Strong uniqueness for Dirichlet operators related to stochastic quantization under exponential/trigonometric interactions on the two-dimensional torus

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Abstract. We consider space-time quantum fields with exponential/trigonometric interactions. In the context of Euclidean quantum field theory, the former and the latter are called the Høegh-Krohn model and the Sine-Gordon model, respectively. The main objective of the present paper is to construct infinite dimensional diffusion processes which solve modified stochastic quantization equations for these quantum fields on the two-dimensional torus by the Dirichlet form approach and to prove strong uniqueness of the corresponding Dirichlet operators.

1. Introduction

In recent years, there has been a growing interest in the study of infinite dimensional stochastic dynamics associated with various models of Euclidean quantum field theory, hydrodynamics and statistical mechanics, see, *e.g.*, [6, 21, 23, 60, 63] and references therein. One of the principal programs in these studies is to obtain Gibbs measures as equilibrium states of stochastic processes taking values in infinite dimensional state spaces.

In Euclidean quantum field theory, the space-time free field in finite volume is given by the Gaussian measure μ_0 on a Sobolev space of negative order $H^{-\delta}(\mathbb{T}^2)$, $\delta > 0$, defined on the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ with zero mean and covariance operator $(1 - \Delta)^{-1}$, Δ being the Laplacian in $L^2(\mathbb{T}^2)$ with periodic

boundary conditions. Heuristically, μ_0 is given by the expression

$$\mu_0(d\phi) \propto \exp\left\{-\frac{1}{2}\int_{\mathbb{T}^2}\left(\phi(x)^2 + |\nabla\phi(x)|_{\mathbb{R}^2}^2\right)dx\right\}\prod_{x\in\mathbb{T}^2}d\phi(x),$$

and it is also called the *massive Gaussian free field*. The ϕ_2^{2m}-*quantum field*, a special case of the $P(\phi)_2$-*quantum field in finite volume*, is one of the most important objects of study in Euclidean quantum field theory (see, *e.g.*, [57, 87]), and is given by the probability measure

$$\mu_{\text{pol}}^{(2m)}(d\phi) = \frac{1}{Z_{\text{pol}}^{(2m)}}\exp\left(-\int_{\mathbb{T}^2} :\phi^{2m}(x): dx\right)\mu_0(d\phi), \quad m = 2, 3, 4\ldots,$$

where $:\phi^{2m}:$ denotes the $2m$-th order Wick power of ϕ with respect to μ_0 and $Z_{\text{pol}}^{(2m)} > 0$ is the normalizing constant given by

$$Z_{\text{pol}}^{(2m)} := \int_{H^{-\delta}(\mathbb{T}^2)}\exp\left(-\int_{\mathbb{T}^2} :\phi^{2m}(x): dx\right)\mu_0(d\phi).$$

Parisi and Wu [81] proposed the above program to find a Markov process or a class of Markov processes so that $\mu_{\text{pol}}^{(2m)}$ is their invariant measure. This program is now called the *stochastic quantization*. (More recently, following [63], such processes with $\mu_{\text{pol}}^{(2m)}$ as an invariant measure have been called *dynamical ϕ_2^{2m}-models*.) In the well-known paper [71], Jona-Lasinio and Mitter firstly provided a rigorous justification of the program for $P(\phi)_2$-quantum fields in finite volume. Strictly speaking, they constructed an infinite dimensional continuous Markov process having the ϕ_2^4-measure $\mu_{\text{pol}}^{(4)}$ (*i.e.*, $P(\phi)_2$-measure in the case $m = 2$) as an invariant measure. Applying the Girsanov transform method to the Ornstein-Uhlenbeck process whose unique invariant measure is the space-time free field measure μ_0, they obtained the resulting Markov process as the unique weak solution of the so-called *(regularized) stochastic quantization equation* (SQE in short)

$$\partial_t X_t(x) = -\frac{1}{2}(1-\Delta)^{1-\gamma}X_t(x) - m(1-\Delta)^{-\gamma} :X_t^{2m-1}(x):$$
$$+ (1-\Delta)^{-\frac{\gamma}{2}}\xi_t(x), \quad t > 0, \ x \in \mathbb{T}^2, \tag{1.1}$$

where the positive constant γ provides a regularizing effect on the nonlinear drift term and on the noise term, $\xi = \{\xi_t = (\xi_t(x))_{x\in\mathbb{T}^2}\}_{t\geq 0}$ being an $\mathbb{R}$-valued Gaussian space-time white noise, that is, ξ is the time derivative of a standard $L^2(\mathbb{T}^2)$-cylindrical Brownian motion $\{B_t = (B_t(x))_{x\in\mathbb{T}^2}\}_{t\geq 0}$.

Since then, there has been a large number of follow-up papers on stochastic quantization, and both theories of SPDEs and Dirichlet forms on infinite dimensional state spaces have been developed intensively. The Dirichlet form approach

was later elaborated in a series of papers [26–28]. In particular, in [27] the closability result was given, in [26] the existence of a diffusion (*i.e.*, a path continuous strong Markov process) was proved and in [28] it was shown that this process indeed solves equation (1.1) in the weak sense for quasi-every initial condition, for $\gamma \geq 0$. In this way the generator of the diffusion process has been identified to be the self-adjoint negative operator $\mathcal{L}$ in $L^2(\mu_{\mathrm{pol}}^{(2m)})$ associated with the classical quasi-regular Dirichlet form given by the ϕ_2^{2m}-measure $\mu_{\mathrm{pol}}^{(2m)}$. The problem of it being uniquely determined by the closure from a minimal domain $\mathfrak{F}C_b^\infty$ (of cylindrical functions) was solved in the sense of Markov uniqueness by [84] for $\gamma > 0$. Strong uniqueness in the strong sense of essential self-adjointness of $\mathcal{L}$ in the relevant L^2-space was solved by [42, 75], still for $\gamma > 0$. A (weaker) property of *restricted* Markov uniqueness has been recently obtained by [85] for $\gamma = 0$. This yields uniqueness within the class of *quasi-regular* Dirichlet forms whose generators extend $(\mathcal{L}, \mathfrak{F}C_b^\infty)$. Moreover these authors show that the martingale problem for $(\mathcal{L}, \mathfrak{F}C_b^\infty)$, up to a certain "$\mu$-equivalence" is unique, in the sense that there is a unique strong Markov process solving that martingale problem (*cf.* [25, 78]). In this way also the uniqueness of probabilistic weak solutions of equation (1.1) with $\gamma = 0$ has been shown. In [85], in addition to [28], also results of [41] are used, yelding strong solutions. For uniqueness of the invariant measures in terms of density under translations, see [86].

Let us mention at this point that the original motivation to study measures like the $P(\phi)_2$-measure came from quantum field theory on Euclidean space-time $\mathbb{R}^d$, in particular where the 2-dimensional torus $\mathbb{T}^2$ is replaced by the 2-dimensional space-time $\mathbb{R}^2$. Such measures are called (infinite volume) $P(\phi)_2$-measures and can indeed be constructed from limits of measures constructed relative to the torus, as shown in the work [62]; see [57, 87] for other constructions of such infinite volume measures. The invariance of these measures under the action of the Euclidean group is important to construct the associated relativistic quantum fields, the original physical motivation for studying Euclidean measures. See, *e.g.*, [2, 8, 14, 46, 57, 65, 87, 88]. The corresponding SQE (1.1) with $\gamma > 0$ has been studied in the sense of associating a Dirichlet form and a Markov semigroup in [38]. The construction of an associated diffusion has been first performed also for $\gamma \geq 0$ in [26] and it was shown in [28] that it solves (1.1) in the infinite volume case. For partial results on the uniqueness problems (corresponding to those we mentioned in the case of the torus model), see [30, 75, 85]. The result in [85] also applies to the infinite volume case. Here recent results of [79], related to methods developed in [23, 39, 59, 60, 63], are used. In [22] ergodicity of the stochastic dynamics associated to the infinite volume $P(\phi)_2$-models has been proven in the situation of "pure phases" (in the sense of [61]). Let us remark in passing that *stable-like non-local* Dirichlet forms have been brought in connection with stochastic quantization of Euclidean fields in recent papers [18, 19, 31].

In the present paper, we initiate the study of the corresponding SQE for a model different from the $P(\phi)_2$-case, but which also leads to interesting relativistic quantum fields, as shown at about the same time as for the $P(\phi)_2$-case. This

quantum field model was introduced by Høegh-Krohn [66] in a Hamiltonian setting, and its Euclidean version was constructed in [11]. In the latter paper [11], the probability measure

$$\mu_{\exp}^{(a)}(d\phi) = \frac{1}{Z_{\exp}^{(a)}} \exp\left(-\int_{\mathbb{T}^2} :\exp(a\phi)(x): dx\right) \mu_0(d\phi)$$

was constructed and shown to yield interesting relativistic quantum fields, where $Z_{\exp}^{(a)} > 0$ is the normalizing constant, $a \in (-\sqrt{4\pi}, \sqrt{4\pi})$ is called the *charge parameter*, and the *Wick exponential* $:\exp(a\phi)(x):$ is formally introduced by the expression

$$:\exp(a\phi)(x): = \exp\left(a\phi(x) - \frac{a^2}{2}\mathbb{E}^{\mu_0}[\phi(x)^2]\right), \qquad x \in \mathbb{T}^2,$$

$\mathbb{E}^{\mu_0}$ standing for the expectation with respect to the massive Gaussian free field μ_0 (given heuristically by (1.1)). This model is called the *Høegh-Krohn model* or the $\exp(\phi)_2$-*model*. The papers [7, 13] study extensions of the model to $\mathbb{R}^d$ for $d \geq 2$, and also show relations with martingale theory and multiplicative noise. We also mention that this model has also been discussed in the framework of nonstandard analysis (*cf.* [14]) and white noise analysis (*cf.* [8, 65]), as well as in connection with relativistic quantum fields on curved spaces [50]. A connection with problems in representation theory of groups of mappings has been discussed in [16], see also [15], and for new developments, [5]. The relevance of this model has also been pointed out in connection with string theory [17, 58], and recently rediscovered in connection with topics like Liouville quantum gravity, where the random measure, called the *Gaussian multiplicative chaos*,

$$\mathcal{M}_\phi^{(a)}(dx) = :\exp(a\phi)(x): dx, \quad x \in \mathbb{T}^2, \phi \in H^{-\delta}(\mathbb{T}^2)$$

plays a central role (*cf.*, *e.g.*, [32, 37, 47, 48, 56, 72, 74, 83]).

The Høegh-Krohn model is also closely connected with the *Sine-Gordon model* (or the $\cos(\phi)_2$-*quantum field model*)

$$\mu_{\cos}^{(a)}(d\phi) = \frac{1}{Z_{\cos}^{(a)}} \exp\left(-\int_{\mathbb{T}^2} :\cos(a\phi)(x): dx\right) \mu_0(d\phi),$$

where $:\cos(a\phi)(x):$ is defined by

$$:\cos(a\phi)(x): = \frac{1}{2}\left(:\exp\left(\sqrt{-1}a\phi\right)(x): + :\exp\left(-\sqrt{-1}a\phi\right)(x):\right).$$

As with the Høegh-Krohn model, this quantum field model and generalizations of these models as $\int :\exp(a\phi)(x): \nu(da)$ and $\int :\cos(a\phi)(x): \nu(da)$ have also been studied for a long period by many authors, where $\nu(da)$ is a positive Radon

measure on the interval $(-\sqrt{4\pi}, \sqrt{4\pi})$ and not necessarily a Dirac measure. See, e.g., [8–11, 51, 53, 54]. All this gives us much motivation to study SQEs and the associated Dirichlet forms for such non-polynomial models. In the case where $\mathbb{R}^2$ and $\mathbb{T}^2$ are replaced by $\mathbb{R}$, this has been discussed in [20], which provides solutions of the SQE and proves strong uniqueness of the corresponding Dirichlet operator for any $\gamma \geq 0$. For the same problem where $\exp(\phi)_1$ is replaced by $P(\phi)_1$, see [70, 73].

The main purpose of the present paper is to study regularized SQEs both for the Høegh-Krohn model

$$
\partial_t X_t(x) = -\frac{1}{2}(1-\Delta)^{1-\gamma} X_t(x) - \frac{a}{2}(1-\Delta)^{-\gamma} : \exp(aX_t(x)):
$$
$$
+ (1-\Delta)^{-\frac{\gamma}{2}} \xi_t(x), \quad t > 0, x \in \mathbb{T}^2, \tag{1.2}
$$

and for the Sine-Gordon model

$$
\partial_t X_t(x) = -\frac{1}{2}(1-\Delta)^{1-\gamma} X_t(x) + \frac{a}{2}(1-\Delta)^{-\gamma} : \sin(aX_t(x)):
$$
$$
+ (1-\Delta)^{-\frac{\gamma}{2}} \xi_t(x), \quad t > 0, x \in \mathbb{T}^2 \tag{1.3}
$$

with the Dirichlet form approach. Furthermore, we prove strong uniqueness of the corresponding Dirichlet operators in the relevant L^p-spaces for all $p \geq 1$. To the best of our knowledge, except for the L^1-uniqueness obtained by [90], there seems to be only a few results on L^p-uniqueness of Dirichlet operators associated with (1.2) and (1.3) for general $p \geq 1$. We mention here that Mihalache [77] first proved the unique existence of the regularized SQE (1.2) in the weak probabilistic sense under restrictive conditions on $0 < \gamma \leq 1$ and the charge parameter a, by combining several properties of the $\exp(\phi)_2$-measure $\mu_{\exp}^{(a)}$ established in [11] with approximation methods in [43]. Quite recently, influenced by the quick development on singular SPDEs (cf. [60, 63]), unique probabilistic strong solutions to the original SQEs (1.2) and (1.3) (i.e., with $\gamma = 0$) were constructed by [56, 67, 68, 80] and [40, 64], respectively. However, these results do not imply strong uniqueness of the corresponding Dirichlet operator in general (see Remark 2.6 below). Let us also mention that a recent approach on stochastic quantization of the $\exp(\phi)_2$-model on $\mathbb{T}^2$ or $\mathbb{R}^2$, and of the $P(\phi)_2$-model on $\mathbb{T}^2$, through elliptic SPDEs, has been developed in [4] (based on previous work on dimensional reduction [3]).

