CRITERION OF THE L^2 BOUNDEDNESS IN DUNKL SETTING

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Abstract. The purpose of this paper is to introduce a new class of singular integral operators in the Dunkl setting which is associated with finite reflection groups on the Euclidean space. The group structures induce two nonequivalent metrics: the Euclidean metric and the Dunkl metric, both of which are involved in the estimates of singular integrals, the heat and Poisson kernels. The main result is the $T1$ theorem, the criterion of the L^2 boundedness in Dunkl setting. The key tools used in this paper are the Meyer-type commutation Lemma and almost orthogonal estimates in the Dunkl setting.

1. Introduction and statement of main results

It is well known that group structures enter in a decisive way in harmonic analysis. The main purpose of this paper is to develop the theory of singular integrals in the Dunkl setting which is associated with finite reflection groups on the Euclidean space. This particular group structure is conducting the analysis. Indeed, in the Dunkl setting, there are corresponding Dunkl transform, translation and convolution operators. However, both Euclidean metric and Dunkl metric are involved in the Dunkl setting. For example, the size and smoothness conditions of kernels of the Dunkl–Riesz transforms involve both Euclidean metric and Dunkl metric. Therefore, the Dunkl–Riesz transforms do not fall in the scope of the classical Calderón–Zygmund theory. Motived by these particular examples, we introduce a new class of singular integral operators in the Dunkl setting. As in the classical case, the $T1$ theorem, that is, the criterion for the boundedness on L^2, is crucial. To provide the $T1$ theorem, we develop the Meyer-type commutation Lemma and almost orthogonal estimates in the Dunkl setting.

We now state the background of the Dunkl setting and main results in more details.

1.1. Background on Dunkl setting.

The classical Fourier transform, initially defined on $L^1(\mathbb{R}^N)$, extends to an isometry of $L^2(\mathbb{R}^N)$ and satisfies certain properties with translation, dilation and rotation groups. Dunkl also introduced a similar transform (now called the Dunkl transform), which enjoys properties similar to the classical Fourier transform. See [12]. Corresponding to this new transform, Dunkl also introduced first order differential-difference operators which play the role similar to the usual partial differentiation associated with the reflection group.

To be precise, denote the standard inner product in the Euclidean space $\mathbb{R}^N$ by $\langle x, y\rangle = \sum_{j=1}^{N} x_j y_j$. Let R be a root system in $\mathbb{R}^N$ normalized so that $\langle \alpha, \alpha\rangle = 2$ for $\alpha \in R$ with R_+ a fixed positive subsystem, and G be the finite reflection group generated by the reflections σ_α $(\alpha \in R)$, where $\sigma_\alpha x = x - \langle \alpha, x\rangle\alpha$ for $x \in \mathbb{R}^N$.

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The Dunkl operators T_j are defined by

$$T_j f(x) = \partial_j f(x) + \sum_{\alpha \in R^+} \frac{\kappa(\alpha)}{2} \langle \alpha, e_j \rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle},$$

where $e_1, \dots, e_N$ are the standard unit vectors of $\mathbb{R}^N$.

The Dunkl Laplacian is defined as $\triangle := \sum\limits_{j=1}^{N} T_j^2$, which is equivalent to

$$\triangle f(x) = \triangle_{\mathbb{R}^N} f(x) + \sum_{\alpha \in R} \kappa(\alpha) \Big(\frac{\partial_\alpha f(x)}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \Big).$$

It generates the heat semigroup

$$H_t f(x) := e^{t\triangle} f(x) := \int_{\mathbb{R}^N} h_t(x,y) f(y) d\omega(y),$$

where the heat kernel $h_t(x,y)$ is a C^∞ function for all $t > 0, x, y \in \mathbb{R}^N$ and satisfies $h_t(x,y) = h_t(y,x) > 0$ and $\int_{\mathbb{R}^N} h_t(x,y) d\omega(y) = 1$.

The Poisson semigroup is given by

$$P_t f(x) = \pi^{-\frac{1}{2}} \int_0^\infty e^{-u} \exp\Big(\frac{t^2}{4u} \triangle \Big) f(x) \frac{du}{u^{\frac{1}{2}}}$$

and $u(x,t) = P_t f(x)$, so-called the Dunkl Poisson integral, solves the boundary value problem

$$\begin{cases} (\partial_t^2 + \triangle_x) u(x,t) = 0, \\ u(x,0) = f(x) \end{cases}$$

in the half-space $\mathbb{R}^N_+$, see [7].

All these tools, the Dunkl transform, Laplacian and Poisson integral together with the Dunkl translation and convolution operators([28]), opened the door for developing the harmonic analysis related to the Dunkl setting. For example, in [7], the Littlewood–Paley theory was established and the Hardy space $H^1(\mathbb{R}^N)$ was characterized by the area integrals, maximal fuction and the Riesz transforms, see also [6]. The atomic decomposition of $H^1_d(\mathbb{R}^N)$ was provided in [14]. The boundedness of singular integral convolution operators and the Hörmander multipliers was given by [15] and [16], respectively. See [1, 2, 3, 4, 5, 11, 13, 25] for other topics related to the Dunkl setting.

1.2. **Statement of main results.**

Let $\|x\| := \Big\{ \sum\limits_{j=1}^{N} |x_j|^2 \Big\}^{\frac{1}{2}}$ be the standard norm in $\mathbb{R}^N$ and let $B(x,r) := \{y \in \mathbb{R}^N : \|x - y\| < r\}$ stands for the Euclidean ball with center $x \in \mathbb{R}^N$ and radius $r > 0$. Let

$$d(x,y) := \min_{\sigma \in G} \|x - \sigma(y)\|$$

be the distance between two G-orbits $\mathcal{O}(x)$ and $\mathcal{O}(y)$, which is also known as the Dunkl metric. Obviously, $d(x,y) \leqslant \|x-y\|$, $d(x,y) = d(y,x)$ and $d(x,y) \leqslant d(x,z)+d(z,y)$ for all $x, y, z \in \mathbb{R}^N$.

A non-negative *multiplicity function* κ defined on R (the given root system in $\mathbb{R}^N$) is fixed throughout this paper. Let $d\omega(x) = \prod_{\alpha \in R} |\langle \alpha, x \rangle|^{\kappa(\alpha)} dx$ be the associated measure in $\mathbb{R}^N$ (see for example [7]), where, here and subsequently, dx stands for the Lebesgue measure in

$\mathbb{R}^N$. We denote by $\mathbf{N} = N + \sum\limits_{\alpha\in R}\kappa(\alpha)$ the homogeneous dimension of the system. Clearly, $\omega(B(tx,tr)) = t^{\mathbf{N}}\omega(B(x,r))$ and $\int_{\mathbb{R}^N} f(x)d\omega(x) = \int_{\mathbb{R}^N}\frac{1}{t^{\mathbf{N}}}f(\frac{x}{t})d\omega(x)$ for $f\in L^1(\mathbb{R}^N,\omega)$, $t>0$.

Observe that for $x\in\mathbb{R}^N$ and $r>0, \omega(B(x,r)) \sim r^N\prod_{\alpha\in R}\big(|\langle\alpha,x\rangle| + r\big)^{\kappa(\alpha)}$ and hence, $\inf\limits_{x\in\mathbb{R}^N}\omega(B(x,1)) \geqslant c > 0$.

Moreover,

$$C^{-1}\Big(\frac{r_2}{r_1}\Big)^N \leqslant \frac{w(B(x,r_2))}{w(B(x,r_1))} \leqslant C\Big(\frac{r_2}{r_1}\Big)^{\mathbf{N}} \qquad \text{for} \quad 0<r_1<r_2. \tag{1.1}$$

This implies that $d\omega(x)$ satisfies the doubling and reverse doubling properties, that is, there is a constant $C>0$ such that for all $x\in\mathbb{R}^N, r>0$ and $\lambda\geqslant 1$,

$$C^{-1}\lambda^N\omega(B(x,r)) \leqslant \omega(B(x,\lambda r)) \leqslant C\lambda^{\mathbf{N}}\omega(B(x,r)). \tag{1.2}$$

We can consider the Dunkl setting $(\mathbb{R}^N,\|\cdot\|,\omega)$ as a space of homogeneous type in the sense of Coifman and Weiss, where the measure ω satisfies the doubling and the reverse doubling properties. Let $C_0^\eta(\mathbb{R}^N), \eta>0$, denote the space of continuous functions f with compact support and

$$\|f\|_\eta := \sup_{x\neq y}\frac{|f(x)-f(y)|}{\|x-y\|^\eta} < \infty. \tag{1.3}$$

Also let $C^\eta(\mathbb{R}^N)$, $\eta>0$, denote the space of continuous functions f on $\mathbb{R}^N$ with the same norm as above.

Inspired by the Dunkl–Riesz transform, we introduce a new class of singular integral operators in the Dunkl setting $(\mathbb{R}^N,\|\cdot\|,\omega)$ as follows.

Definition 1.1. Let T be an operator defined initially as a mapping from $C_0^\eta(\mathbb{R}^N)$ to $(C_0^\eta(\mathbb{R}^N))'$ with $\eta>0$. T is said to be a Dunkl–Calderón–Zygmund singular integral operator if $K(x,y)$, the kernel of T, satisfies the following estimates: for some $0<\varepsilon\leqslant 1$,

$$|K(x,y)| \lesssim \frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{d(x,y)}{\|x-y\|}\Big)^\varepsilon \tag{1.4}$$

for all $x\neq y$;

$$|K(x,y)-K(x,y')| \lesssim \Big(\frac{\|y-y'\|}{\|x-y\|}\Big)^\varepsilon\frac{1}{\omega(B(x,d(x,y)))} \tag{1.5}$$

for $\|y-y'\|\leqslant d(x,y)/2$;

$$|K(x',y)-K(x,y)| \lesssim \Big(\frac{\|x-x'\|}{\|x-y\|}\Big)^\varepsilon\frac{1}{\omega(B(x,d(x,y)))} \tag{1.6}$$

for $\|x-x'\|\leqslant d(x,y)/2$.

Moreover,

$$\langle T(f),g\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}K(x,y)f(x)g(y)d\omega(x)d\omega(y)$$

for all $f,g\in C_0^\eta(\mathbb{R}^N)$ with $\operatorname{supp} f\cap\operatorname{supp} g=\emptyset$.

A Dunkl–Calderón–Zygmund singular integral operator is said to be the Dunkl–Calderón–Zygmund operator if it extends to a bounded operator on $L^2(\mathbb{R}^N,\omega)$.

We remark that the size and regularity conditions of the Dunkl-Calderón–Zygmund singular integral operator are much weaker than the classical Calderón–Zygmund singular integral operators given in space of homogeneous type in the sense of Coifman and Weiss since $d(x,y)$ and $\|x-y\|$ are not equivalent. Therefore, this new class contains all classical Calderón–Zygmund singular integral operators. Moreover, the definition of the Dunkl–Calderón–Zygmund operators can be extended to bounded functions in $C^\eta(\mathbb{R}^N)$. We include the details in Section 2.

The main result of this paper is to establish a criterion of boundedness on L^2 for the Dunkl–Calderón–Zygmund singular integrals, that is, the $T1$ theorem, in the Dunkl setting.

To begin with, we first introduce the *weak boundedness property* (WBP) in the Dunkl setting as follows.

Definition 1.2. The singular integral operator T with the kernel $K(x,y)$ in Definition 1.1 satisfies the weak boundedness property if there exist $\eta>0$ and $C<\infty$ such that

$$|\langle K,f\rangle| \leqslant C\max\{\omega(B(x_0,r)),\omega(B(y_0,r))\}$$

for all $f\in C_0^\eta(\mathbb{R}^N\times\mathbb{R}^N)$ with $\mathrm{supp}(f)\subseteq B(x_0,r)\times B(y_0,r), x_0,y_0\in\mathbb{R}^N, \|f\|_{L^\infty(\mathbb{R}^N,\omega)}\leqslant 1, \|f(\cdot,y)\|_\eta\leqslant r^{-\eta}$ for all $y\in\mathbb{R}^N$ and $\|f(x,\cdot)\|_\eta\leqslant r^{-\eta}$ for all $x\in\mathbb{R}^N$. We denote this by $T\in WBP$.

We remark that the weak boundedness property defined in Definition 1.2 is different from the classical version. Indeed, they are eqiuvalent. The reason to use this version is that the Fourier transform was used in the proof of the Meyer-type commutation Lemma, see [23]. However, to show this lemma on space of homogeneous type in the sense of Coifman and Weiss, this version of WBP was first introduced in [21].

Next, we denote $\mathrm{BMO}(\mathbb{R}^N,\omega)$ by the standard BMO space on $(\mathbb{R}^N,\|\cdot\|,\omega)$. That is,

$$\mathrm{BMO}(\mathbb{R}^N,\omega)=\{b\in L^1_{loc}(\mathbb{R}^N,\omega):\|b\|_*<\infty\},$$

where

$$\|b\|_*=\sup_{B\subset\mathbb{R}^N}\frac{1}{\omega(B)}\int_B|b(x)-b_B|d\omega(x)<\infty$$

with the supremum is taken over all Euclidean balls $B=B(y,r)=\{z\in\mathbb{R}^N:\|z-y\|<r\}$ and

$$b_B=\frac{1}{\omega(B)}\int_B b(x)d\omega(x).$$

Now we can state the $T1$ theorem for the Dunkl–Calderón–Zygmund singular integral operator T by the following

Theorem 1.3. *Suppose that T is a Dunkl–Calderón–Zygmund singular integral operator. Then T extends to a bounded operator on $L^2(\mathbb{R}^N,\omega)$ if and only if* (1) $T(1)\in\mathrm{BMO}(\mathbb{R}^N,\omega)$; (2) $T^*(1)\in\mathrm{BMO}(\mathbb{R}^N,\omega)$; (3) $T\in WBP$.

The key techniques to prove this main result include the Coifman-type approximation to the identity, the Meyer-type commutation Lemma, and new almost orthogonal estimates in the Dunkl setting, which will be provided in Section 2. We would like to point out that the theory of the Dunkl–Calderón–Zygmund singular integral operator will be a crucial tool to develop the wevelet-type decomposition and the Hardy space theory in the Dunkl setting, which will be given by further coming paper.

2. $T1$ Theorem in Dunkl setting

In this section we provide the proof of our main result Theorem 1.3. In Subsection 2.1–2.3 we will first establish several lemmas, particularly, the almost orthogonal estimates in the Dunkl setting.

2.1. **Dunkl–Calderón–Zygmund Operators.**

To begin with, we first point out that the size and regularity conditions of the Dunkl–Calderón–Zygmund operators are modelled from the pointwise size and regularity conditions of the Dunkl–Riesz transform obtained in [19], before which people only got the Hörmander type condition for regularity. The main techniques in [19] is to use functional calculus and apply the Dunkl heat kernel estimate obtained in [15, Theorem 3.1]. We note that there is some recent improvement on the Dunkl heat kernel estimate in [18].

Next, we note that the Dunkl–Calderón–Zygmund operators satisfies the boundedness properties analogous to the classical Calderón–Zygmund operators in the Euclidean setting.

Theorem 2.1. *Suppose that T is a Dunkl–Calderón–Zygmund operator as in Definition 1.1. Then T is bounded on $L^p(\mathbb{R}^N,\omega), 1 < p < \infty$, is of weak type (1,1), from $H^1_d(\mathbb{R}^N,\omega)$ to $L^1(\mathbb{R}^N,\omega)$ and from $L^\infty(\mathbb{R}^N,\omega)$ to* $\mathrm{BMO}(\mathbb{R}^N,\omega)$.

