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A problem about Mahler functions

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In memory of Alf van der Poorten

Abstract. Let K be a field of characteristic zero and k and l be two multiplicatively independent positive integers. We prove the following result that was conjectured by Loxton and van der Poorten during the Eighties: a power series $F(z) \in K[[z]]$ satisfies both a k- and a l-Mahler-type functional equation if and only if it is a rational function.

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1. Introduction

In a series of three papers [27–29] published in 1929 and 1930, Mahler initiated a totally new direction in transcendence theory. *Mahler's method*, a term coined much later by Loxton and van der Poorten, aims at proving transcendence and algebraic independence of values at algebraic points of locally analytic functions satisfying a certain type of functional equations. In its original form, it concerns equations of the form

$$F\left(z^k\right) = R(z, F(z)), \tag{1.1}$$

where $R(z, x)$ denotes a bivariate rational function with coefficients in a number field and $k \geq 2$ is an integer. For instance, using the fact that $F(z) = \sum_{n=0}^{\infty} z^{2^n}$ satisfies the basic functional equation

$$F\left(z^2\right) = F(z) - z,$$

Mahler was able to prove that $F(\alpha)$ is a transcendental number for every algebraic number α with $0 < |\alpha| < 1$. As observed by Mahler himself, his approach allows

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one to deal with functions of several variables and systems of functional equations as well. It also leads to algebraic independence results, transcendence measures, measures of algebraic independence, and so forth. Mahler's method was later developed by various authors, including Becker, Kubota, Loxton and van der Poorten, Masser, Nishioka, Töpfer, among others. For classical aspects of Mahler's theory, we refer the reader to the monograph of Ku. Nishioka [35] and the reference therein. However, a major deficiency of Mahler's method is that, contrary to Siegel E- and G-functions, there is not a single classical transcendental constant that is known to be the value at an algebraic point of an analytic function solution to a Mahler-type functional equation[1]. This may explain why it was somewhat neglected for almost fifty years.

At the beginning of the Eighties, Mahler's method really took on a new significance after Mendès France popularized the fact that some Mahler-type systems of functional equations naturally arise in the study of automata theory (see for instance [31]). Though already noticed in 1968 by Cobham [11], this connection remained relatively unknown at that time, probably because Cobham's work was never published in an academic journal. Cobham claimed that Mahler's method has the following nice consequence for the Hartmanis-Stearns problem about the computational complexity of algebraic irrational real numbers [21]: the expansion of an algebraic irrational number in an integer base cannot be generated by a finite automaton. His idea was to derive this result by applying Mahler's method to systems of functional equations of the form

$$\begin{pmatrix} F_1(z^k) \\ \vdots \\ F_n(z^k) \end{pmatrix} = A(z) \begin{pmatrix} F_1(z) \\ \vdots \\ F_n(z) \end{pmatrix} + B(z)\,, \tag{1.2}$$

where $A(z)$ is an $n \times n$ matrix and $B(z)$ is an n-dimensional vector, both having entries that are rational functions with algebraic coefficients. Though Cobham's conjecture is proved in [1] by means of a completely different approach, it still remained a challenging problem to complete the proof he envisaged. In this direction, a great deal of work has been done by Loxton and van der Poorten [25,26] and a particular attention was then paid to systems of functional equations as in (1.2) (see for instance [9,32,33,35,38]). Very recently, another proof of Cobham's conjecture using Mahler's method is finally obtained in [4,38], thus solving a long-standing problem in Mahler's method.

Let K be a field. We observe that a power series $F(z) \in K[[z]]$ is a component of a vector satisfying a system of functional equations of the form (1.2)[2] if and only

[1] A remarkable discovery of Denis (see [13]), which deserves to be better understood, is that Mahler's method can be also applied to prove transcendence and algebraic independence results involving periods of t-modules which are variants of the more classical periods of abelian varieties, in the framework of the arithmetic of function fields of positive characteristic. For a detailed discussion on this topic, we refer the reader to the recent survey by Pellarin [37], see also [36].

[2] We assume here that the entries of $A(z)$ and $B(z)$ are in $K(z)$.

if the family

$$1,\, F(z),\, F\left(z^k\right),\, F\left(z^{k^2}\right),\, \dots$$

is linearly dependent over the field $K(z)$, that is, if there exist a natural number n and polynomials $Q(z), P_0(z), \dots, P_n(z) \in K[z]$, not all of which are zero, such that

$$Q(z) + \sum_{i=0}^{n} P_i(z) F\left(z^{k^i}\right) = 0. \tag{1.3}$$

Following Loxton and van der Poorten [26], we use the following definition.

Definition 1.1. Let K be a field and $k \geq 2$ be an integer. A power series $F(z) \in K[[z]]$ is a *k-Mahler function*, or for short is *k-Mahler*, if it satisfies a functional equation of the form (1.3).

Beyond transcendence, Mahler's method and automata theory, it is worth mentioning that Mahler functions naturally occur as generating functions in various other topics such as combinatorics of partitions, numeration and the analysis of algorithms (see [15] and the references therein and also dozens of examples in [7,8] and [19, Chapter 14]). A specially intriguing appearance of Mahler functions is related to the study of Siegel G-functions and in particular of diagonals of rational functions[3]. Though no general result confirms this claim, one observes that many generating series associated with the p-adic valuation of the coefficients of G-functions with rational coefficients turn out to be p-Mahler functions.

As a simple illustration, we give the following example. Let us consider the algebraic function

$$\mathfrak{f}(z) := \frac{1}{(1-z)\sqrt{1-4z}} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k} z^n .$$

Note that $\mathfrak{f}$ is a G-function which satisfies the following minimal differential equation:

$$\mathfrak{f}'(z) = \frac{(3-6z)}{(1-z)(1-4z)} \mathfrak{f}(z) .$$

Let us define the sequence

$$a(n) := \nu_3 \left(\sum_{k=0}^{n} \binom{2k}{k} \right) ,$$

where ν_3 denotes the 3-adic valuation. We claim that the function

$$\mathfrak{f}_1(z) := \sum_{n \geq 0} a(n) z^n \in \mathbb{Q}[[z]]$$

[3] See for instance [3] for a discussion of the links between diagonals of rational functions with algebraic coefficients and G-functions.

is a 3-Mahler function[4]. This actually comes from the following nice equality

$$\nu_3\left(\sum_{k=0}^{n}\binom{2k}{k}\right)=\nu_3\left(n^2\binom{2n}{n}\right), \tag{1.4}$$

independently proved by Allouche and Shallit in 1989 (unpublished) and by Zagier [45]. Indeed, setting $\mathfrak{f}_2(z) := \sum_{n\geq 0} a(3n)z^n$ and $\mathfrak{f}_3(z) := \sum_{n\geq 0} a(3n+1)z^n$, we infer from Equality (1.4) that

$$\begin{pmatrix}\mathfrak{f}_1(z^3)\\ \mathfrak{f}_2(z^3)\\ \mathfrak{f}_3(z^3)\end{pmatrix} = A(z)\begin{pmatrix}\mathfrak{f}_1(z)\\ \mathfrak{f}_2(z)\\ \mathfrak{f}_3(z)\end{pmatrix} + B(z)\,,$$

with

$$A(z) := \frac{1}{z^3(1+z+z^2)}\begin{pmatrix} z(1+z+z^2) & -z^2 & -z\\ 0 & z^2(1+z) & -z^4\\ 0 & -z^2 & z^2(1+z)\end{pmatrix}$$

and

$$B(z) := \frac{1}{z^3(1+z+z^2)}\begin{pmatrix}\dfrac{z(2z^2-1)}{z-1}\\ -\dfrac{z^4}{z-1}\\ \dfrac{z^2(1+z)}{z-1}\end{pmatrix}.$$

A simple computation then gives the relation

$$a_0(z)+a_1(z)\mathfrak{f}_1(z)+a_2(z)\mathfrak{f}_1\big(z^3\big)+a_3(z)\mathfrak{f}_1\big(z^9\big)+a_4(z)\mathfrak{f}_1\big(z^{27}\big)=0\,,$$

where

$$\begin{aligned}
a_0(z) &:= z+2z^2-z^3+z^4+3z^5-z^7+3z^8+z^9-z^{11}+3z^{12}-2z^{14}\\
&\quad -z^{15}+2z^{16}-2z^{17}-2z^{18}+2z^{21},\\
a_1(z) &:= -1-z^4-z^8+z^9+z^{13}+z^{17},\\
a_2(z) &:= 1+z+z^2+z^3+z^4+z^5+z^6+z^7+z^8-z^{13}-z^{14}-z^{15}-z^{16}\\
&\quad -z^{17}-z^{18}-z^{19}-z^{20}-z^{21},\\
a_3(z) &:= -z^3-z^6-z^7-z^9-z^{10}-z^{11}-z^{13}-z^{14}+z^{16}-z^{17}+z^{19}\\
&\quad +z^{20}+z^{22}+z^{23}+z^{24}+z^{26}+z^{27}+z^{30},\\
a_4(z) &:= z^{21}-z^{48}\,.
\end{aligned}$$

[4] It would be interesting to know the set of primes p for which $\sum_{n\geq 0}\nu_p\left(\sum_{k=0}^{n}\binom{2k}{k}\right)z^n$ is a p-Mahler function.

Of course, one could produce similar examples associated with transcendental G-functions by considering the Hadamard product (denoted by $\odot$ below) of several well-chosen algebraic functions. For instance, the elliptic integral

$$\mathfrak{g}(z) := \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-16z\sin^2\theta}} = \frac{1}{\sqrt{1-4z}} \odot \frac{1}{\sqrt{1-4z}} = \sum_{n=0}^{\infty} \binom{2n}{n}^2 z^n$$

is a transcendental G-function which satisfies the following minimal differential equation

$$\left(z^2 - 16z^3\right)\mathfrak{g}''(z) + \left(z - 32z^2\right)\mathfrak{g}'(z) - 4z\mathfrak{g}(z) = 0\,,$$

and it is not hard to see that, for every prime p,

$$\mathfrak{g}_p(z) := \sum_{n=0}^{\infty} \nu_p\left(\binom{2n}{n}^2\right) z^n$$

is a p-Mahler function. More precisely, one can show that $\mathfrak{g}_p$ satisfies a relation of the form

$$a_0(z) + a_1(z)\mathfrak{g}_p(z) + a_2(z)\mathfrak{g}_p(z^p) + a_3(z)\mathfrak{g}_p\left(z^{p^2}\right) + a_4(z)\mathfrak{g}_p\left(z^{p^3}\right) = 0\,,$$

where the $a_i(z)$ are polynomials of degree $\mathrm{O}(p^3)$ too long to be reproduced here.

Regarding (1.1), (1.2) or (1.3), it is tempting to ask about the significance of the integer parameter k. Already in 1976, van der Poorten [40] suggested that two solutions of Mahler-type functional equations associated with essentially distinct parameters should be completely different. For instance, one may naturally expect [40] (and it is now proved [34]) that the two functions

$$\sum_{n=0}^{\infty} z^{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} z^{3^n}$$

are algebraically independent over $\mathbb{C}(z)$. This idea was later formalized by Loxton and van der Poorten who made a general conjecture whose one-dimensional version can be stated as follows[5].

Conjecture 1.2 (Loxton and van der Poorten). Let k and l be two multiplicatively independent positive integers and L be a number field. Let $F(z) \in L[[z]]$ be a locally analytic function that is both k- and ℓ-Mahler. Then $F(z)$ is a rational function.

[5] Note that in fact this conjecture does not imply any statement concerning algebraic independence. It does, however, cover linear independence. Indeed, say that $F(z)$ and $G(z)$ are irrational power series such that F is 2-Mahler and G is 3-Mahler, then 1, F and G are linearly independent over $\mathbb{C}(z)$ (otherwise F is at once 2- and 3-Mahler, and thus rational).

We recall that two integers k and l larger than 1 are multiplicatively independent if there is no pair of positive integers (n, m) such that $k^n = \ell^m$, or equivalently, if $\log(k)/\log(\ell) \notin \mathbb{Q}$. Conjecture 1.2 first appeared in a 1987 paper of van der Poorten [41]. Since then it was explicitly studied in a number of different contexts including some papers of Loxton [24], Becker [9], Randé [42], Bell [10] and the monograph of Everest *et al.* [19]. Independently, Zannier also considered a similar question in [46].

In this paper, our aim is to prove the following result, which has been proven independently by Schäfke and Singer [39].

Theorem 1.3. *Let K be a field of characteristic zero and let k and l be two multiplicatively independent positive integers. Then a power series $F(z) \in K[[z]]$ is both k- and ℓ-Mahler if and only if it is a rational function.*

Let us make few comments.

- Taking K to be a number field in Theorem 1.3 gives Conjecture 1.2;
- If k and ℓ denote two multiplicatively dependent natural numbers, then it is easy to see that a power series is k-Mahler if and only if it is also ℓ-Mahler (see Remark 8.2);
- As explained in more detail in Section 2, one motivation for proving Theorem 1.3 is that it provides a far-reaching generalization of one fundamental result in the theory of sets of integers recognizable by finite automata: Cobham's theorem. Loxton and van der Poorten [24, 41] actually guessed that Conjecture 1.2 should be a consequence of some algebraic independence results for Mahler functions of several variables. In particular, they hoped to obtain a totally new proof of Cobham's theorem by using Mahler's method. Note, however, that our proof of Theorem 1.3 follows a totally different way and ultimately relies on Cobham's theorem, so we do not obtain an independent derivation of that result;
- Another important motivation for establishing Theorem 1.3 comes from the fact that this kind of statements, though highly natural and somewhat ubiquitous, are usually very difficult to prove. In particular, similar independence phenomena, involving two multiplicatively independent integers, are expected in various contexts but only very few results have been obtained up to now. As an illustration, we cite below three interesting open problems that rest on such a principle, all of them being widely open[6]. A long-standing question in dynamical systems is the so-called $\times 2\ \times 3$ problem addressed by Furstenberg [20]: prove that the only Borel measures on $[0, 1]$ that are simultaneously ergodic for $T_2(x) = 2x \pmod 1$ and $T_3(x) = 3x \pmod 1$ are the Lebesgue measure and measures supported by those orbits that are periodic for both actions T_2 and T_3. The following problem, sometimes attributed to Mahler, was suggested by Mendès France

[6] In all of these problems, the integers 2 and 3 may of course be replaced by any two multiplicatively independent integers larger than 1.

in [31] (see also [2]): given a binary sequence $(a_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}}$, prove that

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{3^n}$$

are both algebraic numbers only if both are rational numbers. The third problem we mention appeared implicitly in work of Ramanujan (see [44]): prove that both 2^x and 3^x are integers only if x is a natural number. This is a particular instance of the four exponentials conjecture, a famous open problem in transcendence theory [43, Chapter 1, page 15].

The outline of the paper is as follows. In Section 2, we briefly discuss the connection between Theorem 1.3 and Cobham's theorem. In Section 3, we describe our strategy for proving Theorem 1.3. Then the remaining Sections 4-11 are devoted to the different steps of the proof of Theorem 1.3. Throughout this paper, k and l will denote integers larger than or equal to 2.

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2. Connection with finite automata and Cobham's theorem

One motivation for proving Theorem 1.3 is that it provides a far-reaching generalization of a fundamental result in the theory of sets of integers recognizable by finite automata. The aim of this section is to briefly describe this connection. For more details and formal definitions on automatic sets and automatic sequences, we refer the reader to the book of Allouche and Shallit [6].

Let $k \geq 2$ be a natural number. A set $\mathcal{N} \subset \mathbb{N}$ is said to be *k-automatic* if there is a finite automaton (more formally a k-deterministic finite automaton) that accepts as input the expansion of n in base k and outputs 1 if $n \in \mathcal{N}$ and 0 otherwise. For example, the set of Thue-Morse integers $1, 2, 4, 7, 8, 11, 13, \ldots$, formed by the integers whose sum of binary digits is odd, is 2-automatic. The associated automaton is given in Figure 1 below. It has two states. This automaton successively reads the binary digits of n (starting, say, from the most significant digit and the initial state q_0) and thus ends the reading either in state q_0 or in state q_1. The initial state q_0 gives the output 0, while q_1 gives the output 1.

Another typical 2-automatic set of integers is given by the powers of 2: $1, 2, 4, 8, 16, \ldots$. Though these integers have very simple expansions in base 2, one can observe that this is not the case when writing them in base 3. One of the most important results in the theory of automatic sets formalizes this idea. It says

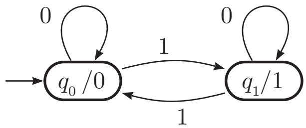

Figure 2.1. The finite-state automaton recognizing the set of Thue-Morse integers.

that only very well-behaved sets of integers can be automatic with respect to two multiplicatively independent numbers. Indeed, in 1969 Cobham [12] proved the following result.

Theorem 2.1 (Cobham). *Let k and ℓ be two multiplicatively independent integers. Then a set $\mathcal{N} \subseteq \mathbb{N}$ is both k- and ℓ-automatic if and only if it is the union of a finite set and a finite number of arithmetic progressions.*

The proof given by Cobham of his theorem is elementary but notoriously difficult, and it remains a challenging problem to find a more natural/conceptual proof (see for instance the comment in Eilenberg [17, page 118]). There are many interesting generalizations of this result. A very recent one is due to Durand [16] and we refer the reader to the introduction of [16] for a brief but complete discussion about such generalizations.

To conclude this section, let us briefly explain why Cobham's Theorem is a consequence of Theorem 1.3. Let us assume that $\mathcal{N} \subseteq \mathbb{N}$ is k-automatic. Set $F(x) := \sum_{n\in\mathcal{N}} x^n \in \mathbb{Z}[[x]]$. Then it is known that $F(x)$ is k-Mahler (see for instance [19, page 232]). In addition, let us assume that $\mathcal{N}$ is also ℓ-automatic where k and ℓ are multiplicatively independent. Then by Theorem 1.3, it follows that $F(x)$ is a rational function and thus the sequence of coefficients of $F(x)$ satisfies a linear recurrence. Since the coefficients of $F(x)$ take only two distinct values (0 and 1), we see that this linear recurrence is ultimately periodic. This exactly means that $\mathcal{N}$ is the union of a finite set and a finite number of arithmetic progressions, as claimed by Cobham's theorem.

