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Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation

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Abstract. We extend the De Giorgi-Nash-Moser theory to a class of kinetic Fokker-Planck equations and deduce new results on the Landau-Coulomb equation. More precisely, we first study the Hölder regularity and establish a Harnack inequality for solutions to a general linear equation of Fokker-Planck type whose coefficients are merely measurable and essentially bounded, *i.e.* assuming no regularity on the coefficients in order to later derive results for non-linear problems. This general equation has the formal structure of the hypoelliptic equations "of type II", sometimes also called ultraparabolic equations of Kolmogorov type, but with rough coefficients: it combines a first-order skew-symmetric operator with a second-order elliptic operator involving derivatives along only part of the coordinates and with rough coefficients. These general results are then applied to the non-negative essentially bounded weak solutions of the Landau equation with inverse-power law $\gamma \in [-d, 1]$ whose mass, energy and entropy density are bounded and mass is bounded away from 0, and we deduce the Hölder regularity of these solutions.

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1. Introduction

1.1. The Landau equation

We consider the Landau equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A[f]\nabla_v f + B[f]f), \tag{1.1}$$

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where

$$\begin{cases} A[f](v) = a_{d,\gamma} \displaystyle\int_{\mathbb{R}^d} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} f(v-w)\,\mathrm{d}w \\ B[f](v) = b_{d,\gamma} \displaystyle\int_{\mathbb{R}^d} |w|^{\gamma} w f(v-w)\,\mathrm{d}w, \end{cases}$$

with $\gamma \in [-d, 0]$ and $a_{d,\gamma} > 0$. We note that the main physical case is that of Coulomb interactions when $\gamma = -d$ and $d = 3$ (giving rise to the *Landau-Coulomb equation* in plasma physics); the other cases are *hard potentials* $\gamma \in (0, 1]$ (not covered here[1]), *Maxwellian molecules* $\gamma = 0$, and *soft potentials* $\gamma \in [-d, 0)$. It can be rewritten as follows:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A[f]\nabla_v f) + B[f] \cdot \nabla_v f + c[f]f, \tag{1.2}$$

where

$$c[f](v) = \begin{cases} c_{d,\gamma} \displaystyle\int_{\mathbb{R}^d} |w|^{\gamma} f(v-w)\,\mathrm{d}w & \text{if } \gamma > -d \\ c_{d,\gamma} f & \text{if } \gamma = -d. \end{cases}$$

We assume that the mass, energy and entropy density of the weak solution f satisfy the following control at a given space-time point (x, t):

$$\mathfrak{C}(x,t) \begin{cases} M_1 = \leq = M(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\,\mathrm{d}v \leq M_0 & \text{(local mass)} \\ E(x,t) = \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^d} f(x,v,t)|v|^2\,\mathrm{d}v \leq E_0 & \text{(local energy)} \\ H(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t) \ln f(x,v,t)\,\mathrm{d}v \leq H_0 & \text{(local entropy).} \end{cases} \tag{1.3}$$

The *weak solutions* to equation (1.1) on $U_x \times U_v \times I$, $U_x \subset \mathbb{R}^d$ open, $U_v \subset \mathbb{R}^d$ open, $I = [a, b]$ with $-\infty < a < b \leq +\infty$, are defined as functions $f \in L^\infty_t(I, L^2_{x,v}(U_x \times U_v))) \cap L^2_{x,t}(U_x \times I, H^1_v(U_v))$ such that $\partial_t f + v \cdot \nabla_x f \in L^2_{x,t}(U_x \times I, H^{-1}_v(U_v))$, f satisfies estimates (1.3) and satisfies the equation in the sense of distributions[2].

Theorem 1.1 (Hölder continuity for the Landau equation). *Assume* $\gamma \in [-d, 0]$. *Let* f *be an essentially bounded weak solution of* (1.2) *in* $B_1 \times B_1 \times (-1, 0]$. *Assume*

[1] Our method would apply as well in this case with no changes, we did not include it only because it requires the additional condition $\sup_x \int_v f(t,x,v)|v|^{2+\gamma}\,\mathrm{d}v < \infty$ on the solution, and we wanted a clean statement.

[2] Observe that the coefficients $A[f]$ and $B[f]$ are controlled under assumption (1.3), thanks to Lemmas A.1 and A.2.

that $\mathfrak{C}(x,t)$ (*equation* (1.3)) *holds true for all* $B_1 \times (-1,0]$. *Then* f *is* α*-Hölder continuous with respect to* $(x,v,t) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2},0]$ *and*

$$\|f\|_{C^\alpha(B_{1/2}\times B_{1/2}\times(-1/2,0])} \leq C\left(\|f\|_{L^2(B_1\times B_1\times(-1,0])} + \|f\|^{1+\frac{|\gamma|}{d}}_{L^\infty(B_1\times B_1\times(-1,0])}\right),$$

for some α *and* C *depending on dimension,* M_1*,* M_0*,* E_0 *and* H_0.

Remark 1.2. After this work was completed, we heard of a nice recent preprint of Cameron, Silvestre and Snelson [11] that establishes *a priori* upper bounds for solutions to the spatially inhomogeneous Landau equation in the case of moderately soft potentials ($\gamma \in [-2,0]$), with arbitrary initial data, under the assumption (1.3). When $\gamma \in [-2,0]$, it thus allows us to remove the L^∞ assumption on the weak solution in Theorem 1.1.

Remark 1.3. Under the assumptions of Theorem 1.1, it is known [23, 56] that the diffusion matrix $A[f]$ is uniformly elliptic and $B[f]$ and $c[f]$ are essentially bounded for bounded velocities (see Lemmas A.1 and A.2 in appendix). In particular, the assumption (1.7) given below, and under which Theorems 1.4 and 1.6 hold true, is satisfied. Let us add that Theorems 1.1 and 1.6 are new, while Theorem 1.4 was proved in [61,62] with a different method; our method of proof however is new and, we believe, elementary, and it includes intermediate results used in our proof of Harnack inequality in Theorem 1.6.

1.2. The studied question and its history

We are also motivated by the study of the following nonlinear kinetic Fokker-Planck equation

$$\partial_t f + v\cdot\nabla_x f = \rho[f]\,\nabla_v\cdot(\nabla_v f + vf), \quad t\geq 0,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \tag{1.4}$$

(with or without periodicity conditions with respect to the space variable) where $d\in\mathbb{N}^*$, $f = f(x,v,t)\geq 0$ and $\rho[f] := \int_{\mathbb{R}^d} f(x,v,t)\,\mathrm{d}v$. The construction of global smooth solutions for such a problem is one motivation of the present paper.

The linear kinetic Fokker-Planck equation $\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(\nabla_v f + vf)$ is sometimes called the Kolmogorov-Fokker-Planck equation, as it was studied by Kolmogorov in the seminal paper [47]. In this note, Kolmogorov explicitely calculated the fundamental solution, which exhibits regularisation in both variables x and v, even though the operator $\nabla_v\cdot(\nabla_v + v) - v\cdot\nabla_x$ shows ellipticity in the v variable only. This inspired Hörmander and his theory of hypoellipticity [42], where the regularisation is recovered by geometric commutator estimates (see also [55]).

Another question which has attracted a lot of attention in calculus of variations and partial differential equations along the 20th century is Hilbert's 19th problem about the analytic regularity of solutions to certain integral variational problems, when the quasilinear Euler-Lagrange equations satisfy ellipticity conditions. Several previous results had established analyticity conditionally to some differentiability properties of the solution, but the full answer came with the landmark works of

De Giorgi [17,18] and Nash [53], where they proved that any solution to these variational problems with square integrable derivative is analytic. More precisely, their key contribution is the following[3]: the derivative f of the solution of the variational problem solves the quasilinear parabolic problem

$$\partial_t f = \nabla_v \left(A(v,t) \nabla_v f \right), \quad t \geq 0, \ v \in \mathbb{R}^d, \tag{1.5}$$

with $f = f(v,t) \geq 0$ and $A = A(v,t)$ satisfying the ellipticity condition $0 < \lambda I \leq A \leq \Lambda I$ for two constants $\lambda, \Lambda > 0$ but is, besides that, merely measurable. Then the solution f is Hölder continuous.

The method has been extended to degenerate cases, like the p-Laplacian, first in the elliptic case by Ladyzhenskaya and Uralt'seva [49]; then degenerate parabolic cases were covered by DiBenedetto [24] (see also DiBenedetto, Gianazza and Vespri [25–27]). More recently, the method has been extended to integral operators, such as fractional diffusion, in [9,10] – see also the work of Kassmann [46] and of Kassmann and Felsinger [29]. Further application to fluid mechanics can be found in [12,36,58].

1.3. Main results

In view of the Landau equation and the nonlinear (quasilinear) equation (1.4), it is natural to ask whether a similar result as the one of De Giorgi-Nash holds for hypoelliptic equations. More precisely, we consider the following kinetic Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A \nabla_v f) + B \cdot \nabla_v f + s, \quad t \in (0,T), \ (x,v) \in \Omega, \tag{1.6}$$

where Ω is an open set of $\mathbb{R}^{2d}$, $f = f(x,v,t)$, B and s are bounded measurable coefficients depending in (x,v,t), and the $d \times d$ real matrices A, B and source term s are measurable and satisfy

$$\begin{cases} 0 < \lambda I \leq A \leq \Lambda I \\ |B| \leq \Lambda \\ s \text{ essentially bounded,} \end{cases} \tag{1.7}$$

for two constants λ, Λ. We establish the Hölder continuity and the Harnack inequality for solutions to this problem. To state the result, one needs cylinders that respect the two invariant transformations of our equation: define the scaling $(x,v,t) \mapsto (r^3 x, rv, r^2 t)$ and the transformation

$$\mathcal{T}_{z_0} : z \mapsto (x_0 + x + t v_0, v_0 + v, t_0 + t). \tag{1.8}$$

[3] We give the parabolic version due to Nash here.

Given $z_0 = (x_0, v_0, t_0) \in \mathbb{R}^{2d+1}$, the cylinder $Q_r(z_0)$ "centered" at z_0 of "radius" r is defined as

$$Q_r(z_0)=\Big\{(x, v, t): |x-x_0-(t-t_0)v_0| < r^3,\ |v-v_0| < r,\ t \in \Big(t_0-r^2, t_0\Big]\Big\}. \quad (1.9)$$

When $z_0 = 0$, we omit to specify the base point: $Q_r := Q_r(0, 0, 0)$.

The *weak solutions* to equation (1.6) on $U_x \times U_v \times I$, $U_x \subset \mathbb{R}^d$ open, $U_v \subset \mathbb{R}^d$ open, $I = [a, b]$ with $-\infty < a < b \leq +\infty$, are defined as functions $f \in L^\infty_t(I, L^2_{x,v}(U_x \times U_v))) \cap L^2_{x,t}(U_x \times I, H^1_v(U_v))$ such that $\partial_t f + v \cdot \nabla_x f \in L^2_{x,t}(U_x \times I, H^{-1}_v(U_v))$ and f satisfies the equation (1.6) in the sense of distributions.

Theorem 1.4 (Hölder continuity). *Let f be a weak solution of* (1.6) *in $Q_{\text{ext}} := Q_{r_0}(z_0)$ and $Q_{\text{int}} := Q_{r_1}(z_0)$ with $r_1 < r_0$. Then f is α-Hölder continuous with respect to (x, v, t) in Q_{int} and*

$$\|f\|_{C^\alpha(Q_{\text{int}})} \leq C \left(\|f\|_{L^2(Q_{\text{ext}})} + \|s\|_{L^\infty(Q_{\text{ext}})}\right)$$

for some universal α (i.e. $\alpha = \alpha(d, \lambda, \Lambda)$) *and $C = C(d, \lambda, \Lambda, Q_{\text{ext}}, Q_{\text{int}})$.*

Remark 1.5. The boundedness of L^2 solutions was first obtained by Pascucci and Polidoro [54]. The Hölder continuity was proved by Wang and Zhang in [61,62] by a different method. See below for further comments.

As a first step, we prove that L^2 sub-solutions are locally bounded; we refer to such a result as an $L^2 - L^\infty$ estimate. We then prove that solutions are Hölder continuous by means of lemma which is an hypoelliptic counterpart of De Giorgi's "isoperimetric lemma". We finally prove a "quantitative version" of the strong maximum principle: a Harnack inequality.

Theorem 1.6 (Harnack inequality). *If f is a non-negative weak solution of* (1.6) *in Q_1, then*

$$\sup_{Q^-} f \leq C \left(\inf_{Q^+} f + \|s\|_{L^\infty(Q_1)}\right), \quad (1.10)$$

where $Q^+ := Q_R$, $Q^- := Q_R(0, 0, -\Delta)$, $C > 1$ and $R, \Delta \in (0, 1)$ are small (*in particular $Q^\pm \subset Q_1$ and they are disjoint*)*, and universal,* i.e. *they only depend on dimension and ellipticity constants.*

Remark 1.7. Applying the transformation $\mathcal{T}_{z_0}(x, v, t) = (x_0 + x + tv_0, v_0 + v, t_0 + t)$, the Harnack inequality holds for cylinders centered at any $z_0 = (x_0, v_0, t_0)$.

1.4. Comments and previously known results

In [54], the authors obtain an $L^2 - L^\infty$ estimate but cannot reach the Hölder continuity estimate. We prove the $L^2 - L^\infty$ estimate using hypoelliptic estimates obtained by Bouchut [7]. Still, the same crucial step is at the core of both proofs: estabilishing hypoelliptic gain of Sobolev regularity for a kinetic Fokker-Planck equation with rough coefficients. Note that in [54], and more recently in [45], this is done by replacing the linear operator with rough coefficients by the corresponding one with constant coefficients and treating the difference as a source error term; while in our paper we use regularisation estimates related to velocity averaging lemma.

Hypoelliptic estimates in [7] are part of the well-developed literature about velocity averaging lemmas and transfer of regularity results. The latter is a smoothing effect for v-averages of solutions to $\partial_t f + v \cdot \nabla_x f = S$ observed for the first time in [1,35] independently, later improved and generalized in [28,34] (no smoothing on f itself can be observed, since the transport operator is hyperbolic and propagates the singularities). And when S is of the form $S = \nabla_v \cdot (A(x, v, t)\nabla_v f) + s$, where s is a given source term in L^2, the smoothing effect of velocity averaging can be combined with the H^1 regularity in the v variable implied by the energy inequality in order to obtain some amount of smoothing on the solution f itself. A first observation of this type (at the level of a compactness argument) can be found in [50]. More recently, Bouchut [7] has obtained more quantitative Sobolev regularity estimates. These estimates are one key ingredient in our proof.