Here $H^1_d(\mathbb{R}^N,\omega)$ is the Dunkl–Hardy space introduced in [7].

The proof of Theorem 2.1 is standard. We can first show that T is of weak type $(1,1)$ via the Calderón–Zygmund decomposition, and the boundedness of T from $H^1_d(\mathbb{R}^N,\omega)$ to $L^1(\mathbb{R}^N,\omega)$ by atomic decomposition, and from $L^\infty(\mathbb{R}^N,\omega)$ to $\mathrm{BMO}(\mathbb{R}^N,\omega)$ by duality. The only thing we would like to mention is that in those atoms are defined in Coifman–Weiss sense defined in terms of $\|x-y\|$ metric see [16, 14].

2.2. **Meyer-type commutation Lemma in Dunkl setting.**

We now establish the Meyer-type commutation Lemma in the Dunkl setting as follows.

Lemma 2.2 (Meyer-type commutation Lemma)**.** *Suppose that T is a Dunkl–Calderón–Zygmund singular integral operator from $C^\eta_0(\mathbb{R}^N)$ to $(C^\eta_0(\mathbb{R}^N))'$ satisfying $T \in WBP$ and $T(1) = 0$. Then for any $M > 1$, there exists a positive constant C_M depending on M such that*

$$\|T\phi\|_{L^\infty(B(x_0,Mr),\omega)} \leqslant C_M$$

whenever there exist $x_0 \in \mathbb{R}^N$ and $r > 0$ such that $\mathrm{supp}\,(\phi) \subseteq B(x_0,r)$ *with $\|\phi\|_{L^\infty(\mathbb{R}^N,\omega)} \leqslant 1$ and $\|\phi\|_\eta \leqslant r^{-\eta}$.*

Proof. Fix a function $\theta \in C^\infty(\mathbb{R})$ with the following properties: $\theta(x) = 1$ for $|x| \le 1$ and $\theta(x) = 0$ for $|x| > 2$. Let $\chi_0(x) = \theta(\frac{d(x,x_0)}{2r})$ and $\chi_1 = 1 - \chi_0$. Then $\phi = \phi\chi_0$ and for all $\psi \in C^\eta_0(\mathbb{R}^N)$ with $\mathrm{supp}\,\psi \subseteq B(x_0,Mr)$,

$$\begin{aligned}\langle T\phi,\psi\rangle &= \langle K(x,y),\phi(y)\psi(x)\rangle = \langle K(x,y),\chi_0(y)\phi(y)\psi(x)\rangle\\ &= \langle K(x,y),\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle + \langle K(x,y),\chi_0(y)\phi(x)\psi(x)\rangle =: p+q,\end{aligned}$$

where $K(x,y)$ is the distribution kernel of T.

To estimate p, let $\lambda_\delta(x,y) = \theta(\frac{\|x-y\|}{\delta})$. Then

$$\begin{aligned}p &= \langle K(x,y),(1-\lambda_\delta(x,y))\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle\\ &\quad + \langle K(x,y),\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle =: p_{1,\delta} + p_{2,\delta}.\end{aligned} \tag{2.1}$$

Since K is locally integrable on $\Omega = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\}$. By the size condition on $K(x,y)$ and the smoothness condition on ϕ together with the fact that if $\chi_0(y) \neq 0, \psi(x) \neq 0$

and $1-\lambda_\delta(x,y)\neq 0$, then $\delta\leqslant\|x-y\|$ and $d(x,y)\leqslant(M+4)r$. Thus, the first term on the right side of (2.1) satisfies

$$\begin{aligned}|p_{1,\delta}| &= \Big|\iint_\Omega K(x,y)(1-\lambda_\delta(x,y))\chi_0(y)[\phi(y)-\phi(x)]\psi(x)d\omega(y)d\omega(x)\Big|\\ &\lesssim \iint_{d(x,y)\leqslant(M+4)r}\frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{d(x,y)}{\|x-y\|}\Big)^\eta\Big(\frac{\|x-y\|}{r}\Big)^\eta|\psi(x)|d\omega(y)d\omega(x)\\ &\lesssim \|\psi\|_{L^1(\mathbb{R}^N,\omega)}.\end{aligned}$$

It remains to show that $\lim_{\delta\to0}p_{2,\delta}=0$, that is,

$$\lim_{\delta\to0}\langle K(x,y),\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle=0. \tag{2.2}$$

The weak boundedness property plays a crucial role. To this end, let $\{y_j\}_{j\in\mathbb{Z}}\in\mathbb{R}^N$ be the maximal collection of points satisfying

$$\frac{1}{2}\delta<\inf_{j\neq k}\|y_j-y_k\|\leqslant\delta. \tag{2.3}$$

By observing that $\{y_j\}_{j\in\mathbb{Z}}$ is a maximal collection, we get that for each $x\in\mathbb{R}^N$ there exists a point y_j such that $\|x-y_j\|\leqslant\delta$. Let $\eta_j(y)=\theta(\frac{\|y-y_j\|}{\delta})$ and $\bar\eta_j(y)=[\sum_{i=1}^\infty\eta_i(y)]^{-1}\eta_j(y)$. To see that $\bar\eta_j$ is well defined, it suffices to show that for any $y\in\mathbb{R}^N$, there are only finitely many η_j with $\eta_j(y)\neq0$. This follows from the following fact: $\eta_j(y)\neq0$ if and only if $\|y-y_j\|\leqslant2\delta$ and hence this implies that $B(y_j,\delta)\subseteq B(y,4\delta)$. Inequality (2.3) shows $B(y_j,\frac{\delta}{4})\cap B(y_k,\frac{\delta}{4})=\emptyset$ for $j\neq k$ and hence there are at most C_0 points $y_j\in\mathbb{R}^N$ such that $B(y_j,\frac{\delta}{4})\subseteq B(y,4\delta)$. Now let $\Gamma=\{j:\bar\eta_j(y)\chi_0(y)\neq0\}$. Then $\#\Gamma\leqslant Cr^N/\delta^N$ since supp $(\chi_0)\subseteq\bigcup_{\sigma\in G}B(\sigma(x_0),2r)$ and supp$(\bar\eta_j)\subseteq B(y_j,2\delta)$. We write

$$\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)=\sum_{j\in\Gamma}\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x),$$

and

$$\langle K(x,y),\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle=\sum_{j\in\Gamma}\langle K(x,y),\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle.$$

It is then easy to check that supp$(\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x))\subseteq B(y_j,4\delta)\times B(y_j,2\delta)$ and

$$\|\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\|_{L^\infty(\mathbb{R}^N,\omega)}\leqslant C\delta^\eta,$$

where C is a constant depending only on θ,ϕ,ψ,x_0, and r but not on δ and j.

We claim that

$$\|\lambda_\delta(\cdot,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(\cdot)]\psi(\cdot)\|_\eta\lesssim1, \tag{2.4}$$

and

$$\|\lambda_\delta(x,\cdot)\bar\eta_j(\cdot)\chi_0(\cdot)[\phi(\cdot)-\phi(x)]\psi(x)\|_\eta\lesssim1. \tag{2.5}$$

Assuming (2.4) and (2.5) for the moment, since $T\in WBP$, we have

$$\begin{aligned}&|\langle K(x,y),\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle|\\ &\quad\leqslant\sum_{j\in\Gamma}|\langle K(x,y),\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle|\end{aligned}$$

$$\begin{aligned}&\leqslant \sum_{j\in\Gamma}\omega(B(y_j,4\delta))\delta^\eta \lesssim \frac{r^N}{\delta^N}\sup_{j\in\Gamma}\omega(B(y_j,1))\delta^N\delta^\eta\\&\lesssim \sup_{j\in\Gamma}\omega(B(y_j,1))r^N\delta^\eta.\end{aligned}$$

Hence, (2.2) holds.

It remains to show (2.4) and (2.5). To check (2.4), it suffices to show that for given $x_1, x_2 \in \mathbb{R}^N$ with $\|x_1-x_2\|\leqslant\delta$,

$$|\bar\eta_j(y)\chi_0(y)||\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)-\lambda_\delta(x_2,y)[\phi(y)-\phi(x_2)]\psi(x_2)|\lesssim\|x_1-x_2\|^\eta,$$

since if $\|x_1-x_2\|\geqslant\delta$, then the expansion on the left above is clearly bounded by

$$\begin{aligned}&|\bar\eta_j(y)\chi_0(y)|\{|\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)|+|\lambda_\delta(x_2,y)[\phi(y)-\phi(x_2)]\psi(x_2)|\}\\&\lesssim\delta^\eta\leqslant\|x_1-x_2\|^\eta.\end{aligned}$$

By the construction of $\bar\eta_j$, it follows that $|\bar\eta_j(y)\chi_0(y)|\lesssim 1$ for all $y\in\mathbb{R}^N$. Thus

$$\begin{aligned}&|\bar\eta_j(y)\chi_0(y)||\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)-\lambda_\delta(x_2,y)[\phi(y)-\phi(x_2)]\psi(x_2)|\\&\quad\lesssim\big|\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)-\lambda_\delta(x_2,y)[\phi(y)-\phi(x_2)]\psi(x_2)\big|\\&\quad\lesssim\big|\lambda_\delta(x_1,y)-\lambda_\delta(x_2,y)|[\phi(y)-\phi(x_1)]\psi(x_1)\big|+\big|\lambda_\delta(x_2,y)[\phi(x_1)-\phi(x_2)]\psi(x_1)\big|\\&\qquad+\big|\lambda_\delta(x_2,y)[\phi(y)-\phi(x_2)]|\psi(x_1)-\psi(x_2)\big|\\&\quad=: I_1+I_2+I_3.\end{aligned}$$

Recall that $\|x_1-x_2\|\leqslant\delta$. If $\|x_1-y\|>4\delta$, then $\lambda_\delta(x_1,y)=\lambda_\delta(x_2,y)=0$, so $I_1=0$. Thus we may assume that $\|x_1-y\|\leqslant 4\delta$,

$$I_1\lesssim\left|\frac{\|x_1-y\|}{\delta}-\frac{\|x_2-y\|}{\delta}\right|\|x_1-y\|^\eta\lesssim\delta^{\eta-1}\|x_1-x_2\|\lesssim\|x_1-x_2\|^\eta,$$

since we may assume $\eta\leq 1$. Terms I_2 and I_3 are easy to estimate:

$$I_2+I_3\lesssim\|x_1-x_2\|^\eta,$$

since we may assume that $\delta<1$.

To check (2.5) it suffices to show that for $y_1,y_2\in\mathbb{R}^N$ with $\|y_1-y_2\|\leqslant\delta$,

$$|\lambda_\delta(x,y_1)\bar\eta_j(y_1)\chi_0(y_1)[\phi(y_1)-\phi(x)]\psi(x)-\lambda_\delta(x,y_2)\bar\eta_j(y_2)\chi_0(y_2)[\phi(y_2)-\phi(x)]\psi(x)|\lesssim|y_1-y_2|^\eta.$$

Similarly, if $\|y_1-y_2\|\geqslant\delta$, then the expansion on the left-hand side above is bounded by

$$\begin{aligned}&|\lambda_\delta(x,y_1)\bar\eta_j(y_1)\chi_0(y_1)[\phi(y_1)-\phi(x)]\psi(x)|+|\lambda_\delta(x,y_2)\bar\eta_j(y_2)\chi_0(y_2)[\phi(y_2)-\phi(x)]\psi(x)|\\&\quad\lesssim\delta^\eta\leqslant\|y_1-y_2\|^\eta.\end{aligned}$$

Hence, suppose $\|y_1-y_2\|\leqslant\delta$ and write

$$\begin{aligned}&|\lambda_\delta(x,y_1)\bar\eta_j(y_1)\chi_0(y_1)[\phi(y_1)-\phi(x)]\psi(x)-\lambda_\delta(x,y_2)\bar\eta_j(y_2)\chi_0(y_2)[\phi(y_2)-\phi(x)]\psi(x)|\\&\leqslant\big|\lambda_\delta(x,y_1)-\lambda_\delta(x,y_2)|\bar\eta_j(y_1)\chi_0(y_1)[\phi(y_1)-\phi(x)]\psi(x)\big|\\&\quad+\big|\lambda_\delta(x,y_2)[\bar\eta_j(y_1)-\bar\eta_j(y_2)]\chi_0(y_1)[\phi(y_1)-\phi(x)]\psi(x)\big|\\&\quad+\big|\lambda_\delta(x,y_2)\bar\eta_j(y_2)[\chi_0(y_1)-\chi_0(y_2)][\phi(y_1)-\phi(x)]\psi(x)\big|\\&\quad+\big|\lambda_\delta(x,y_2)\bar\eta_j(y_2)\chi_0(y_2)[\phi(y_1)-\phi(y_2)]\psi(x)\big|\\&=: II_1+II_2+II_3+II_4.\end{aligned}$$

If $\|x-y_1\|>4\delta$, then $\lambda_\delta(x,y_1)=\lambda_\delta(x,y_2)=0$, so $II_1=II_2=II_3=II_4=0$. Thus we may assume that $\|x-y_1\|\leqslant 4\delta$,

$$II_1\lesssim\left|\frac{\|x-y_1\|}{\delta}-\frac{\|x-y_2\|}{\delta}\right|\|x-y_1\|^\eta\lesssim\delta^{\eta-1}\|y_1-y_2\|\lesssim\|y_1-y_2\|^\eta.$$

And

$$II_2\lesssim\left|\frac{\|y_1-y_j\|}{\delta}-\frac{\|y_2-y_j\|}{\delta}\right|\|y_1-x\|^\eta\lesssim\delta^{\eta-1}\|y_1-y_2\|\lesssim\|y_1-y_2\|^\eta.$$

Similarly,

$$II_3\lesssim\left|\frac{d(y_1,x_0)}{\delta}-\frac{d(y_2,x_0)}{\delta}\right|\|y_1-x\|^\eta\lesssim\delta^{\eta-1}d(y_1,y_2)\lesssim\delta^{\eta-1}\|y_1-y_2\|\lesssim\|y_1-y_2\|^\eta.$$

It is clear that

$$II_4\lesssim\|y_1-y_2\|^\eta.$$

This completes the proofs of (2.4) and (2.5), and thus, we obtain $|p|\lesssim\|\psi\|_{L^1(\mathbb{R}^N,\omega)}$.

