Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. IV (2005), 129-177

Minimal surfaces in pseudohermitian geometry

JIH-HSIN CHENG, JENN-FANG HWANG, ANDREA MALCHIODI
AND PAUL YANG

Abstract. We consider surfaces immersed in three-dimensional pseudohermitian manifolds. We define the notion of (p-)mean curvature and of the associated (p-)minimal surfaces, extending some concepts previously given for the (flat) Heisenberg group. We interpret the p-mean curvature not only as the tangential sublaplacian of a defining function, but also as the curvature of a characteristic curve, and as a quantity in terms of calibration geometry.
As a differential equation, the p-minimal surface equation is degenerate (hyperbolic and elliptic). To analyze the singular set (*i.e.*, the set where the (p-)area integrand vanishes), we formulate some *extension* theorems, which describe how the characteristic curves meet the singular set. This allows us to classify the entire solutions to this equation and to solve a Bernstein-type problem (for graphs over the xy-plane) in the Heisenberg group H_1. In H_1, identified with the Euclidean space $\mathbb{R}^3$, the p-minimal surfaces are classical ruled surfaces with the rulings generated by Legendrian lines. We also prove a uniqueness theorem for the Dirichlet problem under a condition on the size of the singular set in two dimensions, and generalize to higher dimensions without any size control condition.
We also show that there are no closed, connected, C^2 smoothly immersed constant p-mean curvature or p-minimal surfaces of genus greater than one in the standard S^3. This fact continues to hold when S^3 is replaced by a general pseudohermitian 3-manifold.

Mathematics Subject Classification (2000): 35L80 (primary); 35J70, 32V20, 53A10, 49Q10 (secondary).

1. Introduction and statement of the results

Minimal surfaces in a Riemannian manifold play an important role in the study of the topology and the geometry of the ambient manifold. For instance, the positive mass theorem originally was proved with the aid of minimal surface theory ([SY]). In order to study the mass in an analogous manner or to formulate a boundary value problem for prescribing the Webster scalar curvature

Pervenuto alla Redazione il 26 aprile 2004 e in forma definitiva il 26 novembre 2004.

on a domain with boundary in pseudohermitian geometry, we find it necessary to formulate a notion of mean curvature for a surface in a pseudohermitian manifold.

Let Σ be a surface in a 3-dimensional pseudohermitian manifold M. Roughly speaking, M is a manifold having a (contact) plane distribution on which an almost complex structure is defined (see the Appendix for some basic facts in pseudohermitian geometry). A singular point of Σ is a point where the background (contact) plane coincides with the tangent plane of Σ. To every nonsingular point of Σ, we associate a vector field *normal* to Σ, called the Legendrian normal. If for example Σ bounds some smooth set Ω, then the p-area, in analogy with the Riemannian case, is obtained as the variation of the volume of Ω in the direction of the Legendrian normal (see formulas (2.4) and (2.5)).

The p-mean curvature H of Σ is in turn given by the first variation of the p-area. It is easy to see that H equals the negative subdivergence of the Legendrian normal e_2. Suppose ψ is a defining function of Σ such that $e_2 = \nabla_b \psi / |\nabla_b \psi|_G$ (G is the Levi metric. See the definition in Section 2). Then the p-mean curvature equation and the p-minimal surface equation ($H \equiv 0$) read

$$-\operatorname{div}_b(\nabla_b \psi / |\nabla_b \psi|_G) = H \qquad \text{(pMCE)}$$

and

$$\operatorname{div}_b(\nabla_b \psi / |\nabla_b \psi|_G) = 0\,, \qquad \text{(pMSE)}$$

respectively.

Alternatively, having in mind the Gauss map, the mean curvature can be defined in terms of the covariant derivative (with respect to the pseudohermitian connection) along Σ of the *Legendrian tangent* e_1 to Σ (see (2.1)).

Of course the case of the Heisenberg group as a pseudohermitian manifold is one of the most important. Indeed it is the simplest model example, and represents a blow-up limit of general pseudo-hermitian manifolds. In the case of a smooth surface in the Heisenberg group, our definitions coincide with those given in [CDG, DGN], and [Pau]. In particular these notions, especially in the framework of geometric measure theory, have been used to study existence or regularity properties of minimizers for the relative perimeter or extremizers of isoperimetric inequalities (see, e.g., [DGN, GN, LM, LR], and [Pan]). The p-area can also be identified with the 3-dimensional spherical Hausdorff measure of Σ (see, e.g., [B, FSS]).

In this paper, we study the subject mainly from the viewpoint of partial differential equations and from that of differential geometry. Our basic results are the analysis of the singular set (see Section 3). As consequences, we can prove a Bernstein-type theorem (see Section 4 and Theorem A) and the nonexistence of closed hyperbolic p-minimal surfaces (see Section 7 and Theorem E). We also establish a comparison theorem (see Section 5) which is a substitute for the maximum principle and may become a useful tool in the subject.

For a p-minimal graph $(x, y, u(x, y))$ in the Heisenberg group H_1, the above equation (pMSE) reduces to

$$(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy} = 0 \qquad \text{(pMGE)}$$

by taking $\psi = z - u(x, y)$ on the nonsingular domain. This is a degenerate (hyperbolic and elliptic) partial differential equation. It is degenerate hyperbolic (on the nonsingular domain) having only one characteristic direction (note that a 2-dimensional hyperbolic equation has two characteristic directions [Jo]). We call the integral curves of this characteristic direction the characteristic curves. We show that the p-mean curvature is the line curvature of a characteristic curve. Therefore the characteristic curves of (pMGE) are straight lines. Moreover, the value of u along a characteristic curve is determined in a simple way (see (2.22), (2.23)).

The analysis of the singular set is necessary to characterize the solutions. As long as the behavior of H (consider (pMCE) for a graph in H_1) is not too bad (say, bounded), we show that the singular set consists of only isolated points and smooth curves (see Theorem B below). Under a quite weak growth condition on H, a characteristic curve Γ reaches a singular point p_0 in a finite arc-length parameter and has an approximate tangent. From the opposite direction, we find another characteristic curve Γ' reaching also p_0 with the opposite approximate tangent. The union of Γ, p_0, and Γ' forms a smooth curve (see Corollary 3.6 and Theorem 3.10). Making use of such *extension* theorems, we can easily deal with the singular set, in order to study the Bernstein problem. Namely, we study entire p-minimal graphs (a graph or a solution is called entire if it is defined on the whole xy-plane). The following are two families of such examples (cf. [Pau]):

$$u = ax + by + c \text{ (a plane with } a, b, c \text{ being real constants)}; \qquad (1.1)$$

$$u = -abx^2 + (a^2 - b^2)xy + aby^2 + g(-bx + ay) \qquad (1.2)$$

$$(a, b \text{ being real constants such that } a^2 + b^2 = 1 \text{ and } g \in C^2)\,.$$

We have the following classification result (see Section 4).

Theorem A. (1.1) *and* (1.2) *are the only entire* C^2 *smooth solutions to the p-minimal graph equation* (pMGE).

To prove Theorem A, we analyze the characteristic curves and the singular set of a solution for the case $H = 0$. Observe that the characteristic curves are straight lines which intersect at singular points. Let $S(u)$ denote the singular set consisting of all points where $u_x - y = 0$ and $u_y + x = 0$. Let $N(u)$ denote the xy-plane projection of the negative Legendrian normal $-e_2$. It follows that $N(u) = (u_x - y, u_y + x)/D$ where $D = \sqrt{(u_x - y)^2 + (u_y + x)^2}$. On the nonsingular domain, (pMGE) has the form

$$\operatorname{div} N(u) (= H) = 0\,,$$

where div denotes the xy-plane divergence. The following result gives a local description of the singular set (see Section 3).

Theorem B. *Let Ω be a domain in the xy-plane. Let $u \in C^2(\Omega)$ be such that* $\operatorname{div} N(u) = H$ *in* $\Omega \backslash S(u)$. *Suppose* $|H| \leq C\frac{1}{r}$ *near a singular point* $p_0 \in S(u)$ *where* $r(p) = |p - p_0|$ *for* $p \in \Omega$ *and C is a positive constant. Then either p_0 is isolated in $S(u)$ or there exists a small neighborhood of p_0 which intersects with $S(u)$ in exactly a C^1 smooth curve through p_0.*

We show that the restriction on H is necessary by giving a C^∞ smooth counterexample. In additon the blow-up rate $H = C\frac{1}{r}$ is realized by some natural examples (see Section 3). Theorem B follows from a characterization for a singular point to be non-isolated (see Theorem 3.3). We remark that the local result Theorem 3.3 and hence Theorem B continue to hold when the background Heisenberg group is replaced by a general pseudohermitian manifold (see the arguments before the proof of Theorem E in Section 7).

When two characteristic lines meet at a point of a singular curve, they must form a straight line (see Lemma 4.4). So we can describe all possible configurations of characteristic lines as if singular curves are not there. It turns out that there are only two possible configurations of characteristic lines. Either all characteristic lines intersect at one singular point or they are all parallel. In the former case, we are led to the solution (1.1) while for the latter case, (1.2) is the only possible solution.

The characteristic curves on a p-minimal surface are the Legendrian geodesics (see (2.1)). Since the Legendrian geodesics in H_1 are nothing but straight lines, a general complete p-minimal surface is a complete ruled surface generated by Legendrian rulings. We will discuss this and point out that a known complete embedded non-planar p-minimal surface has no singular points (characteristic points in the terminology of some other authors) after (4.10) in Section 4.

Since (pMGE) is also a degenerate elliptic partial differential equation, one can use non-degenerate elliptic equations to approximate it. With this regularization method, Pauls ([Pau]) obtained a $W^{1,p}$ Dirichlet solution and showed that such surfaces are the X-minimal surfaces in the sense of Garofalo and Nhieu ([GN]). In general the solution to the Dirichlet problem may not be unique. However, we can still establish a uniqueness theorem by making use of a structural equality of "elliptic" type (Lemma 5.1). More generally we have the following comparison principle.

Theorem C. *For a bounded domain Ω in the xy-plane, let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy* $\operatorname{div} N(u) \geq \operatorname{div} N(v)$ *in* $\Omega \backslash S$ *and* $u \leq v$ *on* $\partial\Omega$ *where* $S = S(u) \cup S(v)$. *Suppose* $\mathcal{H}_1(\bar{S})$, *the 1-dimensional Hausdorff measure of $\bar{S}$, vanishes. Then* $u \leq v$ *in* Ω.

As an immediate consequence of Theorem C, we have the following uniqueness result for the Dirichlet problem of (pMGE) (see Section 5).

Corollary D. *For a bounded domain Ω in the xy-plane, let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy* $\operatorname{div} N(u) = \operatorname{div} N(v) = 0$ *in $\Omega \backslash S$ and $u = v$ on $\partial\Omega$ where $S = S(u) \cup S(v)$. Suppose $\mathcal{H}_1(\bar{S})$, the 1-dimensional Hausdorff measure of $\bar{S}$, vanishes. Then $u = v$ in Ω.*

We remark that the condition on $\mathcal{H}_1(\bar{S})$ in Corollary D is necessary. A counterexample is given in [Pau] with $\mathcal{H}_1(\bar{S}) \neq 0$. We generalize Theorem C to higher dimensions and for a class of general N (see Section 5). It is noticeable that we do not need the condition on the size control of the singular set for the higher dimensional version of Theorem C (see Theorem C′).

We also study closed p-minimal surfaces in the standard pseudohermitian 3-sphere. A characteristic curve of such a p-minimal surface is part of a Legendrian great circle (see Lemma 7.1). Using this fact, we can describe the *extension* theorems (Corollary 3.6, Theorem 3.10 or Lemma 7.3) in terms of Legendrian great circles, and hence give a direct proof of the nonexistence of hyperbolic p-minimal surfaces embedded in the standard pseudohermitian 3-sphere (part of Corollary F). Then we generalize to the situation that the ambient pseudohermitian 3-manifold is arbitrary and the immersed surface has bounded p-mean curvature (see Section 7).

Theorem E. *Let M be a pseudohermitian 3-manifold. Let Σ be a closed, connected surface, C^2 smoothly immersed in M with bounded p-mean curvature. Then the genus of Σ is less than or equal to 1. In particular, there are no constant p-mean curvature or p-minimal surfaces Σ of genus greater than one in M.*

The idea of the proof for Theorem E goes as follows. A pseudohermitian manifold can be approximated locally by the Heisenberg group using the normal coordinates in the sense of Jerison and Lee ([JL2]). So we can extend the theory of singular set in Section 3 to the general situation (see Section 7). We can then apply (generalized) Lemma 3.8 to conclude that the characteristic line field has index $+1$ at every isolated singular point. Therefore the Euler characteristic number (equals the index sum), hence the genus, of Σ has constraint in view of the Hopf index theorem for a line field.

Corollary F. *There are no closed, connected, C^2 smoothly immersed constant p-mean curvature or p-minimal surfaces of genus ≥ 2 in the standard pseudohermitian 3-sphere.*

Note that in the standard Euclidean 3-sphere, there exist many closed C^∞ smoothly embedded minimal surfaces of genus ≥ 2 ([La]).

On a surface in a pseudohermitian 3-manifold, we define an operator, called the tangential sublaplacian. The p-mean curvature is related to this operator acting on coordinate functions (see (2.19a), (2.19b), (2.19c)) for a graph in H_1. We therefore obtain a "normal form" (see (2.20)) of (pMGE). We also interpret the notion of p-mean curvature in terms of calibration geometry. From this we deduce the area-minimizing property for p-minimal surfaces (see Proposition 6.2). Since the second variation formula is important for later development, we derive it and discuss the stability of a p-minimal surface in Sections 6 and 7.

We remark that, in the preprint ([GP1]), the authors claim the vertical planes are the only complete p-minimal graphs having no singular points (non-characteristic complete minimal graphs in their terminology). This is faulty. For instance, $y = xz$ is a complete (in fact, entire) p-minimal graph over the xz-plane and has no singular points. In [CH], two of us classify all the entire p-minimal graphs over any plane among other things. After our paper was completed, we were informed that the above claim had been corrected and some similar results are also obtained in the new preprint ([GP2]).

ACKNOWLEDGMENTS. We would like to thank Ai-Nung Wang for informative discussion of Monge's ([Mo]) third order equation for ruled surfaces (see Section 4). The first author would also like to thank Yng-Ing Lee for showing him some basic facts in calibration geometry. We began this research during the first author's visit at the IAS, Princeton in the 2001-2002 academic year. He would therefore like to thank the faculty and staff there as their hospitality has greatly facilitated this collaboration.

2. Surfaces in a 3-dimensional pseudohermitian manifold

Let (M, J, Θ) be a 3-dimensional oriented pseudohermitian manifold with a CR structure J and a global contact form Θ (see the Appendix). Let Σ be a surface contained in M. The singular set S_Σ consists of those points where the contact bundle $\xi \equiv \ker \Theta$ coincides with the tangent bundle $T\Sigma$ of Σ. It is easy to see that S_Σ is a closed set. On the nonsingular (open) set $\Sigma \backslash S_\Sigma$, we call the leaves of the 1-dimensional foliation $\xi \cap T\Sigma$ the *characteristic curves*. These curves will play an important role in our study. On ξ, we can associate a natural metric $G = \frac{1}{2} d\Theta(\cdot, J\cdot)$, called the Levi metric. For a vector $v \in \xi$, we define the length of v by $|v|_G = (G(v, v))^{1/2}$. With respect to this metric, we can take a unit vector field $e_1 \in \xi \cap T\Sigma$ on $\Sigma \backslash S_\Sigma$, called the characteristic field. Also associated to (J, Θ) is the so-called pseudohermitian connection, denoted as $\nabla^{p.h.}$ (see (A.2) in the Appendix). We can define a notion of mean curvature for Σ in this geometry as follows. Since $\nabla^{p.h.}$ preserves the Levi metric G, $\nabla^{p.h.} e_1$ is perpendicular to e_1 with respect to G. On the other hand, it is obvious that $G(e_1, e_2) = 0$ where $e_2 = Je_1$. We call e_2 the Legendrian normal or Gauss map. So we have

$$\nabla^{p.h.}_{e_1} e_1 = H e_2 \tag{2.1}$$

for some function H. We call H the p(pseudohermitian)-mean curvature of Σ. Note that if we change the sign of e_1, then e_2 and H change signs accordingly. If $H = 0$, we call Σ a p-minimal surface. In this situation the characteristic curves are nothing but Legendrian (i.e., tangent to ξ) geodesics with respect to the pseudohermitian connection.

