}_Y, \mathscr{O}_{\mathbb{P}^2}(c_1-2d)\big) & \xrightarrow{F^2\cdot(-)} & \operatorname{Ext}^1\big(\mathscr{I}_Y, \mathscr{O}_{\mathbb{P}^2}(2d-c_1)\big) \\
{\scriptstyle \text{Hartshorne-Serre, Fact 2.1}}\big\downarrow & & \big\downarrow{\scriptstyle \text{Hartshorne-Serre, Fact 2.1}} \\
H^0\big(Y, \wedge^2 \mathscr{N}_{X/Y} \otimes \mathscr{O}_{\mathbb{P}^2}(c_1-2d)|_Y\big) & \xrightarrow[F^2\cdot(-)]{} & H^0\big(Y, \wedge^2 \mathscr{N}_{X/Y} \otimes \mathscr{O}_{\mathbb{P}^2}(2d-c_1)|_Y\big).
\end{array}$$

The vertical arrows are injective since $h^1\big(X,\ \mathscr{O}_{\mathbb{P}^2}(c_1 - 2d)\big) = h^1\big(X,\ \mathscr{O}_{\mathbb{P}^2}(2d - c_1)\big) = 0$.

5.2. Proof of Theorem 5.4

Let $\mathscr{E}$ be a rank-two vector bundle on $\mathbb{P}^2$ with arbitrary c_1 and vanishing $c_2 = 0$, as in the formulation of Theorem 5.4. We aim to deform $\mathscr{E}$ to the split bundle $\mathscr{O}_{\mathbb{P}^2}(c_1) \oplus \mathscr{O}_{\mathbb{P}^2}$. To this end we produce three chains of $\mathbb{A}^1$-concordances that together give the desired deformation.

Step 1: Increasing the splitting type. First, we fix an integer $N \gg 0$ satisfying the following three conditions: the sheaf $\mathscr{E}(N-1)$ is globally generated, $N > c_1$ and $2N - c_1 > 0$. We deform $\mathscr{E}$ to a bundle with the same Chern classes and splitting type N. By the Bertini theorem 2.8, the first hypothesis on N guarantees that the vanishing locus of a generic section of $\mathscr{E}(N)$ is smooth; fix a section $\tau \in H^0\big(\mathbb{P}^2,\ \mathscr{E}(N)\big)$ whose vanishing locus Y is a zero-dimensional, smooth scheme.

Applying Construction 5.5, we obtain an $\mathbb{A}^1$-concordance from $\mathscr{E}$ to a bundle $\mathscr{E}_0$ with Chern classes c_1 and 0 and splitting type N. Sequence (5.5.6) then yields that

$$0 = c_2(\mathscr{E}_0) = c_1\big(\mathscr{O}_{\mathbb{P}^2}(N)\big)\cdot c_1\big(\mathscr{O}_{\mathbb{P}^2}(c_1-N)\big)+c_2(\mathscr{I}_Y) \quad \Rightarrow \quad \#Y = N\cdot(N-c_1).$$

Replacing $\mathscr{E}$ by $\mathscr{E}_0$, we are free to make the following assumption for the remainder of the present proof.

Assumption w.l.o.g. 5.7. If N denotes the splitting type of $\mathscr{E}$, then $N > c_1$ and $2N - c_1 > 0$. Further, there exists a reduced, finite subscheme $Y \subset \mathbb{P}^2$ of length $N \cdot (N - c_1)$, a section $s \in H^0\big(\mathbb{P}^2,\ \mathscr{E}(-N)\big)$ vanishing precisely on Y and giving rise to an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \xrightarrow{s} \mathscr{E}(-N) \to \mathscr{I}_Y \otimes \mathscr{O}_{\mathbb{P}^2}(c_1 - 2\cdot N) \to 0. \tag{5.1}$$

Step 2: Deforming the subscheme. Next, we aim to deform the bundle $\mathscr{E}$ to a new bundle with the same Chern classes and splitting type, but for which the associated zero-dimensional subscheme is the intersection of two curves of appropriate degree. To this end, we apply the extension result for vector bundles, Corollary 2.7, which we obtained as a corollary to the Hartshorne-Serre correspondence. More precisely, fix a pair of smooth curves C and D of degrees N and $N - c_1$, respectively, intersecting in a reduced subscheme Y_1 consisting of $N(N - c_1)$ distinct points. Write $X = \mathbb{P}^2 \times \mathbb{A}^1$ and choose a subvariety $Y_X \subset X$ that is the union of $N(N - c_1)$ pairwise disjoint sections, with $Y_X|_{\mathbb{P}^2\times\{0\}} = Y$ and $Y_X|_{\mathbb{P}^2\times\{1\}} = Y_1$. Corollary 2.7 will then allow to find an $\mathbb{A}^1$-concordance between $\mathscr{E}$ and a bundle $\mathscr{E}_1$ that has the splitting type and Chern classes of $\mathscr{E}$, and fits into an exact sequence,

$$0 \to \mathscr{O}_{\mathbb{P}^2} \xrightarrow{s} \mathscr{E}_1(-N) \to \mathscr{I}_{Y_1} \otimes \mathscr{O}_{\mathbb{P}^2}(c_1 - 2\cdot N) \to 0. \tag{5.2}$$

Sequence (5.2) immediately implies that the splitting type of $\mathscr{E}_1$ is N. Replacing $\mathscr{E}$ by $\mathscr{E}_1$, we are free to make the following assumption for the remainder of the present proof.