3. Sketch of proof of Theorem 1.3

In this section we describe the main steps of the proof of Theorem 1.3.

Let R be a ring and $\mathfrak{P}$ be an ideal of R. If $F(x) = \sum_{n=0}^{\infty} f(n)x^n \in R[[x]]$, then we denote by $F_{\mathfrak{P}}(x)$ the reduction of $F(x)$ modulo $\mathfrak{P}$, that is

$$F_{\mathfrak{P}}(x) = \sum_{n=0}^{\infty}(f(n) \bmod \mathfrak{P})x^n \in (R/\mathfrak{P})[[x]]\,.$$

Let K be a field of characteristic zero and $F(x) \in K[[x]]$ be both k- and ℓ-Mahler.

Step 0. This is a preliminary step. In the introduction, we defined Mahler functions as those satisfying Equation (1.3) but it is not always convenient to work with this general form of equations. In Sections 4 and 6 we show that there is no loss of generality to work with some more restricted types of functional equations. Also in Section 8, we prove that one can assume without loss of generality some additional assumptions on k and ℓ; namely that there are primes p and q such that p divides k but does not divide ℓ and q divides ℓ but does not divide k.

Step 1. A first observation, proved in Section 5, is that the coefficients of the formal power series $F(x)$ only belong to some finitely generated $\mathbb{Z}$-algebra $R \subseteq K$. Then we prove the following useful local-global principle: $F(x)$ is a rational function if it has rational reduction modulo a sufficiently large set of maximal ideals of R. Using classical results of commutative algebra about Jacobson rings, we derive from our local-global principle that there is no loss of generality to assume that K is a number field and that R is a localization of the ring of integers of K formed by inverting a positive integer (that is, R is of the form $\mathcal{O}_K[1/N]$ for some positive integer N).

Comment. Our strategy consists now in applying again our local-global principle. Indeed, since R is of the form $\mathcal{O}_K[1/N]$, we have that the quotient ring $R/\mathfrak{P}$ is a finite field for every prime ideal $\mathfrak{P}$ of R. Our plan is thus to exploit the fact that $F_{\mathfrak{P}}(x)$ has coefficients in the finite set $R/\mathfrak{P}$ to prove that $F_{\mathfrak{P}}(x)$ is both a k- and an ℓ-automatic power series (see Section 7 for a definition), for some prime ideals $\mathfrak{P}$. If this is the case, then Cobham's theorem applies and we get that $F_{\mathfrak{P}}(x)$ is a rational function. The local-global principle actually implies that it is enough to prove that $F_{\mathfrak{P}}(x)$ is both k- and ℓ-automatic for infinitely many prime ideals $\mathfrak{P}$ of R.

Step 2. In Section 7, we underline the relation between k-Mahler, k-regular, and k-automatic power series. The latter two notions are defined in that section. In particular, we will use a result of Dumas [14] showing that every k-Mahler power series can be decomposed as

$$F(x) = G(x) \cdot \Pi(x)\,,$$

where $G(x) \in R[[x]]$ is a k-regular power series and $\Pi(x) \in R[[x]]$ is the inverse of an infinite product of polynomials. Since $F(x)$ is also ℓ-Mahler, we also have a similar decomposition

$$F(x) = H(x) \cdot \Pi'(x)\,,$$

where $H(x) \in R[[x]]$ is a ℓ-regular power series and $\Pi'(x) \in R[[x]]$ is the inverse of an infinite product of polynomials. Furthermore, the theory of regular power series implies that $G_{\mathfrak{P}}(x)$ is k-automatic and that $H_{\mathfrak{P}}(x)$ is ℓ-automatic for every prime ideal $\mathfrak{P}$ of R.

In Section 13 we will split both infinite products $\Pi(x)$ and $\Pi'(x)$ and get an expression of the form

$$F(x) = G(x) \cdot \Pi_1(x) \cdot \Pi_2(x) = H(x) \cdot \Pi'_1(x) \cdot \Pi'_2(x)$$

where $\Pi_1(x)$, $\Pi_2(x)$, $\Pi_1'(x)$, $\Pi_2'(x) \in R[[x]]$ are inverses of some other infinite products of polynomials.

Step 3. After proving preliminary results in Sections 9 and 10, we look at the singularities of Mahler functions at roots of unity in Section 11. We use asymptotic techniques to show that one can reduce to the case of considering Mahler equations whose singularities at roots of unity have a restricted form. This ensures, using some results of Section 7, that $\Pi_1(x)$ is k-automatic and that $\Pi_1'(x)$ is ℓ-automatic when reduced modulo every prime ideal $\mathfrak{P}$ of R.

Step 4. In our last step, we use Chebotarev's density theorem in order to ensure the existence of an infinite set $\mathcal{S}$ of prime ideals of R such that $\Pi_2(x)$ is k-automatic and $\Pi_2'(x)$ is ℓ-automatic when reduced modulo every ideal $\mathfrak{P} \in \mathcal{S}$.

Conclusion. Since the product of k-automatic power series is k-automatic, we infer from Steps 2, 3 and 4 that for every prime ideals $\mathfrak{P} \in \mathcal{S}$ the power series $F_{\mathfrak{P}}(x)$ is both k- and ℓ-automatic. By Cobham's theorem, $F_{\mathfrak{P}}(x)$ is rational for every such prime ideal. Then the local-global principle ensures that $F(x)$ is rational, as desired.

4. Preliminary reduction for the form of Mahler equations

In the introduction, we defined k-Mahler functions as power series satisfying a functional equation of the form given in (1.3). In the literature, they are sometimes defined as solutions of a more restricted type of functional equations. We recall here that these apparently stronger conditions on the functional equations actually lead to the same class of functions. In the sequel, it will thus be possible to work without loss of generality with these more restricted type of equations.

Lemma 4.1. *Let us assume that $F(x)$ satisfies a k-Mahler equation as in* (1.3). *Then there exist polynomials $P_0(x), \ldots, P_n(x)$ in $K[x]$, with* $\gcd(P_0(x), \ldots, P_n(x)) = 1$ *and $P_0(x)P_n(x) \neq 0$, and such that*

$$\sum_{i=0}^{n} P_i(x) F\left(x^{k^i}\right) = 0\,. \tag{4.1}$$

Proof. Let us assume that $F(x)$ satisfies a k-Mahler equation as in (1.3). There thus exist some nonnegative integer n and polynomials $A(x), A_0(x), \ldots, A_n(x)$ in $K[x]$, with $A_n(x)$ nonzero, such that

$$\sum_{i=0}^{n} A_i(x) F\left(x^{k^i}\right) = A(x)\,.$$

We first show that we can assume that $A(x) = 0$. Indeed, let us assume that $A(x) \neq 0$. Applying the operator $x \mapsto x^k$ to this equation, we get that

$$\sum_{i=0}^{n} A_i\left(x^k\right) F\left(x^{k^{i+1}}\right) = A\left(x^k\right).$$

Multiplying the first equation by $A(x^k)$ and the second by $A(x)$ and subtracting, we obtain the new equation

$$\sum_{i=0}^{n+1} B_i(x)F\left(x^{k^i}\right) = 0\,,$$

where $B_i(x) := A_i(x)A(x^k) - A_i(x^k)A(x)$ for every integer i, $1 \le i \le n$ and where $B_{n+1}(x) := A_n(x^k)A(x) \neq 0$. We can thus assume without loss of generality that $A(x) = 0$.

Now, among all such nontrivial relations of the form

$$\sum_{i=0}^{n} P_i(x)F\left(x^{k^i}\right) = 0\,, \tag{4.2}$$

we choose one with n minimal. Thus $P_n(x)$ is nonzero. We claim $P_0(x)$ is nonzero. Let us assume this is not the case. Pick the smallest integer j such that $P_j(x)$ is nonzero. By assumption, $j > 0$. Then there is some nonnegative integer a such that the coefficient of x^a in $P_j(x)$ is nonzero. Let b be the unique integer such that $a \equiv b \bmod k$ and $0 \le b < k$. Let us define the operator Λ_b from $K[[x]]$ into itself by

$$\Lambda_b\left(\sum_{i=0}^{\infty} f(i)x^i\right) := \sum_{i=0}^{\infty} f(ki+b)x^i\,.$$

These types of operators are classically used for studying algebraic power series over fields of characteristic $p > 0$, where one takes $k = p$ (see for instance [6, Chapter 12] and the references therein). In this context, these operators are often referred to as Cartier operators. With this definition, every $F(x) \in K[[x]]$ has a unique decomposition as

$$F(x) = \sum_{b=0}^{k-1} x^b \Lambda_b(F)\big(x^k\big)\,,$$

which implies that

$$\Lambda_b\left(F(x)G\big(x^k\big)\right) = \Lambda_b\left(F(x)\right)G(x)$$

for every pair of power series $F(x), G(x) \in K[[x]]$. Applying Λ_b to Equation (4.2), we thus get that

$$0 = \Lambda_b\left(\sum_{i=j}^{n} P_i(x)F\left(x^{k^i}\right)\right) = \sum_{i=j-1}^{n-1} \Lambda_b\left(P_{i+1}(x)\right)F\left(x^{k^i}\right).$$

By construction, $\Lambda_b(P_j(x))$ is nonzero, which shows that this relation is nontrivial. This contradicts the minimality of n. It follows that $P_0(x)$ is nonzero.

Furthermore, if $\gcd(P_0(x), \ldots, P_n(x)) = D(x) \neq 0$, it suffices to divide (4.2) by $D(x)$ to obtain an equation with the desired properties. This ends the proof. $\square$

5. Reduction to the number field case

In this section we show that we may restrict our attention to the case where the base field K is replaced by a number field and more precisely by a localization of the ring of integers of that number field formed by inverting a single integer. This means a ring of the form $\mathcal{O}_K[1/N]$, where K denotes a number field, $\mathcal{O}_K$ the ring of integers of K, and N a positive integer.

Theorem 5.1. *Let us assume that the conclusion of Theorem* 1.3 *holds whenever the field K is replaced by a localization of the ring of integers of a number field of the form $\mathcal{O}_K[1/N]$. Then Theorem* 1.3 *is true.*

We first observe that the coefficients of a Mahler function in $K[[x]]$ actually belong to some finitely generated $\mathbb{Z}$-algebra $R \subseteq K$.

Lemma 5.2. *Let K be a field of characteristic zero, let $k \geq 2$ be an integer, and let $F(x) \in K[[x]]$ be a k-Mahler power series. Then there exists a finitely generated $\mathbb{Z}$-algebra $R \subseteq K$ such that $F(x) \in R[[x]]$.*

Proof. Let $F(x) := \sum_{n=0}^{\infty} f(n)x^n \in K[[x]]$ be a k-Mahler power series. We first infer from Lemma 4.1 that there exist a natural number n and polynomials $P_0(x), \ldots, P_n(x) \in K[x]$ with $P_0(x)P_n(x) \neq 0$ such that

$$\sum_{i=0}^{n} P_i(x)F\left(x^{k^i}\right) = 0\,.$$

Let d be a natural number that is strictly greater than the degrees of the polynomials $P_0(x), \ldots, P_n(x)$. Let R denote the smallest $\mathbb{Z}$-algebra containing:

- The coefficients of $P_0(x), \ldots, P_n(x)$;
- The coefficients $f(0), \ldots, f(d)$;
- The multiplicative inverses of all nonzero coefficients of $P_0(x)$.

By definition, $R \subseteq K$ is a finitely generated $\mathbb{Z}$-algebra. We claim that $F(x) \in R[[x]]$. To see this, suppose that this is not the case. Let n_0 be the smallest nonnegative integer such that $f(n_0) \notin R$. By assumption, $n_0 > d$. Consider the equation

$$P_0(x)F(x) = -\sum_{i=1}^{n} P_i(x)F\left(x^{k^i}\right). \tag{5.1}$$

Let j denote the order of $P_0(x)$ at $x = 0$ and let $c \neq 0$ denote the coefficient of x^j in $P_0(x)$. Then if we extract the coefficient of x^{n_0+j} in Equation (5.1), we see that $cf(n_0)$ can be expressed as an R-linear combination of $f(0), \ldots, f(n_0 - 1)$. Hence $cf(n_0)$ belongs to R by the minimality of n_0. Since $c^{-1} \in R$ we see that $f(n_0) \in R$, a contradiction. This ends the proof. $\square$

We now prove that the height of a rational function which satisfies a Mahler-type equation can be bounded by the maximum of the degrees of the polynomials defining the underlying equation.

Lemma 5.3. *Let K be a field, let n and d be natural numbers, and let $P_0(x), \ldots, P_n(x)$ be polynomials in $K[x]$ of degree at most d with $P_0(x)P_n(x) \neq 0$. Suppose that $F(x) \in K[[x]]$ satisfies the Mahler-type equation*

$$\sum_{i=0}^{n} P_i(x)F(x^{k^i}) = 0\,.$$

If $F(x)$ is rational, then there exist polynomials $A(x)$ and $B(x)$ of degree at most d with $B(0) = 1$ such that $F(x)$ is the power series expansion of $A(x)/B(x)$.

Proof. Without any loss of generality we can assume that $F(x)$ is not identically zero. If $F(x)$ is rational, then there exist two polynomials $A(x)$ and $B(x)$ in $K[x]$ with gcd 1 and with $B(0) = 1$ such that $F(x) = A(x)/B(x)$. Observe that

$$\sum_{i=0}^{n} P_i(x)A\left(x^{k^i}\right)/B\left(x^{k^i}\right) = 0\,.$$

Multiplying both sides of this equation by the product $B(x)B(x^k)\cdots B(x^{k^n})$, we see that $B(x^{k^n})$ divides

$$P_n(x)A\left(x^{k^n}\right)B(x)\cdots B\left(x^{k^{n-1}}\right).$$

Since $\gcd(A(x), B(x)) = 1$ and $A(x)$ is nonzero, we actually have that $B(x^{k^n})$ divides

$$P_n(x)B(x)\cdots B\left(x^{k^{n-1}}\right).$$

Let d_0 denote the degree of $B(x)$. Then we have

$$\begin{aligned} k^n d_0 &\leq \deg(P_n(x)) + \sum_{i=0}^{n-1} \deg\left(B\left(x^{k^i}\right)\right) \\ &\leq d + d_0\left(1 + k + \cdots + k^{n-1}\right) \\ &= d + d_0(k^n - 1)/(k-1)\,. \end{aligned}$$

Thus

$$d_0(k^{n+1} - 2k^n + 1)/(k-1) \leq d\,,$$

which implies $d_0 \leq d$ since $(k^{n+1} - 2k^n + 1)/(k-1) \geq 1$ for every integer $k \geq 2$. A similar argument gives the same upper bound for the degree of $A(x)$. □

We derive from Lemma 5.3 a useful local-global principle for the rationality of Mahler functions with coefficients in a finitely generated $\mathbb{Z}$-algebra.

Lemma 5.4. *Let K be a field, let $k \geq 2$ be an integer, and let $R \subseteq K$ be a ring. Let us assume that $F(x) \in R[[x]]$ has the following properties.*

(i) *There exist a natural number n and polynomials $P_0(x), \dots, P_n(x) \in R[x]$ with $P_0(x)P_n(x) \neq 0$ such that*

$$\sum_{i=0}^{n} P_i(x) F\left(x^{k^i}\right) = 0\,;$$

(ii) *There exists a set $\mathcal{S}$ of maximal ideals of R such that $F(x) \bmod I$ is a rational power series in $(R/I)[[x]]$ for every $I \in \mathcal{S}$;*

(iii) *One has $\bigcap_{I \in \mathcal{S}} I = \{0\}$.*

Then $F(x)$ is a rational function.

Proof. Let d be a natural number greater than the degrees of all polynomials $P_0(x), \dots, P_n(x)$. By (ii), we have that for each maximal ideal I in $\mathcal{S}$, $F(x) \bmod I$ is a rational function. Thus by (i) and Lemma 5.3, we see that for each maximal ideal I in $\mathcal{S}$, there exist two polynomials $A_I(x)$ and $B_I(x) \in (R/I)[x]$ of degree at most d with $B_I(0) = 1$ and such that $F(x) \equiv A_I(x)/B_I(x) \mod I$. In particular, if $F(x) = \sum_{j \geq 0} f(j)x^j$, we see that the sequences in the set $\left\{(f(d+1+i+j) \mod I)_{j \geq 0} \mid i = 0, \dots, d\right\}$ are linearly dependent over R/I. Thus the determinant of each $(d+1) \times (d+1)$ submatrix of the infinite matrix

$$M := \begin{pmatrix} f(d+1) & f(d+2) & f(d+3) & \cdots \\ f(d+2) & f(d+3) & f(d+4) & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ f(2d+1) & f(2d+2) & f(2d+3) & \cdots \end{pmatrix}$$

lies in the maximal ideal I. Since this holds for every maximal ideal I in $\mathcal{S}$, we infer from (iii) that every $(d+1) \times (d+1)$ minor of M vanishes. It follows that M has rank at most d and thus the rows of M are linearly dependent over the field of fractions of R. In particular, there exist $c_0, \dots, c_d \in R$, not all zero, such that

$$\sum_{i=0}^{d} c_i f(d+1+i+j) \;=\; 0$$

for all $j \geq 0$. Letting $B(x) := c_d + c_{d-1}x + \cdots + c_0 x^d$, we see that $B(x)F(x)$ is a polynomial. Hence $F(x)$ is a rational function. This ends the proof. □

We are now ready to prove the main result of this section.

Proof of Theorem 5.1. Let K be a field of characteristic zero and let $F(x) \in K[[x]]$ be a power series that is both k- and ℓ-Mahler for some multiplicatively independent natural numbers k and ℓ. By Lemma 4.1, there are natural numbers n and m and polynomials $P_0(x), \ldots, P_n(x)$ and $Q_0(x), \ldots, Q_m(x)$ with

$$P_0(x)P_n(x)Q_0(x)Q_m(x) \neq 0$$

and such that

$$\sum_{i=0}^{n} P_i(x)F\left(x^{k^i}\right) = \sum_{j=0}^{m} Q_j(x)F\left(x^{\ell^j}\right) = 0\,. \tag{5.2}$$

Then by Lemma 5.2, there is a finitely generated $\mathbb{Z}$-algebra $R \subseteq K$ such that $F(x) \in R[[x]]$. By adjoining all the coefficients of $P_0(x), \ldots, P_n(x)$ and of $Q_0(x), \ldots, Q_m(x)$ to R, we can assume that $P_i(x)$ and $Q_j(x)$ are in $R[x]$ for $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$. By localizing at the multiplicatively closed set consisting of nonzero integers in R, we can assume that R is a finitely generated $\mathbb{Q}$-algebra.