We give two slightly different proofs of the $L^2 - L^\infty$ estimate, one following Moser's approach, the other following De Giorgi's ideas. We emphasize that, in both approaches, the main ingredient is the local gain of integrability of non-negative sub-solutions. This latter is obtained by combining a comparison principle and a Sobolev regularity estimate. We then prove the Hölder continuity through a De Giorgi type argument on the decrease of oscillation. We also derive the Harnack inequality by combining the decrease of oscillation with a result about how the minimum of non-negative solutions deteriorates with time, adapting a scheme Luis Silvestre showed us for elliptic equations.

In [61,62], the authors get a Hölder estimate for weak solutions of so-called ultraparabolic equations, including (1.6). Their proof relies on the construction of cut-off functions and a particular form of weak Poincaré inequality satisfied by non-negative weak sub-solutions. Our paper proposes an alternate method based on hypoelliptic estimates in the presence of rough coefficients, as explained above. It also provides several tools interesting *per se*, *e.g.*, our intermediate-value Lemma 4.1 is adapted to non-local equations of order $2s \in [1, 2)$ in [45].

The C^∞ smoothing of solutions to the Landau equation has been investigated so far in two different settings: either for weak spatially homogeneous solutions (non-negative in L^1 and with finite energy) [6, 20, 22, 60] (see also the related entropy dissipation estimates in [21,23]), or for classical spatially heterogeneous solutions [14,51]. The analytic regularisation of weak spatially homogeneous solutions was investigated in the case of Maxwellian or hard potentials in [13]. Let us also mention that in [56], Silvestre derives an L^∞ bound on the spatially homogeneous solutions for soft potentials without relying on energy methods (which implies as well the smoothing by standard parabolic techniques). Let us also mention works studying modified Landau equations [37,48] and the work [39], which shows that any weak radial solution to the Landau-Coulomb equation belonging to $L^{3/2}$ is automatically bounded and C^2 using barrier arguments. Finally, we highlight the related results of regularisation for the Boltzmann equation without long-range interactions [15, 16, 19], and the related perturbative results for the Landau and (long-range interaction) Boltzmann equation [2–5, 38, 40, 63]. To the best of our knowledge, the regularity of *a priori* non-negative locally L^∞ solutions (under our assumption (1.3)) to the spatially heterogeneous Landau equation has not been investigated so far.

A part of the results of this paper were announced in [33,43].

1.5. Plan of the paper

In Section 2, we prove the universal gain of integrability for non-negative sub-solutions. In Section 3, we derive from this gain of integrability a local upper bound of such non-negative sub-solutions. We give two proofs: one following de Giorgi's approach and the other following Moser's iteration procedure. In Section 4, the Hölder estimate is derived by proving a lemma of "reduction of oscillation". In Section 5 we prove a Harnack inequality for non-negative solutions. In Section 6, we prove a local gain of regularity of sub-solutions. In Section 7, we prove that the velocity gradient of the solution is slightly better than square integrable.

1.6. Notation

We occasionally write $A \lesssim B$ in order to say that $A \le \bar{C} B$ for some constant $\bar{C}$ which only depends on dimension and ellipticity constants λ and Λ. Such a constant $\bar{C}$ is called *universal*.

The inverse transformation $\mathcal{T}_{z_0}^{-1} : z \mapsto z_0^{-1} \circ z$ is defined by

$$\mathcal{T}_{z_0}^{-1}(z) = (x - x_0 - (t - t_0)v_0, v - v_0, t - t_0).$$

The notation $z_0 \circ z$ and z_0^{-1} refers to a Lie group structure associated with the equation.

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2. Local gain of regularity / integrability

We consider the equation (1.6) and we want to establish a local gain of integrability of solutions in order to apply De Giorgi-Moser's iteration and get a local L^∞ bound. Since we will need to perform convex changes of variables, it is necessary to obtain this gain for all (non-negative) sub-solutions. The next theorem is stated in cylinders centered at the origin.

Theorem 2.1 (Gain of integrability for non-negative sub-solutions). *Consider two cylinders* $Q_{\text{int}} := Q_{r_1}$ *and* $Q_{\text{ext}} := Q_{r_0}$ *with* $0 < r_1 < r_0$. *There exists* $p > 2$

(only depending on dimension) such that for all non-negative sub-solution f of (1.6) *in* Q_{ext}*, we have*

$$\|f\|^2_{L^p(Q_{\text{int}})} \leq \bar{C}\left(C^2_{0,1}\|f\|^2_{L^2(Q_{\text{ext}})} + C_{0,1}\int_{Q_{\text{ext}}} |s|^2 \mathbf{1}_{f>0}\right), \tag{2.1}$$

with

$$C_{0,1} = \left(\frac{1}{r_0^2 - r_1^2} + \frac{r_0}{r_0^3 - r_1^3} + \frac{1}{(r_0 - r_1)^2} + 1\right) \quad \textit{and} \quad \bar{C} = \bar{C}(d, \lambda, \Lambda).$$

Remark 2.2. The exponent p is obtained by the Sobolev embedding $H^{\frac{1}{3}}(\mathbb{R}^{2d+1}) \hookrightarrow L^p(\mathbb{R}^{2d+1})$, that is to say $p := 6(2d+1)/(6d+1)$.

This result is a consequence of the comparison principle and the following gain of regularity.

Theorem 2.3 (Gain of regularity for sign-changing solutions). *Consider* $z_0 \in \mathbb{R}^{2d+1}$ *and two cylinders* $Q_{\text{int}} := Q_{r_1}(z_0)$ *and* $Q_{\text{ext}} := Q_{r_0}(z_0)$ *with* $0 < r_1 < r_0$*. Then any (sign-changing) weak solution* f *of* (1.6) *in* Q_{ext} *satisfies*

$$\|f\|^2_{H^{\frac{1}{3}}_{x,v,t}(Q_{\text{int}})} \leq C\left(\|f\|^2_{L^2(Q_{\text{ext}})} + \|s\|^2_{L^2(Q_{\text{ext}})}\right), \tag{2.2}$$

with $C = C(d, \lambda, \Lambda, Q_{\text{ext}}, Q_{\text{int}})$.

Remark 2.4. Using Theorem 2.1 and De Giorgi-Moser's iteration, it is in fact possible to prove that this gain of regularity is also true for non-negative sub-solutions, as we will see in Section 6.

Theorems 2.1 and 2.3 are proved in Section 2.3 below.

2.1. Global estimates and gain of regularity / integrability

Let us remark that our weak solutions in $f \in L^\infty_t(I, L^2_{x,v}(U_x \times U_v))) \cap L^2_{x,t}(U_x \times I, H^1_v(U_v))$ are in $C^0_t(I, L^2_{x,v}(U_x \times U_v) \cap H^{1/2}_t(I, L^2_{x,v}(U_x \times U_v))$, following and adapting respectively the by-now standard arguments in [57] and [30] to the kinetic case. This justifies the calculations performed in our energy estimates in the sequel.

Lemma 2.5 (Global estimate). *Let g be a weak solution of*

$$(\partial_t + v \cdot \nabla_x)g = \nabla_v \cdot (A\nabla_v g) + \nabla_v \cdot H_1 + H_0 \quad \textit{in} \quad \mathbb{R}^{2d+1},$$

with H_1 *and* H_0 *in* $L^2(\mathbb{R}^{2d+1})$ *and* g, H_0 *and* H_1 *supported in* $\mathbb{R}^d \times B(0, r_0) \times \mathbb{R}$*. Then*

$$\|\nabla_v g\|^2_{L^2} + \|D_x^{\frac{1}{3}} g\|^2_{L^2} + \|D_t^{\frac{1}{3}} g\|^2_{L^2} \leq C\left(\|H_1\|^2_{L^2} + \|H_0\|^2_{L^2}\right), \tag{2.3}$$

where $C = \bar{C}(1 + r_0^2)$ *and* $\bar{C} = \bar{C}(d, \lambda, \Lambda)$. *In particular, there exists* $p > 2$ (*only depending on dimension*) *such that*

$$\|g\|_{L^p}^2 \le C\left(\|H_1\|_{L^2}^2 + \|H_0\|_{L^2}^2\right), \tag{2.4}$$

where $C = \bar{C}(1 + r_0^2)$ *and* $\bar{C} = \bar{C}(d, \lambda, \Lambda)$.

Proof. Integrating against $2g$ in $\mathbb{R}^{2d+1}$ yields

$$\begin{aligned} 2\lambda \int_{\mathbb{R}^{2d+1}} |\nabla_v g|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t &\le \int_{\mathbb{R}^{2d+1}} (-2H_1 \cdot \nabla_v g + 2gH_0)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ \le \frac{\lambda}{2}\int_{\mathbb{R}^{2d+1}} |\nabla_v g|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t &+ \frac{2}{\lambda}\int_{\mathbb{R}^{2d+1}} |H_1|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + 2\int_{\mathbb{R}^{2d+1}} |g||H_0|\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t. \end{aligned}$$

Moreover

$$2\int_{\mathbb{R}^{2d+1}} |g||H_0|\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \le \varepsilon \int_{\mathbb{R}^{2d+1}} |g|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \frac{1}{\varepsilon}\int_{\mathbb{R}^{2d+1}} |H_0|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

Since g is supported in $B(0, r_0)$ in the velocity variable, we can use the Poincaré inequality to get

$$\varepsilon \int_{\mathbb{R}^{2d+1}} |g|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \le C_P r_0^2 \varepsilon \int_{\mathbb{R}^{2d+1}} |\nabla_v g|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t,$$

and we choose ε such that $C_P r_0^2 \varepsilon = \lambda/2$. This implies

$$\|\nabla_v g\|_{L^2}^2 \le C\left(\|H_1\|_{L^2}^2 + \|H_0\|_{L^2}^2\right). \tag{2.5}$$

Applying [7, Theorem 1.3] with $p = 2, r = 0, \beta = 1, m = 1, \kappa = 1$ and $\Omega = 1$ yields

$$\begin{aligned} \left\|D_x^{\frac{1}{3}} g\right\|_{L^2}^2 + \left\|D_t^{\frac{1}{3}} g\right\|_{L^2}^2 \lesssim \|g\|_{L^2}^2 &+ \|\nabla_v g\|_{L^2} \left\|\left(1 + |v|^2\right)^{\frac{1}{2}} H_0\right\|_{L^2} \\ &+ \|\nabla_v g\|_{L^2}^{\frac{4}{3}} \left\|\left(1 + |v|^2\right)(H_1 + A\nabla_v g)\right\|_{L^2}^{\frac{2}{3}} \\ &+ \|\nabla_v g\|_{L^2} \left\|\left(1 + |v|^2\right)^{\frac{1}{2}}(H_1 + A\nabla_v g)\right\|_{L^2}. \end{aligned}$$

Using the fact that g, H_0 and H_1 are supported in $\mathbb{R}^d \times B(0, r_0) \times \mathbb{R}$, we get

$$\begin{aligned}\left\| D_x^{\frac{1}{3}} g \right\|_{L^2}^2 + \left\| D_t^{\frac{1}{3}} g \right\|_{L^2}^2 &\lesssim r_0^2 \|\nabla_v g\|_{L^2}^2 + \left(1 + r_0^2\right)^{\frac{1}{2}} \|\nabla_v g\|_{L^2} \|H_0\|_{L^2} \\ &\quad + \left(1 + r_0^2\right)^{\frac{2}{3}} \|\nabla_v g\|_{L^2}^{\frac{4}{3}} \left(\|H_1\|_{L^2}^{\frac{2}{3}} + \|\nabla_v g\|_{L^2}^{\frac{2}{3}} \right) \\ &\quad + \left(1 + r_0^2\right)^{\frac{1}{2}} \|\nabla_v g\|_{L^2} \left(\|H_1\|_{L^2} + \|\nabla_v g\|_{L^2} \right) \\ &\lesssim \left(1 + r_0^2\right) \left(\|\nabla_v g\|_{L^2}^2 + \|H_1\|_{L^2}^2 + \|\nabla_v g\|_{L^2} \|H_0\|_{L^2} \right).\end{aligned}$$

Combining this estimate with (2.5) yields (2.3). The proof is now complete. □

2.2. The local energy estimate

The gain of integrability with respect to v and t is classical; it derives from the natural energy estimate, after truncation. We follow here [52].

Lemma 2.6 (The local energy estimate). *Under the assumptions of Theorems* 2.1 *and* 2.3, *any sub-solution f satisfies*

$$\sup_t \int_{Q_{\text{int}}^t} f^2(\cdot, \cdot, t) + \int_{Q_{\text{int}}} |\nabla_v f|^2 \leq \bar{C} \left(C_{0,1} \int_{Q_{\text{ext}}} f^2 + \int_{Q_{\text{ext}}} |s|^2 \right) \qquad (2.6)$$

for $Q_{\text{int}}^t := \{(x, v) \in \mathbb{R}^{2d} : (x, v, t) \in Q_{\text{int}}\}$, $\bar{C} = \bar{C}(d, \lambda, \Lambda)$ *and*

$$C_{0,1} = \left(\frac{1}{r_0^2 - r_1^2} + \frac{r_0}{r_0^3 - r_1^3} + \frac{1}{(r_0 - r_1)^2} + 1 \right).$$

Moreover, if the sub-solution f is non-negative, then

$$\sup_t \int_{Q_{\text{int}}^t} f^2(\cdot, \cdot, t) + \int_{Q_{\text{int}}} |\nabla_v f|^2 \leq \bar{C} \left(C_{0,1} \int_{Q_{\text{ext}}} f^2 + \int_{Q_{\text{ext}}} |s|^2 \mathbf{1}_{f>0} \right). \quad (2.7)$$

Proof. Consider $\Psi \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{R})$ with $0 \leq \Psi \leq 1$ and integrate the disequation satisfied by f against $2f\Psi^2$ in $\mathcal{R} := \mathbb{R}^{2d} \times [t_1, 0]$ with $t_1 \in (-r_1^2, 0]$, obtaining

$$\begin{aligned}\int_{\mathcal{R}} \partial_t (f^2) \Psi^2 + \int_{\mathcal{R}} v \cdot \nabla_x (f^2) \Psi^2 &\leq 2 \int_{\mathcal{R}} \nabla_v \cdot (A \nabla_v f) f \Psi^2 \\ &\quad + 2 \int_{\mathcal{R}} (B \cdot \nabla_v f) f \Psi^2 + 2 \int_{\mathcal{R}} f s \Psi^2.\end{aligned}$$