Assumption w.l.o.g. 5.8. In addition to the assumptions made in 5.7, we can further assume the reduced scheme Y is the intersection of two smooth curves, say C and D, of degrees N and $N - c_1$, respectively.

The concluding Step 3 of this proof discusses the bundle $\mathscr{E}$ in the context of Construction 5.5. To fix the necessary notation, we briefly discuss the extension class of Sequence (5.1), which determines the isomorphism class of $\mathscr{E}$. The assumptions on the integer N guarantee that the following cohomology groups vanish:

$$H^1\big(X,\ \mathscr{O}_{\mathbb{P}^2}(2N - c_1)\big) = 0 \quad \text{and} \quad H^2\big(X,\ \mathscr{O}_{\mathbb{P}^2}(2N - c_1)\big) = 0. \tag{5.3}$$

Using the Hartshorne-Serre correspondence of Fact 2.1 and Remark 2.2, the splitting type of Sequence (5.1) is thus identified with an element

$$\xi_{\mathscr{E}} \in H^0\big(\mathbb{P}^2,\ \wedge^2\mathscr{N}_{X/Y} \otimes \mathscr{O}_{\mathbb{P}^2}(2N - c_1)|_Y\big) \cong \operatorname{Ext}^1\big(\mathscr{I}_Y,\ \mathscr{O}_{\mathbb{P}^2}(2N - c_1)\big) \tag{5.4}$$

that will later become important.

Step 3*: Deforming to a split bundle*. Generalizing [30, Remark 4.6], we show that $\mathscr{E}$ can be deformed to the split bundle $\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(c_1)$. We begin by showing that the restriction map $r : H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2N - c_1)) \to H^0(Y, \mathscr{O}_{\mathbb{P}^2}(2N - c_1)|_Y)$ is surjective. To this end, factor r as follows:

$$H^0(\mathbb{P}^2,\ \mathscr{O}_{\mathbb{P}^2}(2N-c_1)) \xrightarrow{r_1} H^0(C,\ \mathscr{O}_{\mathbb{P}^2}(2N-c_1)|_C) \xrightarrow{r_2} H^0(Y,\ \mathscr{O}_{\mathbb{P}^2}(2N-c_1)|_Y).$$

Surjectivity of r_1 follows by noting that its cokernel is controlled by $H^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(N-c_1)) = 0$. Surjectivity of r_2 follows by noting that

$$\begin{aligned} H^1(C,\ \mathscr{O}_{\mathbb{P}^2}(2N-c_1)|_C \otimes \mathscr{I}_Y) &\cong H^1(C,\ \mathscr{O}_C((2N-c_1)N - N(N-c_1))) \\ &= H^1(C,\ \mathscr{O}_C(N^2)) \\ &\cong H^0(C,\ \omega_C \otimes \mathscr{O}_C(-N^2))^* \\ &\cong H^0(C,\ \mathscr{O}_C(-3N + N^2 - N^2))^* \\ &= H^0(C,\ \mathscr{O}_C(-3N))^* = 0. \end{aligned}$$

We now appeal to Construction 5.5 to produce an $\mathbb{A}^1$-concordance between $\mathscr{E}$ and the bundle $\mathscr{F} = \mathscr{O}_{\mathbb{P}^2}(c_1) \oplus \mathscr{O}_{\mathbb{P}^2}$. To this end, set $d := N$, and choose a section

$$\begin{aligned} \tau \in H^0(\mathbb{P}^2, \mathscr{F}(N-c_1)) &= H^0(\mathbb{P}^2,\ \mathscr{O}_{\mathbb{P}^2}(N) \oplus \mathscr{O}_{\mathbb{P}^2}(N-c_1)) \\ &= H^0(\mathbb{P}^2,\ \mathscr{O}_{\mathbb{P}^2}(N)) \oplus H^0(\mathbb{P}^2,\ \mathscr{O}_{\mathbb{P}^2}(N-c_1)) \end{aligned}$$

associated with the pair of curves (C, D). The section τ vanishes precisely on Y and gives rise to an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \xrightarrow{\tau} \mathscr{F}(N-c_1) \to \mathscr{I}_Y \otimes \mathscr{O}_{\mathbb{P}^2}(2N-c_1) \to 0. \tag{5.5}$$

Using the Hartshorne-Serre correspondence, Fact 2.1, the extension class $\zeta_{\mathscr{F}}$ associated with (5.5) yields an element

$$\xi_{\mathscr{F}} \in H^0(\mathbb{P}^2,\ \wedge^2 \mathscr{N}_{X/Y} \otimes \mathscr{O}_{\mathbb{P}^2}(c_1 - 2N)|_Y).$$

