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Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity

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Abstract. The chemotaxis-Navier-Stokes system with signal production, as given by

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS(x,n,c) \cdot \nabla c\big) \\ c_t + u \cdot \nabla c = \Delta c - c + n \\ u_t + (u \cdot \nabla) u = \Delta u - \nabla P + n \nabla \phi \\ \nabla \cdot u = 0, \end{cases}$$

is considered in bounded planar convex domains Ω with smooth boundary, where $\phi \in W^{2,\infty}(\Omega)$ and $S \in C^2(\bar{\Omega} \times [0,\infty)^2; \mathbb{R}^{2\times 2})$.

The main results assert that parallel to the case of the corresponding Keller-Segel system obtained on neglecting u, any arbitrarily small algebraic saturation effect in the chemotactic sensitivity at large densities is sufficient to rule out any blow-up phenomenon. Indeed, under the assumption that there exist $S_0 \geq 0$ and $\alpha > 0$ such that

$$|S(x,n,c)| \leq S_0 \cdot (1+n)^{-\alpha} \qquad \text{for all } x \in \bar{\Omega}, n \geq 0 \text{ and } c \geq 0,$$

it is shown that for all suitably regular initial data an associated initial-boundary value problem possesses a globally defined bounded classical solution.

The analysis is based on the consecutive identification of three energy-like functionals, the first among which involves a certain sublinear L^p seminorm of n.

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1. Introduction

In this paper we will investigate an initial-boundary value problem for the chemotaxis-Navier-Stokes system

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot \big(nS(x,n,c) \cdot \nabla c\big) & x \in \Omega, \quad t > 0 \\ c_t + u \cdot \nabla c = \Delta c - c + n & x \in \Omega, \quad t > 0 \\ u_t + (u \cdot \nabla) u = \Delta u - \nabla P + n \nabla \phi & x \in \Omega, \quad t > 0 \\ \nabla \cdot u = 0 & x \in \Omega, \quad t > 0 \end{cases} \tag{1.1}$$

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in a bounded domain in $\mathbb{R}^N$, where our main focus will be on the planar case $N = 2$. Systems of this type arise in the modeling of bacterial populations in which individuals, besides moving randomly, partially adjust their movement according to concentration gradients of a chemical which they produce themselves, where in generalization of the celebrated Keller-Segel chemotaxis system [27], the model (1.1) furthermore accounts for the interaction of bacteria with the surrounding fluid, as indicated by experimental findings to be of substantial relevance to the emergence of large-scale convection patterns, *e.g.*, in populations of *Bacillus subtilis* suspended to sessile water drops ([20, 47], *cf.* also the related setting addressed in [28], and [3] for a concise derivation of chemotaxis-fluid systems of the considered type). Correspondingly, in (1.1) it is assumed that both cells and the chemical, at population density and concentration denoted by n and c, respectively, are transported by the surrounding fluid with the velocity field u, while the bacteria in turn exert a nontrivial influence on the fluid motion by means of buoyant forces in an external gravitational field with potential ϕ, as reflected in a corresponding source term appearing in the Navier-Stokes subsystem of (1.1). In accordance with refined modeling approaches accounting for additional rotational flux components especially near boundaries [64], the chemotactic sensitivity $S = S(x, n, c)$ in (1.1) will be assumed to be a general matrix-valued function possibly containing nondiagonal elements and thus allowing for chemotactic motion not necessarily parallel to the chemoattractive gradient [64].

Motivated by the particular experimental background from [20, 47], most previous studies on chemotaxis-fluid systems related to (1.1) focus on situations when a chemoattractant is consumed by the bacteria, rather than produced as in (1.1). Accordingly, quite an elaborate theory has been established for the corresponding system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(nS(c)\nabla c) & \text{for} \quad x\in\Omega, \quad t>0 \\ c_t + u\cdot\nabla c = \Delta c - nf(c) & \text{for} \quad x\in\Omega, \quad t>0 \\ u_t + (u\cdot\nabla)u = \Delta u - \nabla P + n\nabla\phi & \text{for} \quad x\in\Omega, \quad t>0 \\ \nabla\cdot u = 0 & \text{for} \quad x\in\Omega, \quad t>0 \end{cases} \tag{1.2}$$

under various assumptions on the scalar chemotactic sensitivity S and the signal consumption rate coefficient f. For instance, for a class of systems of type (1.2) including the prototypical choices

$$S \equiv \text{const.} \qquad \text{and} \qquad f(c) = c, \quad c \geq 0, \tag{1.3}$$

it is known that uniquely determined global-in-time smooth solutions exist for all suitably regular initial data in smoothly bounded planar convex domains ([53]; *cf.* also [5, 66] for recent results on the whole space $\mathbb{R}^2$), whereas in the corresponding three-dimensional analogue at least certain global weak solutions can be constructed [59]. For the simplified system obtained from (1.2) on replacing the Navier-Stokes equations therein by the respective Stokes system not containing the nonlinear convection term $(u \cdot \nabla)u$, global existence results for the Cauchy problem in $\mathbb{R}^3$ are available under certain more restrictive assumptions on S and f and a smallness assumption, *e.g.*, on c [11], and also for a corresponding boundary-value problem in bounded convex three-dimensional domains without imposing any such additional requirements [53]. Although with few exceptions [30,49] in three-dimensional versions and further variants of (1.2), the solutions constructed so far are all weak only [5,32,48] and may possibly develop singularities within finite time (see [6] for a detailed discussion on refined extensibility criteria), after all the dissipative effect of the signal absorption mechanism addressed in (1.2) has been identified to be sufficient for solutions to become eventually smooth and to approach spatially homogeneous equilibria in several cases [32,60]. This partially generalizes the quite comprehensive knowledge on global boundedness and smooth stabilization, even at exponential rates, in the two-dimensional version of (1.2) [54,65], and also extends known facts on regularity and large time behavior in the associated fluid-free analogue of (1.2) [41,42]. Quite a number of results on global existence and boundedness properties have also been obtained for the variant of (1.2) obtained on replacing Δn by nonlinear diffusion operators generalizing the porous medium-type choice Δn^m for several ranges of $m > 1$ [10,12,26,33,43,44,48,56].

In contrast to such systems with consumption of chemoattractant, in problems of type (1.1) the signal production mechanism may significantly enhance the destabilizing effects of chemotactic cross-diffusion. Indeed, even in the classical Keller-Segel system without fluid interaction, as obtained on letting $S \equiv 1$ in

$$\begin{cases} n_t = \Delta n - \nabla \cdot (nS(n)\nabla c) & \text{for} \quad x \in \Omega, \quad t > 0 \\ c_t = \Delta c - c + n & \text{for} \quad x \in \Omega, \quad t > 0, \end{cases} \tag{1.4}$$

it is known that for large classes of initial data, solutions blow up with respect to the spatial L^∞ norm of n when either $N \geq 3$ [55], or $N = 2$ and the total mass of cells is large [23,37], while global bounded solutions can be constructed under appropriate smallness conditions on the initial data [38,52]. Only when the chemotactic flux is appropriately limited, *e.g.*, according to suitable saturation effects at large cell densities, such explosion phenomena can entirely be ruled out. In fact, for the Neumann problem associated with (1.4) in smoothly bounded domains $\Omega \subset \mathbb{R}^N$, it is known that if $N \geq 2$ and $S \in C^2([0,\infty))$ is nonnegative and such that

$$S(n) \leq Cn^{-\frac{N-2}{N}-\varepsilon} \qquad \text{for all } n > 0 \tag{1.5}$$

with some $C > 0$ and $\varepsilon > 0$, then global bounded classical solutions can be found for all suitably smooth initial data [24,29], whereas if there exist $C > 0$ and $\varepsilon > 0$ fulfilling

$$S(n) \geq Cn^{-\frac{N-2}{N}+\varepsilon} \qquad \text{for all } n > 0, \tag{1.6}$$

then some solutions may become unbounded [24,51], where in some particular cases and related systems involving nonlinear diffusion, such unboundedness phenomena are even known to occur within finite time ([8,9]; *cf.* also the surveys [22] and [3]).

Main results

To the best of our knowledge, the literature on chemotaxis-fluid systems of the form (1.1) with signal production mechanisms so far concentrates on either Stokes-type simplifications [50], or on the construction of small-data solutions [4,30], or also on systems involving logisitic growth restrictions as an additional dissipative mechanism [13,46]. The purpose of the present work consists in examining the question how far relations of the form in (1.5) and (1.6) continue to determine a critical asymptotic behavior of S with regard to the emergence of singularities also in the full chemotaxis-Navier-Stokes system (1.1) with any such further regularizing prerequisites. Our main results in this direction will reveal that in two-dimensional convex domains Ω with smooth boundary, within the accuracy of algebraic rates, asymptotically constant S indeed will mark a borderline case in this respect, even in the context of matrix-valued sensitivities, thereby extending a recent finding which asserts a similar conclusion in the case when the fluid flow is governed by the linear Stokes system [50]. In view of the mentioned blow-up results for the fluid-free special case (1.4) of (1.1), this amounts to deriving a corresponding statement on global existence and boundedness under an assumption paralleling that in (1.5). To formulate this more precisely, let us suppose that $S \in C^2(\bar{\Omega} \times [0,\infty)^2; \mathbb{R}^{2\times 2})$ has the property that there exist $S_0 \geq 0$ and $\alpha > 0$ fulfilling

$$|S(x,n,c)| \leq S_0 \cdot (1+n)^{-\alpha} \qquad \text{for all } x \in \bar{\Omega}, n \geq 0 \text{ and } c \geq 0, \tag{1.7}$$

where we evidently may assume without loss of generality that herein $\alpha < \frac{1}{2}$. Then assuming moreover for simplicity that $\phi \in W^{2,\infty}(\Omega)$, we will consider (1.1) along with the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x \in \Omega \tag{1.8}$$

and the boundary conditions

$$\big(nS(x,n,c)\cdot\nabla c\big)\cdot\nu = \nabla n \cdot \nu, \quad \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.9}$$

with initial data which are such that

$$\begin{cases} n_0 \in C^{\vartheta}(\bar{\Omega}) & \text{with some } \vartheta \in (0,1), \text{ with } n_0 \geq 0 \text{ and } n_0 \not\equiv 0 \\ c_0 \in W^{1,\infty}(\Omega) & \text{is nonnegative} \\ u_0 \in D(A). \end{cases} \tag{1.10}$$

Here and throughout the sequel, we let $A := -\mathcal{P}\Delta$ denote the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^2)$, defined on its domain

$$D(A) := W^{2,2}(\Omega;\mathbb{R}^2) \cap W_0^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega)$$

with

$$L^2_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega,\mathbb{R}^2)}^{\|\cdot\|_{L^2(\Omega)}} = \overline{\left\{\varphi\in C^\infty_0(\Omega,\mathbb{R}^2);\ \nabla\cdot\varphi=0\right\}}^{\|\cdot\|_{L^2(\Omega)}},$$

and with $\mathcal{P}$ representing the Helmholtz projection of $L^2(\Omega;\mathbb{R}^2)$ onto $L^2_\sigma(\Omega)$.

In this framework, our main results read as follows.

Theorem 1.1. *Let $\Omega\subset\mathbb{R}^2$ be a bounded convex domain with smooth boundary, and let $\phi\in W^{2,\infty}(\Omega)$ and $S\in C^2(\bar\Omega\times[0,\infty)^2;\mathbb{R}^{2\times2})$ be such that* (1.7) *holds with some $S_0\ge 0$ and $\alpha\in(0,\frac12)$. Then for all (n_0,c_0,u_0) satisfying* (1.10)*, there exist functions*

$$\begin{cases} n\in C^0\left(\bar\Omega\times[0,\infty)\right)\cap C^{2,1}\left(\bar\Omega\times(0,\infty)\right)\\ c\in C^0\left(\bar\Omega\times[0,\infty)\right)\cap C^{2,1}\left(\bar\Omega\times(0,\infty)\right)\cap\bigcap_{p>1}L^\infty\left((0,\infty);W^{1,p}(\Omega)\right)\\ u\in C^0\left(\bar\Omega\times[0,\infty);\mathbb{R}^2\right)\cap C^{2,1}\left(\bar\Omega\times(0,\infty);\mathbb{R}^2\right)\\ \qquad\qquad\cap\bigcap_{\beta\in(0,1)}L^\infty\left((0,\infty);D\left(A^\beta\right)\right)\\ P\in C^{1,0}\left(\bar\Omega\times(0,\infty)\right),\end{cases}\tag{1.11}$$

such that n and c are nonnegative in $\Omega\times(0,\infty)$, and such that (n,c,u,P) solves (1.1), (1.9), (1.8) *in the classical sense in $\Omega\times(0,\infty)$. Moreover, this solution is bounded in the sense that for each $p>1$ and any $\beta\in(0,1)$ there exists $C(p,\beta)>0$ with the property that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)}+\|c(\cdot,t)\|_{W^{1,p}(\Omega)}+\|A^\beta u(\cdot,t)\|_{L^2(\Omega)}\le C(p,\beta)\quad \textit{for all } t>0.\tag{1.12}$$

In comparison to the corresponding fluid-free case in (1.4), a major technical challenge originating from the additional fluid interaction in (1.1) seems to consist in the circumstance that the evident mass conservation property $\int_\Omega n\equiv\int_\Omega n_0$, as the only apparent a priori information available, seems insufficient to warrant any useful knowledge on regularity in the Navier-Stokes subsystem of (1.1). This substantially differs also from the correspondingly simplified chemotaxis-Stokes analogue studied in [50], where this temporally uniform spatial L^1 bound on the respective forcing term could be used as a starting point for an appropriate bootstrap procedure.

Our approach will accordingly be based on an entirely alternative strategy, the core of which can be found in Section 5, where a functional of the form

$$-\int_\Omega n^{2\alpha}+C\int_\Omega c^2,\qquad t\ge 0,$$

will be seen to enjoy a certain energy-like property for solutions to a suitably regularized version of (1.1), provided that $C>0$ is chosen appropriately (Lemma 5.1). We note that the use of functionals of such a structure, yet convex according to our choice of signs but growing in a sublinear manner with respect to the crucial quantity n due to our assumption $\alpha < \frac{1}{2}$, seems quite unusual in the context of chemotaxis systems; in the present setting, however, from our analysis of an associated energy-dissipation inequality we shall obtain a favorable further regularity property of n (Lemma 5.2) which will turn out to be sufficient for deriving a space-time L^2 estimate for ∇u through the standard energy inequality associated with the Navier-Stokes subsystem of (1.1) (Lemma 5.3). This will enable us to adequately control the terms stemming from the fluid interaction in the analysis of the evolution of

$$\int_\Omega n \ln n + a \int_\Omega |\nabla c|^2, \qquad t \ge 0,$$

with suitable $a > 0$, so as to derive, inter alia, bounds for $\int_\Omega |\nabla c|^2$ and for $\int_t^{t+1} \int_\Omega n^2$, and in consequence also for $\int_\Omega |\nabla u|^2$ (Lemma 6.2 and Lemma 6.3). Section 7 will thereafter be devoted to an analysis of

$$\frac{1}{p} \int_\Omega n^p + \frac{2}{q} \int_\Omega |\nabla c|^{2q}, \qquad t \ge 0,$$

which by means of our previously gained estimates will be seen to also play the role of a quasi-energy for suitably chosen but arbitrarily large $p > 1$ and $q > 1$. The bounds on $\int_\Omega n^p$ and $\int_\Omega |\nabla c|^{2q}$ thereby obtained will finally be used in a straightforward manner to assert higher order regularity properties in Section 8 and Section 9, and to establish the claimed result on global existence and boundedness through a limit procedure in the approximate problems in Section 10.

