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Generalized stochastic Lagrangian paths for the Navier-Stokes equation

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Abstract. In the note added in proof of the seminal paper [14], Ebin and Marsden introduced the so-called *correct Laplacian* for the Navier-Stokes equation on a compact Riemannian manifold. In the spirit of Brenier's generalized flows for the Euler equation, we introduce a class of semimartingales on a compact Riemannian manifold. We prove that these semimartingales are critical points to the corresponding kinetic energy if and only if its drift term solves weakly the Navier-Stokes equation defined with Ebin-Marsden's Laplacian. We also show that for the case of torus, classical solutions of the Navier-Stokes equation realize the minimum of the kinetic energy in a suitable class.

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1. Introduction

Euler equations describe the velocity of incompressible non-viscous fluids. Considering these equations on a compact Riemannian manifold M without boundary, they are

$$\frac{d}{dt}u_t + (u_t \cdot \nabla)u_t = -\nabla p, \quad \operatorname{div}(u_t) = 0. \tag{1.1}$$

Lagrange's point of view consists in describing the position of the particles: for a solution u to (1.1), it concerns solutions of the ordinary differential equation (ODE)

$$\frac{d}{dt}g_t(x) = u_t(g_t(x)), \quad g_0(x) = x. \tag{1.2}$$

When $(t, x) \to u_t(x)$ is smooth, the ODE (1.2) defines a flow of C^∞-diffeomorphisms g_t. From the position values, we get the velocity by

$$u_t(x) = \left(\frac{d}{dt}g_t\right)\left(g_t^{-1}(x)\right).$$

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In this case, the two points of view are equivalent. Throughout the whole paper we shall consider the interval of time $[0, T]$. Equation (1.2) defines a continuous map

$$g_\cdot : [0, T] \to \mathrm{Diff}(M)$$

from $[0, T]$ to the group of diffeomorphisms of M.

In a famous work [6], V.I. Arnold gave a geometric interpretation to the incompressible Euler equation, saying that u is a solution to (1.1) if and only if $t \to g_t$ is a *geodesic* on the submanifold of $\mathrm{Diff}(M)$ keeping the volume measure invariant, equipped with the L^2 metric. Equivalently, $g_\cdot$ minimizes the action

$$S[\varphi] = \frac{1}{2} \int_0^T \int_M \left| \frac{d}{dt} \varphi_t(x) \right|^2_{T_x M} dx dt \tag{1.3}$$

on $C([0, T], \mathrm{Diff}(M))$, where dx denotes the normalized Lebesgue measure on U or the normalized Riemannian volume on M (see also [14]).

In [7], Y. Brenier gave a probabilistic interpretation to (1.1), by looking for probability measures Σ on the path space $C([0, T], M)$, which minimize the kinetic energy

$$S[\Sigma] = \frac{1}{2} \int_{C([0,T],M)} \left[\int_0^T |\dot\gamma(t)|^2_{T_{\gamma(t)}M} \, dt \right] d\Sigma(\gamma), \tag{1.4}$$

with constraints $(e_t)_* \Sigma = dx$, where $e_t : \gamma \to \gamma(t)$ denotes the evaluation map. Let

$$X(\gamma, t) = \gamma(t).$$

Then under Σ, $\{X(\cdot, t); t \geq 0\}$ is a M-valued stochastic process. Moreover, in [7] as well as in [8], Brenier proved that such a probability measure Σ gives rise to a weak solution of the Euler equation in the sense of Di Perna and Majda [13]. More precisely, define a probability measure μ on $[0, T] \times TM$ by

$$\begin{aligned} &\int_{[0,T]\times TM} f(t, x, v) \, \mu(dt, dx, dv) \\ &= \frac{1}{T} \int_0^T \int_{C([0,T],M)} f(t, \gamma(t), \gamma'(t)) d\Sigma(\gamma) \, dt. \end{aligned}$$

Then μ solves the Euler equation in generalized sense:

$$\int \left[v \cdot w(x) \alpha'(t) + v \cdot (\nabla w(x) \cdot v) \alpha(t) \right] \mu(dt, dx, dv) = 0$$

for any $\alpha \in C^\infty(]0, T[)$ and any smooth vector field w such that $\mathrm{div}(w) = 0$. We also refer to [1] in which the authors used the theory of mass transportation.

In this work, we will be concerned with Navier-Stokes equations. We will be interested in studying, in analogy with Brenier's work for the Euler equation, a possible concept of weak (Lagrangian) solution of the Navier-Stokes equation. The weak solutions that we construct will not, in particular, have necessarily the flow property, contrary to [2], where we considered another notion of weak solution. In a well-known book [28], Temam tried to develop Arnold's paradigm for Navier-Stokes equations on the torus. He solved the ODE (1.2) with coefficients u_t arising from solutions of these equations. However paths coming from ODE (1.2) are not suitable to be seen as Lagrangian paths for Navier-Stokes equations and they are not derived from a variational principle. On the other hand, in [11], P. Constantin and G. Iyer established a probabilistic representation of Navier-Stokes equations, using stochastic flows: this suggests to say that Lagrangian paths for Navier-Stokes equations shoud be semimartingales.

We will deal with Navier-Stokes equations on a compact Riemannian manifold M. There are two natural ways to define the "Laplace" operator on vector fields. The first way is to use the de Rham-Hodge Laplace operator $\Box$ on differential 1-forms, that is $\Box = dd^* + d^*d$. As usual, with respect to the Riemannian metric tensor $\langle\ ,\ \rangle$ given a vector field A, we set $A^\sharp$ for the associated differential 1-form; for a differential 1-form ω, we set $\omega^\flat$ for the corresponding vector field. Then we define

$$\Box A = \left(\Box A^\sharp\right)^\flat .$$

Weitzenböck formula states that

$$-\Box A = \Delta A - \mathrm{Ric}(A) \tag{1.5}$$

where $\Delta A = \mathrm{Trace}(\nabla^2 A)$ and $\mathrm{Ric}(A)$ is the vector field associated to the Ricci tensor $Ricci$ by $\langle \mathrm{Ric}(A), B\rangle = Ricci(A, B)$. Another natural way, following [14], is to use the deformation tensor. More precisely, let A be a vector field on M, the deformation tensor $\mathrm{Def}\, A$ is a symmetric tensor of type $(0, 2)$ such that

$$(\mathrm{Def}\, A)(X, Y) = \frac{1}{2}\left(\langle \nabla_X A, Y\rangle + \langle \nabla_Y A, X\rangle\right). \tag{1.6}$$

Then $\mathrm{Def} : TM \to S^2T^*M$ maps a vector field to a symmetric tensor of type $(0, 2)$. Let $\mathrm{Def}^* : S^2T^*M \to TM$ be the adjoint operator. According to [25], as well as to [27], we define

$$\hat{\Box} = 2\mathrm{Def}^*\mathrm{Def}. \tag{1.7}$$

Let us explain briefly the relation between these two Laplace operators. Let $x \in M$ and $\{e_1, \ldots, e_d\}$ be an orthonormal basis of the tangent space T_xM; then

$$\begin{aligned} ||\mathrm{Def}(A)_x||^2_{T_xM\otimes T_xM} &= \sum_{i,j=1}^{d} \mathrm{Def}(A)_x(e_i, e_j)^2 \\ &= \sum_{i,j=1}^{d} \frac{1}{4}\left(\langle \nabla_{e_i} A, e_j\rangle + \langle \nabla_{e_j} A, e_i\rangle\right)^2 \\ &= \frac{1}{2}\left(||\nabla A||^2_{T_xM\otimes T_xM} + \langle \nabla A, (\nabla A)^*\rangle_{T_xM\otimes T_xM}\right) \end{aligned}$$

where $*$ denotes the transposed operator. Therefore

$$\int_M \langle \hat{\Box} A,\, A\rangle\, dx = \int_M \left(||\nabla A||^2_{T_xM\otimes T_xM} + \langle \nabla A,\, (\nabla A)^*\rangle_{T_xM\otimes T_xM} \right) dx. \quad (1.8)$$

On the other hand, by Weitzenböck formula, we have

$$\int_M |\mathrm{div}(A)|^2 dx + \int_M |dA^\sharp|^2 dx = \int_M ||\nabla A||^2_{T_xM\otimes T_xM} dx + \int_M \langle \mathrm{Ric}(A),\, A\rangle\, dx. \quad (1.9)$$

Notice that

$$\int_M ||\nabla A||^2_{T_xM\otimes T_xM}\, dx - \int_M |dA^\sharp|^2 dx = \int_M \langle \nabla A,\, (\nabla A)^*\rangle_{T_xM\otimes T_xM}\, dx. \quad (1.10)$$

Combining (1.8)-(1.10), we get

$$\int_M \langle \hat{\Box} A,\, A\rangle\, dx = \int_M ||\nabla A||^2_{T_xM\otimes T_xM} + \int_M |\mathrm{div}(A)|^2\, dx - \int_M \langle \mathrm{Ric}(A),\, A\rangle\, dx.$$

By polarization, for any smooth vector field B, we have

$$\int_M \langle A, B\rangle\, dx = \int_M \langle \nabla A, \nabla B\rangle_{T_xM\otimes T_xM} + \int_M \mathrm{div}(A)\mathrm{div}(B)\, dx - \int_M \langle \mathrm{Ric}(A), B\rangle\, dx.$$

It follows that

$$\hat{\Box} A = -\Delta A - \mathrm{Ric}(A) - \nabla \mathrm{div}(A).$$

Then on vector fields of divergence zero A, the following relation holds true (see also [25,26])

$$-\hat{\Box} A = \Delta A + \mathrm{Ric}(A). \quad (1.11)$$

Comparing (1.11) to (1.5), the sign of Ric is opposite.