The characterization of locally frees given in Theorem 2.3 asserts that both $\xi_{\mathscr{E}}$ and $\xi_{\mathscr{F}}$ are nowhere-vanishing on Y. Since Y is finite, we can thus find a section $F_Y \in H^0(Y, \mathscr{O}_{\mathbb{P}^2}(2N - c_1)|_Y)$ such that

$$\xi_{\mathscr{E}} = F_Y^2 \cdot \xi_{\mathscr{F}}. \tag{5.6}$$

Surjectivity of the restriction map r allows us to extend F_Y to a section $F \in H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2N - c_1))$ with vanishing locus disjoint from Y. Feeding these data into Construction 5.5, we obtain an $\mathbb{A}^1$-concordance between $\mathscr{F}$ and a bundle $\mathscr{F}_0$ that appears in a sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{F}_0(-N) \to \mathscr{I}_Y \otimes \mathscr{O}_{\mathbb{P}^2}(c_1 - 2N) \to 0, \tag{5.7}$$

and whose extension class is $F^2 \cdot \zeta_{\mathscr{F}}$. To identify this class, recall the diagram of Remark 5.6, which reads in our context as follows:

$$\begin{array}{ccc} \operatorname{Ext}^1\big(\mathscr{I}_Y, \mathscr{O}_{\mathbb{P}^2}(c_1-2N)\big) & \xrightarrow{F^2\cdot(-)} & \operatorname{Ext}^1\big(\mathscr{I}_Y, \mathscr{O}_{\mathbb{P}^2}(2N-c_1)\big) \\ \big\downarrow & & \big\updownarrow \text{isom. by (5.3), (5.4)} \\ H^0\big(Y, \wedge^2\mathscr{N}_{X/Y}\otimes\mathscr{O}_{\mathbb{P}^2}(c_1-2N)|_Y\big) & \xrightarrow[F^2\cdot(-)]{} & H^0\big(Y, \wedge^2\mathscr{N}_{X/Y}\otimes\mathscr{O}_{\mathbb{P}^2}(2N-c_1)|_Y\big). \end{array}$$

The upper horizontal morphism maps the extension class $\zeta_{\mathscr{F}}$ of (5.5) to the extension class of (5.7). On the other hand, it follows by construction of F that the lower horizontal morphism maps $\xi_{\mathscr{F}}$ to $\xi_{\mathscr{E}}$, the latter being induced by Sequence (5.1). Since the vertical arrow on the right is an isomorphism, we obtain that the extension classes of (5.7) and (5.1) agree. In summary, we have seen that $\mathscr{F}_0 \cong \mathscr{E}$, which concludes the proof of Theorem 5.4. □

5.3. Concluding remarks

We discuss briefly the geometry of the moduli spaces $M(d)$ and the relationship with questions of $\mathbb{A}^1$-chain connectedness. As in [7], the space $M(d)$ is fibered over the Hilbert scheme of local complete intersections of codimension two in $\mathbb{P}^2$. The fibers are complements of hyperplane arrangements in projective spaces $\mathbb{P}H^0(Y, \mathscr{O}_Y)$, typically isomorphic to products of $\mathbb{G}_m$. It is possible to produce explicit deformations of lci subschemes, *cf.* the proof of Theorem 5.4, to show that the Hilbert scheme of lci subschemes of codimension two in $\mathbb{P}^2$ is $\mathbb{A}^1$-chain connected. However, the fibers are generally not $\mathbb{A}^1$-chain-connected. Any morphism $\mathbb{A}^1 \to M(d)$ for $d \geq 2$ is a deformation of the underlying lci subscheme equipped with a constant section. Vector bundles differing only in the extension class and not in the subscheme cannot be connected by an $\mathbb{A}^1$-chain inside $M(d)$.

It is this subtle geometry of the moduli spaces $M(d)$ that prevents us from giving a complete concordance classification of rank-two bundles. While we do not expect the moduli spaces $M(d)$ to be $\mathbb{A}^1$-chain connected, there might exist chains of $\mathbb{A}^1$-concordances through higher splitting types which connect bundles that are not $\mathbb{A}^1$-concordant through bundles of type d. We were not able to settle this question except in the case of "topologically split" bundles presented here.

6. $\mathbb{A}^1$-h-cobordism classification of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$

In this section we discuss the classification of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ up to $\mathbb{A}^1$-h-cobordism. On the one hand, the non-deformability results of Strømme allow to show that there exist many $\mathbb{P}^1$-bundles which can not be connected by direct $\mathbb{A}^1$-h-cobordisms. On the other hand, the explicit $\mathbb{A}^1$-concordances produced in Section 5 allow us to establish that any $\mathbb{P}^1$-bundle deformable to the projectivization of a split vector bundle is in fact already $\mathbb{A}^1$-h-cobordant to the split bundle. This provides an $\mathbb{A}^1$-h-cobordism classification for certain $\mathbb{P}^1$-bundles, and also exhibits how far direct $\mathbb{A}^1$-h-cobordism is from being an equivalence relation.