Let $\mathcal{M} \subseteq \operatorname{Spec}(R)$ denote the collection of maximal ideals of R. Since R is a finitely generated $\mathbb{Q}$-algebra, R is a Jacobson ring and R/I is a finite extension of $\mathbb{Q}$ for every $I \in \mathcal{M}$ (see [18, Theorem 4.19, page 132]). Thus, for each maximal ideal I of R, the quotient field R/I is a number field. If we assume that the conclusion of Theorem 1.3 holds when the base field is a number field, then we get that $F(x)$ mod I is a rational function in $(R/I)[[x]]$ for it is clearly both k- and ℓ-Mahler[7]. Since R is a Jacobson ring that is also a domain, we have that $\bigcap_{I \in \mathcal{M}} I = \{0\}$ (*cf.* [18, page 132]). Then Lemma 5.4 implies that $F(x)$ is a rational function in $R[[x]]$. This shows it is sufficient to prove Theorem 1.3 in the case that K is a number field.

We can thus assume that $F(x) \in K[[x]]$ where K is a number field. Now, if we apply again Lemma 5.2, we see that there is a finitely generated $\mathbb{Z}$-algebra $R \subseteq K$ such that $F(x) \in R[[x]]$. Furthermore, every finitely generated $\mathbb{Z}$-subalgebra of a number field K has a generating set of the form $\{a_1/b, \ldots, a_t/b\}$, where b is a nonzero (rational) integer and $a_1, \ldots, a_t$ are algebraic integers in K. Thus R is a subalgebra of a localization of the ring of integers of a number field formed by inverting a single nonzero integer, that is $R \subseteq \mathcal{O}_K[1/b]$, where $\mathcal{O}_K$ denotes the ring of algebraic integers in K. Thus to establish Theorem 1.3 it is sufficient to prove the following result: let k and ℓ be two multiplicatively independent natural numbers,

[7] Note that since $P_0(0)Q_0(0) \neq 0$, we may assume that $P_0(0) = Q_0(0) = 1$ by multiplying the left side of (5.2) by $1/P_0(0)$ and the right side of (5.2) by $1/Q_0(0)$. This ensures that, for each functional equation, not all the coefficients vanish when reduced modulo a maximal ideal I of R. Hence $F(x)$ mod I is both k- and ℓ-Mahler.

let R be of the form $\mathcal{O}_K[1/b]$ where K is a number field, and let $F(x) \in R[[x]]$, then if $F(x)$ is both k- and ℓ-Mahler it is a rational function. This concludes the proof. $\square$

6. Further reductions for the form of Mahler equations

In this section we refine the results of Section 4. We show that a power series satisfying a Mahler equation of the form given in (4.1) is also solution of a more restricted type of functional equations.

Lemma 6.1. *Let K be a field and $k \geq 2$ be an integer. Let us assume that $F(x) := \sum_{s\geq 0} f(s)x^s \in K[[x]]$ satisfies a k-Mahler equation of the form*

$$\sum_{i=0}^{n} P_i(x)F\left(x^{k^i}\right) = 0\,,$$

where $P_0(x), \ldots, P_n(x) \in K[x]$, $\gcd(P_0(x), \ldots, P_n(x)) = 1$ and $P_0(x)P_n(x) \neq 0$. Then there exists a natural number N such that, for every integer $a > N$ with $f(a) \neq 0$, $F(x)$ can be decomposed as

$$F(x) = T_a(x) + x^a F_0(x)\,,$$

where $T_a(x) \in K[x]$ is the Taylor approximation of $F(x)$ at $x = 0$ up to degree $a-1$ and $F_0(x)$ has nonzero constant term and satisfies a k-Mahler equation

$$\sum_{i=0}^{n+1} Q_i(x)F_0\left(x^{k^i}\right) = 0$$

for some polynomials $Q_0, \ldots, Q_{n+1} \in K[x]$ satisfying the following conditions.

(i) *It holds $Q_0(0) = 1$;*
(ii) *If $\alpha \neq 0$ and $P_0(\alpha) = 0$, then $Q_0(\alpha) = 0$;*
(iii) *If $\alpha \neq 0$, $P_0(\alpha) = 0$ and $\alpha^k = \alpha$, then $Q_j(\alpha) \neq 0$ for some $j \in \{1, \ldots, n+1\}$.*

Proof. By assumption, we have that $F(x)$ satisfies a k-Mahler equation

$$\sum_{i=0}^{n} P_i(x)F\left(x^{k^i}\right) = 0\,,$$

where $P_0(x)P_n(x)$ is nonzero. Let N denote the order of vanishing of $P_0(x)$ at $x = 0$. Suppose that $a \geq N$ and $f(a) \neq 0$. Then we have that

$$F(x) = T_a(x) + x^a F_0(x)\,,$$

where $T_a(x)$ is the Taylor approximation of $F(x)$ up to degree $a-1$ and $F_0(x)$ is a power series with nonzero constant term. Then we have

$$\sum_{i=0}^{n} P_i(x)\left(T_a\left(x^{k^i}\right)+x^{k^i\cdot a}F_0\left(x^{k^i}\right)\right)=0\,,$$

which we can write as

$$\sum_{i=0}^{n} P_i(x)x^{k^i\cdot a}F_0\left(x^{k^i}\right)=C(x)\,, \tag{6.1}$$

where $C(x)$ denotes the polynomial

$$C(x):=-\sum_{i=0}^{n} P_i(x)T_a\left(x^{k^i}\right).$$

Set $S(x):=P_0(x)x^{-N}$. By definition of N, $S(x)$ is a polynomial with $S(0)\neq 0$. Then if we divide both sides of Equation (6.1) by x^{a+N}, we obtain that

$$S(x)F_0(x)+\sum_{i=1}^{n} P_i(x)x^{k^i a-a-N}F_0\left(x^{k^i}\right)=x^{-a-N}C(x)\,. \tag{6.2}$$

Observe that the left side is a power series with constant term $S(0)F_0(0)\neq 0$ and thus $C_0(x):=x^{-a-N}C(x)$ is a polynomial with $C_0(0)\neq 0$. Applying the operator $x\mapsto x^k$, we also obtain that

$$S\big(x^k\big)F_0\big(x^k\big)+\sum_{i=1}^{n} P_i\big(x^k\big)x^{k^{i+1}a-ka-kN}F_0\big(x^{k^{i+1}}\big)=C_0\big(x^k\big)\,. \tag{6.3}$$

Multiplying (6.2) by $C_0(x^k)$ and (6.3) by $C_0(x)$ and then subtracting, we get that

$$\begin{aligned}&C_0\big(x^k\big)S(x)F_0(x)+\sum_{i=1}^{n} C_0\big(x^k\big)P_i(x)x^{k^i a-a-N}F_0\left(x^{k^i}\right)\\&-C_0(x)S\big(x^k\big)F_0\big(x^k\big)-\sum_{i=1}^{n} C_0(x)P_i\big(x^k\big)x^{k^{i+1}a-ka-kN}F_0\left(x^{k^{i+1}}\right)=0\,.\end{aligned}$$

Since $C_0(0)$ and $S(0)$ are nonzero, we see that $F_0(x)$ satisfies a non-trivial k-Mahler equation

$$\sum_{i=0}^{n+1} Q_i(x)F_0\left(x^{k^i}\right)=0\,,$$

where

$$Q_0(x):=\frac{C_0\big(x^k\big)S(x)}{\gcd\left(C_0(x),\,C_0\big(x^k\big)\right)}$$

and

$$Q_1(x) := \frac{C_0\big(x^k\big)P_1(x)x^{ka-a-N} - C_0(x)S\big(x^k\big)}{\gcd\Big(C_0(x), C_0\big(x^k\big)\Big)},$$

and, for $i \in \{2, \ldots, n+1\}$,

$$Q_i(x) := \frac{x^{k^i a - ka - N}\Big(C_0\big(x^k\big)x^{(k-1)a}P_i(x) - C_0(x)P_{i-1}\big(x^k\big)\Big)}{\gcd\Big(C_0(x), C_0\big(x^k\big)\Big)},$$

with the convention that $P_{n+1}(x) := 0$. By construction, $Q_0(0) \neq 0$, which we may assume to be equal to 1 by multiplying our equation by $1/Q_0(0)$. Since $S(x)$ divides $Q_0(x)$, we have that if $P_0(\alpha) = 0$ for some nonzero α then $Q_0(\alpha) = 0$. Finally, suppose that $P_0(\alpha) = 0$ for some nonzero α such that $\alpha^k = \alpha$. We claim that $Q_i(\alpha)$ is nonzero for some $i \in \{1, \ldots, n+1\}$. Note that since $\gcd(P_0(x), \ldots, P_n(x)) = 1$, there is smallest positive integer j such that $P_j(\alpha)$ is nonzero. We claim that $Q_j(\alpha) \neq 0$. Indeed, otherwise α would be a root of $C_0(x)/\gcd(C_0(x), C_0(x^k))$, but this is impossible since $\alpha^k = \alpha$. This ends the proof. □

Corollary 6.2. *Let K be a field and let k and ℓ be multiplicatively independent natural numbers. Let $F(x) := \sum_{s\geq 0} f(s)x^s \in K[[x]]$ be a power series that is both k- and ℓ-Mahler and that is not a polynomial. Then there is a natural number a such that $F(x)$ can be decomposed as*

$$F(x) = T_a(x) + x^a F_0(x),$$

where $T_a(x)$ is the Taylor approximation of $F(x)$ up to degree $a-1$, $F_0(x)$ satisfies a k-Mahler equation as in Lemma 6.1*, and $F_0(x)$ also satisfies an ℓ-Mahler equation of the form*

$$\sum_{i=0}^{r} R_i(x)F_0\Big(x^{\ell^i}\Big) = 0$$

with $R_0(x), \ldots, R_r(x) \in K[x]$ and $R_0(0) = 1$.

Proof. Applying Lemma 6.1 to $F(x)$, viewed as a k-Mahler function, we obtain the existence of a positive integer N_1 (which corresponds to N in Lemma 6.1) for which the conclusion of this lemma holds. Similarly, applying Lemma 6.1 to $F(x)$, viewed as a ℓ-Mahler function, we obtain the existence of a positive integer N_2. Now, we can choose $N_3 := \max(N_1, N_2)$ and pick $a > N_3$ such that $f(a) \neq 0$ to obtain the desired conclusion. □

7. Links with automatic and regular power series

The aim of this section is to emphasize the relation between k-Mahler, k-regular, and k-automatic power series. We gather some useful facts about automatic and regular power series that will turn out to be useful for proving Theorem 1.3. We also recall a result of Dumas [14] showing that every k-Mahler power series can be decomposed as the product of a k-regular power series of a special type and the inverse of an infinite product of polynomials. Such a decomposition will play a key role in the proof of Theorem 1.3.

7.1. Automatic and regular power series

We recall here basic facts about regular power series, which were introduced by Allouche and Shallit [7] (see also [8] and [6, Chapter 16]). They form a distinguished class of k-Mahler power series as well as a natural generalization of k-automatic power series.

A useful way to characterize k-automatic sequences, due to Eilenberg [17], is given in terms of the so-called k-kernel.

Definition 7.1. Let $k \geq 2$ be an integer and let $\mathbf{f} = (f(n))_{n\geq 0}$ be a sequence with values in a set E. The *k-kernel* of $\mathbf{f}$ is defined as the set

$$\left\{(f(k^a n + b))_{n\geq 0} \mid a \geq 0, b \in \{0, \dots, k^a - 1\}\right\} .$$

Theorem 7.2 (Eilenberg). *A sequence is k-automatic if and only if its k-kernel is finite.*

This characterization gives rise to the following natural generalization of automatic sequences introduced by Allouche and Shallit [7].

Definition 7.3. Let R be a commutative Noetherian ring and let $\mathbf{f} = (f(n))_{n\geq 0}$ be a R-valued sequence. Then $\mathbf{f}$ is said to be *k-regular* if the dimension of the R-module spanned by its k-kernel is finite.

In the sequel, we will say that a power series $F(x) \in R[[x]]$ is *k-regular* (respectively *k-automatic*) if its sequence of coefficients is k-regular (respectively k-automatic). Of course, with a subset $\mathcal{E}$ of $\mathbb{N}$, we can associate its characteristic sequence $\chi(n)$, taking values in $\{0, 1\}$, and thus a power series $F_{\mathcal{E}}(x) := \sum \chi(n)x^n \in \mathbb{Z}[[x]]$. When the set $\mathcal{E}$ is k-automatic, $F_{\mathcal{E}}(x)$ is a k-automatic power series. More generally, a power series $F(x) = \sum f(n)x^n$ with coefficients in a finite set S is k-automatic if and only if for every $s \in S$ the set $\{n \in \mathbb{N} \mid f(n) = s\}$ is k-automatic. In the following proposition, we collect some useful general facts about k-regular power series.

Proposition 7.4. *Let R be a commutative ring and $k \geq 2$ be an integer. Then the following properties hold.*

(i) *If $F(x) \in R[[x]]$ is k-regular and I is an ideal of R, then $F(x) \bmod I \in (R/I)[[x]]$ is k-regular;*
(ii) *If $F(x) \in R[[x]]$ is k-regular, then the coefficients of $F(x)$ take only finitely many distinct values if and only if $F(x)$ is k-automatic;*

(iii) *If* $F(x) = \sum_{i\geq 0} f(i)x^i$ *and* $G(x) = \sum_{i\geq 0} g(i)x^i$ *are two* k*-regular power series in* $R[[x]]$*, then the Cauchy product*

$$F(x)G(x) := \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\binom{i}{j} f(j)g(i-j)\right)x^i$$

is k*-regular.*

Proof. The property (i) follows directly from the definition of a k-regular sequence, while (ii) and (iii) correspond respectively to Theorem 16.1.5 and Corollary 16.4.2 in [6]. □

In Section 11, we will need to use that k-regular sequences with complex values do have strict restrictions on the growth of their absolute values, a fact evidenced by the following result.

Proposition 7.5. *Let* $k \geq 2$ *be a natural number and let* $F(x) \in \mathbb{C}[[x]]$ *be a* k*-regular power series. Then* $F(x)$ *is analytic in the open unit disk and there exist two positive real numbers* C *and* m *such that*

$$|F(x)| < C(1-|x|)^{-m},$$

for all $x \in B(0,1)$.

Proof. Let $F(x) = \sum_{i=0}^{\infty} f(i)x^i \in \mathbb{C}[[x]]$ be a k-regular power series. Then there is some positive constant A and some integer $d > 0$ such that

$$|f(i)| \leq A(i+1)^d,$$

for every nonnegative integer i (see [6, Theorem 16.3.1]). This immediately gives that $F(x)$ is analytic in the open unit disk. Moreover, for $x \in B(0,1)$,

$$|F(x)| \leq \sum_{i=0}^{\infty} A(i+1)^d |x|^i \leq \sum_{i=0}^{\infty} Ad!\binom{i+d}{d}|x|^i = Ad!(1-|x|)^{-d-1}.$$

The result follows. □

7.2. Becker power series

Becker [9, Theorem 1] showed that a k-regular power series is necessarily k-Mahler. In addition to this, he proved [9, Theorem 2] the following partial converse (see Theorem 7.6 below). The general converse does not hold. For example, the power series in $\mathbb{Q}[[x]]$ defined by the k-Mahler equation

$$(1-x)F(x) = F\left(x^k\right)$$

and satisfying $F(0) = 1$ is not k-regular. This can easily be shown using Proposition 7.5.

Theorem 7.6 (Becker). *Let K be a field, let k be a natural number ≥ 2, and let $F(x) \in K[[x]]$ be a power series that satisfies a k-Mahler equation of the form*

$$F(x) = \sum_{i=1}^{n} P_i(x) F\left(x^{k^i}\right) \tag{7.1}$$

for some polynomials $P_1(x), \ldots, P_n(x) \in K[x]$. Then $F(x)$ is a k-regular power series.

Definition 7.7. In honour of Becker's result, a power series $F(x) \in K[[x]]$ that satisfies an equation of the form given in Equation (7.1) will be called a *k-Becker* power series.

Theorem 7.6 shows that the set of k-Becker power series is contained in the set of k-regular power series. However, the converse is not true. As an example, we provide the following result that will also be used in Section 13.

Proposition 7.8. *Let k be a natural number, and let $\omega \in \mathbb{C}$ be a root of unity with the property that if $j \geq 1$ then $\omega^{k^j} \neq \omega$. Then*

$$\left(\prod_{j=0}^{\infty}\left(1 - \omega x^{k^j}\right)\right)^{-1}$$

is k-regular but it is not k-Becker.

Proof. Since ω is a root of unity, the sequence $\omega, \omega^k, \omega^{k^2}, \ldots$ is eventually periodic and there is some smallest natural number N such that

$$\omega^{k^{2N}} = \omega^{k^N}.$$

Set $\beta := \omega^{k^N}$ and let us consider the polynomial

$$Q(x) = (1 - \beta x)\left(1 - \beta x^k\right) \cdots \left(1 - \beta x^{k^{N-1}}\right).$$

Then

$$\frac{Q\left(x^k\right)}{Q(x)} = \frac{1 - \beta x^{k^N}}{1 - \beta x}.$$

Since

$$1 - \beta x^{k^N} = 1 - \left(\beta x\right)^{k^N},$$

we see that $Q(x^k)/Q(x)$ is a polynomial.

Since

$$1 - (\beta x)^{k^N} = \frac{Q\left(x^k\right)}{Q(x)} \cdot (1 - \beta x),$$

we get that $(1 - \omega x)$ divides the polynomial $Q(x^k)(1 - \beta x)/Q(x)$. Furthermore, $(1 - \omega x)$ cannot divide $(1 - \beta x)$ since by assumption $\omega \neq \beta$. By Euclid's lemma, we thus obtain that

$$\frac{Q(x^k)}{Q(x)} = (1 - \omega x)S(x)$$

for some polynomial $S(x)$.