2. A family of regularized problems

In order to adequately approximate solutions of the original problem (1.1), (1.8), (1.9) involving a nonlinear boundary condition for the quantity n, we follow an idea from [34] (*cf.* also [4,50,58,61,62]) and introduce an appropriate regularization in which S vanishes near the lateral boundary. More precisely, let us fix $(\rho_\varepsilon)_{\varepsilon\in(0,1)} \subset C_0^\infty(\Omega)$ and $(\chi_\varepsilon)_{\varepsilon\in(0,1)} \subset C^\infty([0,\infty))$ such that

$$0 \le \rho_\varepsilon \le 1 \text{ in } \Omega \quad \text{with} \quad \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon \searrow 0,$$

and that

$$0 \le \chi_\varepsilon \le 1 \text{ in } [0,\infty) \text{ with } \chi_\varepsilon \equiv 0 \text{ in } [\tfrac{1}{\varepsilon},\infty) \text{ and } \chi_\varepsilon \nearrow 1 \text{ in } [0,\infty) \text{ as } \varepsilon \searrow 0. \qquad (2.1)$$

For $\varepsilon \in (0,1)$, we then define

$$S_\varepsilon(x,n,c) := \rho_\varepsilon(x) \cdot \chi_\varepsilon(n) \cdot S(x,n,c), \quad (x,n,c) \in \bar\Omega \times [0,\infty)^2, \qquad (2.2)$$

and observe that $S_\varepsilon \in C^2(\bar\Omega \times [0,\infty)^2; \mathbb{R}^{2\times 2})$, and that evidently (1.7) continues to hold for S_ε, with the values of S_0 and α unchanged. Moreover, upon replacing S by

S_ε we formally obtain that the nonlinear boundary condition for n in (1.9) reduces to a homogeneous Neumann-type condition. Accordingly, in order to construct a global solution to (1.1), (1.8), (1.9) through an appropriate limit procedure, for $\varepsilon \in (0, 1)$ we consider the regularized problems

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot \Big(n_\varepsilon S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \Big) & \text{for } x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon & \text{for } x \in \Omega,\ t > 0, \\ u_{\varepsilon t} + \big(Y_\varepsilon u_\varepsilon \cdot \nabla \big) u_\varepsilon = \Delta u_\varepsilon - \nabla P_\varepsilon + n_\varepsilon \nabla \phi & \text{for } x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0 & \text{for } x \in \Omega,\ t > 0, \\ \dfrac{\partial n_\varepsilon}{\partial \nu} = \dfrac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x),\ c_\varepsilon(x,0) = c_0(x),\ u_\varepsilon(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{2.3}$$

where Y_ε denotes the standard Yosida approximation of the Stokes operator defined by

$$Y_\varepsilon \varphi := (1 + \varepsilon A)^{-1} \varphi \qquad \text{for } \varepsilon \in (0, 1) \text{ and } \varphi \in L^2_\sigma(\Omega). \tag{2.4}$$

A well-known construction on the basis of the contraction mapping principle (see, *e.g.*, [53, Lemma 2.1] and also [1]) asserts local existence of classical solutions to these problems as well as a convenient extensibility criterion.

Lemma 2.1. *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, that $\phi \in W^{1,\infty}(\Omega)$, that $S \in C^2(\bar\Omega \times [0, \infty)^2; \mathbb{R}^{2\times 2})$ satisfies* (1.7) *for some $S_0 \geq 0$ and $\alpha \geq 0$, and that n_0, c_0 and u_0 comply with* (1.10)*. Then for each $\varepsilon \in (0, 1)$ there exist $T_{\max,\varepsilon} \in (0, \infty]$ and functions*

$$\begin{cases} n_\varepsilon \in C^0(\bar\Omega \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\bar\Omega \times (0, T_{\max,\varepsilon})) \\ c_\varepsilon \in C^0(\bar\Omega \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\bar\Omega \times (0, T_{\max,\varepsilon})) \\ \qquad\qquad \cap \bigcap_{p>1} C^0([0, T_{\max,\varepsilon}); W^{1,p}(\Omega)) \\ u_\varepsilon \in C^0(\bar\Omega \times [0, T_{\max,\varepsilon}); \mathbb{R}^2) \cap C^{2,1}(\bar\Omega \times (0, T_{\max,\varepsilon}); \mathbb{R}^2) \\ \qquad\qquad \cap \bigcap_{\beta \in (0,1)} C^0(([0, T_{\max,\varepsilon}); D(A^\beta)) \\ P_\varepsilon \in C^{1,0}(\bar\Omega \times (0, T_{\max,\varepsilon})), \end{cases} \tag{2.5}$$

such that n_ε and c_ε are nonnegative in $\Omega \times (0, T_{\max,\varepsilon})$, that $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (2.3) *classically in $\Omega \times (0, T_{\max,\varepsilon})$, and that*

$$\text{if } T_{\max,\varepsilon} < \infty, \text{ then}$$
$$\limsup_{t \nearrow T_{\max,\varepsilon}} \Big(\|n_\varepsilon\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,p}(\Omega)} + \|A^\beta u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \Big) = \infty \tag{2.6}$$
$$\text{for all } p > 2 \text{ and } \beta \in \Big(\tfrac{1}{2}, 1 \Big).$$

Two basic but important properties of these solutions are immediate.

Lemma 2.2. *For each* $\varepsilon \in (0,1)$*, we have*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \textit{for all } t \in (0, T_{\max,\varepsilon}) \tag{2.7}$$

and

$$\int_\Omega c_\varepsilon(\cdot,t) \le \max\left\{ \int_\Omega c_0\,,\ \int_\Omega n_0 \right\} \qquad \textit{for all } t \in (0, T_{\max,\varepsilon}). \tag{2.8}$$

Proof. The identity (2.7) directly results on integrating the first equation in (2.3). Thereupon, from the second equation in (2.3) we obtain that

$$\frac{d}{dt}\int_\Omega c_\varepsilon + \int_\Omega c_\varepsilon = \int_\Omega n_\varepsilon = \int_\Omega n_0 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}),$$

which implies (2.8) through an ODE comparison argument. □

As a preparation for both Lemma 3.2 and Lemma 6.1 below, let us also include in this preliminary section the following elementary ODE lemma, a proof of which can be found in [63, Lemma 3.4].

Lemma 2.3. *Let* $T > 0$ *and* $y \in C^0([0,T)) \cap C^1(0,T)$ *be such that*

$$y'(t) + ay(t) \le g(t) \qquad \textit{for all } t \in (0,T),$$

where $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ *has the property that*

$$\frac{1}{\tau}\int_t^{t+\tau} g(s)ds \le b \qquad \textit{for all } t \in (0,T)$$

with some $\tau > 0$ *and* $b > 0$*. Then*

$$y(t) \le y(0) + \frac{b\tau}{1-e^{-a\tau}} \qquad \textit{for all } t \in [0,T).$$

3. Estimates for solutions to a regularized Navier-Stokes system

In this section we plan to derive some estimates for the approximate Navier-Stokes subsystem of (2.3) which will be used in several places below firstly in order to assert global existence of the solutions to (2.3), and secondly to allow afterwards for the derivation of ε-independent estimates for the latter. Specifically, for $\varepsilon \in (0, 1)$ we consider the problem

$$\begin{cases} \widetilde{u}_t + \big(Y_\varepsilon \widetilde{u} \cdot \nabla\big)\widetilde{u} = \Delta \widetilde{u} - \nabla \widetilde{P} + f(x,t) & x \in \Omega, \quad t > 0 \\ \nabla \cdot \widetilde{u} = 0 & x \in \Omega, \quad t > 0 \\ \widetilde{u} = 0 & x \in \partial\Omega, \ t > 0 \\ \widetilde{u}(x,0) = \widetilde{u}_0(x) & x \in \Omega, \end{cases} \tag{3.1}$$

where f is a given suitably regular function on $\Omega \times (0, T)$ with some $T \in (0, \infty]$, and where $\widetilde{u}_0 \in L^2_\sigma(\Omega)$.

For frequent later reference, let us first collect some basic properties of the Yosida approximation.

Lemma 3.1. *Let $\varepsilon \in (0, 1)$. Then*

$$\|Y_\varepsilon \varphi\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)} \qquad \textit{for all } \varphi \in L^2_\sigma(\Omega) \tag{3.2}$$

and

$$\|\nabla Y_\varepsilon \varphi\|_{L^2(\Omega)} \le \|\nabla \varphi\|_{L^2(\Omega)} \qquad \textit{for all } \varphi \in W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega). \tag{3.3}$$

Moreover, for all $p > 1$ there exists $C(p) > 0$ such that for all $\varepsilon \in (0, 1)$ we have

$$\|Y_\varepsilon \varphi\|_{L^p(\Omega)} \le C(p) \|\nabla \varphi\|_{L^2(\Omega)} \qquad \textit{for all } \varphi \in W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega). \tag{3.4}$$

Proof. Given $\varphi \in L^2_\sigma(\Omega)$, writing $\psi := Y_\varepsilon \varphi$ we have $(1 + \varepsilon A)\psi = \varphi$ in Ω and hence, by self-adjointness of $A^{\frac{1}{2}}$ and the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_\Omega |\psi|^2 \le \int_\Omega |\psi|^2 + \varepsilon \int_\Omega |A^{\frac{1}{2}} \psi|^2 &= \int_\Omega |\psi|^2 + \varepsilon \int_\Omega \psi \cdot A\psi \\ &= \int_\Omega \varphi \cdot \psi \le \|\varphi\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \end{aligned}$$

for all $\varepsilon \in (0, 1)$. This clearly implies (3.2), whereupon observing that $A^{\frac{1}{2}}$ commutes with Y_ε on $D(A^{\frac{1}{2}})$ we obtain

$$\|\nabla Y_\varepsilon \varphi\|_{L^2(\Omega)} = \|A^{\frac{1}{2}} Y_\varepsilon \varphi\|_{L^2(\Omega)} = \|Y_\varepsilon A^{\frac{1}{2}} \varphi\|_{L^2(\Omega)} \le \|A^{\frac{1}{2}} \varphi\|_{L^2(\Omega)} = \|\nabla \varphi\|_{L^2(\Omega)}$$

for any such φ and ε, because $\|A^{\frac{1}{2}} \tilde{\varphi}\|_{L^2(\Omega)} = \|\nabla \tilde{\varphi}\|_{L^2(\Omega)}$ for all $\tilde{\varphi} \in D(A^{\frac{1}{2}}) = W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega)$.

To verify (3.4), we only need to note that since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ and $Y_\varepsilon(L^2_\sigma(\Omega)) \subset W^{1,2}_0(\Omega; \mathbb{R}^2)$, there exists $C_1 > 0$ such that

$$\|Y_\varepsilon \varphi\|_{L^p(\Omega)} \le C_1 \|\nabla Y_\varepsilon \varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}_0(\Omega; \mathbb{R}^2) \cap L^2_\sigma(\Omega).$$

Therefore, namely, (3.4) results from (3.3). $\square$

Now addressing (3.1), we first identify a mild boundedness property of the source term therein which is sufficient to allow for the natural conclusions obtained from the standard Navier-Stokes energy inequality in the presently considered two-dimensional setting. Here and below, the parameter $\tau > 0$ is included so as to ensure applicability also in cases when no a priori knowledge on the existence time of solutions is available, such as in Lemma 3.4 and Lemma 4.2 below.

Lemma 3.2. *Let $T \in (0,\infty]$ and $\tau \in (0,T)$, and let $\widetilde{u}_0 \in C^0(\bar{\Omega};\mathbb{R}^2) \cap W_0^{1,2}(\Omega;\mathbb{R}^2) \cap L^2_\sigma(\Omega)$. Then for all $p > 1$ and each $b > 0$ there exists $C(p,b,\tau) > 0$ with the property that if $f \in C^0(\bar{\Omega}\times[0,T);\mathbb{R}^2)$, $\widetilde{u} \in C^0(\bar{\Omega}\times[0,T);\mathbb{R}^2) \cap C^0([0,T); W_0^{1,2}(\Omega;\mathbb{R}^2)) \cap C^{2,1}(\bar{\Omega}\times(0,T);\mathbb{R}^2)$ and $\widetilde{P} \in C^{1,0}(\bar{\Omega}\times(0,T))$ are such that (3.1)holds for some $\varepsilon \in (0,1)$, and such that*

$$\frac{1}{\tau}\int_t^{t+\tau} \|f(\cdot,s)\|^2_{L^p(\Omega)} \le b \qquad \textit{for all } t \in (0, T-\tau), \tag{3.5}$$

then

$$\int_\Omega |\widetilde{u}(\cdot,t)|^2 \le C(p,b,\tau) \qquad \textit{for all } t \in (0,T) \tag{3.6}$$

and

$$\int_t^{t+\tau}\int_\Omega |\nabla\widetilde{u}(x,s)|^2 \le C(p,b,\tau) \qquad \textit{for all } t \in (0,T-\tau). \tag{3.7}$$

Proof. Since $W^{1,2}(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$, thanks to a corresponding Poincaré-Sobolev inequality we can find $C_1 > 0$ such that

$$\|\varphi\|_{L^{\frac{p}{p-1}}(\Omega)} \le C_1\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2).$$

Testing (3.1) by $\widetilde{u}$ and using the Hölder inequality and Young's inequality, we accordingly obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\widetilde{u}|^2 + \int_\Omega |\nabla\widetilde{u}|^2 &= \int_\Omega f\cdot\widetilde{u}\\ &\le \|f\|_{L^p(\Omega)}\|\widetilde{u}\|_{L^{\frac{p}{p-1}}(\Omega)} \le C_1\|f\|_{L^p(\Omega)}\|\nabla\widetilde{u}\|_{L^2(\Omega)}\\ &\le \frac{1}{2}\int_\Omega |\nabla\widetilde{u}|^2 + \frac{C_1^2}{2}\|f\|^2_{L^p(\Omega)} \qquad \text{for all } t \in (0,T).\end{aligned}$$

As the standard Poincaré inequality in $W_0^{1,2}(\Omega)$ provides moreover $C_2 > 0$ such that

$$\|\varphi\|_{L^2(\Omega)} \le C_2\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega;\mathbb{R}^2),$$

from this we infer that

$$\frac{d}{dt}\int_\Omega |\widetilde{u}|^2 + \frac{1}{2C_2^2}\int_\Omega |\widetilde{u}|^2 + \frac{1}{2}\int_\Omega |\nabla\widetilde{u}|^2 \le C_1^2\|f\|^2_{L^p(\Omega)} \quad \text{for all } t \in (0,T). \tag{3.8}$$

In view of (3.5), employing Lemma 2.3 we firstly conclude that

$$\int_\Omega |\widetilde{u}(\cdot,t)|^2 \le C_3 := \int_\Omega |\widetilde{u}_0|^2 + \frac{bC_1^2\tau}{1-e^{-\frac{\tau}{2C_2^2}}} \qquad \text{for all } t \in (0,T), \tag{3.9}$$

and thus, secondly, obtain on integrating (3.8) that

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega |\nabla\widetilde{u}|^2 &\le 2\int_\Omega |\widetilde{u}(\cdot,t)|^2 + 2C_1^2\int_t^{t+\tau} \|f(\cdot,s)\|^2_{L^p(\Omega)} \\ &\le 2C_3 + 2C_1^2 b\tau \qquad \text{for all } t \in (0, T-\tau),\end{aligned}$$

which together with (3.9) yields both claimed estimates. □

Next, using the latter result we can show that under a somewhat stronger assumption involving a spatio-temporal L^2 bound for the force in (3.1), solutions even remain bounded in H^1.

Lemma 3.3. *Let $T \in (0,\infty]$ and $\tau \in (0,T)$, and let $\widetilde{u}_0 \in C^0(\bar\Omega;\mathbb{R}^2)\cap W_0^{1,2}(\Omega;\mathbb{R}^2)\cap L^2_\sigma(\Omega)$. Then for all $b>0$ there exists $C(b,\tau)>0$ such that if $f\in C^0(\bar\Omega\times[0,T);\mathbb{R}^2)$, $\widetilde{u}\in C^0(\bar\Omega\times[0,T);\mathbb{R}^2)\cap C^0([0,T);W_0^{1,2}(\Omega;\mathbb{R}^2))\cap C^{2,1}(\bar\Omega\times(0,T);\mathbb{R}^2)$ and $\widetilde{P}\in C^{1,0}(\bar\Omega\times(0,T))$ solve (3.1) for some $\varepsilon\in(0,1)$ and satisfy*

$$\frac{1}{\tau}\int_t^{t+\tau}\int_\Omega |f(x,s)|^2 \le b \qquad \textit{for all } t \in (0,T-\tau), \tag{3.10}$$

then

$$\int_\Omega |\nabla\widetilde{u}(\cdot,t)|^2 \le C(b,\tau) \qquad \textit{for all } t\in(0,T). \tag{3.11}$$

Proof. We first invoke Lemma 3.2 with $p := 2$ to fix $C_1 > 0$ and $C_2 > 0$ such that

$$\int_\Omega |\widetilde{u}(\cdot,t)|^2 \le C_1 \qquad \text{for all } t\in(0,T) \tag{3.12}$$

and

$$\int_{t-\tau}^{t}\int_\Omega |\nabla\widetilde{u}(x,s)|^2 \le C_2 \qquad \text{for all } t\in(\tau,T). \tag{3.13}$$

Then by applying the Helmholtz projection to both sides of the first equation in (3.1), multiplying the resulting identity by $A\widetilde{u}$, integrating by parts and using

Young's inequality we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\widetilde{u}|^2 + \int_\Omega |A\widetilde{u}|^2 &= -\int_\Omega A\widetilde{u}\cdot\mathcal{P}[(Y_\varepsilon\widetilde{u}\cdot\nabla)\widetilde{u}] + \int_\Omega A\widetilde{u}\cdot\mathcal{P}f \\
&\le \frac{1}{4}\int_\Omega |A\widetilde{u}|^2 + \int_\Omega \Big|\mathcal{P}[(Y_\varepsilon\widetilde{u}\cdot\nabla)\widetilde{u}]\Big|^2 + \frac{1}{4}\int_\Omega |A\widetilde{u}|^2 + \int_\Omega |\mathcal{P}f|^2 \qquad (3.14)\\
&\le \frac{1}{2}\int_\Omega |A\widetilde{u}|^2 + \int_\Omega \Big|(Y_\varepsilon\widetilde{u}\cdot\nabla)\widetilde{u}\Big|^2 + \int_\Omega |f|^2 \qquad \text{for all } t\in(0,T),
\end{aligned}$$

because of the orthogonal projection property of $\mathcal{P}$. Here we employ the Cauchy-Schwarz inequality and Young's inequality together with (3.12) to see that with some $C_3>0$ and $C_4>0$ we have

$$\begin{aligned}
&\int_\Omega \Big|(Y_\varepsilon\widetilde{u}\cdot\nabla)\widetilde{u}\Big|^2 \le C_3\|Y_\varepsilon\widetilde{u}\|^2_{L^4(\Omega)}\|\nabla\widetilde{u}\|^2_{L^4(\Omega)}\\
&\le C_4\cdot\Big\{\|\nabla Y_\varepsilon\widetilde{u}\|_{L^2(\Omega)}\|Y_\varepsilon\widetilde{u}\|_{L^2(\Omega)}\Big\}\cdot\Big\{\|A\widetilde{u}\|_{L^2(\Omega)}\|\nabla\widetilde{u}\|_{L^2(\Omega)}\Big\}\\
&\le C_1^{\frac{1}{2}}C_4\|A\widetilde{u}\|_{L^2(\Omega)}\|\nabla\widetilde{u}\|^2_{L^2(\Omega)} \le \frac{1}{2}\int_\Omega |A\widetilde{u}|^2 + \frac{C_1C_4^2}{2}\|\nabla\widetilde{u}\|^4_{L^2(\Omega)} \text{ for all } t\in(0,T),
\end{aligned}$$

where we make use of Lemma 3.1 and the well-known facts that $\|\nabla(\cdot)\|_{L^2(\Omega)}$ and $\|A(\cdot)\|_{L^2(\Omega)}$ constitute norms equivalent to $\|\cdot\|_{W^{1,2}(\Omega)}$ and $\|\cdot\|_{W^{2,2}(\Omega)}$, respectively, on $D(A)$.