The structure of the present paper is as follows. In Section 2, we describe our framework and state the main results. To summarize them, let us stress that both the $\exp(\phi)_2$-measure and the $\cos(\phi)_2$-measure have been constructed for values of the charge constant a in the interval $(-\sqrt{4\pi}, \sqrt{4\pi})$. Our methods permit to cover the whole range of this parameter and show that the pre-Dirichlet form associated with (1.2) (respectively (1.3)), i.e., associated with the $\exp(\phi)_2$-measure $\mu_{\exp}^{(a)}$ (respectively the $\cos(\phi)_2$-measure $\mu_{\cos}^{(a)}$), for all $a \in (-\sqrt{4\pi}, \sqrt{4\pi})$ is closable, and one has L^p-uniqueness of the corresponding Dirichlet operator for all $1 \leq p < \frac{1}{2}(1 + \frac{4\pi\gamma}{a^2})$ with $|a| < \sqrt{4\pi\gamma}$. As a corollary we show the existence of

diffusion processes solving (1.2) and (1.3) weakly for quasi-every starting point in the state space $H^{-\delta}(\mathbb{T}^2)$ for suitable $\delta > 0$. In Section 3, we construct the Wick power, the Wick exponential and the Wick trigonometric mappings, including also a new detailed estimate for the Wick powers (Proposition 3.1) and a proof of the finite dimensional approximation of the Wick exponential and the Wick trigonometric mappings (Theorem 3.4 and Corollary 3.7). In Section 4, we provide the proof of our main result (Theorem 2.4) by combining the tools of Section 3 with the argument in [75]. Finally, in the appendix, we present key estimates on the relevant Green functions, which are used in Sections 3 and 4.

Throughout the present paper, we denote by C positive constants which may change from line to line. When their dependence on some parameters are significant, we specify them as C_p, $C(\varepsilon)$, etc. Besides, we use the notation $a \lesssim b$ if there exists a positive constant C, independent of the variables under consideration, such that $a \leq C \cdot b$.

Note added in proof. Let us signalize that during the printing procedure of the present paper, several papers [34, 35, 45] also concerning models with exponential interactions have appeared on arXiv.

2. Framework and results

We begin by introducing some notation and the objects we will be working with. Let $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ be the 2-dimensional torus equipped with the Lebesgue measure dx. Let $L^2(\mathbb{T}^2)$ be the Hilbert space consisting of all real-valued Lebesgue square integrable functions equipped with the usual inner product

$$(f, g)_{L^2} = \int_{\mathbb{T}^2} f(x)g(x)dx, \quad f, g \in L^2(\mathbb{T}^2).$$

For $k = (k_1, k_2) \in \mathbb{Z}^2$ and $x = (x_1, x_2) \in \mathbb{T}^2$, we write $|k| = (k_1^2 + k_2^2)^{1/2}$, $\|k\| = \max\{|k_1|, |k_2|\}$ and $k \cdot x = k_1 x_1 + k_2 x_2$. We denote by $\mathcal{S}(\mathbb{T}^2)$ the space of real-valued smooth functions on $\mathbb{T}^2$. We then define the space of distributions $\mathcal{S}'(\mathbb{T}^2)$ as the set of linear maps u from $\mathcal{S}(\mathbb{T}^2)$ to $\mathbb{R}$ such that there exist $N \in \mathbb{N}$

and $C > 0$ with

$$|u(\varphi)| \leq C \max_{0 \leq i+j \leq N} \max_{(x_1,x_2) \in \mathbb{T}^2} \left| \frac{\partial^{i+j} \varphi}{\partial x_1^i \partial x_2^j}(x_1, x_2) \right|, \quad \varphi \in \mathcal{S}(\mathbb{T}^2).$$

Since $\mathcal{S}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2) \subset \mathcal{S}'(\mathbb{T}^2)$, the L^2-inner product $(\cdot, \cdot)_{L^2}$ is naturally extended to the pairing of $\mathcal{S}(\mathbb{T}^2)$ and its dual space $\mathcal{S}'(\mathbb{T}^2)$. Although we work in the framework of real-valued functions, it is sometimes easier to do computations by using a system of complex-valued functions $\{\mathbf{e}_k; k \in \mathbb{Z}^2\}$ defined by

$$\mathbf{e}_k(x) = \frac{1}{2\pi} e^{\sqrt{-1}k \cdot x}, \quad k \in \mathbb{Z}^2, \ x \in \mathbb{T}^2.$$

For each $u \in \mathcal{S}'(\mathbb{T}^2)$, we define the k-th Fourier coefficient $\hat{u}(k)$ ($k \in \mathbb{Z}^2$) by

$$\hat{u}(k) := {}_{\mathcal{S}'}\langle u, \mathbf{e}_{-k} \rangle_{\mathcal{S}} = \int_{\mathbb{T}^2} u(x) \mathbf{e}_{-k}(x) dx.$$

Note that, since u is real-valued, $\overline{\hat{u}(k)} = \hat{u}(-k)$ holds for $k \in \mathbb{Z}^2$.

We define the real L^2-Sobolev space of order s with periodic boundary condition by

$$H^s(\mathbb{T}^2) = \left\{ u \in \mathcal{S}'(\mathbb{T}^2); \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s |\hat{u}(k)|^2 < \infty \right\}, \quad s \in \mathbb{R}.$$

This space is a Hilbert space equipped with the inner product

$$(u, v)_{H^s} = \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s \hat{u}(k) \hat{v}(-k), \quad u, v \in H^s(\mathbb{T}^2).$$

Note that $H^0(\mathbb{T}^2)$ coincides with $L^2(\mathbb{T}^2)$ and we regard $H^{-s}(\mathbb{T}^2)$ as the dual space of $H^s(\mathbb{T}^2)$ through the standard chain

$$H^s(\mathbb{T}^2) \subset L^2(\mathbb{T}^2) \subset H^{-s}(\mathbb{T}^2), \quad s \geq 0.$$

We denote by ${}_{H^{-\alpha}}\langle \cdot, \cdot \rangle_{H^\alpha}$ the dualization between $H^\alpha(\mathbb{T}^2)$ and its dual space $H^{-\alpha}(\mathbb{T}^2)$.

Let Δ be the Laplacian with periodic boundary condition acting on $L^2(\mathbb{T}^2)$. It is a self-adjoint negative operator in $L^2(\mathbb{T}^2)$ with $\mathrm{Dom}(\Delta) = H^2(\mathbb{T}^2)$. We set $A := (1 - \Delta)^{-1}$ and $\lambda_k := 1 + |k|^2$ ($k \in \mathbb{Z}^2$). Note that A is a compact and self-adjoint operator acting on $L^2(\mathbb{T}^2)$ and that the spectra of both $1 - \Delta$ and A consist only of eigenvalues. Let $\{e_k; k \in \mathbb{Z}^2\}$ be the usual CONS of $L^2(\mathbb{T}^2)$ associated with $\{\mathbf{e}_k; k \in \mathbb{Z}^2\}$, that is, $e_{(0,0)}(x) = (2\pi)^{-1}$ and

$$e_k(x) = \frac{1}{\sqrt{2}\pi} \begin{cases} \cos(k \cdot x) & k \in \mathbb{Z}_+^2 \\ \sin(k \cdot x) & k \in \mathbb{Z}_-^2 \end{cases} \quad x \in \mathbb{T}^2,$$

where $\mathbb{Z}_+^2 = \{(k_1, k_2) \in \mathbb{Z}^2; k_1 > 0\} \cup \{(0, k_2); k_2 > 0\}$ and $\mathbb{Z}_-^2 = -\mathbb{Z}_+^2$. Then we have that $\{(\lambda_k, e_k); k \in \mathbb{Z}^2\}$ and $\{(\lambda_k^{-1}, e_k); k \in \mathbb{Z}^2\}$ are normalized eigenbasis for $1 - \Delta$ and A, respectively. Namely, we have

$$(1 - \Delta)e_k = \lambda_k e_k, \quad Ae_k = \lambda_k^{-1} e_k, \qquad k \in \mathbb{Z}^2.$$

Setting $e_k^{(s)} := \lambda_k^{-s/2} e_k$ ($s \in \mathbb{R}$), we obtain a CONS $\{e_k^{(s)}; k \in \mathbb{Z}^2\}$ of $H^s(\mathbb{T}^2)$. Then the operator A can be extended to $H^{-s}(\mathbb{T}^2)$ ($s > 0$) by setting $Ae_k^{(-s)} := \lambda_k^{-1} e_k^{(-s)}$ ($k \in \mathbb{Z}^2$). Note that A is not a trace class operator because of

$$\mathrm{Tr}(A) = \sum_{k \in \mathbb{Z}^2} \lambda_k^{-1} = \sum_{k \in \mathbb{Z}^2} \frac{1}{1 + |k|^2} = \infty.$$

On the other hand, it follows from

$$\mathrm{Tr}(A^p) = \sum_{k \in \mathbb{Z}^2} \lambda_k^{-p} = \sum_{k \in \mathbb{Z}^2} \frac{1}{(1 + |k|^2)^p} < \infty, \quad p > 1$$

that A^p is a trace class operator for any $p > 1$.

Throughout the present paper, we fix two parameters $\delta(> 0)$ and γ such that

$$0 < \gamma \le 1, \ \delta + 2\gamma > 2. \tag{2.1}$$

We set $E := H^{-\delta}(\mathbb{T}^2)$, $H := L^2(\mathbb{T}^2)$, $\mathcal{H} := H^\gamma(\mathbb{T}^2)$ and define the space-time free field by $\mu_0 := N(0, A^{1+\delta})$. Namely, μ_0 is the mean zero Gaussian probability measure supported on E such that

$$\int_E (l_1, z)_E (l_2, z)_E \, \mu_0(dz) = \left(A^{1+\delta} l_1, l_2\right)_E, \quad l_1, l_2 \in E. \tag{2.2}$$

We can read (2.2) as

$$\int_E \langle z, l_1\rangle \langle z, l_2\rangle \, \mu_0(dz) = \left((1 - \Delta)^{-1} l_1, l_2\right)_H, \quad l_1, l_2 \in E^* \subset H, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the dualization between E and $E^* = H^\delta(\mathbb{T}^2)$.

Next, we introduce the $\exp(\phi)_2$-*measure* $\mu_{\exp}^{(a)}$ and the $\cos(\phi)_2$-*measure* $\mu_{\cos}^{(a)}$ on E. The details of the rigorous construction of these probability measures as well as various properties of them will be presented in Section 3. For $\Lambda \subset \mathbb{R}^2$, we write $c\Lambda := \{cx; x \in \Lambda\}$ ($c \in \mathbb{R}$). Throughout the present paper, we assume

$$\Lambda \text{ is compact such that } \Lambda = -\Lambda \text{ and } 0 \text{ is an interior point of } \Lambda. \tag{2.4}$$

As typical examples of Λ, we may consider $D(N) = \{x \in \mathbb{R}^2; |x|_{\mathbb{R}^2} \leq N\}$ and $Q(N) = \{x = (x_1, x_2) \in \mathbb{R}^2; |x_1| \leq N, |x_2| \leq N\}$ $(N \in \mathbb{N})$. We define the projection operator Π_Λ on E by

$$(\Pi_\Lambda z)(x) := \sum_{k \in \mathbb{Z}^2} \mathbf{1}_\Lambda(k) \hat{z}(k) \mathbf{e}_k(x) = \sum_{k \in \Lambda \cap \mathbb{Z}^2} \langle z, e_k \rangle e_k(x), \quad z \in E, x \in \mathbb{T}^2, \quad (2.5)$$

where $\mathbf{1}_\Lambda$ stands for the indicator function of Λ. For a cut-off function $g \in H$ and the charge parameter $a \in (-\sqrt{4\pi}, \sqrt{4\pi})$, we define the *Wick exponential* $:\exp(az):(g)$ with respect to μ_0 by

$$:\exp(az):(g) = \lim_{N \to \infty} \int_{\mathbb{T}^2} \exp\left(a(\Pi_{N\Lambda}z)(x) - \frac{a^2}{2}\mathbb{E}^{\mu_0}[(\Pi_{N\Lambda}z)(x)^2]\right) g(x) dx, \quad (2.6)$$

where the right-hand side of (2.6) converges in $L^2(\mu_0)$ and does not depend on the choice of Λ (see Theorem 3.4 for details). We note that the Gaussian multiplicative chaos $\mathcal{M}_z^{(a)}(A)$ $(A \in \mathcal{B}(\mathbb{T}^2))$ coincides with $:\exp(az):(\mathbf{1}_A)$. We may define the *imaginary Wick exponential* $:\exp(\sqrt{-1}az):(g)$ by replacing a by $\sqrt{-1}a$ in (2.6). Using the imaginary Wick exponential $:\exp(\sqrt{-1}az):(g)$, we can also introduce the *Wick cosine* $:\cos(az):(g)$ and the *Wick sine* $:\sin(az):(g)$ by

$$:\cos(az):(g) = \frac{1}{2}\Big(:\exp(\sqrt{-1}az):(g) + :\exp(-\sqrt{-1}az):(g)\Big)$$

$$= \lim_{N \to \infty} \int_{\mathbb{T}^2} \cos\big(a(\Pi_{N\Lambda}z)(x)\big) \exp\left(\frac{a^2}{2}\mathbb{E}^{\mu_0}[(\Pi_{N\Lambda}z)(x)^2]\right) g(x) dx,$$

$$:\sin(az):(g) = \frac{1}{2\sqrt{-1}}\Big(:\exp(\sqrt{-1}az):(g) - :\exp(-\sqrt{-1}az):(g)\Big)$$

$$= \lim_{N \to \infty} \int_{\mathbb{T}^2} \sin\big(a(\Pi_{N\Lambda}z)(x)\big) \exp\left(\frac{a^2}{2}\mathbb{E}^{\mu_0}[(\Pi_{N\Lambda}z)(x)^2]\right) g(x) dx,$$

respectively (see Corollary 3.7 for details). We then define the $\exp(\phi)_2$-measure $\mu_{\exp}^{(a)}$ on E by

$$\mu_{\exp}^{(a)}(dz) := \frac{1}{Z_{\exp}^{(a)}} \exp\left(-:\exp(az):(\mathbf{1}_{\mathbb{T}^2})\right) \mu_0(dz), \quad (2.7)$$

where

$$Z_{\exp}^{(a)} := \int_E \exp\left(-:\exp(az):(\mathbf{1}_{\mathbb{T}^2})\right) \mu_0(dz) \quad (2.8)$$

is the normalization constant. Replacing $:\exp(az):(\mathbf{1}_{\mathbb{T}^2})$ by $:\cos(az):(\mathbf{1}_{\mathbb{T}^2})$ in (2.7) and (2.8), we may also define the $\cos(\phi)_2$-measure $\mu_{\cos}^{(a)}$.