Set

$$F(x) := \left(\prod_{j=0}^{\infty} \left(1 - \omega x^{k^j}\right) \right)^{-1}$$

and $G(x) := Q(x)^{-1} F(x)$. Since $F(x)$ satisfies the k-Mahler recurrence

$$F(x^k) = (1 - \omega x) F(x) ,$$

we see that

$$G(x^k) = Q(x^k)^{-1} (1 - \omega x) Q(x) G(x),$$

or equivalently,

$$G(x) = S(x) G(x^k) .$$

Thus $G(x)$ is a k-Becker power series. By Proposition 7.4, $F(x)$ is k-regular as it is a product of a polynomial (which is k-regular) and a k-regular power series.

On the other hand, $F(x)$ cannot be a k-Becker power series. To see this, suppose that $F(x)$ satisfies an equation of the form

$$F(x) = \sum_{i=1}^{d} P_i(x) F\left(x^{k^i}\right) .$$

Now, dividing both sides by $F(x^k)$, the right side becomes a polynomial in x, while the left side is $(1 - \omega x)^{-1}$, a contradiction. The result follows. □

In Section 11, we will need the following basic result about k-Becker power series.

Lemma 7.9. *Let $k \geq 2$ and let us assume that $F(x) \in K[[x]]$ satisfies a k-Mahler equation of the form*

$$F(x) = \sum_{i=1}^{n} a_i F\left(x^{k^i}\right)$$

for some constants $a_1, \ldots, a_n \in K$. Then $F(x)$ is constant.

Proof. Let us denote by $F(x) = \sum_{i \geq 0} f(i) x^i$ the power series expansion of $F(x)$. If $F(x)$ were non-constant, there would be some smallest positive integer i_0 such that $f(i_0) \neq 0$. Thus $F(x) = \lambda + x^{i_0} F_0(x)$ for some λ in K and some $F_0(x) \in K[[x]]$. But taking the coefficient of x^{i_0} in the right-hand side of the equation

$$F(x) = \sum_{i=1}^{n} a_i F\left(x^{k^i}\right),$$

we see that $f(i_0) = 0$, a contradiction. The result follows. □

Though there are some Mahler functions that are not Becker functions, the following result shows that every k-Mahler power series can be decomposed as the product of a k-Becker power series and the inverse of an infinite product of polynomials. This decomposition will turn out to be very useful to prove Theorem 1.3. This result appears as Theorem 31 in the Thèse de Doctorat of Dumas [14].

Proposition 7.10. *Let k be a natural number, let K be a field, and let $F(x) \in K[[x]]$ be a k-Mahler power series satisfying an equation of the form*

$$\sum_{i=0}^{n} P_i(x) F\left(x^{k^i}\right) = 0,$$

where $P_0(x), \ldots, P_n(x) \in K[x]$ and $P_0(0) = 1$. Then there is a k-Becker power series $G(x)$ such that

$$F(x) = \left(\prod_{i=0}^{\infty} P_0\left(x^{k^i}\right)\right)^{-1} G(x).$$

8. Conditions on k and ℓ

In this section K will denote an arbitrary field. We consider power series in $K[[x]]$ that are both k- and ℓ-Mahler with respect to two multiplicatively independent natural numbers k and ℓ. More specifically, we look at the set of natural numbers m for which such a power series is necessarily m-Mahler.

Proposition 8.1. *Let k and ℓ be two integers ≥ 2 and let $F(x) \in K[[x]]$ be a power series that is both k- and ℓ-Mahler. Let us assume that a and b are integers with the property that $m := k^a \ell^b$ is an integer and $m > 1$. Then $F(x)$ is also m-Mahler.*

Proof. Let V denote the $K(x)$-vector space spanned by all the power series that belong to the set $\left\{F(x^{k^a \ell^b}) \mid a, b \in \mathbb{N}\right\}$. Recall that by Lemma 4.1, we can assume that the corresponding Mahler equations are both homogeneous. Hence there exists some natural number N such that for every integer $n \geq N$ we have $F(x^{k^n}) = \sum_{i=0}^{N-1} P_{i,n}(x) F(x^{k^i})$ and $F(x^{\ell^n}) = \sum_{i=0}^{N-1} Q_{i,n}(x) F(x^{\ell^i})$ for some rational functions $P_{0,n}(x), \ldots, P_{N-1,n}(x), Q_{0,n}(x), \ldots, Q_{n,N-1}(x)$. Thus V is a $K(x)$-vector space of dimension at most N^2.

Suppose that a and b are integers such that $m := k^a\ell^b$ is an integer and $m > 1$. If a and b are nonnegative, then $F(x^{m^j}) \in V$ for every integer $j \geq 0$ and since the dimension of V is finite, we see that $F(x)$ is m-Mahler. Thus we may assume that at least one of a or b is negative. Since $m \geq 1$, at least one of a or b must also be positive. Without loss of generality, we may thus assume that $a > 0$ and $b < 0$.

We are now going to show that $F(x^{m^j}) \in V$ for every nonnegative integer j. To see this, we fix a nonnegative integer j. Then we observe that $m^j\ell^{-bj} = k^{ja}$ and thus $F(x^{m^j l^i})$ belongs to V for every integer $i \geq -bj$. Since $-bj \geq 0$, there exists a smallest nonnegative integer i_0 such that $F(x^{m^j\ell^i}) \in V$ for every integer $i \geq i_0$. If i_0 is zero, then we are done. We assume that i_0 is positive and look for a contradiction. By definition of i_0, we note that $F(x^{m^j\ell^{i_0-1}}) \notin V$. By assumption, $F(x)$ satisfies a ℓ-Mahler equation of the form

$$\sum_{i=0}^{N} P_i(x)F(x^{\ell^i}) = 0,$$

with $P_0(x), \ldots, P_N(x) \in K[x]$ and $P_0(x) \neq 0$. Applying the operator $x \mapsto x^{m^j\ell^{i_0-1}}$, we get that

$$P_0\left(x^{m^j\ell^{i_0-1}}\right) F\left(x^{m^j\ell^{i_0-1}}\right) = -\sum_{i=1}^{N} P_i\left(x^{m^j\ell^{i_0-1}}\right) F\left(x^{m^j\ell^{i_0-1+i}}\right).$$

By definition of i_0, the right side of this equation is in V, and so $F(x^{m^j\ell^{i_0-1}}) \in V$ since $P_0(x)$ is nonzero. This is a contradiction. It follows that $F(x^{m^j}) \in V$ for every nonnegative integer j.

Since V is a $K(x)$-vector space of dimension at most N^2, we see that $F(x), F(x^m), \ldots, F(x^{m^{N^2}})$ are linearly dependent over $K(x)$, which implies that $F(x)$ is m-Mahler. This ends the proof. $\square$

Remark 8.2. Taking $k = \ell$ and $b = 0$ in Proposition 8.1, we see that if a power series $F(x)$ is k-Mahler then it is also k^a-Mahler for every $a \geq 1$. The converse is obvious. Consequently, if k and ℓ are multiplicatively dependent natural numbers, then $F(x)$ is k-Mahler if and only if it is ℓ-Mahler.

Corollary 8.3. *Let k and ℓ be two multiplicatively independent natural numbers and let $F(x) \in K[[x]]$ be a power series that is both k- and ℓ-Mahler. Then there exist two multiplicatively independent positive integers k' and ℓ' such that the following conditions hold.*

(i) *There is a prime number p that divides k' and does not divide ℓ';*
(ii) *There is a prime number q that divides ℓ' and does not divide k';*
(iii) *$F(x)$ is both k'- and ℓ'-Mahler.*

Proof. There exist prime numbers $p_1, \dots, p_m$ and nonnegative integers $a_1, \dots, a_m$, $b_1, \dots, b_m$ such that

$$k = \prod_{i=1}^{m} p_i^{a_i} \quad \text{and} \quad \ell = \prod_{i=1}^{m} p_i^{b_i}.$$

Moreover, we can assume that, for each i, at least one of a_i or b_i is positive.

Note that if there are i and j such that $a_i = 0$ and $b_j = 0$, then we can take $k' := k$ and $\ell' := \ell$ and set $p := p_j$ and $q := p_i$ to obtain the desired result. Thus we can assume without loss of generality that $b_i > 0$ for $i \in \{1, \dots, m\}$. Then there is some $i_0 \in \{1, \dots, m\}$ such that $a_{i_0}/b_{i_0} \le a_j/b_j$ for all $j \in \{1, \dots, m\}$. In particular, $c_j := a_j b_{i_0} - b_j a_{i_0}$ is a nonnegative integer for all $j \in \{1, \dots, m\}$. Hence

$$k' := k^{b_{i_0}} \ell^{-a_{i_0}} = \prod_{j=1}^{m} p_j^{c_j} \in \mathbb{N}.$$

Furthermore, p_{i_0} does not divide k' and since k and ℓ are multiplicatively independent, the c_i's are not all equal to zero.

Now we pick $i_1 \in \{1, \dots, m\}$ such that $c_{i_1}/b_{i_1} \ge c_j/b_j$ for all $j \in \{1, \dots, m\}$. Note that $c_{i_1} > 0$ since the c_i's are not all equal to zero. Set

$$\ell' := \ell^{c_{i_1}} (k')^{-b_{i_1}} = \prod_{j=1}^{m} p_j^{b_j c_{i_1} - b_{i_1} c_j} \in \mathbb{N}.$$

Since $c_{i_0} = 0, c_{i_1} > 0$ and the b_i's are positive, we get that p_{i_0} divides ℓ'. Moreover, p_{i_1} does not divide ℓ' while p_{i_1} divides k' for c_{i_1} is positive. In particular, k' and ℓ' are multiplicatively independent. Furthermore, Proposition 8.1 implies that $F(x)$ is both k'- and ℓ'-Mahler. Setting $q := p_{i_0}$ and $p = p_{i_1}$, we obtain that k' and ℓ' have all the desired properties. This concludes the proof. $\square$

9. Asymptotic estimates for some infinite products

In this section, we study the behaviour around the unit circle of infinite products of the form

$$\left(\prod_{i=0}^{\infty} P(x^{k^i}) \right)^{-1},$$

where $P(x) \in \mathbb{C}[x]$ and $P(0) = 1$. We obtain some asymptotic estimates that will be necessary in Section 11.

We will prove that when α is a root of unity satisfying $\alpha^k = \alpha$ that is not a root of P, then this product is rather well-behaved when approaching α through certain well-chosen sets of points. Throughout Sections 9, 10, and 11, we make use of certain subsets of the unit circle having 1 as a limit point. We define these sets now.

Definition 9.1. Let $\varepsilon \in (0, 1)$ and let $\theta \in [-1, 1]$. Then we define

$$X_{\theta,\varepsilon} := \{\exp((-1 + i\theta)s) \mid s \in (0, \varepsilon)\}. \tag{9.1}$$

We take X_θ to be the set $\{0\} \cup \{\exp((-1 + i\theta)s \mid s \geq 0\}$.

We note that each X_θ is a compact subset of the closed unit disk. In fact, X_θ is homeomorphic to $\mathbb{R}_{\geq 0} \cup \{+\infty\}$.

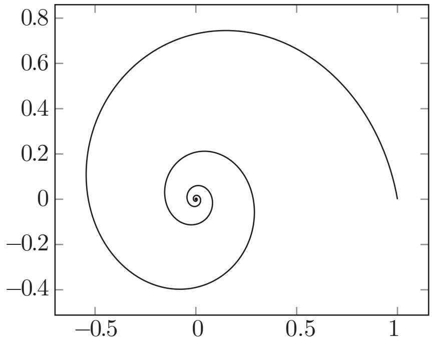

Figure 9.1. This picture of the full set X_θ, with $\theta = 5$, shows the spiral-like structure of the curve.

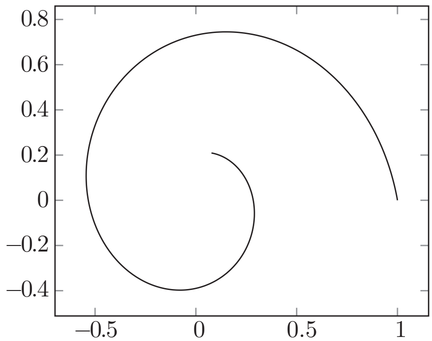

Figure 9.2. This picture shows the set $X_{\theta,\varepsilon}$, where we take $\theta = 5$ and $\epsilon = 1.5$.

Moreover, if $\theta \neq \theta'$, two sets of the form $X_{\theta,\varepsilon}$ and $X_{\theta',\varepsilon'}$ are always disjoint. This can be seen by noting that if $\exp((-1 + i\theta)s) = \exp((-1 + i\theta')s')$, with $\theta, \theta' \in [0, 1]$, then they have the same modulus and hence $s = s'$; next we must have that $\exp(i\theta s) = \exp(i\theta' s)$ and so $(\theta - \theta')s$ must be an integer multiple of 2π, which can only occur if $\theta = \theta'$ since $|\theta - \theta'| \leq 2$ and $0 < s < 1$. Finally, we remark that a set of the form $X_{\theta,\varepsilon}$ has the property that if $y \in X_{\theta,\varepsilon}$ and k is a positive integer then there is a unique point $z \in X_{\theta,\varepsilon}$ such that $z^k = y$.

Proposition 9.2. *Let $k \geq 2$ be a natural number, let α be root of unity that satisfies $\alpha^k = \alpha$, and let $P(x)$ be a nonzero polynomial with $P(0) = 1$ and $P(\alpha) \neq 0$. Then for all but countably many $\theta \in [-1, 1]$, there exist two positive real numbers A and $\varepsilon \in (0, 1)$, depending upon θ, such that*

$$|1-t|^A \ < \ \left| \left(\prod_{j=0}^{\infty} P\Big((t\alpha)^{k^j} \Big) \right)^{-1} \right| \ < \ |1-t|^{-A}$$

whenever $t \in X_{\theta,\epsilon}$.

In contrast, the following result shows that such infinite products behave differently when α is a root of P. In the case where $k = 2$, we point out that a different proof can be found in [5, Théorème 3]. Precise asymptotics for the coefficients of the power series expansion of this infinite product has also been studied by Mahler, de Bruijn, and Dumas and Flajolet (see [15] and the references therein). We give the following proof for the sake of completeness.

Lemma 9.3. *Let $k \geq 2$ be a natural number. Then if $\{t_n\}$ is a sequence of complex numbers with $|t_n| < 1$ for every n such that $t_n \to 1$ as $n \to \infty$ then*

$$\lim_{n\to\infty} \left| \prod_{j=0}^{\infty} \frac{1}{1 - t_n^{k^j}} \right| \cdot |1 - t_n|^A = \infty\,,$$

for every positive real number A.

Proof. By ignoring some initial terms of our sequence, we may assume that $|1 - t_n| \in (0, 1/k^9)$ for every n. Now let $t \in B(0, 1)$ be such that $|1 - t| \in (0, 1/k^9)$. Let $N \geq 2$ be the largest natural number such that $|1 - t| \in (0, k^{-(N+1)^2})$. Then

$$\begin{aligned}
\left| \prod_{j=0}^{\infty} \left(1 - t^{k^j}\right)^{-1} \right| &\geq \left| \prod_{j=0}^{N} \left(1 - t^{k^j}\right)^{-1} \right| \\
&= \left| (1-t)^{-(N+1)} \right| \left| \prod_{j=0}^{N} \left(1 + t + \cdots + t^{k^j - 1}\right)^{-1} \right| \\
&\geq \left| (1-t)^{-(N+1)} \right| \prod_{j=0}^{N} k^{-j} \\
&\geq |1-t|^{-(N+1)} k^{-(N+1)^2} \\
&> |1-t|^{-N}.
\end{aligned}$$

By definition of N, we obtain that $|1 - t| > k^{-(N+2)^2}$, which easily gives that

$$N > \sqrt{\frac{-\log|1-t|}{4\log k}}\,.$$

This ends the proof, for the right-hand side tends to infinity when t tends to 1. $\square$

We are now going to prove Proposition 9.2. We will need the following two auxiliary results.

Lemma 9.4. *Let $k \geq 2$ be a natural number. Then for $t \in (0, 1)$, we have*

$$\sum_{i=1}^{\infty} t^i/i \ \geq \ (1 - 1/k) \sum_{i=0}^{\infty} t^{k^i} .$$

Proof. We have

$$\begin{aligned}
\sum_{i=1}^{\infty} t^i/i &= t + \sum_{i=0}^{\infty} \sum_{j=k^i+1}^{k^{i+1}} t^j/j \\
&\geq t + \sum_{i=0}^{\infty} \sum_{j=k^i+1}^{k^{i+1}} t^{k^{i+1}}/k^{i+1} \\
&= t + \sum_{i=0}^{\infty} t^{k^{i+1}} \left(k^{i+1} - k^i\right)/k^{i+1} \\
&= t + (1 - 1/k) \sum_{i=0}^{\infty} t^{k^{i+1}} \\
&\geq (1 - 1/k) \sum_{i=0}^{\infty} t^{k^i} ,
\end{aligned}$$

which ends the proof. □

Lemma 9.5. *Let $k \geq 2$ be a natural number and let $\lambda \neq 1$ be a complex number. Then for all but countably many $\theta \in [-1, 1]$, there exist two positive real numbers A and $\varepsilon \in (0, 1)$, depending upon θ, such that*

$$|1 - t|^A \ < \ \left| \prod_{j=0}^{\infty} \frac{1}{1 - \lambda t^{k^j}} \right| \ < \ |1 - t|^{-A}$$

whenever $t \in X_{\theta,\varepsilon}$.

Proof. We first prove the inequality on the right-hand side.

We note that for each $j \geq 0$ there are only finitely many complex solutions to the equation $1 - \lambda t^{k^j} = 0$, and thus there are at most countably many solutions as j ranges over all nonnegative integers. As already observed, for $\theta \neq \theta'$ with $\theta, \theta' \in [-1, 1]$ and for $\varepsilon, \varepsilon' \in (0, 1)$, one has $X_{\theta,\varepsilon} \cap X_{\theta',\varepsilon'} = \emptyset$. It follows that for all but countably many values of $\theta \in [-1, 1]$ the equation $1 - \lambda t^{k^j} = 0$ has no solution on $X_{\theta,\varepsilon}$ whenever $\varepsilon \in (0, 1)$. Moreover, since $\lambda \neq 1$, $t = 1$ and $t = 0$ are never a solution, this equation has no solution in X_θ. For the remainder of the proof we assume that $\theta \in [-1, 1]$ has this property.