Notice that on the torus $\mathbb{T}^d$, the Ricci tensor vanishes, so that $\Box$ and $\hat{\Box}$ are the same. However on the sphere $\mathbb{S}^d$, we have $\mathrm{Ric}(A) = (d-1)A$ and these two operators are different.

Now we consider the following Navier-Stokes equation on M

$$\frac{d}{dt} u_t + \nabla_{u_t} u_t + \nu \hat{\Box} u_t = -\nabla p,\ \mathrm{div}(u_t) = 0, \quad (1.12)$$

where $\nu > 0$ is the viscosity coefficient. Since $\mathrm{div}(u_t) = 0$, we have $\int_M \langle \nabla_{u_t} u_t, u_t\rangle\, dx = 0$. Using the relation $\hat{\Box} = \Box - 2\mathrm{Ric}$ and equation (1.12), we get

$$\frac{1}{2}\frac{d}{dt}\int_M |u_t|^2\, dx + \nu \int_M \left(\left| du_t^\sharp \right|^2 + \left| d^* u_t^\sharp \right|^2 \right) dx - 2\nu \int_M \langle \mathrm{Ric}\, u_t, u_t\rangle\, dx = 0. \quad (1.13)$$

When Ric is negative, the above relation yields the existence of Leray's weak solution (see for instance [26]). For the general case, in [27], M. Taylor proved the existence of Leray's weak solution to (1.12), for any initial condtion $u_0 \in L^2$ (see [27, Theorem 4.6, page 498 and page 504]), that is $\int_0^T \int_M |u_t(x)|^2 \, dx \, dt < +\infty$.

The purpose of this work is to construct Lagrangian paths for equation (1.12). A difficulty on a general compact Riemannian manifold is that there does not exist a family of vector fields $\{A_n;\ n \geq 1\}$ such that the sum of squares of A_n defines Laplace operators and associated stochastic flows preserve the Riemannian volume, see for example [16]. Therefore in this context, it is suitable to consider a class of semimartingales on M such that Brownian motions perturbed by a drift belong to this class. Another advantadge of this point of view is that it is intrinsic, independent of the construction of Brownian motions on M.

Comparing to Brenier's generalized flows for Euler equations, the paths $t \to \xi_t$ are never of finite energy in the sense of (1.3). Instead, we shall consider the mean kinetic energy (see Definition 2.9 below). This functional first appeared in stochastic optimal control [19] as well as in connection with quantum mechanics [29].

Roughly speaking, the main result of this paper (see Theorem 2.11 below) says that the semimartingale ξ_t in a suitable class is a critical point to the stochastic kinetic energy (2.15) if and only if its drift term u_t solves Navier-Stokes equation (1.12) in the sense of Leray.

In the recent years the functional (2.15) has been used with success in various contexts (see for example [2–5, 10–12, 15, 20, 23]). In comparison with [2–5], we do not require, in the present work, that martingales have the flow property.

The organisation of the paper is as follows. In Section 2, we shall introduce and study the class of ν-Brownian incompressible semimartingales. We prove that such a semimartingale is a critical point of the corresponding kinetic energy [12] if and only if it solves the Navier-Stokes equation in the sense of Leray. We also prove the existence of a minimum under certain conditions. In Section 3, we shall show, in the case of a torus $\mathbb{T}^d$, that a classical solution to Navier-Stokes equation gives rise to a ν-Brownian incompressible martingale which realizes the minimum of the kinetic energy in a convenient class.

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2. Generalized stochastic paths for the Navier-Stokes equation

In this section, M will denote a connected compact Riemannian manifold without boundary. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t;\ t \geq 0\}$ satisfying the usual conditions.

A M-valued stochastic process ξ_t defined on $(\Omega, \mathcal{F}, P)$ is said to be a semimartingale on M if for any $f \in C^2(M)$, $f(\xi_t)$ is a real valued semimartingale. This notion is independent of the chosen connection on M; however, the corresponding local characteristics are dependent of the choice of connection. In this paper all the semimartingales considered will be continuous semimartingales with absolutely continuous local characteristics. They are usually called Itô semimartingales, and when they are $\mathbb{R}^d$-valued they can always be written in the form (2.1) below. We will write semimartingales for simplicity. For a semimartingale (ξ_t) starting from a point $x \in M$ and given a connection ∇, the stochastic parallel translation $//_t$ along $\xi_\cdot$ can be defined. For the reader's convenience, we give a brief introduction of $//_t$. Let (Γ_{ij}^k) be the Christofell coefficients of ∇, that is, on a local chart $(x_1, \ldots, x_d)$,

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{d} \Gamma_{ij}^k \, \partial_k, \quad \text{where } \partial_i = \frac{\partial}{\partial x_i}.$$

For simplicity, at first, we consider $M = \mathbb{R}^d$, then (ξ_t) is a semimartingale on $\mathbb{R}^d$ which can be written in the form

$$\xi_t = \xi_0 + \int_0^t \xi_s^0 \, ds + \sum_{j=1}^{m} \int_0^t H_j(s) \, dw_j(s), \tag{2.1}$$

where $s \to (w_1(s), \ldots, w_m(s))$ is a standard Brownian motion. Consider the stochastic differential equation (SDE) with respect to (ξ_t) on $\mathbb{R}^d$:

$$dZ_t^i = -\sum_{k,\ell=1}^{d} \Gamma(\xi_t)^i \, Z_t^\ell \circ d\xi_t^k, \quad Z_0 = v \in \mathbb{R}^d, \tag{2.2}$$

which admits a unique strong solution. Then $v \to Z_t$ is called the stochastic parallel transport of v along ξ_t with respect to the connection ∇.

Now we consider the SDE (2.2) on a local chart U of M with $\xi_0 = x \in U$ and $v \in T_x M$. Before the exiting time τ_x of ξ_t from U, $v \to Z_t \in T_{\xi_t} M$ is the parallel transport of v along $\{\xi_t, 0 \le t \le \tau_x\}$. By connecting the path chart by chart, the stochastic parallel transport along $(\xi_t)_{t \ge 0}$ can be defined on M. For a more detailed description, we refer to the book [9, Chapter VIII, Section 1.d, pages 390-394]. The following is taken from [9, Theorem 1.7, page 393]:

Theorem 2.1. *There exists a negligible subset $\mathcal{N}$ of Ω such that for $w \notin \mathcal{N}$, there is a time continuously dependent linear map $//_t$ from $T_{\xi_0} M$ to $T_{\xi_t} M$ such that for any $v \in T_{\xi_0} M$, the stochastic parallel transport Z_t of v defined by the* SDE (2.2) *along $\{\xi_s; 0 \le s \le t\}$ is $//_t v$. Moreover, the process $t \to //_t$ is adapted, and for all $t \ge 0$, $//_t$ is invertible, whose inverse is denoted by $//_t^{-1}$.*

In what follows, we will consider the Levi-Civita connection, still denoted by ∇. Let

$$\zeta_t = \int_0^t //_s^{-1} \circ d\xi_s.$$

Then ζ_t is a T_xM-valued semimartingale. By [9, Theorem 2.8, page 403], there exist processes $(\xi^0(s), H_1(s), \ldots, H_m(s))$ which are adapted to $\mathcal{F}_t$ such that

$$\xi^0(s), H_1(s), \ldots, H_m(s) \in T_{\xi_s}M$$

and ζ_t admits Itô form

$$\zeta_t = \int_0^t //_s^{-1}\xi^0(s)\,ds + \sum_{i=1}^m \int_0^t //_s^{-1} H_i(s)\,dw_s^i \tag{2.3}$$

where $w_t = (w_t^1, \cdots, w_t^m)$ is a standard Brownian motion on $\mathbb{R}^m$. For example, if the semimartingale ξ_t comes from a SDE on M:

$$d\xi_t = X_0(t,\xi_t)dt + \sum_{i=1}^m X_i(t,\xi_t)\circ dw_t^i, \quad \xi_0 = x,$$

then

$$\xi^0(t) = X_0(t,\xi_t) + \frac{1}{2}\sum_{i=1}^m (\nabla_{X_i}X_i)(t,\xi_t).$$

As in [3,12], we consider the operator

$$D_t\xi = //_t \lim_{\varepsilon\to 0} \mathbb{E}\left(\frac{\zeta_{t+\varepsilon}-\zeta_t}{\varepsilon}\Big|\mathcal{F}_t\right), \tag{2.4}$$

which is well-defined and equals $\xi^0(t)$. For a semimartingale ξ_t given by (2.3), Itô's formula has the following form (see [9, page 409]):

$$\begin{aligned} f(\xi_t) = f(\xi_0) &+ \int_0^t \left(\left\langle \nabla f(\xi_s), \xi^0(s)\right\rangle + \frac{1}{2}\sum_{i=1}^m \langle \nabla_{H_i(s)}(\nabla f)(\xi_s), H_i(s)\rangle\right) ds \\ &+ \sum_{i=1}^m \int_0^t \langle \nabla f(\xi_s), H_i(s)\rangle\, dw_s^i. \end{aligned} \tag{2.5}$$

Let $\{g_t(x,\omega);\ t \geq 0, x \in M, \omega \in \Omega\}$ be a family of continuous semimartingales with values in M. Let $\mathbb{P}^g$ denote the law of g in the continuous path space $C([0,T], M)$, that is, for every cylindrical functional F,

$$\begin{aligned} &\int_{C([0,T],M)} F(\gamma(t_1), \ldots, \gamma(t_n))d\mathbb{P}^g(\gamma) \\ &= \int_M \left[\int_{C([0,T],M)} F(g_{t_1}(x), \ldots, g_{t_n}(x))d\mathbb{P}_x^g\right] dx \end{aligned}$$

where $\mathbb{P}^g = \mathbb{P}_x^g \otimes dx$ and under $\mathbb{P}_x^g$, the semimartingale g_t starts from x.