Now we are in a position to introduce a pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$. We put $\mu = \mu_{\exp}^{(a)}, \mu_{\cos}^{(a)}$ and set $K := \text{Span}\{e_k; k \in \mathbb{Z}^2\}$. We mention that $K \subset E^*$ is

a dense linear subspace of E. Let $\mathfrak{F}C_b^\infty := \mathfrak{F}C_b^\infty(K)$ be the space of all smooth cylinder functions on E having the form

$$F(z) = f\big(\langle z, \varphi_1 \rangle, \ldots, \langle z, \varphi_n \rangle\big), \quad z \in E,$$

with $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $\{\varphi_1, \ldots, \varphi_n\} \subset K$. Since we have $\mathrm{supp}(\mu) = E$, two different functions in $\mathfrak{F}C_b^\infty$ represent two different μ-classes. Note that $\mathfrak{F}C_b^\infty$ is dense in $L^p(\mu)$ for all $p \geq 1$. If $F : E \to \mathbb{R}$ is differentiable along every $h \in H$, and the mapping $h \mapsto (\partial F/\partial h)(z)$ is a bounded linear functional on H, the function F is called H-*Gâteaux differentiable* at $z \in E$. Such a mapping is called the H-*Gâteaux derivative* of F at $z \in E$, and denoted by $D_H F(z)$. For a cylinder function $F \in \mathfrak{F}C_b^\infty$ having the above form, the H-Gâteaux derivative $D_H F : E \to H$ is given by

$$D_H F(z) = \sum_{j=1}^n \partial_j f\big(\langle z, \varphi_1 \rangle, \ldots, \langle z, \varphi_n \rangle\big)\varphi_j, \quad z \in E,$$

where ∂_j stands for the j-th partial derivative. For $\varphi \in K$, we also define a function $b(\mu; \varphi) : E \to \mathbb{R}$ by

$$b(\mu; \varphi)(z) := \begin{cases} a : \exp(az) : (\varphi) & \mu = \mu_{\exp}^{(a)} \\ -a : \sin(az) : (\varphi) & \mu = \mu_{\cos}^{(a)}. \end{cases}$$

We consider the pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ which is given by

$$\mathcal{E}(F, G) = \frac{1}{2} \int_E \big(A^\gamma D_H F(z), D_H G(z)\big)_H \mu(dz), \quad F, G \in \mathfrak{F}C_b^\infty.$$

In Section 4 we prove the following:

Proposition 2.1.

$$\mathcal{E}(F, G) = -\int_E \mathcal{L}F(z)G(z)\mu(dz), \quad F, G \in \mathfrak{F}C_b^\infty,$$

where $\mathcal{L}F \in L^p(\mu)$, $1 \leq p < 1 + \frac{4\pi}{a^2}$ is given by

$$\mathcal{L}F(z) = \frac{1}{2}\sum_{i,j=1}^n \partial_i \partial_j f\big(\langle z, \varphi_1 \rangle, \ldots, \langle z, \varphi_n \rangle\big) \cdot (A^\gamma \varphi_i, \varphi_j)_H$$

$$-\frac{1}{2}\sum_{i=1}^n \partial_i f\big(\langle z, \varphi_1 \rangle, \ldots, \langle z, \varphi_n \rangle\big) \cdot \big\{\langle z, A^{\gamma-1}\varphi_j \rangle + b(\mu; A^\gamma \varphi_j)(z)\big\}, \quad z \in E$$

for $F = f(\langle \cdot, \varphi_1 \rangle, \ldots, \langle \cdot, \varphi_n \rangle) \in \mathfrak{F}C_b^\infty$.

Remark 2.2. Since $A^\gamma \varphi \in K$ for any $\gamma \in \mathbb{R}$ and $\varphi \in K$, we easily see that $\mathfrak{F}C_b^\infty$ coincides with the space of all cylinder functions on E having the form

$$F(z) = f\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big), \quad z \in E, \tag{2.9}$$

with $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and $\{\varphi_1, \ldots, \varphi_n\} \subset K$. Here φ_j^* denotes the unique continuous extension of the functional $(\varphi_j, \cdot)_{\mathcal{H}}$ to E, that is, φ_j^* is given by $\varphi_j^*(z) = \langle A^{-\gamma}\varphi_j, z \rangle$ $(z \in E)$. For a cylinder function $F \in \mathfrak{F}C_b^\infty$ having the above form, the map $D_{\mathcal{H}}F : E \to \mathcal{H}$ is defined by

$$D_{\mathcal{H}}F(z) := \sum_{j=1}^n \partial_j f\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big)\varphi_j, \quad z \in E.$$

By the chain rule, it coincides with the usual $\mathcal{H}$-Gâteaux derivative of F, hence the above definition is independent of the representation of F in (2.9). On the other hand, since the H-Gâteaux derivative $D_H F$ is also given by

$$D_H F(z) = \sum_{j=1}^n \partial_j f\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big) \cdot A^{-\gamma}\varphi_j, \quad z \in E,$$

we easily see that the pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ and its associated pre-Dirichlet operator $\mathcal{L}$ are also represented as

$$\mathcal{E}(F, G) = \frac{1}{2} \int_E \big(D_{\mathcal{H}}F(z), D_{\mathcal{H}}G(z)\big)_{\mathcal{H}} \, \mu(dz), \quad F, G \in \mathfrak{F}C_b^\infty,$$

and

$$
\begin{aligned}
\mathcal{L}F(z) = &\frac{1}{2} \sum_{i,j=1}^n \partial_i \partial_j f\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big)(\varphi_i, \varphi_j)_{\mathcal{H}} \\
&- \frac{1}{2} \sum_{i=1}^n \partial_i f\big(\varphi_1^*(z), \ldots, \varphi_n^*(z)\big) \cdot \{\langle z, A^{-1}\varphi_j \rangle + b(\mu; \varphi_j)(z)\}, \quad z \in E,
\end{aligned}
$$

respectively.

Remark 2.3. Since $\inf_{|a| < \sqrt{4\pi}}\left(1 + \frac{4\pi}{a^2}\right) = 2$, we have $\mathcal{L}F \in L^2(\mu)$ for any $|a| < \sqrt{4\pi}$.

Proposition 2.1 means that the operator $(\mathcal{L}, \mathfrak{F}C_b^\infty)$ is the pre-Dirichlet operator which is associated with the pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$. In particular, it implies that $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ is closable in $L^2(\mu)$. We denote by $\mathcal{D}(\mathcal{E})$ the completion of $\mathfrak{F}C_b^\infty$ with respect to the $\mathcal{E}_1^{1/2}$-norm. (Here we use standard notation from the theory of Dirichlet forms, see, *e.g.*, [1, 55, 76].) Then by standard theory (*cf.* [1, 27, 55, 76]), $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form and the operator $\mathcal{L}$ has a self-adjoint extension $(\mathcal{L}_\mu, \text{Dom}(\mathcal{L}_\mu))$, called the *Friedrichs extension*, corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. The semigroup $\{e^{t\mathcal{L}_\mu}\}_{t \geq 0}$ generated by $(\mathcal{L}_\mu, \text{Dom}(\mathcal{L}_\mu))$ in $L^2(\mu)$

is Markovian, *i.e.*, $0 \leq e^{t\mathcal{L}_\mu} F \leq 1$, μ-a.e. whenever $0 \leq F \leq 1$, μ-a.e. Moreover, since $\{e^{t\mathcal{L}_\mu}\}_{t \geq 0}$ is symmetric on $L^2(\mu)$, the Markovian property implies that

$$\int_E e^{t\mathcal{L}_\mu} F(z)\mu(dz) \leq \int_E F(z)\mu(dz), \quad F \in L^2(\mu), \ F \geq 0, \ \mu\text{-a.e.}$$

Hence $\|e^{t\mathcal{L}_\mu} F\|_{L^1(\mu)} \leq \|F\|_{L^1(\mu)}$ holds for $F \in L^2(\mu)$, and $\{e^{t\mathcal{L}_\mu}\}_{t \geq 0}$ can be extended as a family of C_0-semigroup of contractions in $L^p(\mu)$ for all $p \geq 1$. See, *e.g.*, [82, Theorem X.55] for details.

It is a fundamental question whether the Friedrichs extension is the only closed extension of $(\mathcal{L}, \mathfrak{F}C_b^\infty)$ generating a C_0-semigroup on $L^p(\mu)$, $p \geq 1$, *i.e.*, whether we have $L^p(\mu)$-uniqueness for $(\mathcal{L}, \mathfrak{F}C_b^\infty)$. For $p = 2$, this is equivalent to the fundamental problem of essential self-adjointness of $\mathcal{L}$ in quantum physics (*cf.* [82]). Even if $p = 2$, in general there are many lower semi bounded self-adjoint extensions $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ in $L^2(\mu)$ which therefore generate different symmetric strongly continuous semigroups $\{e^{t\widetilde{\mathcal{L}}}\}_{t \geq 0}$ in $L^2(\mu)$. If, however, we have $L^p(\mu)$-uniqueness of $\mathcal{L}$ for some $p \geq 2$, there is hence only one semigroup which is strongly continuous and with generator extending $\mathcal{L}$ in $L^p(\mu)$. Consequently, in this case, only one such L^p-, hence only one such L^2-dynamics exists, associated with μ.

The following theorems are the main results of the present paper. For the notions of "quasi-everywhere" and "capacity", we refer to [1, 55, 76].

Theorem 2.4. *Let $0 < \gamma \leq 1$ be the regularization constant satisfying $\delta + 2\gamma > 2$ and let $a \in \mathbb{R}$ be the charge constant satisfying $|a| < \sqrt{4\pi\gamma}$. Then the pre-Dirichlet operator $(\mathcal{L}, \mathfrak{F}C_b^\infty)$ is $L^p(\mu)$-unique for all $1 \leq p < \frac{1}{2}\left(1 + \frac{4\pi\gamma}{a^2}\right)$. Namely, there exists exactly one C_0-semigroup in $L^p(\mu)$ such that its generator extends $(\mathcal{L}, \mathfrak{F}C_b^\infty)$. In particular, the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the unique extension of $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ such that $\mathfrak{F}C_b^\infty$ is contained in the domain of the associated generator.*

Moreover, by applying [28, Theorem 6.1], we obtain the following result as an immediate corollary of this theorem.

Corollary 2.5. *There exists a conservative diffusion process $\mathbb{M} := (\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \{\mathbb{P}_z\}_{z \in E})$ such that the semigroup $\{P_t\}_{t \geq 0}$ generated by the unique extension of $(\mathcal{L}, \mathfrak{F}C_b^\infty)$ satisfies the following identity for any bounded Borel measurable function $F : E \to \mathbb{R}$, and $t > 0$:*

$$P_t F(z) = \int_\Theta F(X_t(\omega))\mathbb{P}_z(d\omega), \quad \mu\text{-a.s. } z \in E,$$

in the sense that the right-hand side is a μ-version for the $L^2(\mu)$-class $P_t F$. Moreover, $\mathbb{M}$ is the unique μ-symmetric Hunt process with the state space E solving the regularized SQEs (1.2) and (1.3) weakly for $\mathcal{E}$-q.e. $z \in E$, in the cases $\mu = \mu_{\exp}^{(a)}$ and $\mu = \mu_{\cos}^{(a)}$, respectively. Namely, there exist a set $S \subset E$ with

$\mathrm{Cap}(S) = 0$ *and a system of independent one-dimensional* $(\mathcal{F}_t)_{t \geq 0}$*-Brownian motions* $\{(B_t^k)_{t \geq 0}; k \in \mathbb{Z}^2\}$ *with* $B_0^k = 0$ $(k \in \mathbb{Z}^2)$ *defined on the probability space* $(\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_z)$ *such that for any* $z \in E \setminus S$, $\mathbb{P}_z(X_t \in E \setminus S$ *for all* $t \geq 0) = 1$ *and the diffusion process* $X = (X_t)_{t \geq 0}$ *satisfies*

$$\langle X_t, e_k \rangle = \langle z, e_k \rangle - \frac{1}{2} \int_0^t \left\{ (1 + |k|^2)^{1-\gamma} \langle X_s, e_k \rangle + (1 + |k|^2)^{-\gamma} b(\mu; e_k)(X_s) \right\} ds$$
$$+ (1 + |k|^2)^{-\gamma/2} B_t^k \qquad t > 0, \ k \in \mathbb{Z}^2,$$

$\mathbb{P}_z$*-almost surely.*

Remark 2.6. Following arguments in [40, 56, 64, 67, 68, 80], we might construct a unique strong solution for SQE (1.2) (respectively SQE (1.3)) for $\mu_{\exp}^{(a)}$- (respectively $\mu_{\cos}^{(a)}$-) almost every initial datum. However, this does not imply the L^p-uniqueness of the corresponding Dirichlet operator. This is obvious, since a priori the latter might have extensions which generate semigroups which have no probabilistic interpretation as transition probabilities of Markovian processes. For example, there might be an extension generating a non-Markovian semigroup which cannot be the transition probability semigroup of a Markov process or there might be also an extension generating a Markovian semigroup whose associated Dirichlet form is not quasi-regular, hence by the main result in [24] and [76] this semigroup can also not be the transition probability semigroup of a Markov process with càdlàg paths. Therefore, in general, none of the properties L^p-uniqueness of the Dirichlet operator and strong uniqueness of the corresponding SQE implies the other. We refer to [29, Sections 2 and 3] for a detailed discussion of this point.