Observe that X_θ is a compact set that is closed under the map $t \mapsto t^k$ and we have that $1 - \lambda t$ is nonzero for $t \in X_\theta$. By compactness, we see that there exist two positive real numbers ε_0 and c_0, $c_0 < 1$ and depending upon θ, such that

$$\inf\left\{\left|1 - \lambda t^{k^j}\right| : t \in X_{\theta,\varepsilon_0},\ j \geq 0\right\} > c_0\,. \tag{9.2}$$

We fix $t \in X_{\theta,\varepsilon_0}$ and we let $N = N(t)$ to be the largest nonnegative integer such that $|t^{k^N}| \geq 1/2$. Then for $j \geq 1$ we have $\left|t^{k^{N+j}}\right| = \left|(t^{k^{N+1}})^{k^{j-1}}\right| < (1/2)^{k^{j-1}}$. Hence

$$\left|1 - \lambda t^{k^{N+j}}\right| \geq 1 - |\lambda|(1/2)^{k^{j-1}}\,.$$

Since the series $\sum_{j\geq 0}(1/2)^{k^{j-1}}$ converges, we get that the infinite product

$$\prod_{j=N(t)+1}^{\infty} \left|\frac{1}{1 - \lambda t^{k^j}}\right|$$

is uniformly bounded over X_{θ,ε_0} by some constant c_1. (We note that $\lambda \neq 1$ is fixed, $N = N(t)$ depends upon t, $t \in X_{\theta,\varepsilon_0}$, and it is necessary to begin the product at $N + 1$ in order to achieve uniformity in our bound.) Then

$$\begin{aligned}\left|\prod_{j=0}^{\infty}\left(1 - \lambda t^{k^j}\right)^{-1}\right| &= \prod_{j=0}^{N}\left|1 - \lambda t^{k^j}\right|^{-1} \prod_{j=1}^{\infty}\left|1 - \lambda t^{k^{N+j}}\right|^{-1} \\ &\leq (1/c_0)^{N+1} c_1 \\ &= \left(k^{N+1}\right)^{-\log c_0/\log k} c_1\,.\end{aligned}$$

Furthermore, we have by assumption that $\left|t^{k^{N+1}}\right| < 1/2$ and thus $k^{N+1} < -\log 2/\log|t|$. This implies that

$$\left|\prod_{j=0}^{\infty}\left(1 - \lambda t^{k^j}\right)^{-1}\right| \leq c_1\,(-\log 2/\log|t|)^{-\log c_0/\log k}\,.$$

Now we let t tend to 1 along X_{θ,ε_0}, that is we write $t = \exp((-1 + i\theta)s)$ with $s \in (0, \varepsilon_0)$ and we let s tend to 0. Then we have $|t| = \exp(-s)$ and so $\log|t| = -s$. Then when $t \to 1$ along the arc X_{θ,ε_0} we have that $|1 - t|/\log|t|$ tends to

$$\lim_{s\to 0} \frac{|1 - \exp((-1 + i\theta)s)|}{-s} = -|-1 + i\theta| \geq -\sqrt{2}$$

and hence there exists some positive real numbers $\varepsilon < \varepsilon_0$ and c_2 such that

$$c_1\,(-\log 2/\log|t|)^{-\log c_0/\log k} < c_2|1-t|^{\log c_0/\log k}\,,$$

whenever $t \in X_{\theta,\varepsilon}$. Since $c_0 < 1$, we obtain that there exists a positive real number A_1 such that

$$\left|\prod_{j=0}^{\infty}\left(1-\lambda t^{k^j}\right)^{-1}\right| < |1-t|^{-A_1}\,,$$

for all $t \in X_{\theta,\varepsilon}$. This gives the right-hand side bound in the statement of the lemma.

To get the left side, note that for all $t \in X_\theta$,

$$\left|\prod_{j=0}^{\infty}\frac{1}{1-\lambda t^{k^j}}\right| \geq \prod_{j=0}^{\infty}\left|1+|\lambda||t|^{k^j}\right|^{-1} \geq \prod_{j=0}^{\infty}\exp\left(-|\lambda|\cdot|t|^{k^j}\right).$$

By Lemma 9.4, we have

$$\prod_{j=0}^{\infty}\exp\left(-|\lambda|\cdot|t|^{k^j}\right) \geq \exp\left(-|\lambda|(1-1/k)^{-1}\sum_{i=1}^{\infty}|t|^i/i\right) = (1-|t|)^{|\lambda|k/(k-1)}\,.$$

We thus obtain that, for all $t \in X_\theta$,

$$\left|\prod_{j=0}^{\infty}\frac{1}{1-\lambda t^{k^j}}\right| \geq (1-|t|)^{A_2}\,,$$

where $A_2 := \lfloor |\lambda|k/(k-1)\rfloor + 1$. Now we note that, when $t \to 1$ along the arc X_{θ,ε_0}, we have $|1-t|/(1-|t|)$ tends to $|-1+i\theta| \in [1,\sqrt{2}]$, which can be seen by writing $t = \exp((-1+i\theta)s)$ and letting $s \to 0$ and taking limits. Since $\varepsilon < 1$, it follows that there is some positive constant $A_3 > A_2$ for which we have

$$\left|\prod_{j=0}^{\infty}\frac{1}{1-\lambda t^{k^j}}\right| > |1-t|^{A_3}\,,$$

whenever $t \in X_{\theta,\varepsilon}$.

Taking A to be equal to the maximum of A_1 and A_3, we get the desired result. □

Proof of Proposition 9.2. Let $\beta_1,\ldots,\beta_s$ denote the complex roots of P (considered with mutliplicities) so that we may factor $P(x)$ as $P(x) = (1-\beta_1^{-1}x)\cdots(1-\beta_s^{-1}x)$. We thus obtain

$$\left|\prod_{j=0}^{\infty}\frac{1}{P\left((t\alpha)^{k^j}\right)}\right| = \prod_{i=1}^{s}\left|\prod_{j=0}^{\infty}\frac{1}{1-\beta_i^{-1}\alpha t^{k^j}}\right|,$$

where $\beta_i^{-1}\alpha \neq 1$ for every $i \in \{1, \dots, s\}$. Then by Lemma 9.5, there are cocountable[8] subsets Y_i of $[-1, 1]$ such that for a given i and a given $\theta \in Y_i$, there exist a natural number A and a positive real number ε, $0 < \varepsilon < 1$, depending upon θ, such that

$$|1-t|^A < \left| \prod_{j=0}^{\infty} \left(1 - \beta_i^{-1}\alpha t\right)^{-1} \right| < |1-t|^{-A}$$

whenever $t \in X_{\theta,\varepsilon}$. Since the finite intersection of cocountable sets is cocountable, we see that taking $Y = Y_1 \cap \cdots \cap Y_s$, that whenever $\theta \in Y$ we have there exist natural numbers A_i and positive real numbers ε_i, $0 < \varepsilon_i < 1$, depending upon θ, such that

$$|1-t|^{A_i} < \left| \prod_{j=0}^{\infty} \left(1 - \beta_i^{-1}\alpha t\right)^{-1} \right| < |1-t|^{-A_i}$$

whenever $t \in X_{\theta,\varepsilon_i}$. Taking $\varepsilon := \min(\varepsilon_1, \dots, \varepsilon_s)$ and $A := \sum_{i=1}^{s} A_i$, we obtain the desired result. □

10. Asymptotic estimates for solutions of analytic Mahler-type systems

In this section we fix a non-trivial norm $\|\cdot\|$ on $\mathbb{C}^d$. We let $B(x, r)$ (respectively $\overline{B(x, r)}$) denote the open (respectively closed) ball of radius r centered at x. Our results will not depend on the choice of this norm. Throughout this section, we make use of the sets $X_{\theta,\varepsilon}$ and X_θ defined in Definition 9.1.

As defined in Section 7, a Becker function $F(x) \in \mathbb{C}[[x]]$ is an analytic function on the open unit disk satisfying a functional equation of the form:

$$F(x) = \sum_{i=1}^{n} P_i(x) F\left(x^{k^i}\right)$$

for some polynomials $P_1(x), \dots, P_n(x) \in \mathbb{C}[x]$. Of course, such an equation leads to a k-Mahler linear system

$$\begin{pmatrix} F(x) \\ \vdots \\ F(x^{k^{n-1}}) \end{pmatrix} = A(x) \begin{pmatrix} F(x^k) \\ \vdots \\ F(x^{k^n}) \end{pmatrix},$$

where $A(x)$ is an $n \times n$ matrix with polynomial entries. In what follows, we provide an asymptotic lower bound around certain points of the unit circle for solutions of similar systems but associated with more general matrices. Indeed, we consider matrices whose entries are only assumed to be analytic on $B(0, 1)$ and continuous on $\overline{B(0, 1)}$. This result will be used in Section 11.

[8] This means, of course, that the complement of Y_i in $[-1, 1]$ is a countable set.

Proposition 10.1. *Let d and k be two natural numbers, let α be a root of unity such that $\alpha^k = \alpha$ and let $A : \overline{B(0,1)} \to M_d(\mathbb{C})$ be a continuous matrix-valued function. Let us assume that $w(x) \in \mathbb{C}[[x]]^d$ satisfies the equation*

$$w(x) = A(x)w\left(x^k\right)$$

for all $x \in B(0,1)$. Let us also assume that the following properties hold.

(i) *The coordinates of $w(x)$ are analytic in $B(0,1)$ and continuous on $\overline{B(0,1)}$;*
(ii) *The matrix $A(\alpha)$ is not nilpotent;*
(iii) *There exist two positive real numbers ε and M such that $|\det(A(x))| > (1 - |x|)^M$ for every x with $1 - \varepsilon < |x| < 1$;*
(iv) *The set $\{w(x) \mid x \in B(0,1)\}$ is not contained in a proper vector subspace of $\mathbb{C}^d$.*

If ζ is a root of unity such that $\zeta^{k^j} = 1$ for some natural number j and $\theta \in [-1,1]$, then there exist a positive real number C and a subset $S \subseteq X_\theta$ that has 1 as a limit point such that

$$||w(t\alpha\zeta)|| > |1-t|^C$$

for all $t \in S$.

Before proving Proposition 10.1, we will need two auxiliary results.

Lemma 10.2. *Let d and k be two natural numbers, let α be a root of unity such that $\alpha^k = \alpha$, and let $A : \overline{B(0,1)} \to M_d(\mathbb{C})$ be a continuous matrix-valued function. Let us assume that $w(x) \in \mathbb{C}[[x]]^d$ satisfies the equation*

$$w(x) = A(x)w\left(x^k\right)$$

for all $x \in B(0,1)$. Let us also assume that the following properties hold.

(i) *The coordinates of $w(x)$ are analytic in $B(0,1)$ and continuous on $\overline{B(0,1)}$;*
(ii) *The matrix $A(\alpha)$ is not nilpotent;*
(iii) *The set $\{w(x) \mid x \in B(0,1)\}$ is not contained in a proper vector subspace of $\mathbb{C}^d$.*

Then if $\theta \in [-1,1]$, then there exist a positive real number C and a subset $S \subseteq X_\theta$ that has 1 as a limit point such that

$$||w(t\alpha)|| > |1-t|^C$$

for all $t \in S$.

Proof. Since $A(\alpha)$ is not nilpotent, there is some natural number e such that the kernel of $A(\alpha)^e$ and the kernel of $A(\alpha)^{e+1}$ are equal to a same proper subspace of $\mathbb{C}^d$, say W. Then there is a nonzero vector subspace V such that $A(\alpha)(V) \subseteq V$

and $V \oplus W = \mathbb{C}^d$. Moreover, by compactness, there is a positive real number c_0, $c_0 < 1$, such that

$$||A(\alpha)(w)|| \geq c_0 \tag{10.1}$$

whenever $w \in V$ is a vector of norm 1.

Since every vector x has a unique decomposition of the form $v \oplus w$ with v in V and w in W, we see that the map $\pi(x) := v$ gives a continuous linear projection map $\pi : \mathbb{C}^d \to V$ with the property that $u - \pi(u) \in W$ for all $u \in \mathbb{C}^d$. We infer from Inequality (10.1) that

$$||\pi(A(\alpha)(u))|| = \|A(\alpha)(\pi(u))\| \geq c_0||\pi(u)|| \tag{10.2}$$

for all $u \in \mathbb{C}^d$. Since A is continuous on $\overline{B(0,1)}$, Inequality (10.2) implies the existence of a positive constant $\varepsilon > 0$ such that

$$||\pi(A(x)(u))|| > (c_0/2)||\pi(u)||\,,$$

for all $u \in \mathbb{C}^d$ and all $x \in B(\alpha, \varepsilon) \cap \overline{B(0,1)}$. It follows by a simple induction that if $x_1, \ldots, x_m \in B(\alpha, \varepsilon) \cap \overline{B(0,1)}$ then

$$||\pi(A(x_1)\cdots A(x_m)(u))| \geq (c_0/2)^m ||\pi(u)||\,. \tag{10.3}$$

Let $\theta \in [-1, 1]$. We claim that there exists a complex number t_0 such that $t_0 \in X_\theta \cap B(1, \varepsilon)$ and $w(t_0\alpha) \notin W$. Otherwise, there would be a nonzero row vector u such that $u \cdot w(t\alpha) = 0$ for all $t \in X_\theta \cap B(1, \varepsilon)$. But $u \cdot w(x)$ is analytic in $B(0, 1)$ for $w(x)$ is and hence it would be identically zero on $B(0, 1)$ by the identity theorem since $X_\theta \cap B(1, \varepsilon)$ has accumulation points inside the open unit disk. This would contradict assumption (iii).

From now on, we fix a complex number t_0 with this property. For every $i \geq 1$, we then define t_i to be the unique element in X_θ such that $t_i^k = t_{i-1}$. Since $w(t_0\alpha) \notin W$, there exists a positive real number c_1 such that

$$||\pi(w(t_0\alpha))|| = c_1 > 0\,.$$

Furthermore, by construction, the sequence $t_0, t_1, t_2, \ldots$ belongs to $X_\theta \cap B(1, \varepsilon)$. We thus infer from (10.3) that

$$\begin{aligned}
||\pi(w(t_n\alpha))|| &= ||\pi(A(t_n\alpha)A(t_{n-1}\alpha)\cdots A(t_1\alpha)(w(t_0\alpha))|| \\
&\geq (c_0/2)^n||\pi(w(t_0\alpha)|| \\
&\geq c_1(c_0/2)^n\,,
\end{aligned}$$

for all $n \geq 1$. Furthermore, since the projection π is continuous, there is some positive real number c_2 such that $||\pi(u)|| < c_2||u||$ for all $u \in \mathbb{C}^d$. Thus

$$||w(t_n\alpha)|| \geq c_2^{-1}||\pi(w(t_n\alpha))|| \geq c_2^{-1}c_1(c_0/2)^n$$

for all $n \geq 1$.

On the other hand, we note that we have a map $\log : X_\theta \setminus \{0\} \to \mathbb{C}$ given by $\log(\exp((-1+i\theta)s) = (-1+i\theta)s$, and for each positive real number a, we have an a-th power map $X_\theta \to X_\theta$ given by $t \mapsto \exp(a\log(t))$. Since

$$\lim_{a\to 0^+} \frac{t_0^a - 1}{a} = \log(t_0)\,,$$

we have that $|t_0^a - 1|/a|t_0 - 1| \to |\log(t_0)|/|t_0 - 1|$, as $a \to 0^+$. Since t_0 is fixed, we let κ denote the quantity $|\log(t_0)|/|t_0 - 1|$.

Then there exists some $\varepsilon_0 \in (0,1)$ such that

$$|t_0^a - 1| < 2a\kappa|1 - t_0|$$

for $a \in (0, \varepsilon_0)$. Thus if n is large enough, say $n \geq n_0$, then $k^n > 1/\varepsilon_0$ and we have $|t_n - 1| = |(t_0)^{1/k^n} - 1| < 2\kappa|1 - t_0|/k^n$. Hence $k^n > 2\kappa|1 - t_0|/|1 - t_n|$. Then for $n \geq n_0$ we have

$$\begin{aligned} ||w(t_n\alpha)|| &> c_2^{-1}c_1(c_0/2)^n \\ &= c_2^{-1}c_1 k^{n\log_k(c_0/2)} \\ &\geq \left(c_2^{-1}c_1\,(2\kappa|1-t_0|)^{\log_k(c_0/2)}\right)|1-t_n|^{-\log_k(c_0/2)}. \end{aligned}$$

Thus if we take $C := -2\log_k(c_0/2) > 0$, the fact that t_n tends to 1 as n tends to infinity implies the existence of a positive integer $n_1 \geq n_0$ such that

$$||w(t_n\alpha)|| > |1 - t_n|^C\,,$$

for all $n \geq n_1$. Taking $S := \{t_n \mid n \geq n_1\}$, we obtain the desired result. □

Lemma 10.3. *Let $B : \overline{B(0,1)} \to M_d(\mathbb{C})$ be a continuous matrix-valued function whose entries are analytic inside the unit disk and continuous on the closed unit disk. Let us assume that there exist two positive real numbers ε and M such that $|\det(B(x))| > (1 - |x|)^M$ for every x such that $1 - \varepsilon < |x| < 1$. Then there exists a positive real number C such that for every column vector u of norm 1, we have*

$$||B(x)(u)|| \geq (1 - |x|)^C$$

for every x such that $1 - \varepsilon < |x| < 1$.