Add $\int_{\mathcal{R}} f^2 \partial_t(\Psi^2)$, integrate by parts and use the upper bound on A to get

$$\begin{aligned}
&\int_{\mathcal{R}} \partial_t\big(f^2\Psi^2\big) + 2\lambda \int_{\mathcal{R}} |\nabla_v f|^2 \Psi^2 \\
&\leq \int_{\mathcal{R}} f^2(\partial_t + v\cdot\nabla_x)\big(\Psi^2\big) - 4\int_{\mathcal{R}} \Psi A \nabla_v f \cdot f \nabla_v \Psi + 2\int_{\mathcal{R}} (B\cdot\nabla_v f) f \Psi^2 + 2\int_{\mathcal{R}} f s \Psi^2 \\
&\leq \int_{\mathcal{R}} f^2(\partial_t + v\cdot\nabla_x)\big(\Psi^2\big) + 4\Lambda \int_{\mathcal{R}} (|\nabla_v f|\Psi) f (\Psi + |\nabla_v \Psi|) + 2\int_{\mathcal{R}} f s \Psi^2 \\
&\leq \int_{\mathcal{R}} f^2\Big[(\partial_t + v\cdot\nabla_x)\big(\Psi^2\big) + 8\Lambda^2\lambda^{-1}\big(|\nabla_v\Psi|^2 + \Psi^2\big)\Big] + 2\int_{\mathcal{R}} f s \Psi^2 + \lambda \int_{\mathcal{R}} |\nabla_v f|^2 \Psi^2.
\end{aligned}$$

We thus get

$$\begin{aligned}
&\int_{\mathcal{R}} \partial_t\big(f^2\Psi^2\big) + \lambda \int_{\mathcal{R}} |\nabla_v f|^2 \Psi^2 \\
&\leq \bar{C}\left(\|\partial_t \Psi\|_\infty + r_0 \|\nabla_x \Psi\|_\infty + \|\nabla_v \Psi\|_\infty^2 + 1\right) \int_{\mathcal{R}\cap \operatorname{supp}\Psi} f^2 + 2\int_{\mathcal{R}} f s \Psi^2
\end{aligned} \tag{2.8}$$

with $\bar{C} = \bar{C}(d, \lambda, \Lambda)$. Choose next Ψ^2 such that $\Psi(t=0) = 0$ and $\operatorname{supp}\Psi \subset Q_{\mathrm{ext}}$ and get for $t_1 \in \mathbb{R}$:

$$\int_{\mathbb{R}^{2d}} f^2(\cdot,\cdot,t_1)\Psi^2(t_1)\,\mathrm{d}x\,\mathrm{d}v + \lambda \int_{\mathbb{R}^{2d+1}} |\nabla_v f|^2 \Psi^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \leq C\int_{Q_{\mathrm{ext}}} f^2 + 2\int_{Q_{\mathrm{ext}}} |f||s|.$$

If Ψ additionally satisfies $\Psi \equiv 1$ in Q_{int}, we get (2.6). We remark that (2.7) is a simple consequence of (2.6). The proof is now complete. □

2.3. Local gain: proofs

Proof of Theorems 2.1 *and* 2.3. We first remark that if f is a non-negative sub-solution of (1.6), then $f = f\mathbf{1}_{f\geq 0}$ and it is also a sub-solution of the same equation when the source term s is replaced with $s\mathbf{1}_{f\geq 0}$.

For $i = 1, \frac{1}{2}$, consider $f_i = f\chi_i$ where χ_1 and $\chi_{1/2}$ are two truncation functions such that

$$\chi_1 \equiv 1 \text{ in } Q_{\mathrm{int}} \quad \text{and} \quad \chi_1 \equiv 0 \text{ outside } Q_{\mathrm{mid}},$$
$$\chi_{\frac{1}{2}} \equiv 1 \text{ in } Q_{\mathrm{mid}} \quad \text{and} \quad \chi_{\frac{1}{2}} \equiv 0 \text{ outside } Q_{\mathrm{ext}}.$$

The function f_1 now satisfies

$$(\partial_t + v\cdot\nabla_x) f_1 \leq \nabla_v\cdot(A\nabla_v f_1) + \nabla_v \cdot H_1 + H_0 \quad \text{in} \quad \mathbb{R}^{2d+1},$$

with H_1 and H_0 given by

$$\begin{cases} H_1 &= (-A\nabla_v \chi_1) f_{\frac{1}{2}} \\ H_0 &= (B\chi_1 - A\nabla_v \chi_1) \cdot \nabla_v f_{\frac{1}{2}} + \alpha_1 f_{\frac{1}{2}} + s\mathbf{1}_{\{f \geq 0\}} \chi_1, \end{cases}$$

with $\alpha_1 = (\partial_t + v \cdot \nabla_x)\chi_1$. We remark that f_1, H_0 and H_1 are supported in Q_{ext}.

We now consider the solution g of

$$(\partial_t + v \cdot \nabla_x) g = \nabla_v \cdot (A\nabla_v g) + \nabla_v \cdot H_1 + H_0 \quad \text{in} \quad \mathbb{R}^{2d+1}.$$

We remark that g is also supported in Q_{ext}, and since $h := f_1 - g$ is a sub-solution of the equation $\partial_t h + v \cdot \nabla_x h \leq \nabla_v (A\nabla_v h)$ with zero initial data at $t = -r_0^2$, the comparison principle implies that $h \leq 0$ everywhere, and therefore $0 \leq f_1 \leq g$. It can be proved for instance by observing that h_+ is also a sub-solution of the same inequation and the standard energy estimate implies that its $L^2_{x,v}$-norm is non-increasing along the time variable.

Moreover,

$$\begin{cases} \|H_1\|_{L^2}^2 \lesssim \|\nabla_v \chi_1\|_{L^\infty}^2 \|f\|_{L^2(Q_{\text{ext}})}^2 \\ \|H_0\|_{L^2}^2 \lesssim \left(1 + \|\nabla_v \chi_1\|_{L^\infty}^2\right) \|\nabla_v f\|_{L^2(Q_{\text{mid}})}^2 + \|\alpha_1\|_{L^\infty}^2 \|f\|_{L^2(Q_{\text{ext}})}^2 \\ \qquad\qquad + \left\| s\mathbf{1}_{\{f \geq 0\}} \right\|_{L^2(Q_{\text{ext}})}^2. \end{cases}$$

In view of Lemma 2.6, we know that

$$\|\nabla_v f\|_{L^2(Q_{\text{mid}})}^2 \lesssim C_{0,1} \|f\|_{L^2(Q_{\text{ext}})}^2 + \left\| s\mathbf{1}_{\{f \geq 0\}} \right\|_{L^2(Q_{\text{ext}})}^2.$$

Hence,

$$\|H_0\|_{L^2}^2 + \|H_1\|_{L^2}^2 \lesssim \left[\left(1 + \|\nabla_v \chi_1\|_{L^\infty}^2\right)(1 + C_{0,1}) + \|\alpha_1\|_{L^\infty}^2\right] \|f\|_{L^2(Q_{\text{ext}})}^2 + \left(2 + \|\nabla_v \chi_1\|_{L^\infty}^2\right) \|s\mathbf{1}_{\{f \geq 0\}}\|_{L^1(Q_{\text{ext}})}^2.$$

In view of the definition of $C_{0,1}$ in Lemma 2.6, we thus get

$$\|H_0\|_{L^2}^2 + \|H_1\|_{L^2}^2 \lesssim C_{0,1}^2 \|f\|_{L^2(Q_{\text{ext}})}^2 + (r_0 - r_1)^{-2} \left\| s\mathbf{1}_{\{f \geq 0\}} \right\|_{L^1(Q_{\text{ext}})}^2.$$

Lemma 2.5 then yields

$$\|g\|_{L^p(Q_{\text{int}})}^2 \leq \bar{C} \left(C_{0,1}^2 \|f\|_{L^2(Q_{\text{ext}})}^2 + C_{0,1} \int_{Q_{\text{ext}}} |s|^2 \mathbf{1}_{f \geq 0} \right).$$

We then obtain (2.1) by using the fact that $0 \leq f_1 \leq g$. This achieves the proof of Theorem 2.1.

As for Theorem 2.3, Lemma 2.5 can be applied directly to f_1 and the conclusion follows along the same lines, with some simplifications. □

3. Local upper bounds for non-negative sub-solutions

In this section, we prove that non-negative L^2 sub-solutions are in fact locally bounded.

Theorem 3.1 (Upper bounds for non-negative L^2 sub-solutions). *Given two cylinders $Q_{\text{ext}} := Q_{r_0}(z_0)$ and $Q_\infty := Q_{r_\infty}(z_0)$ with $0 < r_\infty < r_0$, let f be a non-negative L^2 sub-solution of* (1.6) *in Q_{ext} with $s \in L^q(Q_{\text{ext}})$ and $q > (2p)/(p-1)$ with p only depending on dimension. Then for any $\mathfrak{g} >$, there exists $\kappa = \kappa(d, \lambda, \Lambda, Q_{\text{ext}}, Q_\infty, \mathfrak{g}, q) > 0$ such that*

$$\left\{ \begin{array}{l} \|s\|_{L^q(Q_{\text{ext}})} \le \mathfrak{g} \\ \|f\|_{L^2(Q_{\text{ext}})} \le \kappa \end{array} \right\} \qquad \Rightarrow \qquad f \le \frac{1}{2} \text{ in } Q_\infty.$$

Remark 3.2. The exponent $p = 6(2d+1)/(6d+1)$ is the one given by the gain of integrability in Theorem 2.1 (see Remark 2.2).

We give two proofs of such a result. The first one sticks to the case $q = +\infty$ with no lower order terms and use Moser's approach. The second one deals with the general case and uses De Giorgi's approach.

3.1. Moser's approach

Proof of Theorem 3.1 *in the case without source term by Moser's iteration.* Using tranformations introduced in Equation (1.8), we reduce to the case $z_0 = 0$.

We first observe that, for all $q > 1$, the function f^q satisfies

$$(\partial_t + v\nabla_x) f^q \le \nabla_v \cdot (A \nabla_v f^q) \quad \text{ in } Q_{r_0}.$$

We now rewrite (2.1) with $s = 0$ from Q_{r_n} to $Q_{r_{n+1}}$ with $r_{n+1} < r_n$ as follows:

$$\left(\int_{Q_{r_{n+1}}(0)} (f^q)^p \right)^{\frac{2}{p}} \le \bar{C} C_n^2 \int_{Q_{r_n}(0)} f^{2q}, \tag{3.1}$$

where $\bar{C} = \bar{C}(d, \lambda, \Lambda)$ and

$$C_n = \left(\frac{1}{r_n^2 - r_{n+1}^2} + \frac{r_n}{r_n^3 - r_{n+1}^3} + \frac{1}{(r_n - r_{n+1})^2} \right) + \|B\|_{L^\infty} + 1. \tag{3.2}$$

Choose now $q = q_n = (p/2)^n$ for $n \in \mathbb{N}$ and write a_n for $(\int_{Q_n} f^{2q_n})^{1/(2q_n)}$. Using that for $\bar{C} = \bar{C}(d, \lambda, \Lambda, Q_{\text{ext}}) \ge 1$ large enough, we have $|Q_{\text{ext}}| \le \bar{C}$, we get from (3.1)

$$a_{n+1} \le (\bar{C})^{\frac{1}{2q_n}} (C_n)^{\frac{1}{q_n}} a_n. \tag{3.3}$$

Finally we choose

$$r_{n+1} = r_n - \frac{1}{a(n+1)^2}$$

for some $a > 0$ (only depending on $r_0 - r_\infty$) so that (3.2) yields $C_n \sim a^2 n^4$ as $n \to +\infty$. In particular, we can choose $\bar{C} = \bar{C}(d, \lambda, \Lambda, \|B\|_{L^\infty})$ large enough so that $C_n \leq \bar{C}^{\frac{1}{2}} a^2 n^4$ and we get from (3.3) that

$$a_{n+1} \leq \left(\bar{C} a^2 n^4\right)^{\frac{1}{q_n}} a_n.$$

The convergence of the following infinite product

$$\prod_{n=0}^{\infty} \left(\bar{C} a^2\right)^{\frac{1}{q_n}} (n^4)^{\frac{1}{q_n}} < +\infty$$

achieves the proof. □

3.2. De Giorgi's approach

Proof of Theorem 3.1 *by De Giorgi's approach*. We again reduce to the case $z_0 = 0$ thanks to the transformation $\mathcal{T}_{z_0}^{-1}$ defined in Equation (1.8). For $n \geq 0$ integer, consider radius r_n, time T_n, cylinder Q_n and constant C_n as follows

$$r_n = r_\infty + (r_0 - r_\infty) 2^{-n}, \quad T_n = t_0 - r_n^2, \quad C_n = \frac{1}{2}(1 - 2^{-n}),$$

and cut-off functions Ψ_n (independent of time) as follows

$$\Psi_n \equiv \begin{cases} 1 & \text{in } Q_{r_n}^0 \\ 0 & \text{outside } Q_{r_{n-1}}^0 \end{cases} \quad \text{and} \quad \begin{cases} \|\nabla_v \Psi_n\|_{L^\infty} \leq \dfrac{1}{r_{n-1} - r_n} \leq C_{0,\infty} 2^n \\ \|\nabla_x \Psi_k\|_{L^\infty} \leq \dfrac{1}{r_{n-1}^3 - r_n^3} \leq C_{0,\infty} 2^n, \end{cases}$$

where $C_{0,\infty} = C(r_0, r_\infty)$ only depends on r_0 and r_∞, and as before

$$Q_r^\tau := \{(x, v) : (x, v, \tau) \in Q_r\}.$$

The energy estimate. Remark that $f_n = (f - C_n)^+$ is a sub-solution of (1.6) in Q_{r_n} with $s_n = s\mathbf{1}_{f \geq C_n}$. Then the energy estimate (2.8) obtained in the proof of Lemma 2.6 yields for all $T_{n-1} \leq \tau \leq T_n \leq t \leq 0$,

$$\begin{aligned} &\int_{Q_{r_n}^t} f_n^2 + \lambda \int_{Q_{r_n}} |\nabla_v f_n|^2 \\ &\leq \int_{Q_{r_n}^\tau} f_n^2 + \left(r_n \|\nabla_x \Psi_n\|_\infty + \|\nabla_v \Psi_n\|_\infty^2 + 1\right) \int_{Q_{r_{n-1}}} f_n^2 + 2 \int_{Q_{r_{n-1}}} f_n |s|. \end{aligned} \tag{3.4}$$

Averaging both sides of the inequality in $\tau \in (T_{n-1}, T_n)$ and using the estimates on the gradients of the cut-off function yields

$$U_n := \sup_{t\in(T_n,0)} \int_{Q^t_{r_n}} f_n^2 \le C4^n \int_{Q_{r_{n-1}}} f_n^2 + 2\int_{Q_{r_{n-1}}} f_n|s|, \tag{3.5}$$

where $C = C(r_0, r_\infty)$. Remark that,

$$U_n \le U_{n-1} \le \cdots \le U_0 \le \kappa \le 1 \tag{3.6}$$

(we choose $\kappa \le 1$).