Therefore, (3.14) shows that $y(t) := \int_\Omega |\nabla\widetilde{u}(\cdot,t)|^2$, $t\in[0,T)$, as well as $a(t) := C_1C_4^2\int_\Omega |\nabla\widetilde{u}(\cdot,t)|^2$ and $h(t) := 2\int_\Omega |f(\cdot,t)|^2$, $t\in(0,T)$, satisfy

$$y'(t)\le a(t)y(t)+h(t) \qquad \text{for all } t\in(0,T), \tag{3.15}$$

where (3.13) and (3.10) ensure that

$$\int_{t-\tau}^t a(s)ds \le C_5 := C_1C_2C_4^2 \qquad \text{for all } t\in(\tau,T) \tag{3.16}$$

and that

$$\int_{t-\tau}^t h(s)ds \le C_6 := 2b\tau \qquad \text{for all } t\in(\tau,T). \tag{3.17}$$

Now given $t\in(0,T)$, again thanks to (3.13) we can pick $t_\star(t)\ge 0$ such that $t_\star(t)\in[t-\tau,t)$ and

$$\int_\Omega |\nabla\widetilde{u}(\cdot,t_\star(t))|^2 \le C_7 := \max\left\{\int_\Omega |\nabla\widetilde{u}_0|^2\,,\ \frac{C_2}{\tau}\right\},$$

so that by integration of (3.15) we infer that

$$\begin{aligned}
y(t) &\le y(t_\star(t))e^{\int_{t_\star(t)}^t a(s)ds} + \int_{t_\star(t)}^t e^{\int_s^t a(\sigma)d\sigma}h(s)ds\\
&\le C_7e^{C_5} + \int_{t_\star(t)}^t e^{C_5}h(s)ds \le C_7e^{C_5} + e^{C_5}C_6,
\end{aligned}$$

and that hence (3.11) holds. □

Finally, if f even belongs to $L^\infty((0,T);L^2(\Omega;\mathbb{R}^2))$, then solutions to (3.1) enjoy certain boundedness and temporal Hölder continuity properties even in the spaces $D(A^\beta)$ for arbitrary $\beta<1$:

Lemma 3.4. *Let $T\in(0,\infty]$, and let $\widetilde{u}_0\in D(A)$. Then for all $\beta\in(\frac{1}{2},1)$ and each $b>0$ there exists $C(\beta,b)>0$ such that if $f\in C^0(\bar\Omega\times[0,T);\mathbb{R}^2)$, $\widetilde{u}\in C^0([0,T);D(A^\beta))\cap C^{2,1}(\bar\Omega\times(0,T);\mathbb{R}^2)$ and $P\in C^{1,0}(\bar\Omega\times(0,T))$ solve* (3.1) *for some $\varepsilon\in(0,1)$ and satisfy*

$$\|f(\cdot,t)\|_{L^2(\Omega)}\le b \qquad \text{for all } t\in(0,T), \tag{3.18}$$

then

$$\|A^\beta\widetilde{u}(\cdot,t)\|_{L^2(\Omega)}\le C(\beta,b) \qquad \text{for all } t\in(0,T) \tag{3.19}$$

and

$$\begin{aligned}\|A^\beta\widetilde{u}(\cdot,t)-A^\beta\widetilde{u}(\cdot,t_0)\|_{L^2(\Omega)}&\le C(\beta,b)\cdot(t-t_0)^{1-\beta}\\ &\text{for all } t_0\in[0,T) \text{ and any } t\in(t_0,T).\end{aligned} \tag{3.20}$$

Proof. Since $\beta<1$, it is possible to choose $p\in(1,2)$ such that

$$p>\frac{2}{3-2\beta}. \tag{3.21}$$

Therefore, in the projected version of (3.1), that is, in the identity

$$\widetilde{u}_t+A\widetilde{u}=\widetilde{f}:=-\mathcal{P}\Big[(Y_\varepsilon\widetilde{u}\cdot\nabla)\widetilde{u}\Big]+\mathcal{P}[f], \qquad x\in\Omega,\ t\in(0,T), \tag{3.22}$$

using the Hölder inequality and the continuity of $\mathcal{P}$ in $L^p(\Omega;\mathbb{R}^2)$ ([15]) as well as (3.18), we see that there exist $C_1>0$ and $C_2>0$ that

$$\begin{aligned}\|\widetilde{f}(\cdot,t)\|_{L^p(\Omega)}&\le C_1\Big\|(Y_\varepsilon\widetilde{u}\cdot\nabla)\widetilde{u}\Big\|_{L^p(\Omega)}+C_1\|f\|_{L^p(\Omega)}\\ &\le C_2\|Y_\varepsilon\widetilde{u}\|_{L^{\frac{2p}{2-p}}(\Omega)}\|\nabla\widetilde{u}\|_{L^2(\Omega)}+C_2 \qquad \text{for all } t\in(0,T).\end{aligned}$$

As $W^{1,2}(\Omega)\hookrightarrow L^{\frac{2p}{2-p}}(\Omega)$, thanks to Lemma 3.3, in view of (3.18) applicable to, *e.g.*, $\tau:=\min\{1,\frac{T}{2}\}$, we obtain therefore $C_3>0$ and $C_4>0$ such that

$$\begin{aligned}\|\widetilde{f}(\cdot,t)\|_{L^p(\Omega)}&\le C_3\|\nabla Y_\varepsilon\widetilde{u}\|_{L^2(\Omega)}\|\nabla\widetilde{u}\|_{L^2(\Omega)}+C_2\\ &\le C_3\|\nabla\widetilde{u}\|^2_{L^2(\Omega)}+C_2\\ &\le C_4 \qquad \text{for all } t\in(0,T),\end{aligned} \tag{3.23}$$

again because $\|\nabla Y_\varepsilon\widetilde{u}\|_{L^2(\Omega)}\le\|\nabla\widetilde{u}\|_{L^2(\Omega)}$ due to Lemma 3.1. Now well-known smoothing properties of the Dirichlet-Stokes semigroup $(e^{-tA})_{t\ge0}$ in Ω [17, page 201], [21] assert the existence of $\lambda>0$ and $C_5>0$ such that, for all $t>0$, we have

$$\|A^\beta e^{-tA}\varphi\|_{L^2(\Omega)}\le C_5t^{-\kappa}e^{-\lambda t}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi\in L^2_\sigma(\Omega) \tag{3.24}$$

with $\kappa := \beta + \frac{1}{p} - \frac{1}{2}$. Therefore, on the basis of a variation-of-constants representation of $\widetilde{u}$ associated with (3.22), using that for each $t > 0$, A^β commutes with e^{-tA} on $D(A^\beta)$, and that e^{-tA} acts as a contraction on $L^2_\sigma(\Omega)$, we can estimate

$$\begin{aligned}
\|A^\beta \widetilde{u}(\cdot,t)\|_{L^2(\Omega)} &\le \|A^\beta e^{-tA}\widetilde{u}_0\|_{L^2(\Omega)} + \int_0^t \|A^\beta e^{-(t-s)A}\widetilde{f}(\cdot,s)\|_{L^2(\Omega)} ds \\
&\le \|e^{-tA}A^\beta \widetilde{u}_0\|_{L^2(\Omega)} + C_5 \int_0^t (t-s)^{-\kappa} e^{-\lambda(t-s)} \|\widetilde{f}(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le \|A^\beta \widetilde{u}_0\|_{L^2(\Omega)} + C_4 C_5 \int_0^t (t-s)^{-\kappa} e^{-\lambda(t-s)} ds \\
&\le \|A^\beta \widetilde{u}_0\|_{L^2(\Omega)} + C_4 C_5 C_6 \qquad \text{for all } t \in (0,T),
\end{aligned}$$

with $C_6 := \int_0^\infty \sigma^{-\kappa} e^{-\lambda\sigma} d\sigma$ being finite due to the fact that as a consequence of (3.21) we have $\kappa < 1$.

Similarly, following a standard argument [14, page 109], for arbitrary $t_0 \in [0,T)$ and $t \in (t_0,T)$ we may use (3.24) to see that

$$\begin{aligned}
&\|A^\beta \widetilde{u}(\cdot,t) - A^\beta \widetilde{u}(\cdot,t_0)\|_{L^2(\Omega)} \le \|A^\beta e^{-tA}\widetilde{u}_0 - A^\beta e^{-t_0 A}\widetilde{u}_0\|_{L^2(\Omega)} \\
&\quad + \int_0^{t_0} \left\| A^\beta [e^{-(t-s)A} - e^{-(t_0-s)A}] \widetilde{f}(\cdot,s) \right\|_{L^2(\Omega)} ds \\
&\quad + \int_{t_0}^t \|A^\beta e^{-(t-s)A}\widetilde{f}(\cdot,s)\|_{L^2(\Omega)} ds \\
&= \left\| -\int_{t_0}^t A^\beta e^{-\tau A} A\widetilde{u}_0 d\tau \right\|_{L^2(\Omega)} + \int_0^{t_0} \left\| -\int_{t_0}^t A^{1+\beta} e^{-(\tau-s)A}\widetilde{f}(\cdot,s) d\tau \right\|_{L^2(\Omega)} ds \\
&\quad + \int_{t_0}^t \|A^\beta e^{-(t-s)A}\widetilde{f}(\cdot,s)\|_{L^2(\Omega)} ds \\
&\le C_5 \cdot \left\{ \int_{t_0}^t \tau^{-\beta} d\tau \right\} \cdot \|A\widetilde{u}_0\|_{L^2(\Omega)} \\
&\quad + C_4 C_5 \int_0^{t_0} \int_{t_0}^t (\tau-s)^{-(1+\beta)} d\tau ds + C_4 C_5 \int_{t_0}^t (t-s)^{-\beta} ds \\
&= \frac{C_5}{1-\beta} \|A\widetilde{u}_0\|_{L^2(\Omega)} \cdot (t^{1-\beta} - t_0^{1-\beta}) \\
&\quad + \frac{C_4 C_5}{\beta(1-\beta)} \cdot \left\{ (t-t_0)^{1-\beta} + t_0^{1-\beta} - t^{1-\beta} \right\} + \frac{C_4 C_5}{1-\beta} \cdot (t-t_0)^{1-\beta} \\
&\le \frac{C_5}{1-\beta} \|A\widetilde{u}_0\|_{L^2(\Omega)} \cdot (t-t_0)^{1-\beta} + \frac{C_4 C_5}{\beta(1-\beta)} \cdot (t-t_0)^{1-\beta} + \frac{C_4 C_5}{1-\beta} \cdot (t-t_0)^{1-\beta}
\end{aligned}$$

and conclude. □

4. Global existence in the approximate problems

With the above regularization properties of the fluid evolution in (2.3) at hand, we are now in the position to make sure that all these approximate problems are in fact globally solvable. To see this, we first combine two standard L^p testing procedures, when applied to the first two equations in (2.3), to obtain that under the assumption that $T_{\max,\varepsilon}$ is finite, both n_ε and c_ε belong to $L^\infty((0,T_{\max,\varepsilon});L^p(\Omega))$ for any finite p.

Lemma 4.1. *Assume that for some $\varepsilon\in(0,1)$ we have $T_{\max,\varepsilon}<\infty$. Then for all $p\ge 2$ there exists $C(p,\varepsilon)>0$ such that*

$$\int_\Omega n_\varepsilon^p(\cdot,t)\le C(p,\varepsilon)\qquad \textit{for all } t\in(0,T_{\max,\varepsilon}) \tag{4.1}$$

and

$$\int_\Omega c_\varepsilon^p(\cdot,t)\le C(p,\varepsilon)\qquad \textit{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.2}$$

Proof. Since $S_\varepsilon(x,\hat n,\hat c)=0$ for all $(x,\hat n,\hat c)\in\bar\Omega\times[0,\infty)^2$ with $\hat n\ge\frac{1}{\varepsilon}$ by (2.1), on testing the first equation in (2.3) by n_ε^{p-1} and using Young's inequality and $\nabla\cdot u_\varepsilon\equiv 0$, we see that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p+(p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 &= (p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\\
&\le (p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2\\
&\quad+\frac{p-1}{4}\int_\Omega n_\varepsilon^p\Big|S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\Big|^2\cdot|\nabla c_\varepsilon|^2\\
&\le (p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2\\
&\quad+\frac{(p-1)S_0^2}{4\varepsilon^p}\int_\Omega|\nabla c_\varepsilon|^2,
\end{aligned}$$

that is,

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p\le\frac{p(p-1)S_0^2}{4\varepsilon^p}\int_\Omega|\nabla c_\varepsilon|^2\qquad\text{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.3}$$

Moreover, multiplying the second equation in (2.3) by $(c_\varepsilon+1)^{p-1}$ and integrating by parts, again using Young's inequality, and noting that u_ε is solenoidal, we obtain

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega(c_\varepsilon+1)^p+(p-1)\int_\Omega(c_\varepsilon+1)^{p-2}|\nabla c_\varepsilon|^2+\int_\Omega c_\varepsilon(c_\varepsilon+1)^{p-1}\\
&=\int_\Omega n_\varepsilon(c_\varepsilon+1)^{p-1}\\
&\le\frac{1}{p}\int_\Omega n_\varepsilon^p+\frac{p-1}{p}\int_\Omega(c_\varepsilon+1)^p
\end{aligned}$$

and hence

$$\frac{S_0^2}{4\varepsilon^p}\frac{d}{dt}\int_\Omega (c_\varepsilon+1)^p + \frac{p(p-1)S_0^2}{4\varepsilon^p}\int_\Omega |\nabla c_\varepsilon|^2$$
$$\le \frac{S_0^2}{4\varepsilon^p}\int_\Omega n_\varepsilon^p + \frac{(p-1)S_0^2}{4\varepsilon^p}\int_\Omega (c_\varepsilon+1)^p \qquad \text{for all } t\in(0,T_{\max,\varepsilon}).$$

Adding this to (4.3) shows that writing $\kappa_\varepsilon := \max\{\frac{S_0^2}{4\varepsilon^p}, p-1\}$ we have

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon^p + \frac{S_0^2}{4\varepsilon^p}\int_\Omega (c_\varepsilon+1)^p\right\} \le \kappa_\varepsilon\cdot\left\{\int_\Omega n_\varepsilon^p + \frac{S_0^2}{4\varepsilon^p}\int_\Omega (c_\varepsilon+1)^p\right\}$$
$$\text{for all } t\in(0,T_{\max,\varepsilon})$$

and thus, on integration,

$$\int_\Omega n_\varepsilon^p + \frac{S_0^2}{4\varepsilon^p}\int_\Omega (c_\varepsilon+1)^p \le \left\{\int_\Omega n_0^p + \frac{S_0^2}{4\varepsilon^p}\int_\Omega (c_0+1)^p\right\}\cdot e^{\kappa_\varepsilon T_{\max,\varepsilon}}$$
$$\text{for all } t\in(0,T_{\max,\varepsilon}),$$

which yields both (4.1) and (4.2). □

In light of Lemma 3.4, this particularly implies two convenient boundedness properties of u_ε.