3. Preliminaries

3.1. A review of the Wick power

We give notation and review some statements on the Wick power based on [43]. In particular, we discuss the convergence of the finite dimensional approximation of the Wick power in a strengthened form.

Let Λ be a subset of $\mathbb{R}^2$ satisfying (2.4) and Π_Λ be the projection operator defined in (2.5). We set

$$\rho_\Lambda := \mathbb{E}^{\mu_0}\big[(\Pi_\Lambda z)(x)^2\big]^{1/2} = \frac{1}{2\pi} \left\{ \sum_{k \in \mathbb{Z}^2} \mathbf{1}_\Lambda(k)\big(1 + |k|^2\big)^{-1} \right\}^{1/2}, \quad x \in \mathbb{T}^2,$$

and define $\eta_{\Lambda,x} \in E^*$ by

$$\eta_{\Lambda,x} := \frac{1}{\rho_\Lambda} \sum_{k \in \mathbb{Z}^2} \mathbf{1}_\Lambda(k) e_k(x) e_k = \frac{1}{\rho_\Lambda} \sum_{k \in \mathbb{Z}^2} \mathbf{1}_\Lambda(k) \overline{\mathbf{e}_k(x)} \mathbf{e}_k, \quad x \in \mathbb{T}^2. \quad (3.1)$$

Then we easily have

$$(\Pi_\Lambda z)(x) = \rho_\Lambda \langle z, \eta_{\Lambda,x} \rangle, \qquad z \in E, \ x \in \mathbb{T}^2.$$

We next consider the following Laplace equation on $\mathbb{T}^2$:

$$(1 - \Delta)^\alpha u = f, \qquad \alpha > 0. \tag{3.2}$$

If $f = \sum_{k \in \mathbb{Z}^2} f_k e_k \in H^s(\mathbb{T}^2)$, then

$$u = (1 - \Delta)^{-\alpha} f = \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^{-\alpha} f_k e_k \in H^{s+2\alpha}(\mathbb{T}^2)$$

solves (3.2), and can be also represented as

$$(1 - \Delta)^{-\alpha} f(x) = \int_{\mathbb{T}^2} G_{\mathbb{T}^2}^{(\alpha)}(x, y) f(y) dy, \qquad x \in \mathbb{T}^2$$

by using the Green function $G_{\mathbb{T}^2}^{(\alpha)}(x, y)$. We mention that

$$G_{\mathbb{T}^2}^{(\alpha)}(x, y) = \sum_{k \in \mathbb{Z}^2} G^{(\alpha)}(x, y + 2\pi k), \qquad x, y \in \mathbb{T}^2. \tag{3.3}$$

See, *e.g.*, [57, Proposition 7.3.1] for details. Combining Proposition A.2 with (3.3), we have

$$K^{(\alpha)}(x - y) := G_{\mathbb{T}^2}^{(\alpha)}(x, y) \leq \begin{cases} C\left(1 + |x - y|_{\mathbb{R}^2}^{2\alpha - 2}\right) & \text{if } 0 < \alpha < 1 \\ -\dfrac{1}{2\pi} \log |x - y|_{\mathbb{R}^2} + C & \text{if } \alpha = 1, \end{cases} \tag{3.4}$$

for some constant $C > 0$. This estimate implies $K^{(1)} \in L^{\infty-}(\mathbb{T}^2) := \cap_{p>1} L^p(\mathbb{T}^2)$ and $K^{(\alpha)} \in L^{1+}(\mathbb{T}^2) := \cup_{p>1} L^p(\mathbb{T}^2)$ for each $0 < \alpha < 1$. We set

$$K_\Lambda^{(\alpha)}(x) := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} \mathbf{1}_\Lambda(k)(1 + |k|^2)^{-\alpha} \mathbf{e}_k(x), \quad x \in \mathbb{T}^2, \ 0 < \alpha \leq 1. \tag{3.5}$$

By [89, Theorems 4.1 and 4.4], for every $0 < \alpha \leq 1$, the N-th cubic partial sum $K_N^{(\alpha)} = K_{Q(N)}^{(\alpha)}$ satisfies

$$K^{(\alpha)}(x) = \lim_{N \to \infty} K_N^{(\alpha)}(x) \quad \text{for almost every } x \in \mathbb{T}^2, \tag{3.6}$$

and

$$\sup_{N \in \mathbb{N}} \|K_N^{(1)}\|_{L^p} \leq C_p \|K^{(1)}\|_{L^p}, \quad p \geq 1, \tag{3.7}$$

for some constant $C_p > 0$, and

$$\sup_{N \in \mathbb{N}} \|K_N^{(\alpha)}\|_{L^{1+\varepsilon}} \leq C_\varepsilon \|K^{(\alpha)}\|_{L^{1+\varepsilon}}, \quad 0 < \alpha < 1, \tag{3.8}$$

where $\varepsilon > 0$ is some sufficiently small constant.

Next, we introduce a mapping $W : E^* \ni f \mapsto W_f \in L^2(\mu_0)$ by

$$W_f(z) := \langle z, f \rangle, \quad z \in E.$$

It follows from (2.3) that $W_f \in L^2(\mu_0)$ and

$$\|W_f\|_{L^2(\mu_0)} = \|f\|_{H^{-1}}, \quad f \in E^*.$$

Therefore the mapping W_f gives rise to an isometric isomorphism from $H^{-1}(\mathbb{T}^2)$ into $L^2(\mu_0)$. We still denote the mapping from $H^{-1}(\mathbb{T}^2)$ into $L^2(\mu_0)$ by W_f and call it the *white noise function*.

Let $\{\mathbb{H}_n\}_{n=0}^\infty$ be the Hermite polynomials defined by

$$\mathbb{H}_n(\xi) := \frac{(-1)^n}{\sqrt{n!}} e^{\xi^2/2} \left(\frac{d}{d\xi} \right)^n e^{-\xi^2/2}, \quad n = 0, 1, 2, \ldots, \ \xi \in \mathbb{R}.$$

For later use, we recall the well-known formula

$$\int_E \mathbb{H}_m(W_f(z))\mathbb{H}_n(W_g(z))\mu_0(dz) = \delta_{m,n}(f, g)_{H^{-1}}^n, \tag{3.9}$$

where $f, g \in H^{-1}(\mathbb{T}^2)$ with $\|f\|_{H^{-1}} = \|g\|_{H^{-1}} = 1$. It follows from (3.9) that $\mathbb{H}_n(W_f) \in L^2(\mu_0)$ for any $f \in H^{-1}(\mathbb{T}^2)$ and $n \in \mathbb{N} \cup \{0\}$.

For a real separable Hilbert space $\mathcal{K}$, we denote by $\mathcal{C}_n(\mathcal{K})$, $n \in \mathbb{N}$, the closed linear subspace of $L^2(\mu_0; \mathcal{K})$ generated by the $\mathcal{K}$-valued random variables $\{\mathbb{H}_n(W_f)k; \ f \in H^{-1}(\mathbb{T}^2), \|f\|_{H^{-1}} = 1, k \in \mathcal{K}\}$ and by $\mathcal{C}_0(\mathcal{K})$ the set of constant $\mathcal{K}$-valued vectors. We set $\mathcal{C}_n := \mathcal{C}_n(\mathbb{R})$ for simplicity. The space $\mathcal{C}_n(\mathcal{K})$ is called the *Wiener chaos* of order n and the following Itô-Wiener decomposition holds:

$$L^2(\mu_0; \mathcal{K}) = \oplus_{n=0}^\infty \mathcal{C}_n(\mathcal{K}).$$

Applying the Nelson hypercontractivity estimate (*cf.* [87, Theorem I.22]), we also have

$$\|F\|_{L^p(\mu_0;\mathcal{K})} \le (p - 1)^{n/2} \|F\|_{L^2(\mu_0;\mathcal{K})}, \qquad F \in \mathcal{C}_n(\mathcal{K}), \ p \ge 2. \tag{3.10}$$

We now introduce the renormalized powers. For any $n \in \mathbb{N} \cup \{0\}, N \in \mathbb{N}$ and $x \in \mathbb{T}^2$, we define the ρ_Λ^n-*regularized Wick power evaluated at* $x \in \mathbb{T}^2$ by

$$\begin{aligned}
\Phi_{n,\Lambda}(z)_x &:= \sqrt{n!}\rho_\Lambda^n \mathbb{H}_n\left(\rho_\Lambda^{-1}(\Pi_\Lambda z)(x) \right) \\
&= \sqrt{n!}\rho_\Lambda^n \mathbb{H}_n\left(\langle z, \eta_{\Lambda,x} \rangle \right) = \sqrt{n!}\rho_\Lambda^n \mathbb{H}_n\left(W_{\eta_{\Lambda,x}}(z) \right), \quad z \in E.
\end{aligned} \tag{3.11}$$

By (3.1) and (3.5), we have

$$\left(\eta_{\Lambda,x}, \eta_{\Lambda',y}\right)_{H^{-1}} = \rho_\Lambda^{-1}\rho_{\Lambda'}^{-1} K_\Lambda^{(1)}(x-y), \quad \|\eta_{\Lambda,x}\|_{H^{-1}} = 1, \quad x, y \in \mathbb{T}^2 \quad (3.12)$$

for any two subsets $\Lambda \subset \Lambda'$ in $\mathbb{R}^2$ satisfying (2.4). Thus it follows from (3.9), (3.11) and (3.12) that

$$\|\Phi_{n,\Lambda}(\cdot)_x\|_{L^2(\mu_0)} = \sqrt{n!}\rho_\Lambda^n, \qquad n \in \mathbb{N} \cup \{0\}, x \in \mathbb{T}^2. \qquad (3.13)$$

We now introduce

$$(\Phi_{n,\Lambda}(\cdot), g)_H := \int_{\mathbb{T}^2} \Phi_{n,\Lambda}(\cdot)_x g(x)dx \quad \text{in } L^2(\mu_0), \quad g \in H. \qquad (3.14)$$

Note that the right-hand side of (3.14) is well-defined in the sense of Bochner integrals because of (3.13). Since $\mathcal{C}_n$ is a closed subspace of $L^2(\mu_0)$ and $\Phi_{n,\Lambda}(\cdot)_x \in \mathcal{C}_n$ for every $x \in \mathbb{T}^2$, we also see that $(\Phi_{n,\Lambda}(\cdot), g)_H \in \mathcal{C}_n$ and thus $\Phi_{n,\Lambda} \in \mathcal{C}_n(H^{-\alpha}(\mathbb{T}^2))$ for all $\alpha \geq 0$.

Although the following proposition might be well-known (*cf.* [87, Section V.6]), we reproduce the proof for the sake of completeness and later purpose.

Proposition 3.1.

(1) *Let $p \geq 1$ and $\alpha > 0$. Then $\{\Phi_{n,N\Lambda}; N \in \mathbb{N}\}$ is a Cauchy sequence in $L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))$, and hence it converges to a mapping $\Phi_n \in L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))$. Furthermore, the limit Φ_n is independent of the choice of Λ. We denote $\Phi_n(z)$, $z \in E$, by $: z^n :$ and call it the Wick power of order n. Moreover, for $p \geq 2$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we have*

$$\|\Phi_n\|_{L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))}$$
$$\leq C(p-1)^{n/2}\sqrt{n!}\left\{\alpha^{-2}\left(\frac{1+\varepsilon}{4\pi\alpha}\right)^{n-1} n! + C^n\alpha^{-1}\left(\frac{1+\varepsilon}{\varepsilon}\right)^{n-1}\right\}^{1/2}, \qquad (3.15)$$

where $C > 0$ is independent of $\varepsilon > 0$;

(2) *For each $g \in H$, the sequence $\{(\Phi_{n,N\Lambda}(\cdot), g)_H; N \in \mathbb{N}\}$ converges to a function $\Phi_n(\cdot)(g)$ in $L^p(\mu_0)$, $p \geq 1$, as $N \to \infty$. Furthermore, the limit $\Phi_n(\cdot)(g)$ is independent of the choice of Λ. We denote $\Phi_n(z)(g)$, $z \in E$, by $: z^n : (g)$ and we have $\Phi_n(\cdot)(g) \in \mathcal{C}_n$, and*

$$\|\Phi_n(\cdot)(g)\|_{L^p(\mu_0)}$$
$$\leq C(p-1)^{n/2}\sqrt{n!}\left\{\left(\frac{1+\varepsilon}{4\pi}\right)^{n-1} n! + C^n\left(\frac{1+\varepsilon}{\varepsilon}\right)^{n-1}\right\}^{1/2} \|g\|_H, \qquad (3.16)$$

where $C > 0$ is independent of $\varepsilon > 0$. Furthermore,

$$\Phi_n(z)(g) = {}_{H^{-\alpha}}\langle :z^n:, g\rangle_{H^\alpha}, \quad \mu_0\text{-a.e. } z \in E, \ g \in H^\alpha(\mathbb{T}^2). \quad (3.17)$$