The non-linearization procedure. Using the (universal) exponent $p > 2$ given by Theorem 2.1, we next estimate the terms in the right-hand side of (3.5) as follows

$$\begin{cases} \int_{Q_{r_{n-1}}} f_n^2 & \le \left(\int_{Q_{r_{n-1}}} f_n^p\right)^{\frac{2}{p}} \left|\{f_n \ge 0\} \cap Q_{r_{n-1}}\right|^{1-\frac{2}{p}} \\ \int_{Q_{r_{n-1}}} f_n|s| & \le \mathfrak{g}\left(\int_{Q_{r_{n-1}}} f_n^p\right)^{\frac{1}{p}} \left|\{f_n \ge 0\} \cap Q_{r_{n-1}}\right|^{1-\frac{1}{p}-\frac{1}{q}}, \end{cases} \tag{3.7}$$

(we used that $\|s\|_{L^q(Q_{\text{ext}})} \le \gamma$) if p and q satisfy

$$1 - \frac{1}{p} - \frac{1}{q} > 0.$$

We remark that $\{f_n \ge 0\} = \{f_{n-1} \ge C_n - C_{n-1} = 2^{-k-1}\}$, which in turn implies

$$|\{f_n \ge 0\} \cap Q_{r_{n-1}}| \le 2^{2n+2} \int_{Q_{r_{n-1}}} f_{n-1}^2 \le \bar{C}4^n U_{n-1}. \tag{3.8}$$

Combining these three estimates with (3.5) yields

$$U_n \le C2^{4n}\left[\left(\int_{Q_{r_{n-1}}} f_{n-1}^p\right)^{\frac{2}{p}} U_{n-1}^{1-\frac{2}{p}} + \|s\|_{L^q(Q_{\text{ext}})}\left(\int_{Q_{r_{n-1}}} f_{n-1}^p\right)^{\frac{1}{p}} U_{n-1}^{1-\frac{1}{p}-\frac{1}{q}}\right] \tag{3.9}$$

(we also used that $f_n \le f_{n-1}$) where $C = C(d, \lambda, \Lambda, r_0, r_\infty)$.

Use of the gain of integrability. In view of Theorem 2.1, we know that

$$\left(\int_{Q_{r_{n-1}}} f_{n-1}^p\right)^{\frac{2}{p}} \le C\left(8^n \int_{Q_{r_{n-2}}} f_{n-1}^2 + 4^n \int_{Q_{r_{n-2}}} s^2 \mathbf{1}_{f_{n-1}>0}\right)$$

with $C = C(d, \lambda, \Lambda, r_0, r_\infty)$. We next estimate the terms in the right-hand side of the previous equation depending of the source term as in (3.7), but with $p = 2$: we use (3.8) to get

$$\int_{Q_{r_{n-2}}} s^2 \mathbf{1}_{f_{n-1}\geq 0} \leq \mathfrak{g}^2 \big|\{f_{n-1} > 0\} \cap Q_{r_{n-2}}\big|^{1-\frac{2}{q}} \leq \mathfrak{g}^2 2^{2n-\frac{4n}{q}} U_{n-2}^{1-\frac{2}{q}}.$$

Hence, we can use (3.6) and $U_0 \leq 1$ again in order to write

$$\left(\int_{Q_{r_{n-1}}} f_{n-1}^p\right)^{\frac{2}{p}} \leq C\left(2^{3n} U_{n-2} + 2^{4n-\frac{4n}{q}} U_{n-2}^{1-\frac{2}{q}}\right) \leq C 2^{4n} U_{n-2}^{1-\frac{2}{q}}$$

with $C = C(d, \lambda, \Lambda, r_0, r_\infty, q, \mathfrak{g})$. Then (3.9) and (3.6) imply

$$U_n \leq C 2^{4n}\left(2^{4n} U_{n-2}^{2-\frac{2}{p}-\frac{2}{q}} + U_{n-2}^{\frac{3}{2}-\frac{1}{p}-\frac{2}{q}}\right) \leq C 2^{8n} U_{n-2}^{\frac{3}{2}-\frac{1}{p}-\frac{2}{q}}.$$

Conclusion. Remark that we can assume that $C \geq 1$. We rewrite it as

$$V_n \leq \beta^n V_{n-1}^{\alpha} \tag{3.10}$$

where $V_n = U_{2n}$, $\beta = 2^8 C$ and $\alpha = \frac{3}{2} - \frac{1}{p} - \frac{2}{q}$. Remark that $\alpha > 1$ as soon as

$$\frac{1}{q} < \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right).$$

Applying (3.10) recursively, we get

$$V_n \leq \beta^{k+\alpha(k-1)+\alpha^2(k-2)+\cdots+\alpha^{k-1}} V_0^{\alpha^k}.$$

Since

$$\begin{aligned} n+\alpha(n-1)+\cdots+\alpha^{n-1} &= n\big(1+\alpha+\cdots+\alpha^{n-1}\big) - \alpha\big(1+2\alpha+\cdots+(n-1)\alpha^{n-2}\big) \\ &= n\frac{\alpha^n-1}{\alpha-1} - \alpha\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\alpha^n-1}{\alpha-1}\right) \\ &= \frac{n(\alpha^n-1)}{\alpha-1} - \alpha\left(\frac{n\alpha^{n-1}(\alpha-1)-(\alpha^n-1)}{(\alpha-1)^2}\right) \\ &= \frac{\alpha(\alpha^n-1)-n(\alpha-1)}{(\alpha-1)^2} \leq \frac{\alpha}{(\alpha-1)^2}\alpha^n, \end{aligned}$$

we have

$$V_n \leq \left(\beta^{\frac{\alpha}{(\alpha-1)^2}} V_0\right)^{\alpha^n}.$$

This implies that $U_{2n} = V_n \to 0$ as $n \to +\infty$ as soon as

$$\beta^{\frac{\alpha}{(\alpha-1)^2}} V_0 \leq (2^8 C)^{\frac{\alpha}{(\alpha-1)^2}} \kappa < 1$$

where $C = C(d, \lambda, \Lambda, r_0, r_\infty, q, \mathfrak{g})$. Hence,

$$U_\infty = \int_{Q_{r_\infty}} \left(f - \frac{1}{2}\right)_+^2 = 0,$$

which means that $f \leq 1/2$ in Q_{r_∞}. This completes the proof of Theorem 3.1. □

4. Intermediate-value lemma and Hölder continuity

4.1. A De Giorgi intermediate-value lemma

An important step in the proof of regularity in De Giorgi's method for elliptic equations is based on an inequality of isoperimetric form (see the proof of [18, Lemma II]). This inequality is a quantitative variant of the well-known fact that no H^1 function can have a jump discontinuity, and can also be understood as a quantitative minimum principle. More precisely, given an H^1 function u with values $[0, 1]$, and which takes the values 0 and 1 on sets of positive measure, De Giorgi's isoperimetric inequality provides a lower bound on the measure of the set of intermediate values $\{0 < u < 1\}$. In the present subsection, we establish an analogue of this inequality adapted to our equation and the combination of the first order transport operator and the second order elliptic operator in the velocity variable.

We prove the core lemma at "unit scale". We recall that $Q_2 = B_8 \times B_2 \times (-4, 0]$ and $Q_1 = B_1 \times B_1 \times (-1, 0]$, $Q_\omega = B_{\omega^3} \times B_\omega \times (-\omega^2, 0]$ and we denote the shifted cube $\hat{Q} := Q_\omega(0, 0, -1) = B_{\omega^3} \times B_\omega \times (-1 - \omega^2, -1]$ (see Figure 4.1).

Lemma 4.1 (A De Giorgi intermediate-value lemma). *Let $\omega = \frac{1}{4}$. For any (universal) constants $\delta_1 \in (0, 1)$, $\delta_2 \in (0, 1)$ there exist $\nu > 0$ and $\theta \in (0, 1)$ (both universal) such that for any sub-solution f of (1.6) in Q_2 with $f \leq 1$, $|s| \leq 1$, and*

$$|\{f \geq 1 - \theta\} \cap Q_\omega| \geq \delta_1 |Q_\omega|,$$
$$|\{f \leq 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}|,$$

we have

$$|\{0 < f < 1 - \theta\} \cap B_1 \times B_1 \times (-2, 0]| \geq \nu.$$

Remark 4.2. While De Giorgi's isoperimetric inequality is based on an explicit computation leading to a precise estimate with effective constants, the proof of Lemma 4.1 is obtained by an argument by contradiction, so that the values of θ and ν are not known explicitly.

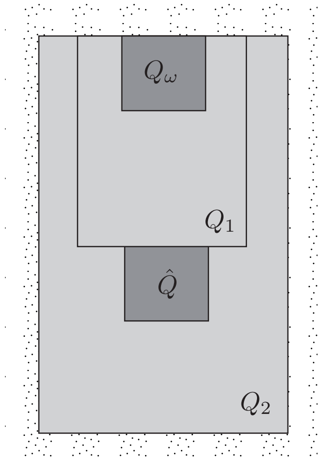

Figure 4.1. Cylinders involved in the statement of the De Giorgi intermediate-value Lemma.

Remark 4.3. The compactness argument used in the proof is reminiscent of one used by Guo in [41] and of one used by the fourth author in [59].

Proof. We argue by contradiction by assuming that there exists a sequence $(f_k)_{k\geq 0}$ of sub-solutions:

$$(\partial_t + v \cdot \nabla_x) f_k \leq \nabla_v \cdot (A_k \nabla_v f_k) + B_k \cdot \nabla_v f_k + s_k, \tag{4.1}$$

such that $f_k \leq 1, |s_k| \leq 1$,

$$\begin{cases} \theta_k \to 0 \\ \alpha_k \to 0 \end{cases} \quad \text{as} \quad k \to +\infty$$

and

$$|\{f_k \geq 1 - \theta_k\} \cap Q_\omega| \geq \delta_1 |Q_\omega|,$$
$$|\{f_k \leq 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}|,$$
$$|\{0 < f_k < 1 - \theta_k\} \cap (Q_1 \cup \hat{Q})| \to 0 \quad \text{as} \quad k \to +\infty.$$

The convexity of $z \mapsto z^+$ together with $|s_k| \leq 1$ implies that the non-negative part f_k^+ of f_k satisfies the same inequation, and therefore

$$(\partial_t + v \cdot \nabla_x) f_k^+ = \nabla_v \cdot \left(A_k \nabla_v f_k^+\right) + B_k \cdot \nabla_v f_k^+ + 1 - \mu_k \tag{4.2}$$

for some non-negative measures μ_k.

A priori estimates for f_k^+. The natural energy estimate is obtained by multiplying the equation with $f_k^+\Psi^2$ with a smooth cut-off function Ψ supported in Q_2, valued in [0, 1] and $\Psi \equiv 1$ in $\tilde{Q}_1$ with

$$\tilde{Q}_1 = B_1 \times B_1 \times (-2, 0).$$

Using the fact that $f_k^+ \le 1$ and $|s_k| \le 1$, we get

$$\begin{aligned}\lambda \int_{\mathbb{R}^{2d+1}} \left|\nabla_v f_k^+\right|^2 \Psi^2 &\le \bar{C} \int_{\mathbb{R}^{2d+1}} \Big(\Psi^2 + |\nabla_v \Psi|^2 + \Psi |(\partial_t + v \cdot \nabla_x)\Psi|\Big) \\ &\quad + \Lambda \int_{\mathbb{R}^{2d+1}} \left|\nabla_v f_k^+\right| f_k^+ \Psi^2 \\ &\le \bar{C} \int_{\mathbb{R}^{2d+1}} \Big(\Psi^2 + |\nabla_v \Psi|^2 + \Psi |(\partial_t + v \cdot \nabla_x)\Psi|\Big) \\ &\quad + \frac{\lambda}{2} \int_{\mathbb{R}^{2d+1}} \left|\nabla_v f_k^+\right|^2 \Psi^2.\end{aligned}$$

Hence

$$\lambda \int_{\mathbb{R}^{2d+1}} \left|\nabla_v f_k^+\right|^2 \Psi^2 \le \bar{C} \int_{\mathbb{R}^{2d+1}} \Big(\Psi^2 + |\nabla_v \Psi|^2 + \Psi |(\partial_t + v \cdot \nabla_x)\Psi|\Big) \tag{4.3}$$

where $\bar{C} = \bar{C}(d, \lambda, \Lambda)$.

We can also multiply the equation by Ψ^2 and get

$$\begin{aligned}-\int_{\mathbb{R}^{2d+1}} f_k^+ (\partial_t + v \cdot \nabla_x)(\Psi^2) = &-\int_{\mathbb{R}^{2d+1}} A_k \nabla_v f_k^+ \cdot \nabla_v (\Psi^2) + \int_{\mathbb{R}^{2d+1}} B_k \cdot \nabla_v f_k^+ \Psi^2 \\ &+ \int_{\mathbb{R}^{2d+1}} \Psi^2 - \int_{\mathbb{R}^{2d+1}} \Psi^2 \, \mathrm{d}\mu_k.\end{aligned}$$

Combining the latter equation with (4.3), we deduce

$$\int_{\mathbb{R}^{2d+1}} \Psi^2 \, \mathrm{d}\mu_k \le \bar{C} \int_{\mathbb{R}^{2d+1}} \Big(\Psi^2 + |\nabla_v \Psi|^2 + \Psi |(\partial_t + v \cdot \nabla_x)\Psi|\Big) \tag{4.4}$$

where $\bar{C} = \bar{C}(d, \lambda, \Lambda)$.