Lemma 4.2. *Suppose that $T_{\max,\varepsilon}<\infty$ for some $\varepsilon\in(0,1)$, and let $\beta\in(\frac12,1)$. Then there exists $C(\beta,\varepsilon)>0$ such that*

$$\|A^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C(\beta,\varepsilon) \qquad \textit{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.4}$$

In particular, there exists $C(\varepsilon)>0$ with the property that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C(\varepsilon) \qquad \textit{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.5}$$

Proof. Since Lemma 4.1 in particular warrants that n_ε belongs to $L^\infty((0,T_{\max,\varepsilon}); L^2(\Omega))$, then (4.4) directly results from Lemma 3.4. Noting that for any such β we have $D(A^\beta)\hookrightarrow L^\infty(\Omega)$ [16,21], from this we immediately obtain (4.5). □

Now in conjunction with the L^p bound for n_ε asserted by Lemma 4.1, the latter L^∞ estimate for u_ε rules out finite-time blow-up of ∇c_ε in any Lebesgue space with finite summability power.

Lemma 4.3. *Assume that $T_{\max,\varepsilon}<\infty$ for some $\varepsilon\in(0,1)$. Then for any $p\ge 2$ one can find $C(p,\varepsilon)>0$ such that*

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C(p,\varepsilon) \qquad \textit{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.6}$$

Proof. We fix any $\beta \in (\frac{1}{2}, 1)$ and $\beta_0 \in (\frac{1}{2}, \beta)$ and then obtain from known results for the associated fractional powers $(-\Delta + 1)^\beta$ and $(-\Delta + 1)^{\beta_0}$ of the sectorial realization of $-\Delta+1$ under homogeneous Neumann boundary conditions in $L^p(\Omega)$ [14,21] that there exist positive constants C_1, C_2 and C_3 such that with $a := \frac{\beta_0}{\beta} \in (0, 1)$ we have

$$\|\nabla\varphi\|_{L^p(\Omega)} \le C_1\|(-\Delta + 1)^{\beta_0}\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in C^2(\bar{\Omega}) \text{ such that } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega \tag{4.7}$$

and

$$\|(-\Delta + 1)^{\beta_0}\varphi\|_{L^p(\Omega)} \le C_2\|(-\Delta + 1)^{\beta}\varphi\|^a_{L^p(\Omega)}\|\varphi\|^{1-a}_{L^p(\Omega)} \qquad \text{for all } \varphi \in C^2(\bar{\Omega}) \text{ such that } \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega, \tag{4.8}$$

as well as

$$\|(-\Delta + 1)^{\beta}e^{t(\Delta-1)}\varphi\|_{L^p(\Omega)} \le C_3 t^{-\beta}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in L^p(\Omega), \tag{4.9}$$

where $(e^{t\Delta})_{t\ge0}$ denotes the correspondingly generated analytic semigroup.

Therefore, if for arbitrary $T \in (\tau, T_{\max,\varepsilon})$ with $\tau := \frac{T_{\max,\varepsilon}}{2}$ we introduce the finite number

$$M_\varepsilon(T) := \sup_{t\in(\tau,T)} \|(-\Delta + 1)^\beta c_\varepsilon(\cdot, t)\|_{L^p(\Omega)},$$

then by means of a variation-of-constants representation of $c_\varepsilon(\cdot, t)$ for $t \in (\tau, T_{\max,\varepsilon})$ and using that $e^{t(\Delta-1)}$ commutes with $(-\Delta + 1)^\beta$ on $D((-\Delta + 1)^\beta)$ and acts as a contraction on $L^p(\Omega)$, we can estimate

$$\begin{aligned}
M_\varepsilon(T) &= \sup_{t\in(\tau,T)} \bigg\|(-\Delta + 1)^\beta e^{(t-\tau)(\Delta-1)}c_\varepsilon(\cdot, \tau) \\
&\quad + \int_\tau^t (-\Delta + 1)^\beta e^{(t-s)(\Delta-1)}n_\varepsilon(\cdot, s)ds \\
&\quad - \int_\tau^t (-\Delta + 1)^\beta e^{(t-s)(\Delta-1)}\Big(u_\varepsilon(\cdot, s) \cdot \nabla c_\varepsilon(\cdot, s)\Big)ds\bigg\|_{L^p(\Omega)} \\
&\le \|(-\Delta + 1)^\beta c_\varepsilon(\cdot, \tau)\|_{L^p(\Omega)} \\
&\quad + C_3 \sup_{t\in(\tau,T)} \int_\tau^t (t-s)^{-\beta}\|n_\varepsilon(\cdot, s)\|_{L^p(\Omega)}ds \\
&\quad + C_3 \sup_{t\in(\tau,T)} \int_\tau^t (t-s)^{-\beta}\Big\|u_\varepsilon(\cdot, s) \cdot \nabla c_\varepsilon(\cdot, s)\Big\|_{L^p(\Omega)}ds \\
&\le C_4 + C_5 \sup_{t\in(\tau,T)} \int_\tau^t (t-s)^{-\beta}\|\nabla c_\varepsilon(\cdot, s)\|_{L^p(\Omega)}ds \quad \text{for all } T \in (\tau, T_{\max,\varepsilon}),
\end{aligned} \tag{4.10}$$

with

$$C_4 := \|(-\Delta+1)^{\beta} c_\varepsilon(\cdot,\tau)\|_{L^p(\Omega)} + C_3 \|n_\varepsilon\|_{L^\infty((\tau,T_{\max,\varepsilon});L^p(\Omega))} \cdot \frac{(T_{\max,\varepsilon}-\tau)^{1-\beta}}{1-\beta}$$

and

$$C_5 := \|u_\varepsilon\|_{L^\infty(\Omega\times(\tau,T_{\max,\varepsilon}))},$$

both being finite due to Lemma 4.1, Lemma 4.2 and the fact that $c_\varepsilon(\cdot,\tau)$ belongs to $C^2(\bar{\Omega})$ according to Lemma 2.1. Now since also $C_6 := \sup_{t\in(\tau,T_{\max,\varepsilon})} \|c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} < \infty$ thanks to Lemma 4.1, by (4.7) and (4.8) we have

$$\begin{aligned} \|\nabla c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} &\le C_1 \|(-\Delta+1)^{\beta_0} c_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \\ &\le C_1 C_2 \|(-\Delta+1)^{\beta} c_\varepsilon(\cdot,s)\|^a_{L^p(\Omega)} \|c_\varepsilon(\cdot,s)\|^{1-a}_{L^p(\Omega)} \\ &\le C_1 C_2 C_6^{1-a} M_\varepsilon^a(T) \qquad \text{for all } s\in(\tau,T), \end{aligned}$$

so that (4.10) yields the inequality

$$M_\varepsilon(T) \le C_4 + C_7 M_\varepsilon^a(T) \qquad \text{for all } T\in(\tau,T_{\max,\varepsilon})$$

with $C_7 := \frac{1}{1-\beta}\big(T_{\max,\varepsilon}-\tau\big)^{1-\beta} C_1C_2C_5C_6^{1-a}$. As $a<1$, this entails that

$$M_\varepsilon(T) \le C_8 := \max\left\{ \left(\frac{C_4}{C_7}\right)^{\frac{1}{a}},\ (2C_7)^{\frac{1}{1-a}} \right\} \qquad \text{for all } T\in(\tau,T_{\max,\varepsilon}),$$

and that hence, again by (4.7) and (4.8),

$$\|\nabla c_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C_1C_2C_8^aC_6^{1-a} \qquad \text{for all } t\in(\tau,T_{\max,\varepsilon}),$$

which in view of the inclusion $\nabla c_\varepsilon \in L^\infty((0,\tau);L^p(\Omega;\mathbb{R}^2))$, as asserted by Lemma 2.1, completes the proof. □

This, inter alia, allows us to pass to the limit $p\to\infty$ in the former statement from Lemma 4.1 by means of a standard recursive argument of Moser type.

Lemma 4.4. *If $T_{\max,\varepsilon}<\infty$ for some $\varepsilon\in(0,1)$, then there exists $C(\varepsilon)>0$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C(\varepsilon) \qquad \textit{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.11}$$

Proof. Writing the first equation in (2.3) in the form

$$n_{\varepsilon t} = \Delta n_\varepsilon + \nabla\cdot f_\varepsilon(x,t), \qquad x\in\Omega,\ t\in(0,T_{\max,\varepsilon}),$$

with

$$f_\varepsilon(x,t) := -n_\varepsilon(x,t)S_\varepsilon(x,n_\varepsilon(x,t),c_\varepsilon(x,t))\cdot\nabla c_\varepsilon(x,t)$$
$$-n_\varepsilon(x,t)u_\varepsilon(x,t), \qquad x\in\Omega,\ t\in(0,T_{\max,\varepsilon}),$$

we observe that according to the local smoothness properties provided by Lemma 2.1 and the estimates provided by Lemma 4.1, Lemma 4.2 and Lemma 4.3, under the hypothesis that $T_{\max,\varepsilon}$ is finite we know that $n_\varepsilon \in L^\infty((0,T_{\max,\varepsilon});L^p(\Omega))$ and $f_\varepsilon \in L^\infty((0,T_{\max,\varepsilon});L^p(\Omega;\mathbb{R}^2))$ for all $p\in(1,\infty)$.

Therefore, (4.11) readily results upon a straightforward application of a Moser-type iteration procedure (*cf.*, *e.g.*, [45, Lemma A.1] for a general result in this direction which precisely covers the present situation). □

We have now gathered all ingredients necessary for our proof of global existence in (2.3).

Lemma 4.5. *For each* $\varepsilon\in(0,1)$*, the solution of* (2.3) *is global in time; that is, in Lemma* 2.1 *we have* $T_{\max,\varepsilon}=\infty$.

Proof. Due to the extensibility criterion (2.6), this is an evident consequence of Lemma 4.4, Lemma 4.3 and Lemma 4.2. □

5. A space-time L^2 bound for ∇u_ε

In order to prepare an appropriate passage to the limit $\varepsilon\to0$ to be performed in Lemma 10.1, a natural next goal consists in establishing suitable ε-independent bounds for the solutions of (2.3). Here unlike in the fluid-free situation obtained on letting $u_\varepsilon\equiv0$, where previous studies have shown that estimates for both n_ε and ∇c_ε in high-power spatial L^p spaces can be obtained in a rather straightforward manner [24,45], in the present context involving fluid interaction such a direct approach seems inadequate due to the lack of sufficient a priori knowledge on regularity properties of u_ε. We will thus firstly be concerned with the derivation of some basic ε-independent information on u_ε, for which in accordance with the results from Section 3, and in particular from Lemma 3.2, it is sufficient to provide integral estimate for the factor n_ε in the forcing term of the fluid equation which involve the integrability power 2 with respect to time, but which fortunately may refer to spatial L^p spaces with an arbitrarily small exponent $p>1$.

To see that a property of this type indeed is enforced by our subcriticality assumption that $\alpha>0$ in (1.7), in the following lemma we will track the time evolution of a certain sublinear functional of n_ε, for which (1.7) entails a favorable quasi-energy property when appropriately combined with $\int_\Omega c_\varepsilon^2$. The use of such functionals with sublinear growth with respect to the unknown seems rather unusual in the context of cross-diffusive systems of the considered class, especially when intended to be a starting point of a series of arguments finally resulting in boundedness and smoothness of solutions; after all, in constructing certain weak solutions to chemotaxis and also to some chemotaxis-fluid systems some precedent studies rely on the analysis of functionals with even only logarithmic growth, but in most cases the correspondingly obtained estimates could not be used for the derivation of substantial further properties beyond [57,58,61].

Lemma 5.1. *There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$ we have*

$$\int_t^{t+1}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 \le C \qquad \textit{for all } t \ge 0 \tag{5.1}$$

and

$$\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^2 \le C \qquad \textit{for all } t \ge 0. \tag{5.2}$$

Proof. Noting that n_ε is positive in $\bar\Omega \times (0, \infty)$ by the strong maximum principle, we may multiply the first equation in (2.3) by $n_\varepsilon^{2\alpha-1}$ and integrate by parts to see, by means of Young's inequality and (1.7), that

$$\begin{aligned}
&-\frac{1}{2\alpha}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\alpha} + (1-2\alpha)\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 \\
&= (1-2\alpha)\int_\Omega n_\varepsilon^{2\alpha-1}\nabla n_\varepsilon \cdot \Big(S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \\
&\le \frac{1-2\alpha}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 + \frac{1-2\alpha}{2}\int_\Omega n_\varepsilon^{2\alpha}\Big|S_\varepsilon(x, n_\varepsilon, c_\varepsilon)\Big|^2\cdot|\nabla c_\varepsilon|^2 \\
&\le \frac{1-2\alpha}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 + \frac{1-2\alpha}{2}S_0^2\int_\Omega n_\varepsilon^{2\alpha}(n_\varepsilon+1)^{-2\alpha}|\nabla c_\varepsilon|^2 \\
&\le \frac{1-2\alpha}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 + C_1\int_\Omega |\nabla c_\varepsilon|^2 \qquad \text{for all } t > 0,
\end{aligned} \tag{5.3}$$

with $C_1 := \frac{1-2\alpha}{2}S_0^2$, where we also have used that $\nabla \cdot u_\varepsilon \equiv 0$. In order to compensate the rightmost summand herein appropriately, we test the second equation in (2.3) by c_ε to obtain, again, due to the solenoidality of u_ε, that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^2 = \int_\Omega n_\varepsilon c_\varepsilon \qquad \text{for all } t > 0. \tag{5.4}$$

Here we fix any $\theta > 1$ satisfying

$$\theta < \frac{1}{1-\alpha} \tag{5.5}$$

and apply the Hölder inequality to find that

$$\begin{aligned}
\int_\Omega n_\varepsilon c_\varepsilon &\le \Big\{\int_\Omega n_\varepsilon^\theta\Big\}^{\frac{1}{\theta}} \cdot \Big\{\int_\Omega c_\varepsilon^{\frac{\theta}{\theta-1}}\Big\}^{\frac{\theta-1}{\theta}} \\
&= \|n_\varepsilon^\alpha\|_{L^{\frac{\theta}{\alpha}}(\Omega)}^{\frac{1}{\alpha}} \|c_\varepsilon\|_{L^{\frac{\theta}{\theta-1}}(\Omega)} \qquad \text{for all } t > 0.
\end{aligned} \tag{5.6}$$

Now, since in the present two-dimensional setting we have $W^{1,2}(\Omega) \hookrightarrow L^{\frac{\theta}{\theta-1}}(\Omega)$, there exists $C_2 > 0$ fulfilling

$$\|c_\varepsilon\|^2_{L^{\frac{\theta}{\theta-1}}(\Omega)} \le C_2\|\nabla c_\varepsilon\|^2_{L^2(\Omega)} + C_2\|c_\varepsilon\|^2_{L^1(\Omega)} \qquad \text{for all } t > 0,$$

so that in view of (2.8) we infer that

$$\|c_\varepsilon\|^2_{L^{\frac{\theta}{\theta-1}}(\Omega)} \le C_2 \int_\Omega |\nabla c_\varepsilon|^2 + C_3 \qquad \text{for all } t > 0$$

with some $C_3 > 0$. By means of Young's inequality, in (5.6) we can therefore proceed to estimate

$$\begin{aligned} \int_\Omega n_\varepsilon c_\varepsilon &\le \frac{1}{2C_2}\|c_\varepsilon\|^2_{L^{\frac{\theta}{\theta-1}}(\Omega)} + \frac{C_2}{2}\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)} \\ &\le \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{C_3}{2C_2} + \frac{C_2}{2}\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)} \qquad \text{for all } t > 0. \end{aligned} \tag{5.7}$$