To finish the proof of Lemma 2.2, we now estimate q. It suffices to show that for $x\in B(x_0,r)$,

$$|T\chi_0(x)|\lesssim 1. \tag{2.6}$$

To see this, it is easy to check that $q=\langle T\chi_0,\phi\psi\rangle$, and hence (2.6) implies

$$|q|\leqslant\|T\chi_0\|_{L^\infty(B(x_0,r),\omega)}\|\phi\psi\|_{L^1(B(x_0,r),\omega)}\lesssim\|\psi\|_{L^1(\mathbb{R}^N,\omega)}.$$

To show (2.6), let $\psi\in C^\eta(\mathbb{R}^N)$ with $\mathrm{supp}(\psi)\subseteq B(x_0,r)$ and $\int_{\mathbb{R}^N}\psi(x)d\omega(x)=0$. By the facts that $T(1)=0$ and $\int_{\mathbb{R}^N}\psi(x)d\omega(x)=0$, we obtain

$$|\langle T\chi_0,\psi\rangle|=\Big|-\langle T\chi_1,\psi\rangle\Big|=\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[K(x,y)-K(x_0,y)]\chi_1(y)\psi(x)d\omega(y)d\omega(x)\Big|.$$

Observe that the supports of χ_1 and ψ imply $d(y,x_0)>2r$ and $\|x-x_0\|\leqslant r$, respectively. The smoothness condition of K yields

$$\begin{aligned}|\langle T\chi_0,\psi\rangle|&\lesssim\iint\limits_{d(y,x_0)>2r\geqslant2\|x-x_0\|}\frac{1}{\omega(B(x,d(y,x_0)))}\Big(\frac{\|x-x_0\|}{\|y-x_0\|}\Big)^\varepsilon d\omega(y)|\psi(x)|d\omega(x)\\&\lesssim\iint\limits_{d(y,x_0)>2r\geqslant2\|x-x_0\|}\frac{1}{\omega(B(x,d(y,x_0)))}\Big(\frac{r}{d(y,x_0)}\Big)^\varepsilon d\omega(y)|\psi(x)|d\omega(x)\\&\lesssim\int_{\mathbb{R}^N}|\psi(x)|d\omega(x).\end{aligned}$$

This implies that $T\chi_0(x)=\alpha+\gamma(x)$ for $x\in B(x_0,r)$ with α is a constant depending on χ_0 and $\|\gamma(x)\|_{L^\infty(\mathbb{R}^N,\omega)}\leqslant C_0$ for some constant C_0 independent of χ_0. To estimate α, choose $\varphi\in C_0^\eta(\mathbb{R}^N)$ with supp $\varphi\subseteq B(x_0,r), 0\leqslant\varphi\leqslant1, \|\varphi\|_\eta\leqslant r^{-\eta}$ and $\int_{\mathbb{R}^N}\varphi(x)d\omega(x)=C_1\omega(B(x_0,r))$, for some constant C_1 independent of r. We then use $T\in WBP$ to get

$$\left|C_1\omega(B(x_0,r))\alpha+\int_{\mathbb{R}^N}\varphi(x)\gamma(x)d\omega(x)\right|=|\langle T\chi_0,\varphi\rangle|\leqslant C\omega(B(x_0,r)),$$

which implies $|\alpha|\leqslant C_0+\frac{C}{C_1}$ and hence, the proof of Lemma 2.2 is complete. □

We remark that the Meyer-type commutation Lemma was first proved in [23]. Then this lemma was used in [10] for proving the Tb Theorem on spaces of homogeneous type in the sense of Coifman and Weiss, and in [29] for giving a new proof of the $T1$ Theorem on non-homogeneous spaces. Moreover, if the operator T and functions ϕ, χ_0 satisfy the conditions as given in Lemma 2.2, then $T\phi(x)$ is a locally bounded function rather than a distribution. This fact will play a crucial role in the proof of Theorem 2.3 below.

2.3. **Boundedness of Dunkl–Calderón–Zygmund operators on smooth molecules.** It is well known that in the classical case, the almost orthogonal estimates are fundamental tools for the proof of the $T1$ theorem. The following result provides such a tool in the Dunkl setting.

Theorem 2.3. *Suppose that T is the singular integral operator as in Definition 1.1. Suppose further that $T(1) = T^*(1) = 0$ and $T \in WBP$. Then T maps $\mathbb{M}(\beta,\gamma,r,x_0)$ to $\widetilde{\mathbb{M}}(\beta,\gamma',r,x_0)$ with $0<\beta<\varepsilon, 0<\gamma'<\gamma<\varepsilon$, where ε is the exponent of the regularity of the kernel of T. Moreover, there exists a constant C such that*

$$\|T(f)\|_{\widetilde{\mathbb{M}}(\beta,\gamma',r,x_0)} \leqslant C\|f\|_{\mathbb{M}(\beta,\gamma,r,x_0)}.$$

Here $\mathbb{M}(\beta,\gamma,r,x_0)$ and $\widetilde{\mathbb{M}}(\beta,\gamma,r,x_0)$ are defined by following.

Definition 2.4. A function $f(x)$ is said to be a smooth molecule for $0<\beta\leqslant 1, \gamma>0, r>0$ and some fixed $x_0\in\mathbb{R}^N$, if $f(x)$ satisfies the following conditions:

$$|f(x)| \leqslant C\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+\|x-x_0\|}\Big)^{\gamma}; \tag{2.7}$$

$$\begin{aligned}|f(x)-f(x')| \leqslant C\Big(\frac{\|x-x'\|}{r}\Big)^{\beta}\Big\{&\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+\|x-x_0\|}\Big)^{\gamma}\\ &+\frac{1}{V(x',x_0,r+d(x',x_0))}\Big(\frac{r}{r+\|x'-x_0\|}\Big)^{\gamma}\Big\};\end{aligned} \tag{2.8}$$

$$\int_{\mathbb{R}^N} f(x)d\omega(x) = 0. \tag{2.9}$$

If $f(x)$ is a smooth molecule, we denote by $f\in\mathbb{M}(\beta,\gamma,r,x_0)$ and define the norm of f by $\|f\|_{\mathbb{M}(\beta,\gamma,r,x_0)} := \inf\{C : (2.7) - (2.8) \text{ hold}\}$.

If the Euclidean metric is replaced by the Dunkl metric in some places, then we have the following.

Definition 2.5. A function $f(x)$ is said to be a weak smooth molecule for $0<\beta\leqslant 1, \gamma>0, r>0$ and some fixed $x_0\in\mathbb{R}^N$, if $f(x)$ satisfies the following conditions:

$$|f(x)| \leqslant C\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}; \tag{2.10}$$

$$\begin{aligned}|f(x)-f(x')| \leqslant C\Big(\frac{\|x-x'\|}{r}\Big)^{\beta}\Big\{&\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}\\ &+\frac{1}{V(x',x_0,r+d(x',x_0))}\Big(\frac{r}{r+d(x',x_0)}\Big)^{\gamma}\Big\};\end{aligned} \tag{2.11}$$

$$\int_{\mathbb{R}^N} f(x)d\omega(x) = 0. \tag{2.12}$$

If $f(x)$ is a weak smooth molecule, we denote $f(x)$ by $f \in \widetilde{\mathbb{M}}(\beta,\gamma,r,x_0)$ and define the norm of f by $\|f\|_{\widetilde{\mathbb{M}}(\beta,\gamma,r,x_0)} := \inf\{C : (2.10) - (2.11) \text{ hold}\}$.

It is easy to see that $\widetilde{\mathbb{M}}(\beta,\gamma,r,x_0) \subset \mathbb{M}(\beta,\gamma,r,x_0)$. In the classical case, to provide the boundedness from $H^p(\mathbb{R}^N)$ to $H^p(\mathbb{R}^N)$, the molecule was first introduced in [9]. In [23], to show the classical Calderón–Zygmund operators form an algebra, Meyer introduced the smooth atom. Indeed, the smooth atom has no compact support. The smooth molecule defined in 2.4 is the version of Meyer's smooth atom in the Dunkl setting. In [14] following estimates were provided:

$$|\partial_t^m \partial_x^\alpha \partial_y^\beta p_t(x,y)| \lesssim t^{-m-|\alpha|-|\beta|} \frac{1}{V(x,y,t+d(x,y))} \frac{t}{t+\|x-y\|}, \tag{2.13}$$

where p_t is the Poisson kernel.

These estimates indicade that $q_t(x,y) = t\frac{\partial}{\partial t}p_t(x,y)$ for all $x,y \in \mathbb{R}^N$ and $t>0$, satisfy the following conditions:

$$\begin{aligned}
&\text{(i)} \quad |q_t(x,y)| \leqslant \frac{1}{V(x,y,t+d(x,y))}\frac{t}{t+\|x-y\|},\\
&\text{(ii)} \quad |q_t(x,y)-q_t(x',y)|\\
&\qquad \leqslant \frac{\|x-x'\|}{t}\Big(\frac{1}{V(x,y,t+d(x,y))}\frac{t}{t+\|x-y\|} + \frac{1}{V(x',y,t+d(x',y))}\frac{t}{t+\|x'-y\|}\Big),\\
&\text{(iii)} \quad |q_t(x,y)-q_t(x,y')|\\
&\qquad \leqslant \frac{\|y-y'\|}{t}\Big(\frac{1}{V(x,y,t+d(x,y))}\frac{t}{t+\|x-y\|} + \frac{1}{V(x,y',t+d(x,y'))}\frac{t}{t+\|x-y'\|}\Big),\\
&\text{(vi)} \quad \int_{\mathbb{R}^N} q_t(x,y)d\omega(y) = \int_{\mathbb{R}^N} q_t(x,y)d\omega(x) = 0.
\end{aligned}$$

And $\psi_t(x,y)$ for all $x,y \in \mathbb{R}^N$, $t>0$ also satisfy the similar conditions as $q_t(x,y)$ but $\psi_t(x,y)$ is supported in $\{(x,y) : d(x,y) \leqslant t\}$, see [14] for more details. Thus, both $q_t(x,y)$ and $\psi_t(x,y)$ are smooth molecules in $\mathbb{M}(1,1,t,y)$ for any fixed y and $\mathbb{M}(1,1,t,x)$ for any fixed x.

Proof of Theorem 2.3. Suppose that $f(x)$ is a smooth molecule in $\mathbb{M}(\beta,\gamma,r,x_0)$, we will show that $\|T(f)\|_{\widetilde{\mathbb{M}}(\beta,\gamma',r,x_0)} \leqslant C\|f\|_{\mathbb{M}(\beta,\gamma,r,x_0)}$, where $0<\beta<\varepsilon, 0<\gamma<\gamma'<\varepsilon$ and ε is the exponent of the regularity of the kernel of T. We first estimate the size condition for $Tf(x)$. To this end, we consider two cases: Case (1): $d(x,x_0) \leqslant 5r$ and Case (2): $d(x,x_0) = R > 5r$.

For the first case, set $1 = \xi(y) + \eta(y)$, where $\xi(y) = \theta\Big(\frac{d(y,x_0)}{10r}\Big)$ with $\theta \in C_0^\infty(\mathbb{R}), \theta(x) = 1$ for $\|x\| \leqslant 1$ and $\theta(x) = 0$ for $\|x\| \geqslant 2$. Applying Lemma 2.2, we write

$$\begin{aligned}
Tf(x) &= \langle K(x,y), (\xi(y)+\eta(y))f(y)\rangle\\
&= \int_{\mathbb{R}^N} K(x,y)\xi(y)(f(y)-f(x))d\omega(y) + f(x)\langle K(x,y),\xi(y)\rangle\\
&\qquad + \int_{\mathbb{R}^N} K(x,y)\eta(y)f(y)d\omega(y)\\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Applying the size condition for the kernel $K(x,y)$ in (1.4) and the smoothness condition for f in (2.8), we have

$$\begin{aligned}|I_1| &\lesssim \int_{d(x,y)\leqslant 20r} |K(x,y)|\cdot|f(y)-f(x)|d\omega(y)\\ &\lesssim \int_{d(x,y)\leqslant 20r} \frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{d(x,y)}{\|x-y\|}\Big)^{\beta}\Big(\frac{\|x-y\|}{r}\Big)^{\beta}\Big\{\frac{1}{V(y,x_0,r+d(y,x_0))}\\ &\qquad\times\Big(\frac{r}{r+\|y-x_0\|}\Big)^{\gamma}+\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+\|x-x_0\|}\Big)^{\gamma}\Big\}d\omega(y).\end{aligned}$$

Note that if $d(y,x)\leqslant 20r$ and $d(x,x_0)\leqslant 5r$, then $\omega(B(y,r+d(x,x_0)))\sim\omega(B(x,r+d(x,x_0)))$. Thus, we obtain

$$\begin{aligned}|I_1| &\lesssim \frac{1}{r^{\beta}}\frac{1}{V(x,x_0,r+d(x,x_0))}\int_{d(x,y)\leqslant 20r}\frac{1}{\omega(B(x,d(x,y)))}d(x,y)^{\beta}d\omega(y)\\ &\lesssim \frac{1}{V(x,x_0,r+d(x,x_0))}\lesssim\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.\end{aligned}$$

Similar to the proof of (2.6) in Lemma 2.2, we can get $|T(\xi)(x)|\lesssim 1$ and thus

$$I_2\lesssim|f(x)|\lesssim\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.$$

For the last term I_3, observing that $d(x,x_0)\leqslant 5r$ and the support of $\eta(y)$ is contained in $\{y\mid d(y,x_0)\geqslant 10r\}$, so $d(x,y)\geqslant 5r$ and $d(x,y)\sim d(y,x_0)$, and thus,

$$\begin{aligned}|I_3| &\lesssim \int_{\substack{d(y,x_0)\geqslant 10r\\ d(x,y)\geqslant 5r}}\frac{1}{\omega(B(x,d(x,y)))}\frac{1}{V(y,x_0,r+d(y,x_0))}\Big(\frac{r}{r+\|y-x_0\|}\Big)^{\gamma}d\omega(y)\\ &\lesssim \frac{1}{\omega(B(x,r))}\int_{d(y,x_0)\geqslant 10r}\frac{1}{\omega(B(x_0,d(y,x_0)))}\Big(\frac{r}{d(y,x_0)}\Big)^{\gamma}d\omega(y)\\ &\lesssim \frac{1}{\omega(B(x,r))}\lesssim\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.\end{aligned}$$

Case 2. $d(x,x_0)=R>5r$.

Set $1=I(y)+J(y)+L(y)$, where $I(y)=\theta\big(\frac{16d(y,x)}{R}\big)$, $J(y)=\theta\big(\frac{16d(y,x_0)}{R}\big)$ and $f_1(y)=I(y)f(y)$, $f_2(y)=J(y)f(y)$ and $f_3(y)=L(y)f(y)$.

Observing that, if y is in the support of $f_1(y)$, then $d(y,x_0)\sim d(x,x_0)=R$, and thus,

(i) $|f_1(y)|\lesssim|I(y)|\dfrac{1}{V(y,x_0,r+d(y,x_0))}\Big(\dfrac{r}{r+\|y-x_0\|}\Big)^{\gamma}\lesssim\dfrac{1}{V(x,x_0,R)}\Big(\dfrac{r}{R}\Big)^{\gamma}.$

(ii) $\displaystyle\int_{\mathbb{R}^N}|f_1(y)|d\omega(y)\lesssim\int_{d(y,x_0)\geqslant\frac{7R}{8}}\frac{1}{V(y,x_0,d(y,x_0))}\Big(\frac{r}{d(y,x_0)}\Big)^{\gamma}d\omega(y)\lesssim\Big(\frac{r}{R}\Big)^{\gamma}.$

(iii) $|f_1(y)-f_1(x)|\lesssim\Big(\dfrac{\|y-x\|}{r}\Big)^{\beta}\dfrac{1}{V(x,x_0,d(x,x_0))}\Big(\dfrac{r}{R}\Big)^{\gamma}.$

(iv) $\displaystyle\int_{\mathbb{R}^N}|f_3(y)|\omega(y)dy\lesssim\int_{d(y,x_0)\geqslant\frac{R}{16}}\frac{1}{V(y,x_0,r+d(y,x_0))}\Big(\frac{r}{d(y,x_0)}\Big)^{\gamma}d\omega(y)\lesssim\Big(\frac{r}{R}\Big)^{\gamma}.$

By the fact $\int_{\mathbb{R}^N}f(y)d\omega(y)=0$, we have

(v) $\displaystyle\Big|\int_{\mathbb{R}^N}f_2(y)d\omega(y)\Big|=\Big|-\int_{\mathbb{R}^N}f_1(y)d\omega(y)-\int_{\mathbb{R}^N}f_3(y)d\omega(y)\Big|\lesssim\Big(\frac{r}{R}\Big)^{\gamma}.$

We first estimate $Tf_1(x)$ as follows.