6.1. Preliminaries on $\mathbb{A}^1$-h-cobordisms

For the reader's convenience, we briefly recall the definition of an $\mathbb{A}^1$-h-cobordism.

Definition 6.1 ($\mathbb{A}^1$-h-cobordism, [4, Definition 3.1.1]). Given two smooth, proper abstract varieties X_0 and X_1, an $\mathbb{A}^1$-h-cobordism between X_0 and X_1 is a proper, surjective morphism of smooth abstract varieties, $f : \mathcal{X} \to \mathbb{A}^1$, such that the following holds.

(6.1.1) The fibers $f^{-1}(0)$ and $f^{-1}(1)$ are isomorphic to X_0 and X_1, respectively.
(6.1.2) The natural closed immersions $i_0 : X_0 \to \mathcal{X}$ and $i_1 : X_1 \to \mathcal{X}$ are $\mathbb{A}^1$-weak equivalences.

We say that X_0 and X_1 are directly $\mathbb{A}^1$-h-cobordant if there exists an $\mathbb{A}^1$-h-cobordism between X_0 and X_1. We say that X_0 and X_1 are $\mathbb{A}^1$-h-cobordant if they are equivalent under the equivalence relation generated by direct $\mathbb{A}^1$-h-cobordisms.

Remark 6.2 (Smoothness of the cobordism map). In the setting of Definition 6.1, we conclude as in Remark 2.12 that f is smooth over a Zariski-open neighborhood $V \subseteq \mathbb{A}^1$ that contains 0 and 1.

Remark 6.3 (Restrictions of Picard groups). Maintaining the assumptions of Remark 6.2, write $\mathcal{X}_V := f^{-1}(V)$. We obtain a commutative diagram of groups,

$$\begin{array}{ccc} [\mathcal{X},\ B\mathbb{G}_m]_{\mathbb{A}^1} & \xleftrightarrow{\ \cong\ } & [X_0,\ B\mathbb{G}_m]_{\mathbb{A}^1} \\ {\scriptstyle\cong}\updownarrow & & \updownarrow{\scriptstyle\cong} \\ \operatorname{Pic}(\mathcal{X}) & \xrightarrow[\text{restriction}]{} & \operatorname{Pic}(X_0). \end{array}$$

The isomorphism on top follows from Assumption (6.1.2). The left and right bijections follow from smoothness of $\mathcal{X}$ and X_0, respectively. In particular, the natural restriction map $\operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\mathcal{X}_V) \to \operatorname{Pic}(X_0)$ is bijective, and the restriction $\operatorname{Pic}(\mathcal{X}_V) \to \operatorname{Pic}(X_0)$ is surjective. The same holds for restrictions to X_1.

6.2. $\mathbb{A}^1$-h-cobordism of projective bundles

Given any two $\mathbb{A}^1$-concordant vector bundles over a smooth projective variety X, we will show in Lemma 6.4 that the associated projectivized bundles are $\mathbb{A}^1$-h-cobordant. At the moment, this is the only source of $\mathbb{A}^1$-h-cobordisms between projective bundles at our disposal. As an immediate corollary, we obtain in Proposition 6.5 an $\mathbb{A}^1$-h-cobordism classification of those projective bundles that are deformable to split bundles.

Lemma 6.4 (Construction of direct $\mathbb{A}^1$-h-cobordisms from concordances). *Assume X is a smooth projective abstract variety. Let $\mathscr{E} \to X \times \mathbb{A}^1$ be a direct $\mathbb{A}^1$-concordance between vector bundles $\iota_0^*\mathscr{E}$ and $\iota_1^*\mathscr{E}$. Then, projectivization induces a direct $\mathbb{A}^1$-h-cobordism*

$$\mathbb{P}_{X\times\mathbb{A}^1}(\mathscr{E}) \xrightarrow{g} X \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$$

between the projective bundles $\mathbb{P}_X(\iota_0^*\mathscr{E})$ and $\mathbb{P}_X(\iota_1^*\mathscr{E})$.

Proof. This is a special case of [4, Proposition 3.1.5], with $Y = \mathbb{P}^1$, $Z = \mathbb{P}_{X\times\mathbb{A}^1}(\mathscr{E})$, and $U = \bigsqcup U_i$ a suitable open affine cover of X. To apply the result, we need to check condition (LT) in loc.cit. Starting with an arbitrary open, affine cover $V_i \to X$ of X, the pullback of g along $v_i \times \mathrm{Id} : V_i \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ has the form $g_{v_i} : \mathbb{P}_{V_i\times\mathbb{A}^1}(\mathscr{E}|_{V_i\times\mathbb{A}^1}) \to V_i \times \mathbb{A}^1$. In other words, g_{v_i} is the projectivization of a rank-two vector bundle over the affine scheme $V_i \times \mathbb{A}^1$. By the homotopy invariance results of Quillen-Suslin [28] and Lindel [21], any such vector bundle is the pullback of a rank-two vector bundle from V_i. We can then further refine V to a covering U of X such that the restriction of $\mathscr{E}$ to $U \times \mathbb{A}^1$ is in fact the trivial rank-two bundle. For the Zariski covering U of X, the condition (LT) in [4, Proposition 3.1.5] is satisfied, which proves the claim. □