We finally make sure that here the last summand can essentially be absorbed by the dissipated quantity in (5.3). Indeed, invoking the Gagliardo-Nirenberg inequality and recalling (2.7), we find $C_4 > 0$ and $C_5 > 0$ such that

$$\begin{aligned} \frac{C_2}{2}\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{\theta}{\alpha}}(\Omega)} &\le C_4\|\nabla n_\varepsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)}\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha\theta}}_{L^{\frac{1}{\alpha}}(\Omega)} + C_4\|n_\varepsilon^\alpha\|^{\frac{2}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} \\ &\le C_5\|\nabla n_\varepsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)} + C_5 \qquad \text{for all } t > 0, \end{aligned}$$

so that due to (5.7), from (5.4) we obtain the inequality

$$\frac{d}{dt}\int_\Omega c_\varepsilon^2 + \int_\Omega |\nabla c_\varepsilon|^2 + 2\int_\Omega c_\varepsilon^2 \le 2C_5\|\nabla n_\varepsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)} + C_6 \qquad \text{for all } t > 0$$

with $C_6 := \frac{C_3}{C_2} + 2C_5$. By taking an appropriate linear combination of this with (5.3), we thus infer that

$$\begin{aligned} &\frac{d}{dt}\left\{-\frac{1}{2\alpha}\int_\Omega n_\varepsilon^{2\alpha} + 2C_1\int_\Omega c_\varepsilon^2\right\} + (1-2\alpha)\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 \\ &\quad + 2C_1\int_\Omega |\nabla c_\varepsilon|^2 + 4C_1\int_\Omega c_\varepsilon^2 \\ &\le \frac{1-2\alpha}{2}\int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 + C_1\int_\Omega |\nabla c_\varepsilon|^2 \\ &\quad + 4C_1C_5\|\nabla n_\varepsilon^\alpha\|^{\frac{2(\theta-1)}{\alpha\theta}}_{L^2(\Omega)} + 2C_1C_6 \qquad \text{for all } t > 0, \end{aligned}$$

that is,

$$\frac{d}{dt}\left\{-\frac{1}{2\alpha}\int_\Omega n_\varepsilon^{2\alpha}+2C_1\int_\Omega c_\varepsilon^2\right\}+\frac{1-2\alpha}{2\alpha^2}\int_\Omega|\nabla n_\varepsilon^\alpha|^2 + C_1\int_\Omega|\nabla c_\varepsilon|^2+4C_1\int_\Omega c_\varepsilon^2 \le 4C_1C_5\|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^{\frac{2(\theta-1)}{\alpha\theta}}+2C_1C_6 \qquad \text{for all } t>0. \tag{5.8}$$

Here since our restriction (5.5) warrants that

$$\frac{2(\theta-1)}{\alpha\theta}=\frac{2}{\alpha}\cdot\Big(1-\frac{1}{\theta}\Big)<\frac{2}{\alpha}\cdot\Big(1-(1-\alpha)\Big)=2,$$

we may once again employ Young's inequality to find $C_7>0$ such that

$$4C_1C_5\|\nabla n_\varepsilon^\alpha\|_{L^2(\Omega)}^{\frac{2(\theta-1)}{\alpha\theta}}\le\frac{1-2\alpha}{4\alpha^2}\int_\Omega|\nabla n_\varepsilon^\alpha|^2+C_7 \qquad \text{for all } t>0. \tag{5.9}$$

Thus, if we write

$$y(t):=-\frac{1}{2\alpha}\int_\Omega n_\varepsilon^{2\alpha}(\cdot,t)+2C_1\int_\Omega c_\varepsilon^2(\cdot,t), \qquad t\ge 0,$$

and

$$g(t):=\frac{1-2\alpha}{4\alpha^2}\int_\Omega|\nabla n_\varepsilon^\alpha(\cdot,t)|^2+C_1\int_\Omega|\nabla c_\varepsilon(\cdot,t)|^2, \qquad t>0,$$

then, since $4C_1\int_\Omega c_\varepsilon^2\ge 2y(t)$ for all $t>0$, from (5.8) and (5.9) we obtain that

$$y'(t)+2y(t)+g(t)\le 2C_1C_6+C_7 \qquad \text{for all } t>0. \tag{5.10}$$

As g is nonnegative, by an ODE comparison this firstly implies that

$$y(t)\le C_8:=\max\left\{-\frac{1}{2\alpha}\int_\Omega n_0^{2\alpha}+2C_1\int_\Omega c_0^2\,,\ \frac{2C_1C_6+C_7}{2}\right\} \qquad \text{for all } t>0, \tag{5.11}$$

and thereafter we secondly conclude from (5.10) on integration that

$$\int_t^{t+1}g(s)ds\le y(t)-y(t+1)-2\int_t^{t+1}y(s)ds+2C_1C_6+C_7 \quad \text{for all } t\ge 0.$$

Since our overall assumption $\alpha<\frac{1}{2}$ ensures that by the Hölder inequality and (2.7) we have

$$-y(t)\le\frac{1}{2\alpha}\int_\Omega n_\varepsilon^{2\alpha}(\cdot,t)\le\frac{|\Omega|^{1-2\alpha}}{2\alpha}\cdot\left\{\int_\Omega n_\varepsilon(\cdot,t)\right\}^{2\alpha} = C_9:=\frac{|\Omega|^{1-2\alpha}}{2\alpha}\cdot\left\{\int_\Omega n_0\right\}^{2\alpha} \qquad \text{for all } t\ge 0,$$

together with (5.11) this implies that

$$\frac{1-2\alpha}{4\alpha^2}\int_t^{t+1}\int_\Omega |\nabla n_\varepsilon^\alpha|^2 + C_1\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^2$$
$$= \int_t^{t+1} g(s)ds \le C_8 + 3C_9 + 2C_1C_6 + C_7 \qquad \text{for all } t \ge 0$$

and thereby establishes both (5.1) and (5.2). □

Upon interpolation with the mass conservation property (2.7), the estimate (5.1) entails the following.

Lemma 5.2. *For all $p > 1$, there exists $C(p) > 0$ with the property that for all $\varepsilon \in (0,1)$,*

$$\int_t^{t+1} \|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)}^{\frac{2p\alpha}{p-1}} ds \le C(p) \qquad \textit{for all } t \ge 0.$$

In particular, one can find $C > 0$ such that

$$\int_t^{t+1} \|n_\varepsilon(\cdot,s)\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} ds \le C \qquad \textit{for all } t \ge 0 \tag{5.12}$$

whenever $\varepsilon \in (0,1)$.

Proof. Since the Gagliardo-Nirenberg inequality, in conjunction with (2.7), shows that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}\int_t^{t+1} \|n_\varepsilon(\cdot,s)\|_{L^p(\Omega)}^{\frac{2p\alpha}{p-1}} ds &= \int_t^{t+1} \|n_\varepsilon^\alpha(\cdot,s)\|_{L^{\frac{p}{\alpha}}(\Omega)}^{\frac{2p}{p-1}} ds \\ &\le C_1\int_t^{t+1} \|\nabla n_\varepsilon^\alpha(\cdot,s)\|^2_{L^2(\Omega)} \|n_\varepsilon^\alpha(\cdot,s)\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{2}{p-1}} ds \\ &\quad + C_1\int_t^{t+1} \|n_\varepsilon^\alpha(\cdot,s)\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{2p}{p-1}} ds \\ &\le C_2\int_t^{t+1} \|\nabla n_\varepsilon^\alpha(\cdot,s)\|^2_{L^2(\Omega)} ds + C_2 \qquad \text{for all } t \ge 0,\end{aligned}$$

the claim directly results from Lemma 5.1. □

Now thanks to Lemma 5.2, we thereby obtain a basic regularity property of u_ε in the following form.

Lemma 5.3. *There exists $C > 0$ such that*

$$\int_t^{t+1}\int_\Omega |\nabla u_\varepsilon|^2 \le C \qquad \textit{for all } t \ge 0$$

whenever $\varepsilon \in (0,1)$.

Proof. In view of (5.12), we only need to apply Lemma 3.2 to $\tau := 1$ and $p := \frac{1}{1-\alpha}$. □

6. A time-independent spatial L^2 bound for ∇u_ε

Based on the above information, we proceed to derive further regularity properties of the solution to (2.3), and in particular of u_ε. In this direction, the main outcome of this section will improve Lemma 5.3 in Lemma 6.3 by providing a bound for ∇u_ε in $L^2(\Omega;\mathbb{R}^2)$ which is independent of $t>0$. In view of Lemma 3.3, for this it will be sufficient to achieve a bound for $\int_t^{t+1}\int_\Omega n_\varepsilon^2$, which will be accomplished in Lemma 6.2 on the basis of the following a priori information on c_ε that is widely independent of any regularity property of u_ε but strongly relies on Lemma 5.2.

Lemma 6.1. *For each $p\ge 2$ one can find $C(p)>0$ such that for any choice of $\varepsilon\in(0,1)$,*

$$\int_\Omega c_\varepsilon^p(\cdot,t)\le C(p) \qquad \textit{for all } t>0. \tag{6.1}$$

Proof. Once more we use c_ε^{p-1} as a test function in the second equation of (2.3); this time we estimate the term on the right by using the Hölder inequality with the exponent $\frac{p-2\alpha}{p-4\alpha}$, which is positive since $\alpha<\frac12$ and $p\ge 2$. Accordingly, we obtain

$$\begin{aligned}\frac1p\frac{d}{dt}\int_\Omega c_\varepsilon^p+(p-1)\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2+\int_\Omega c_\varepsilon^p&=\int_\Omega n_\varepsilon c_\varepsilon^{p-1}\\ &\le \|n_\varepsilon\|_{L^{\frac{p-2\alpha}{p-4\alpha}}(\Omega)}\cdot\left\{\int_\Omega c_\varepsilon^{\frac{(p-1)(p-2\alpha)}{2\alpha}}\right\}^{\frac{2\alpha}{p-2\alpha}} \qquad \text{for all } t>0.\end{aligned} \tag{6.2}$$

Here, by means of the Gagliardo-Nirenberg inequality and (2.8), we can find $C_1>0$ and $C_2>0$ such that

$$\begin{aligned}\left\{\int_\Omega c_\varepsilon^{\frac{(p-1)(p-2\alpha)}{2\alpha}}\right\}^{\frac{2\alpha}{p-2\alpha}}&=\|c_\varepsilon^{\frac p2}\|_{L^{\frac{(p-1)(p-2\alpha)}{p\alpha}}(\Omega)}^{\frac{2(p-1)}{p}}\\ &\le C_1\|\nabla c_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^{\frac{2(p-2\alpha-1)}{p-2\alpha}}\|c_\varepsilon^{\frac p2}\|_{L^{\frac2p}(\Omega)}^{\frac{4\alpha}{p(p-2\alpha)}}+C_1\|c_\varepsilon^{\frac p2}\|_{L^{\frac2p}(\Omega)}^{\frac{2(p-1)}{p}}\\ &\le C_2\|\nabla c_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^{\frac{2(p-2\alpha-1)}{p-2\alpha}}+C_2\\ &\le 2C_2\left\{\|\nabla c_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^2+1\right\}^{\frac{p-2\alpha-1}{p-2\alpha}} \qquad \text{for all } t>0,\end{aligned}$$

so that Young's inequality provides $C_3>0$ fulfilling

$$\begin{aligned}&\|n_\varepsilon\|_{L^{\frac{p-2\alpha}{p-4\alpha}}(\Omega)}\cdot\left\{\int_\Omega c_\varepsilon^{\frac{(p-1)(p-2\alpha)}{2\alpha}}\right\}^{\frac{2\alpha}{p-2\alpha}}\\ &\le\frac{4(p-1)}{p^2}\cdot\left\{\|\nabla c_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^2+1\right\}+C_3\|n_\varepsilon\|_{L^{\frac{p-2\alpha}{p-4\alpha}}(\Omega)}^{p-2\alpha}\\ &=(p-1)\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2+C_3\|n_\varepsilon\|_{L^{\frac{p-2\alpha}{p-4\alpha}}(\Omega)}^{p-2\alpha}+\frac{4(p-1)}{p^2}\end{aligned}$$

for all $t > 0$. Therefore, (6.2) shows that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega c_\varepsilon^p + \int_\Omega c_\varepsilon^p \le C_3 \|n_\varepsilon\|^{p-2\alpha}_{L^{\frac{p-2\alpha}{p-4\alpha}}(\Omega)} + \frac{4(p-1)}{p^2} \qquad \text{for all } t > 0,$$

where observing that

$$\frac{2 \cdot \frac{p-2\alpha}{p-4\alpha} \cdot \alpha}{\frac{p-2\alpha}{p-4\alpha} - 1} = p - 2\alpha,$$

we may invoke Lemma 5.2 to find $C_4 > 0$ such that

$$\int_t^{t+1} \left\{ C_3 \|n_\varepsilon(\cdot,s)\|^{p-2\alpha}_{L^{\frac{p-2\alpha}{p-4\alpha}}(\Omega)} + \frac{4(p-1)}{p^2} \right\} ds \le C_4 \qquad \text{for all } t \ge 0.$$

This enables us to apply Lemma 2.3 to conclude that

$$\int_\Omega c_\varepsilon^p \le \int_\Omega c_0^p + \frac{pC_4}{1-e^{-p}} \qquad \text{for all } t > 0$$

whenever $\varepsilon \in (0,1)$. □

We are now prepared to trace the evolution of the superlinear functional $\int_\Omega n_\varepsilon \ln n_\varepsilon$, which has turned out to be fruitful in numerous previous works on chemotaxis systems, especially, but not exclusively, in two-dimensional situations [3,7,38]. In the present context, we shall see that when suitably combined with $\int_\Omega |\nabla c_\varepsilon|^2$, this functional indeed plays the role of a quasi-energy, where in estimating the respective destabilizing contributions of the taxis and the fluid interaction terms arising during the corresponding testing procedure, substantial use will be made of both Lemma 6.1 and Lemma 5.3.

Lemma 6.2. *There exists $C > 0$ such that for any $\varepsilon \in (0,1)$ we have*

$$\int_t^{t+1}\int_\Omega n_\varepsilon^2 \le C \qquad \textit{for all } t \ge 0 \tag{6.3}$$

and

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2 \le C \qquad \textit{for all } t > 0. \tag{6.4}$$

Proof. Again using that n_ε is positive in $\bar\Omega \times (0,\infty)$, we may multiply the first equation in (2.3) by $\ln n_\varepsilon$ to see on integrating by parts and employing Young's inequality, as well as (1.7), that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} &= \int_\Omega \nabla n_\varepsilon \cdot \Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot \nabla c_\varepsilon \Big) \\
&\le \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{1}{2}\int_\Omega n_\varepsilon \Big| S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\Big|^2 \cdot |\nabla c_\varepsilon|^2 \\
&\le \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{S_0^2}{2}\int_\Omega n_\varepsilon (1+n_\varepsilon)^{-2\alpha}|\nabla c_\varepsilon|^2 \qquad \text{for all } t > 0
\end{aligned}$$

and hence

$$\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \le \frac{S_0^2}{2}\int_\Omega n_\varepsilon^{1-2\alpha}|\nabla c_\varepsilon|^2 \qquad \text{for all } t>0. \tag{6.5}$$

In order to control the term on the right-hand side herein from above and to estimate the second summand on the left from below, we use the Gagliardo-Nirenberg inequality together with (2.7) to find $C_1>0$ and $C_2>0$ such that

$$\begin{aligned}\int_\Omega n_\varepsilon^2 &= \|\sqrt{n_\varepsilon}\|_{L^4(\Omega)}^4\\ &\le C_1\|\nabla\sqrt{n_\varepsilon}\|_{L^2(\Omega)}^2\|\sqrt{n_\varepsilon}\|_{L^2(\Omega)}^2 + C_1\|\sqrt{n_\varepsilon}\|_{L^2(\Omega)}^4\\ &\le C_2\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_2 \qquad \text{for all } t>0.\end{aligned}$$

Therefore, we firstly have

$$\frac{1}{2}\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \ge \frac{1}{2C_2}\int_\Omega n_\varepsilon^2 - \frac{1}{2} \qquad \text{for all } t>0,$$

and by means of Young's inequality we secondly obtain $C_3>0$ such that

$$\frac{S_0^2}{2}\int_\Omega n_\varepsilon^{1-2\alpha}|\nabla c_\varepsilon|^2 \le \frac{1}{4C_2}\int_\Omega n_\varepsilon^2 + C_3\int_\Omega |\nabla c_\varepsilon|^{\frac{4}{1+2\alpha}} \qquad \text{for all } t>0,$$

whence (6.5) implies that

$$\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{4C_2}\int_\Omega n_\varepsilon^2 \le \frac{1}{2} + C_3\int_\Omega |\nabla c_\varepsilon|^{\frac{4}{1+2\alpha}} \qquad \text{for all } t>0. \tag{6.6}$$