We are going to give a variational formulation for the p-mean curvature H. First let us find a candidate area integral. Suppose Ω is a smooth domain in M with boundary $\partial\Omega = \Sigma$. Consider $V(\Omega)$, the volume of Ω, given by

$$V(\Omega) = \frac{1}{2}\int_{\Omega} \Theta \wedge d\Theta$$

($\frac{1}{2}$ is a normalization constant. For $\Omega \subset H_1$, this volume is just the usual Euclidean volume). Take Legendrian fields $e_1, e_2 = Je_1 \in \xi$, orthonormal with respect to G, wherever defined in a neighborhood of Σ (note that we do not require e_1 to be characteristic, i.e. tangent along Σ here). We consider a variation of the surface Σ in the direction fe_2 where f is a suitable function with compact support in $\Sigma \backslash S_\Sigma$. The vector field fe_2 generates a flow φ_t for t close to 0. We compute

$$\begin{aligned} \delta_{fe_2} V(\Omega) &= \left.\frac{d}{dt}\right|_{t=0} V(\varphi_t(\Omega)) = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \int_{\Omega} \varphi_t^*(\Theta \wedge d\Theta) \\ &= \frac{1}{2}\int_{\Omega} L_{fe_2}(\Theta \wedge d\Theta)\,. \end{aligned} \tag{2.2}$$

It follows from the formula $L_X = d \circ i_X + i_X \circ d$ (i_X denotes the interior product in the direction X) that

$$L_{fe_2}(\Theta \wedge d\Theta) = d \circ i_{fe_2}(\Theta \wedge d\Theta)\,. \tag{2.3}$$

Substituting (2.3) in (2.2) and making use of Stokes' theorem, we obtain

$$\delta_{fe_2} V(\Omega) = \frac{1}{2}\int_{\Sigma} i_{fe_2}(\Theta \wedge d\Theta) = \int_{\Sigma} f\Theta \wedge e^1\,. \tag{2.4}$$

Here e^1 together with e^2, Θ form a dual basis of (e_1, e_2, T) where T is the Reeb vector field (uniquely determined by $\Theta(T) = 1$ and $i_T d\Theta = 0$). Note that $d\Theta = 2e^1 \wedge e^2$ (see (A.1r)). For e_1 being a characteristic field, we define the p-area of a surface Σ to be the surface integral of the 2-form $\Theta \wedge e^1$:

$$\text{p-Area } (\Sigma) = \int_{\Sigma} \Theta \wedge e^1\,. \tag{2.5}$$

Note that $\Theta \wedge e^1$ continuously extends over the singular set S_Σ and vanishes on S_Σ. In fact, we can write e^1 with respect to a dual orthonormal basis $\{\hat{e}^1, \hat{e}^2\}$ of ξ, which is smooth near a singular point, say p_0, as follows: $e^1 = \cos\beta\hat{e}^1 + \sin\beta\hat{e}^2$. Here β may not be continuous at p_0. Now $\Theta \wedge \hat{e}^1$ and $\Theta \wedge \hat{e}^2$ tend to 0 on Σ as $p \in \Sigma$ tends to p_0 since Θ vanishes on $T_{p_0}\Sigma = \xi_{p_0}$. It follows that $\Theta \wedge e^1$ tends to 0 on Σ as $p \in \Sigma$ tends to p_0 since $\cos\beta$ and $\sin\beta$ are bounded by 1.

We can recover the p-mean curvature H from the first variation formula of the p-area functional (2.5). We compute

$$\delta_{fe_2}\int_\Sigma \Theta\wedge e^1 = \int_\Sigma L_{fe_2}(\Theta\wedge e^1) = \int_\Sigma i_{fe_2}\circ d(\Theta\wedge e^1)\,. \tag{2.6}$$

Here we have used the formula $L_X = d\circ i_X + i_X\circ d$ and the condition that f is a function with compact support away from the singular set and the boundary of Σ. From the equations $d\Theta = 2e^1\wedge e^2$ and $de^1 = -e^2\wedge\omega \bmod \Theta$ (see (A.1r), (A.3r)), we compute

$$d(\Theta\wedge e^1) = d\Theta\wedge e^1 - \Theta\wedge de^1 = \Theta\wedge e^2\wedge\omega\,. \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain by the definition of the interior product that

$$\begin{aligned}\delta_{fe_2}\int_\Sigma \Theta\wedge e^1 &= \int_\Sigma f(-\Theta\wedge\omega + \omega(e_2)\Theta\wedge e^2)\\ &= \int_\Sigma -f\omega(e_1)\Theta\wedge e^1\\ &\qquad (\Theta\wedge e^2 = 0 \text{ on } \Sigma \text{ since } e_1 \text{ is tangent along } \Sigma)\\ &= -\int_\Sigma fH\Theta\wedge e^1\,.\end{aligned} \tag{2.8}$$

In the last equality, we have used the fact that $H = \omega(e_1)$ (obtained by comparing (2.1) with (A.2r)). Similarly we can also compute the first variation of (2.5) with respect to the field gT where g is a function with compact support away from the singular set and the boundary of Σ. Together with (2.8), the result reads

$$\delta_{fe_2+gT}\int_\Sigma \Theta\wedge e^1 = -\int_\Sigma (f-\alpha g)H\Theta\wedge e^1\,. \tag{2.8$'$}$$

Here we define the function α on $\Sigma\backslash S_\Sigma$ such that $\alpha e_2 + T\in T\Sigma$. We leave the deduction of (2.8$'$) to the reader. Let ψ be a defining function of Σ. It follows that the subgradient $\nabla_b\psi = (e_1\psi)e_1 + (e_2\psi)e_2 = (e_2\psi)e_2$ since $e_1\in T\Sigma$, hence $e_1\psi = 0$. So $\nabla_b\psi/|\nabla_b\psi|_G = e_2$ (change the sign of ψ if necessary). Next we compute the subdivergence of e_2. Since $\nabla^{p.h.}$ preserves the Levi metric G, we can easily obtain $G(\nabla^{p.h.}e_2, e_2) = 0$ and $G(\nabla^{p.h.}_{e_1}e_2, e_1) = -G(e_2, \nabla^{p.h.}_{e_1}e_1) = -H$ by (2.1). Therefore $\mathrm{div}_b(e_2)\equiv G(\nabla^{p.h.}_{e_1}e_2, e_1) + G(\nabla^{p.h.}_{e_2}e_2, e_2) = -H$. We have derived the p-mean curvature equation (pMCE) and the p-minimal surface equation (pMSE):

$$-\mathrm{div}_b(\nabla_b\psi/|\nabla_b\psi|_G) = H \tag{pMCE}$$

and

$$\mathrm{div}_b(\nabla_b\psi/|\nabla_b\psi|_G) = 0\,, \tag{pMSE}$$

respectively.

For a graph $(x, y, u(x, y))$ in the 3-dimensional Heisenberg group H_1, we can take $\psi = z - u(x, y)$. Then (at a nonsingular point) up to sign, e_1 is uniquely determined by the following equations:

$$\Theta_0 \equiv dz + xdy - ydx = 0\,, \tag{2.9a}$$

$$d\psi = d(z - u(x, y)) = 0\,, \tag{2.9b}$$

$$G(e_1, e_1) = \frac{1}{2} d\Theta_0(e_1, Je_1) = 1\,. \tag{2.9c}$$

Using (2.9a), (2.9c), we can write $e_1 = f\hat{e}_1 + g\hat{e}_2$ with $f^2 + g^2 = 1$, in which $\hat{e}_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$, $\hat{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ are standard left-invariant Legendrian vector fields in H_1 (see the Appendix). Applying (2.9b) to this expression, we obtain $(u_x - y)f + (u_y + x)g = 0$. So $(f, g) = \pm(-(u_y + x), u_x - y)/[(u_x - y)^2 + (u_y + x)^2]^{1/2}$ (positive sign so that $\nabla_b\psi/|\nabla_b\psi|_G = e_2 = -[(u_x - y)\hat{e}_1 + (u_y + x)\hat{e}_2]/D$ where $D = [(u_x - y)^2 + (u_y + x)^2]^{1/2}$). Now from (pMCE) we obtain a formula for H through a direct computation:

$$H = D^{-3}\{(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy}\}\,. \tag{2.10}$$

At a nonsingular point, the equation (pMSE) reduces to the p-minimal graph equation (pMGE):

$$(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy} = 0\,. \tag{pMGE}$$

In fact, if u is C^2 smooth, the p-mean curvature H in (2.10) vanishes on the nonsingular domain (where $D \neq 0$) if and only if (pMGE) holds on the whole domain. We can also compute $e^1 = D^{-1}\{-(u_y + x)dx + (u_x - y)dy\}$ and express the p-area 2-form as follows:

$$\Theta \wedge e^1 = Ddx \wedge dy = [(u_x - y)^2 + (u_y + x)^2]^{1/2} dx \wedge dy\,. \tag{2.11}$$

At a singular point, the contact form Θ is proportional to $d\psi$ (see (2.9a), (2.9b)). Therefore $u_x - y = 0$, $u_y + x = 0$ describe the xy-plane projection $S(u)$ of the singular set S_Σ:

$$S(u) = \{(x, y) \in R^2 : u_x - y = 0, u_y + x = 0\}\,. \tag{S}$$

From (2.11) we see that the p-area form $\Theta \wedge e^1$ is degenerate on $S(u)$ or S_Σ.

Let e_2 be the Legendrian normal of a family of deformed surfaces foliating a neighborhood of Σ. We define the tangential subgradient ∇_b^t of a function f defined near Σ by the formula: $\nabla_b^t f = \nabla_b f - G(\nabla_b f, e_2)e_2$ (see (A.8) for the definition of $\nabla_b f$) and the tangential pseudohermitian connection $\nabla^{t.p.h.}$ of a Legendrian (i.e. in ξ) vector field X by $\nabla^{t.p.h.}X = \nabla^{p.h.}X - G(\nabla^{p.h.}X, e_2)e_2$. Then we define the tangential sublaplacian Δ_b^t of f by

$$\Delta_b^t f = \mathrm{div}_b^t(\nabla_b^t f) \tag{2.12}$$

where $\mathrm{div}_b^t(X)$ is defined to be the trace of $\nabla^{t.p.h.}X$ considered as an endomorphism of $\xi : v \to \nabla_v^{t.p.h.}X$. Now for an orthonormal basis e_1, e_2 of ξ with respect to G, we have

$$\mathrm{div}_b^t(X) = G(\nabla_{e_1}^{t.p.h.}X, e_1) + G(\nabla_{e_2}^{t.p.h.}X, e_2) = G(\nabla_{e_1}^{t.p.h.}X, e_1) \tag{2.13}$$

since $\nabla_{e_2}^{t.p.h.}X$ is proportional to e_1. On the other hand, we write $\nabla_b f = (e_1 f)e_1 + (e_2 f)e_2$ (cf. (A.8r)). It follows that $\nabla_b^t f = (e_1 f)e_1$. Setting $X = \nabla_b^t f = (e_1 f)e_1$ in (2.13) and noting that $\nabla_{e_1}^{t.p.h.}e_1 = 0$ by (2.1) gives $\mathrm{div}_b^t(\nabla_b^t f) = (e_1)^2 f + (e_1 f)G(\nabla_{e_1}^{t.p.h.}e_1, e_1) = (e_1)^2 f$. Substituting this in (2.12), we obtain

$$\Delta_b^t f = (e_1)^2 f\,. \tag{2.14}$$

Note that (2.14) holds for a general surface Σ contained in an arbitrary pseudohermitian 3-manifold. We also note that $\Delta_b^t + 2\alpha e_1$ is self-adjoint with respect to the p-area form $\Theta \wedge e^1$ as shown by the following integral formula:

$$\int_\Sigma [f\Delta_b^t g - g\Delta_b^t f]\Theta \wedge e^1 = 2\int_\Sigma [g(e_1 f) - f(e_1 g)]\alpha\Theta \wedge e^1$$

for smooth functions f, g with compact support away from the singular set. The proof is left to the reader (Hint: observe that the adjoint of e_1 is $-e_1 - 2\alpha$ by noting that $\Theta \wedge e^2 = 0$ and $e^1 \wedge e^2 = \alpha e^1 \wedge \Theta$ on the surface). When Σ is a graph $(x, y, u(x, y))$ in H_1, we can relate $(e_1)^2 f$ for $f = x, y$, or u to the p-mean curvature H. Denote the projection of $-e_2$ ($-e_1$, respectively) onto the xy-plane by $N(u)$ or simply N ($N^\perp(u)$ or simply $N^\perp$, respectively). Recall that $e_1 = [-(u_y + x)\hat{e}_1 + (u_x - y)\hat{e}_2]/D$ where $D = [(u_x - y)^2 + (u_y + x)^2]^{1/2}$ (see the paragraph after (2.9)). So $N^\perp = [(u_y + x)\partial_x - (u_x - y)\partial_y]/D$. Write $(u_y + x)D^{-1} = \sin\theta$, $(u_x - y)D^{-1} = \cos\theta$ for some local function θ. Then we can write

$$N = (\cos\theta)\partial_x + (\sin\theta)\partial_y\,, \tag{2.15a}$$

$$N^\perp = (\sin\theta)\partial_x - (\cos\theta)\partial_y\,. \tag{2.15b}$$

First note that from (pMCE) we can express the p-mean curvature H as follows:

$$H = \mathrm{div}\,N = (\cos\theta)_x + (\sin\theta)_y = -(\sin\theta)\theta_x + (\cos\theta)\theta_y\,. \tag{2.16}$$

Now starting from (2.15b) and using (2.16), we can deduce

$$\begin{aligned}(N^\perp)^2 = {} & \sin^2\theta\partial_x^2 - 2\sin\theta\cos\theta\partial_x\partial_y + \cos^2\theta\partial_y^2 \\ & - (\cos\theta)H\partial_x - (\sin\theta)H\partial_y\,.\end{aligned} \tag{2.17}$$

On the other hand, we can write (2.10) in the following form:

$$H = D^{-1}(\sin^2\theta\partial_x^2 u - 2\sin\theta\cos\theta\partial_x\partial_y u + \cos^2\theta\partial_y^2 u)\,. \tag{2.18}$$

Applying (2.17) to $x, y, u(x, y)$, respectively and making use of (2.18), we obtain

$$\Delta_b^t x = (e_1)^2 x = (N^\perp)^2 x = -(\cos\theta)H\,, \tag{2.19a}$$

$$\Delta_b^t y = (e_1)^2 y = (N^\perp)^2 y = -(\sin\theta)H\,, \tag{2.19b}$$

$$\begin{aligned}\Delta_b^t u &= (e_1)^2 u = (N^\perp)^2 u = D^{-1}(xu_y - yu_x + x^2 + y^2)H \\ &= H(x\sin\theta - y\cos\theta) = H\det\begin{bmatrix} x & y \\ \cos\theta & \sin\theta \end{bmatrix}\end{aligned} \tag{2.19c}$$

(here "det" means *determinant*). Formula (2.19c) gives the following *normal form* of (pMGE):

Lemma 2.1. *For a C^2 smooth p-minimal graph $u = u(x, y)$ in the 3-dimensional Heisenberg group H_1, we have the equation*

$$\Delta_b^t u = (e_1)^2 u = (N^\perp)^2 u = 0 \tag{2.20}$$

on the nonsingular domain.

Also from (2.19a) and (2.19b), we have $\Delta_b^t x = \Delta_b^t y = 0$ on a p-minimal graph $\Sigma = \{(x, y, u(x, y))\}$. Together with (2.20), we have $\Delta_b^t(x, y, z) \equiv (\Delta_b^t x, \Delta_b^t y, \Delta_b^t z) = (0, 0, 0)$ (i.e., Δ_b^t annihilates the coordinate functions) on Σ. This is a property analogous to that for (Euclidean) minimal surfaces in $\mathbb{R}^3$ ([Os]). In general, we have

$$\Delta_b^t(x, y, z) = He_2\,.$$

We will often call the xy-plane projection of characteristic curves for a graph $(x, y, u(x, y))$ in H_1 still characteristic curves if no confusion occurs. Note that the integral curves of $N^\perp$ are just the xy-plane projection of integral curves of e_1. So they are characteristic curves. Along a characteristic curve $(x(s), y(s))$ where s is a unit-speed parameter, we have the equations

$$\frac{dx}{ds} = \sin\theta, \quad \frac{dy}{ds} = -\cos\theta \tag{2.21}$$

by (2.15b). Noting that $u_x = (\cos\theta)D + y$, $u_y = (\sin\theta)D - x$, we compute

$$\begin{aligned}\frac{du}{ds} &= u_x\frac{dx}{ds} + u_y\frac{dy}{ds} = [(\cos\theta)D + y]\sin\theta + [(\sin\theta)D - x](-\cos\theta) \\ &= x\cos\theta + y\sin\theta\,,\end{aligned} \tag{2.22}$$

$$\frac{d\theta}{ds} = \theta_x\frac{dx}{ds} + \theta_y\frac{dy}{ds} = \theta_x\sin\theta - \theta_y\cos\theta = -H \tag{2.23}$$

by (2.16). In general, we can consider H as a function of x, y, u, θ in view of the O.D.E. system (2.21), (2.22) and (2.23). From (2.21) and (2.23), we compute

$$\frac{d^2x}{ds^2} = -H\cos\theta, \quad \frac{d^2y}{ds^2} = -H\sin\theta\,. \tag{2.24}$$

Observe that $(\cos\theta, \sin\theta)$ is the unit normal to the unit tangent $(\frac{dx}{ds}, \frac{dy}{ds}) = (\sin\theta, -\cos\theta)$. So $-H$ is just the curvature of a characteristic curve. In particular, when $H = 0$, characteristic curves are nothing but straight lines or line segments (see Proposition 4.1 and Corollary 4.2).