We shall say that the semimartingale g_t is *incompressible* if, for each $t > 0$,

$$\mathbb{E}_{\mathbb{P}^g}[f(g_t)] = \int_M f(x)dx, \quad \text{for all } f \in C(M) \tag{2.6}$$

the expectation being taken with respect to the law $\mathbb{P}^g$ of g.

Let $\nu > 0$; we shall say that g_t is a *ν-Brownian semimartingale* if, under $\mathbb{P}^g$, there exists a time-dependent adapted random vector field u_t over g_t such that

$$M_t^f = f(g_t) - f(g_0) - \int_0^t \Big(\nu \Delta f(g_s) + \langle u_s, \nabla f(g_s)\rangle\Big)ds, \tag{2.7}$$

is a local continuous martingale with the quadratic variation given by

$$\left\langle M_t^{f_1}, M_t^{f_2}\right\rangle = 2\nu \int_0^t \langle \nabla f_1, \nabla f_2\rangle(g_s)ds.$$

For a semimartingale ξ_t given by (2.3), if $\{H_1(s), \ldots, H_m(s)\}$ is a system such that for any vector $v \in T_{\xi_s}M$, $\sum_{i=1}^m \langle v, H_i(s)\rangle^2 = 2\nu|v|^2$, then it is a ν-Brownian semimartingale.

Example 2.2. In the flat case $\mathbb{R}^d$, such a semimartingale admits the following form

$$dg_t(w) = \sqrt{2\nu}\, dW_t + u_t(w)\, dt, \tag{2.8}$$

where (W_t) is a Brownian motion on $\mathbb{R}^d$ and $\{u_t; t \geq 0\}$ is an adapted $\mathbb{R}^d$-valued process such that $\int_0^T |u_t(w)|^2\, dt < +\infty$ almost surely.

Example 2.3. For the general case of a compact Riemannian manifold M, we consider the bundle of orthonormal frames $O(M)$. Let $(V_t)_{t\in[0,T]}$ be a family of C^1 vector fields such that the dependence $t \to V_t$ is C^1. Denote by $\tilde{V}_t$ the horizontal lift of V_t to $O(M)$. Let $\mathrm{div}(V_t)$ and $\mathrm{div}(\tilde{V}_t)$ be respectively the divergence operators on M and on $O(M)$; they are linked by (see [17, page 595])

$$\mathrm{div}(\tilde{V}_t) = \mathrm{div}(V_t) \circ \pi,$$

where $\pi : O(M) \to M$ is the canonical projection. It follows that if $\mathrm{div}(V_t) = 0$, then $\mathrm{div}(\tilde{V}_t) = 0$. Consider the horizontal diffusion r_t on $O(M)$ defined by the SDE

$$dr_t = \sqrt{2\nu} \sum_{i=1}^d H_i(r_t) \circ dW_t^i + \tilde{V}_t(r_t)\, dt, \quad r_0 \in O(M), \tag{2.9}$$

where $\{H_1, \cdots, H_d\}$ are the canonical horizontal vector fields on $O(M)$. Let dr be the Liouville measure on $O(M)$; then the stochastic flow $r_0 \to r_t(r_0)$ leaves dr invariant. Set

$$\xi(t, x) = \pi(r_t(r_0)), \quad r_0 \in \pi^{-1}(x). \tag{2.10}$$

For any continuous function f on M:

$$\int_M \mathbb{E}(f(\xi(t,x))\,dx = \int_M f(x)\,dx.$$

Then ξ is an incompressible ν-Brownian diffusion, with $D_t\xi(x) = V_t(\xi(t,x))$.

Remark 2.4. Let P_t be the semigroup associated to $\frac{1}{2}\Delta_M + V_t$ with $\operatorname{div}(V_t) = 0$; then for any $f \in C^2(M)$,

$$\frac{d}{dt}\int_M P_t f(x)\,dx = \int_M \left(\frac{1}{2}\Delta_M P_t f + V_t P_t f\right) dx = 0.$$

It follows that for any continuous function $f : M \to \mathbb{R}$:

$$\int_M P_t f(x)\,dx = \int_M f(x)\,dx.$$

Therefore any SDE on M defining a Brownian motion with drift V gives rise to an incompressible ν-Brownian diffusion ξ with $D_t\xi(x) = V_t(\xi(t,x))$.

Example 2.5. Let $\mathbb{Z}^2$ be the set of two dimensional lattice points and define $\mathbb{Z}^2_0 = \mathbb{Z}^2 \setminus \{(0,0)^*\}$. For $k \in \mathbb{Z}^2_0$, we consider the vector $k^\perp = (k_2, -k_1)^*$ and the vector fields

$$A_k(\theta) = \sqrt{\frac{\nu}{\nu_0}}\frac{\cos(k\cdot\theta)}{|k|^\beta}k^\perp, \quad B_k(\theta) = \sqrt{\frac{\nu}{\nu_0}}\frac{\sin(k\cdot\theta)}{|k|^\beta}k^\perp, \quad \theta \in \mathbb{T}^2,$$

where $\beta > 1$ is some constant.

Let $\tilde{\mathbb{Z}}^2_0$ the subset of $\mathbb{Z}^2_0$ where we identify vectors k, k' such that $k + k' = 0$ and let

$$\nu_0 = \sum_{k\in\tilde{\mathbb{Z}}^2_0} \frac{1}{2|k|^{2\beta}}.$$

The family $\{A_k, B_k : k \in \mathbb{Z}^2_0\}$ constitutes an orthogonal basis of the space of divergence free vector fields on $\mathbb{T}^2$ and satisfies

$$\sum_{k\in\tilde{\mathbb{Z}}^2_0}\Big(\langle A_k, v\rangle^2 + \langle B_k, v\rangle^2\Big) = \nu\,|v|^2_{T_\theta\mathbb{T}^2}, \quad v \in T_\theta\mathbb{T}^2,$$

and

$$\sum_{k\in\tilde{\mathbb{Z}}^2_0} \nabla_{A_k}A_k = 0, \quad \sum_{k\in\tilde{\mathbb{Z}}^2_0} \nabla_{B_k}B_k = 0.$$

Consider the SDE on $\mathbb{T}^2$,

$$d\xi_t = \sum_{k\in\tilde{\mathbb{Z}}^2_0}\Big(A_k(\xi_t)\circ dW^k_t + B_k(\xi_t)\circ d\tilde{W}^k_t\Big) + u(t,\xi_t)\,dt, \quad \theta_0 = \theta \in \mathbb{T}^2, \quad (2.11)$$

where $\{W_t^k, \tilde{W}_t^k;\ k \in \mathbb{Z}_0^2\}$ are independent standard Brownian motions on $\mathbb{R}$, and $u(t,\cdot)$ is a family of divergence free vector fields in $H^1(\mathbb{T}^2)$, such that,

$$\int_0^T \int_{\mathbb{T}^2} \left(|u|^2 + |\nabla u|^2\right) dxdt < +\infty.$$

Then by [12,17], for $\beta \geq 3$, the SDE (2.11) defines a stochastic flow of measurable maps which preserves the Haar measure dx on $\mathbb{T}^2$. More precisely, for almost every w, the map

$$x \to \xi_t(x,w) \text{ solution to (2.11) with initial condition } x$$

leaves dx invariant; this property is stronger than incompressibility.

In what follows, we shall denote by $\mathcal{S}$ the set of incompressible semimartingales, by $\mathcal{S}_\nu$ the set of incompressible ν-Brownian semimartingales and by $\mathcal{D}_\nu$ the set of incompressible ν-Brownian diffusions. Clearly we have

$$\mathcal{D}_\nu \subset \mathcal{S}_\nu \subset \mathcal{S}.$$

Proposition 2.6. *Let $g \in \mathcal{S}_\nu$; then, for any $f \in C^2(M)$,*

$$\mathbb{E}_{\mathbb{P}^g}(\langle \nabla f(g_t), u_t\rangle) = 0. \tag{2.12}$$

Proof. Taking the expectation with respect to $\mathbb{P}^g$ in (2.7), we have

$$\mathbb{E}_{\mathbb{P}^g}(f(g_t)) - \mathbb{E}_{\mathbb{P}^g}(f(g_0)) = \nu \int_0^t \mathbb{E}_{\mathbb{P}^g}(\Delta f(g_s))\, ds + \int_0^t \mathbb{E}_{\mathbb{P}^g}(\langle \nabla f(g_s), u_s\rangle)\, ds.$$

It follows that

$$\nu \int_0^t \int_M \Delta f(x)\, dx\, ds + \int_0^t \mathbb{E}_{\mathbb{P}^g}(\langle \nabla f(g_s), u_s\rangle)\, ds = 0.$$