Now the summand on the right can essentially be absorbed by making use of Lemma 6.1 and the dissipative properties of the second equation in (2.3). Indeed testing the latter by $-\Delta c_\varepsilon$ shows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 &= -\int_\Omega n_\varepsilon \Delta c_\varepsilon + \int_\Omega (u_\varepsilon\cdot\nabla c_\varepsilon)\Delta c_\varepsilon\\ = -\int_\Omega n_\varepsilon\Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon)&\\ = -\int_\Omega n_\varepsilon\Delta c_\varepsilon - \int_\Omega \nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \quad \text{for all } t>0,&\end{aligned} \tag{6.7}$$

because

$$\int_\Omega \nabla c_\varepsilon\cdot(D^2 c_\varepsilon\cdot u_\varepsilon) = \frac{1}{2}\int_\Omega u_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^2 = 0 \qquad \text{for all } t>0$$

due to the fact that $\nabla \cdot u_\varepsilon \equiv 0$. Here since combining the Gagliardo-Nirenberg inequality with well-known elliptic regularity theory [19], we can pick $C_3 > 0$ such that

$$\|\nabla c_\varepsilon\|_{L^4(\Omega)}^2 \le C_3 \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \qquad \text{for all } t > 0,$$

using the Cauchy-Schwarz inequality and Young's inequality we can estimate

$$\begin{aligned}
&-\int_\Omega \nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) \le \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^4(\Omega)}^2 \\
&\le C_3 \|\nabla u_\varepsilon\|_{L^2(\Omega)} \|\Delta c_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^2(\Omega)} \\
&\le \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + C_3^2 \left\{ \int_\Omega |\nabla u_\varepsilon|^2 \right\} \cdot \left\{ \int_\Omega |\nabla c_\varepsilon|^2 \right\} \qquad \text{for all } t > 0.
\end{aligned}$$

As the Cauchy-Schwarz inequality furthermore warrants that

$$-\int_\Omega n_\varepsilon \Delta c_\varepsilon \le \frac{1}{4} \int_\Omega |\Delta c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 \qquad \text{for all } t > 0,$$

from (6.7) we thus infer that

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 + 2 \int_\Omega |\nabla c_\varepsilon|^2 \\
&\le 2 \int_\Omega n_\varepsilon^2 + 2C_3^2 \left\{ \int_\Omega |\nabla u_\varepsilon|^2 \right\} \cdot \left\{ \int_\Omega |\nabla c_\varepsilon|^2 \right\} \qquad \text{for all } t > 0
\end{aligned}$$

and that hence, by (6.6), writing $a := \frac{1}{16C_2}$ we have

$$\begin{aligned}
&\frac{d}{dt} \left\{ \int_\Omega n_\varepsilon \ln n_\varepsilon + a \int_\Omega |\nabla c_\varepsilon|^2 \right\} + 2a \int_\Omega n_\varepsilon^2 + a \int_\Omega |\Delta c_\varepsilon|^2 + 2a \int_\Omega |\nabla c_\varepsilon|^2 \\
&\le \frac{1}{2} + C_3 \int_\Omega |\nabla c_\varepsilon|^{\frac{4}{1+2\alpha}} + 2aC_3^2 \left\{ \int_\Omega |\nabla u_\varepsilon|^2 \right\} \cdot \left\{ \int_\Omega |\nabla c_\varepsilon|^2 \right\} \quad \text{for all } t > 0.
\end{aligned} \tag{6.8}$$

We now once more apply the Gagliardo-Nirenberg inequality together with elliptic regularity estimates and Young's inequality to see that, in view of Lemma 6.1 applied to $p := \frac{1}{\alpha}$, we can find positive constants C_4, C_5 and C_6 fulfilling

$$\begin{aligned}
\frac{1}{2} + C_3 \int_\Omega |\nabla c_\varepsilon|^{\frac{4}{1+2\alpha}} &= \frac{1}{2} + C_3 \|\nabla c_\varepsilon\|_{L^{\frac{4}{1+2\alpha}}(\Omega)}^{\frac{4}{1+2\alpha}} \\
&\le \frac{1}{2} + C_4 \|\Delta c_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{1+2\alpha}} \|c_\varepsilon\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{2}{1+2\alpha}} + C_4 \|c_\varepsilon\|_{L^{\frac{1}{\alpha}}(\Omega)}^{\frac{4}{1+2\alpha}} \\
&\le C_5 \|\Delta c_\varepsilon\|_{L^2(\Omega)}^{\frac{2}{1+2\alpha}} + C_5 \\
&\le a \int_\Omega |\Delta c_\varepsilon|^2 + C_6 \qquad \text{for all } t > 0.
\end{aligned}$$

In light of this, from (6.8) we thus infer that

$$y(t) := \int_\Omega n_\varepsilon(\cdot,t)\ln n_\varepsilon(\cdot,t) + a\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2, \qquad t \ge 0,$$

as well as

$$g(t) := 2a\int_\Omega n_\varepsilon^2(\cdot,t) \quad \text{and} \quad h(t) := 2C_3^2\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2, \qquad t > 0,$$

satisfy

$$y'(t) + g(t) \le h(t)\left\{y(t) + \frac{|\Omega|}{e}\right\} + C_6 \qquad \text{for all } t > 0, \tag{6.9}$$

because for all $t > 0$ we have

$$-\int_\Omega n_\varepsilon \ln n_\varepsilon \le \frac{|\Omega|}{e}, \tag{6.10}$$

due to the fact that $\xi\ln\xi \ge -\frac{1}{e}$ for all $\xi > 0$. In order to integrate this appropriately, we recall that according to Lemma 5.3, Lemma 5.2 and Lemma 5.1 we can find $C_7 > 0$ and $C_8 > 0$ such that

$$\int_{t-1}^t h(s)ds = 2C_3^2\int_{t-1}^t \left\{\int_\Omega |\nabla u_\varepsilon|^2\right\} \le C_7 \qquad \text{for all } t \ge 1 \tag{6.11}$$

and

$$\int_{t-1}^t \left\{\|n_\varepsilon(\cdot,s)\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} + \|\nabla c_\varepsilon(\cdot,s)\|^2_{L^2(\Omega)}\right\}ds \le C_8 \qquad \text{for all } t \ge 1,$$

so that, for each fixed $t > 0$, we can find $t_\star(t) \equiv t_\star(t;\varepsilon) \ge 0$ such that $t_\star(t) \in (t-1,t)$ and

$$\begin{aligned}&\|n_\varepsilon(\cdot,t_\star(t))\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} + \|\nabla c_\varepsilon(\cdot,t_\star(t))\|^2_{L^2(\Omega)}\\ &\le C_9 := \max\left\{C_8\,,\ \|n_0\|^2_{L^{\frac{1}{1-\alpha}}(\Omega)} + \|\nabla c_0\|^2_{L^2(\Omega)}\right\}.\end{aligned}$$

By validity of the elementary inequality $\xi\ln\xi \le \frac{1-\alpha}{\alpha e}\xi^{\frac{1}{1-\alpha}}$ for all $\xi > 0$, this in particular implies that

$$\int_\Omega n_\varepsilon(\cdot,t_\star(t))\ln n_\varepsilon(\cdot,t_\star(t)) \le \frac{1-\alpha}{\alpha e}\int_\Omega n_\varepsilon^{\frac{1}{1-\alpha}}(\cdot,t_\star(t)) \le C_{10} := \frac{1-\alpha}{\alpha e}\cdot C_9^{\frac{1}{2(1-\alpha)}}$$

and that hence

$$y(t_\star(t)) \le C_{11} := C_{10} + aC_9.$$

As g is nonnegative, integrating (6.9) and using (6.11), we therefore see that

$$\begin{aligned} y(t) &\le \left\{ y(t_\star(t)) + \frac{|\Omega|}{e} \right\} \cdot e^{\int_{t_\star(t)}^t h(s)ds} + \int_{t_\star(t)}^t e^{\int_s^t h(\sigma)d\sigma} \cdot C_6 ds \\ &\le \left\{ C_{11} + \frac{|\Omega|}{e} \right\} \cdot e^{C_7} + \int_{t_\star(t)}^t e^{C_7} \cdot C_6 ds \\ &\le C_{12} := \left\{ C_6 + C_{11} + \frac{|\Omega|}{e} \right\} \cdot e^{C_7} \qquad \text{for all } t > 0, \end{aligned} \tag{6.12}$$

and that thus, again by (6.9) and (6.11),

$$\begin{aligned} \int_t^{t+1} g(s)ds &\le y(t) - y(t+1) + \int_t^{t+1} h(s) \left\{ y(s) + \frac{|\Omega|}{e} \right\} ds + C_6 \\ &\le C_{12} + \frac{|\Omega|}{e} + C_7 \left\{ C_{12} + \frac{|\Omega|}{e} \right\} + C_6 \qquad \text{for all } t \ge 0, \end{aligned} \tag{6.13}$$

because for all $t > 0$ we have $-y(t) \le -\int_\Omega n_\varepsilon(\cdot, t) \ln n_\varepsilon(\cdot, t) \le |\Omega| e^{-1}$ due to (6.10). Whereas (6.12) in conjunction with (6.10) entails (6.4), from (6.13) we directly obtain (6.3). □

In consequence of the space-time L^2 estimate for n_ε contained in the latter, recalling Lemma 3.3 we directly obtain the following:

Lemma 6.3. *There exists $C > 0$ such that*

$$\int_\Omega |\nabla u_\varepsilon(\cdot, t)|^2 \le C \qquad \text{for all } t > 0 \tag{6.14}$$

whenever $\varepsilon \in (0, 1)$.

Proof. According to the boundedness of $\nabla\phi$ in Ω, the spatio-temporal estimate (6.3) for n_ε from Lemma 6.2 ensures that

$$\sup_{\varepsilon \in (0,1)} \sup_{t \ge 0} \int_t^{t+1} \int_\Omega |n_\varepsilon \nabla\phi|^2 < \infty.$$

Therefore, an application of Lemma 3.3 to $\tau := 1$ directly yields (6.14). □

We also note an evident by-product thereof, becoming useful in the derivation of an L^∞ bound for n_ε in Lemma 8.1.

Lemma 6.4. *For all $p > 1$ one can find $C(p) > 0$ fulfilling*

$$\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \le C(p) \qquad \text{for all } t > 0 \tag{6.15}$$

and each $\varepsilon \in (0, 1)$.

Proof. Since $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, this is an immediate consequence of Poincaré's inequality and Lemma 6.3. □

7. Bounds for n_ε and ∇c_ε in $L^p(\Omega)$

Now our knowledge on regularity of u_ε is sufficient to allow for the derivation of L^p bounds for both n_ε and ∇c_ε by means of an analysis of a functional combining $\int_\Omega n_\varepsilon^p$ with $\int_\Omega |\nabla c_\varepsilon|^{2q}$ with arbitrarily large p and suitably chosen $q = q(p)$, thus following an approach well-established in the context of semilinear and also quasilinear chemotaxis systems [25,45].

Lemma 7.1. *For all $p > 1$ there exists $C(p) > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C(p) \qquad \text{for all } t > 0 \tag{7.1}$$

and

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^p \le C(p) \qquad \text{for all } t > 0. \tag{7.2}$$

Moreover, there exists $C > 0$ such that

$$\int_t^{t+1} \int_\Omega |\nabla n_\varepsilon|^2 \le C \qquad \text{for all } t \ge 0 \tag{7.3}$$

whenever $\varepsilon \in (0,1)$.

Proof. Without loss of generality assuming that $p \ge 2$, since $\alpha > 0$ it is possible to choose $q > \frac{3}{2}$ such that $q \ge \frac{p}{2}$ and

$$\frac{p}{1+4\alpha} < q < p. \tag{7.4}$$

Now, using the first two equations in (2.3), we compute

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + (p-1)\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \\ &= (p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon \cdot \Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \qquad \text{for all } t > 0 \end{aligned} \tag{7.5}$$

and

$$
\begin{aligned}
\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^{2q} &= \int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla\Delta c_\varepsilon - \int_\Omega |\nabla c_\varepsilon|^{2q} \\
&\quad + \int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla n_\varepsilon \\
&\quad - \int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) \\
&= \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2q-2}\Delta|\nabla c_\varepsilon|^2 - \int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2 c_\varepsilon|^2 \\
&\quad - \int_\Omega |\nabla c_\varepsilon|^{2q} \\
&\quad - \int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2q-2}\Delta c_\varepsilon - \int_\Omega n_\varepsilon\nabla c_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^{2q-2} \\
&\quad - \int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
&\le -\frac{2(q-1)}{q^2}\int_\Omega \Big|\nabla|\nabla c_\varepsilon|^q\Big|^2 - \int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2 c_\varepsilon|^2 \\
&\quad - \int_\Omega |\nabla c_\varepsilon|^{2q} \\
&\quad - \int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2q-2}\Delta c_\varepsilon - 2(q-1) \\
&\quad \cdot\int_\Omega n_\varepsilon|\nabla c_\varepsilon|^{2q-4}\nabla c_\varepsilon\cdot(D^2 c_\varepsilon\cdot\nabla c_\varepsilon) \\
&\quad - \int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \quad \text{for all } t>0,
\end{aligned}
\tag{7.6}
$$

where we have used that $\nabla c_\varepsilon\cdot\nabla\Delta c_\varepsilon = \frac{1}{2}\Delta|\nabla c_\varepsilon|^2 - |D^2 c_\varepsilon|^2$ and $\nabla|\nabla c_\varepsilon|^2 = 2D^2 c_\varepsilon\cdot\nabla c_\varepsilon$ in $\Omega\times(0,\infty)$, that $\frac{\partial|\nabla c_\varepsilon|^2}{\partial\nu}\le 0$ on $\partial\Omega\times(0,\infty)$ by convexity of Ω and the identity $\frac{\partial c_\varepsilon}{\partial\nu}=0$ on $\partial\Omega\times(0,\infty)$ [36], and that $\nabla\cdot u_\varepsilon\equiv 0$, which namely implies that

$$
\begin{aligned}
-\int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot\nabla(u_\varepsilon\cdot\nabla c_\varepsilon) + \int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon) \\
= -\int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon\cdot(D^2 c_\varepsilon\cdot u_\varepsilon) \\
= -\frac{1}{2q}\int_\Omega u_\varepsilon\cdot\nabla|\nabla c_\varepsilon|^{2q} \\
= 0 \qquad \text{for all } t>0.
\end{aligned}
$$

Now on the right-hand side of (7.5), we recall (1.7) and employ Young's inequality to see that for all $t > 0$,

$$\begin{aligned} &(p-1)\int_\Omega n_\varepsilon^{p-1}\nabla n_\varepsilon\cdot\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big) \\ &\le \frac{p-1}{2}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2+\frac{(p-1)S_0^2}{2}\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2, \end{aligned} \tag{7.7}$$

where invoking the Cauchy-Schwarz inequality and the Gagliardo-Nirenberg inequality, thanks to (2.7) and the outcome of Lemma 6.2, we obtain that with some positive constants C_1 and C_2 we have

$$\begin{aligned} &\frac{(p-1)S_0^2}{2}\int_\Omega n_\varepsilon^{p-2\alpha}|\nabla c_\varepsilon|^2 \\ &\le \frac{(p-1)S_0^2}{2}\Big\{\int_\Omega n_\varepsilon^{2(p-2\alpha)}\Big\}^{\frac12}\cdot\Big\{\int_\Omega|\nabla c_\varepsilon|^4\Big\}^{\frac12} \\ &= \frac{(p-1)S_0^2}{2}\|n_\varepsilon^{\frac p2}\|_{L^{\frac{4(p-2\alpha)}{p}}(\Omega)}^{\frac{2(p-2\alpha)}{p}}\Big\||\nabla c_\varepsilon|^q\Big\|_{L^{\frac4q}(\Omega)}^{\frac2q} \\ &\le C_1\cdot\Big\{\|\nabla n_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^{\frac{2(p-2\alpha)}{p}\cdot(1-\frac{1}{2(p-2\alpha)})}\|n_\varepsilon^{\frac p2}\|_{L^{\frac2p}(\Omega)}^{\frac1p}+\|n_\varepsilon^{\frac p2}\|_{L^{\frac2p}(\Omega)}^{\frac{2(p-2\alpha)}{p}}\Big\} \\ &\quad\times\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac1q}\Big\||\nabla c_\varepsilon|^q\Big\|_{L^{\frac2q}(\Omega)}^{\frac1q}+\Big\||\nabla c_\varepsilon|^q\Big\|_{L^{\frac2q}(\Omega)}^{\frac2q}\Big\} \\ &\le C_2\cdot\Big\{\|\nabla n_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^{\frac{2(p-2\alpha)}{p}\cdot(1-\frac{1}{2(p-2\alpha)})}+1\Big\}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac1q}+1\Big\} \end{aligned} \tag{7.8}$$

for all $t > 0$. Here, since

$$\frac{2(p-2\alpha)}{p}\cdot\Big(1-\frac{1}{2(p-2\alpha)}\Big)=\frac{2p-4\alpha-1}{p},$$

we see, again by Young's inequality, that with some $C_3 > 0$ and $C_4 := 2^{\frac{2q}{2q-1}-1}C_3$ we have

$$\begin{aligned} &C_2\cdot\Big\{\|\nabla n_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^{\frac{2(p-2\alpha)}{p}\cdot(1-\frac{1}{2(p-2\alpha)})}+1\Big\}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac1q}+1\Big\} \\ &\le \frac{q-1}{q^2}\cdot 2^{1-2q}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac1q}+1\Big\}^{2q}+C_3\cdot\Big\{\|\nabla n_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^{\frac{2p-4\alpha-1}{p}}+1\Big\}^{\frac{2q}{2q-1}} \\ &\le \frac{q-1}{q^2}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^2+1\Big\} \\ &\quad+C_4\cdot\Big\{\|\nabla n_\varepsilon^{\frac p2}\|_{L^2(\Omega)}^{\frac{2p-4\alpha-1}{p}\cdot\frac{2q}{2q-1}}+1\Big\}\qquad\text{for all } t>0, \end{aligned} \tag{7.9}$$

and since the left inequality in (7.4) warrants that

$$\frac{2q}{2q-1} = \frac{1}{1-\frac{1}{2q}} < \frac{1}{1-\frac{1+4\alpha}{2p}} = \frac{2p}{2p-4\alpha-1}$$

and that thus

$$\frac{2p-4\alpha-1}{p} \cdot \frac{2q}{2q-1} < 2,$$

Young's inequality again becomes applicable so as to provide $C_5 > 0$ fulfilling

$$\begin{aligned} C_4 \cdot \left\{ \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{2p-4\alpha-1}{p}\cdot\frac{2q}{2q-1}} + 1 \right\} &\le \frac{p-1}{p^2} \|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + C_5 \\ &= \frac{p-1}{4} \int_\Omega n_\varepsilon^{p-2} |\nabla n_\varepsilon|^2 + C_5 \qquad \text{for all } t > 0, \end{aligned}$$

which, combined with (7.5), (7.7), (7.8) and (7.9), shows that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{p-1}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \\ &\le \frac{q-1}{q^2}\int_\Omega \Big|\nabla|\nabla c_\varepsilon|^q\Big|^2 + C_6 \quad \text{for all } t > 0 \end{aligned} \tag{7.10}$$

with $C_6 := \frac{q-1}{q^2} + C_5$.