3. The singular set-proof of Theorem B

Let Ω be a domain (connected open subset) in the xy-plane, and let $u \in C^2(\Omega)$. Let $\Sigma = \{(x, y, u(x, y)) | (x, y) \in \Omega\} \subset H_1$. In this section, we want to analyze $S(u)$ (still called the singular set), the xy-plane projection of the singular set S_Σ, for the graph Σ.

First for $a, b \in R$, $a^2 + b^2 = 1$, we define $F_{a,b} \equiv a(u_x - y) + b(u_y + x)$ and $\Gamma_{a,b} \equiv \{(x, y) \in \Omega \mid F_{a,b}(x, y) = 0\}$. Then it is easy to see that $\Gamma_{a,b} = S(u) \cup \{(x, y) \in \Omega \mid N(u)(x, y) = \pm(b, -a)\}$. Let

$$U = \begin{bmatrix} u_{xx} & u_{xy} - 1 \\ u_{xy} + 1 & u_{yy} \end{bmatrix}. \tag{U}$$

Lemma 3.1. *Let $u \in C^2(\Omega)$ and let $p_0 \in S(u) \subset \Omega$. Then there exists a small neighborhood of p_0, whose intersection with $S(u)$ is contained in a C^1 smooth curve.*

Proof. Compute the gradient of $F_{a,b}$:

$$\begin{aligned} \nabla F_{a,b} &= (au_{xx} + b(u_{xy} + 1), a(u_{xy} - 1) + bu_{yy}) \\ &= (a, b)\begin{bmatrix} u_{xx} & u_{xy} - 1 \\ u_{xy} + 1 & u_{yy} \end{bmatrix} = (a, b)U\,. \end{aligned} \tag{3.1}$$

Note that U is never a zero matrix since $u_{xy}+1$ and $u_{xy}-1$ can never be zero simultaneously. So there exists at most one unit eigenvector (a_0, b_0) of eigenvalue 0 up to a sign for U at p_0. For $(a, b) \neq \pm(a_0, b_0)$, $\nabla F_{a,b} \neq 0$ at p_0. Then by the implicit function theorem, there exists a small neighborhood of p_0, in which $\Gamma_{a,b}$ at least for $(a, b) \neq \pm(a_0, b_0)$ forms a C^1 smooth curve. On the other hand, it is obvious that $S(u)$ is contained in $\Gamma_{a,b}$. We are done. □

Lemma 3.2. *Let $u \in C^2(\Omega)$ and suppose $p_0 \in S(u) \subset \Omega$ is not isolated, i.e., there exists a sequence of distinct points $p_j \in S(u)$ approaching p_0. Then* $\det U(p_0)$ *is zero.*

Proof. Let $\Gamma_{a,b}$ be the C^1 smooth curve as in the proof of Lemma 3.1. Since $\Gamma_{a,b}$ is C^1 smooth near p_0, we can take a parameter s of unit speed for $\Gamma_{a,b}$ near p_0, and find a subsequence of p_j, still denoted as p_j, $p_j = (x, y)(s_j)$, $p_0 = (x, y)(s_0)$ such that s_j tends to s_0 monotonically. Since $(u_x - y)(p_j) = (u_x - y)(x, y)(s_j) = 0$ for all j, there exists $\tilde{s}_j$ between s_j and s_{j+1} such that $d(u_x - y)/ds = 0$ at $\tilde{s}_j$. So by the chain rule we obtain

$$u_{xx}\frac{dx}{ds} + (u_{xy} - 1)\frac{dy}{ds} = 0 \tag{3.2a}$$

at $\tilde{s}_j$ and at s_0 by letting $\tilde{s}_j$ go to s_0. Starting from $(u_y + x)(p_j) = (u_y + x)(x, y)(s_j) = 0$, we also obtain by a similar argument that

$$(u_{xy} + 1)\frac{dx}{ds} + u_{yy}\frac{dy}{ds} = 0 \tag{3.2b}$$

at s_0. But $(\frac{dx}{ds}, \frac{dy}{ds}) \neq 0$ since s is a unit-speed parameter. It follows from (3.2a), (3.2b) that $\det U(p_0) = 0$. □

Note that we do not assume any condition on H in Lemma 3.1 and Lemma 3.2. If we make a restriction on H, we can obtain the converse of Lemma 3.2.

Theorem 3.3. *Let Ω be a domain in the xy- plane. Let $u \in C^2(\Omega)$ be such that* $\operatorname{div} N(u) = H$ *in $\Omega \backslash S(u)$. Suppose $|H| \leq C\frac{1}{r}$ near a singular point $p_0 \in S(u)$ where $r(p) = |p - p_0|$ for $p \in \Omega$ and C is a positive constant. Then the following are equivalent*:

(1) *p_0 is not isolated in $S(u)$,*
(2) $\det U(p_0) = 0$,
(3) *there exists a small neighborhood of p_0 which intersects with $S(u)$ in exactly a C^1 smooth curve through p_0.*

Proof. (1)⇒(2) by Lemma 3.2. (3)⇒(1) is obvious. It suffices to show that (2)⇒(3). Suppose $\det U(p_0) = 0$. Note that $U(p_0) \neq 0$-matrix since the off-diagonal terms $u_{xy} + 1$ and $u_{xy} - 1$ in (U) can never be zero simultaneously. Therefore $\operatorname{rank}(U(p_0)) = 1$, and there exists a unique (a_0, b_0), up to sign, such that $(a_0, b_0)U = 0$ at p_0. So for any $(a, b) \neq \pm(a_0, b_0)$, $F_{a,b} \equiv a(u_x - y) + b(u_y + x)$ satisfies $\nabla F_{a,b} \neq 0$ at p_0 and for $(a_1, b_1) \neq \pm(a_0, b_0)$, $(a_2, b_2) \neq \pm(a_0, b_0)$ and $(a_1, b_1) \neq \pm(a_2, b_2)$

$$\nabla F_{a_1,b_1} = c\nabla F_{a_2,b_2} \tag{3.3}$$

at p_0 where c is a nonzero constant. Therefore Γ_{a_1,b_1} and Γ_{a_2,b_2} are C^1 smooth curves in a neighborhood of p_0 (recall that $\Gamma_{a,b}$ is defined by $F_{a,b} = 0$). Also Γ_{a_1,b_1} and Γ_{a_2,b_2} are tangent at p_0. Thus we can take unit-speed arc-length parameters s, t for Γ_{a_1,b_1}, Γ_{a_2,b_2} described by $\gamma_1(s)$, $\gamma_2(t)$, respectively, so that $\gamma_1(0) = \gamma_2(0) = p_0$ and $\gamma_1'(0) = \gamma_2'(0)$. Observe that

$$(r\circ\gamma_1)'(0+) = (r\circ\gamma_2)'(0+) = 1$$
$$(r\circ\gamma_1)'(0-) = (r\circ\gamma_2)'(0-) = -1\,.$$

Therefore we can find a small $\epsilon > 0$ such that $(r\circ\gamma_1)'(s), (r\circ\gamma_2)'(t) > 0$ for $s, t > 0$ ($(r\circ\gamma_1)'(s), (r\circ\gamma_2)'(t) < 0$ for $s, t < 0$, respectively) whenever $\gamma_1(s), \gamma_2(t) \in B_\epsilon(p_0)$, a ball of radius ϵ and center p_0. Also note that $S(u) \cap B_\epsilon(p_0) \subset \Gamma_{a_i,b_i}, i = 1, 2$. We can write

$$(\Gamma_{a_1,b_1} \cap B_\epsilon(p_0))\backslash S(u) = \cup_{j=1}^{\infty}\gamma_1(]s_j^+, \tilde{s}_j^+[) \cup \cup_{j=1}^{\infty}\gamma_1(]s_j^-, \tilde{s}_j^-[)$$
$$(\Gamma_{a_2,b_2} \cap B_\epsilon(p_0))\backslash S(u) = \cup_{j=1}^{\infty}\gamma_2(]t_j^+, \tilde{t}_j^+[) \cup \cup_{j=1}^{\infty}\gamma_2(]t_j^-, \tilde{t}_j^-[)$$

where $]s_j^+, \tilde{s}_j^+[,]s_j^-, \tilde{s}_j^-[$, etc. denote open intervals and we have arranged

$$\begin{aligned} s_1^- < \tilde{s}_1^- \le \ldots s_j^- < \tilde{s}_j^- \le s_{j+1}^- < \tilde{s}_{j+1}^- \le \ldots \\ \le 0 \le \ldots \le s_{j+1}^+ < \tilde{s}_{j+1}^+ \le s_j^+ < \tilde{s}_j^+ \le \ldots \le s_1^+ < \tilde{s}_1^+ \\ t_1^- < \tilde{t}_1^- \le \ldots t_j^- < \tilde{t}_j^- \le t_{j+1}^- < \tilde{t}_{j+1}^- \le \ldots \\ \le 0 \le \ldots \le t_{j+1}^+ < \tilde{t}_{j+1}^+ \le t_j^+ < \tilde{t}_j^+ \le \ldots \le t_1^+ < \tilde{t}_1^+ \end{aligned}$$

(if there are only finite number of open intervals for negative (positive, respectively) s, then $\tilde{s}_j^- = s_j^- = \tilde{s}_m^-$ ($s_j^+ = \tilde{s}_j^+ = s_m^+$, respectively) for $j \ge m$, a certain integer, by convention. Note that $\tilde{s}_j^-$ or s_j^+ may equal 0 for some finite j. In this case, all the s with subindex larger than j equal 0. We apply the same convention to the parameter t). Since $r\circ\gamma_1$ and $r\circ\gamma_2$ are monotonic (increasing for positive parameters and decreasing for negative parameters), we actually have $\gamma_1(s_j^\pm) = \gamma_2(t_j^\pm)$, $\gamma_1(\tilde{s}_j^\pm) = \gamma_2(\tilde{t}_j^\pm)$ except possibly $\gamma_1(s_1^-) \ne \gamma_2(t_1^-)$ or $\gamma_1(\tilde{s}_1^+) \ne \gamma_2(\tilde{t}_1^+)$. Let $e_j = \gamma_1(s_j^+)$ and $\tilde{e}_j = \gamma_1(\tilde{s}_j^+)$. Then either Γ_{a_1,b_1} and Γ_{a_2,b_2} meet at $\tilde{e}_1$ or $\tilde{e}_2$, or they do not meet in $B_\epsilon(p_0)\backslash\{p_0\}$ for positive parameters. In the former situation, we need to show that e_i, $i \ge 1$, does not converge to p_0 as $i \to \infty$ (then there is a smaller ball $B_{\epsilon'}(p_0)$ such that $S(u) \cap B_{\epsilon'}(p_0)$ contains $\gamma_1([0, \bar{s}^+[)$ for a small $\bar{s}^+ > 0$). Suppose it is not so. Let Ω_i be the region surrounded by Γ_{a_1,b_1} and Γ_{a_2,b_2} from e_i to $\tilde{e}_i$ for all $i \ge 1$ or 2 (if $\gamma_1(\tilde{s}_1^+) \ne \gamma_2(\tilde{t}_1^+)$) (note that the curves Γ_{a_1,b_1} and Γ_{a_2,b_2} only meet at singular points since $(a_1, b_1) \ne \pm(a_2, b_2)$, i.e., points of the arcs $\gamma_1([\tilde{s}_{i+1}^+, s_i^+]) = \gamma_2([\tilde{t}_{i+1}^+, t_i^+])$.) Observe that Γ_{a_1,b_1} and Γ_{a_2,b_2} asymptotically approximate the same straight line by (3.3). So the distance function $r(p) \equiv |p - p_0|$ is one-to-one for $p \in \Gamma_{a_1,b_1}(\Gamma_{a_2,b_2}$, respectively) near p_0 with parameter $s > 0$ ($t > 0$, respectively) since $(r\circ\gamma_1)'(s), (r\circ\gamma_2)'(t) > 0$ for $s, t > 0$ as shown previously. Now we want to compare both sides of

$$\oint_{\partial\Omega_i} N(u)\cdot\nu ds = \iint_{\Omega_i} \operatorname{div} N(u) dxdy = \iint_{\Omega_i} H dxdy \tag{3.4}$$

where ν is the unit outward normal to Γ_{a_1,b_1} and Γ_{a_2,b_2}. On $\Gamma_{a_1,b_1}(\Gamma_{a_2,b_2}$, respectively), $N(u) \perp (a_1, b_1)$ ((a_2, b_2), respectively). So $N(u)$ is a constant unit vector field along Γ_{a_1,b_1} (Γ_{a_2,b_2}, respectively). On the other hand, ν approaches a fixed unit vector $\nu_0 = \nabla F_{a_1,b_1}(p_0)/|\nabla F_{a_1,b_1}(p_0)| = \nabla F_{a_2,b_2}(p_0)/|\nabla F_{a_2,b_2}(p_0)|$

as $e_i, \tilde{e}_i$ tend to p_0 for i large. It follows that $N(u) \cdot \nu$ tends to a constant $c_1 = \pm(-b_1, a_1) \cdot \nu_0$ ($c_2 = \pm(-b_2, a_2) \cdot \nu_0$, respectively) along $\Gamma_{a_1,b_1}(\Gamma_{a_2,b_2}$, respectively) as i goes to infinity. We choose (in advance) $(a_1, b_1) \neq \pm(a_2, b_2)$ (also both $\neq \pm(a_0, b_0)$) such that $c_1 \neq c_2$. Thus we can estimate

$$\left| \oint_{\partial\Omega_i} N(u) \cdot \nu ds \right| \geq \frac{| c_1 - c_2 |}{2} | r(\tilde{e}_i) - r(e_i) | \tag{3.5}$$

for large i. On the other hand, we can make Ω_i contained in a fan-shaped region of angle θ_i with vertex p_0 so that $\theta_i \to 0$ as $i \to \infty$. We estimate

$$\left| \iint_{\Omega_i} H dx dy \right| \leq \left| \int_{r(e_i)}^{r(\tilde{e}_i)} \left(\int_0^{\theta_i} \frac{C}{r} d\theta \right) r dr \right| \leq C\theta_i | r(\tilde{e}_i) - r(e_i) | . \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$\frac{| c_1 - c_2 |}{2} | r(\tilde{e}_i) - r(e_i) | \leq C\theta_i | r(\tilde{e}_i) - r(e_i) | .$$

Hence $\frac{|c_1 - c_2|}{2} \leq C\theta_i$. But θ_i tending to 0 gives $c_1 = c_2$, a contradiction. Another situation is that Γ_{a_1,b_1} and Γ_{a_2,b_2} do not meet in a small neighborhood of p_0 except at p_0 for $s, t > 0$. In this case, let Ω_i be the region surrounded by Γ_{a_1,b_1}, Γ_{a_2,b_2}, and $\partial B_{r_i}(p_0)$ for large i and contained in a fan-shaped region of angle θ_i with vertex p_0 so that $r_i \to 0$, $\theta_i \to 0$ as $i \to \infty$. Observing that the arc length of $\bar{\Omega}_i \cap \partial B_{r_i}(p_0)$ is bounded by $\theta_i r_i$, we can reach a contradiction by a similar argument as above. For $s, t < 0$, we apply a similar argument to conclude that there is a small $\epsilon'' > 0$ such that $S(u) \cap B_{\epsilon''}(p_0)$ contains $\gamma_1(]\bar{s}^-, 0])$ for a small $\bar{s}^- < 0$. Now an even smaller ball of radius $< \min(\epsilon', \epsilon'')$ and center p_0 will serve our purpose. □

Proof of Theorem B. It is an immediate consequence of Theorem 3.3. □

We remark that Theorem B does not hold if we remove the condition on H.