Proof. Without loss of generality, we may assume $n \in \mathbb{N}$ and $p \geq 2$. (There is nothing to prove in the case $n = 0$.) By (3.6), (3.7) and (3.8), we have

$$\lim_{M \to \infty} \int_{\mathbb{T}^2} K_M^{(\alpha)}(x) K_\Lambda^{(1)}(x)^n dx = \int_{\mathbb{T}^2} K^{(\alpha)}(x) K_\Lambda^{(1)}(x)^n dx, \quad (3.18)$$

$$\lim_{N \to \infty} \int_{\mathbb{T}^2} K^{(\alpha)}(x) K_N^{(1)}(x)^n dx = \int_{\mathbb{T}^2} K^{(\alpha)}(x) K^{(1)}(x)^n dx. \quad (3.19)$$

It follows from the elementary inequality

$$(a + b)^n \leq (1 + \varepsilon)^{n-1} a^n + \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{n-1} b^n, \quad a, b, \varepsilon > 0$$

and (3.4) that

$$\begin{aligned}
&\int_{\mathbb{T}^2} K^{(\alpha)}(x) K^{(1)}(x)^n dx \\
&\leq C^n + C(1 + \varepsilon)^{n-1} \int_{|x|_{\mathbb{R}^2} \leq 1} |x|_{\mathbb{R}^2}^{2\alpha-2} \left(-\frac{1}{2\pi} \log |x|_{\mathbb{R}^2}\right)^n dx \\
&\quad + C^{n+1} \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{n-1} \int_{|x|_{\mathbb{R}^2} \leq 1} |x|_{\mathbb{R}^2}^{2\alpha-2} dx \qquad (3.20) \\
&\leq C^n + C \left(\frac{1 + \varepsilon}{2\pi}\right)^{n-1} (2\alpha)^{-(n+1)} n! + C^{n+1} \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{n-1} \cdot \left(\frac{\pi}{\alpha}\right) \\
&\leq C\alpha^{-2} \left(\frac{1 + \varepsilon}{4\pi\alpha}\right)^{n-1} n! + C^{n+1}\alpha^{-1} \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{n-1}, \qquad \varepsilon > 0.
\end{aligned}$$

Now we prove item (1). Combining (3.9), (3.11) with (3.12), we have

$$\begin{aligned}
&\int_E (\Phi_{n,M\Lambda}(z), f)_H (\Phi_{n,N\Lambda}(z), g)_H \mu_0(dz) \\
&= n! \rho_{M\Lambda}^n \rho_{N\Lambda}^n \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} f(x) g(y) (\eta_{M\Lambda,x}, \eta_{N\Lambda,y})_{H^{-1}}^n dx dy \qquad (3.21) \\
&= n! \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K_{N\Lambda}^{(1)}(x - y)^n f(x) g(y) dx dy, \qquad M \geq N, \ f, g \in H.
\end{aligned}$$

Then by (3.5), (3.18) and (3.21), we have

$$
\begin{aligned}
&\left\|\Phi_{n,N\Lambda}\right\|^2_{L^2(\mu_0;H^{-\alpha}(\mathbb{T}^2))} \\
&= \int_E \sum_{k\in\mathbb{Z}^2} \left\{(1+|k|^2)^{-\alpha}\widehat{\Phi_{n,N\Lambda}(z).}(k)\widehat{\Phi_{n,N\Lambda}(z).}(-k)\right\}\mu_0(dz) \\
&= n! \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-\alpha} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_{N\Lambda}^{(1)}(x-y)^n \mathbf{e}_k(x)\mathbf{e}_{-k}(y)\,dxdy \\
&= \frac{n!}{2\pi} \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-\alpha} \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K_{N\Lambda}^{(1)}(x-y)^n \mathbf{e}_k(x-y)\,dxdy \\
&= \frac{n!}{2\pi} \lim_{M\to\infty} \sum_{\|k\|\le M} (1+|k|^2)^{-\alpha}(2\pi)^2 \int_{\mathbb{T}^2} K_{N\Lambda}^{(1)}(x)^n \mathbf{e}_k(x)\,dx \\
&= 4\pi^2 n! \lim_{M\to\infty} \int_{\mathbb{T}^2} K_M^{(\alpha)}(x) K_{N\Lambda}^{(1)}(x)^n\,dx \\
&= 4\pi^2 n! \int_{\mathbb{T}^2} K^{(\alpha)}(x) K_{N\Lambda}^{(1)}(x)^n\,dx.
\end{aligned}
$$

On the other hand, for any $M, N \in \mathbb{N}$, we also have

$$
\begin{aligned}
&\int_{\mathbb{T}^2} K_M^{(\alpha)}(x) K_{N\Lambda}^{(1)}(x)^n\,dx \\
&= \left(\frac{1}{2\pi}\right)^{n+1} \int_{\mathbb{T}^2} \left\{\sum_{\|k\|\le M}(1+|k|^2)^{-\alpha}\mathbf{e}_k(x)\right\}\cdot\left\{\sum_{l\in\mathbb{Z}^2}\mathbf{1}_{N\Lambda}(l)(1+|l|^2)^{-1}\mathbf{e}_l(x)\right\}^n dx \\
&= \left(\frac{1}{2\pi}\right)^{n+1} \sum_{\|k\|\le M}(1+|k|^2)^{-\alpha} \sum_{l(1)\in\mathbb{Z}^2}\cdots\sum_{l(n)\in\mathbb{Z}^2}\left\{\prod_{j=1}^n \mathbf{1}_{N\Lambda}(l(j))(1+|l(j)|^2)^{-1}\right\} \\
&\quad\times \int_{\mathbb{T}^2} \mathbf{e}_{k+l(1)+\cdots+l(n)}(x)\,dx \\
&= \left(\frac{1}{2\pi}\right)^{n-1} \sum_{\|k\|\le M}(1+|k|^2)^{-\alpha} \sum_{l(1)\in\mathbb{Z}^2}\cdots\sum_{l(n)\in\mathbb{Z}^2} \\
&\quad\times \left\{\prod_{j=1}^n \mathbf{1}_{N\Lambda}(l(j))(1+|l(j)|^2)^{-1}\right\} \mathbf{1}_{k+l(1)+\cdots+l(n)=0} \\
&\le \left(\frac{1}{2\pi}\right)^{n-1} \sum_{k\in\mathbb{Z}^2}(1+|k|^2)^{-\alpha} \sum_{l(1)\in\mathbb{Z}^2}\cdots\sum_{l(n)\in\mathbb{Z}^2}\left\{\prod_{j=1}^n(1+|l(j)|^2)^{-1}\right\}\mathbf{1}_{k+l(1)+\cdots+l(n)=0} \\
&= \lim_{N\to\infty}\left(\lim_{M\to\infty}\int_{\mathbb{T}^2} K_M^{(\alpha)}(x) K_N^{(1)}(x)^n\,dx\right) = \int_{\mathbb{T}^2} K^{(\alpha)}(x) K^{(1)}(x)^n\,dx,
\end{aligned}
$$

where we used (3.18) and (3.19) for the final line. Thus we obtain $\Phi_{n,N\Lambda} \in \mathcal{C}_n(H^{-\alpha}(\mathbb{T}^2))$ and

$$
\begin{aligned}
&\left\| \Phi_{n,N\Lambda} \right\|^2_{L^p(\mu_0;H^{-\alpha}(\mathbb{T}^2))} \\
&\leq (p-1)^n \left\| \Phi_{n,N\Lambda} \right\|^2_{L^2(\mu_0;H^{-\alpha}(\mathbb{T}^2))} \\
&\leq 4\pi^2 n! (p-1)^n \int_{\mathbb{T}^2} K^{(\alpha)}(x) K^{(1)}(x)^n dx, \qquad N \in \mathbb{N}, \ p \geq 2,
\end{aligned}
\tag{3.22}
$$

where we used (3.10) for the first line.

Recalling (3.21) again, and repeating the same arguments as above, for any two integers $M \geq N$, we also have

$$
\begin{aligned}
&\left\| \Phi_{n,M\Lambda} - \Phi_{n,N\Lambda} \right\|^2_{L^p(\mu_0;H^{-\alpha}(\mathbb{T}^2))} \\
&\leq 4\pi^2 n! (p-1)^n \int_{\mathbb{T}^2} K^{(\alpha)}(x) \left(K^{(1)}_{M\Lambda}(x)^n - K^{(1)}_{N\Lambda}(x)^n \right) dx.
\end{aligned}
\tag{3.23}
$$

Now we fix a constant $\mathfrak{p} = \mathfrak{p}_\alpha > 1$ sufficiently small such that $\mathfrak{p}(1-\alpha) < 1$ and let $\mathfrak{q} = \mathfrak{q}_\alpha (\gg 1)$ be the conjugate constant of $\mathfrak{p}$. Combining (3.4) with Lemma A.1, we have

$$
\begin{aligned}
&\int_{\mathbb{T}^2} K^{(\alpha)}(x) \left(K^{(1)}_{M\Lambda}(x)^n - K^{(1)}_{N\Lambda}(x)^n \right) dx \\
&\leq \left\| K^{(\alpha)} \right\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \sum_{j=0}^{n-1} \left\| (K^{(1)}_{M\Lambda} - K^{(1)}_{N\Lambda})(K^{(1)}_{M\Lambda})^j (K^{(1)}_{N\Lambda})^{n-1-j} \right\|_{L^{\mathfrak{q}}(\mathbb{T}^2)} \\
&\leq \left\| K^{(\alpha)} \right\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \sum_{j=0}^{n-1} \left\| K^{(1)}_{M\Lambda} - K^{(1)}_{N\Lambda} \right\|_{L^{n\mathfrak{q}}(\mathbb{T}^2)} \left\| K^{(1)}_{M\Lambda} \right\|^j_{L^{n\mathfrak{q}}(\mathbb{T}^2)} \left\| K^{(1)}_{N\Lambda} \right\|^{n-1-j}_{L^{n\mathfrak{q}}(\mathbb{T}^2)} \\
&\leq \left\| K^{(\alpha)} \right\|_{L^{\mathfrak{p}}(\mathbb{T}^2)} \cdot n \left\{ \sum_{k \in \mathbb{Z}^2} \left(\frac{1}{1+|k|^2} \right)^{\frac{n\mathfrak{q}}{n\mathfrak{q}-1}} \right\}^{\frac{(n\mathfrak{q}-1)(n-1)}{n\mathfrak{q}}} \\
&\quad \times \left\{ \sum_{k \in \mathbb{Z}^2} \mathbf{1}_{M\Lambda \setminus N\Lambda}(k) \left(\frac{1}{1+|k|^2} \right)^{\frac{n\mathfrak{q}}{n\mathfrak{q}-1}} \right\}^{\frac{n\mathfrak{q}-1}{n\mathfrak{q}}},
\end{aligned}
\tag{3.24}
$$

where we used the elementary binomial equality

$$
a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1} \right)
$$

for the second line, and the generalized Hölder inequality for $(n+1)$-factors by noting that

$$
\frac{1}{\mathfrak{p}} + \frac{1}{n\mathfrak{q}} + \frac{j}{n\mathfrak{q}} + \frac{n-1-j}{n\mathfrak{q}} = 1, \qquad j = 0,1,\ldots,n-1
$$

for the third line. By summarizing (3.23), (3.24), and letting $M, N \to \infty$, we have shown that $\{\Phi_{n,N\Lambda}\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))$. Letting $N \to \infty$ on both sides of (3.22) and combining it with (3.20), we obtain the desired estimate (3.15).

For a general subset $\Lambda \subset \mathbb{R}^2$ satisfying (2.4), we can find some $r > 0$ and $R \in \mathbb{N}$ such that $Q(r) \subset \Lambda \subset Q(R)$. Repeating the same argument above, we obtain

$$\|\Phi_{n,N\Lambda} - \Phi_{n,Q(NR)}\|_{L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))}^2$$

$$\lesssim \|K^{(\alpha)}\|_{L^p(\mathbb{T}^2)} \left\{ \sum_{[Nr] < \|k\| \le NR} \left(\frac{1}{1 + |k|^2} \right)^{\frac{nq}{nq-1}} \right\}^{\frac{nq-1}{nq}} \to 0 \quad \text{as } N \to \infty.$$

Thus we have completed the proof of item (1).

Next, we prove item (2). Applying Young's inequality for convolution, we have

$$\left\| \Phi_{n,M\Lambda}(\cdot)(g) - \Phi_{n,N\Lambda}(\cdot)(g) \right\|_{L^2(\mu_0)}^2$$

$$= n! \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left(K_{M\Lambda}^{(1)}(x-y)^n - K_{N\Lambda}^{(1)}(x-y)^n \right) g(x)g(y)dxdy$$

$$= n! \left(\left\{ \left(K_{M\Lambda}^{(1)} \right)^n - \left(K_{N\Lambda}^{(1)} \right)^n \right\} * g, g \right)_{L^2}$$

$$\le n! \left\| \left(K_{M\Lambda}^{(1)} \right)^n - \left(K_{N\Lambda}^{(1)} \right)^n \right\|_{L^1} \|g\|_{L^2}^2, \qquad M > N.$$

Then by following the proof of item (1), we can easily show $L^p(\mu_0)$-convergence of the sequence $\{(\Phi_{n,N\Lambda}(\cdot), g)_H; N \in \mathbb{N}\}$. The fact that $\Phi_n(\cdot)(g) \in \mathcal{C}_n$ and the estimate (3.16) are clear.