Since $\int_M \Delta f(x)\, dx = 0$, we get the result. □

Proposition 2.7. *Let g_t be a semimartingale on M satisfying*

$$dg_t(x) = \sum_{i=1}^m A_i(g_t(x)) \circ dW_t^i + u_t(w,x)\, dx,$$

where $A_1, \cdots, A_m$ are C^2 divergence free vector fields on M and $u_t(w,x) \in T_{g_t(x)}M$ is adapted such that $\int_M \mathbb{E}_x(\int_0^T |u_t(w,x)|^2 dt)\, dx < +\infty$; if g is incompressible, then for any $f \in C^2(M)$

$$\mathbb{E}_{\mathbb{P}^g}(\langle \nabla f(g_t), u_t\rangle) = 0.$$

Proof. Let $f \in C^2(M)$; then by Itô's formula (2.5),

$$f(g_t) = f(g_0) + M_t^f + \frac{1}{2}\sum_{i=1}^m \int_0^t \left(\langle \nabla_{A_i}(\nabla f), A_i\rangle + \langle \nabla f, \nabla_{A_i} A_i\rangle\right) ds + \int_0^t \langle \nabla f(g_s), u_s\rangle\, ds,$$

where M_t^f is the martingale part. Note that $\langle \nabla_{A_i}(\nabla f), A_i\rangle + \langle \nabla f, \nabla_{A_i} A_i\rangle = \mathcal{L}_{A_i}\mathcal{L}_{A_i} f$ where $\mathcal{L}_A$ denotes the Lie derivative with respect to A ; then taking the expectation under $\mathbb{E}_{\mathbb{P}}$, we get

$$\frac{1}{2}\sum_{i=1}^m \left(\int_M \mathcal{L}_{A_i}\mathcal{L}_{A_i} f\, dx\right) + \mathbb{E}_{\mathbb{P}^g}(\langle \nabla f(g_t), u_t\rangle) = 0.$$

Since for each i, $\int_M \mathcal{L}_{A_i}\mathcal{L}_{A_i} f\, dx = 0$, the result follows. □

In general it is not clear whether the incompressibility condition implies the relation (2.12). However, the following is true:

Proposition 2.8. *Let $A_1, \cdots, A_m$ be $C^{2+\alpha}$ vector fields on M and A_0 be a $C^{1+\alpha}$ vector field with some $\alpha > 0$; consider*

$$d\xi_t(x) = \sum_{i=1}^m A_i(\xi_t(x)) \circ dW_t^i + A_0(\xi_t(x))\, dt, \quad \xi_0 = x. \tag{2.13}$$

Then for almost all w, the map $x \to \xi_t(x)$ preserves the measure dx if and only if $\operatorname{div}(A_i) = 0$ *for* $i = 0, 1, \cdots, m$.

Proof. We give a sketch of proof (see [18] for more discussions). By [21], $x \to \xi_t(x)$ is a diffeomorphism of M and the push forward measure $(\xi_t^{-1})_{\#}(dx)$ of dx by the inverse map of ξ_t admits the density K_t which is given by (see [22]):

$$K_t(x) = \exp\left(-\sum_{i=1}^m \int_0^t \operatorname{div}(A_i)(\xi_s(x)) \circ dW_t^i - \int_0^t \operatorname{div}(A_0)(\xi_s(x))\, ds\right). \tag{2.14}$$

If $\operatorname{div}(A_i) = 0$ for $i = 0, 1, \cdots, m$, it is clear that $K_t = 1$ and $x \to \xi_t(x)$ preserves dx.

Conversely, if $K_t(x) = 1$ for any $x \in M$ and $t \geq 0$ then:

$$\sum_{i=1}^m \int_0^t \operatorname{div}(A_i)(\xi_s(x)) \circ dW_t^i + \int_0^t \operatorname{div}(A_0)(\xi_s(x))\, ds = 0,$$

or in Itô form:

$$\sum_{i=1}^m \int_0^t \operatorname{div}(A_i)(\xi_s(x)) dW_t^i + \int_0^t \left[\frac{1}{2}\sum_{i=1}^m \mathcal{L}_{A_i}\operatorname{div}(A_i) + \operatorname{div}(A_0)\right](\xi_s(x))\, ds = 0.$$

The first term of above equality is of finite quadratic variation, while the second one is of finite variation; so that for each $i = 1, \cdots, m$, $\mathrm{div}(A_i)(\xi_s(x)) = 0$ and also

$$\left[\frac{1}{2}\sum_{i=1}^{m}\mathcal{L}_{A_i}\mathrm{div}(A_i) + \mathrm{div}(A_0)\right](\xi_s(x)) = 0.$$

It follows that, almost everywhere,

$$\mathrm{div}(A_i)(\xi_s(x)) = 0 \quad \text{for} \quad i = 0, 1, \cdots, m;$$

so that $\mathrm{div}(A_i) = 0$ for $i = 0, 1, \cdots, m$. □

According to [12], as well as [4, 15, 20], we introduce the following action functional on semimartingales.

Definition 2.9. Let

$$S(g) = \frac{1}{2}\mathbb{E}_{\mathbb{P}^g}\left(\int_0^T |D_t g|^2 dt\right). \tag{2.15}$$

We say that g has *finite energy* if $S(g) < \infty$.

In what follows, we shall denote more precisely $D_t g(x)$ for $D_t g$ under the law $\mathbb{P}_x^g$. Then the action defined in (2.15) can be rewritten in the following form:

$$S(g) = \frac{1}{2}\int_M \mathbb{E}_{\mathbb{P}_x^g}\left(\int_0^T |D_t g(x)|^2 dt\right) dx. \tag{2.16}$$

We first recall briefly known results about the calculus of stochastic variation (see [4, 10, 12]). Let $u_t(x)$ be a smooth vector field on a compact manifold (or on $\mathbb{R}^d$) which, for every t, is of divergence zero. Consider an incompressible diffusion $g_t(x)$ with covariance a such that $a(x, x) = 2\mu\mathfrak{g}^{-1}(x)$ where $\mathfrak{g}$ is the metric tensor and time-dependent drift $u(t, \cdot)$. It defines a flow of diffeomorphisms preserving the volume measure. We have $D_t g(x) = u_t(g_t(x))$ and

$$S(g) = \frac{1}{2}\int_{\mathbb{T}^d} \mathbb{E}_{\mathbb{P}_x^g}\left(\int_0^T |u_t(g_t(x))|^2 \, dt\right) dx.$$

There are two manners to perform the perturbation.

First type of perturbation of identity. Let w be a smooth divergence free vector field and $\alpha \in C^1(]0, T[)$. Consider, for $\varepsilon > 0$, the ODE

$$\frac{d\Phi_t^\varepsilon(x)}{dt} = \varepsilon\,\alpha'(t)\, w(\Phi_t^\varepsilon(x)), \quad \Phi_0^\varepsilon(x) = x. \tag{2.17}$$

For each $t > 0$, Φ_t^ε is a perturbation of the identity map id. By Itô's formula, for each fixed $\varepsilon > 0$, $t \to \Phi_t^\varepsilon(g_t(x))$ is a semimartingale starting from x. Note that g and $\Phi^\varepsilon(g)$ are defined on the same probability space. It was proved in [4, 12] that u

is a weak solution to Navier-Stokes equation if and only if g is a critical point of S. More precisely, $\frac{d}{d\varepsilon} S(\Phi^\varepsilon(g))\Big|_{\varepsilon=0} = 0$ if and only if

$$\int_{\mathbb{T}^d}\int_0^T \left\langle u_t, \alpha'(t)w + \alpha(t)\,\nabla w \cdot u_t - \nu\,\alpha(t)\Box w \right\rangle dtdx = 0. \tag{2.18}$$

Here and in the following, $\cdot$ denotes the duality pairing.

Second type of perturbation of identity. Note that in [20], the perturbation of the identity was defined in a different way. For each fixed $t > 0$, the author of [20] considered the ODE

$$\frac{d\Psi_s^t}{ds} = \alpha(t)w(\Psi_s^t), \quad \Psi_0^t(x) = x. \tag{2.19}$$

Since $\alpha(0) = \alpha(T) = 0$, we have $\Psi_s^0 = \Psi_s^T =$ Identity. Set

$$\Psi(g)_t^\varepsilon(x) = \Psi_\varepsilon^t(g_t(x)).$$

Remarks 2.10. We see that Ψ^ε sends a semimartingale g to another one $\Psi^\varepsilon(g)$, leaving the initial and final data g_0 and g_T fixed. Therefore the variational principle holds among flows with fixed initial and final states with this perturbation. However with first type of perturbation of identity, the final states are not fixed.

Our perturbations do not preserve covariations of processes. The perturbations presented in Eyink [15] preserve covariations and can be generalized to manifolds with methods similar to calculus of variations in Wiener space. For these perturbations, we are able to prove that smooth solutions to Navier Stokes equation are critical points of the energy functional. But we are not able to prove the converse and we believe that the argument in [15] is incomplete.

By [20], $\frac{d}{d\varepsilon} S(\Psi(g)^\varepsilon)\Big|_{\varepsilon=0} = 0$ if and only if the equation (2.18) holds.

Now we deal with the general case of compact Riemannian manifolds.

Let $(u_t)_{t\in[0,T]}$ be a family of divergence free vector fields on M, belonging to the Sobolev space $\mathbb{D}_1^2$ and such that

$$\int_0^T \int_M \left(|u_t(x)|^2 + |\nabla u_t(x)|^2\right) dxdt < +\infty. \tag{2.20}$$

Consider the SDE (2.9) and (2.10) with such a $u_t = V_t$ for $t \in [0, T]$. First we notice that in [17, Proposition 4.3], the condition $q > 2$ is used only to insure the tightness of a family of probability measures; this condition can be relaxed to $q = 2$ using Meyer-Zheng tightness results (see the proof of Theorem 2.12 below). Therefore by [17, Theorem 6.4], equations (2.9) and (2.10) define a diffusion process g, which is in $\mathcal{D}_\nu$.