Proposition 6.5. ($\mathbb{A}^1$-h-cobordism classification of projective bundles deformable to split ones). *Fix integers c_1 and c_2, and assume that there exists $d \in \mathbb{N}$ such that $d^2 - dc_1 + c_2 = 0$. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be two rank-two bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2. Then $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_1)$ and $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_2)$ are $\mathbb{A}^1$-h-cobordant.*

Proof. It suffices to show that the projectivization $\mathscr{E}_1$ is $\mathbb{A}^1$-h-cobordant to the projectivization of a split bundle. Under the hypothesis on the Chern classes, the bundle $\mathscr{E}_1(-d)$ has first Chern class $c_1 - d$ and second Chern class 0. As a consequence, Theorem 5.4 shows that $\mathscr{E}_1(-d)$ is $\mathbb{A}^1$-concordant to the split bundle $\mathscr{O} \oplus \mathscr{O}(c_1 - d)$. Tensoring the chain of $\mathbb{A}^1$-concordances guaranteed by Theorem 5.4 with $\mathscr{O}(d)$, one obtains a chain of $\mathbb{A}^1$-concordances between $\mathscr{E}_1$ and a split bundle. Applying Lemma 6.4 to these $\mathbb{A}^1$-concordances provides the required chain of $\mathbb{A}^1$-h-cobordisms. □

6.3. Direct $\mathbb{A}^1$-h-cobordisms of projective bundles

While Proposition 6.5 might suggest that there are large classes of projectivized vector bundles that are all $\mathbb{A}^1$-h-cobordant, the following theorem asserts that few are in fact directly $\mathbb{A}^1$-h-cobordant.

Theorem 6.6 (Non-existence of direct $\mathbb{A}^1$-h-cobordisms). *Fix integers $c_1, c_2 \in \mathbb{Z}$. There are infinitely many rank-two vector bundles $(\mathscr{E}_j)_{j\in\mathbb{N}}$ on $\mathbb{P}^2$ with Chern classes c_1 and c_2 such that no two of the projectivizations $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_j)$ are directly $\mathbb{A}^1$-h-cobordant.*

Before proving Theorem 6.6 in Section 6.3 below, we draw an immediate corollary and add a few comments.

Corollary 6.7. *Direct $\mathbb{A}^1$-h-cobordism fails to be an equivalence relation.*

Proof. Fix integers c_1, c_2 and assume there exists an integer d such that $d^2 - dc_1 + c_2 = 0$. By Proposition 6.5, any two bundles with these Chern classes will be $\mathbb{A}^1$-h-cobordant. However, by Theorem 6.6, there is an infinite set of bundles no two of which are directly $\mathbb{A}^1$-h-cobordant. □

Remark 6.8. Note that h-cobordism of smooth manifolds *is* an equivalence relation: the obvious composition of two h-cobordisms is not a smooth manifold and not parametrized by the unit interval, but it can be smoothed and re-parametrized. The above shows that such a smoothing is not, in general, possible in algebraic geometry.

Question 6.9.

(6.9.1) If X is any smooth projective variety that is $\mathbb{A}^1$-h-cobordant to a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, does X have the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$? It seems likely to us that the answer is no: the examples coming to mind are non-trivial rank three vector bundles over $\mathbb{P}^1$ deformable to the trivial one.

(6.9.2) Do there exist varieties that have the $\mathbb{A}^1$-homotopy type of a projective bundle but are not $\mathbb{A}^1$-h-cobordant to a projective bundle?

(6.9.3) The techniques developed here do not provide an $\mathbb{A}^1$-h-cobordism classification for projective space bundles that are not deformable to split bundles since we do not know the $\mathbb{A}^1$-concordance classification of such bundles. However, non-deformability results for vector bundles, would not imply non-existence of $\mathbb{A}^1$-h-cobordisms: the $\mathbb{A}^1$-h-cobordisms could go through singular fibers or simply fibers which have no projective bundle structure, and there are presently no methods to prove that a map with singular fibers is an $\mathbb{A}^1$-h-cobordism.

Proof of Theorem 6.6

The proof of Theorem 6.6 relies on Strømme's results concerning deformations of vector bundles, as outlined in Section 2.5.

Step 1*: Simplification of Chern numbers*. Given any rank-two bundle $\mathscr{E}$ on $\mathbb{P}^2$ and any invertible sheaf $\mathscr{L} \in \operatorname{Pic}(\mathbb{P}^2)$, then $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$ and $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E} \otimes \mathscr{L})$ are canonically isomorphic. To prove Theorem 6.6, it will therefore suffice to consider the case where $c_1 \in \{0, -1\}$.