Example 1. Let $u = xg(y)$. Then $u_x = g(y), u_y = xg'(y)$. It follows that the singular set $S(u) = \{g(y) = y \text{ and } g'(y) + 1 = 0\} \cup \{g(y) = y \text{ and } x = 0\}$. Take $g(y) = \exp(-\frac{1}{y^2}) \sin(-\frac{1}{y}) + y$. Compute $g'(y) = 2\exp(-\frac{1}{y^2}) y^{-3} \sin(-\frac{1}{y}) + \exp(-\frac{1}{y^2}) y^{-2} \cos(-\frac{1}{y}) + 1$. So for y small, $g'(y) + 1 = 0$ has no solution. Therefore $S(u)$ (when y is small) $= \{g(y) = y, x = 0\} = \{\sin(-\frac{1}{y}) = 0, x = 0\}$ has infinitely-many points near $(0, 0)$. Note that g, hence u, is C^∞ smooth. This example shows that even for $u \in C^\infty$, $S(u)$ may contain non-isolated points which do not belong to curves in $S(u)$. On the other hand, the p-mean curvature $H(u)$ has a blow-up rate $\exp(\frac{1}{r^2})$ for a certain sequence of points (x_j, y_j) satisfying $x_j = \exp(-\frac{1}{y_j^2})$ and converging to $(0, 0)$.

Example 2. Let $u = \pm\frac{1}{2}(x^2 + y^2)$. Then $u_x = \pm x$, $u_y = \pm y$. So $S(u) = \{(0, 0)\}$. By (2.10) we compute the p-mean curvature $H = \pm 2^{-1/2} r^{-1}$ where $r = \sqrt{x^2 + y^2}$. This is the case that the equality of the condition on H holds.

Example 3. The following example shows that in Theorem 3.3, (2) does not imply (1) if we remove the condition on H. Let

$$u = \frac{1}{2}(x^2 - y^2) + \frac{1}{\beta}(sgn(x)|x|^\beta - sgn(y)|y|^\beta)$$

for $\beta > 2$. Compute $u_x = x + |x|^{\beta-1}$, $u_y = -y - |y|^{\beta-1}$. It follows that $(u_x - y) - (u_y + x) = |x|^{\beta-1} + |y|^{\beta-1}$. So $S(u) \subset \{|x|^{\beta-1} + |y|^{\beta-1} = 0\}$, and hence $S(u) = \{(0, 0)\}$. This means that $(0, 0)$ is an isolated singular point. Compute $u_{xx} = 1 + (\beta - 1)sgn(x)|x|^{\beta-2}$, $u_{xy} = 0$ and $u_{yy} = -1 - (\beta - 1)sgn(y)|y|^{\beta-2}$. It is easy to see that $|u_{xx}| \le \beta$ and $|u_{yy}| \le \beta$ for $|x| \le 1$, $|y| \le 1$. So by (2.10) we can estimate

$$H \le \frac{|u_{xx}| + |u_{yy}|}{D} \le \frac{\beta + \beta}{(|x|^{\beta-1} + |y|^{\beta-1})/\sqrt{2}} \le C\frac{1}{r^{\beta-1}}$$

near $(0, 0)$ where $r = \sqrt{x^2 + y^2}$. In the second inequality above, we have used the following estimate

$$\begin{aligned} D^2 &\equiv (u_x - y)^2 + (u_y + x)^2 = (x + |x|^{\beta-1} - y)^2 + (-y - |y|^{\beta-1} + x)^2 \\ &\ge \frac{1}{2}(|x|^{\beta-1} + |y|^{\beta-1})^2 \end{aligned}$$

(by $2(a^2 + b^2) \ge (a - b)^2$). On the other hand, we observe that $U = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ at $(0, 0)$. It follows that $\det U = 0$ at $(0, 0)$.

According to Theorem B, $S(u)$ may contain some C^1 smooth curves, called singular curves. We will study the behavior of $N(u)$ near a point of a singular curve. First we show that for a graph $t = u(x, y)$ the p-minimal graph equation (pMGE) is rotationally invariant.

Lemma 3.4. *Suppose $u \in C^2$. Let $x = a\tilde{x} - b\tilde{y}$, $y = b\tilde{x} + a\tilde{y}$ where $a^2 + b^2 = 1$, and let $\tilde{u}(\tilde{x}, \tilde{y}) = u(x(\tilde{x}, \tilde{y}), y(\tilde{x}, \tilde{y}))$. Then we have*

$$N(\tilde{u}) = N(u) \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

and hence $\widetilde{\operatorname{div}} N(\tilde{u}) = \operatorname{div} N(u)$ where $\widetilde{\operatorname{div}}$ denotes the plane divergence with respect to $(\tilde{x}, \tilde{y})$.

Proof. First we observe that both ∇u (viewed as a row vector) and $(-y, x)$ satisfy the following transformation law (for a plane *vector*):

$$\tilde{\nabla}\tilde{u} = (\nabla u)\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad (-\tilde{y}, \tilde{x}) = (-y, x)\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

It follows that $N(u) = [\nabla u + (-y, x)]/D$ also obeys the same transformation law by noting that D is invariant. Then a direct computation shows that $\widetilde{\operatorname{div}} N(\tilde{u}) = \operatorname{div} N(u)$. □

Let Γ_s be a singular curve contained in $S(u)$ for a C^2 smooth u (defined on a certain domain). Let $p_0 \in \Gamma_s$. Suppose there exists a ball $B_\epsilon(p_0)$ such that $\Gamma_s \cap B_\epsilon(p_0)$ divides $B_\epsilon(p_0)$ into two disjoint nonsingular parts (this is true if $|H| \leq C\frac{1}{r}$ near p_0 in view of Theorem B). That is to say, $B_\epsilon(p_0)\backslash(\Gamma_s \cap B_\epsilon(p_0)) = B^+ \cup B^-$ where B^+ and B^- are disjoint domains (proper open and connected).

Proposition 3.5. *Suppose we have the situation as described above. Then both* $N(u)(p_0^+) \equiv \lim_{p \in B^+ \to p_0} N(u)(p)$ *and* $N(u)(p_0^-) \equiv \lim_{p \in B^- \to p_0} N(u)(p)$ *exist. Moreover,* $N(u)(p_0^+) = -N(u)(p_0^-)$.

Proof. By Lemma 3.4 we may assume the x-axis through $p_0 = (x_0, y_0)$ is transverse to Γ_s. Moreover, we may assume either $u_{xx}(x_0, y_0) \neq 0$ or $(u_{xy} + 1)(x_0, y_0) \neq 0$. (Note that in (3.2a) $\frac{dy}{ds} \neq 0$ at p_0. So if $u_{xx}(x_0, y_0) = 0$, then $(u_{xy} - 1)(x_0, y_0) = 0$. It follows that $(u_{xy} + 1)(x_0, y_0) = 2 \neq 0$.) Let $\Gamma_s \cap \{y = c\} = \{(x_c, c)\}$ for c close to y_0. Since $(x_c, c) \in \Gamma_s$, we have $(u_y + x)(x_c, c) = 0$ and $(u_x - y)(x_c, c) = 0$. So if $u_{xx}(x_0, y_0) \neq 0$, we compute

$$\begin{aligned} \frac{(u_y + x)(x, c)}{(u_x - y)(x, c)} &= \frac{(u_y + x)(x, c) - (u_y + x)(x_c, c)}{(u_x - y)(x, c) - (u_x - y)(x_c, c)} \\ &= \frac{(x - x_c)(u_{xy} + 1)(x_c^1, c)}{(x - x_c)u_{xx}(x_c^2, c)} \qquad (3.7) \\ &\quad \text{(for } x_c^1, x_c^2 \text{ between } x_c \text{ and } x \text{ by the mean value theorem)} \\ &= \frac{(u_{xy} + 1)(x_c^1, c)}{u_{xx}(x_c^2, c)}. \end{aligned}$$

Letting (x, c) go to (x_0, y_0) in (3.7), we obtain

$$\lim_{p \in B^+ \to p_0} \frac{u_y + x}{u_x - y} = \frac{u_{xy} + 1}{u_{xx}}(p_0) = \lim_{p \in B^- \to p_0} \frac{u_y + x}{u_x - y}. \qquad (3.8)$$

Therefore by (3.8) two limits of the unit vector $N(u) = (u_x - y, u_y + x)D^{-1}$ from both sides exist, and their values can only be different by a sign. That is to say,

$$N(u)(p_0^+) = \pm N(u)(p_0^-). \qquad (3.9)$$

Now we observe that $(u_x - y)(x, y_0) - (u_x - y)(x_0, y_0) = u_{xx}(\eta, y_0)(x - x_0)$ for some η between x_0 and x. Since $(u_x - y)(x_0, y_0) = 0$, $(u_x - y)(x, y_0)$ and $u_{xx}(x_0, y_0)$ have the same (different, respectively) sign for $x > x_0$ ($x < x_0$, respectively) and x being close enough to x_0. Thus we should have the negative sign in (3.9). We are done. In case $u_{xx}(x_0, y_0) = 0$, we have $(u_{xy}+1)(x_0, y_0) \neq 0$. So we compute $\frac{(u_x-y)(x,c)}{(u_y+x)(x,c)}$ instead of $\frac{(u_y+x)(x,c)}{(u_x-y)(x,c)}$, and get $\frac{u_{xx}}{u_{xy}+1}(p_0)$ instead of $\frac{u_{xy}+1}{u_{xx}}(p_0)$ in (3.8). Then we still conclude (3.9). Instead of $(u_x - y)(x, y_0)$, we consider $(u_y + x)(x, y_0)$. A similar argument as above shows that $(u_y + x)(x, y_0)$ will have the same (different, respectively) sign as $(u_{xy} + 1)(x_0, y_0)$ for $x > x_0$ ($x < x_0$, respectively) and x being close enough to x_0. So we still take the negative sign in (3.9). □

Note that we do not assume any condition on H in Proposition 3.5. We will study how two characteristic curves meet at a point of a singular curve. We say a characteristic curve $\Gamma \subset B^+$ or B^- touches Γ_s at p_0 if $p_0 \in \bar{\Gamma}$, the closure of Γ in the xy-plane, and touches transversally if, furthermore, p_0 is the only intersection point of the tangent line of Γ_s at p_0 and the tangent line of Γ at p_0 (which makes sense in view of Proposition 3.5).

Corollary 3.6. *Suppose we have the same assumptions as in Proposition* 3.5. *Then there is a unique characteristic curve $\Gamma_+ \subset B^+$ touching Γ_s at p_0 transversally (with $N^\perp(u)(p_0^+)$ being the tangent vector of $\bar{\Gamma}_+$ at p_0). Similarly there exists a unique characteristic curve $\Gamma_- \subset B^-$ touching Γ_s also at p_0 so that $\Gamma_- \cup \{p_0\} \cup \Gamma_+$ forms a C^1 smooth curve in $B_\epsilon(p_0)$.*

First note that we can change the sign of $N^\perp(u)$ if necessary to make a C^0 characteristic (i.e. tangent to integral curves of $N^\perp(u)$ where $N^\perp(u)$ is defined) vector field $\check{N}^\perp(u)$ in $B_\epsilon(p_0)$ in view of Proposition 3.5. So $\bar{\Gamma}_+$ of any characteristic curve $\Gamma_+ \subset B^+$ touching Γ_s at p_0 must have the tangent vector $N^\perp(u)(p_0^+)$ at p_0. To show the uniqueness of Γ_+ and the transversality of $N^\perp(u)(p_0^+)$ to Γ_s in Corollary 3.6, we observe that

Lemma 3.7. *Suppose we have the same assumptions as in Proposition* 3.5. *Then there hold*

(1) $N^\perp(u)(p_0^+)U(p_0) = 0$,
(2) $(c, d)U^T(p_0) = 0$

for a nonzero vector (c, d) tangent to Γ_s at p_0, where we view $N^\perp(u)(p_0^+)$ as a row vector and U^T denotes the transpose of U.

Proof. Let $(x(s), y(s))$ describe Γ_s so that $p_0 = (x(0), y(0))$ and $(x'(0), y'(0)) = (c, d)$. Since $u_x - y = 0$, $u_y + x = 0$ on Γ_s, we differentiate these two equations to get (3.2a) and (3.2b) along Γ_s by the chain rule. At $s = 0$, we obtain (2). From the proof of Proposition 3.5, we learn that $N(u)(p_0^+)$ is proportional to $(u_{xx}, u_{xy} + 1)(p_0)$ (if this is not a zero vector). A similar argument (L'Hôpital's rule in the y-direction) shows that $N(u)(p_0^+)$ is also proportional

to $(u_{xy}-1, u_{yy})(p_0)$ (if this is not a zero vector). Therefore $N^{\perp}(u)(p_0^+)$ is perpendicular to $(u_{xx}, u_{xy}+1)(p_0)$ and $(u_{xy}-1, u_{yy})(p_0)$, hence (1) follows. □

We will give a proof of Corollary 3.6 after the proof of Theorem 3.10.

Remark. If u is not of class C^2, the extension theorem (Corollary 3.6) may fail as the following example shows. Consider the function

$$u(x, y) = \begin{cases} -xy, & y \geq 0 \\ -xy + y^2 \cot \vartheta, & y < 0 \end{cases}$$

where $0 < \vartheta < \frac{\pi}{2}$. There holds

$$N^{\perp}(u) = \begin{cases} (0, 1), & y > 0 \\ (\cos \vartheta, \sin \vartheta), & y < 0\,. \end{cases}$$

Note that the function u is of class $C^{1,1}$ on R^2 and satisfies $\operatorname{div} N(u) = 0$ in $R^2 \backslash \{y = 0\}$ where u is of class C^2.

Next we want to analyze the configuration of characteristic curves near an isolated singularity. First observe that for a C^2 smooth u defined on a domain Ω, characteristic curves are also the integral curves of the C^1 smooth vector field $N^{\perp}D = (u_y + x, -u_x + y)$ which vanishes at singular points. We think of $N^{\perp}D$ as a mapping: $\Omega \subset R^2 \to R^2$, so that the differential $d(N^{\perp}D)_{p_0}$: $R^2 \to R^2$ is defined for $p_0 \in \Omega$.

Lemma 3.8. *Let $u \in C^2(\Omega)$. Suppose $|H| = o(\frac{1}{r})$ (little "o") near an isolated singular point $p_0 \in \Omega$ where $r(p) = |p - p_0|$. Then $d(N^{\perp}D)_{p_0}$ is the identity linear transformation and the index of $N^{\perp}D$ at the isolated zero p_0 is $+1$. Moreover, $u_{xx} = u_{xy} = u_{yy} = 0$ at p_0.*

Proof. In view of (2.10), we write $H = D^{-3}P(u)$ where

$$P(u) \equiv (u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy}\,. \tag{3.10}$$

Let $p - p_0 = (\triangle x, \triangle y)$, $r = r(p) = |p - p_0| = [(\triangle x)^2 + (\triangle y)^2]^{1/2}$. Noting that $(u_y + x)(p_0) = (u_x - y)(p_0) = 0$ by the definition of a singular point, we can express

$$(u_y + x)(p) = (u_{yx} + 1)(p_0) \triangle x + u_{yy}(p_0) \triangle y + o(r)\,, \tag{3.11}$$

$$(u_x - y)(p) = u_{xx}(p_0) \triangle x + (u_{xy} - 1)(p_0) \triangle y + o(r) \tag{3.12}$$

for p near p_0. Let $a = (u_{yx} + 1)(p_0)$, $b = u_{yy}(p_0)$, $c = u_{xx}(p_0)$, and $d = (u_{xy} - 1)(p_0)$. Substituting (3.11) and (3.12) in (3.10), we compute the highest order term:

$$\begin{aligned} P(u) &= (a \triangle x + b \triangle y)^2 c \\ &\quad - (a \triangle x + b \triangle y)(c \triangle x + d \triangle y)(a + d) + (c \triangle x + d \triangle y)^2 b + o(r^2) \\ &= (bc - ad)\left\{(\triangle x \triangle y)\begin{bmatrix} c & a \\ d & b \end{bmatrix}\begin{pmatrix} \triangle x \\ \triangle y \end{pmatrix}\right\} + o(r^2)\,. \end{aligned} \tag{3.13}$$

On the other hand, substituting (3.11), (3.12) into $D^3 \equiv [(u_x-y)^2+(u_y+x)^2]^{3/2}$, we obtain

$$D^3 = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} \triangle x \\ \triangle y \end{pmatrix} \right|^3 + o(r^3) . \tag{3.14}$$

Note that $bc - ad = \det U(p_0) \neq 0$ by Theorem 3.3. Letting $\triangle y = 0$ and assuming $c \neq 0$, we estimate the highest order term of $H = D^{-3}P(u)$:

$$\frac{(bc-ad)c(\triangle x)^2}{[(a^2+c^2)(\triangle x)^2]^{3/2}} = \frac{(bc-ad)c(\triangle x)^2}{[a^2+c^2]^{3/2}(\triangle x)^3} = \frac{(bc-ad)c}{[a^2+c^2]^{3/2}\triangle x} .$$