For $g \in H^\alpha(\mathbb{T}^2)$, we have

$$(\Phi_{n,N}(z), g)_H = {}_{H^{-\alpha}}\langle \Phi_{n,N}(z), g \rangle_{H^\alpha}, \quad z \in E, N \in \mathbb{N}. \tag{3.25}$$

Then by taking the limit $N \to \infty$ on both sides of (3.25), we obtain (3.17). This completes the proof. $\qquad\square$

Proposition 3.2. *Let $p \ge 2$ and $0 < \alpha \le 1$. If $|a| < \sqrt{\frac{4\pi\alpha}{p-1}}$, the mapping $\sum_{n=0}^{\infty} \frac{a^n}{n!} \Phi_n(\cdot)$ converges as an element of $L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))$. Furthermore, for each $g \in H$, the function $\sum_{n=0}^{\infty} \frac{a^n}{n!} (\Phi_n(\cdot), g)_H$ converges as an element of $L^p(\mu_0)$ provided that $|a| < \sqrt{\frac{4\pi}{p-1}}$.*

Proof. To prove the first assertion, it is sufficient to check

$$\sum_{n=0}^{\infty} \frac{|a|^n}{n!} \|\Phi_n\|_{L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))} < \infty, \quad \text{if} \quad |a| < \sqrt{\frac{4\pi\alpha}{p-1}}. \tag{3.26}$$

Recalling (3.15), we estimate the summand in (3.26) by

$$\frac{1}{n!}\|\Phi_n\|_{L^p(\mu_0;H^{-\alpha}(\mathbb{T}^2))} \lesssim (p-1)^{n/2}\left(\frac{1+\varepsilon}{4\pi\alpha}\right)^{\frac{n-1}{2}}$$
$$+ \left\{\frac{C(p-1)}{n!}\right\}^{n/2}\left(\frac{1+\varepsilon}{\varepsilon}\right)^{\frac{n-1}{2}} =: k(n) + l(n),$$

then by

$$\lim_{n\to\infty}\left(\frac{k(n+1)}{k(n)}\right) = \sqrt{\frac{(p-1)(1+\varepsilon)}{4\pi\alpha}}, \quad \lim_{n\to\infty}\left(\frac{l(n+1)}{l(n)}\right) = 0,$$

and letting $\varepsilon \searrow 0$, we see that (3.26) holds. Replacing (3.15) by (3.16) in the above argument, we obtain the second assertion. This completes the proof. $\square$

3.2. Finite dimensional approximation of the Wick exponential

In this subsection, we construct the Wick exponential mapping:$\exp(a\cdot)$ through the finite dimensional approximation introduced in the previous subsection.

First, we take an arbitrary number $a \in \mathbb{R}$ and $\Lambda \subset \mathbb{R}^2$ satisfying (2.4). For any $z \in E$, we define the smooth non-negative function $\exp_\Lambda(az)_\bullet : \mathbb{T}^2 \to [0,\infty)$ by

$$\exp_\Lambda(az)_x := \exp\left\{a(\Pi_\Lambda z)(x) - \frac{(a\rho_\Lambda)^2}{2}\right\}, \quad x \in \mathbb{T}^2,$$

where Π_Λ is the projection operator defined in (2.5). Substituting $t = a\rho_\Lambda$ and $\xi = \rho_\Lambda^{-1}(\Pi_\Lambda z)(x)$ into the formula

$$\exp\left(-\frac{t^2}{2} + \xi t\right) = \sum_{n=0}^{\infty}\frac{a^n}{\sqrt{n!}}\mathbb{H}_n(\xi),$$

we easily deduce

$$\exp_\Lambda(az)_x = \sum_{n=0}^{\infty}\frac{a^n}{n!}\Phi_{n,\Lambda}(z)_x, \quad z \in E, x \in \mathbb{T}^2, \tag{3.27}$$

with $\Phi_{n,\Lambda}(z)_x$ as in (3.11). Furthermore, we have the following.

Proposition 3.3. *For any $\alpha \geq 0$, we have $\exp_\Lambda(a\cdot) \in L^2(\mu_0;H^{-\alpha}(\mathbb{T}^2))$ and*

$$\mathcal{P}_n\Big(\exp_\Lambda(a\cdot)\Big) = \frac{a^n}{n!}\Phi_{n,\Lambda}(\cdot), \quad n = 0,1,2,\ldots, \tag{3.28}$$

where $\mathcal{P}_n : L^2(\mu_0;H^{-\alpha}(\mathbb{T}^2)) \to \mathcal{C}_n(H^{-\alpha}(\mathbb{T}^2))$ is the orthogonal projection onto the Wiener chaos of order n.

Proof. Recalling that

$$\int_E e^{W_f(z)} \mu_0(dz) = \exp\left(\frac{1}{2}\|f\|_{H^{-1}}^2\right), \quad f \in H^{-1}(\mathbb{T}^2),$$

and $\|\eta_{\Lambda,x}\|_{H^{-1}} = 1$ $(x \in \mathbb{T}^2)$, we have

$$\int_E \|\exp_\Lambda(az)\|_H^2 \mu_0(dz)$$

$$= e^{-(a\rho_\Lambda)^2} \int_{\mathbb{T}^2} \left(\int_E \exp\left(2a\rho_\Lambda W_{\eta_{\Lambda,x}}(z)\right)\mu_0(dz)\right) dx$$

$$= e^{-(a\rho_\Lambda)^2} \int_{\mathbb{T}^2} \exp\left(2a^2\rho_\Lambda^2 \|\eta_{\Lambda,x}\|_{H^{-1}}^2\right) dx = 4\pi^2 e^{a^2\rho_\Lambda^2},$$

which means $\exp_\Lambda(a\cdot) \in L^2(\mu_0; H^{-\alpha}(\mathbb{T}^2))$ $(\alpha \geq 0)$. Combining this and the fact that $\Phi_{n,\Lambda} \in \mathcal{C}_n(H^{-\alpha}(\mathbb{T}^2))$ with (3.27), we easily get (3.28). This completes the proof. $\qquad\square$

As an immediate corollary of Proposition 3.3, we easily obtain

$$\left(\exp_\Lambda(a\cdot), g\right)_H = \sum_{n=0}^\infty \frac{a^n}{n!}(\Phi_{n,\Lambda}(\cdot), g)_H \quad \text{in } L^2(\mu_0), \ g \in H.$$

The following theorem plays a crucial role in the proof of our main results. The reader is refered to [87, Theorem V. 24 and Proposition VIII. 43] for a similar result in the case of space-time cut off quantum fields with interactions of exponential type.

Theorem 3.4. *Let $p \geq 2$, $g \in H$ and $\Lambda \subset \mathbb{R}^2$ be a subset of $\mathbb{R}^2$ satisfying (2.4). Then we have the following:*

(1) *Let $a \in (-\sqrt{4\pi\alpha}, \sqrt{4\pi\alpha})$. Then*

$$\lim_{N\to\infty} \exp_{N\Lambda}(a\cdot) = \sum_{n=0}^\infty \frac{a^n}{n!}\Phi_n(\cdot) \quad in \ L^2\left(\mu_0; H^{-\alpha}(\mathbb{T}^2)\right). \qquad (3.29)$$

We call the right-hand side of (3.29) the Wick exponential mapping and denote it by :exp(az): $(z \in E)$;

(2) *Let $a \in (-\sqrt{4\pi}, \sqrt{4\pi})$. Then for each $g \in H$,*

$$\lim_{N\to\infty} \left(\exp_{N\Lambda}(a\cdot), g\right)_H = \sum_{n=0}^\infty \frac{a^n}{n!}\Phi_n(\cdot)(g) \quad in \ L^2(\mu_0). \qquad (3.30)$$

We denote the the right-hand side of (3.30) by :exp(az):(g) $(z \in E)$. *Furthermore, if $g \in H^\alpha(\mathbb{T}^2)$ and $a \in (-\sqrt{4\pi\alpha}, \sqrt{4\pi\alpha})$, we have*

$$:\exp(az):(g) = {}_{H^{-\alpha}}\langle :\exp(az):, g\rangle_{H^\alpha}, \quad \mu_0\text{-a.e. } z \in E;$$

(3) *If $|a| < \sqrt{\frac{4\pi\alpha}{p-1}}$, the Wick exponential* :$\exp(a\cdot)$: *belongs to $L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))$.*
Furthermore, if $|a| < \sqrt{\frac{4\pi}{p-1}}$, the function :$\exp(a\cdot)$: (g) *belongs to $L^p(\mu_0)$.*

Proof. By (3.22) and (3.26)

$$\sum_{n=0}^{\infty} \frac{|a|^n}{n!} \left(\sup_{N \in \mathbb{N}} \|\Phi_{n,N\Lambda}\|_{L^2(\mu_0; H^{-\alpha}(\mathbb{T}^2))} \right) < \infty, \quad \text{if} \quad |a| < \sqrt{4\pi\alpha}.$$

Thus we obtain item (1) as

$$\lim_{N \to \infty} \exp_{N\Lambda}(a\cdot) = \lim_{N \to \infty} \sum_{n=0}^{\infty} \frac{a^n}{n!} \Phi_{n,N\Lambda}(\cdot)$$
$$= \sum_{n=0}^{\infty} \frac{a^n}{n!} \left(\lim_{N \to \infty} \Phi_{n,N\Lambda}(\cdot) \right)$$
$$= \sum_{n=0}^{\infty} \frac{a^n}{n!} \Phi_n(\cdot) \quad \text{in } L^2\big(\mu_0; H^{-\alpha}(\mathbb{T}^2)\big).$$

In the same way, we can prove the former part of item (2). For $g \in H^\alpha(\mathbb{T}^2)$, we have

$$\int_E \left| {}_{H^{-\alpha}}\langle \exp_N(az), g \rangle_{H^\alpha} - {}_{H^{-\alpha}}\langle :\exp(az):, g \rangle_{H^\alpha} \right|^2 \mu_0(dz)$$
$$\leq \|g\|_{H^\alpha}^2 \int_E \big\| \exp_N(az) - :\exp(az): \big\|_{H^{-\alpha}}^2 \mu_0(dz) \to 0 \quad \text{as } N \to \infty.$$

Hence, by taking the limit $N \to \infty$ on both sides of

$$(\exp_N(az), g)_H = {}_{H^{-\alpha}}\langle \exp_N(az), g \rangle_{H^\alpha}, \quad z \in E, N \in \mathbb{N},$$

we obtain the latter part of item (2).

Since item (3) is a straightforward consequence of items (1), (2) and Proposition 3.2, our proof is complete. □

Remark 3.5. In [67, Theorem 2.2], it is shown that the sequence $\{\exp_{2^N\Lambda}(az)\}_{N=0}^{\infty}$ converges both in $H^{-\alpha}(\mathbb{T}^2)$ μ_0-almost surely and in $L^2(\mu_0; H^{-\alpha}(\mathbb{T}^2))$ with $\frac{a^2}{4\pi} < \alpha < 1$.

Finally, we mention a significant property of the Wick exponential, which, however, seems to be a well-known fact. For the sake of completeness, we give a proof. See also, *e.g.*, [87, Theorem V.24] and [67, Corollary 2.3].

Proposition 3.6. *For all $|a| < \sqrt{4\pi}$, the function $\exp\left(-:\exp(a\cdot):(\mathbf{1}_{\mathbb{T}^2})\right)$ belongs to $L^\infty(\mu_0)$ and*

$$Z_{\exp}^{(a)} := \int_E \exp\left(-:\exp(az):(\mathbf{1}_{\mathbb{T}^2})\right)\mu_0(dz) > 0.$$

Proof. Since $\left(\exp_\Lambda(az), \mathbf{1}_{\mathbb{T}}^2\right)_H \geq 0$ for every $z \in E$, we easily see $:\exp(az):$ $(\mathbf{1}_{\mathbb{T}^2}) \geq 0$ for μ_0-a.e. $z \in E$. This implies that the function $\exp\left(-:\exp(az):\right.$ $\left.(\mathbf{1}_{\mathbb{T}^2})\right)$ belongs to $L^\infty(\mu_0)$. On the other hand, it follows from Jensen's inequality that

$$\int_E \exp\left(-:\exp(az):(\mathbf{1}_{\mathbb{T}^2})\right)\mu_0(dz) \geq \exp\left(-\int_E :\exp(az):(\mathbf{1}_{\mathbb{T}^2})\,\mu_0(dz)\right)$$

$$\geq \exp\left(-\|:\exp(az):(\mathbf{1}_{\mathbb{T}^2})\|_{L^\infty(\mu_0)}\right) > 0.$$

This completes the proof. □

3.3. Finite dimensional approximation of the Wick trigonometric mappings

In this subsection, we construct the Wick trigonometric mappings $:\cos(a\cdot):$ and $:\sin(a\cdot):$ for later use.