Step 2*: Construction of bundles $\mathscr{E}_j$*. Fix a number $c_2 \in \mathbb{N}$, and recall from Proposition 2.33 that there exists a (large) number $D \gg 3 + c_1$, with the following property. Given any number $j \in \mathbb{N}$, there exists a rank-two vector bundle $\mathscr{E}_j$ on $\mathbb{P}^2$ with Chern classes c_1 and c_2 and with splitting type $d(j) := D + j$ that does not appear as a fiber in any family of bundles on $\mathbb{P}^2$ that is generically of type less than $d(j)$. Choose such D and $(\mathscr{E}_j)_{j\in\mathbb{N}}$ and fix that choice for the remainder of the proof.

We aim to show that no two of the $\mathbb{P}^1$-bundles $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_j)$ are directly $\mathbb{A}^1$-h-cobordant. We argue by contradiction and assume that the following holds.

Assumption 6.10. There exist two distinct numbers $a_0, a_1 \in \mathbb{N}$ and a direct $\mathbb{A}^1$-h-cobordism $f : \mathcal{X} \to \mathbb{A}^1$ where $X_0 := f^{-1}(0) \cong \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{a_0})$ and $X_1 := f^{-1}(1) \cong \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{a_1})$.

Remark 6.11. Since $a_0 \neq a_1$, it follows from construction that $d(a_0) \neq d(a_1)$.

Step 3*: Extending the bundles to open neighborhoods*. If $i \in \{0, 1\}$ is any given index, it follows from the deformation rigidity of $\mathbb{P}^1$-bundles over $\mathbb{P}^2$, Theorem 2.11, that there exist open neighborhoods $U_i = U(i) \subseteq \mathbb{A}^1$, rank-two vector bundles $\mathscr{E}_{U_i}$ over $\mathbb{P}^2 \times U_i$ and commutative diagrams as follows,

$$\begin{array}{ccccc}
\mathcal{X}_{U_i} & & \xrightarrow{\quad f_{U_i} \quad} & & U_i \\
{\scriptstyle \phi_{U_i}}\updownarrow{\scriptstyle \cong} & & & & \updownarrow{\scriptstyle =} \\
\mathbb{P}_{\mathbb{P}^2\times U_i}(\mathscr{E}_{U_i}) & \xrightarrow[\text{bundle map}]{\alpha_{U_i}} & \mathbb{P}^2 \times U_i & \xrightarrow[\text{projection}]{\beta_{U_i}} & U_i \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{a_i}) & \xrightarrow[\text{bundle map}]{\alpha_i} & \mathbb{P}^2 & \xrightarrow[\text{constant}]{\beta_i} & \{i\}
\end{array}$$

where $\mathcal{X}_{U_i} := f^{-1}(U_i)$ and $f_{U_i} := f|_{\mathcal{X}_{U_i}}$.

Using the semicontinuity of splitting types, Fact 2.26, and using the assumption that the bundles $\mathscr{E}_{a_i}$ do not appear as a fiber in any family that is generically of type less than $d(a_i)$, we are free to shrink the open sets U_i and assume that the following holds in addition.

Assumption w.l.o.g. 6.12. The generic splitting type is constant in the families $\mathscr{E}_{U_i}$. More precisely, given any closed point $t \in U_i$, then the bundle $\mathscr{E}_{U_i}|_{\mathbb{P}^2\times\{t\}}$ has generic splitting type $d(a_i)$.

Step 4*: End of proof*. Choose any closed point $t \in U_0 \cap U_1$. We obtain identifications $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{a_0}) \cong X_t \cong \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{a_1})$. Now, since the splitting types $d(a_i)$ are larger than $3 + c_1$, if follows from the uniqueness of the bundle structure, Theorem 2.10, that there exists an automorphism $\psi \in \mathrm{Aut}(\mathbb{P}^2)$ and a commutative diagram,

$$\begin{array}{ccccc}
\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{a_0}) & \xleftrightarrow{\cong} & X_t & \xleftrightarrow{\cong} & \mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{a_1}) \\
{\scriptstyle \text{bundle map}}\downarrow & & & & \downarrow{\scriptstyle \text{bundle map}} \\
\mathbb{P}^2 & & \xrightarrow[\psi]{\cong} & & \mathbb{P}^2.
\end{array}$$

It follows that the bundle $\mathscr{E}_{a_0}$ and the pull-back $\psi^*\mathscr{E}_{a_1}$ differ only by the twist with a suitable line bundle, say $\mathscr{L} \in \mathrm{Pic}(\mathbb{P}^2)$. However, since the Chern classes of the two bundles agree, it follows that $\mathscr{L} \cong \mathscr{O}_{\mathbb{P}^2}$, hence $\mathscr{E}_{a_0} \cong \psi^*\mathscr{E}_{a_1}$. In particular, we obtain that the generic splitting types of $\mathscr{E}_{a_0}$ and $\mathscr{E}_{a_1}$ agree. By Assumption 6.12, this means that $d(a_0) = d(a_1)$, which contradicts Remark 6.11. This finishes the proof of Theorem 6.6. □

7. Complex realization

In the final section, we specialize to the case $k = \mathbb{C}$ and compare the algebraic classification results proven in this paper to their complex-geometric counterparts.