The assumption $|H| = o(\frac{1}{r})$ forces $c = 0$. On the other hand, letting $\triangle x = 0$ will force $b = 0$ by a similar argument. Now we can write

$$H = \frac{-ad(a+d)\triangle x \triangle y}{[a^2(\triangle x)^2 + d^2(\triangle y)^2]^{3/2}} + o\left(\frac{1}{r}\right) .$$

It follows from the assumption $|H| = o(\frac{1}{r})$ again that $a + d = 0$ (note that $ad = ad - bc = -\det U(p_0) \neq 0$). We have proved that $u_{xx}(p_0) = c = 0$, $u_{yy}(p_0) = b = 0$, $u_{xy}(p_0) = \frac{a+d}{2} = 0$. Therefore in matrix form,

$$d(N^{\perp}D)_{p_0} = \begin{bmatrix} u_{yx}+1 & u_{yy} \\ -u_{xx} & -u_{xy}+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Note that $\det(d(N^{\perp}D)_{p_0}) = 1 > 0$. So the index of $N^{\perp}D$ at p_0 is $+1$ (see, e.g., Lemma 5 in Section 6 in [Mil]). □

Lemma 3.9. *Let $u \in C^2(\Omega)$. Suppose $|H| = o(\frac{1}{r})$ (little "o") near an isolated singular point $p_0 \in \Omega$ where $r(p) = |p - p_0|$. Then there exists a small neighborhood $V \subset \Omega$ of p_0 such that the characteristic curve through a point in $V \setminus \{p_0\}$ reaches p_0 (towards the $-N^{\perp}$ direction) in finite unit-speed parameter.*

Proof. Write $p - p_0 = (\triangle x, \triangle y) = (r\cos\varphi, r\sin\varphi)$ in polar coordinates. At p, we express

$$N^{\perp} = \alpha(\cos\varphi, \sin\varphi) + \beta(-\sin\varphi, \cos\varphi) \tag{3.15}$$

where

$$\alpha = N^{\perp} \cdot (\cos\varphi, \sin\varphi), \quad \beta = N^{\perp} \cdot (-\sin\varphi, \cos\varphi) . \tag{3.16}$$

Noting that $u_{xx} = u_{xy} = u_{yy} = 0$ at p_0 by Lemma 3.8, we obtain that

$$u_y + x = \triangle x + o(r), \quad -u_x + y = \triangle y + o(r) \tag{3.17}$$

near p_0 by (3.11) and (3.12). It follows that $D = [(\triangle x)^2 + (\triangle y)^2]^{1/2} + o(r)$ $= r + o(r)$ near p_0. From this and (3.17), we can estimate $N^{\perp} = (\frac{\triangle x}{r} + o(1), \frac{\triangle y}{r} + o(1))$. Substituting this and $(\cos\varphi, \sin\varphi) = (\frac{\triangle x}{r}, \frac{\triangle y}{r})$ in (3.16) gives

$$\alpha = \frac{(\triangle x)^2 + (\triangle y)^2}{r^2} + o(1) = 1 + o(1) . \tag{3.18}$$

Now let $(x(s), y(s))$ describe a characteristic curve in the $-N^\perp$ direction, i.e., $\frac{d(x(s),y(s))}{ds} = -N^\perp$. We compute

$$\begin{aligned}\frac{d(x(s), y(s))}{ds} &= \frac{d(r(s)\cos\varphi(s), r(s)\sin\varphi(s))}{ds} \\ &= \frac{dr}{ds}(\cos\varphi, \sin\varphi) + r\frac{d\varphi}{ds}(-\sin\varphi, \cos\varphi)\,.\end{aligned}$$

Comparing with (3.15), we obtain

$$\frac{dr}{ds} = -\alpha, \quad r\frac{d\varphi}{ds} = -\beta\,. \tag{3.19}$$

Observe that α tends to 1 as r goes to 0 by (3.18) (hence $\beta = o(1)$ since $\alpha^2 + \beta^2 = 1$). So from (3.19) we can find a small neighborhood V of p_0 so that the distance between p_0 and the characteristic curve Γ through a point $p_1 \in V\backslash\{p_0\}$ is decreasing towards the $-N^\perp$ direction. Let s denote a unit-speed parameter of Γ and $p_1 = \Gamma(s_1)$. Then from the following formula

$$r(s) - r(s_1) = \int_{s_1}^{s} \frac{dr}{ds} ds = \int_{s_1}^{s} (-\alpha) ds\,,$$

we learn that $r(s)$ reaches 0 for a finite s. □

Let $B_r(p_0) = \{p \in \Omega \mid |p - p_0| < r\}$. Define $H_M(r) = \max_{p\in\partial B_r(p_0)} |H(p)|$.

Theorem 3.10. *Let $u \in C^2(\Omega)$. Suppose $|H| = o(\frac{1}{r})$ (little "o") near an isolated singular point $p_0 \in \Omega$ where $r(p) = |p - p_0|$. Moreover, suppose there is $r_0 > 0$ such that*

$$\int_0^{r_0} H_M(r) dr < \infty\,. \tag{3.20}$$

Then for any unit tangent vector v at p_0, there exists a unique characteristic curve Γ touching p_0 (i.e. $p_0 \in \bar{\Gamma}$, the closure of Γ) such that $N^\perp(u)(p_0^+)$, the limit of $N^\perp(u)$ at p_0 along Γ, equals v. Moreover, there exists a neighborhood V of p_0 such that $V\backslash\{p_0\}$ is contained in the union of all such Γ's.

Proof. Take $\delta > 0$ small enough so that all characteristic curves through points on $\partial B_\delta(p_0)$ reach p_0 in finite unit-speed parameter in view of Lemma 3.9. Let Γ denote the characteristic curve through a point $p_1 \in \partial B_\delta(p_0)$. Let $(s_0, s_1]$ be the interval of unit-speed paramater describing points of Γ between p_0 and p_1. Take a sequence of points $p_j \in \Gamma \to p_0$ with parameter $s_j \to s_0$. We compute

$$\theta(p_j) - \theta(p_k) = \int_{s_k}^{s_j} \frac{d\theta}{ds} ds = -\int_{s_k}^{s_j} H ds$$

by (2.23) (recall that θ is defined by $N = N(u) = (\cos\theta, \sin\theta)$). It follows from $\frac{dr}{ds} \to -1$ as $s \to s_0$ or $r \to 0$ and (3.20) that

$$|\theta(p_j) - \theta(p_k)| \le \int_{r_k}^{r_j} H_M(r) \left|\frac{ds}{dr}\right| dr \le 2\int_{r_k}^{r_j} H_M(r) dr \to 0$$

as $p_j, p_k \to p_0$. This means that $\{\theta(p_j)\}$ is a Cauchy sequence. Therefore it converges to some number, denoted as $\theta(p_0; p_1)$. Define a map $\Psi : \partial B_\delta(p_0) \to S^1$ by $\Psi(q) = \theta(p_0; q)$. We claim that Ψ is a homeomorphism. Take a sequence of points $q_j \in \partial B_\delta(p_0)$ converging to $\hat{q}$. We want to show that $\theta(p_0; q_j)$ converges to $\theta(p_0; \hat{q})$. Without loss of generality, we may assume all $q_j's$ are sitting on one side of $\hat{q}$ so that $\varphi(q_1) > \varphi(q_2) > \ldots > \varphi(\hat{q})$ where φ is the angle in polar coordinates centered at p_0 ranging in $[0, 2\pi)$. Observe that

$$\begin{gathered}\theta(p_0; q_j) \geq \theta(p_0; q_{j+1}) \geq \ldots \geq \theta(p_0; \hat{q}) \\ \text{(letting } \theta \text{ take values in } [0, 2\pi))\end{gathered} \tag{3.21}$$

for j large since two distinct characteristic curves can not intersect in $B_\delta(p_0)\backslash\{p_0\}$. Let $\hat{\theta}$ be the limit of $\theta(p_0; q_j)$ as $j \to \infty$. Now suppose $\hat{\theta} \neq \theta(p_0; \hat{q})$ (hence $\hat{\theta} > \theta(p_0; \hat{q})$). Let Γ_j ($\hat{\Gamma}$, resp.) denote the characteristic curve connecting q_j ($\hat{q}$, resp.) and p_0. Then we can find two rays emitting from p_0 with angle smaller than $\hat{\theta} - \theta(p_0; \hat{q})$ and a small positive $\hat{\delta} < \delta$ so that Γ_j and $\hat{\Gamma}$ do not meet a fan-shaped region $\hat{\Omega}$ surrounded by these two rays and $\partial B_{\hat{\delta}}(p_0)$ for j large. Take a point $\check{p} \in \hat{\Omega}$. Consider the characteristic curve $\check{\Gamma}$ through $\check{p}$. Then $\check{\Gamma}$ must intersect $\partial B_\delta(p_0)$ at a point $\check{q}$ while reaching p_0 with $\theta = \theta(p_0; \check{q})$. Since $\check{\Gamma}$ does not intersect with any Γ_j, we have $\theta(p_0; q_j) > \theta(p_0; \check{q})$ for large j. On the other hand, $\check{q}$ must coincide with $\hat{q}$ for the same reason. So $\check{\Gamma} = \hat{\Gamma}$, an obvious contradiction. Thus $\hat{\theta} = \theta(p_0; \hat{q})$. We have proved the continuity of Ψ.

Next we claim that Ψ is surjective. If not, $S^1\backslash\Psi(\partial B_\delta(p_0))$ is a nonempty open set. Then by a similar fan-shaped region argument as shown above, we can reach a contradiction. Let Γ_1, Γ_2 be two characteristic curves through q_1, $q_2 \in \partial B_\delta(p_0)$ touching p_0 with $\theta(p_0; q_1) = \theta(p_0; q_2)$. We want to show that $q_1 = q_2$. Suppose $q_1 \neq q_2$. So Γ_1 and Γ_2 are distinct (with empty intersection in $B_\delta(p_0)\backslash\{p_0\}$) and tangent at p_0. Let Ω_r denote the smaller domain, surrounded by Γ_1, Γ_2, and $\partial B_r(p_0)$ for small $r > 0$. Then Ω_r is contained in a fan-shaped region with vertex p_0 and angle θ_r such that $\theta_r \to 0$ as $r \to 0$. Let $\Gamma_r \equiv \partial\Omega_r \cap \partial B_r(p_0)$. It follows that

$$|\Gamma_r| \leq r\theta_r \tag{3.22}$$

where $|\Gamma_r|$ denotes the length of the arc Γ_r. Since $N^\perp(u)$ is perpendicular to the unit outward normal ν ($= \pm N(u)$) on Γ_1 and Γ_2, we obtain

$$g(r) \equiv \oint_{\partial\Omega_r} (u_y + x, -u_x + y) \cdot \nu ds = \int_{\Gamma_r} (u_y + x, -u_x + y) \cdot \nu ds\,. \tag{3.23}$$

Observing that $(u_y + x, -u_x + y) = p - p_0 + o(r)$ by (3.17) and $\nu = \frac{p-p_0}{r}$ on Γ_r, we deduce from (3.23) that

$$g(r) = [r + o(r)]|\Gamma_r|\,. \tag{3.24}$$

On the other hand, the divergence theorem tells us that

$$\begin{aligned} g(r) &= \iint_{\Omega_r} [(u_y + x)_x + (-u_x + y)_y] dx \wedge dy \\ &= 2 \iint_{\Omega_r} dx \wedge dy = 2 \int_0^r |\Gamma_\tau| d\tau \,. \end{aligned} \tag{3.25}$$

It follows from (3.25) that $g'(r) = 2|\Gamma_r|$. Comparing this with (3.24), we obtain $\frac{g'(r)}{g(r)} = \frac{2}{r} + o(\frac{1}{r})$. Therefore $g(r) = cr^2 + o(r^2)$ for some constant $c > 0$. However, inserting (3.22) into (3.24) shows that $g(r) = o(r^2)$, i.e., $\frac{g(r)}{r^2} \to 0$ as $r \to 0$. We have reached a contradiction. So $q_1 = q_2$ and hence Ψ is injective. Next we will show that Ψ^{-1} is continuous. Suppose this is not true. Then we can find a sequence of $q_j \in \partial B_\delta(p_0)$ converging to $\check{q} \neq \hat{q}$ while $\theta(p_0; q_j)$ converges to $\theta(p_0; \hat{q})$ (may assume monotonicity (3.21) or reverse order for large j). Take a point $\bar{q} \in \partial B_\delta(p_0)$, $\bar{q} \neq \check{q}$ and $\bar{q} \neq \hat{q}$, such that $\theta(p_0; q_j) \geq \theta(p_0; \bar{q}) \geq \theta(p_0; \hat{q})$ for all large j. Since $\lim \theta(p_0; q_j) = \theta(p_0; \hat{q})$, we must have $\theta(p_0; \bar{q}) = \theta(p_0; \hat{q})$ contradicting the injectivity of Ψ. Altogether we have shown that Ψ is a homeomorphism. The theorem follows from this fact. □

Proof of Corollary 3.6. Let $N^\perp(p_0^+)$ denote $N^\perp(u)(p_0^+)$ for simplicity. First we claim that $N^\perp(p_0^+)$ can not be tangent to Γ_s at p_0. If yes, we have $N^\perp(p_0^+)\ (U(p_0) - U^T(p_0)) = 0$ by Lemma 3.7 (1), (2). However, $U - U^T = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$. It follows that $N^\perp(p_0^+) = 0$, a contradiction. So $N^\perp(p_0^+)$ is transversal to Γ_s, and $N^\perp(p_0^+)U^T(p_0) \neq 0$. In fact, we have

$$|N^\perp(p_0^+)U^T(p_0)| = 2 \tag{3.26}$$

by noting that the unit-length vector $N^\perp(p_0^+)$ is proportional to $(u_{xy} + 1, -u_{xx})$ (if $\neq 0$) and $(u_{yy}, -u_{xy} + 1)$ (if $\neq 0$) from the proof of Lemma 3.7 or Proposition 3.5. Now suppose Γ_+ and Γ'_+ are two distinct (never intersect in $B_\epsilon(p_0)\backslash\{p_0\}$ for some small ϵ) characteristic curves contained in B^+ touching p_0 (hence with the same tangent vector $N^\perp(p_0^+)$ at p_0). Let Ω_r denote the domain, surrounded by Γ_+, Γ'_+, and $\partial B_r(p_0)$ for $0 < r < \epsilon$. Then Ω_r is contained in a fan-shaped region with vertex p_0 and angle θ_r such that $\theta_r \to 0$ as $r \to 0$. Let $\Gamma_r \equiv \partial\Omega_r \cap \partial B_r(p_0)$. Then (3.22) holds.

On the other hand, $(u_x - y, u_y + x) = (\triangle x, \triangle y)U^T(p_0) + o(r^2)$ by (3.11) and (3.12) while $(\triangle x, \triangle y) = rN^\perp(p_0^+) + o(r)$ and $\nu = (\triangle x, \triangle y)r^{-1}$ on Γ_r tends to $N^\perp(p_0^+)$ as $r \to 0$. So from (3.23) we compute

$$\begin{aligned} g(r) &= \int_{\Gamma_r} (u_y + x, -u_x + y) \cdot \nu ds \\ &= \int_{\Gamma_r} DN^\perp \cdot \nu ds \\ &= (2r + o(r))|\Gamma_r| \,. \end{aligned} \tag{3.27}$$

Here we have used (3.26) in the last equality. Now a similar argument as in the proof of Theorem 3.10 by comparing (3.25) with (3.27) gives $g(r) = cr + o(r)$ for a positive constant c. However, substituting (3.22) into (3.27) shows that $g(r) = o(r^2)$. We have reached a contradiction. Therefore Γ_+ must coincide with Γ'_+. Similarly we have a unique characteristic curve $\Gamma_- \subset B^-$ touching Γ_s also at p_0 so that $\Gamma_- \cup \{p_0\} \cup \Gamma_+$ forms a C^1 smooth curve in $B_\epsilon(p_0)$. □

The line integral in (3.23) has a geometric interpretation. Recall that the standard contact form in the Heisenberg group H_1 is $\Theta_0 = dz + xdy - ydx$. Let $\tilde{u}$ denote the map: $(x, y) \to (x, y, u(x, y))$. It is easy to see that $\tilde{u}^*\Theta_0 = (u_y + x)dy - (y - u_x)dx$. Now it is clear that the line integral in (3.23) is exactly the line integral of $\tilde{u}^*\Theta_0$. Note that $\tilde{u}^*\Theta_0$ vanishes along any characteristic curve. If we remove the condition (3.20) in Theorem 3.10, $\theta(p_0; p_1)$ will not exist as shown in the following example.