Proof. Our assumption implies that $B(x)$ is invertible for every x such that $1 - \varepsilon < |x| < 1$. Let $\Delta(x)$ denote the determinant of $B(x)$. Using the classical adjoint formula for the inverse of $B(x)$, we see that $B(x)^{-1}$ has entries $c_{i,j}(x)$ that have the property that they are expressible (up to sign) as the ratio of the determinant of a submatrix of $B(x)$ and $\Delta(x)$. Since the entries of $B(x)$ are continuous on $\overline{B(0,1)}$, each determinant of a submatrix of $B(x)$ is also continuous on $\overline{B(0,1)}$. By compactness, we see that there is a positive real number κ such that

$$|c_{i,j}(x)| \leq \kappa/|\Delta(x)| \leq \kappa(1 - |x|)^{-M}$$

for every $(i, j) \in \{1, \dots, d\}^2$ and every x such that $1 - \varepsilon < |x| < 1$. Thus there exists a positive real number C such that

$$\|B(x)^{-1}\| \leq (1 - |x|)^{-C}$$

for every x such that $1 - \varepsilon < |x| < 1$. It follows that if u is a vector of norm 1, then

$$\|B(x)(u)\| \geq (1 - |x|)^{C},$$

for every x such that $1 - \varepsilon < |x| < 1$. The result follows. □

Proof of Proposition 10.1. Let $\theta \in [-1, 1]$. Since $A(\alpha)$ is not nilpotent, we first infer from Lemma 10.2 that there exist a positive real number C_0 and a sequence $t_n \in X_\theta$, which tends to 1, such that $||w(t_n\alpha)|| > |1 - t_n|^{C_0}$ for every integer $n \geq 1$. Let $s_n \in X_\theta$ be such that $s_n^{k^j} = t_n$. Then

$$w(s_n\alpha\zeta) = A(s_n\alpha\zeta)A\Big(s_n^k\alpha\zeta^k\Big)\cdots A\Big(s_n^{k^{j-1}}\alpha\zeta^{k^{j-1}}\Big)(w(t_n\alpha)).$$

By assumption there exists a positive real number M such that $|\det(A(x))| > (1 - |x|)^M$ for every x with $1 - \varepsilon < |x| < 1$. Set

$$B(x) := A(x\alpha\zeta)A\Big(x^k\alpha\zeta^k\Big)\cdots A\Big(x^{k^{j-1}}\alpha\zeta^{k^{j-1}}\Big).$$

Then there is a positive real number C_1 such that if $(1 - \varepsilon)^{1/k^{j-1}} < |x| < 1$ then

$$\det(B(x)) > (1 - |x|)^M \cdots \Big(1 - |x|^{k^{j-1}}\Big)^M \geq (1 - |x|)^{jM}.$$

It follows from Lemma 10.3 that there exists a positive real number C_1 such that for n sufficiently large we have

$$\begin{aligned} ||w(s_n\alpha\zeta)|| = ||B(s_n)(w(t_n\alpha))|| &> (1 - |s_n|)^{C_1}||w(t_n\alpha)|| \\ &> (1 - |s_n|)^{C_1}|1 - t_n|^{C_0}. \end{aligned}$$

We have that $t_n = \exp((-1 + i\theta)u_n)$ where u_n is a sequence of positive numbers tending to 0. Taking limits, we then see that $|1 - t_n|/(1 - |s_n|) \to |-1 + i\theta| \cdot k^j$ and $|1 - s_n|/(1 - |s_n|) \to |-1 + i\theta|$ as $n \to \infty$. Hence there exists a positive real number C such that

$$||w(s_n\alpha\zeta)|| \geq |1 - s_n|^C$$

for all n sufficiently large. The result follows. □

11. Elimination of singularities at certain roots of unity

In this section we look at the singularities of k-Mahler functions at roots of unity of a certain form. Strictly speaking, we do not necessarily eliminate singularities, and so the section title is perhaps misleading. We do, however, show that one can reduce to the case of considering Mahler equations whose singularities at roots of unity have a restricted form.

Assumption-Notation 11.1. Throughout this section we make the following assumptions and use the following notation.

(a) We assume that k and l are integers, $k, l \geq 2$, for which: there exists a prime p such that $p|k$ and p does not divide ℓ, and there exists a prime q such that $q|\ell$ and q does not divide k. In particular, k and ℓ are two multiplicatively independent integers;

(b) We assume that $F(x)$ is a k-Mahler complex power series that satisfies an equation of the form

$$\sum_{i=0}^{a} A_i(x) F\left(x^{k^i}\right) = 0$$

with $A_0, \ldots, A_a \in \mathbb{C}[x]$ and $A_0(0) \neq 0$;

(c) We assume that $F(x)$ is an ℓ-Mahler complex power series that satisfies an equation of the form

$$\sum_{i=0}^{b} B_i(x) F\left(x^{\ell^i}\right) = 0$$

with $B_0, \ldots, B_b \in \mathbb{C}[x]$ and $B_0(0) \neq 0$.

In this section our aim is to prove the following result. It will be a key result for proving Theorem 1.3.

Theorem 11.2. *Let $F(x) \in \mathbb{C}[[x]]$ be a power series that satisfies Assumption-Notation* 11.1 *and that is not a polynomial. Then $F(x)$ satisfies a non-trivial k-Mahler equation of the form*

$$\sum_{i=0}^{c} P_i(x) F\left(x^{k^i}\right) = 0,$$

with the property that $P_0(0) = 1$ and $P_0(\alpha) \neq 0$ if α is a root of unity satisfying $\alpha^{k^j} = \alpha$ for some positive integer j.

Though this result is of a purely algebraic nature, our proof relies heavily on analytic methods. One may ask whether a purely algebraic proof exists.

Strategy of proof. Using Assumptions (b) and (c), Proposition 7.10 leads to two different expressions for F:

$$F(x) = \left(\prod_{j=0}^{\infty} A_0\left(x^{k^j}\right)\right)^{-1} H(x) \quad \text{and} \quad F(x) = \left(\prod_{j=0}^{\infty} B_0\left(x^{\ell^j}\right)\right)^{-1} G(x),$$

where H is k-Becker and G is ℓ-Becker. This gives:

$$\left(\prod_{j=0}^{\infty} A_0\left(x^{k^j}\right)\right)^{-1} = \left(\prod_{j=0}^{\infty} B_0\left(x^{\ell^j}\right)\right)^{-1} G(x)H(x)^{-1}.$$

We want to argue by contradiction assuming that A_0 has a root α satisfying $\alpha^{k^{i_0}} = \alpha$ for some positive integer i_0. The main idea is to use the asymptotics of Sections 7, 9, and 10 in order to show that the absolute values of the left-hand side and the right-hand side of the above Equality behave really differently in some neighbourhood of α, providing a contradiction. However, there are several technical difficulties and the proof will be divided into seven steps, as briefly described below.

In Step 1, we will first replace, for technical reasons, F by some function F_0 and the Equality above will be consequently replaced by

$$\left(\prod_{j=0}^{\infty} \widetilde{Q}_0\left(x^{k^{i_0 j}}\right)\right)^{-1} = \left(\prod_{j=0}^{\infty} R_0\left(x^{\ell^j}\right)\right)^{-1} G(x)H(x)^{-1}, \tag{11.1}$$

where $\widetilde{Q}_0$ is a polynomial satisfying $\widetilde{Q}_0(\alpha) = 0$ and $\alpha^{k^{i_0}} = \alpha$. Again for technical reasons, we will also have to replace the point α by $\alpha\zeta_0$, where ζ_0 is some well-chosen p^n-th root of unity (the choice of ζ_0 is made in Step 3). Here, p denotes the prime from Assumption (a).

At this point, one could use the results of Sections 7 and 9 to derive upper bounds showing that both $\left|\left(\prod_{j=0}^{\infty} R_0(x^{\ell^j})\right)^{-1}\right|$ and $|G(x)|$ do not grow too fast in some neighbourhood of the point $\alpha\zeta_0$. In contrast, it follows from Lemma 9.3 and Proposition 9.2 that $\left|\left(\prod_{j=0}^{\infty} \widetilde{Q}_0(x^{k^{i_0 j}})\right)^{-1}\right|$ becomes much bigger at certain well-chosen points near this point since $\widetilde{Q}_0(\alpha) = 0$ and $\alpha^{k^{i_0}} = \alpha$. This would be enough to derive a contradiction if we were able to obtain a lower bound for $|H(x)|$ around $\alpha\zeta_0$. Since H is a k-Becker function, it is easy to obtain a general upper bound (as we will do for G in Step 5), but we cannot obtain a suitable lower bound because the matrix associated with the underlying linear system of functional equations could be nilpotent.

In order to overcome this difficulty, we will replace H by the function $L(x) := H(x)\left(\prod_{j=0}^{\infty}(1-\alpha^{-1}x^{k^{i_0 j}})^r\right)$, for some well-chosen rational parameter r. The choice of the parameter r will be given at Step 2. Once this last modification is made, one obtains, instead of Equality (11.1), an equality of the form:

$$\left|\prod_{j=0}^{\infty}\left(1-\alpha^{-1}x^{k^{i_0 j}}\right)^{-b}\right| = \left|\left(\prod_{j=0}^{\infty} R_0\left(x^{\ell^j}\right)\right)^{-1} G(x)\left(\prod_{j=0}^{\infty} S_0\left(x^{k^{i_0 j}}\right)\right) L(x)^{-1}\right|,$$

where S_0 is some polynomial and b is positive. It corresponds to Equality (11.8) in the proof.

In step 3, we will show that our choice of r allows to derive a suitable lower bound for $|L(x)|$ around $\alpha\zeta_0$ by applying Proposition 10.1. On the other hand, in Steps 4, 5, and 6, we will use the results of Sections 7 and 9 in order to provide suitable upper bounds for $\left|\left(\prod_{j=0}^{\infty} R_0(x^{\ell^j})\right)^{-1}\right|$, $|G(x)|$, and $\left|\prod_{j=0}^{\infty} S_0(x^{k^{i_0 j}})\right|$ around $\alpha\zeta_0$.

In step 7, we will finally gather all the bounds obtained in Steps 3, 4, 5, and 6 in order to deduce that, around $\alpha\zeta_0$, the right-hand side of Equality (11.8) is much smaller than the left-hand side should be according to Lemma 9.3. This will provide the desired contradiction.

With the preliminary results of Sections 6, 7, 9, and 10, we are now almost ready to prove Theorem 11.2. Before doing this, we give the following simple lemma. We recall that the Kronecker symbol $\delta_{i,j}$ is defined, as usual, by $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

Lemma 11.3. *Let d be a natural number and let A be a $d \times d$ complex matrix whose (i, j)-entry is $\delta_{i,j+1}$ if $i \geq 2$. If there is an integer r such that the $(1, r)$-entry of of A is nonzero, then A is not nilpotent.*

Proof. Let $(a_1, \ldots, a_d)$ denote the first row of A. Then by the theory of companion matrices, A has characteristic polynomial $x^d - a_1x^{d-1} - a_2x^{d-2} - \cdots - a_d$. But if A is nilpotent, its characteristic polynomial must be x^d and hence the first row of A must be zero. □

Proof of Theorem 11.2. Consider the set I of all polynomials $P(x) \in \mathbb{C}[x]$ for which there exist positive integers a and b with $0 < a < b$ such that

$$P(x)F(x) \in \sum_{j=a}^{b} \mathbb{C}[x]F\left(x^{k^j}\right).$$

We note that I is an ideal of $\mathbb{C}[x]$. Let $P_0(x)$ be a generator for I. It follows from assumption (b) that $P_0(0) \neq 0$ and we can assume without loss of generality that $P_0(0) = 1$. Let us assume that α is a root of $P_0(x)$ with the property that $\alpha^{k^{i_0}} = \alpha$ for some positive integer i_0. We will obtain a contradiction from this assumption, which will prove the theorem.

Step 1 (*preliminaries*). Since $F(x)$ is k-Mahler, it is also k^{i_0}-Mahler and hence $F(x)$ satisfies a non-trivial polynomial equation of the form

$$\sum_{j=0}^{d} Q_j(x) F\Big(x^{k^{i_0 j}}\Big) = 0$$

with $Q_0, \ldots, Q_d$ polynomials and $Q_0(x) Q_d(x) \neq 0$. We pick such a nontrivial relation with Q_0 nonzero and the degree of Q_0 minimal. By assumption P_0 divides Q_0 and so α is a root is of $Q_0(x)$. The minimality of the degree of Q_0 also implies that $\gcd(Q_0(x), \ldots, Q_d(x)) = 1$. By Lemma 6.1, there exists some natural number N such that $F(x)$ can be decomposed as $F(x) = T(x) + x^N F_0(x)$, where $T(x)$ is a polynomial of degree $N - 1$ and $F_0(x)$ is a power series with nonzero constant term such that $F_0(x)$ satisfies a k^{i_0}-Mahler equation of the form

$$\sum_{j=0}^{e} \widetilde{Q}_j(x) F_0\Big(x^{k^{i_0 j}}\Big) = 0 \tag{11.2}$$

with $\widetilde{Q}_0(0) = 1$, $\widetilde{Q}_0(\alpha) = 0$ and $\widetilde{Q}_{j_0}(\alpha) \neq 0$ for some integer j_0, $0 < j_0 \leq e$. Moreover, by picking N sufficiently large, we may assume that $F_0(x)$ satisfies a nontrivial ℓ-Mahler equation

$$\sum_{j=0}^{f} R_j(x) F_0\Big(x^{\ell^j}\Big) = 0$$

for some polynomials $R_j(x)$ with $R_0(0) = 1$. Now, we infer from Proposition 7.10 that there is some ℓ-Becker power series $G(x)$ such that

$$F_0(x) = \left(\prod_{j=0}^{\infty} R_0\Big(x^{\ell^j}\Big)\right)^{-1} G(x), \tag{11.3}$$

and that there is some k-Becker power series $H(x)$ such that

$$F_0(x) = \left(\prod_{j=0}^{\infty} \widetilde{Q}_0\Big(x^{k^{i_0 j}}\Big)\right)^{-1} H(x)\,. \tag{11.4}$$

Step 2 (*Choice of the parameter* r). For $j = 0, \ldots, e$, we let c_j denote the order of vanishing of $\widetilde{Q}_j(x)$ at α, with the convention that $c_j = \infty$ if $\widetilde{Q}_j(x) = 0$. We note that by assumption $0 < c_0 < \infty$ and $c_{j_0} = 0 < c_0$. Let

$$b := \max\left\{ \frac{c_0 - c_j}{j} \mid j = 1, \ldots, e \right\}. \tag{11.5}$$

Since at least one of $c_1, \dots, c_d$ is strictly less than c_0, we have that b is positive. Moreover, by definition there is some $j_1 \in \{1, \dots, e\}$ such that $c_{j_1} + bj_1 - c_0 = 0$. Then, for $j \in \{0, \dots, e\}$, we set

$$S_j(x) := \widetilde{Q}_j(x) \left(\prod_{n=0}^{j-1} \left(1 - \alpha^{-1} x^{k^{i_0 n}}\right)^b \right) \left(1 - \alpha^{-1} x\right)^{-c_0}. \tag{11.6}$$

Note that (11.5) implies that $S_0(x)$ is a polynomial in $\mathbb{C}[x]$ such that $S_0(0) = 1$ and $S_0(\alpha) \neq 0$.

Now, we set

$$L(x) := H(x) \left(\prod_{j=0}^{\infty} \left(1 - \alpha^{-1} x^{k^{i_0 j}}\right)^{b-c_0} \right). \tag{11.7}$$

In other words, we choose $r := b - c_0$. Then we infer from Equalities (11.3), (11.4), (11.6), and (11.7) that

$$\prod_{j=0}^{\infty} \left(1 - \alpha^{-1} x^{k^{i_0 j}}\right)^{-b} = \left(\prod_{j=0}^{\infty} R_0\left(x^{\ell^j}\right) \right)^{-1} G(x) \left(\prod_{j=0}^{\infty} S_0\left(x^{k^{i_0 j}}\right) \right) L(x)^{-1}. \tag{11.8}$$

Step 3 (*Upper bound for* $|L(x)|^{-1}$). We first infer from (11.2) and (11.7) that the function L satisfies the following relation:

$$\sum_{n=0}^{e} \widetilde{Q}_n(x) \left(\prod_{j=n}^{\infty} S_0\left(x^{k^{i_0 j}}\right)^{-1} \right) \left(\prod_{j=n}^{\infty} \left(1 - \alpha^{-1} x^{k^{i_0 j}}\right)^{-b} \right) L\left(x^{k^{i_0 n}}\right) = 0,$$

which gives by (11.6) that

$$\begin{aligned} L(x) &= -\sum_{n=1}^{e} \left(\widetilde{Q}_n(x) \widetilde{Q}_0(x)^{-1} \prod_{j=0}^{n-1} S_0\left(x^{k^{i_0 j}}\right) \prod_{j=0}^{n-1} \left(1 - \alpha^{-1} x^{k^{i_0 j}}\right)^b \right) L\left(x^{k^{i_0 n}}\right) \\ &= -\sum_{n=1}^{e} \left(S_n(x) \prod_{j=1}^{n-1} S_0\left(x^{k^{i_0 j}}\right) \right) L\left(x^{k^{i_0 n}}\right). \end{aligned}$$

Let $A(x)$ denote the $e \times e$ matrix whose (i, j)-entry is $\delta_{i,j+1}$ if $i \geq 2$ and whose $(1, j)$-entry is

$$C_j(x) := -S_n(x) \prod_{j=1}^{n-1} S_0\left(x^{k^{i_0 j}}\right)$$

for $j = 1, \dots, e$. Then the previous computation gives us the following functional equation:

$$\left[L(x), L\left(x^{k^{i_0}}\right), \dots, L\left(x^{k^{i_0(e-1)}}\right)\right]^T = A(x)\left[L\left(x^{k^{i_0}}\right), \dots, L\left(x^{k^{i_0 e}}\right)\right]^T, \tag{11.9}$$

where ^{T} denotes the transpose.

In order to obtain the desired upper bound (namely, Inequality (11.11) that will be stated in the sequel), we are going to apply Proposition 10.1. We thus start by showing that the vector $[L(x), L(x^{k^{i_0}}), \ldots, L(x^{k^{i_0(e-1)}})]^T$ and the matrix $A(x)$ satisfy the assumptions (i), (ii), (iii), and (iv) of this proposition. We first note that $L(x)$ is not identically zero since $F(x)$ is not a polynomial. Furthermore, we assume that L is not a nonzero constant since otherwise the desired upper bound (11.11) would be immediately satisfied.