For any $z \in E$ and $a \in \mathbb{R}$, we define the smooth functions $\cos_\Lambda(az)_\bullet$ and $\sin_\Lambda(az)_\bullet$ by

$$\cos_\Lambda(az)_x := \frac{1}{2}\left(\exp_\Lambda\left(\sqrt{-1}az\right)_x + \exp_\Lambda\left(-\sqrt{-1}az\right)_x\right)$$

$$= \cos\left(a(\Pi_\Lambda z)(x)\right)\exp\left\{\frac{(a\rho_\Lambda)^2}{2}\right\},$$

$$\sin_\Lambda(az)_x := \frac{1}{2\sqrt{-1}}\left(\exp_\Lambda\left(\sqrt{-1}az\right)_x - \exp_\Lambda\left(-\sqrt{-1}az\right)_x\right)$$

$$= \sin\left(a(\Pi_\Lambda z)(x)\right)\exp\left\{\frac{(a\rho_\Lambda)^2}{2}\right\}, \qquad x \in \mathbb{T}^2.$$

Repeating the same argument as in the previous subsection, we easily obtain the following:

Corollary 3.7. *Let $p \geq 2$, $g \in H$ and $\Lambda \subset \mathbb{R}^2$ be a symmetric and compact set as introduced in the beginning of this section.*

(1) *Let $a \in (-\sqrt{4\pi\alpha}, \sqrt{4\pi\alpha})$. Then*

$$\lim_{N\to\infty} \cos_{N\Lambda}(a\cdot) = \sum_{n=0}^\infty (-1)^n \frac{a^{2n}}{(2n)!}\Phi_{2n}(\cdot), \tag{3.31}$$

$$\lim_{N\to\infty} \sin_{N\Lambda}(a\cdot) = \sum_{n=0}^\infty (-1)^n \frac{a^{2n+1}}{(2n+1)!}\Phi_{2n+1}(\cdot) \tag{3.32}$$

in $L^2(\mu_0; H^{-\alpha}(\mathbb{T}^2))$. *We call the right-hand side of* (3.31) *and* (3.32) *the Wick cosine mapping and the Wick sine mapping. We denote them by* :cos($a\cdot$): *and* :sin($a\cdot$):, *respectively;*

(2) *Let $a \in (-\sqrt{4\pi}, \sqrt{4\pi})$. Then for each $g \in H$,*

$$\lim_{N \to \infty} \left(\cos_{N\Lambda}(a\cdot), g\right)_H = \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n}}{(2n)!} \Phi_{2n}(\cdot)(g), \qquad (3.33)$$

$$\lim_{N \to \infty} \left(\sin_{N\Lambda}(a\cdot), g\right)_H = \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n+1}}{(2n+1)!} \Phi_{2n+1}(\cdot)(g) \qquad (3.34)$$

in $L^2(\mu_0)$. We denote the the right-hand side of (3.33) *and* (3.34) *by* :cos($a\cdot$):(g) *and* :sin($a\cdot$):(g), *respectively. Furthermore, if $g \in H^\alpha(\mathbb{T}^2)$ and $a \in (-\sqrt{4\pi\alpha}, \sqrt{4\pi\alpha})$, we have*

$$:\cos(az):(g) = {}_{H^{-\alpha}}\langle:\cos(az):, g\rangle_{H^\alpha},$$
$$:\sin(az):(g) = {}_{H^{-\alpha}}\langle:\sin(az):, g\rangle_{H^\alpha}, \quad \mu_0\text{-a.e. } z \in E;$$

(3) *If $|a| < \sqrt{\frac{4\pi\alpha}{p-1}}$, both the Wick cosine* :cos($a\cdot$): *and the Wick sine* :sin($a\cdot$): *belong to $L^p(\mu_0; H^{-\alpha}(\mathbb{T}^2))$. Furthermore, if $|a| < \sqrt{\frac{4\pi}{p-1}}$, both the functions* :cos($a\cdot$):(g) *and* :sin($a\cdot$):(g) *also belong to $L^p(\mu_0)$.*

Before closing this subsection, we present the exponential integrability of the Wick trigonometric functions which plays a crucial role in the proof of the main result. Although this property was first shown by Fröhlich [51, 52], we provide a proof for the sake of completeness. See also [12, 53].

Proposition 3.8. *For all $|a| < \sqrt{4\pi}$, the function $\exp\left(-:\cos(a\cdot):(\mathbf{1}_{\mathbb{T}^2})\right)$ belongs to $L^{\infty-}(\mu_0)$. Namely, for any $p \geq 1$,*

$$\int_E \exp\left(-p:\cos(az):(\mathbf{1}_{\mathbb{T}^2})\right) \mu_0(dz) < \infty. \qquad (3.35)$$

Furthermore, we have

$$Z_{\cos}^{(a)} := \int_E \exp\left(-:\cos(az):(\mathbf{1}_{\mathbb{T}^2})\right) \mu_0(dz) > 0. \qquad (3.36)$$

If we replace : cos($a\cdot$):($\mathbf{1}_{\mathbb{T}^2}$) *by* : sin($a\cdot$):($\mathbf{1}_{\mathbb{T}^2}$), *these properties still hold.*

Proof. Since (3.36) can be shown in the same way as in the proof of Proposition 3.6, we aim to prove (3.35). First, we fix $\Lambda \subset \mathbb{R}^2$ satisfying (2.4) and set $\Psi_N^+(z) = \left(\exp_{N\Lambda}(\sqrt{-1}az), \mathbf{1}_{\mathbb{T}^2}\right)_H$ and $\Psi_N^-(z) = \left(\exp_{N\Lambda}(-\sqrt{-1}az), \mathbf{1}_{\mathbb{T}^2}\right)_H$.

Noting that $\overline{\Psi_N^+(z)} = \Psi_N^-(z)$ and $\left(\cos_{N\Lambda}(az), \mathbf{1}_{\mathbb{T}^2}\right)_H = \frac{1}{2}\left(\Psi_N^+(z) + \Psi_N^-(z)\right)$, we have

$$
\begin{aligned}
&\int_E \exp\left\{-p\left(\cos_{N\Lambda}(az), \mathbf{1}_{\mathbb{T}^2}\right)_H\right\} \mu_0(dz) \\
&\leq 2\int_E \cosh\left\{p\left(\cos_{N\Lambda}(az), \mathbf{1}_{\mathbb{T}^2}\right)_H\right\} \mu_0(dz) \\
&= 2\sum_{n=0}^\infty \frac{p^{2n}}{(2n)!} \int_E \left(\cos_{N\Lambda}(az), \mathbf{1}_{\mathbb{T}^2}\right)_H^{2n} \mu_0(dz) \\
&= 2\sum_{n=0}^\infty \frac{p^{2n}}{(2n)!} \int_E \left\{\frac{1}{2}\left(\Psi_N^+(z) + \Psi_N^-(z)\right)\right\}^{2n} \mu_0(dz) \\
&\leq 2\sum_{n=0}^\infty \frac{p^{2n}}{(2n)!} \int_E \frac{1}{2^{2n}} \sum_{m=0}^{2n} \binom{2n}{m} |\Psi_N^+(z)|^m |\Psi_N^-(z)|^{2n-m} \mu_0(dz) \\
&\leq 2\sum_{n=0}^\infty \frac{p^{2n}}{(n!)^2} \int_E \left(\Psi_N^+(z)\right)^n \left(\Psi_N^-(z)\right)^n \mu_0(dz),
\end{aligned}
\tag{3.37}
$$

where we used

$$
|\Psi_N^+(z)|^m |\Psi_N^-(z)|^{2n-m} = |\Psi_N^+(z)|^{2n} = \left(\Psi_N^+(z)\Psi_N^-(z)\right)^n, \quad m = 0, \ldots, 2n
$$

and the identities $\sum_{m=0}^{2n} \binom{2n}{m} = 2^{2n}$ and $(2n)! \geq (n!)^2$ for the final line in the above estimate.

For given $i \in \{1, \ldots, 2n\}$, we set $\varepsilon_i = 1$ if i is even and $\varepsilon_i = -1$ if i is odd. Then

$$
\begin{aligned}
&\int_E \left(\Psi_N^+(z)\right)^n \left(\Psi_N^-(z)\right)^n \mu_0(dz) \\
&= \exp\left(a^2 \rho_{N\Lambda}^2 n\right) \int_{(\mathbb{T}^2)^{2n}} dx_1 \cdots dx_{2n} \\
&\quad \times \int_E \exp\left\{\sqrt{-1}a \sum_{i=1}^n \left((\Pi_{N\Lambda}z)(x_{2i}) - (\Pi_{N\Lambda}z)(x_{2i-1})\right)\right\} \mu_0(dz) \\
&= \exp\left(a^2 \rho_{N\Lambda}^2 n\right) \\
&\quad \times \int_{(\mathbb{T}^2)^{2n}} \exp\left(-\frac{(a\rho_{N\Lambda})^2}{2} \left\|\sum_{i=1}^n \left(\eta_{N\Lambda, x_{2i}} - \eta_{N\Lambda, x_{2i-1}}\right)\right\|_{H^{-1}}^2\right) dx_1 \cdots dx_{2n} \\
&= \int_{(\mathbb{T}^2)^{2n}} \exp\left(-a^2 \sum_{1\leq i<j\leq 2n} \varepsilon_i \varepsilon_j K_{N\Lambda}^{(1)}(x_i - x_j)\right) dx_1 \cdots dx_{2n} \\
&\leq \int_{(\mathbb{T}^2)^{2n}} \exp\left(-a^2 \sum_{1\leq i<j\leq 2n} \varepsilon_i \varepsilon_j K^{(1)}(x_i - x_j)\right) dx_1 \cdots dx_{2n},
\end{aligned}
\tag{3.38}
$$

where we used that the Green functions $K_{N\Lambda}^{(1)}$ and $K^{(1)}$ satisfy $K_{N\Lambda}^{(1)} \le K^{(1)}$ as quadratic forms on $\mathbb{T}^2$ for the final line. (Note that the final line can be regarded as the *conditioning property* mentioned in [52, Lemma IV.3].) Recalling (3.3) and [33, (4.2)] (see also Proposition A.2 below), we obtain

$$K^{(1)}(x) = -\frac{1}{2\pi} \log |x|_{\mathbb{R}^2} + R(x), \quad x \in \mathbb{T}^2 \setminus \{0\}, \tag{3.39}$$

for some smooth function R. Combining (3.39) with [51, Lemma 2.1], we also have

$$
\begin{aligned}
&\int_{(\mathbb{T}^2)^{2n}} \exp\left(-a^2 \sum_{1 \le i < j \le 2n} \varepsilon_i \varepsilon_j K^{(1)}(x_i - x_j)\right) dx_1 \cdots dx_{2n} \\
&\le \exp(2a^2 Cn) \int_{(\mathbb{T}^2)^{2n}} \prod_{1 \le i < j \le 2n} |x_i - x_j|_{\mathbb{R}^2}^{\varepsilon_i \varepsilon_j a^2/2\pi} dx_1 \cdots dx_{2n}.
\end{aligned}
\tag{3.40}
$$

Furthermore, it follows from [44, (1.5), (2.7) and (2.8)] that

$$
\begin{aligned}
&\int_{(\mathbb{T}^2)^{2n}} \prod_{1 \le i < j \le 2n} |x_i - x_j|_{\mathbb{R}^2}^{\varepsilon_i \varepsilon_j a^2/2\pi} dx_1 \cdots dx_{2n} \\
&\le (4\pi^2)^{2n + \frac{a^2}{4\pi} \sum_{1 \le i < j \le 2n} \varepsilon_i \varepsilon_j} (n!)^{\max\{a^2/2\pi, 1\}} (Q_1^*)^n \\
&= C^n (n!)^{\max\{a^2/2\pi, 1\}},
\end{aligned}
\tag{3.41}
$$

where the positive constant Q_1^* is given by

$$Q_1^* = \frac{\pi^2 \Gamma(2 - a^2/2\pi)}{(1 - a^2/8\pi)(1 - a^2/4\pi)\Gamma(1 - a^2/4\pi)\Gamma(2 - a^2/4\pi)}.$$

Inserting (3.38), (3.40) and (3.41) into (3.37), we obtain

$$
\begin{aligned}
&\sup_{N \in \mathbb{N}} \int_E \exp\left(-p\big(\cos_{N\Lambda}(az), \mathbf{1}_{\mathbb{T}^2}\big)_H\right) \mu_0(dz) \\
&\le \sum_{n=0}^{\infty} \frac{C^n}{(n!)^{\min\{2 - a^2/2\pi, 1\}}} (p^2)^n.
\end{aligned}
\tag{3.42}
$$

Noting that the condition $|a| < \sqrt{4\pi}$ implies $2 - a^2/2\pi > 0$, we see that the right-hand side of (3.42) converges absolutely for every $p \ge 1$. This estimate implies that $\big\{\exp\big(-p\big(\cos_{N\Lambda}(a\cdot), \mathbf{1}_{\mathbb{T}^2}\big)_H\big)\big\}_{N \in \mathbb{N}}$ is uniformly integrable. Hence, by combining this property with Corollary 3.7, we finally obtain the desired estimate (3.35). $\square$

4. Proof of main results

In this section we only consider the case $\mu = \mu_{\cos}^{(a)}$ and give a proof of Proposition 2.1 and Theorem 2.4. Since we have $\exp(-:\exp(a\cdot):(\mathbf{1}_{\mathbb{T}^2})) \in L^\infty(\mu_0)$ (see Proposition 3.6), which is a stronger property than $\exp(-:\cos(a\cdot):(\mathbf{1}_{\mathbb{T}^2})) \in L^{\infty-}(\mu_0)$ (see Proposition 3.8), the proof in the case of $\mu = \mu_{\exp}^{(a)}$ goes in a very similar way with only slight modifications.

Proof of Proposition 2.1. Applying a standard Gaussian integration by parts formula (*cf.* [57, page 207]), we easily have

$$
\int_E \big(D_H F(z), \varphi\big)_H \frac{1}{Z_N^{(a)}} \exp\Big(- \big(\cos_{D(N)}(az), \mathbf{1}_{\mathbb{T}^2}\big)_H \Big) \mu_0(dz)
$$
$$
= \int_E F(z) \Big\{ \langle z, (1-\Delta)\varphi \rangle + a\big(\sin_{D(N)}(az), \varphi\big)_H \Big\} \tag{4.1}
$$
$$
\times \frac{1}{Z_N^{(a)}} \exp\Big(- \big(\cos_{D(N)}(az), \mathbf{1}_{\mathbb{T}^2}\big)_H \Big) \mu_0(dz)
$$

for all $F \in \mathfrak{F}C_b^\infty, \varphi \in K$ and $N \in \mathbb{N}$, where

$$
Z_N^{(a)} := \int_E \exp\Big(- \big(\cos_{D(N)}(az), \mathbf{1}_{\mathbb{T}^2}\big)_H \Big) \mu_0(dz).
$$

Combining Corollary 3.7 with Proposition 3.8 (in particular, (3.42)), we can take a subsequence $\{N(j)\}_{j=1}^\infty$ with $N(j) \nearrow \infty$ as $j \to \infty$ such that

$$
\lim_{j\to\infty} \big(\cos_{D(N)}(az), \varphi\big)_H = :\cos(az):(\varphi),
$$
$$
\lim_{j\to\infty} \big(\sin_{D(N)}(az), \varphi\big)_H = :\sin(az):(\varphi), \quad \mu_0\text{-a.s. } z \in E,
$$

and $\lim_{j\to\infty} Z_N^{(a)} = Z_{\cos}^{(a)}$ provided that $|a| < \sqrt{4\pi}$. Thus, taking the limit $j \to \infty$ on both sides of (4.1), we obtain

$$
\int_E \big(D_H F(z), \varphi\big)_H \mu_{\cos}^{(a)}(dz) = \int_E F(z) \Big\{ \langle z, (1-\Delta)\varphi \rangle + a :\sin(az):(\varphi) \Big\} \mu_{\cos}^{(a)}(dz),
$$

and this leads us to the desired integration by parts formula (2.9).