Passage to the limit. On the one hand, Banach-Alaoglu theorem implies that

$$f_k^+ \overset{*}{\rightharpoonup} F \text{ in } L^\infty(\tilde{Q}_1)$$

and

$$\nabla_v f_k^+ \rightharpoonup \nabla_v F \quad \text{and} \quad \begin{cases} A_k \nabla_v f_k^+ \rightharpoonup H_1 \\ B_k \cdot \nabla_v f_k^+ \rightharpoonup H_0 \end{cases} \text{in } L^2(\tilde{Q}_1), \tag{4.5}$$

for some weak limit $F \in L^\infty(\tilde{Q}_1) \cap (L^2_{x,t} H^1_v)(\tilde{Q}_1)$. In particular, (4.3) implies that

$$\int_Q |\nabla_v F|^2 \lesssim_Q 1 \tag{4.6}$$

for all $Q \Subset Q_2$, with a control depending on Q. On the other hand, the bound (4.4) implies that

$$\mu_k \rightharpoonup \mu \quad \text{in } \mathcal{M}(\tilde{Q}_1).$$

We thus have

$$(\partial_t + v \cdot \nabla_x) F = \nabla_v H_1 + H_0 + 1 - \mu. \tag{4.7}$$

By velocity averaging (see [8, Theorem 1.8]), together with the bound (4.3), we deduce the strong convergence

$$f_k^+ \to F \text{ in } L^p(\tilde{Q}_1) \quad \text{for } 1 \leq p < +\infty.$$

This implies the convergence in probability, and the function F thus satisfies

$$|\{F = 1\} \cap Q_\omega| \geq \delta_1 |Q_\omega|, \tag{4.8}$$

$$|\{F = 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}|, \tag{4.9}$$

$$|\{0 < F < 1\} \cap (B_1 \times B_1 \times (-2, 0])| = 0.$$

In view of (4.6), since indicator functions are not in H^1 unless they are constant, we have that for almost every $(x,t) \in B_1 \times (-1, 0)$,

$$\begin{cases} \text{either for almost every } v \in B_1, & F(x,v,t) = 0 \\ \text{or for almost every } v \in B_1, & F(x,v,t) = 1. \end{cases}$$

In other words, $F(x,v,t) = \mathbf{1}_P(x,t)$ for some measurable set $P \subset B_1 \times (-1,0)$. In view of (4.8) and (4.9), P satisfies

$$\begin{cases} |P \cap B_{\omega^3} \times (-\omega^2, 0)| > 0 \\ |B_{\omega^3} \times (-1-\omega^2, -1) \setminus P| > 0. \end{cases} \tag{4.10}$$

Propagation. We thus get from (4.7)

$$\partial_t F + v \cdot \nabla_x F \leq \nabla_v H_1 + H_0 + 1 \quad \text{in } \tilde{Q}_1.$$

Consider a cut-off funtion $\xi \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \zeta(z)\, \mathrm{d}z = 1, \quad \zeta(z) = \zeta(-z), \quad \operatorname{supp} \zeta \subset B_{\frac{1}{2}}.$$

Given $v_0 \in B_{\frac{1}{2}}$, since F only depends on (t,x), we can use a test-function of the form $\zeta(v-v_0)$, and get for all $v_0 \in B_{\frac{1}{2}}$,

$$\partial_t F + v_0 \cdot \nabla_x F \leq \int_{\mathbb{R}^d} \Big[|H_1(x,v,t)\nabla_v \zeta(v-v_0)| + |H_0(x,v,t)\zeta(v-v_0)| \Big] \mathrm{d}v + 1$$

in $(x,t) \in B_1 \times (-2,0)$. Since F is an indicator function and $H_0, H_1 \in L^2(\tilde{Q}_1)$, this implies for $v_0 \in B_{\frac{1}{2}}$,

$$\partial_t F + v_0 \cdot \nabla_x F \leq 0 \quad \text{in } B_1 \times (-2,0). \tag{4.11}$$

We next remark that

$$\begin{cases} \text{for all } (x,t) \in B_{\omega^3} \times (-\omega^2, 0) \text{ and } (x_0,t_0) \in B_{\omega^3} \times (-1-\omega^2, -1) \\ \text{there exists } v_0 \in B_\omega \text{ so that } (x_0,v_0,t_0) \in \hat{Q} \text{ and } (x,t) = (x_0+sv_0, t_0+s). \end{cases} \tag{4.12}$$

Indeed, the time shift s is fixed by $t = t_0 + s$ and belongs to $(1-\omega^2, 1+\omega^2)$. Then the velocity v_0 is fixed by $x = x_0 + sv_0$ and satisfies

$$|v_0| = \frac{|x-x_0|}{t-t_0} < \frac{2\omega^3}{1-\omega^2} \leq \omega,$$

since $\omega = \frac{1}{4} \leq \frac{1}{\sqrt{3}}$. We can use (4.11) and (4.12) and the second inequality in (4.10)) and conclude that $F \equiv 0$ in Q_ω, and contradicts the first inequality in (4.10). The proof is complete. ☐

4.2. Improvement of oscillation

It is a classical fact that Hölder continuity is a consequence of the decrease of the oscillation of the solution "at unit scale".

Lemma 4.4 (Improvement of oscillation). *There exist $\lambda_0 \in (0,1)$, $\omega \in (0,1/2)$ and $\beta > 0$ (all universal) such that any f solution of* (1.6) *in Q_2 with* $\operatorname{osc}_{Q_2} f \leq 2$ *and $|s| \leq \beta$ satisfies*

$$\operatorname{osc}_{Q_{\frac{\omega}{2}}} f \leq 2 - \lambda_0.$$

This lemma is a consequence of the following one.

Lemma 4.5 (A measure-to-pointwise estimate). *Given $\delta_2 > 0$, there exist $\lambda_0 \in (0,1)$, $\omega \in (0,1/2)$ and $\beta > 0$ (depending on δ_2 but not on the sub-solution) such that any f sub-solution of* (1.6) *in Q_2 with $f \leq 1$ and $|s| \leq \beta$ such that $|\{f \leq 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}|$ satisfies*

$$f \leq 1 - \lambda_0 \quad \textit{a.e. in} \quad Q_{\frac{\omega}{2}}. \tag{4.13}$$

Proof of Lemma 4.4. Let f be a solution of (1.6) in Q_2 with $\operatorname{osc}_{Q_2} f \leq 2$ and $|s| \leq \beta$. We can reduce to the case where $|f| \leq 1$. Indeed, we remark that there exists a constant C such that $\tilde{f} = f - C$ satisfies (1.6) in $Q_2(0)$ with $|\tilde{f}| \leq 1$ and the same source term.

If $|\{f \leq 0\} \cap \hat{Q}| \geq |\hat{Q}|/2$, then apply Lemma 4.5 with $\delta_2 = 1/2$.

In the other case, considering $-f$ implies that the essential infimum of f is raised. In both cases, we get the desired improvement of the oscillation of f. This completes the proof of the lemma. □

We now turn to the proof of Lemma 4.5.

Proof of Lemma 4.5. The proof proceeds in several steps.

Choice of parameters. Theorem 3.1 provides us with κ correponding to the upper bound $\mathfrak{g} = 1$ on the source term and $Q_{\mathrm{ext}} = Q_\omega$ and $Q_\infty = Q_{\frac{\omega}{2}}$. Lemma 4.1 applied with δ_2 and $\delta_1 = \sqrt{\kappa}/|Q_\omega|$ provides us with ν and θ universal. We choose next k_0 the smallest positive integer such that

$$k_0 \nu > |B_1 \times B_1 \times (-2, 0)|.$$

We finally choose β such that $\beta \leq \theta^{k_0}$.

Iteration. We define $f_0 = f$ and

$$f_{k+1} = \frac{1}{\theta}(f_k - (1-\theta)) = \theta^{-k}\big(f - (1-\theta^k)\big).$$

They satisfy $f_k \leq 1$ and

$$(\partial_t + v \cdot \nabla_x) f_k \leq \nabla_v \cdot (A \nabla_v f_k) + B \cdot \nabla_v f_k + s_k$$

with $s_k = \theta^{-k} s$. In particular $|s_k| \leq \theta^{-k_0}\beta \leq 1$, which allows us to apply Theorem 3.1 with the upper bound $\mathfrak{g} = 1$ as above. Remark that

$$|\{f_0 \leq 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}| \quad \text{and} \quad \{f_{k+1} \leq 0\} \supset \{f_k \leq 0\}. \tag{4.14}$$

Our goal is to prove that there exists at least one index $k \in \{1, \ldots, k_0\}$ such that

$$|\{f_k \geq 0\} \cap Q_\omega| \leq \delta_1 |Q_\omega|.$$

Indeed, observing that for such an index k_1

$$\|(f_{k_1})_+\|_{L^2(Q_\omega)} \leq \left[\big|\{f_{k_1} \geq 0\} \cap Q_\omega\big|\right]^{\frac{1}{2}} \leq \sqrt{\delta_1 |Q_\omega|} \leq \kappa,$$

Theorem 3.1 then implies that

$$f \leq 1 - \frac{1}{2}\theta^{k_1} \leq 1 - \frac{1}{2}\theta^{k_0} \quad \text{in} \quad Q_{\frac{\omega}{2}},$$

which concludes the proof.

Let us prove the claim by contradiction. Assume that for all $k = 1, \dots, k_0$,

$$|\{f_k \ge 0\} \cap Q_\omega| \ge \delta_1 |Q_\omega|.$$

Since $f_{k+1} = \frac{1}{\theta}(f_k - (1-\theta))$, this also implies for $k = 0, \dots, k_0 - 1$,

$$|\{f_k \ge 1-\theta\} \cap Q_\omega| \ge \delta_1 |Q_\omega|.$$

But (4.14) also implies that for all $k \ge 0$,

$$|\{f_k \le 0\} \cap \hat{Q}| \ge \delta_2 |\hat{Q}|.$$

Hence Lemma 4.1 implies that for $k = 0, \dots, k_0 - 1$,

$$|\{0 \le f_k \le 1-\theta\} \cap (B_1 \times B_1 \times (-2,0))| \ge \nu.$$

Now observe that

$$\begin{aligned} |\{f_{k+1} \le 0\} \cap (B_1 \times B_1 \times (-2,0))| &= |\{f_k \le 0\} \cap (B_1 \times B_1 \times (-2,0))| \\ &\quad + |\{0 \le f_k \le 1-\theta\} \cap (B_1 \times B_1 \times (-2,0))| \\ &\ge |\{f_k \le 0\} \cap (B_1 \times B_1 \times (-2,0))| + \nu. \end{aligned}$$

In particular

$$|B_1 \times B_1 \times (-2,0)| \ge \left|\{f_{k_0} \le 0\} \cap (B_1 \times B_1 \times (-2,0))\right| \ge k_0 \nu,$$

which is impossible for k_0 as chosen above. The proof is now complete. □

4.3. Proof of the Hölder estimate

Proof of Theorem 1.4. Consider an L^2 solution f of Equation (1.6) in a cylinder $Q_{\text{ext}} = Q_{r_0}(z_0)$. By Theorem 3.1, we know that f is locally bounded in Q_{ext}. In particular, f is bounded in $Q_{\text{mid}} = Q_{\frac{r_0+r_1}{2}}(z_0)$ and

$$\|f\|_{L^\infty(Q_{\text{mid}})} \le C_0 \left(\|f\|_{L^2(Q_{\text{ext}})} + \|s\|_{L^\infty(Q_{\text{ext}})} \right)$$

for some constant $C_0 = C(d, \lambda, \Lambda, Q_{\text{ext}}, Q_{\text{mid}})$. If $f \equiv 0$ in Q_{ext}, there is nothing to prove. If f is not identically 0, recalling that β is given by Lemma 4.4, we assume that

$$\|f\|_{L^\infty(Q_{\text{mid}})} \le 1 \quad \text{and} \quad \|s\|_{L^\infty(Q_{\text{ext}})} \le \beta,$$

by considering, if necessary,

$$\tilde{f} = \frac{f}{C_0 \left(\|f\|_{L^2(Q_{\text{ext}})} + \|s\|_{L^\infty(Q_{\text{ext}})} \right) + \beta^{-1} \|s\|_{L^\infty(Q_{\text{ext}})}}.$$

Let $z_1 \in Q_{\rm int} := Q_{r_1}(z_0)$. We want to prove that for all $r > 0$ such that $Q_{2r}(z_1) \subset Q_{\rm mid}$,

$$\operatorname{osc}_{Q_r(z_1)} f \le C r^\alpha \tag{4.15}$$

for some universal $\alpha \in (0,1)$ and some constant $C = C(d, \lambda, \Lambda, r_0, r_1)$. Let $\tilde r > 0$ denote the largest $r \in (0,1)$ such that $Q_{2r}(z_0) \subset Q_{\rm mid}$. We remark that for $r \in (0, \tilde r)$, $Q_{2r}(z_1) = \mathcal{T}_{z_1}^{-1}(Q_{2r})$ where $\mathcal{T}_{z_1}$ is defined in Equation (1.8) and $\bar f = f \circ \mathcal{T}_{z_1}$ satisfies (1.6) in $Q_{2\tilde r}$ with the source term $\bar s := s \circ \mathcal{T}_{z_1}$ and the coefficients $\bar A := A \circ \mathcal{T}_{z_1}$ and $\bar B := B \circ \mathcal{T}_{z_1}$. In particular $\bar f$ and $\bar s$ satisfy

$$\|\bar f\|_{L^\infty(Q_{2\tilde r})} \le 1 \quad \text{and} \quad \|\bar s\|_{L^\infty(Q_{2\tilde r})} \le \beta,$$

and (4.15) is equivalent to: for all $r \in (0, \tilde r)$,

$$\operatorname{osc}_{Q_r} \bar f \le C r^\alpha. \tag{4.16}$$

We recall how to scale solutions. For all $r \in (0, \tilde r)$, the function

$$\bar f_r(x, v, t) = \bar f(r^3 x, r v, r^2 t)$$

is defined in Q_2 and satisfies (1.6) with

$$\begin{cases} \bar B_r(x,v,t) = r \bar B(r^3 x, r v, r^2 t) \\ \bar s_r(x,v,t) = r^2 \bar s(r^3 x, r v, r^2 t). \end{cases}$$

Since $\operatorname{osc}_{Q_{2\tilde r}} \bar f \le 2$, we have $\operatorname{osc}_{Q_2} \bar f_{\tilde r} \le 2$ and Lemma 4.4 implies that

$$\operatorname{osc}_{Q_{\frac{\omega}{2}}} \bar f_{\tilde r} = \operatorname{osc}_{Q_{\frac{\omega}{2}\tilde r}} \bar f \le 2\theta$$

with $\theta = 1 - \lambda_0/2$ (we used the fact that $\tilde r \le 1$ to ensure that $\|\bar s_{\tilde r}\|_{L^\infty(Q_2)} \le \beta$). We remark that we can assume that $\theta \ge 1/2$ and we recall that $\omega \in (0, 1/2)$. We next apply Lemma 4.4 to $\theta^{-1} \bar f_{\tilde r_1}$ with $\tilde r_1 = (\omega/4)\tilde r$, which rescales the L^∞ bound on the source term by a factor $(\omega/4)^2 \theta^{-1} < 1$ as compared to $\|\bar s_{\tilde r}\|_{L^\infty(Q_2)} \le \beta$. Hence the assumed bounds are still valid and we get

$$\operatorname{osc}_{Q_{\tilde r_2}} \bar f \le 2\theta^2$$

with $\tilde r_2 = (\omega/2)\tilde r_1$. Inductively, we deduce that

$$\operatorname{osc}_{Q_{\tilde r_k}} \bar f \le 2\theta^k$$

with $\tilde r_k = (\omega/2)^k \tilde r/2$. This yields (4.16) for $r = \tilde r_k$ with

$$\alpha = \frac{\ln \theta}{\ln(\omega/2)} \quad \text{and} \quad C = 2\left(\frac{2}{\tilde r}\right)^\alpha.$$

If now $r \in [\tilde{r}_{k+1}, \tilde{r}_k]$, then

$$\text{osc}_{Q_r} \bar{f} \leq \text{osc}_{Q_{r_k}} \bar{f} \leq C\tilde{r}_k^\alpha = C\left(\frac{2}{\omega}\right)^\alpha \tilde{r}_{k+1}^\alpha \leq \tilde{C}r^\alpha$$

with $\tilde{C} = C(2/\omega)^\alpha$. Observe finally that the constant C and $\tilde{C}$ are uniformly bounded above as z_0 varies in Q_{int} since $\tilde{r} \geq r_1 - r_0$. The proof is now complete. □

5. Harnack inequality

In this section, we derive Harnack inequality for solutions to Equation (1.6). We use here an approach that Luis Silvestre explained to us in the elliptic setting: we start with Hölder continuous solutions and we consider expanding cylinders to control the spreading of the lower bound of non-negative solutions (see Lemma 5.5). The Harnack inequality is a consequence of the decrease of oscillation we proved earlier and a so-called "doubling property" that estimates how the minimum of a solution propagates with time. Let us first recall the decrease of oscillation proposition.