Set $u(y)=\theta\Big(\frac{2d(y,x)}{R}\Big)$. Then $f_1(y)=u(y)f_1(y)$. By using Lemma 2.2, we have

$$\begin{aligned}Tf_1(x)&=\langle K(x,y)u(y)f_1(y)\rangle=\int_{\mathbb{R}^N}K(x,y)u(y)[f_1(y)-f_1(x)]d\omega(y)+f_1(x)\langle K(x,\cdot),u(\cdot)\rangle\\&=:I+II.\end{aligned}$$

Similar to the proof of (2.6) in Lemma 2.2, we can get $|T(u)(x)|\lesssim 1$ and thus

$$|II|\lesssim|f(x)|\lesssim\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.$$

For the term I, we write it in two parts.

$$\begin{aligned}I&=\int_{d(x,y)\leqslant r}K(x,y)u(y)[f_1(y)-f_1(x)]d\omega(y)+\int_{r<d(x,y)\leqslant R}K(x,y)u(y)[f_1(y)-f_1(x)]d\omega(y)\\&=:I_1+I_2.\end{aligned}$$

Applying the size condition on the kernel $K(x,y)$ and the property (iii) above implies that

$$\begin{aligned}|I_1|&\lesssim\int_{d(x,y)\leqslant r}\frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{d(x,y)}{\|x-y\|}\Big)^{\beta}\Big(\frac{\|x-y\|}{r}\Big)^{\beta}\frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{R}\Big)^{\gamma}d\omega(y)\\&=\frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{R}\Big)^{\gamma}\int_{d(x,y)\leqslant r}\frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{d(x,y)}{r}\Big)^{\beta}d\omega(y)\\&\lesssim\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.\end{aligned}$$

Applying the size conditions for the kernel $K(x,y)$ and property (i) above, we obtain that for $\delta=\gamma-\gamma'$,

$$\begin{aligned}|I_2|&\lesssim\int_{r<d(x,y)\leqslant\frac{R}{4}}\frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{d(x,y)}{\|y-x\|}\Big)^{\delta}\Big[|f_1(y)|+|f_1(x)|\Big]d\omega(y)\\&\lesssim\frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{R}\Big)^{\gamma}\Big(\frac{1}{r}\Big)^{\delta}\int_{d(x,y)\leqslant\frac{R}{4}}\frac{1}{\omega(B(x,d(x,y)))}d(x,y)^{\delta}d\omega(y)\\&\lesssim\Big(\frac{R}{r}\Big)^{\delta}\Big(\frac{r}{R}\Big)^{\gamma}\frac{1}{V(x,x_0,d(x,x_0))}\\&\lesssim\frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma'}.\end{aligned}$$

To estimate $Tf_2(x)$, we decompose it in two parts.

$$Tf_2(x)=\int_{\mathbb{R}^N}[K(x,y)-K(x,x_0)]f_2(y)d\omega(y)dy+K(x,x_0)\int_{\mathbb{R}^N}f_2(y)d\omega(y)=:II_1+II_2.$$

By the estimate in (v) above,

$$|II_2|\lesssim|K(x,x_0)|\Big(\frac{r}{R}\Big)^{\gamma}\lesssim\frac{1}{\omega(B(x,d(x,x_0)))}\Big(\frac{r}{R}\Big)^{\gamma}\lesssim\frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.$$

For the term II_1, we write it by

$$II_1=\Big(\int_{\|y-x_0\|\leqslant\frac{R}{4}}+\int_{d(y,x_0)\leqslant\frac{R}{4}\leqslant\|y-x_0\|}\Big)[K(x,y)-K(x,x_0)]f_2(y)d\omega(y)=:II_{11}+II_{12}.$$

Applying the size condition for f_2 and the smoothness condition on the kernel $K(x,y)$ in (1.5) with $\|y-x_0\|\leqslant\frac{1}{2}d(x,x_0)$ for term II_{11} implies that

$$\begin{aligned}|II_{11}| &\lesssim \int_{d(y,x_0)\leqslant\frac{R}{8}} \frac{1}{\omega(B(x,d(x,x_0)))}\Big(\frac{\|y-x_0\|}{\|x-x_0\|}\Big)^{\gamma'}\Big(\frac{r}{r+\|y-x_0\|}\Big)^{\gamma}\frac{1}{V(y,x_0,r+d(y,x_0))}d\omega(y)\\ &\lesssim \frac{1}{\omega(B(x,d(x,x_0)))}\Big(\frac{r}{R}\Big)^{\gamma'}\int_{d(y,x_0)\leqslant\frac{R}{8}}\Big(\frac{r}{r+\|y-x_0\|}\Big)^{\gamma-\gamma'}\frac{1}{V(y,x_0,r+d(y,x_0))}d\omega(y)\\ &\lesssim \frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma'}.\end{aligned}$$

For the term II_{12}, since $d(y,x_0)\leqslant\frac{R}{4}$ implies $d(x,y)\geqslant d(x,x_0)-d(y,x_0)\geqslant\frac{3}{4}d(x,x_0)$. Applying the size conditions for the kernel $K(x,y)$ and $K(x,x_0)$ yields

$$\begin{aligned}|II_{12}| &\lesssim \int_{d(y,x_0)\leqslant\frac{R}{4}\leqslant\|y-x_0\|}\Big\{\frac{1}{\omega(B(x,d(x,y)))}+\frac{1}{\omega(B(x,d(x,x_0)))}\Big\}\\ &\quad\times\Big(\frac{r}{r+\|y-x_0\|}\Big)^{\gamma}\frac{1}{V(y,x_0,r+d(y,x_0))}d\omega(y)\\ &\lesssim \frac{1}{\omega(B(x,d(x,x_0)))}\Big(\frac{r}{R}\Big)^{\gamma'}\int_{R^N}\Big(\frac{r}{r+d(y,x_0)}\Big)^{\gamma-\gamma'}\frac{1}{V(y,x_0,r+d(y,x_0))}d\omega(y)\\ &\lesssim \frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma'}.\end{aligned}$$

Finally,

$$\begin{aligned}|Tf_3(y)| &\lesssim \int_{\substack{d(y,x)\geqslant\frac{R}{16},\\ d(y,x_0)\geqslant\frac{R}{16}}}\frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{r}{r+\|y-x_0\|}\Big)^{\gamma}\frac{1}{V(y,x_0,d(y,x_0))}d\omega(y)\\ &\lesssim \frac{1}{\omega(B(x,d(x,x_0)))}\int_{d(y,x_0)\geqslant\frac{R}{16}}\Big(\frac{r}{d(y,x_0)}\Big)^{\gamma}\frac{1}{V(y,x_0,d(y,x_0))}d\omega(y)\\ &\lesssim \frac{1}{\omega(B(x,d(x,x_0)))}\Big(\frac{r}{R}\Big)^{\gamma}\\ &\lesssim \frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.\end{aligned}$$

It remains to show the regularity of $T(f)$, that is, the following estimate:

$$\begin{aligned}|Tf(x)-Tf(x')| \lesssim \Big(\frac{\|x-x'\|}{r}\Big)^{\beta}\Big\{&\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma'}\\ &+\frac{1}{V(x',x_0,r+d(x',x_0))}\Big(\frac{r}{r+d(x',x_0)}\Big)^{\gamma'}\Big\}.\end{aligned}$$

Observing that we only need to consider the case where $\|x-x'\|\leqslant\frac{1}{20}r$. Indeed, if $\|x-x'\|\geqslant\frac{1}{20}r$, by the size estimate of $T(f)$,

$$\begin{aligned}|Tf(x)-Tf(x')| \lesssim &\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma'}\\ &+\frac{1}{V(x',x_0,r+d(x',x_0))}\Big(\frac{r}{r+d(x',x_0)}\Big)^{\gamma'},\end{aligned}$$

which gives the desired regularity estimate of $T(f)$.

Set $\|x-x'\|=\delta\leqslant\frac{1}{20}r$. We will consider it in the following two cases: $d(x,x_0)=R\geqslant 10r$ and $d(x,x_0)<10r$.

Case (1): $d(x,x_0)=R\geqslant 10r$. Let $I(y)=\theta(\frac{8d(y,x)}{R})$, $J(y)=1-I(y)$. Denote $f_1(y)=I(y)f(y)$, $f_2(y)=J(y)f(y)$. Let $u(y)=\theta(\frac{d(y,x)}{2\delta})$ and $v(y)=1-u(y)$. Write

$$\begin{aligned}Tf_1(x)&=\int_{\mathbb{R}^N}K(x,y)u(y)[f_1(y)-f_1(x)]d\omega(y)+\int_{\mathbb{R}^N}K(x,y)[v(y)f_1(y)+u(y)f_1(x)]d\omega(y)\\&=:p(x)+q(x).\end{aligned}$$

Since $u(y)$ is supported in $\{y:d(x,y)\leqslant 4\delta\}$, we have

$$\begin{aligned}|p(x)|&\lesssim\int_{d(x,y)\leqslant 4\delta}\frac{1}{\omega(B(x,d(y,x)))}\Big(\frac{d(x,y)}{\|x-y\|}\Big)^{\beta}\Big(\frac{\|x-y\|}{r}\Big)^{\beta}\\&\quad\times\Big\{\frac{1}{V(y,x_0,r+d(y,x_0))}\Big(\frac{r}{r+d(y,x_0)}\Big)^{\gamma}+\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}\Big\}d\omega(y)\\&\lesssim\frac{1}{r^{\beta}}\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}\int_{d(x,y)\leqslant 4\delta}\frac{1}{\omega(B(x,d(y,x)))}\big(d(x,y)\big)^{\beta}d\omega(y)\\&\lesssim\Big(\frac{\delta}{r}\Big)^{\beta}\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma},\end{aligned}$$

since $d(y,x)\leqslant\frac{1}{4}R$ and $d(x,x_0)=R$, so $d(y,x_0)\sim d(x,x_0)$.

If replacing x by x', we still have

$$|p(x')|\lesssim\Big(\frac{\delta}{r}\Big)^{\beta}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}\frac{1}{V(x,x_0,r+d(x,x_0))}.$$

Therefore,

$$|p(x)-p(x')|\lesssim\Big(\frac{\delta}{r}\Big)^{\beta}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}\frac{1}{V(x,x_0,r+d(x,x_0))}.$$

Observing that by $T(1)=0$, we can write

$$\begin{aligned}q(x)-q(x')&=\int_{d(x,y)\geqslant 2\delta}[K(x,y)-K(x',y)]v(y)[f_1(y)-f_1(x)]d\omega(y)\\&\quad+[f_1(x)-f_1(x')]\int_{\mathbb{R}^N}K(x',y)u(y)d\omega(y)\\&=:I+II.\end{aligned}$$

Similar to the proof of (2.6) in Lemma 2.2, we can get $|T(u)(x')|\lesssim 1$ and thus

$$\begin{aligned}II&\lesssim|f_1(x)-f_1(x')|\\&\lesssim\Big(\frac{\delta}{r}\Big)^{\beta}\Big\{\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}+\frac{1}{V(x',x_0,r+d(x',x_0))}\Big(\frac{r}{r+d(x',x_0)}\Big)^{\gamma}\Big\}.\end{aligned}$$

For term I, applying the smoothness condition of $K(x,y)$ with $\|x-x'\|=\delta\leqslant\frac{1}{2}d(x,y)$ and the smoothness condition for f_1 implies that

$$\begin{aligned}|I|&\lesssim\int_{d(x,y)\geqslant 2\delta}\frac{1}{\omega(B(x,d(y,x)))}\Big(\frac{\|x-x'\|}{\|x-y\|}\Big)^{\varepsilon}\Big(\frac{\|y-x\|}{r}\Big)^{\beta}\frac{1}{V(x,x_0,d(x,x_0))}\Big(\frac{r}{R}\Big)^{\gamma}d\omega(y)\\&\lesssim\frac{\delta^{\varepsilon}}{r^{\beta}}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}\frac{1}{V(x,x_0,r+d(x,x_0))}\int_{d(x,y)\geqslant 2\delta}\frac{1}{\omega(B(x,d(y,x)))}\frac{1}{(d(x,y))^{\varepsilon-\beta}}d\omega(y)\end{aligned}$$

$$\lesssim \Big(\frac{\delta}{r}\Big)^{\beta}\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma},$$

since $d(y,x_0)\sim d(x,x_0)$. The estimates of I and II gives the desired estimate for $Tf_1(x)-Tf_1(x')$.

To see the estimate for $Tf_2(x)-Tf_2(x')$, note that if $f_2(y)\neq 0$, then $d(x,y)\geqslant \frac{1}{8}R\geqslant 2\|x-x'\|$. Therefore,

$$\begin{aligned}
&|Tf_2(x)-Tf_2(x')|\\
&\leqslant \int_{d(y,x)\geqslant\frac{3}{4}R\geqslant 2\delta}|K(x,y)-K(x',y)|\cdot|f_2(y)|d\omega(y)\\
&\lesssim \int_{d(y,x)\geqslant\frac{3}{4}R}\frac{1}{\omega(B(x,d(y,x)))}\Big(\frac{\|x-x'\|}{\|x-y\|}\Big)^{\varepsilon}\frac{1}{V(y,x_0,r+d(y,x_0))}\Big(\frac{r}{r+d(y,x_0)}\Big)^{\gamma}d\omega(y)\\
&\lesssim \Big(\frac{\delta}{R}\Big)^{\varepsilon}\frac{1}{\omega(B(x,d(x,x_0)))}\int_{\mathbb{R}^N}\frac{1}{V(y,x_0,r+d(y,x_0))}\Big(\frac{r}{r+d(y,x_0)}\Big)^{\gamma}d\omega(y)\\
&\lesssim \Big(\frac{\delta}{r}\Big)^{\varepsilon}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\varepsilon}\frac{1}{\omega(B(x,d(x,x_0)))}\\
&\lesssim \Big(\frac{\delta}{r}\Big)^{\beta}\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.
\end{aligned}$$

Cases 2: $d(x,x_0)<10r$. The proof of this case is easier. Indeed, set $1=\xi(y)+\eta(y)$, where $\xi(y)=\theta\Big(\frac{d(y,x)}{5\delta}\Big)$ and again write $Tf(x)=p(x)+q(x)$, where $p(x)=\int_{\mathbb{R}^N}K(x,y)[f(y)-f(x)]\xi(y)d\omega(y)$ and

$$q(x)=\int_{\mathbb{R}^N}K(x,y)f(y)\eta(y)d\omega(y)+f(x)\int_{\mathbb{R}^N}K(x,y)\xi(y)d\omega(y).$$

Applying the size condition for $K(x,y)$ and the smoothness condition for f implies that

$$\begin{aligned}
|p(x)|&\lesssim \int_{d(x,y)\leqslant 10\delta}\frac{1}{\omega(B(x,d(x,y)))}\Big(\frac{d(x,y)}{\|x-y\|}\Big)^{\beta}\Big(\frac{\|x-y\|}{r}\Big)^{\beta}\\
&\qquad\times\Big\{\frac{1}{V(y,x_0,r+d(y,x_0))}\Big(\frac{r}{r+\|y-x_0\|}\Big)^{\gamma}\\
&\qquad\qquad+\frac{1}{V(x,x_0,r+d(x,x_0)))}\Big(\frac{r}{r+\|x-x_0\|}\Big)^{\gamma}\Big\}d\omega(y)\\
&\lesssim \frac{1}{r^{\beta}}\frac{1}{(V(x,x_0,r))}\int_{d(x,y)\leqslant 10\delta}\frac{1}{\omega(B(x,d(x,y)))}\big(d(x,y)\big)^{\beta}d\omega(y)\\
&\lesssim \Big(\frac{\delta}{r}\Big)^{\beta}\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.
\end{aligned}$$