To show that $\mathcal{L}F \in L^p(\mu_{\cos}^{(a)})$, it is sufficient to check that $:\sin(az): (\varphi) \in L^p(\mu_{\cos}^{(a)})$. Applying Corollary 3.7 and Proposition 3.8 again, we easily see that $:\sin(az):(\varphi) \in L^p(\mu_{\cos}^{(a)})$, provided that $|a| < \sqrt{\frac{4\pi}{p-1}}$. This completes the proof. $\square$

Proof of Theorem 2.4. First, we introduce $\mathcal{H}_+ := H^{2\gamma+\delta}(\mathbb{T}^2)$ and $\mathcal{H}_- := E = H^{-\delta}(\mathbb{T}^2)$. By virtue of condition (2.1), the embeddings $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ are

Hilbert-Schmidt. Note that $\mathcal{H}_+$ is regarded as the dual space of $\mathcal{H}_-$ if we iden-
tify $\mathcal{H}$ with its dual. Let $_-(\cdot,\cdot)_+$ stand for this dualization between $\mathcal{H}_-$ and $\mathcal{H}_+$.
Noting the identity

$$_-(z,\varphi)_+ = (z,\varphi)_{\mathcal{H}}, \quad \varphi \in \mathcal{H}_+, z \in \mathcal{H},$$

we may rewrite $\mathcal{L}F$ given in Proposition 2.1 as

$$\mathcal{L}F(z) = \frac{1}{2}\Big\{\mathrm{Tr}(D_{\mathcal{H}}^2 F(z)) +_-\big(\beta(z), D_{\mathcal{H}}F(z)\big)_+\Big\}, \quad F \in \mathfrak{F}C_b^\infty,$$

where $\beta(z) := \beta_{\mathrm{OU}}(z) + \beta_{\cos}^{(a)}(z)$ is given by

$$\beta_{\mathrm{OU}}(z) = -\sum_{k\in\mathbb{Z}^2} \lambda_k^{-\delta/2-\gamma+1}\langle z, e_k\rangle e_k^{(-\delta)},$$

$$\beta_{\cos}^{(a)}(z) = a\sum_{k\in\mathbb{Z}^2} \lambda_k^{-\gamma/2} \sum_{n=0}^\infty (-1)^n \frac{a^{2n+1}}{(2n+1)!} \Phi_{2n+1}(z)(e_k)\, e_k^{(\gamma)}.$$

Thanks to [75, Theorem 3], it is sufficient to check that $\beta_{\mathrm{OU}} \in L^{2p}(\mu_{\cos}^{(a)}; \mathcal{H}_-)$ and
$\beta_{\cos}^{(a)} \in L^{2p}(\mu_{\cos}^{(a)}; \mathcal{H})$ for the L^p-uniqueness of the Dirichlet operator $\mathcal{L}$. By using
the Hölder inequality and remembering the condition $\delta + 2\gamma > 2$, we can estimate
as follows:

$$\int_E \big\|\beta_{\mathrm{OU}}(z)\big\|_E^{2p} \mu_{\cos}^{(a)}(dz)$$

$$\leq \mathrm{Tr}\big(A^{\delta+2\gamma-1}\big)^{p-1} \int_E \sum_{k\in\mathbb{Z}^2} \lambda_k^{-(\delta+2\gamma-p-1)}\langle z, e_k\rangle^{2p} \mu_{\cos}^{(a)}(dz)$$

$$\lesssim \mathrm{Tr}\big(A^{\delta+2\gamma-1}\big)^{p-1} \sum_{k\in\mathbb{Z}^2} \lambda_k^{-(\delta+2\gamma-p-1)}$$

$$\times \left(\int_E \langle z, e_k\rangle^{2p} \exp\big(-:\cos(az):(\mathbf{1}_{\mathbb{T}^2})\big)\mu_0(dz)\right)$$

$$= \mathrm{Tr}\big(A^{\delta+2\gamma-1}\big)^{p-1} \sum_{k\in\mathbb{Z}^2} \lambda_k^{-(\delta+2\gamma-1)}$$

$$\times \left(\int_E \Big\langle z, e_k^{(-1)}\Big\rangle^{2p} \exp\big(-:\cos(az):(\mathbf{1}_{\mathbb{T}^2})\big)\mu_0(dz)\right)$$

$$= \mathrm{Tr}\big(A^{\delta+2\gamma-1}\big)^p \left\{\int_{\mathbb{R}} x^{4p}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\right)\right\}^{1/2}$$

$$\times \left(\int_E \exp\big(-2:\cos(az):(\mathbf{1}_{\mathbb{T}^2})\big)\mu_0(dz)\right)^{1/2} < \infty,$$

where we used $\langle z, e_k^{(-1)}\rangle \sim N(0,1)$ $(k \in \mathbb{Z}^2)$ for the final line.

On the other hand, using the triangle inequality and recalling Proposition 3.8, we obtain

$$\left(\int_E \left\| \beta_{\cos}^{(a)}(z) \right\|_{\mathcal{H}}^{2p} \mu_{\cos}^{(a)}(dz) \right)^{1/2p}$$

$$\leq |a| \sum_{n=0}^{\infty} \frac{|a|^{2n+1}}{(2n+1)!} \left[\int_E \left\{ \sum_{k \in \mathbb{Z}^2} \left(\lambda_k^{-\gamma/2} \Phi_{2n+1}(z)(e_k) \right)^2 \right\}^p \mu_{\cos}^{(a)}(dz) \right]^{1/2p}$$

$$\lesssim |a| \sum_{n=0}^{\infty} \frac{|a|^{2n+1}}{(2n+1)!} \left(\int_E \left\| \Phi_{2n+1} \right\|_{H^{-\gamma}}^{2p(1+\varepsilon)} \mu_0(dz) \right)^{1/2p(1+\varepsilon)} \tag{4.2}$$

$$\times \left(\int_E \exp \left(-\frac{1+\varepsilon}{\varepsilon} :\cos(az):(\mathbf{1}_{\mathbb{T}^2}) \right) \mu_0(dz) \right)^{\varepsilon/2p(1+\varepsilon)}$$

$$\lesssim |a| \sum_{n=0}^{\infty} \frac{|a|^{2n+1}}{(2n+1)!} \left\| \Phi_{2n+1} \right\|_{L^{2p(1+\varepsilon)}(\mu_0; H^{-\gamma}(\mathbb{T}^2))}$$

for any $\varepsilon > 0$. By virtue of Corollary 3.7, the right-hand side of (4.2) is finite under the condition $2p(1+\varepsilon) < 1 + \frac{4\pi\gamma}{a^2}$. Since we may take $\varepsilon > 0$ sufficiently small, this completes the proof. $\qquad \square$

Appendix A. Auxiliary results

In this appendix, for the reader's convenience, we collect auxiliary results which are used in the proofs of the main results.

Lemma A.1. *Let Λ and Λ' be subsets of $\mathbb{R}^2$ satisfying (2.4), and $K_{\Lambda}^{(1)}$ be the approximating Green kernel of $(1 - \Delta)$ defined in (3.5). Then for any $n \in \mathbb{N}$, $q \geq 2$, we have*

$$\left\| K_{\Lambda}^{(1)} \right\|_{L^{qn}(\mathbb{T}^2)} \leq \left(\frac{1}{2\pi} \right)^{\frac{qn-2}{qn}} \left\{ \sum_{k \in \mathbb{Z}^2} \mathbf{1}_{\Lambda}(k) \left(\frac{1}{1+|k|^2} \right)^{\frac{qn}{qn-1}} \right\}^{\frac{qn-1}{qn}}, \tag{A.1}$$

$$\left\| K_{\Lambda'}^{(1)} - K_{\Lambda}^{(1)} \right\|_{L^{qn}(\mathbb{T}^2)} \leq \left(\frac{1}{2\pi} \right)^{\frac{qn-2}{qn}} \left\{ \sum_{k \in \mathbb{Z}^2} \mathbf{1}_{\Lambda' \setminus \Lambda}(k) \left(\frac{1}{1+|k|^2} \right)^{\frac{qn}{qn-1}} \right\}^{\frac{qn-1}{qn}}, \tag{A.2}$$

$$\Lambda \subset \Lambda'.$$

Proof. By recalling the Hausdorff-Young inequality (*cf.* [91, Theorem 2.8 in Chapter XII]), we have

$$\| f \|_{L^r(\mathbb{T}^2)} \leq \left(\frac{1}{2\pi} \right)^{\frac{2}{s}-1} \left(\sum_{k \in \mathbb{Z}^2} |\hat{f}(k)|^s \right)^{1/s}, \qquad f \in L^r(\mathbb{T}^2), \quad r \geq 2, \tag{A.3}$$

where $1 \leq s \leq 2$ is the conjugate index of r. Now we set $f = K_\Lambda^{(1)}$. Replacing r and s by $qn (\geq 2)$ and $\frac{qn}{qn-1}$, respectively, we easily see that (A.3) implies estimate (A.1). Similarly, estimate (A.2) is obtained by putting $f = K_{\Lambda'} - K_\Lambda$. $\qquad\square$

Proposition A.2. *Let* $G^{(\alpha)}(x, y) = G^{(\alpha)}(x - y)$ $(0 < \alpha \leq 1, x, y \in \mathbb{R}^2)$ *be the Green function for* $(1 - \Delta)^\alpha$ *on* $\mathbb{R}^2$. *Then* $G^{(\alpha)}$ *is smooth on* $\mathbb{R}^2 \setminus \{0\}$ *and strictly positive. Furthermore, there exist some constants* $C_1, C_2, C_3 > 0$ *such that the following holds: if* $0 < \alpha < 1$, *then*

$$G^{(\alpha)}(x - y) \leq \begin{cases} C_1 |x - y|_{\mathbb{R}^2}^{2\alpha - 2} & \text{for } |x - y|_{\mathbb{R}^2} < 1 \\ C_2 \exp(-C_3 |x - y|_{\mathbb{R}^2}) & \text{for } |x - y|_{\mathbb{R}^2} \geq 1, \end{cases} \tag{A.4}$$

and if $\alpha = 1$, *then*

$$G^{(1)}(x - y) \leq \begin{cases} -\dfrac{1}{2\pi} \log |x - y|_{\mathbb{R}^2} + C_1 & \text{for } |x - y|_{\mathbb{R}^2} < 1 \\ C_2 \exp(-C_3 |x - y|_{\mathbb{R}^2}) & \text{for } |x - y|_{\mathbb{R}^2} \geq 1. \end{cases} \tag{A.5}$$

Proof. One can find a proof of (A.5), *e.g.*, in [33, Section 4], [87, Proposition V.23] and [57, Proposition 7.2.1]. Although the case $0 < \alpha < 1$ is also treated in [33, Section 4], we give a short proof of (A.4) here to make the present paper more self-contained.

We first recall an integral representation of the Green function

$$G^{(\alpha)}(x - y) = \frac{1}{4\pi \Gamma(\alpha)} \int_0^\infty e^{-s} \exp\left(-\frac{|x - y|_{\mathbb{R}^2}^2}{4s}\right) s^{\alpha - 2} ds, \quad 0 < \alpha \leq 1, \ x, y \in \mathbb{R}^2.$$

Setting $s = \frac{|x - y|_{\mathbb{R}^2}^2}{t}$, we have

$$G^{(\alpha)}(x - y) = \frac{|x - y|_{\mathbb{R}^2}^{2\alpha - 2}}{4\pi \Gamma(\alpha)} I(|x - y|_{\mathbb{R}^2}), \tag{A.6}$$

where

$$I(r) = \int_0^\infty t^{-\alpha} \exp\left(-\frac{t}{4} - \frac{r^2}{t}\right) dt, \qquad r \geq 0.$$

Combining (A.6) with

$$I(r) \leq \int_0^\infty t^{-\alpha} e^{-t/4} dt = 4^{1-\alpha} \Gamma(1 - \alpha),$$

we obtain the desired estimate in the case $|x - y|_{\mathbb{R}^2} < 1$.

Next we consider the case $r = |x - y|_{\mathbb{R}^2} \geq 1$. We set $\tau = \frac{\sqrt{t}}{2} - \frac{r}{\sqrt{t}}$, in other word, $t = (\tau + \sqrt{\tau^2 + 2r})^2$. Then we have

$$
\begin{aligned}
I(r) &= 2e^{-r} \int_{\mathbb{R}} \frac{\left(\tau + \sqrt{\tau^2 + 2r}\right)^{2-2\alpha}}{\sqrt{\tau^2 + 2r}} e^{-\tau^2} d\tau \\
&= 2e^{-r} r^{\frac{1-2\alpha}{2}} \int_{\mathbb{R}} \frac{\left(\frac{\tau}{\sqrt{r}} + \sqrt{\frac{\tau^2}{r} + 2}\right)^{2-2\alpha}}{\sqrt{\frac{\tau^2}{r} + 2}} e^{-\tau^2} d\tau \\
&\leq \sqrt{2} e^{-r} r^{\frac{1-2\alpha}{2}} \int_{\mathbb{R}} \left(\tau + \sqrt{\tau^2 + 2}\right)^{2-2\alpha} e^{-\tau^2} d\tau \lesssim e^{-r} r^{\frac{1-2\alpha}{2}},
\end{aligned}
$$

where we used $r \geq 1$ and $2 - 2\alpha > 0$ for the third line. Combining this estimate with (A.6), we finally obtain

$$
G^{(\alpha)}(x - y) \leq C \exp\left(-|x - y|_{\mathbb{R}^2}\right) |x - y|_{\mathbb{R}^2}^{\alpha - \frac{3}{2}} \leq C \exp\left(-|x - y|_{\mathbb{R}^2}\right),
$$

where we used $|x - y|_{\mathbb{R}^2}^{\alpha - \frac{3}{2}} \leq 1$. This completes the proof.   □

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