7.1. Comparison maps

We have the following diagram of sets of equivalence classes of projectivized rank-two bundles on $\mathbb{P}^2$, where the left column contains sets of algebraic varieties modulo algebraic equivalence relations and the right column contains complex manifolds modulo complex-geometric equivalence relations:

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{projective bundles } \mathbb{P}_{\mathbb{P}^2(\mathbb{C})}(\mathscr{E})\\ \text{modulo } \mathbb{A}^1\text{-}h\text{-cobordism}\end{array}\right\} & \xrightarrow[\text{Prop. 6.5}]{\phi_1} & \left\{\begin{array}{c}\text{complex manifolds } \mathbb{P}_{\mathbb{CP}^2}(\mathscr{E})\\ \text{modulo deformation equivalence}\end{array}\right\} \\
 & & \Big\downarrow {\scriptstyle \text{Props. 7.4, 7.5}\ |\ \phi_4} \\
\Big\downarrow {\scriptstyle \phi_3\ |\ \text{Prop. 6.5}} & & \left\{\begin{array}{c}\text{complex manifolds } \mathbb{P}_{\mathbb{CP}^2}(\mathscr{E})\\ \text{modulo diffeomorphism}\end{array}\right\} \\
 & & \Big\downarrow {\scriptstyle \text{Prop. 7.5}\ |\ \phi_5} \\
\left\{\begin{array}{c}\text{projective bundles } \mathbb{P}_{\mathbb{P}^2(\mathbb{C})}(\mathscr{E})\\ \text{modulo } \mathbb{A}^1\text{-weak equivalence}\end{array}\right\} & \xrightarrow[\text{Prop. 7.2}]{\phi_2} & \left\{\begin{array}{c}\text{complex manifolds } \mathbb{P}_{\mathbb{CP}^2}(\mathscr{E})\\ \text{modulo homotopy equivalence}\end{array}\right\}
\end{array}
\tag{7.1}
$$

Explanation 7.1. The two horizontal arrows ϕ_1 and ϕ_2 are both induced by complex realization $X \mapsto X(\mathbb{C})$, while the vertical arrows ϕ_3, ϕ_4 and ϕ_5 are induced by the identity map. We briefly show that each of these maps is in fact well-defined.

(7.1.1) If $f : X \to \mathbb{A}^1$ is any $\mathbb{A}^1$-h-cobordism, then f is a smooth morphism over a Zariski neighborhood U of $\{0, 1\} \subseteq \mathbb{A}^1$, see Remark 2.12. The complex realization $f(\mathbb{C}) : X(\mathbb{C}) \to \mathbb{C}$, restricted to $U(\mathbb{C})$ provides a deformation from $X_0(\mathbb{C})$ to $X_1(\mathbb{C})$. It follows that ϕ_1 is well-defined.

(7.1.2) Recall that the assignment that sends a smooth k-scheme X to $X(\mathbb{C})$ equipped with its usual structure of a complex manifold extends to a "complex realization" functor $\mathfrak{R}_\iota$ from the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ to the usual homotopy category of topological spaces, [25, Section 3.3]. Using a slightly different model structure on $\mathcal{S}\sqrt{\ }\rfloor_k$, Dugger and Isaksen showed in [9, Theorem 5.2] that the complex realization functor between homotopy categories is actually part of a Quillen adjunction. In particular, in their model structure, $\mathbb{A}^1$-weak equivalences between smooth schemes are sent to weak equivalences of the associated topological spaces. It follows that ϕ_2 is well-defined.

(7.1.3) If $f : X \to \mathbb{A}^1$ is an $\mathbb{A}^1$-h-cobordism, then the inclusions $f^{-1}(0) \to X$ and $f^{-1}(1) \to X$ are $\mathbb{A}^1$-weak equivalences by definition. In particular, $f^{-1}(0)$ and $f^{-1}(1)$ have the same weak $\mathbb{A}^1$-homotopy type. It follows that ϕ_3 is well-defined.

(7.1.4) Deformations as complex manifolds induce diffeomorphisms, by Ehresmann's fibration theorem. This shows that ϕ_4 is well-defined.

(7.1.5) Diffeomorphism of complex manifolds are homotopy equivalences, hence ϕ_5 is well-defined.

7.2. Homotopy classification

Our results imply that the homotopy classification results in the above diagram agree. In other words, we show that the $\mathbb{A}^1$-homotopy invariants do not contain any more information than the classical algebraic-topological invariants for $\mathbb{P}^1$-bundles over $\mathbb{P}^2$.