Proposition 5.1 (Decrease of oscillation). *There exist* $\delta \in (0, 1)$ *and* $\omega \in (0, 1/2)$ (*both universal*) *such that for any* $r \in (0, 1)$ *and any solution* f *of* (1.6) *in some cylinder* $Q_{2r}(z)$ *satisfies*

$$\text{osc}_{Q_{\frac{\omega}{4}r}(z)} f \leq (1-\delta)\left(\text{osc}_{Q_r(z)} f + 2\beta^{-1}\|s\|_{L^\infty}\right).$$

Remark 5.2. The conclusion of the proposition is equivalent to

$$\text{osc}_{Q_{\frac{\omega}{4}r}} f \circ \mathcal{T}_z \leq (1-\delta)\left(\text{osc}_{Q_r} f \circ \mathcal{T}_z + 2\beta^{-1}\|s\|_{L^\infty}\right)$$

with $\mathcal{T}_z(y, w, s) = (x + y + sv, v + w, t + s)$ where $z = (x, v, t)$.

Proof. By considering

$$\tilde{f} = \frac{f \circ \mathcal{T}_z}{\text{osc}_{Q_{2r}(z)} f/2 + \|s\|_{L^\infty}/\beta},$$

and a rescaling $\tilde{f}_r$, we can assume that $z = 0$ and $\text{osc}_{Q_2} \tilde{f}_r \leq 2$ and $\|s\|_{L^\infty} \leq \beta$ (we use here that $r \leq 1$). We then apply Lemma 4.4 to $\tilde{f}_r$ and get the desired result with $1 - \delta = 1 - \lambda_0/2$. □

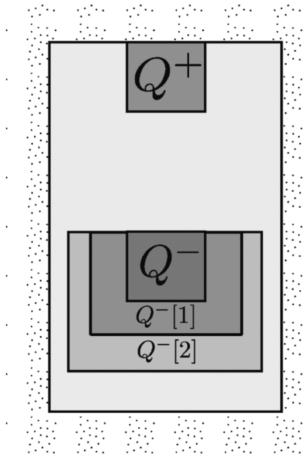

Figure 5.1. The cylinders Q^+, Q^-, $Q^-[1]$ and $Q^-[2]$. Harnack inequality relates the supremum of a solution over Q^- and its infimum over Q^+. The proof consists in constructing a sequence of points lying in $Q^-[1]$ and whose corresponding values explode. Neighborhoods of points included in $Q^-[2]$ are also considered.

5.1. How minima propagate with time

The goal of this subsection is to prove the forthcoming Proposition 5.3. In order to state it, we introduce two cylinders which contain Q^-:

$$Q^- \subset Q^-[1] \subset Q^-[2] \subset Q_1.$$

See Figure 5.1. We recall that $Q^+ = Q_R$ and $Q^- = Q_R(0, 0, -\Delta)$ and $R, \Delta \in (0, 1)$ are small so that in particular $Q^\pm \subset Q_1$ and they are disjoint. We let $Q^-[i]$ be equal to $Q_{\rho_i}(0, 0, -\Delta)$ with $R < \rho_1 < \rho_2 < 1$.

In the following propositions, we introduce *elongated* cylinders Q^{el} where the time is stretched longer in the past than what the scaling would induce:

$$Q_1^{el} = B_{(\omega/4)^3} \times B_{\omega/4} \times (-1, 0]$$
$$Q_r^{el}(z) = \mathcal{T}_z\big(B_{(\omega/4)^3 r^3} \times B_{(\omega/4)r} \times (-r^2, 0]\big).$$

Proposition 5.3 (The propagation of minima). *Assume that f is a non-negative super-solution of* (1.6) *in Q_1 with a non-negative source term s. There exists $r_0 > 0$, $R > 0$ (universal) such that for any $r \in (0, r_0)$ and $z \in Q^-$ such that $Q_r^{el}(z) \subset Q^-[2]$, we have*

$$\min_{Q_r^{el}(z)} f \le C_{\rm pm}\, r^{-q} \min_{Q^+} f$$

for some universal constants $C_{\rm pm}$ and $q > 0$.

We first derive from Lemma 4.5 the following doubling property at the origin. In the two next lemmas, we conveniently assume that 0 is the final time of the first cylinder.

Lemma 5.4 (The doubling property at the origin). *There exists* $\mathfrak{h} \in (0,1)$ (*universal*) *such that for any non-negative super-solution* f *of* (1.6) *in* $B_8 \times B_2 \times (-1, 4]$ *with* $s \geq 0$, *we have*

$$\inf_{Q^1} f \geq \mathfrak{h} \inf_{Q^0} f$$

with $Q^1 = Q_2^{el}(0,0,4)$ *and* $Q^0 = Q_1^{el}$.

Proof. We first notice that since $s \geq 0$, the function f is a super-solution of (1.6) with $s = 0$. We first prove that

$$\inf_{Q_{\omega/2}(0,0,1)} f \geq \mathfrak{h}_0 \inf_{Q_{\omega/4}} f \tag{5.1}$$

for some universal constant $\mathfrak{h}_0$; see Figure 5.2.

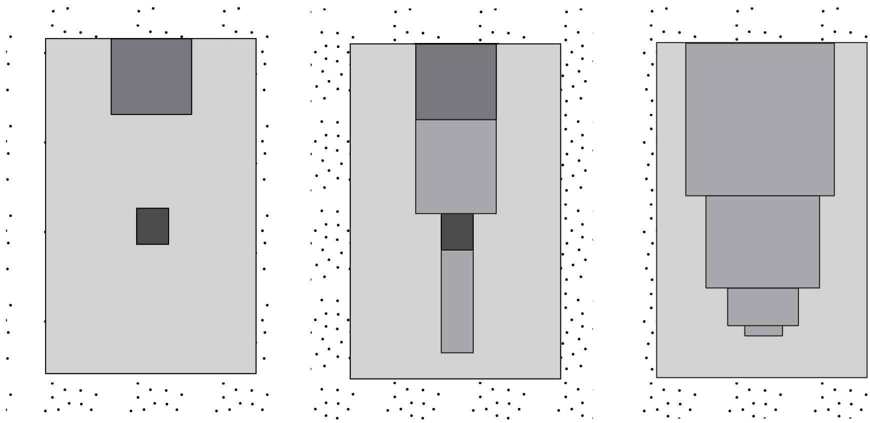

Figure 5.2. The doubling property. On the left, the cylinders $Q_{\omega/4}$ and $Q_{\omega/2}(0,0,1)$. In the middle, the elongated cylinders Q^0 and Q^1. On the right, the iterated cylinders $Q^0, \ldots, Q^N$ (Lemma 5.5).

If $\inf_{Q_{\omega/4}} f = 0$, there is nothing to prove. If not, the function

$$g = \frac{f}{\inf_{Q_{\omega/4}} f}$$

satisfies (1.6) in Q_2 (up to translation in time – this is where we use that $s = 0$) and

$$|\{g \geq 1\} \cap Q_\omega| \geq |Q_{\omega/4}| = \delta_2 |Q_\omega|$$

for some universal δ_2, where Q_ω plays the role of $\hat{Q}$ in Lemma 4.5. We then apply Lemma 4.5 (with time shifted by $+1$) to $\tilde{g} = 1 - g \leq 1$, we get $g \geq \mathfrak{h}_0$ in $B_{(\omega/2)^3} \times B_{\omega/2} \times (1 - (\omega/2)^2, 1]$, that is to say, (5.1) indeed holds true.

Apply now the result to $\tilde{f}(x, v, t) = f(x, v, t - T)$ for $T \in [0, 1 - \omega^2]$ and get

$$\inf_{B_{(\omega/2)^3} \times B_{\omega/2} \times (0,1]} f \geq \mathfrak{h}_0 \inf_{Q^0} f. \tag{5.2}$$

By applying (5.2) on time intervals $(1, 2]$, $(2, 3]$ and $(3, 4]$, we propagate the infimum till time $t = 4$ and get the desired result for $\mathfrak{h} = \mathfrak{h}_0^4$. □

Applying iteratively the previous lemma, we obtain straightforwardly the following lemma whose proof is omitted.

Lemma 5.5 (The iterated doubling property at the origin). *There exists* $\mathfrak{h} > 0$ (*universal*) *such that for any non-negative super-solution* f *of* (1.6) *in* $B_{2^{3N}} \times B_{2^N} \times (-1, T^N)$, *we have*

$$\inf_{Q^N} f \geq \mathfrak{h}^N \inf_{Q^0} f \tag{5.3}$$

with

$$Q^k = B_{R_k^3} \times B_{R_k} \times (T_{k-1}, T_k] \quad \textit{for} \quad k \geq 1,$$

where $R_k = (\omega/4)2^k$ *and* $T_k = \frac{4}{3}(4^k - 1)$ *for* $k \geq 0$.

Remark 5.6. In [44], a measure estimate is also applied iteratively to prove a Harnack inequality for fully nonlinear parabolic equations in non-divergence form.

We can now prove Proposition 5.3.

Proof of Proposition 5.3. In the following proof, we need iterated cylinders that are not centered at the origin and with arbitrary radius.

$$Q_r^k(z) := \mathcal{T}_z\big(r\,Q^k\big).$$

The cylinder Q^k is first scaled by r (this is $r\,Q^k$) and then centered around z (this is $\mathcal{T}_z\big(r\,Q^k\big)$).

Let $z_\infty \in Q^+$ be such that $\min_{Q^+} f = f(z_\infty)$.

Lemma 5.7. *There exist* R, Δ, r_0 (*small, universal*) *such that*

a) *For all* $r \in (0, r_0)$ *and* $z \in Q^-$, *the iterated cylinders* $Q_r^k(z)$ $(k \in \mathbb{N})$ *which are included in* $\{t \leq 0\}$ *are in fact included in* $Q_1(0)$;
b) *The union of the iterated cylinders* $\bigcup_{k=1}^{+\infty} Q_r^k(z)$ *contains* Q^+.

The proof is elementary but tedious. It is given in appendix.

Applying Lemma 5.5, we get

$$\inf_{Q_r^{el}(z)} f \leq \mathfrak{h}^{-N} \inf_{Q_r^N(z)} f \leq \mathfrak{h}^{-N} \min_{Q^+} f,$$

with N such that $z_\infty \in Q_r^N(z)$, *i.e.* $r^{-1}(z^{-1} \circ z_\infty) \in Q^N$. In particular, $r^{-2}(t_\infty - t) \in [T^{N-1}, T^N]$. Since $z_\infty \in Q^+$ and $z \in Q^-$, we know that

$$4^{N-1} \le T^{N-1} \le \frac{t_\infty - t}{r^2} \le \frac{1/2 + R^2}{r^2}.$$

In particular,

$$\mathfrak{h}^{-N} \le \left(\frac{1/2 + R^2}{4}\right)^{\frac{q}{2}} r^{-q},$$

where $q = -\ln\gamma/\ln 2 > 0$. We get the desired inequality with $C_{\mathrm{pm}} = ((1/2 + R^2)/4)^{\frac{q}{2}}$. The proof of the proposition is thus complete. □

5.2. Proof of the Harnack inequality

We can now turn to the proof of Theorem 1.6.

Proof of Theorem 1.6. We first remark that replacing $f(x,v,t)$ with $f(x,v,t) + \|s\|_{L^\infty} t$ if necessary, we can assume that $s \ge 0$. By dividing f by $2\beta^{-1}\|s\|_{L^\infty}$ if necessary, we can assume that $\|s\|_{L^\infty} = \beta/2$ (if $s \not\equiv 0$).

We are going to find a universal constant $C = C_H$ such that (1.10) cannot be false. In other words, we are going to find a universal C_H such that

$$m + 1 \le C_H M \tag{5.4}$$

entails a contradiction where

$$M := \sup_{Q^-} f = f(z_0) \quad \text{and} \quad m := \inf_{Q^+} f = f(z_\infty)$$

for some $z_0 \in Q^-$ and $z_\infty \in Q^+$. We used here the fact that u is (Hölder) continuous.

Our goal is to construct by induction a sequence $(z_k)_{k\ge 0}$ in $Q^-[1]$ (we recall that $Q^- \subset Q^-[1] \subset Q^-[2] \subset Q_1$, see Figure 5.1) such that

$$f(z_k) \ge (1-\delta')^{-k} M \tag{5.5}$$

for some universal $\delta' \in (0,1)$. This implies in particular that $f(z_k) \to +\infty$ as $k \to +\infty$ which is absurd since f is bounded in Q^-.