Example. Let $p_0 = (0, 0)$. Let $u = -\frac{r^2}{\log r^2}$ $(= 0$ at $p_0)$ where $r^2 = x^2 + y^2$. Write $u = f(r^2)$. A direct computation shows that

$$u_x = 2xf'(r^2), \quad u_y = 2yf'(r^2)\,, \tag{3.28}$$

$$D = r\sqrt{1 + 4(f')^2}\,, \tag{3.29}$$

$$u_{xx} = 2f' + 4x^2 f'', u_{xy} = 4xyf'', u_{yy} = 2f' + 4y^2 f''\,. \tag{3.30}$$

It is easy to see that p_0 is an isolated singularity (for a general f). Also u is C^2 at p_0 and $u_{xx} = u_{xy} = u_{yy} = 0$ at p_0 (for $f(t) = -\frac{t}{\log t}$, $t = r^2$). Therefore $d(N^\perp D)_{p_0}$ is the identity transformation and the index of $N^\perp D$ is $+1$. Noting that $(-\sin\varphi, \cos\varphi) = (-y, x)r^{-1}$, we compute $\beta = D^{-1}r^{-1}(u_y, -u_x) \cdot (-y, x) = -2f'/\sqrt{1 + 4(f')^2}$ by (3.16), (3.28), and (3.29). Therefore along a characteristic curve reaching p_0 in the $-N^\perp$ direction, we can estimate

$$\begin{aligned} \frac{d\varphi}{ds} &= -\frac{1}{r}\beta \text{ (by (3.19))} \\ &= \frac{2f'}{r\sqrt{1 + 4(f')^2}} \approx \frac{2}{-r\log r^2} \end{aligned} \tag{3.31}$$

as $r \to 0$ for $f(t) = -\frac{t}{\log t}$, $t = r^2$. Since $\frac{dr}{ds} \to -1$ by (3.18) and

$$\int_0^{r_1} \frac{2}{-r\log r^2} dr = \infty\,,$$

we conclude from (3.31) that $\varphi \to \infty$, hence $\theta \to \infty$ as the point on the characteristic curve approaches p_0. (Observing that $\alpha \to 1$ and $\beta \to 0$ in (3.15) as $r \to 0$, we have the limit of θ equal to the limit of φ plus $\pi/2$ if one of the limits exists, hence another limit exists too.)

Next substituting (3.28) and (3.30) into (3.10) gives

$$P(u) = 2r^2[f' + 4(f')^3 + 2r^2 f'']\,. \tag{3.32}$$

By (2.10), (3.29), and (3.32), we obtain the p-mean curvature

$$H = \frac{2(f' + 4(f')^3 + 2r^2 f'')}{r(1 + 4(f')^2)^{3/2}}\,. \tag{3.33}$$

For $f(t) = -\frac{t}{\log t}$, $t = r^2$, $f' \approx -\frac{1}{\log t}$, $2r^2 f'' \approx \frac{2}{(\log t)^2}$ near $r = 0$. Inserting these estimates into (3.33) gives $H \approx -\frac{1}{r \log r}$. It is now a straightforward computation to verify that $H = o(\frac{1}{r})$ and the integral in (3.20) for such an H diverges.

4. A Bernstein-type theorem and properly embedded p-minimal surfaces

Recall that the characteristic curves for a p-minimal surface Σ in a pseudohermitian 3-manifold M are Legendrian geodesics in M by (2.1). For $M = H_1$, we have

Proposition 4.1. *The Legendrian geodesics in H_1, identified with $\mathbb{R}^3$, with respect to $\nabla^{p.h.}$ are straight lines.*

Proof. Write a unit Legendrian vector field $e_1 = f\hat{e}_1 + g\hat{e}_2$ with $f^2 + g^2 = 1$. Since $\nabla^{p.h.}\hat{e}_1 = \nabla^{p.h.}\hat{e}_2 = 0$, the geodesic equation $\nabla^{p.h.}_{e_1} e_1 = 0$ implies that $e_1 f = e_1 g = 0$. This means that $f = c_1, g = c_2$ for some constants c_1, c_2 along a geodesic Γ (integral curve) of e_1. We compute

$$\begin{aligned} e_1 &= c_1\hat{e}_1 + c_2\hat{e}_2 = c_1(\partial_x + y\partial_z) + c_2(\partial_y - x\partial_z) \\ &= c_1\partial_x + c_2\partial_y + (c_1 y - c_2 x)\partial_z\,. \end{aligned} \tag{4.1}$$

So Γ is described by the following system of ordinary differential equations:

$$\frac{dx}{ds} = c_1\,, \tag{4.2a}$$

$$\frac{dy}{ds} = c_2\,, \tag{4.2b}$$

$$\frac{dz}{ds} = c_1 y - c_2 x\,. \tag{4.2c}$$

By (4.2a), (4.2b) we get $x = c_1 s + d_1$, $y = c_2 s + d_2$ for some constants d_1, d_2. Substituting into (4.2c) gives $z = (c_1 d_2 - c_2 d_1)s + d_3$ for some constant d_3. So Γ is a straight line in $\mathbb{R}^3$. □

Corollary 4.2. *The characteristic curves of a p-minimal surface in H_1 are straight lines or line segments. In particular, a characteristic curve (line) of a p-minimal surface in H_1 through a point q is contained in the contact plane through q.*

Recall that $F_{a,b} \equiv a(u_x - y) + b(u_y + x)$ for real constants a, b with $a^2 + b^2 = 1$.

Lemma 4.3. *Suppose $u \in C^2$ defines a p-minimal graph near $p \in S(u)$, an isolated singular point. Then $F_{a,b} = 0$, for $a, b \in R$, $a^2 + b^2 = 1$, define all straight line segments passing through p which are all characteristic curves in a neighborhood of p with p deleted.*

Proof. First we claim $\nabla F_{a,b}(p) \neq 0$ for all (a, b) with $a^2 + b^2 = 1$. If not, there exists (a_0, b_0) such that $\nabla F_{a_0,b_0}(p) = 0$. So $\det U(p) = 0$ (see the paragraph after (3.1)). It follows from the proof of Theorem B that there is a small neighborhood of p which intersects with $S(u)$ in exactly a C^1 smooth curve through p. This contradicts p being an isolated singular point. We have shown that $\nabla F_{a,b}(p) \neq 0$ for all (a, b) with $a^2 + b^2 = 1$. Therefore $F_{a,b} = 0$ defines a C^1 smooth curve through p for all (a, b).

In a neighborhood of p with p deleted, we observe that $F_{a,b}D^{-1} \equiv \sin\theta_0 \cos\theta - \cos\theta_0 \sin\theta$. Here we write $a = \sin\theta_0, b = -\cos\theta_0$. Recall that $(u_x - y)D^{-1} = \cos\theta$, $(u_y + x)D^{-1} = \sin\theta$ (see Section 2). So $\theta = \theta_0$ on $\{F_{a,b} = 0\}$, and hence by (2.18b) $N^\perp = (\sin\theta, -\cos\theta) = (\sin\theta_0, -\cos\theta_0)$ is a constant unit vector field along $\{F_{a,b} = 0\}$. On the other hand, $\nabla(F_{a,b}D^{-1}) = (-a\sin\theta + b\cos\theta)\nabla\theta$ is parallel to $N = (\cos\theta, \sin\theta)$ since $N^\perp \cdot \nabla\theta = 0$ is our equation. It follows that $\{F_{a,b} = 0\}$ is a straight line segment and an integral curve of $N^\perp$ in a p-deleted neighborhood. □

We remark that Lemma 4.3 provides a more precise description of Theorem 3.10 in the case of $H = 0$.

Since the characteristic curves are straight lines, we will often call them characteristic lines (line segments). From Corollary 3.6, we know that a characteristic line keeps straight after it *goes through* a singular curve. Note that two characteristic line segments Γ_1, Γ_2 can not touch a singular curve at the same point p_0 unless they lie on a straight line by Proposition 3.5 (the limits of $N(u)$ at p_0 along Γ_1, Γ_2 must be either the same or different by a sign). We say a graph is entire if it is defined on the whole xy-plane.

Lemma 4.4. *Suppose $u \in C^2$ defines an entire p-minimal graph. Then $S(u)$ contains no more than one isolated singular point.*

Proof. Suppose we have two such points $p_1, p_2 \in S(u)$. Then there exist two distinct straight lines passing through p_1, p_2, respectively and intersecting at a third point q such that $q \notin S(u)$ in view of Theorem B and Corollary 3.6. From the proof of Lemma 4.3 and Corollary 3.6, these two straight lines are characteristic curves in the complement of $S(u)$, namely integral curves of $N^\perp(u)$. But then at q, $N^\perp(u)$ has two values, a contradiction. □

On the other hand, remember that we can change the sign of $N^\perp(u)$ if necessary to make a C^0 characteristic (i.e. tangent to integral curves of $N^\perp(u)$ where

$N^{\perp}(u)$ is defined) vector field $\check{N}^{\perp}(u)$ on the whole xy-plane except possibly one isolated singular point in view of Proposition 3.5. Moreover, we have a unique characteristic curve "going through" a point of a singular curve by Corollary 3.6. So we can conclude that the following result holds.

Lemma 4.5. *Suppose $u \in C^2$ defines an entire p-minimal graph and $S(u)$ contains no isolated singular point. Then all integral curves (restrict to characteristic lines of $N^{\perp}(u)$) of $\check{N}^{\perp}(u)$ are parallel.*

Proof of Theorem A. According to Lemma 4.4, we have the following two cases.

Case 1. $S(u)$ contains one isolated singular point.

In this case, we claim the solution u is nothing but (1.1). Let p_0 be the singular point. Let r, ϑ denote the polar coordinates with center p_0. We can write $\pm\check{N}^{\perp}(u) = \frac{\partial}{\partial r}$ in view of Lemma 4.3. By (2.20) we have the equation

$$u_{rr} = \frac{\partial^2 u}{\partial r^2} = 0 \tag{4.3}$$

defined on the whole xy-plane except p_0. It follows from (4.3) that $u = rf(\vartheta) + g(\vartheta)$ for some C^2 functions f, g in ϑ. Since u is continuous at $p_0 = (x_0, y_0)$ (where $r = 0$), $u(x_0, y_0) = g(\vartheta)$ for all ϑ. So g is a constant function, say $g = c$. Also $f(\vartheta) = f(\vartheta + 2\pi)$ implies that we can write $f(\vartheta) = \tilde{f}(\cos\vartheta, \sin\vartheta)$ where $\tilde{f}$ is C^2 in $\alpha = \cos\vartheta$ and $\beta = \sin\vartheta$. Compute $u_x = u_r r_x + u_\vartheta \vartheta_x = \alpha\tilde{f} + \beta^2 \tilde{f}_\alpha - \alpha\beta\tilde{f}_\beta$ in which $\tilde{f}_\alpha = \partial\tilde{f}/\partial\alpha$, $\tilde{f}_\beta = \partial\tilde{f}/\partial\beta$, etc. and we have used $\vartheta_x = -(\sin\vartheta)/r$. Similarly we obtain $u_y = \beta\tilde{f} + \alpha^2\tilde{f}_\beta - \alpha\beta\tilde{f}_\alpha$. Since u_x and u_y are continuous at (x_0, y_0), we immediately have the following identities:

$$\beta^2\tilde{f}_\alpha - \alpha\beta\tilde{f}_\beta + \alpha\tilde{f} = a \tag{4.4}$$

$$-\alpha\beta\tilde{f}_\alpha + \alpha^2\tilde{f}_\beta + \beta\tilde{f} = b \tag{4.5}$$

for all α, β. Here $a = u_x(x_0, y_0)$, $b = u_y(x_0, y_0)$. Multiplying (4.4), (4.5) by α, β, respectively and adding the resulting identities, we obtain $(\alpha^2 + \beta^2)\tilde{f} = a\alpha + b\beta$. It follows that $\tilde{f} = a\alpha + b\beta$ since $\alpha^2 + \beta^2 = 1$. We have shown that $u(x, y) = r(a\cos\vartheta + b\sin\vartheta) + c = a(x - x_0) + b(y - y_0) + c_0 = ax + by + (c - ax_0 - by_0) = ax + by + c$. (In fact $(x_0, y_0) = (-b, a)$ from the definition of a singular point and the plane $\{(x, y, u(x, y)\}$ is just the contact plane passing through (x_0, y_0).) We can also give a geometric proof for Case 1 as follows. Let ξ_0 denote the standard contact bundle over H_1 (see the Appendix). Let Σ denote the p-minimal surface defined by u. Observe that the union of all characteristic lines "going through" p_0, the isolated singular point, (together with p_0) constitutes the contact plane $\xi_0(p_0)$ in view of Corollary 4.2 and Lemma 4.3. It follows that $\xi_0(p_0) \subset \Sigma$. So $\Sigma = \xi_0(p_0)$, an entire plane, since Σ is also an entire graph. We are done.

Case 2. $S(u)$ contains no isolated singular point.
In this case we claim u is nothing but (1.2). By Lemma 4.5 and Lemma 3.4 we can find a rotation $\tilde{x} = ax + by$, $\tilde{y} = -bx + ay$ with $a^2 + b^2 = 1$ such that

$$\check{N}^{\perp}(u) = \pm \frac{\partial}{\partial \tilde{x}} . \tag{4.6}$$

By (2.20) our equation reads $\tilde{u}_{\tilde{x}\tilde{x}} = 0$ where $\tilde{u}(\tilde{x}, \tilde{y}) = u(x, y)$. It follows that

$$\tilde{u} = \tilde{x}\tilde{y} + g(\tilde{y}) , \tag{4.7}$$

for some C^2 smooth functions f, g. From (4.6) we know $N(u) = (0, \pm 1)$. By the definition of $N(u)$ we obtain $\tilde{u}_{\tilde{x}} - \tilde{y} = 0$. So $f(\tilde{y}) = \tilde{y}$. Substituting this into (4.7) gives $\tilde{u} = \tilde{x}\tilde{y} + g(\tilde{y})$, and hence $u = -abx^2 + (a^2 - b^2)xy + aby^2 + g(-bx + ay)$. □

We remark that the singular curve in Case 2 is defined by $\tilde{x} = -g'(\tilde{y})/2$, and this curve has only one connected component.

Next we will describe a general properly embedded p-minimal surface in H_1, which may not be a graph. According to Proposition 4.1, such a surface must be a properly embedded ruled surface with Legendrian (tangent to contact planes) rulings when we view H_1 as $\mathbb{R}^3$. We call a ruled surface with Legendrian rulings a Legendrian ruled surface. Conversely, we claim that a properly embedded Legendrian ruled surface is a properly embedded p-minimal surface. First observe that a straight line L through $p_0 = (x_0, y_0, z_0)$ pointing in a contact direction $c_1\hat{e}_1(p_0) + c_2\hat{e}_2(p_0)$, $c_1^2 + c_2^2 = 1$, is tangent to the contact plane everywhere. Here $\hat{e}_1(p_0) = \partial_x + y_0\partial_z$ or $(1, 0, y_0)$ and $\hat{e}_2(p_0) = \partial_y - x_0\partial_z$ or $(0, 1, -x_0)$. In fact we can parametrize any point $p = (x, y, z) \in L$ as follows:

$$(x, y, z) = (x_0, y_0, z_0) + s[c_1\hat{e}_1(p_0) + c_2\hat{e}_2(p_0)] \tag{4.8}$$

for some $s \in R$. The tangent vector at p is just $c_1\hat{e}_1(p_0) + c_2\hat{e}_2(p_0)$ which exactly equals $c_1\hat{e}_1(p) + c_2\hat{e}_2(p)$ by a simple computation. So it is a vector in the contact plane at p. And L is a Legendrian line. A Legendrian ruled surface is generated by such Legendrian lines with its characteristic field $e_1(p) = c_1\hat{e}_1(p_0) + c_2\hat{e}_2(p_0) = c_1\hat{e}_1(p) + c_2\hat{e}_2(p)$ with c_1, c_2 being constant along the characteristic line (or line segment) through a nonsingular point p. It follows that $\nabla^{p.h.}_{e_1} e_1 = c_1\nabla^{p.h.}_{e_1}\hat{e}_1 + c_2\nabla^{p.h.}_{e_1}\hat{e}_2 = 0$ since $\nabla^{p.h.}\hat{e}_j = 0$, $j = 1, 2$. By (2.1) the p-mean curvature H vanishes. So we have shown that a Legendrian ruled surface is a p-minimal surface. Also an immersed Legendrian ruled surface is the union of a family of curves of the form (4.8), and has the following expression:

$$(x_0(\tau), y_0(\tau), z_0(\tau)) + s[\sin\theta(\tau)(1, 0, y_0(\tau)) - \cos\theta(\tau)(0, 1, -x_0(\tau))] . \tag{4.9}$$

Here $(x_0(\tau), y_0(\tau), z_0(\tau))$ is a curve transverse to rulings, and we have written $c_1(\tau) = \sin\theta(\tau)$ and $c_2(\tau) = -\cos\theta(\tau)$.