Proposition 7.2. *The map ϕ_2 in Diagram* (7.1) *is a bijection.*

Proof. Surjectivity of ϕ_2 follows by GAGA, since every projectivization of a holomorphic vector bundle over $\mathbb{CP}^2$ has an algebraic structure. Injectivity can be seen as follows. Given two varieties $\mathbb{P}_{\mathbb{CP}^2}(\mathscr{E}_1)$, $\mathbb{P}_{\mathbb{CP}^2}(\mathscr{E}_2)$ whose complex realizations are homotopy equivalent, then their cohomology rings are isomorphic. In the situation at hand, the cycle class maps are isomorphisms, so we have an isomorphism of Chow rings. As in the proof of Theorem 4.5, the isomorphism of Chow rings implies that the Chern classes of $\mathscr{E}_1$ and $\mathscr{E}_2$ are in the same $\mathrm{Pic}(\mathbb{P}^2)$-orbit, and hence $\mathbb{P}_{\mathbb{CP}^2}(\mathscr{E}_1)$ and $\mathbb{P}_{\mathbb{CP}^2}(\mathscr{E}_2)$ are $\mathbb{A}^1$-weakly equivalent. □

Remark 7.3. Alternatively, one can use a proof as in Section 4 together with Peterson's classification of complex rank n vector bundles on projective spaces, [27], to see that projectivizations of holomorphic vector bundles over $\mathbb{CP}^2$ are classified up to homotopy equivalence by the exact same $\mathbb{Z}$-orbits of $\mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot H^2$.

7.3. Complex-geometric classification

The vertical maps ϕ_4 and ϕ_5 of Diagram (7.1) are also bijections. In other words, all relevant equivalence relations agree on the set of isomorphism classes of $\mathbb{CP}^1$-bundles over $\mathbb{CP}^2$: cohomology ring isomorphisms, homotopy equivalence, diffeomorphism and deformation equivalence.

Proposition 7.4. *The composition $\phi_5 \circ \phi_4$ the maps in Diagram* (7.1) *is a bijection.*

Proof. Surjectivity is clear because $\phi_5 \circ \phi_4$ is induced from the identity map. If two bundles $\mathbb{P}_{\mathbb{CP}^2}(\mathscr{E}_1), \mathbb{P}_{\mathbb{CP}^2}(\mathscr{E}_2)$ are homotopy equivalent, then their Chern classes are in the same $\mathrm{Pic}(\mathbb{P}^2)$-orbit, *cf.* Proposition 7.2 and its proof. Tensoring by a line bundle, we can assume that the Chern classes of $\mathscr{E}_1$ and $\mathscr{E}_2$ agree. Proposition 2.34 asserts that any two rank-two vector bundles over $\mathbb{P}^2$ whose Chern classes agree are equivalent by the equivalence relation generated by deformations over an irreducible base. This implies the claim. □

Proposition 7.5. *The map ϕ_5 in Diagram* (7.1) *is a bijection.*

Proof. The easiest way to see this is to note that the map from projective bundles modulo deformation equivalence to projective bundles modulo diffeomorphism is surjective.

We also outline how the bijectivity can be deduced from the work of Okonek and van de Ven [26]. First note that from the cohomology ring computation in Computation 4.3 we see that the integral homology of $\mathbb{P}_{\mathbb{CP}^2}(\mathscr{E})$ is torsion free, and the only non-trivial Betti numbers are $b_2 = b_4 = 2$ and $b_0 = b_6 = 1$. In the case of a complex manifold, w_2 can be obtained as the mod 2 reduction of c_1, and the second Chern class determines p_1 via $c_2 = \frac{1}{2}(c_1^2 - p_1)$, *cf.* [26, Proposition 8]. The computations of [26, Proposition 15] show that the diffeomorphism invariants of the projective bundle are determined completely by the Chern classes of the projective bundle. Finally, [26, Proposition 17] (or rather its proof) shows that all invariants are realizable by projective bundles. Summing up, the results of Okonek and van de Ven cited above show that diffeomorphism classes of $\mathbb{CP}^1$-bundles over $\mathbb{CP}^2$ are described by the same invariants as the homotopy equivalence classes of such bundles. This proves the claim. □

Remark 7.6. The generic splitting type of an unstable vector bundle can also be seen from the corresponding projective bundle. Given a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, its restriction to a projective line $\ell \subseteq \mathbb{P}^2$ is a Hirzebruch surface $\mathbb{F}_a$. Generically, it is the Hirzebruch surface $\mathbb{F}_{c_1-2d}$ if d is the generic splitting type. There are some lines where we get a different Hirzebruch surface $\mathbb{F}_e$, with $e \equiv c_1 - 2d \mod 2$. These lines, when viewed as points of the dual projective plane, form a curve, the *curve of jumping lines*. The above description provides one explanation why no difference between such projective bundles is visible in $\mathbb{A}^1$-homotopy or the diffeomorphism type - the differences between the Hirzebruch surfaces $\mathbb{F}_d$ and $\mathbb{F}_{d'}$ for $d \equiv d' \mod 2$ are not visible in either setting.

7.4. Relation to $\mathbb{A}^1$-h-cobordism classification

In the case of bundles that are deformable to split bundles, that is, in the case where there exists an integer d with $d^2 - dc_1 + c_2 = 0$, Proposition 6.5 shows that the two remaining arrows in the diagram are bijections as well. The $\mathbb{A}^1$-h-cobordism classification agrees with the $\mathbb{A}^1$-homotopy classification as well as all the complex-geometric equivalence relations. It is, however, not clear (to us) what happens for bundles that are not deformable to split bundles.

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