Remark first that (5.5) holds true for $k = 0$. Let us assume that we already constructed $z_0, \dots, z_k$ and let us construct z_{k+1}. Let $z_k = (x_k, v_k, t_k)$. We choose $r_k > 0$ such that

$$f(z_k) = r_k^{-2q} m, \tag{5.6}$$

where q is given by Proposition 5.3. Inequality (5.4) and the induction hypothesis (5.5) imply

$$r_k^{2q} \le C_H (1-\delta')^k. \tag{5.7}$$

From the decrease of oscillation (Proposition 5.1), we know that

$$1 + \operatorname{osc}_{Q_{r_k}} f \geq (1-\delta)^{-1} \operatorname{osc}_{q_k} f$$

(recall $2\beta^{-1}\|s\|_{L^\infty} = 1$) with

$$Q_k = Q_{r_k}(z_k) \quad \text{and} \quad q_k = Q_{\omega r_k/4}(z_k).$$

In particular, $z_k \in q_k$. Let $z_{k+1} \in Q_k$ be such that

$$\max_{Q_k} f = f(z_{k+1}).$$

Then we get

$$1 + f(z_{k+1}) \geq (1-\delta)^{-1} \left(f(z_k) - \min_{q_k} f \right). \tag{5.8}$$

Recall that $z_k \in Q^-[1]$. Choosing C_H small, we can ensure through (5.7) that $Q_{r_k}(z_k) \subset Q^-[2]$. We also remark that

$$q_k \supset Q^{el}_{(\omega/4)^2 r_k}(z_k).$$

We thus can apply Proposition 5.3 and get

$$\min_{q_k} f \leq \min_{Q^{el}_{(\omega/4)^2 r_k}(z_k)} f \leq \tilde{C}_{\mathrm{pm}} r_k^{-q} m,$$

with $\tilde{C}_{\mathrm{pm}} = C_{\mathrm{pm}}(4/\omega)^q$. The use of (5.6) in the previous inequality yields

$$\min_{q_k} f \leq \tilde{C}_{\mathrm{pm}} r_k^q f(z_k) \leq \tilde{C}_{\mathrm{pm}} \sqrt{C_H} f(z_k). \tag{5.9}$$

Now combining (5.8) and (5.9), we get

$$1 + f(z_{k+1}) \geq (1-\delta)^{-1} \left(1 - \tilde{C}_{\mathrm{pm}} \sqrt{C_H}\right) f(z_k).$$

Use next that $1 \leq C_H M$ (this is a consequence of (5.4)) and the induction hypothesis and get

$$\begin{aligned} f(z_{k+1}) &\geq (1-\delta)^{-1} \left(1 - \tilde{C}_{\mathrm{pm}} \sqrt{C_H}\right) (1-\delta')^{-k} M - C_H M \\ &\geq \mathrm{j}(1-\delta')^{-k} M, \end{aligned}$$

with

$$\mathfrak{j} = (1-\delta)^{-1}\left(1 - \tilde{C}_{\mathrm{pm}}\sqrt{C_H}\right) - C_H.$$

We thus choose δ' such that

$$(1-\delta')^{-1} = \mathfrak{j}$$

and we can choose C_H small enough so that $\delta' \in (0,1)$. In particular we get

$$f(z_{k+1}) \geq (1-\delta')^{-k-1} M$$

which is the desired inequality.

We are left with proving that the sequence $\{z_k\}$ stays in $Q^-[1]$. The fact that z_{k+1} lies in $Q_{r_k}(z_k) = \mathcal{T}_{z_k}(Q_{r_k}(0))$ implies in particular that $|v_{k+1} - v_k| \leq r_k$ which in turn yields

$$|v_k - v_0| \leq \sum_{l \geq 0} r_l \leq C_H^{1/(2q)} \sum_{l \geq 0} (1-\delta')^{\frac{k}{2q}} = \frac{C_H^{1/(2q)}}{1-(1-\delta')^{1/(2q)}}.$$

Using now that the fact that δ' is explicitely given as a function of δ and C_H (see above), we conclude that $|v_k - v_0|$ can be arbitrarily small uniformly in k. We can argue in the same spirit for $|x_k - x_0|$ and $|t_k - t_0|$. Since $z_0 \in Q^-$, we conclude that we can indeed ensure that z_k lies in $Q^-[1]$. The proof of the theorem is now complete. □

6. Local gain of regularity for sub-solutions

In this section, we investigate the regularity of *sub-solutions* to Equation (1.6) beyond the gain of integrability proved above. Observe that, on the one hand, Theorem 2.1 applies to sub-solutions but only concludes to the gain of integrability. On the other hand, Theorem 2.3 proves a gain of Sobolev regularity but only applies to *solutions* (not sub-solutions). It might seem, at first glance, that the lack of ellipticity in all directions means the gain of regularity of solutions is false, since in the elliptic and parabolic case it is entirely based on the energy estimate. However we show here that, using the local upper bound proved above by the De Giorgi–Moser iteration, and refined averaging lemmas, this result still holds in essence for our equation, even though the gain of regularity is only H^s with $s > 0$ small. We prove the following result:

Theorem 6.1 (Gain of regularity for non-negative sub-solutions). *Consider $z_0 \in \mathbb{R}^{2d+1}$ and two cylinders $Q_{\mathrm{int}} := Q_{r_1}(z_0)$ and $Q_{\mathrm{ext}} := Q_{r_0}(z_0)$ with $0 < r_1 < r_0$. Then there is some $\mathfrak{s} \in (0, 1/3)$ such that any weak non-negative sub-solution f*

of (1.6) *in* Q_{ext} *satisfies*

$$\|f\|_{H^{\mathfrak{s}}_{x,v,t}(Q_{\text{int}})} \le C\left(\|f\|_{L^2(Q_{\text{ext}})} + \|s\|_{L^\infty(Q_{\text{ext}})}\right) \tag{6.1}$$

with $C = C(d, \lambda, \Lambda, Q_{\text{ext}}, Q_{\text{int}})$.

Remark 6.2. Since f is sub-solution of (1.6), a non-positive measure appears as a source term. Since such measure is arbitrary, we necessarily gain *strictly* less derivatives in x than what was obtained in Theorem 2.3; this can be seen for instance by considering the Dirac mass belonging to $W^{1/p-1-0,p}$ and exploiting the optimality of the regularisation results in [7].

Proof. We define Q_{mid} in between Q_{int} and Q_{ext} and the same truncation functions as before. Theorem 3.1 implies that

$$\|f\|_{L^\infty(Q_{\text{mid}})} \lesssim \|f\|_{L^2(Q_{\text{ext}})} + \|s\|_{L^\infty(Q_{\text{ext}})}.$$

We want to apply [7, Theorem 1.3] on f in Q_{mid}. However since f is only a sub-solution it satisfies the equation

$$\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(A\nabla_v f) + B\cdot\nabla_v f + s - \mu \qquad \text{in } Q_{\text{ext}},$$

where we have included the defect non-negative measure $\mu \ge 0$ accounting for the inequation. We can now repeat the reasoning from the proof of Lemma 4.1 and reduce to the case

$$\partial_t g + v\cdot\nabla_x g = \nabla_v\cdot(A\nabla_v g) + \nabla_v\cdot H_1 + H_0 - \tilde{\mu} \qquad \text{in } \mathbb{R}^{2d+1},$$

with $g \equiv f$ in Q_{int} and g, the measure $\tilde{\mu} \ge 0$, H_0 and H_1 supported in Q_{mid}, and with g, $\nabla_v g$, H_0 and H_1 bounded in L^2 on Q_{mid}. Then by integrating in x, v, t we deduce that $\tilde{\mu}$ has bounded variation in terms of the previous bounds. Since for $q > (4d+2)$, the space $W^{\frac{1}{2},q}_{x,v,t}$ embeds into the space of continuous bounded functions of x, v, t, we deduce that the space of measures is included in $W^{-\frac{1}{2},q^*}_{x,v,t}$ and therefore

$$\tilde{\mu} = \left(1-\Delta_{x,t}\right)^{\frac{1}{4}}(1-\Delta_v)\,h \quad \text{with} \quad h \in L^{q^*}(Q_{\text{mid}}) \tag{6.2}$$

and that the bound on the $L^{q^*}(Q_{\text{mid}})$ depends on the previous bounds above, where $q^* = 1/(1-1/q)$ is the conjugate exponent of q. Observe that q^* is strictly smaller than 2 and close to one, for instance $q^* \in (1, 14/13)$ in dimension $d = 3$. We then apply [7, Theorem 1.3] with $\kappa = 1$, $r = \frac{1}{2}$, $m = 2$, $\beta = 1$, $p = q^*$: we deduce that g belongs to $W^{\frac{1}{8},p}_{x,t} L^p_v$ (observe that we use a full Laplacian derivative in v in Equation (6.2) in order to be in the framework of [7, Theorem 1.3], even though $(1-\Delta_v)^{1/4}$ would have been enough for the purpose of having $h \in L^{q^*}$). By interpolation with the L^∞ estimate, we obtain then that $g \in H^{\mathfrak{s}}_{x,t} L^2_v$ for some $\mathfrak{s} \in (0, \frac{1}{8})$ small enough. Finally, we combine the latter estimate with the energy estimate $g \in L^2_{x,t} H^1_v$ to conclude with $g \in H^{\mathfrak{s}}_{x,v,t}$. Since the truncation function is equal to one on the smaller cube Q_{int}, it translates into $f \in H^{\mathfrak{s}}_{x,v,t}$ on Q_{int} and this concludes the proof. □

7. Gain of integrability of the velocity gradient

This section is devoted to the proof of the following theorem.

Theorem 7.1 (Gain of integrability for $\nabla_v f$). *Let f be a solution of (1.6) without lower order terms ($B \equiv 0$ and $s \equiv 0$) in some cylinder $Q_{r_0}(z_0)$. There exists a universal $\varepsilon > 0$ such that for all $Q[i] = Q_{r_i}(z_0)$, $i = 0, 1, 2$ with $r_2 < r_1 < r_0$, $\nabla_v f \in L^{2+\varepsilon}(Q_2)$*

$$\int_{Q[2]} |\nabla_v f|^{2+\varepsilon} \, dz \leq C \left(\int_{Q[1]} |\nabla_v f|^2 \, dz \right)^{\frac{2+\varepsilon}{2}}, \tag{7.1}$$

with $C = C(d, \lambda, \Lambda, Q_2, Q_{\text{int}}, Q_{\text{ext}})$.

Remark 7.2. We decided to remove lower order terms B and s in order to simplify the presentation of the proof. We believe there is no additional technical difficulties dealing with bounded B and s, and that such a gain of integrability is true for solutions of the Landau-Coulomb equation under the assumption of Theorem 1.1.

The proof follows along the lines of the one of [32, Theorem 2.1]. It consists in deriving an almost reverse Hölder inequality which in turn implies the result thanks to the analogous of [32, Proposition 1.3]. The following measure-theoretical lemma will be used as a black box in the proof of Theorem 7.1. It implies the use of cylinders with different shape:

$$\mathcal{Q}(z_0, r) = \left\{ z = (x, v, t) : |x_i - x_i^0| < r^3, |v_i - v_i^0| < r, -r^2 < t - t_0 \leq 0 \right\},$$

where $x = (x_1, \ldots, x_d)$ and $v = (v_1, \ldots, v_d)$. The scaling of the equation preserves this family of cylinders but not the Lie group action $\mathcal{T}_z$.

Lemma 7.3 (A Gehring lemma). *Let $g \geq 0$ in $\mathcal{Q}$ such that there exists $q > 1$ such that for all $z_0 \in \mathcal{Q}$ and R such that $\mathcal{Q}_{4R}(z_0) \subset \mathcal{Q}$,*

$$\fint_{\mathcal{Q}_R(z_0)} g^q \, dz \leq b \left(\fint_{\mathcal{Q}_{4R}(z_0)} g \, dz \right)^q + \theta \fint_{\mathcal{Q}_{4R}(z_0)} g^q \, dz$$

for some $\theta > 0$. *There exists* $\theta_0 = \theta_0(q, d)$ *such that if* $\theta < \theta_0$, *then* $g \in L^p_{\mathrm{loc}}(Q)$ *for* $p \in [q, q + \varepsilon)$ *and*

$$\left(⨍_{Q_R} g^p \,\mathrm{d}z\right)^{\frac{1}{p}} \leq c_p \left(⨍_{Q_{4R}} g^q \,\mathrm{d}z\right)^{\frac{1}{q}},$$

the constants $\varepsilon > 0$ *depends only on* b, q, θ *and dimension, and* c_p *further depends on* p.

The proof of Lemma 7.3 is an easy adaptation of the one of [31, Proposition 5.1], by changing Euclidean cubes with cylinders Q_R.

The proof of Theorem 7.1 is a consequence of some estimates involving weighted means of the solution. Given $z_0 \in \mathbb{R}^{2d+1}$, they are defined as follows:

$$\tilde{f}_{2R}(t) = \frac{1}{cR^{4d}} \int_{\mathbb{R}^{2d}} f(t, x, v)\chi_{2R}(x, v, t) \,\mathrm{d}x \,\mathrm{d}v$$

(for some c defined below) where χ_{2R} is a cut-off function such that

$$\chi_{2R}(x, v, t) = \prod_{i=1}^{d} \phi_{R^3}\left(x_i - x_i^0\right)\phi_R\left(v_i - v_i^0\right),$$

with $\phi_R(a) = \phi(a/R)$ for some ϕ such that $\sqrt{\phi} \in C^\infty(\mathbb{R})$ and $\phi \equiv 1$ in $[-1, 1]$ and $\operatorname{supp} \phi \subset [-2, 2]$. We remark that $\chi_{2R} \equiv 1$ in Q_R and $\chi_{2R} \equiv 0$ outside Q_{2R}.

Lemma 7.4. *Let* f *be a solution of* (1.6) *in* Q_0. *Then for* $Q_{3R}(z_0) \subset Q_0$,

$$\int_{Q_R(z_0)} |\nabla_v f|^2 \,\mathrm{d}z \leq C R^{-2} \int_{Q_{2R}(z_0)} |f - \tilde{f}_{2R}|^2 \,\mathrm{d}z, \tag{7.2}$$

$$\sup_{t \in (t_0 - R^2, t_0]} \int_{Q^t_R(z_0)} |f(t) - \tilde{f}_R(t)|^2 \,\mathrm{d}x \,\mathrm{d}v \leq C \int_{Q_{3R}(z_0)} |\nabla_v f|^2 \,\mathrm{d}z, \tag{7.3}$$

where $Q^t_R(z_0) = \{(x, v) : (t, x, v) \in Q_R(z_0)\}$.

Remark 7.5. This lemma corresponds to [32, Lemmas 2.1 and 2.2].