Example. In (4.9) we take $\gamma(\tau) \equiv (x_0(\tau), y_0(\tau), z_0(\tau)) = (\cos\tau, \sin\tau, 0)$ and $\theta(\tau) = \tau$, $0 \le \tau < 2\pi$. It is easy to see that $e_1(\tau) = (\sin\tau, -\cos\tau, 1)$ (note that e_1 is independent of s). Compute $e_1(\tau_1) \times e_1(\tau_2) \cdot (\gamma(\tau_2) - \gamma(\tau_1)) = (\sin\tau_2 - \sin\tau_1)^2 + (\cos\tau_2 - \cos\tau_1)^2$. So $e_1(\tau_1) \times e_1(\tau_2) \cdot (\gamma(\tau_2) - \gamma(\tau_1)) = 0$ if and only if $\tau_1 = \tau_2$. Now it is easy to see that this Legendrian ruled surface is embedded. Let us write down the x, y, z components as follows:

$$x(\tau, s) = \cos\tau + (\sin\tau)s,\ y(\tau, s) = \sin\tau - (\cos\tau)s,\ z(\tau, s) = s\,. \tag{4.10}$$

So $\partial_\tau(x, y, z) = (\frac{\partial x}{\partial\tau}, \frac{\partial y}{\partial\tau}, \frac{\partial z}{\partial\tau}) = (-\sin\tau + (\cos\tau)s, \cos\tau + (\sin\tau)s, 0)$ and $\Theta_0\ (\partial_\tau(x, y, z)) = 1 + s^2 \neq 0$. This means that the tangent vector $\partial_\tau(x, y, z))$ is not annihilated by the contact form Θ_0. Therefore (4.10) defines a properly embedded p-minimal surface in H_1 with no singular points, which is not a vertical plane (i.e. having the equation $ax + by = c$). In fact, eliminating the parameters τ and s in (4.10) gives the equation $z^2 = x^2 + y^2 - 1$.

For a Legendrian ruled surface of graph type, we can have an alternative approach to show that it is p-minimal. Let $(x, y, u(x, y))$ describe such a Legendrian ruled surface. Suppose we can take x as the parameter of the rulings (straight lines) for simplicity. Then $d^2/dx^2\{u(x, y(x))\} = 0$ along a ruling. By the chain rule we have

$$r + 2sa + ta^2 = 0 \tag{4.11}$$

where $a = \frac{dy}{dx}$, $r = u_{xx}$, $s = u_{xy}$, and $t = u_{yy}$. On the other hand, along a Legendrian line, we have the contact form $dz + xdy - ydx = 0$. It follows that $\frac{dz}{dx} + xa - y = p + qa + xa - y = 0$ where $p = u_x$, $q = u_y$. So $a = -\frac{p-y}{q+x}$ (if $q + x = 0$, then $p - y = 0$). Substituting this into (4.11) gives

$$(q + x)^2 r - 2(q + x)(p - y)s + (p - y)^2 t = 0$$

which is exactly (pMGE). We remark that a general ruled surface satisfies a third order partial differential equation (see page 225 in [Mo]. Solving (4.11) for "a" in terms of r, s, t, and substituting the result into $d^3/dx^3\{u(x, y(x))\} = 0$ give such an equation).

5. Comparison principle and uniqueness for the Dirichlet problem

Let Ω be a domain (connected and proper open subset) in the xy-plane. Let $u, v : \Omega \to R$ be two C^1 functions. Recall the singular set $S(u) = \{(x, y) \in \Omega \mid u_x - y = 0,\ u_y + x = 0\}$ and $N(u) = [\nabla u + (-y, x)]D_u^{-1}$ where $D_u = \sqrt{(u_x - y)^2 + (u_y + x)^2}$ (e.g. see (2.15a)).

Lemma 5.1. *Suppose we have the situation described above. Then the equality*

$$(\nabla u - \nabla v) \cdot (N(u) - N(v)) = \frac{D_u + D_v}{2} \mid N(u) - N(v) \mid^2 \tag{5.1}$$

holds on $\Omega\backslash(S(u) \cup S(v))$. *In particular,* $(\nabla u - \nabla v) \cdot (N(u) - N(v)) = 0$ *if and only if* $N(u) = N(v)$.

Proof. Let $\vec{\alpha} = \nabla u + (-y, x)$, $\vec{\beta} = \nabla v + (-y, x)$. Noting that $N(u) = \frac{\vec{\alpha}}{|\vec{\alpha}|}$, $N(v) = \frac{\vec{\beta}}{|\vec{\beta}|}$ ($D_u = |\vec{\alpha}|$, $D_v = |\vec{\beta}|$), we compute

$$\begin{aligned}(\nabla u - \nabla v) \cdot (N(u) - N(v)) &= (\vec{\alpha} - \vec{\beta}) \cdot \left(\frac{\vec{\alpha}}{|\vec{\alpha}|} - \frac{\vec{\beta}}{|\vec{\beta}|} \right) \\ &= |\vec{\alpha}| + |\vec{\beta}| - \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\beta}|} - \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}|} \\ &= (|\vec{\alpha}| + |\vec{\beta}|)(1 - \cos \vartheta)\end{aligned} \tag{5.2}$$

in which $\cos \vartheta = \frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}||\vec{\beta}|}$. On the other hand, we compute $| N(u) - N(v) |^2 = | \frac{\vec{\alpha}}{|\vec{\alpha}|} - \frac{\vec{\beta}}{|\vec{\beta}|} |^2 = 2 - 2\frac{\vec{\alpha} \cdot \vec{\beta}}{|\vec{\alpha}||\vec{\beta}|} = 2(1 - \cos \vartheta)$. Substituting this into the right-hand side of (5.2) gives (5.1). □

Remark. For the prescribed mean curvature equation $\operatorname{div} Tu = H$ in R^n where $Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, we have the following structural inequality:

$$\begin{aligned}(\nabla u - \nabla v) \cdot (Tu - Tv) &\geq \frac{\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2}}{2} |Tu - Tv|^2 \\ &\geq |Tu - Tv|^2 .\end{aligned}$$

The above inequality was discovered by Miklyukov [Mik], Hwang [Hw1], and Collin-Krust [CK] independently. Here we have adopted Hwang's method to prove Lemma 5.1.

Next let $u \in C^0(\bar{\Omega} \backslash S_1)$, $v \in C^0(\bar{\Omega} \backslash S_2)$, i.e., u, v are not defined (may blow up) on sets S_1, $S_2 \subset \Omega$, respectively. Let $S \equiv S_1 \cup S_2 \cup S(u) \cup S(v)$ where $S(u) \subset \Omega \backslash S_1$, $S(v) \subset \Omega \backslash S_2$.

Theorem 5.2. *Suppose Ω is a bounded domain in the xy-plane and $\mathcal{H}_1(\bar{S})$, the 1-dimensional Hausdorff measure of $\bar{S}$, vanishes. Let $u \in C^0(\bar{\Omega} \backslash S_1) \cap C^2(\Omega \backslash S)$, $v \in C^0(\bar{\Omega} \backslash S_2) \cap C^2(\Omega \backslash S)$ such that*

$$\operatorname{div} N(u) \geq \operatorname{div} N(v) \quad \text{in } \Omega \backslash S , \tag{5.3}$$

$$u \leq v \qquad \text{on } \partial\Omega \backslash S . \tag{5.4}$$

Then $N(u) = N(v)$ in $\Omega^+ \backslash S$ where $\Omega^+ \equiv \{p \in \Omega \mid u(p) - v(p) > 0\}$.

Proof. First $\mathcal{H}_1(\bar{S}) = 0$ means that given any $\epsilon > 0$, we can find countably many balls $B_{j,\epsilon}$, $j = 1, 2, \ldots$ such that $\bar{S} \subset \cup_{j=1}^{\infty} B_{j,\epsilon}$ and $\Sigma_{j=1}^{\infty}$ length $(\partial B_{j,\epsilon}) < \epsilon$ and we can arrange $\cup_{j=1}^{\infty} B_{j,\epsilon_1} \subset \cup_{j=1}^{\infty} B_{j,\epsilon_2}$ for $\epsilon_1 < \epsilon_2$. Since $\bar{S}$ is compact, we can find finitely many $B_{j,\epsilon}$'s, say $j = 1, 2, \ldots, n(\epsilon)$, still covering $\bar{S}$. Suppose $\Omega^+ \neq \emptyset$. Then by Sard's theorem there exists a sequence of positive number δ_i converging to 0 as i goes to infinity, such that $\Omega_i^+ \equiv \{p \in \Omega \mid u(p) - v(p) > \delta_i\}$

$\neq \emptyset$ and $\partial\Omega_i^+ \backslash S$ is C^2 smooth. Note that $\partial\Omega_i^+ \cap \partial\Omega \subset S$ by (5.4). Now we consider

$$I_\epsilon^i = \oint_{\partial(\Omega_i^+ \backslash \cup_{j=1}^{n(\epsilon)} B_{j,\epsilon})} \tan^{-1}(u-v)(N(u)-N(v)) \cdot \nu ds$$

where ν, s denote the outward unit normal vector and the arc length parameter, respectively. By the divergence theorem we have

$$\begin{aligned} I_\epsilon^i = \iint_{\Omega_i^+ \backslash \cup_{j=1}^{n(\epsilon)} B_{j,\epsilon}} \Bigg\{ & \frac{1}{1+(u-v)^2}(\nabla u - \nabla v) \cdot (N(u)-N(v)) \\ & + \tan^{-1}(u-v)\,\mathrm{div}(N(u)-N(v)) \Bigg\} dxdy\,. \end{aligned} \tag{5.5}$$

Observe that the second term in the right hand side of (5.5) is nonnegative by (5.3). It follows from (5.1) and (5.5) that

$$I_\epsilon^i \geq \iint_{\Omega_i^+ \backslash \cup_{j=1}^{n(\epsilon)} B_{j,\epsilon}} \left\{ \frac{1}{1+(u-v)^2} \left(\frac{D_u + D_v}{2} \right) |N(u)-N(v)|^2 \right\} dxdy\,. \tag{5.6}$$

On the other hand, we can estimate

$$\begin{aligned} I_\epsilon^i \leq (\tan^{-1}\delta_i) & \int_{\partial\Omega_i^+ \backslash (\cup_{j=1}^{n(\epsilon)} B_{j,\epsilon})} (N(u)-N(v)) \cdot \nu ds \\ & + \frac{\pi}{2} \cdot 2 \cdot \Sigma_{j=1}^{n(\epsilon)} \text{ length } (\partial B_{j,\epsilon}) \leq \pi \cdot \Sigma_{j=1}^{\infty} \text{ length } (\partial B_{j,\epsilon}) < \pi\epsilon \end{aligned} \tag{5.7}$$

by noting that $\nu = -\frac{\nabla(u-v)}{|\nabla(u-v)|}$ and hence $(N(u)-N(v)) \cdot \nu \leq 0$ by (5.1). If $N(u) \neq N(v)$ at some point p in $\Omega^+ \backslash \bar{S}$, then $N(u) \neq N(v)$ in an open neighborhood V of p, contained in Ω_i^+ for all large i. Observe that the measure of $V \backslash \cup_{j=1}^{\infty} B_{j,\epsilon}$ is bounded from below by a positive constant independent of small enough ϵ and i. Thus from (5.6) $I_\epsilon^i \geq c$, a positive constant independent of small enough ϵ and large enough i. Letting ϵ go to 0 in (5.7) will give us a contradiction. So $N(u) = N(v)$ in $\Omega^+ \backslash \bar{S}$ and hence in $\Omega^+ \backslash S$ by continuity. □

Remark. Theorem 5.2 is an analogue of Concus and Finn's general comparison principles for the prescribed mean curvature equation (cf. Theorem 6 in [CF]). In [Hw2] Hwang invoked the "$\tan^{-1}$" technique to simplify the proof of [CF]. Here we have followed the idea of Hwang in [Hw2] to prove Theorem 5.2.

Lemma 5.3. *Let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ where Ω is a bounded domain in the xy-plane. Suppose $N(u) = N(v)$ in $\Omega \backslash (S(u) \cup S(v))$, $u = v$ on $\partial\Omega$. Then $u = v$ in Ω.*

Proof. Suppose $u \neq v$ in Ω. We may assume the set $\{p \in \Omega | u(p) > v(p)\} \neq \emptyset$ (otherwise, interchange u and v). By Sard's theorem (e.g., [St], noting that C^2 is essential), there exists $\epsilon > 0$ such that $\Omega_\epsilon \equiv \{p \in \Omega \mid u(p) - v(p) - \epsilon > 0\} \neq \emptyset$ and $\Gamma_\epsilon \equiv \{p \in \Omega \mid u(p) - v(p) = \epsilon\}$ is C^2 smooth. Note that $\bar{\Gamma}_\epsilon \cap \partial\Omega = \emptyset$ since $u = v$ on $\partial\Omega$ by assumption. Also Γ_ϵ is closed and bounded, hence compact. Therefore Γ_ϵ is the union of (finitely-many) C^2 smooth loops. Choose one of them, and denote it as Γ'_ϵ. We claim

$$\frac{du}{ds} + x\frac{dy}{ds} - y\frac{dx}{ds} = 0 \tag{5.8}$$

on Γ'_ϵ where s is a unit-speed parameter of Γ'_ϵ. For $p \in \Gamma'_\epsilon \cap S(u)$, (5.8) holds by the definition of a singular point. For $p \in \Gamma'_\epsilon \backslash (S(u) \cup S(v))$, we compute $N^\perp(u)u$ ($N^\perp(u)$ as an operator acting on u) as follows:

$$\begin{aligned} N^\perp(u)u &= N^\perp(u) \cdot \nabla u = D^{-1}\{(\nabla u)^\perp + (x, y)\} \cdot \nabla u \\ &= D^{-1}(x, y) \cdot \nabla u = D^{-1}(x, y) \cdot \{\nabla u + (-y, x)\} \\ &= (x, y) \cdot N(u)\,. \end{aligned} \tag{5.9}$$

Similarly we can show

$$N^\perp(v)v = (x, y) \cdot N(v)\,. \tag{5.10}$$

Since $N(u) = N(v)$, hence $N^\perp(u) = N^\perp(v)$ at p, we conclude that $N^\perp(u)(u - v) = 0$ at p by (5.9) and (5.10). This means that $N^\perp(u)$ is tangent to Γ'_ϵ at p. So (5.8) holds at p. For $p \in (\Gamma'_\epsilon \backslash S(u)) \cap S(v)$, $\vec{\beta}$ $(\equiv \nabla v + (-y, x)) = 0$. We observe that $\vec{\alpha}$ $(\equiv \nabla u + (-y, x)) = \vec{\alpha} - \vec{\beta} = \nabla(u - v)$. This means that $N(u)$ $(\equiv \frac{\vec{\alpha}}{|\vec{\alpha}|})$ is normal to Γ'_ϵ. So again $N^\perp(u)$ is tangent to Γ'_ϵ at p. Thus (5.8) holds at p. We have shown that (5.8) holds for all $p \in \Gamma'_\epsilon$. Γ'_ϵ bounds a domain, denoted as Ω'_ϵ. Now integrating (5.8) over Γ'_ϵ, we obtain that the area of Ω'_ϵ vanishes by the divergence theorem, an obvious contradiction. □

Proof of Theorem C. It follows from Theorem 5.2 and Lemma 5.3. □

We can generalize Lemma 5.1 in the following form. Let Ω be a domain in R^n. Let $u, v : \Omega \to R$ be two C^1 functions. Let $\vec{F}$ be a C^0 vector field in R^n. Define $S(u, \vec{F}) = \{p \in \Omega \mid \nabla u + \vec{F} = 0 \text{ at } p\}$ and $S(v, \vec{F})$ similarly.

Lemma 5.1′. *On* $\Omega \backslash [S(u, \vec{F}) \cup S(v, \vec{F})]$, *we have the following identity*:

$$(\nabla u - \nabla v) \cdot \left(\frac{\vec{\alpha}}{|\vec{\alpha}|} - \frac{\vec{\beta}}{|\vec{\beta}|}\right) = \left(\frac{|\vec{\alpha}| + |\vec{\beta}|}{2}\right)\left|\frac{\vec{\alpha}}{|\vec{\alpha}|} - \frac{\vec{\beta}}{|\vec{\beta}|}\right|^2$$

where $\vec{\alpha} = \nabla u + \vec{F}$, $\vec{\beta} = \nabla v + \vec{F}$.