Next, in quite a similar manner we estimate the summands on the right of (7.6) which contain n_ε. Indeed, using that $|\Delta c_\varepsilon| \le \sqrt{2}|D^2 c_\varepsilon|$ in $\Omega \times (0, \infty)$, we first employ Young's inequality to separate the highest-order contributions, according to

$$\begin{aligned} &-\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2q-2}\Delta c_\varepsilon - 2(q-1)\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2q-4}\nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \\ &\le \frac{1}{4}\int_\Omega |\nabla c_\varepsilon|^{2q-2}|\Delta c_\varepsilon|^2 + \int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} \\ &+ \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2 c_\varepsilon|^2 + 2(q-1)^2\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2q-2} \\ &\le \int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2 c_\varepsilon|^2 + \Big(1 + 2(q-1)^2\Big)\int_\Omega n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} \quad \text{for all } t > 0, \end{aligned} \tag{7.11}$$

and thereafter we use the Cauchy-Schwarz inequality and the Gagliardo- Nirenberg inequality along with (2.7) and Lemma 6.2 to infer the existence of $C_7 > 0$ and

$C_8 > 0$ such that

$$\begin{aligned} &\Big(1+2(q-1)^2\Big)\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^{2q-2} \\ &\le \Big(1+2(q-1)^2\Big)\Big\{\int_\Omega n_\varepsilon^4\Big\}^{\frac{1}{2}}\cdot\Big\{\int_\Omega|\nabla c_\varepsilon|^{4q-4}\Big\}^{\frac{1}{2}} \\ &= \Big(1+2(q-1)^2\Big)\|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{8}{p}}(\Omega)}^{\frac{4}{p}}\Big\||\nabla c_\varepsilon|^q\Big\|_{L^{\frac{4q-4}{q}}(\Omega)}^{\frac{2q-2}{q}} \\ &\le C_7\cdot\Big\{\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{3}{p}}\|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{1}{p}}+\|n_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{4}{p}}\Big\} \\ &\quad\times\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac{2q-3}{q}}\Big\||\nabla c_\varepsilon|^q\Big\|_{L^{\frac{2}{q}}(\Omega)}^{\frac{1}{q}}+\Big\||\nabla c_\varepsilon|^q\Big\|_{L^{\frac{2}{q}}(\Omega)}^{\frac{2q-2}{q}}\Big\} \\ &\le C_8\cdot\Big\{\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{3}{p}}+1\Big\}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac{2q-3}{q}}+1\Big\} \end{aligned} \tag{7.12}$$

for all $t > 0$, where we rely on our assumption that $q > \frac{3}{2}$ which guarantees that indeed $\frac{4q-4}{q} \ge \frac{2}{q}$. Now since also $p \ge 2$, we may invoke Young's inequality to find $C_9 > 0$ satisfying

$$\begin{aligned} &C_8\cdot\Big\{\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{3}{p}}+1\Big\}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac{2q-3}{q}}+1\Big\} \\ &\le \frac{p-1}{2p^2}\cdot 2^{1-\frac{2p}{3}}\cdot\Big\{\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{3}{p}}+1\Big\}^{\frac{2p}{3}}+C_9\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac{2q-3}{q}}+1\Big\}^{\frac{2p}{2p-3}} \\ &\le \frac{p-1}{2p^2}\cdot\Big\{\|\nabla n_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2}+1\Big\}+C_{10}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac{2q-3}{q}\cdot\frac{2p}{2p-3}}+1\Big\}\quad\text{for all } t>0 \end{aligned} \tag{7.13}$$

with $C_{10} := 2^{\frac{2p}{2p-3}-1}C_9$, and since moreover the right inequality in (7.4) asserts that

$$\frac{2q-3}{q}\cdot\frac{2p}{2p-3} < 2,$$

another application of Young's inequality shows that there exists $C_{11} > 0$ fulfilling

$$C_{10}\cdot\Big\{\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^{\frac{2q-3}{q}\cdot\frac{2p}{2p-3}}+1\Big\} \le \frac{q-1}{2q^2}\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|_{L^2(\Omega)}^2+C_{11}\quad\text{for all } t>0.$$

Therefore, (7.11), (7.12) and (7.13) imply that for all $t > 0$,

$$\begin{aligned} & -\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2q-2}\Delta c_\varepsilon - 2(q-1)\int_\Omega n_\varepsilon |\nabla c_\varepsilon|^{2q-4}\nabla c_\varepsilon \cdot (D^2 c_\varepsilon \cdot \nabla c_\varepsilon) \\ \le & \int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2 c_\varepsilon|^2 \\ & + \frac{p-1}{2p^2}\cdot \Big\{ \|\nabla n_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + 1 \Big\} + \frac{q-1}{2q^2}\Big\|\nabla|\nabla c_\varepsilon|^q\Big\|^2_{L^2(\Omega)} + C_{11} \qquad (7.14) \\ = & \int_\Omega |\nabla c_\varepsilon|^{2q-2}|D^2 c_\varepsilon|^2 + \frac{p-1}{8}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \\ & + \frac{q-1}{2q^2}\int_\Omega \Big|\nabla|\nabla c_\varepsilon|^q\Big|^2 + C_{12} \end{aligned}$$

with $C_{12} := \frac{p-1}{2p^2} + C_{11}$.

Finally, in the rightmost summand in (7.6) we apply the Cauchy-Schwarz inequality and make use of Lemma 6.3 to find $C_{13} > 0$ such that

$$\begin{aligned} -\int_\Omega |\nabla c_\varepsilon|^{2q-2}\nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon) &\le \Big\{ \int_\Omega |\nabla u_\varepsilon|^2 \Big\}^{\frac12} \cdot \Big\{ \int_\Omega |\nabla c_\varepsilon|^{4q} \Big\}^{\frac12} \\ &\le C_{13}\Big\{ \int_\Omega |\nabla c_\varepsilon|^{4q} \Big\}^{\frac12} \qquad \text{for all } t > 0, \end{aligned}$$

where again, by the Gagliardo-Nirenberg inequality, Lemma 6.2 and Young's inequality, we see that with some $C_{14} > 0$, $C_{15} > 0$ and $C_{16} > 0$ we have

$$\begin{aligned} C_{13}\Big\{ \int_\Omega |\nabla c_\varepsilon|^{4q} \Big\}^{\frac12} &= C_{13}\Big\| |\nabla c_\varepsilon|^q \Big\|^2_{L^4(\Omega)} \\ &\le C_{14}\Big\| \nabla|\nabla c_\varepsilon|^q \Big\|^{\frac{2q-1}{q}}_{L^2(\Omega)} \Big\| |\nabla c_\varepsilon|^q \Big\|^{\frac1q}_{L^{\frac2q}(\Omega)} \\ &\quad + C_{14}\Big\| |\nabla c_\varepsilon|^q \Big\|^2_{L^{\frac2q}(\Omega)} \qquad (7.15) \\ &\le C_{15}\Big\| \nabla|\nabla c_\varepsilon|^q \Big\|^{\frac{2q-1}{q}}_{L^2(\Omega)} + C_{15} \\ &\le \frac{q-1}{8q^2}\int_\Omega \Big|\nabla|\nabla c_\varepsilon|^q\Big|^2 + C_{16} \qquad \text{for all } t > 0. \end{aligned}$$

We now collect (7.14) and (7.15) to see that (7.6) combined with (7.10) entails that

$$\begin{aligned} & \frac{d}{dt}\Big\{ \frac1p \int_\Omega n_\varepsilon^p + \frac{1}{2q}\int_\Omega |\nabla c_\varepsilon|^{2q} \Big\} + \frac{p-1}{8}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 \\ & + \int_\Omega |\nabla c_\varepsilon|^{2q} \le C_6 + C_{12} + C_{16} \end{aligned}$$

for all $t > 0$, so that since a Poincaré-type inequality together with (2.7) provides $C_{17} > 0$ and $C_{18} > 0$ such that

$$\begin{aligned}\int_\Omega n_\varepsilon^p &\le C_{17}\int_\Omega |\nabla n_\varepsilon^{\frac{p}{2}}|^2 + C_{17}\cdot\left\{\int_\Omega n_\varepsilon\right\}^p\\ &\le \frac{p^2C_{17}}{4}\int_\Omega n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + C_{18} \qquad \text{for all } t > 0.\end{aligned}$$

It follows that the fuction $y(t)$ defined by

$$y(t) := \frac{1}{p}\int_\Omega n_\varepsilon^p(\cdot,t) + \frac{1}{2q}\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^{2q}, \qquad t \ge 0,$$

and the fuction $g(t)$ defined by

$$g(t) := \frac{p-1}{16}\int_\Omega n_\varepsilon^{p-2}(\cdot,t)|\nabla n_\varepsilon(\cdot,t)|^2, \qquad t > 0,$$

satisfy

$$y'(t) + C_{19}y(t) + g(t) \le C_{20} \qquad \text{for all } t > 0, \tag{7.16}$$

with $C_{19} := \min\left\{\frac{p-1}{4pC_{17}}\, ,\, 2q\right\}$ and $C_{20} := C_6 + C_{12} + C_{16} + \frac{(p-1)C_{18}}{4p^2C_{17}}$. Since Lemma 2.1 warrants that y is continuous at $t = 0$, an ODE comparison argument becomes applicable so as to assert that (7.16) entails the inequality

$$y(t) \le C_{21} := \max\left\{\frac{1}{p}\int_\Omega n_0^p + \frac{1}{2q}\int_\Omega |\nabla c_0|^{2q}\, ,\, \frac{C_{20}}{C_{19}}\right\} \qquad \text{for all } t > 0$$

and thereby proves both (7.1) and (7.2), because $2q \ge p$ according to our choice of q. Furthermore, restricting ourselves to the particular case $p = 2$ and integrating (7.16), we see that

$$\int_t^{t+1} g(s)ds \le y(t) + C_{20} \le C_{21} + C_{20} \qquad \text{for all } t \ge 0,$$

and that hence also (7.3) holds. □

8. Hölder estimates for n_ε and u_ε

We next aim at the derivation of bounds for the components n_ε and u_ε in spaces of Hölder continuous functions. Firstly, $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\Omega\times(0,\infty))$ thanks to Lemma 7.1 and Lemma 6.4:

Lemma 8.1. *There exists $C > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{8.1}$$

Proof. Since Lemma 7.1, together with Lemma 6.4 warrants that the family $(n_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,\infty); L^p(\Omega))$ and that $\big(-n_\varepsilon S_\varepsilon(\cdot, n_\varepsilon, c_\varepsilon)\cdot\nabla c_\varepsilon - n_\varepsilon u_\varepsilon\big)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,\infty); L^p(\Omega;\mathbb{R}^2))$ for any finite $p>1$, similar by to the claim from Lemma 4.4 this can be seen by means of an L^p iteration of Moser-type ([45, Lemma A.1]). □

Again due to the regularity properties asserted by Lemma 7.1 and Lemma 6.4, and due to the assumed Hölder continuity of n_0, it follows from standard parabolic theory that n_ε even satisfies estimates in appropriate Hölder spaces:

Lemma 8.2. *There exist $\theta\in(0,1)$ and $C>0$ such that, for any $\varepsilon\in(0,1)$,*

$$\|n_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t\ge 0. \tag{8.2}$$

Proof. We interpret the first equation in (2.3) as saying that

$$n_{\varepsilon t} = \nabla\cdot a_\varepsilon(x,t,\nabla n_\varepsilon), \qquad x\in\Omega,\ t>0,$$

with

$$a_\varepsilon(x,t,\xi) := \xi + f_\varepsilon(x,t), \qquad x\in\Omega,\ t>0,\ \xi\in\mathbb{R}^2,$$

where as in Lemma 4.4, $f_\varepsilon := -n_\varepsilon S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon - n_\varepsilon u_\varepsilon$. Since by Young's inequality,

$$a_\varepsilon(x,t,\xi)\cdot\xi = |\xi|^2 + f_\varepsilon\cdot\xi \ge \frac{1}{2}|\xi|^2 - \frac{1}{2}|f_\varepsilon|^2 \quad \text{for all } (x,t,\xi)\in\Omega\times(0,\infty)\times\mathbb{R}^2$$

and

$$|a_\varepsilon(x,t,\xi)| \le |\xi| + |f_\varepsilon| \qquad \text{for all } (x,t,\xi)\in\Omega\times(0,\infty)\times\mathbb{R}^2,$$

again using that Lemma 7.1 and Lemma 6.4 guarantee boundedness of $(f_\varepsilon)_{\varepsilon\in(0,1)}$ in any space $L^\infty((0,\infty); L^p(\Omega;\mathbb{R}^2))$ for arbitrary $p>1$, we obtain from a standard result on Hölder regularity in scalar parabolic equations [39, Theorem 1.3, Remark 1.4], relying on the boundedness property asserted by Lemma 8.1, that (8.2) holds with some $\theta\in(0,\vartheta]$ and $C>0$, with $\vartheta\in(0,1)$ taken from (1.10). □

Independently of the latter two lemmata, the following estimates for the fluid velocity field directly result on applying Lemma 3.4 on the basis of Lemma 7.1.

Lemma 8.3. *Given any $\beta\in(\frac{1}{2},1)$, one can find $\theta(\beta)\in(0,1)$ and $C(\beta)>0$ such that whenever $\varepsilon\in(0,1)$,*

$$\|A^\beta u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C(\beta) \qquad \textit{for all } t>0 \tag{8.3}$$

and

$$\|A^\beta u_\varepsilon(\cdot,t) - A^\beta u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)} \le C_1(t-t_0)^\theta \qquad \textit{for all } t_0\ge 0 \textit{ and } t>t_0. \tag{8.4}$$

Proof. As a particular consequence of Lemma 7.1 and the boundedness of $\nabla\phi$ in Ω, we know that $(n_\varepsilon\nabla\phi)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,\infty);L^2(\Omega;\mathbb{R}^2))$. Therefore Lemma 3.4 applies so as to yield both (8.3) and (8.4). □

This inter alia implies an ε-independent space-time Hölder continuity property also of u_ε. For later reference in Lemma 9.1, we also note a consequence of Lemma 8.3 on integrability of ∇u_ε in higher-power L^p spaces.

Lemma 8.4. *There exist $\theta\in(0,1)$ and $C>0$ with the property that for each $\varepsilon\in(0,1)$ we have*

$$\|u_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])}\le C \qquad \text{for all } t\ge 0. \tag{8.5}$$

Moreover, for all $p>1$ there exists $C(p)>0$ satisfying

$$\|\nabla u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\le C(p) \qquad \text{for all } t>0 \tag{8.6}$$

and arbitrary $\varepsilon\in(0,1)$.