Proof. For the sake of clarity, we put $z_0 = 0$ and $R = 1$. Consider $\tau_2 \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \tau_2 \leq 1$, $\tau_2 \equiv 0$ in $(-\infty, -2^2]$ and $\tau_2 \equiv 1$ in $[-1, 0]$. Use $2(f - \tilde{f}_2)\chi_2\tau_2$ as a test function for (1.6) and get

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \left(f(0) - \tilde{f}_2(0)\right)^2 \chi_2 \,\mathrm{d}x \,\mathrm{d}v + 2 \int_{\mathbb{R}^{2d+1}} (A\nabla_v f \cdot \nabla_v f)\chi_2\tau_2 \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t \\
&= \int_{\mathbb{R}^{2d+1}} \left(f - \tilde{f}_2\right)^2 \chi_2(\partial_t \tau_2) \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t - \int_{\mathbb{R}^{2d+1}} v \cdot \nabla_x \left[\left(f - \tilde{f}_2\right)^2\right] \chi_2\tau_2 \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t \\
&\quad - 2 \int_{\mathbb{R}^{2d+1}} (f - \tilde{f}_2) A\nabla_v f \cdot \nabla_v \chi_2\tau_2 \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t.
\end{aligned}$$

Note that the definition of $\tilde{f}_2$ implies that the remaining term

$$-2\int_{\mathbb{R}^{2d+1}} \big(\partial_t \tilde{f}_2\big)\big(f - \tilde{f}_2\big)\chi_2\tau_2$$

vanishes. This equality yields

$$\int_{\mathbb{R}^{2d}} \big(f(0) - \tilde{f}_2(0)\big)^2 \chi_2 \,\mathrm{d}x\,\mathrm{d}v + \lambda \int_{\mathbb{R}^{2d+1}} |\nabla_v f|^2 \chi_2\tau_2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$
$$\leq \int_{\mathbb{R}^{2d+1}} \big(f - \tilde{f}_2\big)^2 \left(\chi_2|\partial_t\tau_2| + |v\cdot\nabla_x\chi_2|\tau_2 + \frac{\Lambda^2}{\lambda}|\nabla_v\sqrt{\chi_2}|^2\tau_2\right) \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

which in turn yields (7.2). Changing the final time, we also get

$$\sup_{t\in(-1,0]} \int_{\mathbb{R}^{2d}} \big[f(t) - \tilde{f}_2(t)\big]^2 \chi_2(t)\,\mathrm{d}x\,\mathrm{d}v \leq C\int_{\mathcal{Q}_2} |f - \tilde{f}_2|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

Now the function $F = f - \tilde{f}_2$ is such that $\int F(x,v,t)\,\mathrm{d}x\,\mathrm{d}v = 0$. In particular, we have

$$\int_{\mathcal{Q}_2} \big(f - \tilde{f}_2\big)^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \leq C\int_{\mathcal{Q}_2} \left(|\nabla_v f|^2 + \left|D_x^{\frac{1}{3}} f\right|^2\right) \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

Observe that if there are no lower order terms ($B = 0$ and $s = 0$), then we have for all $q \in (1,2]$,

$$\int_{\mathcal{Q}_2} \left|D_x^{\frac{1}{3}} f\right|^q \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \leq C\int_{\mathcal{Q}_3} |\nabla_v f|^q \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t. \tag{7.4}$$

Indeed, in view of the proof of (2.2), it is enough to apply [7, Theorem 1.3] with such a q and use the Poincaré inequality (assuming the cutoff functions to have convex super-level sets).

Combining the three previous estimates yields

$$\sup_{t\in(-1,0]} \int_{\mathcal{Q}_1^t} \big(f(t) - \tilde{f}_2(t)\big)^2 \chi_2(t)\,\mathrm{d}x\,\mathrm{d}v \leq C\int_{\mathcal{Q}_3} |\nabla_v f|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

Finally, we write for $t \in (-1,0]$

$$\begin{aligned}
\frac{1}{2}\int_{\mathcal{Q}_1^t} \big(f(t) - \tilde{f}_1(t)\big)^2\chi_2(t) &\leq \int_{\mathcal{Q}_1^t} \big(f(t) - \tilde{f}_2(t)\big)^2\chi_2(t) + \int_{\mathcal{Q}_1^t} \big(\tilde{f}_2(t) - \tilde{f}_R(t)\big)^2\chi_2(t) \\
&\leq \int_{\mathcal{Q}_1^t} (f(t) - \tilde{f}_2(t))^2\chi_2(t) \\
&\quad + |\mathcal{Q}_1^t|\left(\frac{1}{c}\int_{\mathcal{Q}_1^t} (f - \tilde{f}_2(t))\chi_1(x,v,t)\,\mathrm{d}x\,\mathrm{d}v\right)^2 \\
&\leq C\int_{\mathcal{Q}_1^t} (f(t) - \tilde{f}_2(t))^2\chi_2(t),
\end{aligned}$$

and we get the second desired estimate since $\chi_2 \equiv 1$ in $\mathcal{Q}_1$. □

We now turn to the proof of Theorem 7.1. The use of (7.4) is the main difference with [32].

Proof of Theorem 7.1. Pick $p>2$ and let q denote its conjugate exponent: $\frac{1}{q}+\frac{1}{p}=1$. We follow [32] in writing (omitting the center of cylinders z_0), thanks to (7.2),

$$\begin{aligned}
\fint_{Q_1} |\nabla_v f|^2 &\lesssim \int_{Q_2} |f-\tilde{f}_2|^2 \\
&\le \sup_{t\in(t_0-4,t_0]} \left(\int_{Q_2^t} |f-\tilde{f}_2|^2\right)^{\frac{1}{2}} \int_{t_0-4}^{t_0} dt \left(\int_{Q_2^t} |f-\tilde{f}_2|^2\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{Q_4} |\nabla_v f|^2\right)^{\frac{1}{2}} \int_{t_0-4}^{t_0} dt \left(\int_{Q_2^t} |f-\tilde{f}_2|^q\right)^{\frac{1}{2q}} \left(\int_{Q_2^t} |f-\tilde{f}_2|^p\right)^{\frac{1}{2p}}
\end{aligned}$$

where (7.3) and Hölder inequality are used successively.

We now use Sobolev inequalities and Hölder inequality (twice) successively to get

$$\begin{aligned}
\fint_{Q_1} |\nabla_v f|^2 &\lesssim \left(\int_{Q_4} |\nabla_v f|^2\right)^{\frac{1}{2}} \times \left[\int_{t_0-4}^{t_0} \left(\int_{Q_2^t} |\nabla_v f|^q + |D_x^{1/3} f|^q\right)^{\frac{1}{2q}} \mathrm{d}t\right] \\
&\qquad \times \left(\int_{Q_2^t} |\nabla_v f|^2 + |D_x^{1/3} f|^2\right)^{\frac{1}{4}} \\
&\lesssim \left(\int_{Q_4} |\nabla_v f|^2\right)^{\frac{1}{2}} \left(\int_{Q_2} |\nabla_v f|^q + |D_x^{1/3} f|^q\right)^{\frac{1}{2q}} \\
&\qquad \times \left(\int_{t_0-4}^{t_0} \left(\int_{Q_2^t} |\nabla_v f|^2 + |D_x^{1/3} f|^2\right)^{\frac{q}{2(2q-1)}} \mathrm{d}t\right)^{\frac{2q-1}{2q}} \\
&\lesssim \left(\int_{Q_4} |\nabla_v f|^2\right)^{\frac{1}{2}} \times \left(\int_{Q_2} |\nabla_v f|^q + |D_x^{1/3} f|^q\right)^{\frac{1}{2q}} \\
&\qquad \times \left(\int_{Q_2} |\nabla_v f|^2 + |D_x^{1/3} f|^2\right)^{\frac{1}{4}}.
\end{aligned}$$

We now use (7.4) and get

$$\begin{aligned}\fint_{Q_1} |\nabla_v f|^2 &\lesssim \left(\int_{Q_4} |\nabla_v f|^2\right)^{\frac{1}{2}} \left(\int_{Q_2} |\nabla_v f|^q\right)^{\frac{1}{2q}} \left(\int_{Q_2} |\nabla_v f|^2\right)^{\frac{1}{4}} \\ &\lesssim \left(\int_{Q_4} |\nabla_v f|^2\right)^{\frac{3}{4}} \left(\int_{Q_2} |\nabla_v f|^q\right)^{\frac{1}{2q}}.\end{aligned}$$

Now use and get for all $\varepsilon > 0$,

$$\begin{aligned}\fint_{Q_1} |\nabla_v f|^2 &\lesssim \left(\fint_{Q_4} |\nabla_v f|^2\right)^{\frac{3}{4}} \left(\fint_{Q_4} |\nabla_v f|^q\right)^{\frac{1}{2q}} \\ &\lesssim \left(\fint_{Q_4} |\nabla_v f|^2\right)^{\frac{3}{4}} \left(\fint_{Q_4} |\nabla_v f|^q\right)^{\frac{1}{2q}}.\end{aligned}$$

After rescaling, we get the following:

$$\begin{aligned}\fint_{Q_R} |\nabla_v f|^2 &\lesssim \left(\fint_{Q_{4R}} |\nabla_v f|^2\right)^{\frac{3}{4}} \left(\fint_{Q_{4R}} |\nabla_v f|^q\right)^{\frac{1}{2q}} \\ &\lesssim \varepsilon \fint_{Q_{4R}} |\nabla_v f|^2 + c_\varepsilon \left(\fint_{Q_{4R}} |\nabla_v f|^q\right)^{\frac{2}{q}}.\end{aligned}$$

Apply now Proposition 7.3 in order to achieve the proof of Theorem 7.1. □

Appendix

A. Known estimates for the Landau equation

Lemma A.1 (Lower bound – [23,56]). *Assume there exist positive constants M_1, M_0, E_0 and H_0 such that* (1.3) *holds true. Then*

$$\det A[f] \geq c(1 + |v|)^\kappa$$

with

$$\kappa = \begin{cases} (d-1)(\gamma+2)+\gamma & \text{if } \gamma \in [-2, 0] \\ 3\gamma + 2 & \text{if } \gamma \in [-d, -2), \end{cases}$$

where c only depends on dimension, γ, M_0, M_1, E_0 and H_0.

Lemma A.2 (Upper bounds - [23,56]). *Assume there exist positive constants* M_1*,* M_0*,* E_0 *and* H_0 *such that* (1.3) *holds true. Assume that* $f \in L^\infty(\mathbb{R}^d)$*. Then*

$$|A[f]| \le \begin{cases} C(1+|v|)^{\gamma+2} & \text{if } \gamma \in [-2,0] \\ C\|f\|_\infty^{\frac{|\gamma+2|}{d}} & \text{if } \gamma \in [-d,-2), \end{cases}$$

$$|B[f]| \le \begin{cases} C(1+|v|)^{\gamma+1} & \text{if } \gamma \in [-1,0] \\ C\|f\|_\infty^{\frac{|\gamma+1|}{d}} & \text{if } \gamma \in [-d,-1), \end{cases}$$

$$|c[f]| \le \begin{cases} C & \text{if } \gamma = 0 \\ C\|f\|_\infty^{\frac{|\gamma|}{d}} & \text{if } \gamma \in [-d,0), \end{cases}$$

where C *only depends on dimension,* γ*,* M_0*,* E_0*.*

B. Proof of a technical lemma

Proof of Lemma 5.7. To justify a) and b), we remark that

$$\mathcal{P}^- \subset \bigcup_{k=1}^{+\infty} Q^k \subset \mathcal{P}^+,$$

where

$$\mathcal{P}^- := \left\{(y,w,s) : s \ge \frac{4}{3}\left(\frac{4^2}{\omega^2}\rho^2 - 1\right), |y| \le \rho^3, |w| \le \rho\right\},$$

$$\mathcal{P}^+ := \left\{(y,w,s) : s \ge \frac{4}{3}\left(\frac{4}{\omega^2}\rho^2 - 1\right), |y| \le \rho^3, |w| \le \rho\right\},$$

see Figure B.1.

In what follows, R and r_0 are chosen as functions of Δ. In particular,

$$R \le \sqrt{\Delta} \quad \text{and} \quad r_0 \le \sqrt{\Delta}.$$

As far as a) is concerned, we should ensure that for all $z \in Q^-$ and $r \in (0, r_0)$,

$$\left(z \circ r\mathcal{P}^+\right) \cap \{t \le 0\} \subset Q_1(0).$$

If $z = (x^-, v^-, t^-)$ and $z^+ = (x^+, v^+, t^+) \in r\mathcal{P}^+$ are such that $z \circ z^+ \in \{t \le 0\}$, we have

$$\begin{aligned} 0 &\ge t^- + t^+ \\ &\ge \left(-\Delta - R^2\right) + \frac{4}{3}\left((4/\omega^2)\rho^2 - r^2\right) \\ &\ge -4\Delta + \left(4^2/3\omega^2\right)\rho^2, \end{aligned}$$

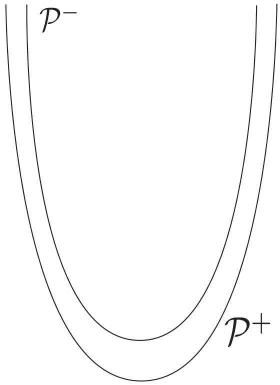

Figure B.1. Paraboloids containing/contained in the union of iterated cylinders.

where $\rho = |v^+|$. This implies in particular

$$\rho^2 \leq \frac{3\omega^2}{4}\Delta.$$

In particular, for $\Delta \in (0, 1)$,

$$\begin{aligned} \left|v^- + v^+\right| &\leq R + \rho \\ &\leq \left(1 + \sqrt{3}\omega/2\right)\sqrt{\Delta} \\ \left|x^- + x^+ + t^+ v^-\right| &\leq R^3 + \rho^3 + R \\ &\leq \left(1 + (\sqrt{3}\omega/2)^3\right)\Delta^{3/2} + \sqrt{\Delta} \\ &\leq \left(2 + (\sqrt{3}\omega/2)^3\right)\sqrt{\Delta}. \end{aligned}$$

We thus can choose Δ small enough (recall $\omega = 1/\sqrt{5}$) to ensure a).

As far as b) is concerned, notice that for $z^+ \in Q^+$ and $z \in Q^-$, we have

$$z^{-1} \circ z^+ = \left(t^+ - t, x^+ - x - (t^+ - t)v, v^+ - v\right).$$

Choosing $R^2 \leq \Delta \leq \frac{1}{2}$ we have $2R \leq (4R)^{\frac{1}{3}}$ and we get

$$\begin{aligned} &\left|v^+ - v\right| \leq 2R \leq (4R)^{\frac{1}{3}}, \\ &\left|x^+ - x - (t^+ - t)v\right| \leq 2R^3 + \left(\Delta + R^2\right)R = 3R^3 + \Delta R \leq 4R, \end{aligned}$$

(since $R \leq 1$ and $\Delta \leq 1$) and

$$t^+ - t \geq \Delta - R^2.$$

In particular $z^{-1} \circ z^+ \in r\mathcal{P}^-$ if

$$\Delta - R^2 \geq \frac{4}{3}\left(\frac{4^2}{\omega^2}(4R)^{\frac{1}{3}} - r^2\right).$$

This is in turn implied by

$$\Delta \geq R^2 + \frac{4^3}{3\omega^2}(4R)^{\frac{1}{3}}.$$

Hence, for Δ given, we can choose $R = R(\Delta)$ small enough to get the desired inequality and in turn point b). ☐

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