Repeating the same proof implies that

$$|p(x')|\lesssim \Big(\frac{\delta}{r}\Big)^{\beta}\frac{1}{V(x,x_0,r+d(x,x_0))}\Big(\frac{r}{r+d(x,x_0)}\Big)^{\gamma}.$$

Similarly, by $T(1)=0$, we have $q(x)-q(x')=:I+II$ with

$$I=\int_{\mathbb{R}^N}[K(x,y)-K(x',y)]\eta(y)[f(y)-f(x)]d\omega(y),$$

$$II = [f(x) - f(x')] \int_{\mathbb{R}^N} K(x', y)\xi(y) d\omega(y).$$

Observing that if $d(y, x) \geqslant 5\delta$, then

$$|K(x, y) - K(x', y)| \lesssim \Big(\frac{\delta}{\|x - y\|}\Big)^{\varepsilon} \frac{1}{\omega(B(x, d(y, x)))}$$

and

$$\begin{aligned}|f(y) - f(x)| \lesssim \Big(\frac{\|x - y\|}{r}\Big)^{\beta} \Big\{ &\frac{1}{V(y, x_0, r + d(y, x_0))} \Big(\frac{r}{r + \|y - x_0\|}\Big)^{\gamma} \\ &+ \frac{1}{V(x, x_0, r + d(x, x_0)))} \Big(\frac{r}{r + \|x - x_0\|}\Big)^{\gamma} \Big\}.\end{aligned}$$

Note that $r + d(x, x_0) \lesssim r + d(y, x_0)$, therefore

$$\begin{aligned}|I| &\lesssim \frac{\delta^{\varepsilon}}{r^{\beta}} \frac{1}{V(x, x_0, r + d(x, x_0)))} \int_{d(y,x)\geqslant 5\delta} \frac{1}{d(y, x)^{\varepsilon - \beta}} \frac{1}{\omega(B(x, d(y, x)))} d\omega(y) \\ &\lesssim \Big(\frac{\delta}{r}\Big)^{\beta} \frac{1}{V(x, x_0, r + d(x, x_0)))} \Big(\frac{r}{r + d(x, x_0)}\Big)^{\gamma}.\end{aligned}$$

Similar to the proof of (2.6) in Lemma 2.2, we can get $|T(\xi)(x')| \lesssim 1$ and thus

$$\begin{aligned}|II| \lesssim |f(x) - f(x')| \lesssim \Big(\frac{\delta}{r}\Big)^{\beta} \Big\{ &\frac{1}{V(x, x_0, r + d(x, x_0)))} \Big(\frac{r}{r + d(x, x_0)}\Big)^{\gamma} \\ &+ \frac{1}{V(x', x_0, r + d(x', x_0)))} \Big(\frac{r}{r + d(x', x_0)}\Big)^{\gamma} \Big\}.\end{aligned}$$

Finally, by the fact $T^*(1) = 0$, then $\int_{\mathbb{R}^N} T(f)(x) d\omega(x) = 0$.

The proof of Theorem 2.3 is complete. □

2.4. **Proof of the $T1$ Theorem.**

To show Theorem 1.3, the $T1$ theorem, observe that the necessary conditions of the $T1$ theorem follow from Theorem 2.1, namely $T(1), T^*(1) \in BMO(\mathbb{R}^N, \omega)$ and $T \in WBP$ by the definition of the weak boundedness of property.

To show the sufficent conditions of Theorem 1.3, we need to first extend T to a continuous linear operator from $\Lambda^s(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, \omega)$ into $(C_0^s(\mathbb{R}^N))'$, where $\Lambda^s(\mathbb{R}^N)$ denotes the closure of $C_0^{\eta}(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_s, 0 < s < \eta$, given in (1.3). To be precise, given $g \in C_0^s(\mathbb{R}^N), 0 < s < 1$, with the support contained in a ball $B(x_0, r)$, and set $\theta \in C_0^s(\mathbb{R}^N)$ with $\theta(x) = 1$ for $d(x, x_0) \leqslant 2r$ and $\theta(x) = 0$ for $d(x, x_0) \geqslant 4r$. Given $f \in \Lambda^s(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, \omega)$, we write

$$\langle Tf, g\rangle = \langle T(\theta f), g\rangle + \langle T((1 - \theta)f), g\rangle.$$

The first term on the right side above makes sense since $\theta f \in C_0^s(\mathbb{R}^N)$. To see that the second term is also well defined, by the size condition of $K(x, y)$ and the fact $f \in L^2(\mathbb{R}^N, \omega)$ together with the doubling and the reverse doubling conditions of the measure ω, we first write

$$\langle T((1 - \theta)f), g\rangle = \int_{\mathbb{R}^N} g(x) \int_{\{y: d(x,y)>r\}} K(x, y)(1 - \theta(y))f(y) d\omega(y)\ d\omega(x).$$

By Hölder's inequality,

$$\Big| \int_{\{y: d(x,y)>r\}} K(x, y)(1 - \theta(y))f(y) d\omega(y) \Big|$$

$$
\begin{aligned}
&\lesssim \|f\|_{L^2(\mathbb{R}^N,\omega)}\Big(\int_{\{y:d(x,y)>r\}} |K(x,y)|^2 d\omega(y)\Big)^{\frac{1}{2}}\\
&\lesssim \|f\|_{L^2(\mathbb{R}^N,\omega)}\Big(\sum_{j=0}^{\infty}\int_{\{2^jr\leqslant d(x,y)\leqslant 2^{j+1}r\}} \frac{1}{\omega(B(x,d(x,y)))^2} d\omega(y)\Big)^{\frac{1}{2}}\\
&\lesssim \|f\|_{L^2(\mathbb{R}^N,\omega)}\Big(\sum_{j=0}^{\infty}\int_{\{2^jr\leqslant d(x,y)\leqslant 2^{j+1}r\}} \frac{1}{\omega(B(x,2^jr))^2} d\omega(y)\Big)^{\frac{1}{2}}\\
&\lesssim \|f\|_{L^2(\mathbb{R}^N,\omega)}\Big(\sum_{j=0}^{\infty} 2^{-jN}\frac{1}{\omega(B(x,r))}\Big)^{\frac{1}{2}} < \infty,
\end{aligned}
$$

where the last inequality follows from the fact that $\inf\limits_{x} \omega(B(x,r)) > 0$.

This implies that $\langle T((1-\theta)f), g\rangle$ is well defined. Moreover, this extension is independent of the choice of θ.

We now describe the properties of Coifman's approximation to the identity acting on $\Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$. Let's begin with considering $(\mathbb{R}^N, \|\cdot\|, \omega)$ as space of homogeneous type in the sense of Coifman and Weiss. Note that the measure ω satisfies the doubling and the reverse doubling properties. Therefore, in this case, the Littlewood–Paley theory has already established in [20]. We recall main results for $(\mathbb{R}^N, \|\cdot\|, \omega)$ ([20]). Here and throughout, $V_k(x)$ always denotes the measure $\omega(B(x,2^{-k}))$ for $k\in\mathbb{Z}$ and $x\in\mathbb{R}^N$. We also denote by $V(x,y)=\omega(B(x,\|x-y\|))$ for $x,y\in\mathbb{R}^N$. Let $\theta:\mathbb{R}\mapsto[0,1]$ be a smooth function which is 1 for $\|x\|\leqslant 1$ and vanishes for $\|x\|\geqslant 2$. Applying the construction of Coifman's approximation to the identity, we define $T_k(f)(x)=\int_{\mathbb{R}^N}\theta(2^k\|x-y\|)f(y)d\omega(y), k\in\mathbb{Z}$. Let M_k be the operator of multiplication by $M_k(x):=\frac{1}{T_k(1)(x)}$ and let W_k be the operator of multiplication by $W_k(x):=\big[T_k\big(\frac{1}{T_k(1)}\big)(x)\big]^{-1}$. Coifman's approximation to the identity is constructed by $S_k=M_kT_kW_kT_kM_k$, where the kernel of S_k is

$$
S_k(x,y)=\int_{\mathbb{R}^n} M_k(x)\theta(2^k\|x-z\|)W_k(z)\theta(2^k\|z-y\|)M_k(y)d\omega(z).
$$

In [20], it was proved that kernels $S_k(x,y)$ defined on $\mathbb{R}^N\times\mathbb{R}^N$ satisfy the following properties.

(i) $S_k(x,y)=S_k(y,x)$;

(ii) $S_k(x,y)=0$ if $\|x-y\|>2^{4-k}$ and $|S_k(x,y)|\leqslant \dfrac{C}{V_k(x)+V_k(y)}$;

(iii) $|S_k(x,y)-S_k(x',y)|\leqslant C\dfrac{2^k\|x-x'\|}{V_k(x)+V_k(y)}$ for $\|x-x'\|\leqslant 2^{8-k}$;

(iv) $|S_k(x,y)-S_k(x,y')|\leqslant C\dfrac{2^k\|y-y'\|}{V_k(x)+V_k(y)}$ for $\|y-y'\|\leqslant 2^{8-k}$;

(v) $\big|[S_k(x,y)-S_k(x',y)]-[S_k(x,y')-S_k(x',y')]\big|\leqslant C\dfrac{2^k\|x-x'\|2^k\|y-y'\|}{V_k(x)+V_k(y)}$ for $\|x-x'\|\leqslant 2^{8-k}$ and $\|y-y'\|\leqslant 2^{8-k}$;

(vi) $\displaystyle\int_{\mathbb{R}^N} S_k(x,y)d\omega(x)=1$ for all $y\in\mathbb{R}^N$;

(vii) $\displaystyle\int_{\mathbb{R}^N} S_k(x,y)d\omega(y)=1$ for all $x\in\mathbb{R}^N$.

Coifman's decomposition of the identity on $L^2(\mathbb{R}^N,\omega)$ is given as follows. Let $D_k := S_k - S_{k-1}$. The identity operator I on $L^2(\mathbb{R}^N,\omega)$ can be written as

$$I = \sum_{k=-\infty}^{\infty} D_k = \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} D_kD_j = T_M + R_M,$$

where $T_M = \sum_{k\in\mathbb{Z}} D_kD_k^M$ with $D_k^M = \sum_{\{j\in\mathbb{Z}:|j|\leqslant M\}} D_{k+j}$ and $R_M = \sum_{\{j,k\in\mathbb{Z}:\,|k-j|>M\}} D_kD_j$. It is known, see [20], that there exists a constant C such that

$$|D_jD_k(x,y)| \leqslant C2^{-|j-k|}\frac{1}{V_{j\wedge k}(x)+V_{j\wedge k}(y)}, \tag{2.14}$$

where $j\wedge k = \min\{j,k\}$.

This estimate implies $\|D_jD_k\|_{L^2(\mathbb{R}^N,\omega)\mapsto L^2(\mathbb{R}^N,\omega)} \lesssim 2^{-|j-k|}$. By the Cotlar–Stein Lemma we obtain

$$\|R_M(f)\|_{L^2(\mathbb{R}^N,\omega)} \leqslant C2^{-M}\|f\|_{L^2(\mathbb{R}^N,\omega)}$$

and then for a fixed large M, T_M^{-1}, the inverse of T_M, is bounded on $L^2(\mathbb{R}^N,\omega)$. This yields that T_M converges to the identity in the $L^2(\mathbb{R}^N,\omega)$-norm and moreover,

$$I = T_M^{-1}T_M = \sum_{k=-\infty}^{\infty} T_M^{-1}D_k^MD_k = T_MT_M^{-1} = \sum_{k=-\infty}^{\infty} D_k^MD_kT_M^{-1} \qquad \text{in } L^2(\mathbb{R}^N,\omega).$$

The following lemma describes the properties of operators T_M acting on Λ^s.

Lemma 2.6. *Suppose* $0<s<\frac{1}{2}$. *Then (i)* $T_M = \sum_{k=-\infty}^{\infty} D_kD_k^M$ *converges uniformly and in the norm of* $\Lambda^s(\mathbb{R}^N)$; *(ii)* T_M *is bounded on* $\Lambda^s(\mathbb{R}^N)$; *(iii)* $\|T_M - I\|_s \to 0$ *as* $M\to+\infty$.

To prove Lemma 2.6, we need the following estimates for D_k and D_k^M.

Lemma 2.7. *Let* $0<s<1$. *Then (i)* $\|D_kf\|_{L^\infty(\mathbb{R}^N,\omega)} \lesssim 2^{-ks}\|f\|_s$; *(ii)* $\|D_kf\|_s \lesssim 2^{ks}\|f\|_{L^\infty(\mathbb{R}^N,\omega)}$; *(iii)* $\|D_kf\|_\beta \lesssim 2^{k(\beta-s)}\|f\|_s$ *if* $0<s\leqslant\beta<1$; *(iv)* $\|D_k^Mf\|_s \lesssim M\|f\|_s$.

Proof. For (i), the cancellations of D_k gives

$$D_kf(x) = \int_{\mathbb{R}^N} D_k(x,y)[f(y)-f(x)]d\omega(y).$$

Since $D_k(x,y)=0$ for $\|x-y\|\geqslant 2^{4-k}$, the size condition of D_k and the smoothness condition of f yield

$$|D_kf(x)| \lesssim \|f\|_s \int_{\|x-y\|\leqslant 2^{4-k}} \frac{1}{V_k(x)+V_k(y)}\|x-y\|^s d\omega(y) \lesssim 2^{-ks}\|f\|_s.$$

For (ii), the smoothness condition of D_k gives

$$\begin{aligned}|D_kf(x)-D_kf(y)| &= \left|\int_{\mathbb{R}^N}(D_k(x,z)-D_k(y,z))f(z)d\omega(z)\right|\\ &\leqslant \|f\|_{L^\infty(\mathbb{R}^N,\omega)}\left(\int_{\|x-z\|\leqslant 2^{4-k}} + \int_{\|y-z\|\leqslant 2^{4-k}}\right)\frac{(2^k\|x-y\|)^s}{V_k(x)+V_k(z)}d\omega(z),\end{aligned}$$

which implies $\|D_kf\|_s \lesssim 2^{ks}\|f\|_{L^\infty(\mathbb{R}^N,\omega)}$.

To estimate (iii), if $\|x-y\| \leqslant 2^{6-k}$, using the the cancellations of D_k and the smoothness condition of f, we get

$$\begin{aligned}|D_kf(x)-D_kf(y)| &= \left|\int_{\mathbb{R}^N}[D_k(x,z)-D_k(y,z)][f(z)-f(x)]d\omega(z)\right|\\ &\lesssim \left(\int_{\|x-z\|\leqslant 2^{4-k}}+\int_{\|y-z\|\leqslant 2^{4-k}}\right)\frac{(2^k\|x-y\|)^\beta}{V_k(x)+V_k(z)}\|x-z\|^s\|f\|_s d\omega(z)\\ &\lesssim \|x-y\|^\beta 2^{k(\beta-s)}\|f\|_s.\end{aligned}$$

When $\|x-y\|>2^{6-k}$, (i) gives $|D_kf(x)-D_kf(y)|\lesssim 2^{-ks}\|f\|_s\lesssim \|x-y\|^\beta 2^{k(\beta-s)}\|f\|_s$. (iii) follows from these estimates. The estimate of (iv) follows from (iii) with $\beta=s$. ☐

We now show Lemma 2.6.