We have the following result,

Theorem 2.11. *Let $g \in D_v$ be given as above. Then g is a critical point of the action functional S if and only if u_t solves weakly the Navier-Stokes equation in Leray's sense, that is,*

$$\int_M \int_0^T \left\langle u_t, \alpha'(t)w + \alpha(t)\,\nabla w \cdot u_t - \nu\,\alpha(t)\hat{\Box} w \right\rangle dt dx = 0 \tag{2.21}$$

for all $\alpha \in C_c^1(]0,T[)$ and all smooth vector fields w such that $\operatorname{div}(w) = 0$.

Proof. Let Ψ_ε^t be the perturbation of identity defined in (2.19). Set $\eta_t^\varepsilon = \Psi_\varepsilon^t(g_t(x))$. Then $\{\eta_t^\varepsilon, t \geq 0\}$ is a semimartingale on M. We denote by $(\xi^0(s), H_1(s), \ldots, H_m(s))$ the local characteristics of $g_t(x)$. Notice that they depend on x. In order to state Itô's formula for manifold-valued functions, for the moment, consider a C^2 map $f : M \to M$. Let $x \in M$ and two tangent vectors $u, v \in T_xM$ be given. Let $x(t) \in M$ be a smooth curve such that $x(0) = x$, $x'(0) = u$, and $Y_t \in T_{x_t}M$ such that $Y_0 = v$. Define $Q(f)(x) : T_xM \times T_xM \to T_{f(x)}M$ by

$$Q(f)(x)(u,v) = \left.\frac{d}{dt}\right|_{t=0} \left[//_t^{-1}(df(x_t)\cdot Y_t)\right] - df(x)\cdot \nabla_u v. \tag{2.22}$$

Then by Itô's formula (see [9, page 408]), the drift term in local characteristics of η_t^ε is given by

$$\begin{aligned} D_t\Psi_\varepsilon^t(g_t(x)) = {} & \frac{\partial}{\partial t}\Psi_\varepsilon^t(g_t(x)) + d\Psi_\varepsilon^t(g_t(x))\cdot \xi_t^0 \\ & + \frac{1}{2}\sum_{i=1}^m Q(\Psi_\varepsilon^t(g_t(x)))(H_i(t), H_i(t)). \end{aligned} \tag{2.23}$$

Let $\varphi(\varepsilon, t) = D_t\Psi_\varepsilon^t(g_t(x)) \in T_{\eta_t^\varepsilon}M$; then

$$S(\Psi_\varepsilon(g)) = \frac{1}{2}\mathbb{E}_{\mathbb{P}^g}\left(\int_0^T |\varphi(\varepsilon,t)|^2\, dt\right).$$

We have $\varphi(0,t) = D_t g(x) = u_t(g_t(x))$. Let

$$\begin{aligned} \varphi_1(\varepsilon,t) &= \frac{\partial}{\partial t}\Psi_\varepsilon^t(g_t(x)), \\ \varphi_2(\varepsilon,t) &= d\Psi_\varepsilon^t(g_t(x))\cdot\xi_t^0, \\ \varphi_3(\varepsilon,t) &= \frac{1}{2}\sum_{i=1}^m Q(\Psi_\varepsilon^t(g_t(x)))(H_i(t),H_i(t)). \end{aligned}$$

Since the connection is torsion-free, we have

$$\left.\frac{D}{d\varepsilon}\varphi_1(\varepsilon,t)\right|_{\varepsilon=0} = \left.\frac{D}{d\varepsilon}\frac{d}{dt}\Psi_\varepsilon^t(g_t(x))\right|_{\varepsilon=0} = \left.\frac{D}{dt}\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\Psi_\varepsilon^t(g_t(x)) = \alpha'(t)w(g_t(x)).$$

In order to compute the derivative of φ_2, consider a smooth curve $\beta(s) \in M$ such that $\beta(0) = g_t(x)$, $\beta'(0) = D_t g(x)$. Then

$$d\Psi_\varepsilon^t(g_t(x)) \cdot \xi_t^0 = \frac{d}{ds}\bigg|_{s=0} \Psi_\varepsilon^t(\beta(s)).$$

Therefore

$$\begin{aligned}\frac{D}{d\varepsilon}\bigg|_{\varepsilon=0} \varphi_2(\varepsilon, t) &= \frac{D}{ds}\bigg|_{s=0} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi_\varepsilon^t(\beta(s)) = \frac{D}{ds}\bigg|_{s=0} [\alpha(t) w(\beta(s))] \\ &= \alpha(t)\, (\nabla w)(g_t(x)) \cdot D_t g(x).\end{aligned}$$

Let $\beta(s) \in M$ be a smooth curve such that $\beta(0) = g_t(x)$ and $\beta'(0) = H_i(t)$ and $\{Y_s; s \geq 0\}$ be a family of tangent vectors along $\{\beta(s); s \geq 0\}$ such that $Y_0 = H_i(t)$. Set

$$\gamma(\varepsilon, s) = \Psi_\varepsilon^t(\beta(s)) \quad \text{and} \quad X(\varepsilon, s) = d\Psi_\varepsilon^t(\beta(s)) \cdot Y_s.$$

If R denotes be the curvature tensor on M, the following commutation relation holds,

$$\frac{D}{d\varepsilon}\frac{D}{ds} X(\varepsilon, s) = \frac{D}{ds}\frac{D}{d\varepsilon} X(\varepsilon, s) + R\left(\frac{\partial \gamma}{\partial \varepsilon}, \frac{\partial \gamma}{\partial s}\right) X(\varepsilon, s).$$

We have $X(0, 0) = H_i(t)$, $\frac{\partial \gamma}{\partial \varepsilon}(0, 0) = \alpha(t) w(x)$, and $\frac{\partial \gamma}{\partial s}(0, 0) = H_i(t)$; therefore

$$\left[R\left(\frac{\partial \gamma}{\partial \varepsilon}, \frac{\partial \gamma}{\partial s}\right) X(\varepsilon, s)\right]\bigg|_{\varepsilon=0, s=0} = \alpha(t)\, R(w(g_t(x)), H_i(t)) H_i(t).$$

Now let $c(\tau) \in M$ be a smooth curve such that $c(0) = \beta(s)$, $c'(0) = Y_s$. We have

$$\begin{aligned}\frac{D}{d\varepsilon}\bigg|_{\varepsilon=0} X(\varepsilon, s) &= \left[\frac{D}{d\tau}\frac{d}{d\varepsilon} \Psi_\varepsilon^t(c(\tau))\right](0, 0) \\ &= \alpha(t) \frac{D}{d\tau}\bigg|_{\tau=0} w(c(\tau)) = \alpha(t)\, (\nabla_{Y_s} w)(\beta(s)),\end{aligned}$$

and

$$\frac{D}{ds}\bigg|_{s=0} (\nabla_{Y_s} w)(\beta(s)) = \langle \nabla_{H_i(t)} \nabla w, H_i(t)\rangle + \langle \nabla w, \nabla_{H_i(t)} H_i(t)\rangle.$$

Note that

$$\frac{D}{d\varepsilon}\bigg|_{\varepsilon=0} d\Psi_\varepsilon^t(g_t(x)) \cdot \nabla_{H_i(t)} H_i(t) = \alpha(t)\, \langle \nabla w, \nabla_{H_i(t)} H_i(t)\rangle.$$

Using (2.22), we finally get

$$\frac{D}{d\varepsilon}\bigg|_{\varepsilon=0} \varphi_3(\varepsilon, t) = \frac{1}{2}\alpha(t) \sum_{i=1}^m \left[\langle \nabla_{H_i(t)} \nabla w, H_i(t)\rangle + R(w, H_i(t)) H_i(t)\right].$$

Note that $\sum_{i=1}^m \langle H_i(t), v\rangle^2 = 2\nu\,|v|^2$ holds for any $v \in T_{g_t(x)}M$. This implies that

$$\sum_{i=1}^m \langle H_i(t), v_1\rangle\langle H_i(t), v_2\rangle = 2\nu\,\langle v_1, v_2\rangle, \quad \text{for } v_1, v_2 \in T_{g_t(x)}M.$$

Let $\{e_1, \cdots, e_d\}$ be an orthonormal basis of $T_{g_t(x)}M$. We put

$$H_i(t) = \sum_{k=1}^d \langle H_i(t), e_k\rangle e_k.$$

Then

$$\begin{aligned}\sum_{i=1}^m \langle \nabla_{H_i(t)}\nabla w, H_i(t)\rangle &= \sum_{k,\ell=1}^d \sum_{i=1}^m \langle \nabla_{e_k}\nabla w, e_\ell\rangle\langle H_i(t), e_k\rangle\langle H_i(t), e_\ell\rangle \\ &= \sum_{k=1}^d \langle \nabla_{e_k}\nabla w, e_k\rangle = \Delta w(g_t(x)).\end{aligned}$$

In the same way we get $\sum_{i=1}^m R(w, H_i(t))H_i(t) = (\mathrm{Ric}\, w)(g_t(x))$.