(i) By definition,

$$S_n(x) = \widetilde{Q}_n(x) \left(\prod_{j=0}^{n-1} \left(1 - \alpha^{-1} x^{k^{i_0 j}}\right)^b \right) (1 - \alpha^{-1} x)^{-c_0} .$$

Moreover, a simple computation gives that

$$\prod_{j=0}^{n-1} \left(1 - \alpha^{-1} x^{k^{i_0 j}}\right)^b = \left(1 - \alpha^{-1} x\right)^{bn} P_n(x)^b ,$$

for some polynomial $P_n(x)$ that does not vanish at α. By definition of c_n, this shows that

$$S_n(x) = \left(1 - \alpha^{-1} x\right)^{c_n + bn - c_0} P_n(x)^b R_n(x) , \tag{11.10}$$

where $P_n(x)$ and $R_n(x)$ are two polynomials that do not vanish at α. By the definition of b in (11.5), we have $c_n + bn - c_0 \geq 0$ for $n \in \{0, \ldots, e\}$, and thus $S_n(x)$ is analytic in the open unit disk and continuous on the closed unit disk. Since the finite product $\prod_{j=1}^{n-1} S_0(x^{k^{i_0 j}})$ is a polynomial, this shows that the entries of the matrix $A(x)$ are analytic on $B(0, 1)$ and continuous on $\overline{B(0, 1)}$;

(ii) As already observed, there is some integer j_1, $1 \leq j_1 \leq e$, such that $c_{j_1} + b j_1 - c_0 = 0$. Since $P_{j_1}(\alpha) R_{j_1}(\alpha) \neq 0$, Equation (11.10) implies that $S_{j_1}(\alpha) \neq 0$. On the other hand, we have that $\prod_{j=0}^{j_1-1} S_0(x^{k^{i_0 j}})$ does not vanish at α since $S_0(\alpha) \neq 0$ and $\alpha^{k^{i_0}} = \alpha$. We thus obtain that the $(1, j_1)$-entry of $A(\alpha)$ is nonzero. By Lemma 11.3, this implies that $A(\alpha)$ is not nilpotent;

(iii) By definition of the matrix A, we get that

$$\det A(x) = (-1)^e C_e(x) = (-1)^{e+1} S_e(x) \prod_{n=1}^{e-1} S_0\left(x^{k^{i_0 n}}\right) .$$

By (11.10), we have that $S_e(x) = (1 - \alpha^{-1} x)^{c_e + be - c_0} P_e(x)^b R_e(x)$, where $P_e(x)$ and $R_e(x)$ are polynomials. It follows that there exist two positive real numbers δ and M such that

$$|\det A(x)| > (1 - |x|)^M$$

for every x such that $1 - \delta < |x| < 1$;

(iv) We claim that

$$\left\{\left[L(x), L\left(x^{k^{i_0}}\right), \dots, L\left(x^{k^{i_0(e-1)}}\right)\right]^T \mid x \in B(0,1)\right\}$$

cannot be contained in a proper subspace of $\mathbb{C}^e$. Indeed, if it were, then there would exist some nonzero row vector u such that

$$u\left[L(x), L\left(x^{k^{i_0}}\right), \dots, L\left(x^{k^{i_0(e-1)}}\right)\right]^T = 0$$

for all $x \in B(0,1)$. But this would give that $L(x), \dots, L(x^{k^{i_0(e-1)}})$ are linearly dependent over $\mathbb{C}$, and hence by Lemma 7.9, we would obtain that $L(x)$ is a constant function, a contradiction.

It follows from (i), (ii), (iii) and (iv) that we can apply Proposition 10.1 to the vector $[L(x), L(x^{k^{i_0}}), \dots, L(x^{k^{i_0(e-1)}})]^T$. From now on, we fix a positive integer N_0 that will be assume to be large enough in step 4. Let μ be a primitive p^n-th root of unity with $n \geq N_0 + i_0(e-1)\nu_p(k)$. Here, $\nu_p(k)$ denotes the p-adic valuation of k and p is the prime number from assumption (a). By Proposition 10.1, for every $\theta \in [-1,1]$, there exist a positive integer M_0 and an infinite sequence $(t_\theta(n))_{n\geq 0} \in X_\theta \setminus \{1\}$ (denoted by $(t(n))_{n\geq 0}$ for short) which tends to 1 such that

$$\left\|\left[L(t(n)\alpha\mu), L\left(t(n)^{k^{i_0}}\alpha\mu^{k^{i_0}}\right), \dots, L\left(t(n)^{k^{i_0(e-1)}}\alpha\mu^{k^{i_0(e-1)}}\right)\right]^T\right\| > |1-t(n)|^{M_0},$$

for every nonnegative integer n. By the pigeonhole principle, we can find an integer $n_0 \geq N_0$, a primitive p^{n_0}-th root of unity ζ_0, such that for every $\theta \in [-1,1]$ there exist a sequence $(s_\theta(n))_{n\geq 0}$ in $X_\theta \setminus \{1\}$ which tends to 1, and a positive integer A_1 (depending upon θ) satisfying

$$|L(s_\theta(n)\alpha\zeta_0)|^{-1} < |1-s_\theta(n)|^{-A_1} \tag{11.11}$$

for every positive integer n.

Remark 11.4. We fix the p^{n_0}-th root of unity ζ_0 once for all.

Step 4 (*Upper bound for* $|(\prod_{j\geq 0} R_0(x^{\ell^j}))^{-1}|$). From assumption (a), we get that if N_0 is large enough, then $R_0((\alpha\zeta_0)^{\ell^j}) \neq 0$ for every $j \geq 0$. Let n_1 and n_2, $n_1 < n_2$, be two positive integers such that

$$(\alpha\zeta_0)^{\ell^{n_1}} = (\alpha\zeta_0)^{\ell^{n_2}}. \tag{11.12}$$

Then for every $\theta \in [-1,1]$ and $t \in X_\theta \setminus \{0,1\}$ we have

$$\prod_{j=0}^{\infty} R_0\left((t\alpha\zeta_0)^{\ell^j}\right) = \prod_{j=0}^{n_1-1} R_0\left((t\alpha\zeta_0)^{\ell^j}\right) \prod_{i=n_1}^{n_2-1}\prod_{j=0}^{\infty} R_0\left(\left((t\alpha\zeta_0)^{\ell^i}\right)^{\ell^{j(n_2-n_1)}}\right).$$

Note that $\prod_{j=0}^{n_1-1} R_0(x^{\ell^j})$ is a polynomial that does not vanish at any point of the finite set $\left\{(\alpha\zeta_0)^{\ell^j}) \mid j \geq 0\right\}$. It follows that, for every $\theta \in [-1, 1]$, there exist two positive real numbers C_1 and ε_1 such that

$$\left| \left(\prod_{j=0}^{n_1-1} R_0\Big(t\alpha\zeta_0)^{\ell^j}\Big) \right)^{-1} \right| < C_1 \,,$$

for all $t \in X_{\theta,\varepsilon_1}$. Furthermore, Equality (11.12) implies that for every integer i, $n_1 \leq i \leq n_2 - 1$, we have

$$\left((\alpha\zeta_0)^{\ell^i} \right)^{\ell^{j(n_2-n_1)}} = \left((\alpha\zeta_0)^{\ell^i} \right) .$$

Thus, for every integer i, $n_1 \leq i \leq n_2 - 1$, we can apply Proposition 9.2 to the infinite product

$$\left(\prod_{j=0}^{\infty} R_0 \left(\left((t\alpha\zeta_0)^{\ell^i} \right)^{\ell^{j(n_2-n_1)}} \right) \right)^{-1} .$$

This implies the existence of a cocountable subset Y_1 of $[-1, 1]$ such that for each $\theta \in Y_1$, there is a positive real number ε_2 and a positive integer A_2, both of which depend upon θ, such that

$$\left| \left(\prod_{j=0}^{\infty} R_0 \left((t\alpha\zeta_0)^{\ell^j} \right) \right)^{-1} \right| < |1 - t|^{-A_2} \tag{11.13}$$

for every $t \in X_{\theta,\varepsilon_2}$.

Step 5 (*Upper bound for* $|G(x)|$). Note first that, since $G(x)$ is a ℓ-Becker power series, Theorem 7.6 implies that $G(x)$ is ℓ-regular. By Proposition 7.5, there exist two positive real numbers C and m such that

$$|G(x)| < C(1 - |x|)^{-m} \,,$$

for every complex number x in the open unit disk. This implies that there exist two positive real numbers ε_3 and A_3 such that

$$|G(x)| < (1 - |x|)^{-A_3} \tag{11.14}$$

for every complex number x with $1 - \varepsilon_3 < 1 - |x| < 1$.

Step 6 (*Upper bound for* $|\prod_{j\geq 0} S_0(x^{k^{i_0 j}})|$). First note that since $\alpha^{k^{i_0}} = \alpha$, $S_0(0) = 1$ and α is not a root of S_0, we can apply Proposition 9.2. We thus obtain the existence of a cocountable subset $Y_2 \subseteq [-1, 1]$ such that for each $\theta \in Y_2$, there is some positive real numbers δ_0 and a positive integer M_0 such that

$$\left| \prod_{j=0}^{\infty} S_0\Big((t\alpha)^{k^{i_0 j}} \Big) \right| < |1-t|^{M_0} \tag{11.15}$$

for every $t \in X_{\theta,\delta_0}$. Henceforth, we assume that we have selected $\theta \in Y_1 \cap Y_2$ and we assume that Equations (11.13) and (11.15) hold – this holds precisely when $t \in X_{\theta,\varepsilon_2} \cap X_{\theta,\delta_0} = X_{\theta,\min(\epsilon_2,\delta_0)}$.

We also note that $(\alpha\zeta_0)^{k^{i_0 j}} = \alpha$ for all $j \geq n_0$. This implies that

$$\prod_{j=0}^{\infty} S_0\Big((t\alpha\zeta_0)^{k^{i_0 j}} \Big) = R(t) \prod_{j=0}^{\infty} S_0\Big((t\alpha)^{k^{i_0 j}} \Big), \tag{11.16}$$

where

$$R(t) = \left(\prod_{j=0}^{n_0-1} S_0\Big((t\alpha\zeta_0)^{k^{i_0 j}} \Big) \right) \left(\prod_{j=0}^{n_0-1} S_0\Big((t\alpha)^{k^{i_0 j}} \Big) \right)^{-1}.$$

Since $\alpha^{k^{i_0 j}} = \alpha$ and $S_0(\alpha) \neq 0$, then, for every $\theta \in Y_2$, there are two positive real numbers δ_1 and C_2 such that

$$|R(t)| < C_2 \tag{11.17}$$

for every $t \in X_{\theta,\delta_1}$.

We thus infer from (11.15), (11.16), and (11.17) that for every $\theta \in Y_2$ there exist a positive real number ε_4 and a positive integer A_4, both of which depend upon θ, such that

$$\left| \prod_{j=0}^{\infty} S_0\Big((t\alpha\zeta_0)^{\ell^j} \Big) \right| < |1-t|^{-A_4} \tag{11.18}$$

for $t \in X_{\theta,\varepsilon_4}$.

Step 7 (*Conclusion*). Set

$$\Pi(x) := \left(\prod_{j=0}^{\infty} R_0\Big(x^{\ell^j} \Big) \right)^{-1} G(x) \left(\prod_{j=0}^{\infty} S_0\Big(x^{k^{i_0 j}} \Big) \right) L(x)^{-1}.$$

Let us fix a real number $\theta \in Y_1 \cap Y_2$. Collecting all the upper bounds obtained in (11.14), (11.13), (11.18), and (11.11), we obtain that

$$|\Pi(s_\theta(n))\alpha\zeta_0)| < |1 - s_\theta(n))|^{-(A_1+A_2+A_3+A_4)}$$

for every integer n large enough. We thus infer from Equality (11.8) that

$$\left|\prod_{j=0}^{\infty}\left(1-(s_\theta(n)\zeta_0)^{k^{i_0 j}}\right)^{-b}\right| = |\Pi(s_\theta(n)\alpha\zeta_0)| < |1-s_\theta(n)|^{-(A_1+A_2+A_3+A_4)}$$

for every integer n large enough. But this contradicts Lemma 9.3, since $\zeta_0^{k^j} = 1$ for all sufficiently large j. This concludes the proof. □

12. Existence of good prime ideals

In this section we prove the following result.

Theorem 12.1. *Let R be a ring of the form $\mathcal{O}_K[1/M]$, where K denotes a number field and M denotes a positive integer. Let $P(x), Q(x) \in R[x]$ be two polynomials with $P(0) = Q(0) = 1$ and such that none of the zeros of $P(x)Q(x)$ are roots of unity. Let k and l be two integers, $k, l \geq 2$, for which: there exists a prime p such that $p|k$ and p does not divide ℓ, and there exists a prime q such that $q|\ell$ and q does not divide k. Then there are infinitely many prime ideals $\mathfrak{P}$ in R such that*

$$\left(\prod_{i=0}^{\infty} P\left(x^{k^i}\right)\right)^{-1} \bmod \mathfrak{P} \quad \text{and} \quad \left(\prod_{i=0}^{\infty} Q\left(x^{\ell^i}\right)\right)^{-1} \bmod \mathfrak{P}$$

are respectively k- and ℓ-automatic power series in $(R/\mathfrak{P})[[x]]$.

We do not know whether the conclusion to the statement of Theorem 12.1 holds if we allow P or Q to vanish at roots of unity, but we suspect that the statement is false in this setting.

Our proof is based on Chebotarev's density theorem for which we refer the reader for example to [22] and to the informative survey [23]. We first prove three auxiliary results.

Lemma 12.2. *Let K be a number field and let α be a nonzero element in K that is not a root of unity. Then for all sufficiently large natural numbers n the equation $\beta^n = \alpha$ has no solution $\beta \in K$.*

Proof. Let $\mathcal{O}_K$ be the ring of integers of K. Each nonzero prime ideal π of $\mathcal{O}_K$ gives rise to a rank one discrete valuation ν_π of the field K. Notice that if $\beta^n = \alpha$ then $\nu_\pi(\alpha) = n\nu_\pi(\beta)$. In particular, if there exists some prime π for which $\nu_\pi(\alpha)$ is nonzero then we see that, in the equation $\beta^n = \alpha$, n must divide $\nu_\pi(\alpha)$ and we get the result. We may write $\alpha = a/b$ with $a, b \in \mathcal{O}_K$, nonzero. Notice that since $\mathcal{O}_K$ is a Dedekind domain, the ideals (a) and (b) must factor into prime ideals. Now if (a) or (b) are different ideals, then there must be some nonzero prime ideal π of R for which the induced valuation of $\alpha = a/b$ is nonzero. The previous remark thus shows that we must have $(\alpha) = \alpha\mathcal{O}_K = \mathcal{O}_K$. We thus may assume without loss of generality that α is a unit in R. But if $\beta^n = \alpha$ then, since $\mathcal{O}_K$ is integrally closed, we must have $\beta \in \mathcal{O}_K$ and β must be a unit. By Dirichlet's unit theorem, the group of units of $\mathcal{O}_K$ is a finitely generated abelian group. Hence if $\beta^n = \alpha$ for infinitely many n, then α must be a torsion element of the units group. That is, α must be a root of unity, which ends the proof. □

Lemma 12.3. *Let m be a natural number and let $d_1, \ldots, d_m$ be positive integers. Suppose that H is a subgroup of $\prod_{i=1}^m (\mathbb{Z}/d_i\mathbb{Z})$ with the property that there exist natural numbers $r_1, \ldots, r_m$ with*

$$1/r_1 + \cdots + 1/r_m < 1$$

such that for each $i \in \{1, \ldots, m\}$, there is an element $h_i \in H$ whose i-th coordinate has order r_i. Then there is an element $h \in H$ such that no coordinate of h is equal to zero.

Proof. For each $i \in \{1, \ldots, m\}$, we let

$$\pi_i : \prod_{i=1}^m (\mathbb{Z}/d_i\mathbb{Z}) \to \mathbb{Z}/d_i\mathbb{Z}$$

denote the projection onto the i-th coordinate. Given $(x_1, \ldots, x_m) \in \mathbb{Z}^m$ we have that $x_1h_1 + \cdots + x_mh_m \in H$. Observe that the density of integers y for which

$$\pi_i\left(\sum_{j\neq i} x_jh_j + yh_i\right) = 0$$

is equal to $1/r_i$. Since this holds for all $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) \in \mathbb{Z}^{m-1}$, we see that the density of $(x_1, \ldots, x_m) \in \mathbb{Z}^m$ for which

$$\pi_i\left(\sum_{j=1}^m x_jh_j\right) = 0$$

is equal to $1/r_i$. Thus the density of $(x_1, \ldots, x_m) \in \mathbb{Z}^m$ for which

$$\pi_i\left(\sum_{j=1}^m x_jh_j\right) = 0$$

holds for some $i \in \{1, \ldots, m\}$ is at most

$$1/r_1 + \cdots + 1/r_m < 1\,.$$

In particular, we see that there is some $(x_1, \ldots, x_m) \in \mathbb{Z}^m$ such that the element $h := x_1h_1 + \cdots + x_mh_m \in H$ has no coordinate equal to zero. □

Lemma 12.4. *Let $k \geq 2$ be an integer, let R be a ring of the form $\mathcal{O}_K[1/M]$, where K denotes a number field and M denotes a positive integer, let $\mathfrak{P}$ be a nonzero prime ideal of R, and let a be an element of R. Suppose that for some natural number n, the polynomial $1 - ax^{k^n} \bmod \mathfrak{P}$ has no roots in $R/\mathfrak{P}$. Then the infinite product*

$$\left(\prod_{j=0}^{\infty}\left(1 - ax^{k^j}\right)\right)^{-1} \bmod \mathfrak{P}$$

is a k-automatic power series in $(R/\mathfrak{P})[[x]]$.