Proof of Lemma 2.6. We first show that $T_M(f)(x)=\sum\limits_{k=-\infty}^{\infty}D_kD_k^M(f)(x)$ is well defined on $\Lambda^s(\mathbb{R}^N)$. To this end, let $f\in C_0^\eta$ with $\eta>s$ and set $G_kf(x):=D_kD_k^Mf(x)$. Observe that if $f\in C_0^\eta$ then $f\in L^\infty(\mathbb{R}^N,\omega)$ and hence, $\|D_k^M(f)\|_{L^1(\mathbb{R}^N,\omega)}\leqslant CM\|f\|_{L^\infty(\mathbb{R}^N,\omega)}$. By (iv), $\|D_k^M(f)\|_\eta\lesssim M\|f\|_\eta$. Therefore,

$$\begin{aligned}|G_k(f)(x)| &= \left|\int_{\mathbb{R}^N}D_k(x,y)D_k^M(f)(y)d\omega(y)\right|\\ &\leqslant C\frac{1}{V_k(x)}\|f\|_{L^\infty(\mathbb{R}^N,\omega)}\lesssim 2^{\mathbf{N}k}\frac{1}{V_0(x)}\|f\|_{L^\infty(\mathbb{R}^N,\omega)}\lesssim 2^{\mathbf{N}k}\|f\|_{L^\infty(\mathbb{R}^N,\omega)}\end{aligned}$$

since $\inf\limits_x V_0(x)\geqslant C>0$. And

$$\begin{aligned}|G_k(f)(x)| &= \left|\int_{\mathbb{R}^N}D_k(x,y)D_K^M(f)(y)d\omega(y)\right| = \left|\int_{\mathbb{R}^N}D_k(x,y)[D_K^M(f)(y)-D_K^M(f)(x)]d\omega(y)\right|\\ &\lesssim \|f\|_\eta\int_{\mathbb{R}^N}|D_k(x,y)|\|x-y\|^\eta d\omega(y)\lesssim 2^{-k\eta}\|f\|_\eta.\end{aligned}$$

These two estimates imply that if $f\in C_0^\eta(\mathbb{R}^N)$ then the series $\sum\limits_{k=-\infty}^{\infty}D_kD_k^M(f)(x)$ converges uniformly. Moreover, for given $x,y\in\mathbb{R}^N$, choose $k_0\in\mathbb{Z}$ such that $2^{-k_0}\leqslant\|x-y\|\leqslant 2^{-k_0+1}$. Then by using Lemma 2.7 and by splitting the sum over k into the sum over $k\geqslant k_0$ and $k\leqslant k_0$, we obtain that

$$\left|\sum_{k=-\infty}^{\infty}[G_kf(x)-G_kf(y)]\right|\lesssim 2^{-k_0s}\|f\|_s+2^{k_0(\beta-s)}\|x-y\|^\beta\|f\|_s\lesssim\|x-y\|^s\|f\|_s. \tag{2.15}$$

Hence if $f\in C_0^\eta(\mathbb{R}^N)$ with $\eta>s$, then the series $\sum\limits_k D_kD_k^Mf$ converges in $\Lambda^s(\mathbb{R}^N)$ norm. Observe that $C_0^\eta(\mathbb{R}^N)$ with $\eta>s$ is dense in $\Lambda^s(\mathbb{R}^N)$. This implies that T_M extends to $\Lambda^s(\mathbb{R}^N)$. Indeed, if $f\in\Lambda^s(\mathbb{R}^N)$, then there exists a sequence $f_n\in C_0^\eta,\eta>s$, such that $\|f_n-f\|_s$ tends to zero as n tends to ∞. Let $T_M(f)(x)=\lim\limits_{n\to\infty}T_M(f_n)(x)$. Then T_M is bounded on $\Lambda^s(\mathbb{R}^N)$ and moreover, $\|T_M(f)\|_s\lesssim\|f\|_s$ for $f\in\Lambda^s(\mathbb{R}^N)$.

To show $\|T_M-I\|_s\to 0$ as $M\to+\infty$, it is sufficient to prove the operator norm

$$\|R_M\|_{\Lambda^s(\mathbb{R}^N)\mapsto\Lambda^s(\mathbb{R}^N)}\to 0$$

as $M \to +\infty$. To this end, we rewrite

$$\begin{aligned} R_M f &= \sum_{k=-\infty}^{\infty} \sum_{\{j\in\mathbb{Z}:\, |k-j|>M\}} D_k D_j = \sum_{\{\ell\in\mathbb{Z}:\, |\ell|>M\}} \sum_{k=-\infty}^{\infty} D_k D_{k+\ell} f \\ &= \sum_{k=-\infty}^{\infty} D_k (I - S_{k+M}) f + \sum_{k=-\infty}^{\infty} D_k S_{k-M-1} f. \end{aligned}$$

Since $\int_{\mathbb{R}^N} S_k(x,y)d\omega(y) = 1$ for $k \in \mathbb{Z}$, we have

$$|(I - S_{k+M})f(x)| = \left| \int_{\mathbb{R}^N} S_{k+M}(x,y)[f(x) - f(y)]d\omega(y) \right| \leqslant C2^{-(M+k)s}\|f\|_s$$

and hence $\|(I - S_{k+M})f\|_{L^\infty(\mathbb{R}^N,\omega)} \leqslant C2^{-(M+k)s}\|f\|_s$. This estimate together with Lemma 2.7 and applying the same proof for (2.15) imply that $\big\| \sum_{k=-\infty}^{\infty} D_k(I - S_{k+M})f \big\|_s \leqslant 2^{-Ms}\|f\|_s$, which gives $\big\| \sum_{k=-\infty}^{\infty} D_k(I - S_{k+M}) \big\|_{\Lambda^s(\mathbb{R}^N)\mapsto\Lambda^s(\mathbb{R}^N)} \to 0$ as $M \to +\infty$.

To estimate $\sum_{k=-\infty}^{\infty} D_k S_{k-M-1} f$, the second term of $R_M f$, let $H_k f = D_k S_{k-M-1} f$ and denote $H_k(x,y)$ by the kernel of H_k. Then $\int_{\mathbb{R}^N} H_k(x,y)d\omega(y) = D_k S_{k-M-1}(1) = D_k(1) = 0$ and $H_k(x,y) = 0$ if $\|x-y\| \geqslant 2^{6-(k-M)}$. By the cancellation of D_k and the smothness of S_{k-M-1},

$$\begin{aligned} |H_k(x,y)| &= \left| \int_{\mathbb{R}^N} D_k(x,z)\big[S_{k-M-1}(z,y) - S_{k-M-1}(x,y)\big]d\omega(z) \right| \\ &\leqslant C \int_{|x-z|\leqslant 2^{2-k}} (V_k(x))^{-1} \frac{2^{k-M-1}\|x-z\|}{V_{k-M-1}(x)} d\omega(z) \\ &\leqslant C2^{-M}(V_{k-M-1}(x))^{-1}. \end{aligned}$$

Thus, for $f \in \Lambda^s(\mathbb{R}^N)$,

$$\begin{aligned} |H_k f(x)| &= \left| \int_{\mathbb{R}^N} H_k(x,y)[f(y) - f(x)]d\omega(y) \right| \\ &\leqslant C \int_{|x-y|\leqslant 2^{6-(k-M)}} 2^{-M}(V_{k-M-1}(x))^{-1}\|x-y\|^s\|f\|_s d\omega(y) \\ &\leqslant C2^{-M}2^{-(k-M)s}\|f\|_s. \end{aligned}$$

This implies that

$$\|H_k f\|_{L^\infty(\mathbb{R}^N,\omega)} \lesssim 2^{-M}2^{-(k-M)s}\|f\|_s. \tag{2.16}$$

If $\|x - x'\| \leqslant 2^{6-(k-M)}$, then

$$\begin{aligned} &|H_k(x,y) - H_k(x',y)| \\ &= \left| \int_{\mathbb{R}^N} \big[D_k(x,z) - D_k(x',z)\big] S_{k-M-1}(z,y)d\omega(z) \right| \\ &\leqslant C \int_{\substack{\{\|x-z\|\leqslant 2^{2-k} \\ \|x'-z\|\leqslant 2^{2-k}\}}} \frac{2^k\|x-x'\|}{V_k(x) + V_k(z)} (V_{k-M-1}(y))^{-1} d\omega(z) \\ &\leqslant C2^k\|x-x'\|(V_{k-M-1}(y))^{-1}. \end{aligned} \tag{2.17}$$

For $\|x-y\|\leqslant 2^{6-(k-M)}$, applying (2.17) yields

$$|H_kf(x)-H_kf(y)| = \Big|\int_{\mathbb{R}^N}[H_k(x,z)-H_k(y,z)][f(z)-f(x)]d\omega(z)\Big|$$
$$\lesssim \int_{\substack{\|x-z\|\leqslant 2^{6-(k-M)}\\ \|y-z\|\leqslant 2^{6-(k-M)}}} 2^k\|x-y\|(V_{k-M-1}(y))^{-1}\|x-z\|^s\|f\|_s d\omega(z)$$
$$\lesssim 2^k 2^{-(k-M)s}\|x-y\|\|f\|_s.$$

For $\|x-y\|>2^{6-(k-M)}$, the estimate (2.16) implies

$$|H_kf(x)-H_kf(y)|\lesssim 2^{-M}2^{-(k-M)s}\|f\|_s\lesssim 2^k2^{-(k-M)s}\|x-y\|\|f\|_s.$$

These estimates imply that $H_k(f)(x)$ is in $\Lambda^1(\mathbb{R}^N)$ with the norm bounded by

$$\|H_kf\|_1\lesssim 2^k2^{-(k-M)s}\|f\|_s. \tag{2.18}$$

Using the fact that $\|f\|_\beta\leqslant \|f\|^{1-\beta}_{L^\infty(\mathbb{R}^N,\omega)}\|f\|_1^\beta, 0<\beta<1$, the estimates (2.16) and (2.18) yield

$$\|H_kf\|_\beta\lesssim 2^{-M(1-2\beta)}2^{-(k-M)(s-\beta)}\|f\|_s. \tag{2.19}$$

Given $x,y\in\mathbb{R}^N$, choose $k_1\in\mathbb{Z}$ such that $2^{-k_1}\leqslant\|x-y\|\leqslant 2^{-k_1+1}$. The estimates in (2.16) and (2.19) imply that for $s<\beta$,

$$\Big|\sum_{k=-\infty}^{\infty}[H_kf(x)-H_kf(y)]\Big|$$
$$\leqslant \sum_{\{k:k\geqslant k_1\}}|H_kf(x)-H_kf(y)| + \sum_{\{k:k<k_1\}}|H_kf(x)-H_kf(y)|$$
$$\lesssim \sum_{\{k:k\geqslant k_1\}} 2\|H_kf\|_{L^\infty(\mathbb{R}^N,\omega)} + \sum_{\{k:k<k_1\}}\|x-y\|^\beta\|H_kf\|_\beta$$
$$\lesssim \sum_{\{k:k\geqslant k_1\}} 2^{-M}2^{-(k-M)s}\|f\|_s + \sum_{\{k:k<k_1\}}\|x-y\|^\beta 2^{-M(1-2\beta)}2^{-(k-M)(s-\beta)}\|f\|_s$$
$$\lesssim 2^{-k_1s}2^{-M+Ms}\|f\|_s + 2^{-M(1-2\beta)}2^{M(s-\beta)}2^{k_1(\beta-s)}\|x-y\|^\beta\|f\|_s$$
$$\lesssim 2^{-M(1-2\beta)}\Big(2^{M(s-2\beta)}+2^{M(s-\beta)}\Big)\|x-y\|^s\|f\|_s$$
$$\lesssim 2^{-M(1-2\beta)}\|x-y\|^s\|f\|_s.$$

Therefore, we have

$$\Big\|\sum_{k=-\infty}^{\infty}H_kf\Big\|_s\lesssim 2^{-M(1-2\beta)}\|f\|_s \qquad \text{for} \quad s<\beta<1.$$

If $s<\frac{1}{2}$, we can choose β so that $2^{-M(1-2\beta)}\to 0$ as $M\to+\infty$. The proof of Lemma 2.6 is finished. □

We are ready to give the proof of sufficient conditions of the $T1$ theorem under the assumptions that $T(1)=T^*(1)=0$. Notice that T_M converges strongly on $L^2(\mathbb{R}^N,\omega)$ since, by the almost orthogonal estimates and the Cotlar–Stein Lemma,

$$\sup_{L_1,L_2}\Big\|\sum_{k=L_1}^{L_2}D_kD_k^M\Big\|_{L^2(\mathbb{R}^N,\omega)\mapsto L^2(\mathbb{R}^N,\omega)}<+\infty.$$

Thus, by Lemma 2.6, T_M converges strongly on $\Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$.

It is clear that $\Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$ is dense in $L^2(\mathbb{R}^N,\omega)$. To prove the sufficient condition of Theorem 1.3, it suffices to show that

$$\big|\langle g_0, Tf_0\rangle\big| \leqslant C\|g_0\|_{L^2(\mathbb{R}^N,\omega)}\|f_0\|_{L^2(\mathbb{R}^N,\omega)}$$

for any $g_0, f_0\in \Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$ with compact supports. For given $f_0\in \Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$ with compact support, by Lemma 2.6, set $f_1 = T_M^{-1}f_0 \in \Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$ and let

$$U_{L_1,L_2}=\sum_{k=L_1}^{L_2} D_kD_k^M.$$

By Lemma 2.6, $\lim\limits_{\substack{L_1\to-\infty\\ L_2\to+\infty}} U_{L_1,L_2}f_1 = f_0$ in $\Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$. Observe that operator T extends to a continuous linear operator from $\Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$ into $(C_0^s(\mathbb{R}^N))'$. Hence, for each $g_0\in \Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$ with compact support,

$$\langle g_0, Tf_0\rangle = \lim_{\substack{L_1\to-\infty\\ L_2\to+\infty}}\langle g_0, TU_{L_1,L_2}f_1\rangle.$$

Similarly, let $g_1 = T_M^{-1}g_0$. Then $g_1\in \Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$ and $\lim\limits_{\substack{L_1'\to-\infty\\ L_2'\to+\infty}} U_{L_1',L_2'}g_1 = g_0$ in $\Lambda^s(\mathbb{R}^N)\cap L^2(\mathbb{R}^N,\omega)$. Thus,

$$\langle g_0, Tf_0\rangle = \lim_{\substack{L_1\to-\infty\\ L_2\to+\infty}}\lim_{\substack{L_1'\to-\infty\\ L_2'\to+\infty}}\langle U_{L_1',L_2'}g_1, TU_{L_1,L_2}f_1\rangle.$$

Observe that

$$\langle U_{L_1',L_2'}g_1, TU_{L_1,L_2}f_1\rangle = \sum_{k=L_1}^{L_2}\sum_{k'=L_1'}^{L_2'}\Big\langle D_{k'}^M g_1, D_{k'}^*TD_kD_k^M f_1\Big\rangle.$$

The following almost orthogonal estimates in the Dunkl setting are crucial to estimate $\langle D_{k'}^M g_1, D_{k'}^*TD_kD_k^M f_1\rangle$.

Lemma 2.8. *Let T be a Dunkl–Calderón–Zygmund singular integral satisfying $T(1)=T^*(1)=0$ and $T\in WBP$. Then*

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D_k(x,u)K(u,v)D_j(v,y)d\omega(u)d\omega(v)\Big|\\ &\quad\lesssim 2^{-|k-j|\varepsilon'}\frac{1}{V(x,y,2^{(-j)\vee(-k)}+d(x,y))}\Big(\frac{2^{(-j)\vee(-k)}}{2^{(-j)\vee(-k)}+d(x,y)}\Big)^{\gamma},\end{aligned}$$

where $\gamma,\varepsilon'\in(0,\varepsilon)$ and ε is the regularity exponent of the kernel of T and $a\vee b = max\{a,b\}$.