Therefore

$$\left.\frac{D}{d\varepsilon}\right|_{\varepsilon=0} \varphi_3(\varepsilon, t) = \nu\alpha(t)\,(\Delta w + \mathrm{Ric}\, w)(g_t(x)),$$

which, due to (1.11), is equal to

$$\nu\alpha(t)\,\left(-\hat{\Box} w)(g_t(x)\right).$$

In conclusion, $\frac{d}{d\varepsilon} S(\Psi_\varepsilon(g))\Big|_{\varepsilon=0} = 0$ yields

$$\begin{aligned}\mathbb{E}_{\mathbb{P}g} \int_0^T \Big[&\alpha'(t)\, w(g_t)\cdot u_t(g_t) + \alpha(t)\,(\nabla_{u_t} w)(g_t)\cdot u_t(g_t) \\ &- \nu\alpha(t)\,\hat{\Box} w(g_t)\cdot u_t(g_t)\Big]\, dt = 0.\end{aligned} \tag{2.24}$$

This is nothing but (2.21). □

Note that in [4, Theorem 3.2], a variational principle was established by using the first type of perturbations of identity, defined by (2.17); on the other hand the manifold M was supposed there to be a symmetric space in order to insure the existence of semimartingales having flow properties. A variational principe on a quite general Lie groups framework was derived in [3] (*cf.* also [10]).

In [7], generalized flows with prescribed initial and final configuration were introduced; more precisely, for any probability measure η on $M \times M$, the minimizer Σ to (1.4) satisfies the constraint:

$$\int_{C([0,T],M)} f(\gamma(0), \gamma(T))\, d\Sigma(\gamma) = \int_{M\times M} f(x, y)\, d\eta(x, y)$$

for any continuous function f on $M \times M$. It is quite difficult to construct incompressible semimartingales with given prescriptions. In order to emphasize the contrast with the situation in [7], let us see the example of a Brownian bridge $g_t^{x,y}$ on $\mathbb{R}$ over [0, 1]. It is known that for $t < 1$, $g_t^{x,y}$ solves the following SDE:

$$dg_t^{x,y} = dw_t - \frac{g_t^{x,y} - y}{1-t}\, dt, \quad g_0^{x,y} = x. \tag{2.25}$$

Then $g_t^{x,y} \to y$ as $t \to 1$, and we have

$$\mathbb{E}\left(\int_0^1 |D_t g^{x,y}|^2\, dt\right) = +\infty. \tag{2.26}$$

Let η be a probability measure on $M \times M$ having dx as two marginals; we shall say that the incompressible semimartingale $\{g_t\}$ has η as final configuration if

$$\mathbb{E}_{\mathbb{P}^g}(f(g_0, g_T)) = \int_{M\times M} f(x, y)\, d\eta(x, y), \quad f \in C(M \times M). \tag{2.27}$$

This means that the joint law of (g_0, g_T) is η. If g_t is as in Example 2.3, then

$$\mathbb{E}_{\mathbb{P}^g}(f(g_0, g_T)) = \int_{M\times M} f(x, y) p_T(x, y)\, dxdy,$$

where $p_t(x, y)$ is the heat kernel associated to (g_t). Conversely if $(\rho_t(x, y))$ is solution to the following Fokker-Planck equation

$$\frac{d}{dt}\rho_t(x, y) = \nu\, \Delta_x \rho_t(x, y) + \langle u_t(x), \nabla_x \rho_t(x, y)\rangle,$$

with $\lim_{t\to 0} \rho_t = \delta_x$, for $u \in L^2([0, T], \mathbb{D}_1^2(M))$ with $\mathrm{div}(u_t) = 0$, it has been constructed in [17] an incompressible ν-Brownian semimartingale which has $\rho_T(x, y)dxdy$ as final configuration.

In order to prove the existence of minimizers to the energy functional (2.16), we have to consider a larger class than the class D_ν of incompressible ν-Brownian diffusions, that is the class S_ν of incompressible ν-Brownian semimartingales. We have the following existence result:

Theorem 2.12. *Let η be a probability measure as above. If there exists an incompressible ν-Brownian semimartingale g on M of finite energy $S(g)$ such that η is its final configuration, then there exists one that minimizes the energy among all incompressible ν-Brownian semimartingales having η as final configuration.*

Proof. Let $J : M \to \mathbb{R}^N$ be an isometric embedding; then $dJ(x) : T_xM \to \mathbb{R}^N$ is such that for each $x \in M$ and $v \in T_xM$, $|dJ(x) \cdot v|_{\mathbb{R}^N} = |v|_{T_xM}$. Denote by $(dJ(x))^* : \mathbb{R}^N \to T_xM$ the adjoint operator of $dJ(x)$, that is,

$$\langle (dJ(x))^* a, v\rangle_{T_xM} = \langle dJ(x)v, a\rangle_{\mathbb{R}^N}, \quad a \in \mathbb{R}^N, v \in T_xM.$$

Let $\{\varepsilon_1, \ldots, \varepsilon_N\}$ be an orthonormal basis of $\mathbb{R}^N$ and set

$$A_i(x) = (dJ(x))^* \varepsilon_i, \quad i = 1, \ldots, N.$$

Then it is well-known that the vector fields $\{A_1, \ldots, A_N\}$ enjoy the following properties:

(i) For any $v \in T_xM$, $|v|^2_{T_xM} = \sum_{i=1}^N \langle A_i(x), v\rangle^2_{T_xM}$;
(ii) $\sum_{i=1}^N \nabla_{A_i} A_i = 0$.

Combining (i) and (ii) gives that $\Delta_M f = \sum_{i=1}^N \mathcal{L}^2_{A_i} f$ for any $f \in C^2(M)$. On the other hand, let $J(x) = (J_1(x), \ldots, J_N(x))$; then:

$$\langle dJ(x)v, \varepsilon_i\rangle = dJ_i(x) \cdot v = \langle \nabla J_i(x), v\rangle_{T_xM}, \quad \text{for any } v \in T_xM.$$

It follows that

$$A_i = \nabla J_i, \quad i = 1, \cdots, N. \tag{2.28}$$

Let $f \in C^2(M)$; then there exists $\bar{f} \in C^2(\mathbb{R}^N)$ such that $f(x) = \bar{f}(J(x))$. We have

$$\begin{aligned} \mathcal{L}_{A_i} f &= \sum_{j=1}^N \frac{\partial \bar{f}}{\partial x_j}(J(x)) \, \langle \nabla J_j(x), A_i(x)\rangle \\ &= \sum_{j=1}^N \frac{\partial \bar{f}}{\partial x_j}(J(x)) \, \langle A_j(x), A_i(x)\rangle. \end{aligned} \tag{2.29}$$

Therefore

$$\begin{aligned} \Delta_M f =& \sum_{i=1}^N \sum_{j,k=1}^N \frac{\partial^2 \bar{f}}{\partial x_j \partial x_k}(J(x)) \, \langle A_j, A_i\rangle\langle A_k, A_i\rangle \\ &+ \sum_{i=1}^N \sum_{j=1}^N \frac{\partial \bar{f}}{\partial x_j}(J(x)) \mathcal{L}_{A_i}\langle A_j, A_i\rangle. \end{aligned}$$

Notice that

$$\sum_{i=1}^{N} \mathcal{L}_{A_i} \langle A_j, A_i \rangle = \operatorname{div}(A_j) = \Delta_M J_j,$$

and according to property (i),

$$\sum_{i=1}^{N} \langle A_j, A_i \rangle \langle A_k, A_i \rangle = \langle A_j, A_k \rangle.$$

Finally the Laplacian Δ_M on M can be expressed by

$$\Delta_M f = \sum_{j,k=1}^{N} \frac{\partial^2 \bar{f}}{\partial x_j \partial x_k}(J(x)) \langle A_j, A_k \rangle + \sum_{j=1}^{N} \frac{\partial \bar{f}}{\partial x_j}(J(x)) \Delta_M J_j. \tag{2.30}$$

Having these preparations, we prove now the existence of a $g \in \mathcal{S}_\nu$ such that the minimum of action functinal S is attained at g in the class of those in $\mathcal{S}_\nu$ having η as final configuration. Let

$$K = \inf_{g \in S_\nu} S(g).$$

There is a minimizing sequence $g^n \in \mathcal{S}_\nu$, that is, $\lim_{n \to +\infty} S(g^n) = K$. Consider the canonical decomposition:

$$J(g_t^n) = J(g_0^n) + M_t^n + \int_0^t b^n(s)\, ds.$$

Let $M_t^n = (M_t^{n,1}, \cdots, M_t^{n,N})$; then

$$\left\langle M_t^{n,i}, M_t^{n,j} \right\rangle = 2\nu \int_0^t \langle \nabla J_i, \nabla J_j \rangle (g_s^n)\, ds. \tag{2.31}$$

By Itô's formula, we have

$$b^n(t) = dJ(g_t^n) \cdot D_t g^n + \nu\, \Delta J\left(g_t^n\right). \tag{2.32}$$

It follows that

$$\mathbb{E}\left(\int_0^T |b^n(t)|^2\, dt \right) \leq 2S(g^n) + 2T\nu\, ||\Delta J||_\infty.$$

Therefore $\int_0^T |b^n(t)|^2\, dt$ is bounded in L^2. We can use [30, Theorem 3] to conclude that the joint law $\hat{P}_n$ of

$$(J(g_\cdot^n), M_\cdot^n, B_\cdot^n, U_\cdot^n)$$

in $C([0,T],\mathbb{R}^N)\times C([0,T],\mathbb{R}^N)\times C([0,T],\mathbb{R}^N)\times C([0,T],\mathbb{R}^{N\times N})$ is a tight family, where

$$B_t^n=\int_0^t b^n(s)\,ds,\quad U_t^n=\left(\left\langle M_t^{n,i},M_t^{n,j}\right\rangle\right)_{1\le i,j\le N}.$$

Let $\hat{P}$ be a limit point; up to a subsequence, we suppose that $\hat{P}_n$ converges weakly to $\hat{P}$.