Proof. Set $F(x) := \prod_{j=0}^{\infty}(1 - ax^{k^j})^{-1} \bmod \mathfrak{P}$. Without loss of generality we can assume that a does not belong to $\mathfrak{P}$. Let us first note that the sequence $a, a^k, a^{k^2}, \ldots$ is necessarily eventually periodic modulo $\mathfrak{P}$. However, it cannot be periodic, as otherwise the polynomial $1 - ax^{k^n}$ would have a root for every natural number n. Thus there exists a positive integer N such that

$$a \not\equiv a^{k^N} \equiv a^{k^{2N}} \mod \mathfrak{P}.$$

Set $b := a^{k^N}$ and let us consider the polynomial

$$Q(x) := (1 - bx)\big(1 - bx^k\big) \cdots \left(1 - bx^{k^{N-1}}\right).$$

Now arguing exactly as in the proof of Proposition 7.8, we see that there exists a polynomial $S(x) \in R[x]$ such that $G(x) := Q(x)^{-1}F(x)$ satisfies the equation

$$G(x) \equiv S(x)G\big(x^k\big) \mod \mathfrak{P}.$$

Thus Theorem 7.6 implies that $G(x) \bmod \mathfrak{P}$ is a k-regular power series in $(R/\mathfrak{P})[[x]]$. By Proposition 7.4, we see that $F(x) \bmod \mathfrak{P}$ is a k-regular power series since it is a product of a polynomial (which is k-regular) and a k-regular power series. Since the base field is finite, Proposition 7.4 gives that $F(x) \bmod \mathfrak{P}$ is actually a k-automatic power series. This ends the proof. □

Proof of Theorem 12.1. By assumption R is of the form $\mathcal{O}_K[1/M]$, where K denotes a number field and M denotes a positive integer. Let L be the Galois extension of K generated by all complex roots of the polynomial $P(x)Q(x)$. Thus there are $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_e \in L$ such that $P(x) = (1-\alpha_1 x)\cdots(1-\alpha_d x)$ and $Q(x) = (1-\beta_1 x)\cdots(1-\beta_e x)$. By assumption there is a prime p that divides k but does not divide ℓ and a prime q that divides ℓ but does not divide k. Let s be a natural number such that p^s and q^s are both larger than $d+e$. Since by assumption none of the roots of $P(x)Q(x)$ is a root of unity, Lemma 12.2 implies that, for $1 \leq i \leq d$ and $1 \leq j \leq e$, there are largest nonnegative integers n_i and m_j with the property that we can write $\alpha_i = \gamma_i^{p^{n_i}} u_i$ and $\beta_j = \delta_j^{q^{m_j}} v_j$ for some elements $\gamma_i, \delta_j \in L(e^{2\pi i/(p^s q^s)})$ and u_i, v_j roots of unity in $L(e^{2\pi i/(p^s q^s)})$.

Next let n denote a natural number that is strictly larger than the maximum of the n_i and the m_j for i and j such that $1 \le i \le d$ and $1 \le j \le e$. Set $E := L(e^{2\pi i/(p^n q^n)})$ and let F denote the Galois extension of E generated by all complex roots of the polynomial

$$\prod_{i=1}^{d}\prod_{j=1}^{e}\Big(x^{p^n}-\gamma_i\Big)\Big(x^{q^n}-\delta_j\Big).$$

For each i, $1 \le i \le d$, we pick a root $\gamma_{i,0}$ of $x^{p^n}-\gamma_i$, and for each j, $1 \le j \le e$, we pick a root $\delta_{j,0}$ of $x^{q^n}-\delta_j$.

Claim. We claim that for every integer i, $1 \le i \le d$, there is an automorphism σ_i in $\mathrm{Gal}(F/E)$ such that

$$\sigma_i(\gamma_{i,0}) = \gamma_{i,0}u\,,$$

with u a primitive p^r-th root of unity for some r greater than or equal to s. Similarly, for every integer j, $1 \le j \le e$, there is an automorphism τ_j in $\mathrm{Gal}(F/E)$ that such that

$$\tau_j(\delta_{j,0}) = \delta_{j,0}u'\,,$$

for some primitive $q^{r'}$-th root of unity u' with r' greater than or equal to s.

Proof of the claim. Note that

$$\left\{\frac{\sigma(\gamma_{i,0})}{\gamma_{i,0}} \ \mid\ \sigma \in \mathrm{Gal}(F/E)\right\}$$

forms a subgroup of the p^n-th roots of unity. To prove the claim we just have to prove that this group cannot be contained in the group of p^{s-1}-th roots of unity. Let us assume that this is the case. Then the product of the Galois conjugates of $\gamma_{i,0}$ must be $\tilde{\gamma}_i := \gamma_{i,0}^{p^t}v$ for some $t < s$ and some $p^{(s-1)}$-th root of unity v. Moreover, $\tilde{\gamma}_i$ lies in $L(e^{2\pi i/(p^n q^n)})$. Note that the Galois group of $L(e^{2\pi i/(p^n q^n)})$ over $L(e^{2\pi i/(p^s q^s)})$ has order dividing $\phi(p^n q^n)/\phi(p^s q^s) = p^{n-s}q^{n-s}$. Since all conjugates of $\tilde{\gamma}_i$ are equal to $\tilde{\gamma}_i$ times some root of unity, we see that the relative norm of $\tilde{\gamma}_i$ with respect to the subfield $L(e^{2\pi i/(p^s q^s)})$ is of the form $\tilde{\gamma}_i{}^d v'$ for some divisor d of $p^{n-s}q^{n-s}$ and some root of unity v'. Moreover,

$$\tilde{\gamma}_i{}^d v' \in L\Big(e^{2\pi i/(p^s q^s)}\Big).$$

Note that the gcd of d and p^{n-t} is equal to p^{n-s_0} for some integer $s_0 \ge s$. Since $\gamma_{i,0}^{p^n} = \tilde{\gamma}_i{}^{p^{n-t}} v^{-p^{n-t}} \in L(e^{2\pi i/(p^s q^s)})$, we see by expressing p^{n-s_0} as an integer linear combination of d and p^{n-t} that

$$\tilde{\gamma}_i{}^{p^{n-s_0}}\omega = \gamma_{i,0}^{p^{n-s_0+t}}\omega' \in L\Big(e^{2\pi i/(p^s q^s)}\Big)$$

for some roots of unity ω and ω' and some $s_0 \geq s$. But $s_0 - t \geq 1$ and so we see that α_i is equal to a root of unity times

$$\left(\gamma_{i,0}^{p^{n-s_0+t}}\omega'\right)^{p^{s_0-t+n_i}},$$

contradicting the maximality of n_i. This confirms the claim. □

For an integer m, we let $\mathbb{U}_m$ denote the subgroup of $\mathbb{C}^*$ consisting of all m-th roots of unity. Note that we can define a group homomorhpism Φ from $\mathrm{Gal}(F/E)$ to $(\mathbb{U}_{p^n})^d \times (\mathbb{U}_{q^n})^e$ by

$$\Phi(\sigma) := (\sigma(\gamma_{1,0})/\gamma_{1,0}, \ldots, \sigma(\gamma_{d,0})/\gamma_{d,0}, \sigma(\delta_{1,0})/\delta_{1,0}, \ldots, \sigma(\delta_{e,0})/\delta_{e,0}).$$

We see that Φ is a group homomorphism since each $\sigma \in \mathrm{Gal}(F/E)$ fixes the p^n-th and q^n-th roots of unity. Set $H := \Phi(\mathrm{Gal}(F/E))$. The claim implies that the i-th coordinate in $(\mathbb{U}_{p^n})^d$ of $\Phi(\sigma_i)$ has order at least equal to p^s. Similarly, it also implies that the j-th coordinate in $(\mathbb{U}_{q^n})^e$ of $\Phi(\tau_j)$ has order at least equal to q^s. Since p^s and q^s are both greater than $d+e$, we have

$$d/p^s + e/q^s < 1.$$

Now, since $(\mathbb{U}_{p^n})^d \times (\mathbb{U}_{q^n})^e \cong (\mathbb{Z}/p^n\mathbb{Z})^d \times (\mathbb{Z}/q^n\mathbb{Z})^e$, we infer from Lemma 12.3 that there exists an element h in H such that every coordinate of h is different from the identity element. In other words, this means that there exists some element τ of $\mathrm{Gal}(F/E)$ that fixes no element in the set

$$\{\gamma_{i,0} \mid 1 \leq i \leq d\} \cup \{\delta_{j,0} \mid 1 \leq j \leq e\}.$$

Since by definition τ fixes all p^n-th and q^n-th roots of unity, we see more generally that no root of the polynomial

$$\prod_{i=1}^{d}\prod_{j=1}^{e}\left(x^{p^n} - \gamma_i\right)\left(x^{q^n} - \delta_j\right)$$

is fixed by τ. Since τ belongs to $\mathrm{Gal}(F/E)$, we can see τ as an element of $\mathrm{Gal}(F/K)$ that fixes all elements of E. We have thus produce an element τ of $\mathrm{Gal}(F/K)$ that fixes all roots of $P(x)Q(x)$ but that that does not fix any of the roots of the polynomial

$$\prod_{i=1}^{d}\prod_{j=1}^{e}\left(x^{p^n} - \gamma_i\right)\left(x^{q^n} - \delta_j\right).$$

It follows from Chebotarev's density theorem (see for instance the discussion in [23]) that there is an infinite set of nonzero prime ideals $\mathcal{S} \subseteq \mathrm{Spec}(R)$ such that if

$\mathfrak{P} \in \mathcal{S}$ then $P(x)Q(x) \bmod \mathfrak{P}$ factors into linear terms while the minimal polynomial of

$$\prod_{i=1}^{d}\prod_{j=1}^{e}\left(x^{p^n}-\gamma_i\right)\left(x^{q^n}-\delta_j\right)$$

over K has no root modulo $\mathfrak{P}$. In particular, there is a natural number N larger than n such that for all such prime ideals $\mathfrak{P}$, the polynomial $P(x)Q(x) \bmod \mathfrak{P}$ splits into linear factors, while the polynomial $P(x^{p^N})Q(x^{q^N}) \bmod \mathfrak{P}$ does not have any roots in $R/\mathfrak{P}$.

For such a prime ideal $\mathfrak{P}$, there thus exist $a_1,\dots,a_d,b_1,\dots,b_e$ in the finite field $R/\mathfrak{P}$ such that

$$P(x) \equiv (1-a_1x)\cdots(1-a_dx) \bmod \mathfrak{P}$$

and

$$Q(x) \equiv (1-b_1x)\cdots(1-b_dx) \bmod \mathfrak{P}\,.$$

Then

$$\left(\prod_{j=0}^{\infty} P\left(x^{k^j}\right)\right)^{-1} \equiv \prod_{i=1}^{d}\left(\prod_{j=0}^{\infty}\left(1-a_ix^{k^j}\right)\right)^{-1} \bmod \mathfrak{P}\,.$$

By Lemma 12.4 the right side is a product of k-automatic power series and hence, by Proposition 7.4, is k-automatic. Thus the infinite product

$$\left(\prod_{j=0}^{\infty} P\left(x^{k^j}\right)\right)^{-1} \bmod \mathfrak{P}$$

is a k-automatic power series in $R/\mathfrak{P}[[x]]$. Similarly, we get that

$$\left(\prod_{j=0}^{\infty} Q\left(x^{\ell^j}\right)\right)^{-1} \equiv \prod_{i=1}^{e}\left(\prod_{j=0}^{\infty}\left(1-b_ix^{\ell^j}\right)\right)^{-1} \bmod \mathfrak{P}\,,$$

which implies that the infinite product

$$\left(\prod_{j=0}^{\infty} Q\left(x^{\ell^j}\right)\right)^{-1} \bmod \mathfrak{P}$$

is a ℓ automatic power series in $R/\mathfrak{P}[[x]]$. This concludes the proof. $\square$

13. Proof of Theorem 1.3

We are now ready to prove our main result.

Proof of Theorem 1.3. Let K be a field of characteristic zero and let k and l be two multiplicatively independent positive integers.

We first note that if $F(x) \in K[[x]]$ is a rational function, then for every integer $m \geq 2$, it obviously satisfies a functional equation as in (1.3) with $n = 0$. Hence, $F(x)$ is m-Mahler, which gives a first implication.

To prove the converse implication, we fix $F(x) \in K[[x]]$ that is both k- and ℓ-Mahler and we aim at proving that $F(x)$ is a rational function. Of course, if $F(x)$ is a polynomial, there is nothing to prove. From now on, we thus assume that $F(x)$ is not a polynomial. By Corollary 8.3, we can assume that there are primes p and q such that p divides k while p does not divide ℓ and such that q divides ℓ while q does not divide k. By Theorem 5.1, we can assume that there is a ring R of the form $\mathcal{O}_K[1/M]$ (where K is a number field and M is a positive integer), such that $F(x) \in R[[x]]$ and satisfies the equations

$$\sum_{i=0}^{n} P_i(x)F(x^{k^i}) = 0$$

with $P_0, \ldots, P_d \in R[x]$ and

$$\sum_{i=0}^{m} Q_i(x)F\left(x^{\ell^i}\right) = 0$$

with $Q_0, \ldots, Q_e \in R[x]$. Without loss of generality, we can assume that all complex roots of $P_0(x)$ and $Q_0(x)$ belong to R (otherwise we could just enlarge R by adjoining these numbers). Furthermore, we can assume that $P_0(x)Q_0(x) \neq 0$. By Corollary 6.2, we can also assume that $P_0(0) = 1$ and that $Q_0(0) = 1$, for otherwise we could just replace $F(x)$ by the power series $F_0(x)$ given there. We choose a ring embedding of R in $\mathbb{C}$ and for the moment we regard $F(x)$ as a complex power series. By Theorem 11.2, we can assume that if α is a root of unity such that $\alpha^{k^j} = \alpha$ for some positive integer j, then $P_0(\alpha) \neq 0$. Similarly, we can assume that if β is a root of unity such that $\beta^{\ell^j} = \beta$ for some positive integer j, then $Q_0(\beta) \neq 0$.

By Proposition 7.10, we can write

$$F(x) = \left(\prod_{j=0}^{\infty} P_0\left(x^{k^j}\right)\right)^{-1} G(x)\,,$$

for some k-regular power series $G(x) \in R[[x]]$. Furthermore, we can decompose $P_0(x)$ as $P_0(x) = S_0(x)S_1(x)$, where $S_0(x)$ and $S_1(x)$ are two polynomials, the zeros of $S_0(x)$ are all roots of unity, none of the zeros of $S_1(x)$ is a root of unity, and $S_0(0) = S_1(0) = 1$. Since by assumption all roots of $P_0(x)$ lie in R, we get that both $S_0(x)$ and $S_1(x)$ belong to $R[x]$. By assumption if α is a root of $S_0(x)$ then for every positive integer j, one has $\alpha^{k^j} \neq \alpha$. Then, it follows from Proposition 7.8 that

$$\left(\prod_{j=0}^{\infty} S_0\left(x^{k^j}\right)\right)^{-1} \in R[[x]]$$

is a k-regular power series. Set $H(x) := \left(\prod_{j=0}^{\infty} S_0(x^{k^j})\right)^{-1} G(x)$. We infer from part (iii) of Proposition 7.4 that $H(x)$ is a k-regular power series. Moreover, one has

$$F(x) = \left(\prod_{j=0}^{\infty} S_1\left(x^{k^j}\right)\right)^{-1} H(x). \tag{13.1}$$

Similarly, by Proposition 7.10, we can write

$$F(x) = \left(\prod_{j=0}^{\infty} Q_0\left(x^{k^j}\right)\right)^{-1} I(x),$$

for some k-regular power series $I(x) \in R[[x]]$. As previously, we can decompose $Q_0(x)$ as $Q_0(x) = T_0(x)T_1(x)$, where $T_0(x)$ and $T_1(x)$ belong to $R[x]$, the zeros of $T_0(x)$ are all roots of unity, none of the zeros of $T_1(x)$ are roots of unity, and $T_0(0) = T_1(0) = 1$. By assumption if β is a root of $T_0(x)$ then for every positive integer j, one has $\beta^{\ell^j} \neq \beta$. Then it follows from Proposition 7.8 that

$$\left(\prod_{j=0}^{\infty} T_0\left(x^{\ell^j}\right)\right)^{-1} \in R[[x]]$$

is a ℓ-regular power series. Set $J := \prod_{j=0}^{\infty} T_0(x^{k^j})^{-1} I(x)$. Again, we see by Proposition 7.4 that $J(x)$ is ℓ-regular. Moreover, one has

$$F(x) = \left(\prod_{j=0}^{\infty} T_1\left(x^{k^j}\right)\right)^{-1} J(x). \tag{13.2}$$

By Theorem 12.1, there is an infinite set of nonzero prime ideals $\mathcal{S}$ of R such that, for every prime ideal $\mathfrak{P}$ in $\mathcal{S}$,

$$\left(\prod_{j=0}^{\infty} S_1\left(x^{k^j}\right)\right)^{-1} \bmod \mathfrak{P}$$

is a k-automatic power series in $(R/\mathfrak{P})[[x]]$ and

$$\left(\prod_{j=0}^{\infty} T_1\left(x^{\ell^j}\right)\right)^{-1} \bmod \mathfrak{P}$$

is a ℓ-automatic power series in $(R/\mathfrak{P})[[x]]$. Then we infer from Equalities (13.1) and (13.2) that, for $\mathfrak{P} \in \mathcal{S}$, $F(x) \bmod \mathfrak{P}$ is k-regular for it is the product of two k-regular power series. Similarly, $F(x) \bmod \mathfrak{P}$ is a ℓ-regular power series.

We recall that since R is of the form $\mathcal{O}_K[1/M]$, it is a *Dedekind domain*; that is, it is a Noetherian normal domain of Krull dimension one. In particular, all nonzero prime ideals are maximal. Now since R is a finitely generated $\mathbb{Z}$-algebra and $\mathfrak{P}$ is a maximal ideal, the quotient ring $R/\mathfrak{P}$ is a finite field (see [18, Theorem 4.19, page 132]). By Proposition 7.4, this implies that $F(x) \bmod \mathfrak{P}$ is actually both k- and ℓ-automatic. By Cobham's theorem, we obtain that the sequence of coefficients of $F(x) \bmod \mathfrak{P}$ is eventually periodic and hence $F(x) \bmod \mathfrak{P}$ is a rational function.

Note that since $\mathcal{S}$ is infinite, the intersection of all ideals in $\mathcal{S}$ is the zero ideal (see [18, Lemma 4.16, page 130]). Moreover, $F(x) \bmod \mathfrak{P}$ is rational for every prime ideal $\mathfrak{P} \in \mathcal{S}$. Applying Lemma 5.4, we obtain that $F(x)$ is a rational function. This ends the proof. $\square$

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