Assuming Lemma 2.8 for the moment, then

$$\|D_k^*TD_{k'}\|_{L^2(\mathbb{R}^N,\omega)\mapsto L^2(\mathbb{R}^N,\omega)}\lesssim 2^{-|k-k'|\varepsilon'}.$$

Applying the Cotlar–Stein lemma yields

$$|\langle U_{L_1,L_2}g_1, TU_{L_1',L_2'}f_1\rangle|\lesssim \|f_1\|_{L^2(\mathbb{R}^N,\omega)}\|g_1\|_{L^2(\mathbb{R}^N,\omega)}\lesssim \|f_0\|_{L^2(\mathbb{R}^N,\omega)}\|g_0\|_{L^2(\mathbb{R}^N,\omega)}$$

for all L_1, L_2, L_1' and L_2'. Hence,

$$\big|\langle g_0, Tf_0\rangle\big| \leqslant C\|g_0\|_{L^2(\mathbb{R}^N,\omega)}\|f_0\|_{L^2(\mathbb{R}^N,\omega)}.$$

The proof of Theorem 1.3 with the assumptions $T(1)=T^*(1)=0$ is complete.

To show Lemma 2.8, we need the following almost orthogonal estimates in the Dunkl setting.

Lemma 2.9. *Let $x, y \in \mathbb{R}^N$ and $\varepsilon_0, S \geqslant s > 0$. Suppose that $f_S(x, \cdot)$ is a weak smooth molecule function in $\widetilde{\mathbb{M}}(\varepsilon_0, \varepsilon_0, S, x)$ and $g_s(\cdot, y)$ is a smooth molecule function in $\mathbb{M}(\varepsilon_0, \varepsilon_0, s, y)$. Then for any $0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0$, there exists $C > 0$ depending on $\varepsilon_0, \varepsilon_1, \varepsilon_2$, such that for all $S \geqslant s > 0$,*

$$\int_{\mathbb{R}^N} f_S(x,u) g_s(u,y) d\omega(u) \leqslant C \Big(\frac{s}{S}\Big)^{\varepsilon_1} \frac{1}{V(x,y,S+d(x,y))} \Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_2}. \tag{2.20}$$

If $f_t(x,\cdot)$ and $g_s(\cdot, y)$ both are smooth molecule functions in $\mathbb{M}(\varepsilon_0, \varepsilon_0, S, x)$ and $\mathbb{M}(\varepsilon_0, \varepsilon_0, s, y)$, respectively, then for any $0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0$, there exists $C > 0$ depending on $\varepsilon_0, \varepsilon_1, \varepsilon_2$, such that for all $t, s > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} f_t(x,u) g_s(u,y) d\omega(u) \leqslant C \Big(\frac{s}{t} \wedge \frac{t}{s}\Big)^{\varepsilon_1} & \frac{1}{V(x,y,(t\vee s)+d(x,y))} \\ & \Big(\frac{t\vee s}{(t\vee s)+\|x-y\|}\Big)^{\varepsilon_2}, \end{aligned} \tag{2.21}$$

where $a \vee b = \max\{a, b\}$.

We first recall a classical estimate and state it in the Dunkl setting as below, whose proof is standard and can be found, for example, in [21].

Lemma 2.10. *For any $\varepsilon, t > 0, y \in \mathbb{R}^N$, there exists a constant C depending on ε such that,*

$$\int_{\mathbb{R}^N} \frac{1}{\omega(x, t+d(x,y))} \Big(\frac{t}{t+d(x,y)}\Big)^{\varepsilon} d\omega(x) \leqslant C.$$

Before proving Lemma 2.9, we give the following lemma.

Lemma 2.11. *For any $\varepsilon_1, \varepsilon_2, t, s > 0$, Let*

$$T = \int_{\mathbb{R}^N} \frac{1}{V(x,z,t+d(x,z))} \Big(\frac{t}{t+d(x,z)}\Big)^{\varepsilon_1} \frac{1}{V(z,y,s+d(z,y))} \Big(\frac{s}{s+d(z,y)}\Big)^{\varepsilon_2} d\omega(z),$$

then there exists a constant C depending on $\varepsilon_1, \varepsilon_2$ such that,

$$T \leqslant \frac{C}{V(x,y,(t\vee s)+d(x,y))}.$$

Proof. Without loss of generality, we can assume $t \geqslant s$. We consider

Case 1: $d(x,y) \leqslant t$, then

$$\begin{aligned} T &\lesssim \int_{\mathbb{R}^N} \frac{1}{V(x,z,t)} \frac{1}{V(z,y,s+d(z,y))} \Big(\frac{s}{s+d(z,y)}\Big)^{\varepsilon_2} d\omega(z) \\ &\lesssim \frac{1}{\omega(B(x,t))} \int_{\mathbb{R}^N} \frac{1}{V(z,y,s+d(z,y))} \Big(\frac{s}{s+d(z,y)}\Big)^{\varepsilon_2} d\omega(z). \end{aligned}$$

By the condition $d(x,y) \leqslant t$ and Lemma 2.10, we have $T \lesssim \frac{1}{V(x,y,t+d(x,y))}$.

Case 2: $d(x,y) \geqslant t$, since $d(x,z) + d(y,z) \geqslant d(x,y)$, we have

$$\begin{aligned} T \leqslant & \int_{d(x,z)\geqslant \frac{1}{2}d(x,y)} \frac{1}{V(x,z,t+d(x,z))} \Big(\frac{t}{t+d(x,z)}\Big)^{\varepsilon_1} \frac{1}{V(z,y,s+d(z,y))} \Big(\frac{s}{s+d(z,y)}\Big)^{\varepsilon_2} d\omega(z) \\ & + \int_{d(y,z)\geqslant \frac{1}{2}d(x,y)} \frac{1}{V(x,z,t+d(x,z))} \Big(\frac{t}{t+d(x,z)}\Big)^{\varepsilon_1} \frac{1}{V(z,y,s+d(z,y))} \Big(\frac{s}{s+d(z,y)}\Big)^{\varepsilon_2} d\omega(z) \\ & =: T_1 + T_2. \end{aligned}$$

For term T_1, we have

$$T_1 \lesssim \int_{d(x,z)\geqslant \frac{1}{2}d(x,y)} \frac{1}{\omega(B(x,t+d(x,y)))} \frac{1}{V(z,y,s+d(z,y))}\Big(\frac{s}{s+d(z,y)}\Big)^{\varepsilon_2} d\omega(z)$$
$$\lesssim \frac{1}{V(x,y,t+d(x,y))} \int_{\mathbb{R}^N} \frac{1}{V(z,y,s+d(z,y))}\Big(\frac{s}{s+d(z,y)}\Big)^{\varepsilon_2} d\omega(z).$$

By Lemma 2.10, we have $T_1 \lesssim \frac{1}{V(x,y,t+d(x,y))}$.

For term T_2,

$$T_2 \lesssim \int_{d(y,z)\geqslant \frac{1}{2}d(x,y)} \frac{1}{V(x,z,t+d(x,z))}\Big(\frac{t}{t+d(x,z)}\Big)^{\varepsilon_1} \frac{1}{\omega(B(y,d(x,y)))} d\omega(z)$$
$$= \frac{1}{\omega(B(y,d(x,y)))} \int_{\mathbb{R}^N} \frac{1}{V(x,z,t+d(x,z))}\Big(\frac{t}{t+d(x,z)}\Big)^{\varepsilon_1} d\omega(z).$$

By the condition $d(x,y) \geqslant t$ and Lemma 2.10, we have $T_2 \lesssim \frac{1}{V(x,y,d(x,y))} \lesssim \frac{1}{V(x,y,t+d(x,y))}$.
This complete the proof of Lemma 2.11. □

Now we prove Lemma 2.9.

Proof. We begin with the estimate (2.20). Let $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, then we only need to show that

$$\int_{\mathbb{R}^N} f_S(x,u) g_s(u,y) d\omega(u) \leqslant C\Big(\frac{s}{S}\Big)^{\varepsilon} \frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon}, \quad \text{for } S \geqslant s.$$

we write

$$A = \int_{\mathbb{R}^N} f_S(x,u) g_s(u,y) d\omega(u) = \int_{\mathbb{R}^N} \Big(f_S(x,u) - f_S(x,y)\Big) g_s(u,y) d\omega(u).$$

Then

$$\begin{aligned} |A| &\leqslant \int_{\|u-y\|\leqslant S} |f_S(x,u) - f_S(x,y)| \cdot |g_s(u,y)| d\omega(u) \\ &\quad + \int_{\|u-y\|> S} \big(|f_S(x,u)| + |f_S(x,y)|\big) \cdot |g_s(u,y)| d\omega(u) \\ &=: I + II, \end{aligned}$$

where

$$\begin{aligned} I \lesssim \int_{\|u-y\|\leqslant S} &\Big(\frac{\|u-y\|}{S}\Big)^{\varepsilon_0} \Big(\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0} \\ &+ \frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_0}\Big) \times \frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0} d\omega(u) \end{aligned}$$

and

$$\begin{aligned} II \lesssim \int_{\|u-y\|> S} &\Big(\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0} + \frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_0}\Big) \\ &\times \frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0} d\omega(u). \end{aligned}$$

For term I, since $\|u-y\|\leqslant S$, we have

$$\Big(\frac{\|u-y\|}{S}\Big)^{\varepsilon_0}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0}\leqslant\left(\frac{\|u-y\|}{S}\right)^{\varepsilon}\left(\frac{s}{\|u-y\|}\right)^{\varepsilon}\left(\frac{s}{s+\|u-y\|}\right)^{\varepsilon_0-\varepsilon}$$
$$=\left(\frac{s}{S}\right)^{\varepsilon}\left(\frac{s}{s+\|u-y\|}\right)^{\varepsilon_0-\varepsilon}.$$

Thus

$$I\lesssim\left(\frac{s}{S}\right)^{\varepsilon}\int_{\|u-y\|\leqslant S}\left(\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0}+\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_0}\right)$$
$$\times\frac{1}{V(u,y,s+d(u,y))}\left(\frac{s}{s+\|u-y\|}\right)^{\varepsilon_0-\varepsilon}d\omega(u).$$

Let

$$I_1=\int_{\|u-y\|\leqslant S}\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0}\frac{1}{V(u,y,s+d(u,y))}\left(\frac{s}{s+\|u-y\|}\right)^{\varepsilon_0-\varepsilon}d\omega(u)$$

and

$$I_2=\int_{\|u-y\|\leqslant S}\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_0}\frac{1}{V(u,y,s+d(u,y))}\left(\frac{s}{s+\|u-y\|}\right)^{\varepsilon_0-\varepsilon}d\omega(u),$$

then $I\lesssim\big(\frac{s}{S}\big)^{\varepsilon}\cdot(I_1+I_2)$. Note that

$$I_2\leqslant\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_0}\int_{\mathbb{R}^N}\frac{1}{V(u,y,s+d(u,y))}\left(\frac{s}{s+d(u,y)}\right)^{\varepsilon_0-\varepsilon}d\omega(u)$$
$$\lesssim\frac{1}{V(x,y,S+d(x,y))}\left(\frac{S}{S+d(x,y)}\right)^{\varepsilon},$$

where we apply Lemma 2.10 in the last inequality above.

And

$$I_1\leqslant\int_{d(u,y)\leqslant S}\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0}\frac{1}{V(u,y,s+d(u,y))}\left(\frac{s}{s+d(u,y)}\right)^{\varepsilon_0-\varepsilon}d\omega(u).$$

We estimate I_1 in two cases: $d(x,y)\leqslant 2S$ and $d(x,y)>2S$.

Case 1: If $d(x,y)\leqslant 2S$, applying Lemma 2.11, we have

$$I_1\lesssim\frac{1}{V(x,y,S+d(x,y))}\lesssim\frac{1}{V(x,y,S+d(x,y))}\left(\frac{S}{S+d(x,y)}\right)^{\varepsilon}.$$

Case 2: If $d(x,y)>2S$, by the conditions $d(u,y)\leqslant S<\frac{1}{2}d(x,y)$ and $d(x,u)+d(y,u)\geqslant d(x,y)$, we get $d(x,u)\geqslant\frac{1}{2}d(x,y)$. And hence,

$$I_1\leqslant\int_{d(u,y)\leqslant S}\frac{1}{V(x,u,S+d(x,u))}\left(\frac{S}{S+d(x,u)}\right)^{\varepsilon}\left(\frac{S}{S+d(x,u)}\right)^{\varepsilon_0-\varepsilon}$$
$$\times\frac{1}{V(u,y,s+d(u,y))}\left(\frac{s}{s+d(u,y)}\right)^{\varepsilon_0-\varepsilon}d\omega(u)$$
$$\lesssim\int_{\mathbb{R}^N}\frac{1}{V(x,u,S+d(x,u))}\left(\frac{S}{S+d(x,y)}\right)^{\varepsilon}\left(\frac{S}{S+d(x,u)}\right)^{\varepsilon_0-\varepsilon}$$

$$\times \frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+d(u,y)}\Big)^{\varepsilon_0-\varepsilon}d\omega(u)$$
$$\lesssim \frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon},$$

where the last inequality above follows from Lemma 2.11.

Therefore

$$I\lesssim \Big(\frac{s}{S}\Big)^{\varepsilon}\cdot(I_1+I_2)\lesssim \Big(\frac{s}{S}\Big)^{\varepsilon}\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon}.$$

For term II, Let

$$II_1=\int_{\|u-y\|>S}\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0}\frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0}d\omega(u)$$

and

$$II_2=\int_{\|u-y\|>S}\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_0}\frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0}d\omega(u)$$

Note that

$$II_2=\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon_0}\int_{\|u-y\|>S}\frac{1}{V(u,y,s+d(u,y))}$$
$$\times\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0-\varepsilon}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon}d\omega(u)$$
$$\lesssim \frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon}\int_{\mathbb{R}^N}\frac{1}{V(u,y,s+d(u,y))}$$
$$\times\Big(\frac{s}{s+d(u,y)}\Big)^{\varepsilon_0-\varepsilon}\Big(\frac{s}{S}\Big)^{\varepsilon}d\omega(u)$$
$$\lesssim (\frac{s}{S})^{\varepsilon}\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon},$$

where again the last inequality follows from Lemma 2.10.

For term II_1, since $d(x,u)+d(y,u)\geqslant d(x,y)$, we have

$$II_1\leqslant \int_{\substack{d(x,u)\geqslant\frac12 d(x,y)\\ \|u-y\|>S}}\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0}\frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0}d\omega(u)$$
$$+\int_{\substack{d(y,u)\geqslant\frac12 d(x,y)\\ \|u-y\|>S}}\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0}\frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0}d\omega(u)$$
$$=: II_{11}+II_{12}.$$

Note that

$$II_{11}\leqslant \int_{\mathbb{R}^N}\frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0-\varepsilon}$$
$$\times\frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{S}\Big)^{\varepsilon}\Big(\frac{s}{s+d(u,y)}\Big)^{\varepsilon_0-\varepsilon}d\omega(u)$$
$$\lesssim (\frac{s}{S})^{\varepsilon}\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon},$$

where the last inequality above follows from Lemma 2.11.