Proof. The estimate (8.5) directly results upon an application of 8.3 and 8.4 to an arbitrary $\beta\in(\frac12,1)$, because a known embedding property of the domain of the corresponding fractional power of the Stokes operator says that for any such β we have $D(A^\beta)\hookrightarrow C^{\theta_1}(\bar\Omega;\mathbb{R}^2)$ whenever $\theta_1\in(0,2\beta-1)$ [16,21]. To verify (8.6), given $p>1$ we fix $\beta\in(\frac12,1)$ suitably large fulfilling $\beta>1-\frac1p$. Then since $D(A^\beta)\hookrightarrow W^{1,p}(\Omega;\mathbb{R}^2)$ (see [16,21]), (8.6) becomes a consequence of (8.3). □

9. Estimates in $C^{2+\theta,1+\frac{\theta}{2}}$ for u_ε and c_ε. Hölder continuity of c_ε

In order to complete our preparations for passing to the limit $\varepsilon\to0$ in Lemma 10.1, we finally derive Hölder estimates for the quantities u_ε and c_ε and their derivatives up to the respective order relevant to (1.1), possibly local in time due to a lack of corresponding regularity at the initial time. As for the component u_ε, we firstly make use of Lemma 8.4 in applying a well-known result on maximal Sobolev regularity in the Stokes system to obtain the following.

Lemma 9.1. *For all $p>1$ and $\tau>0$ there exist $C(p,\tau)>0$ fulfilling*

$$\|u_\varepsilon\|_{L^p((t,t+1);W^{2,p}(\Omega))}+\|u_{\varepsilon t}\|_{L^p(\Omega\times(t,t+1))}\le C(p,\tau) \qquad \text{for all } t\ge\tau \tag{9.1}$$

whenever $\varepsilon\in(0,1)$.

Proof. Given $p>1$, by using Lemma 7.1 along with the boundedness of $\nabla\phi$, we can find $C_1>0$ such that for all $\varepsilon\in(0,1)$ we have

$$\|n_\varepsilon(\cdot,t)\nabla\phi\|_{L^p(\Omega)}\le C_1 \qquad \text{for all } t>0.$$

Moreover, according to the Cauchy-Schwarz inequality and Lemma 3.1, the outcome of Lemma 8.4 ensures that with some positive constants C_2, C_3 and C_4 we have

$$\begin{aligned}\Big\|(Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon\Big\|_{L^p(\Omega)} &\le C_2\|Y_\varepsilon u_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)}\\ &\le C_3\|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\|\nabla u_\varepsilon(\cdot,t)\|_{L^{2p}(\Omega)}\\ &\le C_4 \qquad \text{for all } t>0\end{aligned}$$

whenever $\varepsilon\in(0,1)$. Therefore, (9.1) is a straightforward consequence of maximal Sobolev regularity estimates for the Stokes evolution equation [18]. □

As the latter warrants a favorable Hölder continuity property of the convective term in the third equation in (2.3), in view of Lemma 8.2 classical Schauder theory for the Stokes system becomes now applicable to establish the desired $C^{2+\theta,1+\frac{\theta}{2}}$ estimate for u_ε.

Lemma 9.2. *Let $\tau>0$. Then exist $\theta=\theta(\tau)\in(0,1)$ and $C(\tau)$ such that for any choice of $\varepsilon\in(0,1)$*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])}\le C(\tau) \qquad \textit{for all } t\ge\tau. \tag{9.2}$$

Proof. From Lemma 8.2 and our regularity assumptions on ϕ we know that there exist $\theta_1\in(0,1)$ and $C_1>0$ such that

$$\|n_\varepsilon\nabla\phi\|_{C^{\theta_1,\frac{\theta_1}{2}}(\bar\Omega\times[t,t+2])}\le C_1 \qquad \text{for all } t\ge 0, \tag{9.3}$$

whereas applying Lemma 9.1 to some suitably large $p>1$, in view of a known embedding result [2], we can find $\theta_2\in(0,1)$ with the property that for all $\tau>0$ there exists $C_2>0$ fulfilling

$$\|u_\varepsilon\|_{C^{1+\theta_2,\theta_2}(\bar\Omega\times[t,t+2])}\le C_2 \qquad \text{for all } t\ge\frac{\tau}{2}. \tag{9.4}$$

Moreover, fixing an arbitrary $\beta\in(\frac12,1)$, from Lemma 8.3 we obtain $\theta_3\in(0,1)$ and $C_3>0$ satisfying

$$\|A^\beta u_\varepsilon(\cdot,t)-A^\beta u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)}\le C_3(t-t_0)^{\theta_3} \qquad \text{for all } t_0\ge 0 \text{ and } t>t_0,$$

which in view of the embedding $D(A^\beta)\hookrightarrow C^{\theta_4}(\bar\Omega;\mathbb{R}^2)$ for arbitrary fixed $\theta_4\in(0,2\beta-1)$ implies that

$$\begin{aligned}\|Y_\varepsilon u_\varepsilon(\cdot,t)-Y_\varepsilon u_\varepsilon(\cdot,t_0)\|_{C^{\theta_4}(\bar\Omega)} &\le C_4\|A^\beta Y_\varepsilon u_\varepsilon(\cdot,t)-A^\beta Y_\varepsilon u_\varepsilon(\cdot,t_0)\|_{L^2(\Omega)}\\ &= C_4\Big\|Y_\varepsilon A^\beta\Big[u_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t_0)\Big]\Big\|_{L^2(\Omega)}\\ &\le C_4\Big\|A^\beta\Big[u_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t_0)\Big]\Big\|_{L^2(\Omega)}\\ &\le C_3C_4(t-t_0)^{\theta_3} \quad \text{for all } t_0\ge 0 \text{ and } t>t_0\end{aligned} \tag{9.5}$$

with some $C_4 > 0$, once more due to Lemma 3.1 and the fact that Y_ε commutes with A^β on $D(A^\beta)$.

Now, combining (9.4) with (9.5) shows that there exists $\theta_5 \in (0,1)$ such that, for each $\tau > 0$, one can find $C_5 > 0$ satisfying

$$\left\|(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon\right\|_{C^{\theta_5,\frac{\theta_5}{2}}(\bar{\Omega}\times[t,t+2])} \le C_5 \qquad \text{for all } t \ge \frac{\tau}{2},$$

which together with (9.3) yields (9.2) according to classical Schauder estimates for the Stokes evolution problem [40]. □

Similarly, the regularity properties of n_ε, c_ε and u_ε collected so far imply the following.

Lemma 9.3. *There exists $\theta \in (0,1)$ with the property that one can find $C > 0$ such that for any $\varepsilon \in (0,1)$,*

$$\|c_\varepsilon\|_{C^{\theta,\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} \le C \qquad \text{for all } t \ge 0, \tag{9.6}$$

and that for arbitrary $\tau > 0$ one can pick $C(\tau) > 0$ fulfilling

$$\|c_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} \le C(\tau) \qquad \text{for all } t \ge \tau \tag{9.7}$$

whenever $\varepsilon \in (0,1)$.

Proof. Interpreting the second equation in (2.3) as the inhomogeneous linear heat equation

$$c_{\varepsilon t} = \Delta c_\varepsilon + f_\varepsilon(x,t), \qquad x \in \Omega,\ t > 0, \tag{9.8}$$

with $f_\varepsilon := -c_\varepsilon + n_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon$, collecting the estimates from Lemma 7.1, Lemma 6.1 and Lemma 8.3 we first obtain that for all $p > 1$ there exists $C_1 > 0$ such that

$$\|f_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le C \qquad \text{for all } t > 0. \tag{9.9}$$

Accordingly, (9.6) follows from well-known results from parabolic Hölder regularity theory [39, Theorem 1.3, Remark 1.4]. Moreover, (9.9) together with maximal Sobolev regularity estimates for the Neumann problem associated with the heat equation implies that, for any $p > 1$ and $\tau > 0$, there exists $C_2 > 0$ such that

$$\|c_\varepsilon\|_{L^p((t,t+2);W^{2,p}(\Omega))} + \|c_{\varepsilon t}\|_{L^p(\Omega\times(t,t+2))} \le C_2 \qquad \text{for all } t \ge \frac{\tau}{2}.$$

Again by means of a corresponding embedding property [2], an application thereof to appropriately large $p > 1$ shows that there exists $\theta_1 \in (0,1)$ such that for each $\tau > 0$ we can pick $C_3 > 0$ fulfilling

$$\|c_\varepsilon\|_{C^{1+\theta_1,\theta_1}(\bar{\Omega}\times[t,t+2])} \le C_3 \qquad \text{for all } t \ge \frac{\tau}{2}.$$

Together with the results of Lemma 8.2 and Lemma 8.4, this warrants the existence of $\theta_2 \in (0,1)$ with the property that for any $\tau > 0$ one can pick $C_4 > 0$ such that

$$\|f_\varepsilon\|_{C^{\theta_2,\frac{\theta_2}{2}}(\bar{\Omega}\times[t,t+2])} \le C_4 \qquad \text{for all } t \ge \frac{\tau}{2},$$

whereby (9.7) becomes a consequence of well-known Schauder theory for scalar parabolic equations [31]. □

10. Global existence in the original problem. Proof of Theorem 1.1

By means of a straightforward extraction procedure on the basis of the Arzelà-Ascoli theorem, the above estimates now enable us to construct a limit which, according to a well-known additional regularity argument for the limit component n, in fact can be seen to enjoy the desired smoothness properties and to solve (1.1), (1.8), (1.9) in the classical sense in $\Omega \times (0,\infty)$.

Lemma 10.1. *There exist $\theta \in (0,1)$, $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ and functions*

$$\begin{cases} n \in C^{\theta,\frac{\theta}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times[0,\infty)\big) \cap C^{2+\theta,1+\frac{\theta}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times(0,\infty)\big) \\ c \in C^{\theta,\frac{\theta}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times[0,\infty)\big) \cap C^{2+\theta,1+\frac{\theta}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times(0,\infty)\big) \\ u \in C^{\theta,\frac{\theta}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times[0,\infty);\mathbb{R}^2\big) \cap C^{2+\theta,1+\frac{\theta}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times(0,\infty);\mathbb{R}^2\big) \\ P \in C^{1,0}(\bar{\Omega}\times(0,\infty)) \end{cases} \tag{10.1}$$

such that $n \ge 0$ and $c \ge 0$ in $\Omega\times(0,\infty)$, that $\varepsilon_k \searrow 0$ as $k\to\infty$, that

$$\begin{cases} n_\varepsilon \to n & \text{in } C^0_{\mathrm{loc}}(\bar{\Omega}\times[0,\infty)) \\ c_\varepsilon \to c & \text{in } C^0_{\mathrm{loc}}(\bar{\Omega}\times[0,\infty)) \\ u_\varepsilon \to u & \text{in } C^0_{\mathrm{loc}}(\bar{\Omega}\times[0,\infty);\mathbb{R}^2) \end{cases} \tag{10.2}$$

as $\varepsilon = \varepsilon_k \searrow 0$, and that (n,c,u,P) solves (1.1), (1.8), (1.9) in the classical sense in $\Omega\times(0,\infty)$.

Proof. Collecting Lemma 8.2, Lemma 7.1, Lemma 9.3, Lemma 8.4 and Lemma 9.2, in view of the Arzelà-Ascoli theorem we obtain $\theta_1 \in (0,1)$, $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ and

$$\begin{cases} n \in C^{\theta_1,\frac{\theta_1}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times[0,\infty)\big) \cap L^2_{\mathrm{loc}}\big([0,\infty);W^{1,2}(\Omega)\big) \\ c \in C^{\theta_1,\frac{\theta_1}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times[0,\infty)\big) \cap C^{2+\theta_1,1+\frac{\theta_1}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times(0,\infty)\big) \\ u \in C^{\theta_1,\frac{\theta_1}{2}}_{\mathrm{loc}}(\bar{\Omega}\times[0,\infty);\mathbb{R}^2) \cap C^{2+\theta_1,1+\frac{\theta_1}{2}}_{\mathrm{loc}}\big(\bar{\Omega}\times(0,\infty);\mathbb{R}^2\big) \end{cases} \tag{10.3}$$

such that $\varepsilon_k \searrow 0$ as $k \to \infty$, and such that (10.2) as well as

$$\begin{cases} \nabla n_\varepsilon \rightharpoonup \nabla n & \text{in } L^2_{loc}\big(\Omega \times [0,\infty); \mathbb{R}^2\big), \\ \nabla c_\varepsilon \rightharpoonup \nabla c & \text{in } L^2_{loc}\big(\Omega \times [0,\infty); \mathbb{R}^2\big), \\ c_\varepsilon \to c & \text{in } C^{2,1}_{loc}\big(\bar\Omega \times (0,\infty)\big) \\ u_\varepsilon \to u & \text{in } C^{2,1}_{loc}\big(\bar\Omega \times (0,\infty)\big) \end{cases} \tag{10.4}$$

hold as $\varepsilon = \varepsilon_k \searrow 0$. Therefore, from the nonnegativity of n_ε and c_ε and from (2.3) it can readily be deduced by means of well-known arguments that $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, \infty)$ and that with some $P \in C^{1,0}(\bar\Omega \times (0,\infty))$, the respective second and third equations in (1.1), (1.8), (1.9) are satisfied in the claimed classical pointwise sense (*cf.*, *e.g.*, [59] for a detailed reasoning concerning a limit procedure in approximate Navier-Stokes systems involving the presently used regularization of the nonlinear convective term therein).

As for the first sub-problem in (1.1), (1.8), (1.9), in view of the comparatively poor approximation properties of the component n_ε, as expressed in (10.3) and appearing rather natural in view of our cut-off procedure applied to S near $\partial\Omega$, following, *e.g.*, [4,34] we first verify a respective weak solution property by showing that

$$\begin{aligned} &-\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) - \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \\ &= -\int_0^\infty \int_\Omega \nabla n \cdot \nabla\varphi + \int_0^\infty \int_\Omega n\Big(S(x,n,c)\cdot\nabla c\Big)\cdot\nabla\varphi \end{aligned} \tag{10.5}$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$. To this end, we fix any such φ and then obtain from (10.3) that in the identity

$$\begin{aligned} &-\int_0^\infty \int_\Omega n_\varepsilon\varphi_t - \int_\Omega n_0\varphi(\cdot,0) - \int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\varphi \\ &= -\int_0^\infty \int_\Omega \nabla n_\varepsilon \cdot \nabla\varphi + \int_0^\infty \int_\Omega n_\varepsilon\Big(S_\varepsilon(x,n_\varepsilon,c_\varepsilon)\cdot\nabla c_\varepsilon\Big)\cdot\nabla\varphi, \end{aligned} \tag{10.6}$$

valid for all $\varepsilon \in (0,1)$ according to (2.3), we may let $\varepsilon = \varepsilon_k \searrow 0$ in each of the term on the left-hand side to find that

$$\begin{aligned} &-\int_0^\infty \int_\Omega n_\varepsilon\varphi_t - \int_\Omega n_0\varphi(\cdot,0) - \int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\varphi \\ &\qquad \to -\int_0^\infty \int_\Omega n\varphi_t - \int_\Omega n_0\varphi(\cdot,0) - \int_0^\infty \int_\Omega nu \cdot \nabla\varphi \end{aligned}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Moreover, the first property in (10.4) warrants that

$$-\int_0^\infty \int_\Omega \nabla n_\varepsilon \cdot \nabla\varphi \to -\int_0^\infty \int_\Omega \nabla n \cdot \nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0,$$

whereas the uniform convergence features of n_ε and c_ε asserted in (10.3), in conjunction with the continuity of S, ensure that by the dominated convergence theorem

$$S_\varepsilon(\cdot, n_\varepsilon, c_\varepsilon) \to S(\cdot, n, c) \quad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0,$$

and that hence

$$\int_0^\infty \int_\Omega n_\varepsilon \Big(S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \Big) \cdot \nabla\varphi \to \int_0^\infty \int_\Omega n \Big(S(x,n,c) \cdot \nabla c \Big) \cdot \nabla \varphi \quad \text{as } \varepsilon = \varepsilon_k \searrow 0,$$

due to the second convergence property in (10.4). Therefore, (10.5) follows from (10.6), meaning that n is a generalized solution, in the natural weak sense consistent with those, *e.g.*, in [31] and [35], of the respective initial-boundary value sub-problem of (1.1),(1.8),(1.9). Since n, ∇c and u are already known to be Hölder continuous in $\bar\Omega \times (0,\infty)$ by (10.3), a well-known result on gradient Hölder regularity in scalar parabolic equations ([35]) thus warrants that for some $\theta_2 \in (0,1)$ we have $n \in C^{1+\theta_2, \theta_2}_{loc}(\bar\Omega \times (0,\infty))$. Thereupon, standard parabolic Schauder theory applies so as to yield $\theta_3 \in (0,1)$ with the property that $n \in C^{2+\theta_3, 1+\frac{\theta_3}{2}}_{loc}(\bar\Omega \times (0,\infty))$, and that hence also the first equations in (1.1) and (1.9) are satisfied in the classical sense. □

This in fact already contains the major part of our announced main result:

Proof of Theorem 1.1. The statement on global solvability actually is a by-product of Lemma 10.1. In view of the approximation properties in (10.2) and a standard argument based, *e.g.*, on lower semicontinuity of norms in reflexive spaces with respect to weak convergence therein, the estimate in (1.12) is an evident consequence of the corresponding bounds provided by Lemma 8.1, Lemma 7.1 and Lemma 8.3. □

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