Again by [30, Theorem 3], under $\hat{P}$, the coordinate process

$$(X_t,M_t,B_t,U_t)$$

has the following properties:

(1) $M_0=B_0=0, U_0=0$;
(2) (M_t) is a local martingale such that $U_t=(\langle M_t^i,M_t^j\rangle)_{1\le i,j\le N}$;
(3) $B_t=\int_0^t b(s)\,ds$ with $\int_0^T|b(s)|^2\,ds<+\infty$ almost surely.

Since $J(M)$ is closed in $\mathbb{R}^N$ we see that $X_t\in J(M)$. Let

$$X_t=J(g_t).$$

For any $f\in C^2(M)$, remark that $f(g_t)=\bar{f}(X_t)$ therefore $f(g_t)$ is a real-valued semimartingale. In other words, $\{g_t;\ t\ge 0\}$ is a semimartingale on M. Let $f\in C(M)$; the map $f\circ J^{-1}: J(M)\to\mathbb{R}$ can be extended as a bounded continuous function on $\mathbb{R}^N$; therefore letting $n\to\infty$, we get

$$\int_M f(x)\,dx=\mathbb{E}\left(f\left(g^n(t)\right)\right)=\mathbb{E}\left(f\circ J^{-1}\left(J\left(g_t^n\right)\right)\right)\to\mathbb{E}\left(f\circ J^{-1}(X_t)\right)=\mathbb{E}(f(g_t)).$$

In the same way, for $f\in C(M\times M)$, we have

$$\int_{M\times M} f(x,y)\,d\eta(x,y)=\mathbb{E}\left(f\left(g^n(0),g^n(T)\right)\right)=\mathbb{E}\left(f\left(J^{-1}J\left(g^n(0)\right),J^{-1}J\left(g^n(T)\right)\right)\right)$$

which goes to, as $n\to+\infty$,

$$\mathbb{E}(f(g(0),g(T))).$$

So g is incompressible and has η as final configuration.

Besides, by (2.31), we have

$$\left(\left\langle M_t^i,M_t^j\right\rangle\right)_{1\le i,j\le N}=2\nu\int_0^t\langle\nabla J_i,\nabla J_j\rangle(g_s)\,ds. \tag{2.33}$$

Let $f \in C^2(M)$; denote by M_t^f the martingale part of $f(g_t)$. Then, by Itô's formula,

$$dM_t^f = \sum_{j=1}^{N} \frac{\partial \bar{f}}{\partial x_j}(X_t)\, dM_t^j.$$

Therefore for $f_1, f_2 \in C^2(M)$, according to (2.33), we have

$$\left\langle dM_t^{f_1}, dM_t^{f_2} \right\rangle = \sum_{j,k=1}^{N} \frac{\partial \bar{f}_1}{\partial x_j}(X_t) \frac{\partial \bar{f}_2}{\partial x_k}(X_t)\, 2\nu \langle A_j, A_k \rangle_{g_t}\, dt.$$

On the other hand, using relation (2.29) and property (i), we have

$$\langle \nabla f_1, \nabla f_2 \rangle = \sum_{\alpha=1}^{N} \mathcal{L}_{A_\alpha} f_1\, \mathcal{L}_{A_\alpha} f_2 = \sum_{j,k=1}^{N} \frac{\partial \bar{f}_1}{\partial x_j} \frac{\partial \bar{f}_2}{\partial x_k} \langle A_j, A_k \rangle.$$

Combining the above two equalities, we finally get

$$\left\langle dM_t^{f_1}, dM_t^{f_2} \right\rangle = 2\nu \langle \nabla f_1, \nabla f_2 \rangle_{g_t}\, dt. \tag{2.34}$$

Since $X_t = J(g_t)$, we have

$$dB_t = dJ(g_t) \cdot D_t g\, dt + \frac{1}{2} \mathrm{Hess} J(g_t)\, dg_t \otimes dg_t.$$

Relation (2.34) implies that $\frac{1}{2}\mathrm{Hess} J(g_t)\, dg_t \otimes dg_t = \nu \Delta_M J(g_t)\, dt$. Therefore we get

$$B_t = \int_0^t dJ(g_s) \cdot D_s g\, ds + \nu \int_0^t \Delta_M J(g_s)\, ds, \tag{2.35}$$

which implies (2.7). In conclusion $\{g_t;\ t \geq 0\}$ is a ν-Brownian semimartingale on M, that is, $g \in \mathcal{S}_\nu$.

We want to see that $K = S(g)$. Firstly using the relation (2.32), for any $t \in [0, T]$,

$$\int_0^t dJ\left(g_s^n\right) \cdot D_s g^n\, ds = B_t^n - \nu \int_0^t \Delta J\left(g_s^n\right) ds.$$

Let $\phi : C([0,T], \mathbb{R}^N) \to \mathbb{R}$ be a bounded continuous function, consider $\varphi : C([0,T], \mathbb{R}^N) \times C([0,T], \mathbb{R}^N) \to \mathbb{R}$ defined by

$$\varphi(B, g) = \phi\left(B_\cdot - \nu \int_0^\cdot \Delta J(g_s)\, ds \right).$$

Then φ is a bounded continuous function on $C([0,T], \mathbb{R}^N) \times C([0,T], \mathbb{R}^N)$. It follows that $\int_0^\cdot dJ(g_s^n) \cdot D_s g^n\, ds$ converges in law to $\int_0^\cdot dJ(g_s) \cdot D_s g\, ds$. Let $\varepsilon > 0$; for n big enough,

$$\mathbb{E}\left(\int_0^T |dJ(g_s^n) \cdot D_s g^n|^2\, ds \right) \leq K + \varepsilon.$$

Now by [24, Theorem 10],

$$\mathbb{E}\left(\int_0^T |dJ(g_s)\cdot D_s g|^2\,ds\right)\le K+\varepsilon,$$

that is $\mathbb{E}\left(\int_0^T |D_s g|^2\,ds\right)\le K+\varepsilon$. Letting $\varepsilon\to 0$ gives $S(g)\le K$. So $S(g)=K$. □

Remarks 2.13. Although the second type of perturbation Ψ^ε of identity defined in (2.19) preserve the semimartingales in S with the prescribed final configuration, Ψ^ε does not preserve the class of Brownian ν-semimartingales. Therefore it is not clear whether the minimizer is a critical point to the functional (2.16).

3. Classical solutions and generalized paths

The purpose of this section is to try to reconcile minimizers and critical points in a convenient way. Indeed one may ask whether classical solutions are in any way related to minimizers of action functionals; the answer is affirmative, within the class S_ν with fixed boundary conditions and modulo a time change in the equation.

In this section, M will be a torus: $M=\mathbb{T}^d$. Let $g\in\mathcal{D}_\nu$ be the solution of the following SDE on $\mathbb{T}^d$:

$$dg_t=\sqrt{2\nu}\,dw_t-u(T-t,g_t)\,dt,\quad g_0\in\mathbb{T}^d, \tag{3.1}$$

where g_0 is a random variable having dx as law, w_t is the standard Brownian motion on $\mathbb{R}^d$, and $\{u(t,x);\,t\in[0,T]\}$ is a family of C^2 vector fields on $\mathbb{T}^d$, identified to vector fields on $\mathbb{R}^d$ which are 2π-periodic with respect to each space component. Suppose that u is a strong solution to the Navier-Stokes equation

$$\frac{\partial}{\partial t}u(t,x)+\nabla u(t,x)\cdot u(t,x)-\nu\,\Delta u(t,x)=-\nabla p(T-t,x).$$

By Itô's formula,

$$\begin{aligned}du(T-t,g_t)&=-\left(\frac{\partial u}{\partial t}\right)(T-t,g_t)-\nabla u(T-t,g_t)\cdot u(T-t,g_t)\\&\quad+\nu\,\Delta u(T-t,g_t)+\sqrt{2\nu}\,\nabla u(T-t,g_t)\cdot dw_t\\&=\nabla p(t,g_t)\,dt+\sqrt{2\nu}\,\nabla u(T-t,g_t)\cdot dw_t.\end{aligned} \tag{3.2}$$

According to definition (2.4), $D_t g=-u(T-t,g_t)$ and

$$D_t D_t g=-\nabla p(t,g_t). \tag{3.3}$$

In what follows, we shall consider

$$\mathcal{G}=\left\{g^*\in\mathcal{S}_\nu;\ dg_t^*=\sqrt{2\nu}\,dw_t+D_t g^*\,dt,\ g^*(0)=g(0),\,g^*(T)=g(T)\right\}. \tag{3.4}$$

Note that semimartingales in $\mathcal{G}$ are defined on a same probability space.

Example 3.1. Let α be a real continuous function on $\mathbb{R}^d$ and set

$$\beta(w,t) = \sin\left(\frac{\pi t}{T}\right)\int_0^t \alpha(w_s)\,ds, \quad c(w,t) = \frac{d}{dt}\beta(w,t).$$

Let $a \in \mathbb{R}^d$ be fixed. Consider $v(w,t) = c(w,t)a$; then v is an adapted vector field on $\mathbb{T}^d$. Define

$$g_t^* = g_t + \int_0^t v(w,s)\,ds.$$

Then $g^* \in \mathcal{G}$.

We have the following result.