Moreover,

$$II_{12} = \int_{\substack{d(y,u)\geqslant\frac{1}{2}d(x,y)\\ \|u-y\|>S}} \frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0} \times \frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon}\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon_0-\varepsilon} d\omega(u).$$

Since $d(y,u)\geqslant \frac{1}{2}d(x,y)$ and $\|u-y\|>S$, we have $\|u-y\|\geqslant\frac{1}{2}(S+\|u-y\|)\geqslant\frac{1}{2}(S+d(u,y))\geqslant \frac{1}{4}(S+d(x,y))$.

Therefore

$$\Big(\frac{s}{s+\|u-y\|}\Big)^{\varepsilon} \lesssim \Big(\frac{s}{S+d(x,y)}\Big)^{\varepsilon} = \Big(\frac{s}{S}\Big)^{\varepsilon}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon},$$

which implies

$$\begin{aligned} II_{12} &\lesssim \Big(\frac{s}{S}\Big)^{\varepsilon}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon}\int_{\mathbb{R}^N} \frac{1}{V(x,u,S+d(x,u))}\Big(\frac{S}{S+d(x,u)}\Big)^{\varepsilon_0} \\ &\quad\times \frac{1}{V(u,y,s+d(u,y))}\Big(\frac{s}{s+d(u,y)}\Big)^{\varepsilon_0-\varepsilon} d\omega(u) \\ &\lesssim (\frac{s}{S})^{\varepsilon}\frac{1}{V(x,y,S+d(x,y))}\Big(\frac{S}{S+d(x,y)}\Big)^{\varepsilon}, \end{aligned}$$

where again the last inequality follows from Lemma 2.11.

This completes the proof of the estimate (2.20). The proof of the estimate (2.21) is almost the same. To be precise, replacing $d(x,u)$ by $\|x-u\|$ for all fractions $\frac{S}{S+d(x,u)}$ and $d(x,y)$ by $\|x-y\|$ for all fractions $\frac{S}{S+d(x,y)}$, respectively, yields the proof of the estimate (2.21). We leave the details to the reader. ☐

We return to the proof of Lemma 2.8, that is, if T is a Dunkl–Calderón–Zygmund singular integral satisfying $T(1)=T^*(1)=0$ and $T\in WBP$, then

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D_k(x,u)K(u,v)D_j(v,y)d\omega(u)d\omega(v)\Big| \\ &\lesssim 2^{-|k-j|\varepsilon'}\frac{1}{V(x,y,2^{-j\vee -k}+d(x,y))}\Big(\frac{2^{-j\vee -k}}{2^{-j\vee -k}+d(x,y)}\Big)^{\gamma}, \end{aligned}$$

where $\gamma,\varepsilon'\in(0,\varepsilon)$ and ε is the regularity exponent of the kernel of T.

To this end, we may assume $k\leqslant j$. Observe that $D_k(x,\cdot)$ is a smooth molecule in $\mathbb{M}(1,1,S,x)$ with $S=2^{-k}, x\in\mathbb{R}^N$ and $D_j(\cdot,y)$ is a smooth molecule in $\mathbb{M}(1,1,s,y)$ with $s=2^{-j}$ and $y\in\mathbb{R}^N$. Set $\widetilde{D}_k(x,v)=\int_{\mathbb{R}^N} D_k(x,u)K(u,v)d\omega(u)$. We hereby recall that

$$\int \widetilde{D}_k(x,v)d\omega(v)=0$$

thanks to the conditions $T^*(1)=0$.

By Theorem 2.3, for any $0<\varepsilon_0<1$, $\widetilde{D}_k(x,\cdot)$ is a weak smooth molecule in $\widetilde{\mathbb{M}}(\varepsilon_0,\varepsilon_0,S,x)$ with $S=2^{-k}$.

Note that when $k \leqslant j$, then $S \geqslant s$. Applying the estimate (2.20) in Lemma 2.9 yields

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D_k(x,u)K(u,v)D_j(v,y)d\omega(u)d\omega(v)\Big| = \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \widetilde{D}_k(x,v)D_j(v,y)d\omega(v)\Big|$$
$$\lesssim 2^{(k-j)\varepsilon'}\frac{1}{V(x,y,2^{-k}+d(x,y))}\Big(\frac{2^{-k}}{2^{-k}+d(x,y)}\Big)^{\gamma},$$

where $\varepsilon', \gamma < \varepsilon_0$.

Similarly, if $j \leqslant k$, then $\int_{\mathbb{R}^N} K(u,v)D_j(v,y)d\omega(v)$ is a weak smooth molecule and repeating the same proof gives the desired estimate.

Finally, to finish the proof of Theorem 1.3, it remains to consider the general case: $T(1) \in BMO(\mathbb{R}^N,\omega)$ and $T^*(1) \in BMO(\mathbb{R}^N,\omega)$. To handle this case, we recall the classical para-product operators on space of homogeneous type. We begin with the following definition of the test functions in space of homogeneous type $(\mathbb{R}^N, \|\cdot\|, \omega)$:

Definition 2.12. A function $f(x)$ defined on $\mathbb{R}^N$ is said to be a test function if there exits a constant C such that for $0<\beta\leqslant 1, \gamma>0, r>0$ and $x_0 \in \mathbb{R}^N$,

(i) $|f(x)| \leqslant \dfrac{C}{V(x,r+\|x-x_0\|)}\Big(\dfrac{r}{r+\|x-x_0\|}\Big)^{\gamma}$;

(ii) $|f(x)-f(x')| \leqslant C\Big(\dfrac{\|x-x'\|}{r+\|x-x_0\|}\Big)^{\beta}\dfrac{1}{V(x,r+\|x-x_0\|)}\Big(\dfrac{r}{r+\|x-x_0\|}\Big)^{\gamma}$,
for $\|x-x'\| \leqslant \frac{1}{2}(r+\|x-x_0\|)$;

(iii) $\displaystyle\int_{\mathbb{R}^N} f(x)d\omega(x) = 0.$

We denote such a test function by $f \in \mathcal{M}(\beta,\gamma,r,x_0)$ and $\|f\|_{\mathcal{M}(\beta,\gamma,r,x_0)}$, the norm in $\mathcal{M}(\beta,\gamma,r,x_0)$, is defined by the smallest C satisfying the above conditions (i) and (ii).

Applying Coifman's decomposition for the identity operator and the Calderón–Zygmund operator theory, the discrete Calderón reproducing formula in space of homogeneous type is given by the following

Theorem 2.13. *Let $\{S_k\}_{k\in\mathbb{Z}}$ be a Coifman's approximations to the identity and set $D_k := S_k - S_{k-1}$. Then there exists a family of operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$ such that*

$$f(x) = \sum_{k=-\infty}^{\infty}\sum_{Q\in Q^k}\omega(Q)\widetilde{D}_k(x,x_Q)D_k(f)(x_Q),$$

where Q^k is the collection of all dyadic cubes in $\mathbb{R}^N$ with the side length 2^{-M-k} for some fixed large M and x_Q is the center of Q. The series converges in $L^p(\mathbb{R}^N,\omega)$, $1<p<\infty$, $\mathcal{M}(\beta,\gamma,r,x_0)$ and in $(\mathcal{M}(\beta,\gamma,r,x_0))'$, the dual of in $\mathcal{M}(\beta,\gamma,r,x_0)$. Moreover, the kernels of the operators $\widetilde{D}_k$ satisfy the the following conditions: for $0<\varepsilon<1$,

(i) $|\widetilde{D}_k(x,y)| \leqslant C\dfrac{1}{V_k(x)+V_k(y)+V(x,y)}\Big(\dfrac{2^{-k}}{2^{-k}+\|x-y\|}\Big)^{\varepsilon}$;

(ii) $|\widetilde{D}_k(x,y)-\widetilde{D}_k(x',y)| \leqslant C\Big(\dfrac{\|x-x'\|}{2^{-k}+\|x-x'\|}\Big)^{\varepsilon}\dfrac{1}{V_k(x)+V_k(y)+V(x,y)}\Big(\dfrac{2^{-k}}{2^{-k}+\|x-y\|}\Big)^{\varepsilon}$,
for $\|x-x'\| \leqslant (2^{-k}+\|x-y\|)/2$;

(iii) $\displaystyle\int_{\mathbb{R}^N}\widetilde{D}_k(x,y)d\omega(x) = 0$ *for all* $y\in\mathbb{R}^N$;

(iv) $\int_{\mathbb{R}^N} \widetilde{D}_k(x,y)d\omega(y) = 0 \qquad$ *for all* $x \in \mathbb{R}^N$.

Similarly, there exists a family of linear operators $\{\widetilde{\widetilde{D}}_k\}_{k\in\mathbb{Z}}$ *such that*

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{Q\in Q^k} \omega(Q) D_k(x, x_Q) \widetilde{\widetilde{D}}_k(f)(x_Q),$$

where the kernels of $\widetilde{\widetilde{D}}_k$ *satisfy the above conditions* (i), (iii), (iv) *and* (ii) *with* x *and* y *interchanged.*

The paraproduct operator is defined by

Definition 2.14. Suppose that $\{S_k\}, \{D_k\}$ and $\{\widetilde{\widetilde{D}}_k\}$ are same as defined above. The paraproduct operator of $f \in \mathcal{M}(\beta,\gamma,r,x_0)'$ is defined by

$$\Pi_b(f)(x) = \sum_{k=-\infty}^{\infty} \sum_{Q\in Q^k} \omega(Q) D_k(x, x_Q) \widetilde{\widetilde{D}}_k(b)(x_Q) S_k(f)(x_Q),$$

where $b \in \mathrm{BMO}(\mathbb{R}^N, \omega)$.

We need the following classical result:

Theorem 2.15. *The paraproduct operator is the Calderón–Zygmund operator. Moreover,* $\Pi_b(1) = b$ *in the topology* $(H^1, \mathrm{BMO}), (\Pi_b)^*(1) = 0$ *and there exists a constant* C *such that for* $1 < p < \infty$,

$$\|\Pi_b(f)\|_{L^p(\mathbb{R}^N,\omega)} \leqslant C\|b\|_{\mathrm{BMO}(\mathbb{R}^N,\omega)} \|f\|_{L^p(\mathbb{R}^N,\omega)}.$$

See [20] for all these results and the details of the proofs.

Observe that the classical Calderón–Zygmund operator on $(\mathbb{R}^N,\omega)$ is also the Dunkl–Calderón–Zygmund operator. Suppose now that both $T(1)$ and $T^*(1)$ belong to $\mathrm{BMO}(\mathbb{R}^N,\omega)$. Set $\widetilde{T} = T - \Pi_{T(1)} - (\Pi_{T^*(1)})^*$. Then $\widetilde{T}$ is a Dunkl–Calderón–Zygmund singular integral operator. Moreover, $\widetilde{T}(1) = (\widetilde{T})^*(1) = 0$ and $\widetilde{T} \in WBP$. Therefore, $\widetilde{T}$ is bounded on $L^2(\mathbb{R}^N,\omega)$ and hence, T is also bounded on $L^2(\mathbb{R}^N,\omega)$. The proof of Theorem 1.3 is completed.

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References

[1] B. Amri, J. Ph. Anker and M. Sifi, *Three results in Dunkl theory*, Colloq. Math., **118** (2010), 299–312.

[2] B. Amri, A. Gasmi and M. Sifi, *Linear and bilinear multiplier operators for Dunkl transform*, Mediterr. J. Math., **7** (2010), 503–521.

[3] B. Amri and A. Hammi, Dunkl-Schrödinger operators, *Complex Anal. Oper. Theory*, **13** (2019), 1033–1058.

[4] B. Amri and M. Sifi, *Riesz transforms for Dunkl transform*, Ann. Math. Blaise Pascal, **19** (2012), 247–262.

[5] J.-Ph. Anker, *An introduction to Dunkl theory and its analytic aspects*, Analytic, Algebraic and Geometric Aspects of Differential Equations, 3-58, Trends Math.,Birkhäuser, Chem, 2017.

[6] J.-Ph. Anker, N. Ben Salem, J. Dziubański and N. Hamda, *The Hardy space H^1 in the rational Dunkl setting*, Constr. Approx., **42** (2015), 93–128.

[7] J.-Ph. Anker, J. Dziubański and A. Hejna, *Harmonic functions, conjugate harmonic functions and the Hardy H^1 in rational Dunkel setting*, J. Fourier Anal. Appl., **25** (2019), no. 5, 2356–2418.

[8] R. R. Coifman, G. Weiss, "Analyse Harmonique Non-commutative sur Certains Espaces Homogènes", Lecture Notes in Math., **242**, Springer-Verlag, Berlin (1971).

[9] R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. (1977) 83: 569-645.

[10] G. David, J. L. Journé, S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions paraaccrétives et interpolation*, Rev. Mat. Iberoam., **1** (1985), 1–56.

[11] C.F. Dunkl, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z., **197** (1) (1988), 33–60.

[12] C.F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math., **311** (1989), no. 1,167–183.

[13] C.F. Dunkl, *Integral kernels with reflection group invariance.* Canad. J. Math., **43** (1991) no. 6 1213–1227.

[14] J. Dziubański and A. Hejna, *Remark on atomic decompositions for the Hardy space H^1 in the rational Dunkl setting*, Studia Math., **251** (2020), 89–110.

[15] J. Dziubański and A. Hejna, *Hörmander multiplier theorem for the Dunkl transform*, J. Funct. Anal., **277** (2019), 2133–2159.

[16] J. Dziubański and A. Hejna, *Singular integrals in the rational Dunkl setting*, Revista Matematica Complutense, (2021) DOI10.1007/s13163-021-00402-1.

[17] J. Dziubański and A. Hejna, *Upper and lower bounds for Littlewood-Paley square functions in the Dunkl setting*, Studia Math. 262 (2022), no. 3, 275–303.

[18] J. Dziubański and A. Hejna, *Upper and lower bounds for the Dunkl heat kernel*, Calc. Var. 62, 25 (2023).

[19] Y. Han, M.-Y. Lee, J. Li and B. D. Wick, *Riesz transform and commutators in the Dunkl setting*, arXiv:2105.11275.

[20] Y. Han, D. Müller and D. Yang, *A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot–Carathéodory spaces*, Abstr. Appl. Anal., Vol. 2008, Article ID 893409. 250 pages.

[21] Y. Han and E. T. Sawyer, "Littlewood–Paley theory on spaces of homogeneous type and classical function spaces", Mem. Amer. Math. Soc., **110** (1994), no. 530, 1–126.

[22] H. Li and M. Zhao, *Square function estimates for Dunkl operators*, 2020, arXiv:2003.11843.

[23] Y. Meyer, *Les nouveaux opérateurs de Calderón-Zygmund*, Astérisque, tome **131** (1985), 237–254.

[24] Y. Meyer and R. Coifman, "Wavelets Calderón–Zygmund and multilinear operators", Translated by David Salinger, Cambridge University Press, 1997.

[25] M. Rösler, *Asymototic analysis for the Dunkl transform and its applications*, J. Approx. Theory, **119** (1) (2002), 110–126.

[26] F. Soltani, *Littlewood-Paley g-function in the Dunkl analysis on $\mathbb{R}^d$*, JIPAM. J. Inequal. Pure Appl. Math. 6 (2005), no. 3, Article 84, 13.

[27] F. Soltani, *Littlewood-Paley operators associated with the Dunkl operator on $\mathbb{R}$*, J. Funct. Anal. 221 (2005), no. 1, 205–225. MR 2124902

[28] S. Thangvelu and Yuan Xu, *Convolution operator and maximal function for the Dunkl transform* J. Anal. Math., **97** (2005), 25–55.

[29] X. Tolsa, *Littlewood–Paley Theory and the $T(1)$ Theorem with Non-doubling Measures*, Adv. in Math., **164** (2001), 57–116.

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