Theorem 3.2. *Let $g \in \mathcal{D}_v$ be given in* (3.1). *Assume that the process g is associated with the Navier-Stokes equation in the sense that*

$$D_t D_t g = -\nabla p(t, g_t)$$

a.s. for a regular pression p such that $\nabla^2 p(t,x) \le R\,\mathrm{Id}$, with $RT^2 \le \pi^2$. Then g minimizes the energy S in the class $\mathcal{G}$.

Proof. We define the following:

$$B(g) = \frac{1}{2}\int_0^T |D_t g|^2 dt - \int_0^T p(t, g(t))dt. \tag{3.5}$$

Notice that the function $b(x,y)$ defined in [7, page 243] has no meaning in our setting (*cf.* (2.25) and (2.26)). Let $g^* \in \mathcal{G}$; we shall prove that

$$\mathbb{E}(B(g)) \le \mathbb{E}(B(g^*)). \tag{3.6}$$

Consider the function

$$\phi(t,x) = \frac{R}{2}|x|^2 - p(t,x).$$

For each $t \ge 0$, the function $x \to \phi(t,x)$ is convex on $\mathbb{R}^d$ as $\nabla^2 p(t,x) \le R\,\mathrm{Id}$. By Itô formula

$$d\,(D_t g \cdot g_t) = d(D_t g)\cdot g_t + \sqrt{2\nu}\,D_t g \cdot dw_t + |D_t g|^2\,dt + d(D_t g)\cdot dg_t.$$

Analogously,

$$d(D_t g \cdot g_t^*) = d(D_t g)\cdot g_t^* + \sqrt{2\nu}\,D_t g \cdot dw_t + D_t g \cdot D_t g^* + d(D_t g)\cdot dg_t^*.$$

Remarking that $d(D_t g) \cdot dg_t = d(D_t g) \cdot dg_t^*$, and making the substraction of the above two equalities, we obtain

$$d\left(D_t g \cdot (g_t^* - g_t)\right) = d(D_t g) \cdot (g_t^* - g_t) + \left(D_t g \cdot D_t g^* - |D_t g|^2\right) dt.$$

It follows that

$$\begin{aligned} &D_T g \cdot \left(g_T^* - g_T\right) - D_0 g \cdot \left(g_0^* - g_0\right) \\ &= \int_0^T d(D_t g) \cdot \left(g_t^* - g_t\right) + \int_0^T \left(D_t g \cdot D_t g^* - |D_t g|^2\right) dt. \end{aligned}$$

Notice that $g_0^* = g_0$, $g_T^* = g_T$, and using (3.1), we have

$$\begin{aligned} &\int_0^T \left(-D_t g \cdot D_t g^* + |D_t g|^2\right) dt = \int_0^T d(D_t g) \cdot (g_t - g_t^*) \\ &= \int_0^T \left(g_t^* - g_t\right) \cdot \left(-\sqrt{2\nu}\, \nabla u(T-t, g_t) dw_t - \nabla p(t, g_t) dt\right). \end{aligned} \tag{3.7}$$

Using the convexity, of ϕ, we have

$$\phi\left(t, g_t^*\right) - \phi(t, g_t) \geq \left(R g_t - \nabla p(t, g_t)\right) \cdot \left(g_t^* - g_t\right). \tag{3.8}$$

From (3.7) and (3.8), we get

$$\begin{aligned} &\int_0^T \left(-D_t g \cdot D_t g^* + |D_t g|^2 + R g_t \cdot \left(g_t^* - g_t\right)\right) dt \\ &\leq -\sqrt{2\nu} \int_0^T \left(g_t^* - g_t\right) \cdot \nabla u(T-t, g_t) dw_t + \int_0^T \left(\phi\left(t, g_t^*\right) - \phi(t, g_t)\right) dt. \end{aligned} \tag{3.9}$$

We have $g_t^* - g_t = \int_0^t (D_s g^* - D_s g)\, ds$. Since $g_0^* - g_0 = g_T^* - g_T = 0$, by Poincaré inequaliy on the circle to get

$$\int_0^T \left|g_t^* - g_t\right|^2 dt \leq \left(\frac{T}{\pi}\right)^2 \int_0^T \left|D_t g^* - D_t g\right|^2 dt.$$

Since $(\frac{T}{\pi})^2 \leq \frac{1}{R}$, we have

$$\frac{R}{2} \int_0^T \left|g_t^* - g_t\right|^2 dt \leq \frac{1}{2} \int_0^T \left|D_t g^* - D_t g\right|^2 dt. \tag{3.10}$$

Remark that the inequality, for $x, y, a, b \in \mathbb{R}$

$$x^2 - xy - Rb^2 + Rab \geq \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{R}{2}b^2 + \frac{R}{2}a^2$$

holds if and only if

$$\frac{1}{2}(x-y)^2 \geq \frac{R}{2}(b-a)^2.$$

Therefore by (3.10), we have

$$\begin{aligned} &\int_0^T \left(|D_t g|^2 - D_t g \cdot D_t g^* - R|g_t|^2 + R g_t \cdot g_t^* \right) dt \\ &\geq \int_0^T \left(\frac{1}{2}|D_t g|^2 - \frac{1}{2}|D_t g^*|^2 - \frac{R}{2}|g_t|^2 + \frac{R}{2}|g_t^*|^2 \right) dt. \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11), we get

$$\begin{aligned} &\int_0^T \left(\frac{1}{2}|D_t g|^2 - \frac{1}{2}\left|D_t g^*\right|^2 - \frac{R}{2}|g_t|^2 + \frac{R}{2}\left|g_t^*\right|^2 \right) dt \\ &\leq -\sqrt{2\nu} \int_0^T \left(g_t^* - g_t\right) \cdot \nabla u(T-t, g_t) dw_t + \int_0^T \left(\phi\left(t, g_t^*\right) - \phi(t, g_t) \right) dt, \end{aligned}$$

from which we deduce

$$\begin{aligned} &\int_0^T \left(\frac{1}{2}|D_t g|^2 - \frac{R}{2}|g_t|^2 + \phi(t, g_t) \right) dt \\ &\leq -\sqrt{2\nu} \int_0^T \left(g_t^* - g_t\right) \cdot \nabla u(T-t, g_t) dw_t \\ &\quad + \int_0^T \left(\frac{1}{2}\left|D_t g^*\right|^2 - \frac{R}{2}\left|g_t^*\right|^2 + \phi\left(t, g_t^*\right) \right) dt, \end{aligned}$$

or

$$\begin{aligned} &\int_0^T \left(\frac{1}{2}|D_t g|^2 - p(t, g_t) \right) dt \\ &\leq -\sqrt{2\nu} \int_0^T \left(g_t^* - g_t\right) \cdot \nabla u(T-t, g_t) dw_t + \int_0^T \left(\frac{1}{2}|D_t g^*|^2 - p\left(t, g_t^*\right) \right) dt. \end{aligned}$$

Using definition (3.5),

$$B(g) \leq -\sqrt{2\nu} \int_0^T \left(g_t^* - g_t\right) \cdot \nabla u(T-t, g_t) dw_t + B\left(g^*\right).$$

Taking the expectation of this inequality, we obtain (3.6). Notice that

$$\int_0^T \mathbb{E}(p(t, g_t))\, dt = \int_0^T \mathbb{E}\left(p\big(t, g_t^*\big)\right)\, dt;$$

then (3.6) yields $\mathbb{E}(S(g)) \leq \mathbb{E}(S(g^*))$. □

The following result provides a perturbation in a natural way and illustrates Theorem 3.2.

Proposition 3.3. *Let $v(w, t)$ be the vector field constructed in Example* 3.1. *Consider the following perturbation of g_t given by* (3.1)*:*

$$dg_t^\varepsilon = \sqrt{2\nu} dw_t - u(T - t, g_t)\, dt + \varepsilon\, v(w, t)\, dt, \quad g_0^\varepsilon = x.$$

Then we have

$$\frac{d}{d\varepsilon} S(g^\varepsilon)\bigg|_{\varepsilon=0} = 0.$$

Proof. We see that $\{g^\varepsilon;\ \varepsilon \geq 0\} \subset \mathcal{G}$. We have

$$S(g^\varepsilon) = \frac{1}{2}\mathbb{E}\left(\int_0^T |u(T - t, g_t) - \varepsilon\, v(w, t)|^2\, dt\right).$$

Therefore

$$\frac{d}{d\varepsilon} S(g^\varepsilon)\bigg|_{\varepsilon=0} = -\mathbb{E}\left(\int_0^T \langle u(T - t, g_t), v(w, t)\rangle\, dt\right).$$

Let $V_t = \int_0^t v_s\, ds$. By construction of v, $V_T = 0$. Now by integration by parts,

$$-\int_0^T \left\langle u(T - t, g_t), \dot{V}(w, t)\right\rangle dt = \int_0^T \langle d(u(T - t, g_t)), V(w, t)\rangle\ dt$$

which, using (3.2), is equal to $\int_0^T \langle \nabla p(t, g_t), V(w, t)\rangle\, dt$. Therefore

$$\begin{aligned}\frac{d}{d\varepsilon} S(g^\varepsilon)\bigg|_{\varepsilon=0} &= \int_0^T \mathbb{E}\left(\int_{\mathbb{T}^d} \langle \nabla p(t, g_t(x)), \beta(w, t)a\rangle\, dx\right) dt \\ &= \int_0^T \mathbb{E}(\beta(w, t))\left(\int_{\mathbb{T}^d} \langle \nabla p(t, x), a\rangle\, dx\right) dt = 0.\end{aligned}$$

□

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