In general a contact form $dz + \Sigma_{j=1}^{j=n} f_j dx^j$ in R^{n+1} gives rise to an $\vec{F} = (f_1, f_2, \ldots, f_n)$ such that $\nabla u + \vec{F}$ is the R^n-projection of the Legendrian normal

to the graph $z = u(x^1, x^2, \dots, x^n)$. To generalize Theorem 5.2 to a domain Ω in R^n and replace $N(u)$, $N(v)$ by $\frac{\vec{\alpha}}{|\vec{\alpha}|}$, $\frac{\vec{\beta}}{|\vec{\beta}|}$, we will use $S_{\vec{F}} = S_1 \cup S_2 \cup S(u, \vec{F}) \cup S(v, \vec{F})$ instead of S.

Theorem 5.2′. *Suppose Ω is a bounded domain in R^n and $\mathcal{H}_{n-1}(\bar{S}_{\vec{F}}) = 0$. Let $u \in C^0(\bar{\Omega}\backslash S_1) \cap C^2(\Omega\backslash S_{\vec{F}})$, $v \in C^0(\bar{\Omega}\backslash S_2) \cap C^2(\Omega\backslash S_{\vec{F}})$, and $\vec{F} \in C^1(\Omega) \cap C^0(\bar{\Omega})$ such that*

$$\operatorname{div}\left(\frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}\right) \geq \operatorname{div}\left(\frac{\nabla v + \vec{F}}{|\nabla v + \vec{F}|}\right) \quad \text{in} \quad \Omega\backslash S_{\vec{F}}\,,$$

$$u \leq v \quad \text{on} \quad \partial\Omega\backslash S_{\vec{F}}\,.$$

Then $\frac{\nabla u+\vec{F}}{|\nabla u+\vec{F}|} = \frac{\nabla v+\vec{F}}{|\nabla v+\vec{F}|}$ in $\Omega^+\backslash S_{\vec{F}}$ where $\Omega^+ \equiv \{p \in \Omega \mid u(p) - v(p) > 0\}$.

The proof of Lemma 5.1′ (Theorem 5.2′, respectively) is similar to that of Lemma 5.1 (Theorem 5.2, respectively). We can also generalize Lemma 5.3. Let Ω be a bounded domain in R^{2m}, $m \geq 1$. Take two real functions $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Let $\vec{\alpha} \equiv \nabla u + \vec{F}$ where $\vec{F} = (f_1, f_2, \dots, f_{2m})$ is a C^1 smooth vector field on Ω. Define $\vec{F}^* \equiv (f_2, -f_1, f_4, -f_3, \dots, f_{2m}, -f_{2m-1})$. Denote $\frac{\vec{\alpha}}{|\vec{\alpha}|}$ by $N_{\vec{F}}(u)$ and the set $\{p \in \Omega \mid \vec{\alpha} = 0\}$ by $S_{\vec{F}}(u)$.

Lemma 5.3′. *Suppose we have the situation as described above. Suppose $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ in $\Omega\backslash(S_{\vec{F}}(u) \cup S_{\vec{F}}(v))$, $u = v$ on $\partial\Omega$, and $\operatorname{div} \vec{F}^* > 0$ a.e. in Ω. Then $u = v$ on $\bar{\Omega}$.*

Proof. Suppose the conclusion is not true. We may assume the set $\{p \in \Omega \mid u(p) > v(p)\} \neq \emptyset$. By Sard's theorem we can find a small $\epsilon > 0$ such that $\Omega_\epsilon \equiv \{p \in \Omega \mid u(p) - v(p) - \epsilon > 0\} \neq \emptyset$ and $\Gamma_\epsilon \equiv \{p \in \Omega \mid u(p) - v(p) = \epsilon\} = \partial\Omega_\epsilon$ is C^2 smooth. Let $\vec{\alpha}^* \equiv (u_{y_1}, -u_{x_1}, u_{y_2}, -u_{x_2}, \dots, u_{y_m}, -u_{x_m}) + \vec{F}^*$ (so that $\vec{\alpha}^* \cdot \vec{\alpha} = 0$). Let $\nu = -\frac{\nabla(u-v)}{|\nabla(u-v)|}$ denote the outward unit normal to Γ_ϵ. We claim

$$\vec{\alpha}^* \cdot \nu = 0 \tag{5.11}$$

on Γ_ϵ. Note that $\vec{\alpha} = 0$ if and only if $\vec{\alpha}^* = 0$. So it is obvious that (5.11) holds for $p \in S_{\vec{F}}(u)$. Let $N^*_{\vec{F}}(u) \equiv \frac{\vec{\alpha}^*}{|\vec{\alpha}^*|}$ for $p \in \Omega\backslash S_{\vec{F}}(u)$. In case $p \in \Gamma_\epsilon\backslash(S_{\vec{F}}(u) \cup S_{\vec{F}}(v))$, we can generalize (5.9), (5.10) as follows:

$$N^*_{\vec{F}}(u)u = \vec{F}^* \cdot N_{\vec{F}}(u)\,, \tag{5.9′}$$

$$N^*_{\vec{F}}(v)v = \vec{F}^* \cdot N_{\vec{F}}(v)\,. \tag{5.10′}$$

Since $N_{\vec{F}}(u) = N_{\vec{F}}(v)$ by assumption, and hence $N^*_{\vec{F}}(u) = N^*_{\vec{F}}(v)$, we deduce from (5.9′) and (5.10′) that $N^*_{\vec{F}}(u)(u - v) = 0$. So $N^*_{\vec{F}}(u)$ is tangent to Γ_ϵ (at p). This implies (5.11). For $p \in (\Gamma_\epsilon\backslash S_{\vec{F}}(u)) \cap S_{\vec{F}}(v)$, we still have (5.11) by a similar argument as in the proof of Lemma 5.3. We have proved (5.11)

for all $p \in \Gamma_\epsilon$. Let dA denote the volume element of Γ_ϵ, induced from R^{2m}. Now we compute

$$\begin{aligned} 0 &= \int_{\Gamma_\epsilon} \vec{\alpha}^* \cdot \nu dA \text{ (by (5.11))} \\ &= \int_{\Omega_\epsilon} \operatorname{div}(\vec{\alpha}^*)\ d \text{ (volume) (by the divergence theorem)} \qquad (5.12) \\ &= \int_{\Omega_\epsilon} \operatorname{div} \vec{F}^* d(\text{volume}) > 0 \end{aligned}$$

by assumption. We have reached a contradiction. □

We remark that the condition $\operatorname{div} \vec{F}^* > 0$ is essential in Lemma 5.3′. Consider the case $\vec{F} = 0$. Let $\Omega = B_2 - \bar{B}_1$ where B_r denotes the open ball of radius r. Let $u = f(r), v = g(r)$, and $f \neq g$ with the properties that $f(1) = g(1)$, $f(2) = g(2)$, and $f' > 0, g' > 0$ for $1 \leq r \leq 2$. It follows that $S_{\vec{F}}(u) = \{\nabla u = 0\} = \phi$, $S_{\vec{F}}(v) = \{\nabla u = 0\} = \phi$, and $\nabla u = f'(r)\nabla r$, $\nabla v = g'(r)\nabla r$. Therefore we have

$$\frac{\nabla u}{|\nabla u|} = \nabla r = \frac{\nabla v}{|\nabla v|}$$

by noting that $|\nabla r| = 1$. We have constructed a counterexample for the statement of Lemma 5.3′ if $\operatorname{div} \vec{F}^* > 0$ is not satisfied.

For $\vec{F} = (-y_1, x_1, -y_2, x_2, \ldots, -y_m, x_m)$, we have $\vec{F}^* = (x_1, y_1, x_2, y_2, \ldots, x_m, y_m)$. In this case, we can view the integrand in (5.12) geometrically: ($\widehat{dx_j \wedge dy_j}$ means deleting $dx_j \wedge dy_j$)

$$\begin{aligned} \vec{\alpha}^* \cdot \nu dA &= \Sigma_{j=1}^{j=m} [(u_{y_j} + x_j) dy_j + (u_{x_j} - y_j) dx_j] \\ &\quad \wedge dx_1 \wedge dy_1 \wedge \ldots \wedge \widehat{dx_j \wedge dy_j} \wedge \ldots \wedge dx_m \wedge dy_m \\ &= [du + \Sigma_{j=1}^{j=m} (x_j dy_j - y_j dx_j)] \\ &\quad \wedge (\Sigma_{j=1}^{j=m} dx_1 \wedge dy_1 \wedge \ldots \wedge \widehat{dx_j \wedge dy_j} \wedge \ldots \wedge dx_m \wedge dy_m) \\ &= \frac{1}{2^{m-1}(m-1)!} \Theta_{(m)} \wedge (d\Theta_{(m)})^{m-1}. \end{aligned}$$

Here $\Theta_{(m)} \equiv du + \Sigma_{j=1}^{j=m} (x_j dy_j - y_j dx_j)$ is the standard contact form of the $2m+1$-dimensional Heisenberg group, restricted to the hypersurface $\{(x_1, y_1, x_2, y_2, \ldots, x_m, y_m, u(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)\}$. Integrating the above form gives

$$\begin{aligned} \int_{\Gamma_\epsilon} \vec{\alpha}^* \cdot \nu dA &= \frac{1}{2^{m-1}(m-1)!} \int_{\partial\Omega_\epsilon} \Theta_{(m)} \wedge (d\Theta_{(m)})^{m-1} \\ &= \frac{1}{2^{m-1}(m-1)!} \int_{\Omega_\epsilon} (d\Theta_{(m)})^m \text{ (by Stokes' theorem)} \\ &= \frac{1}{2^{m-1}(m-1)!} 2^m m! \text{ Volume } (\Omega_\epsilon) = 2m \text{ Volume } (\Omega_\epsilon). \end{aligned}$$

In the last equality, we have used $d\Theta_{(m)} = 2\Sigma_{j=1}^{j=m} dx_j \wedge dy_j$ and hence $(d\Theta_{(m)})^m = 2^m m! dx_1 \wedge dy_1 \wedge \ldots \wedge dx_m \wedge dy_m$. Note that div $\vec{F}^* = 2m$ in this case. In general, let $\Theta_{\vec{F}} \equiv dz + \Sigma_{j=1}^{j=2m} f_j dx_j$ for $\vec{F} = (f_1, f_2, \ldots, f_{2m})$. We can easily compute

$$d\Theta_{\vec{F}} \wedge \Sigma_{j=1}^{j=m} (dx_1 \wedge dx_2 \wedge \ldots \wedge \widehat{dx_{2j-1} \wedge dx_{2j}} \wedge \ldots \wedge dx_{2m-1} \wedge dx_{2m})$$
$$= (\operatorname{div} \vec{F}^*) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{2m-1} \wedge dx_{2m} .$$

Note that in case $\Theta_{\vec{F}} = \Theta_{(m)}$ (with u, x_j, y_j replaced by z, x_{2j-1}, x_{2j}, respectively), we have

$$(d\Theta_{\vec{F}})^{m-1} = (d\Theta_{(m)})^{m-1}$$
$$= 2^{m-1}(m-1)! \Sigma_{j=1}^{j=m} (dx_1 \wedge dx_2 \wedge \ldots \wedge \widehat{dx_{2j-1} \wedge dx_{2j}} \wedge \ldots \wedge dx_{2m-1} \wedge dx_{2m}) .$$

We can generalize Theorem C to higher dimensions without the condition on the singular set. Let $N(u) = N_{\vec{F}}(u)$ and $S(u) = S_{\vec{F}}(u)$ for $\vec{F} = (-y_1, x_1, -y_2, x_2, \ldots, -y_m, x_m)$.

Theorem C′. *For a bounded domain Ω in R^{2m}, $m \geq 2$, let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy* div $N(u) \geq$ div $N(v)$ *in $\Omega \backslash S$ and $u \leq v$ on $\partial\Omega$ where $S = S(u) \cup S(v)$. Then $u \leq v$ in Ω.*

First we observe the following size control of the singular set in general dimensions.

Lemma 5.4. *Suppose $u \in C^2(\Omega)$ where $\Omega \subset R^{2m}$. Then for any singular point $p \in S(u)$, there exists an open neighborhood $V \subset \Omega$ such that the m-dimensional Hausdorff measure of $S(u) \cap V$ is finite, and hence $\mathcal{H}_{2m-1}(S(u)) = 0$ for $m \geq 2$.*

Proof. Consider the map $G : p \in \Omega \to (\nabla u + \vec{F})(p)$. Computing the differential dG of G at a singular point p (where $G(p) = 0$), we obtain

$$(u_{ij}) + \begin{pmatrix} 0 & -1 & 0 & 0 & . & . \\ 1 & 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & -1 & . & . \\ 0 & 0 & 1 & 0 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}$$

in matrix form, where (u_{ij}) is the Hessian. Let $(dG)^T$ denote the transpose of dG. It is easy to see that $2m = \operatorname{rank}(dG - (dG)^T)$ since $u_{ij} = u_{ji}$. On the other hand, $\operatorname{rank}(dG - (dG)^T) \leq \operatorname{rank}(dG) + \operatorname{rank}(-dG)^T = 2\operatorname{rank}(dG)$. Therefore $\operatorname{rank}(dG) \geq m$. Hence the kernel of dG has dimension $\leq m$. It follows by the implicit function theorem that there exists an open neighborhood V of p such that $G^{-1}(0) \cap V = S(u) \cap V$ is a submanifold of V, having dimension $\leq m$. □

Proof of Theorem C′. Observe that the condition $\mathcal{H}_{2m-1}(\bar{S}) = 0$ (the dimension $n = 2m$) in the proof of Theorem 5.2 (and Theorem 5.2′) can be replaced by the following condition: for any subdomain $O \subset\subset \Omega$, i.e., $\bar{O} \subset \Omega$, $\mathcal{H}_{2m-1}(\bar{O} \cap \bar{S}) = 0$. Since $S = S(u) \cup S(v)$ is closed in the compact set $\bar{O}$, $\bar{O} \cap \bar{S} = \bar{O} \cap S$. Now Theorem C′ follows from Theorem 5.2′ (with the size control condition on $\bar{S}$ replaced by the above-mentioned one), Lemma 5.3′, and Lemma 5.4. □

6. Second variation formula and area-minimizing property

In this section we will derive the second variation formula for the p-area functional (2.5) and examine the p-mean curvature H from the viewpoint of calibration geometry ([HL]). As a result we can prove the area-minimizing property for a p-minimal graph in H_1.

We follow the notation in Section 2. We assume the surface Σ is p-minimal. Let f, g be functions with compact support away from the singular set and the boundary of Σ. Recall T denotes the Reeb vector field of Θ (see Section 2 or the Appendix). We compute the second variation of (2.5) in the direction $V = fe_2 + gT$:

$$\delta_V^2 \int_\Sigma \Theta \wedge e^1 = \int_\Sigma L_V^2(\Theta \wedge e^1) = \int_\Sigma i_V \circ d\{i_V \circ d(\Theta \wedge e^1)\}\,. \tag{6.1}$$

Here we have used Stokes' theorem and the formula $L_V = i_V \circ d + d \circ i_V$ and $d^2 = 0$. By (2.7) and $H = \omega(e_1)$, we get

$$d(\Theta \wedge e^1) = -H\Theta \wedge e^1 \wedge e^2\,. \tag{6.2}$$

We recall (see Section 2) to define a function α on $\Sigma \backslash S_\Sigma$ such that $\alpha e_2 + T \in T\Sigma$. Observe that $\{\alpha e_2 + T, e_1\}$ is a basis of $T(\Sigma \backslash S_\Sigma)$. So on $\Sigma \backslash S_\Sigma$ we have

$$e^2 \wedge e^1 = \alpha\Theta \wedge e^1\,. \tag{6.3}$$

From (6.2) it is easy to see that $i_V \circ d(\Theta \wedge e^1) = gHe^2 \wedge e^1 - fH\Theta \wedge e^1$. Then taking $i_V \circ d$ of this expression and making use of (A.1r), (A.3r), (6.3) and $H = 0$ on Σ, we obtain

$$\begin{aligned} i_V \circ d\{i_V \circ d(\Theta \wedge e^1)\} &= (g\alpha - f)(gT + fe_2)(H)\Theta \wedge e^1 \\ &= -(g\alpha - f)^2 e_2(H)\Theta \wedge e^1 \end{aligned} \tag{6.4}$$

on Σ. For the last equality we have used $T(H) = -\alpha e_2(H)$ since $\alpha e_2 + T \in T\Sigma$ and $H = 0$ on Σ. Expanding the left-hand side of (A.5r) gives

$$e_2(H) = 2W + e_1(\omega(e_2)) + 2\omega(T) + (\omega(e_2))^2\,. \tag{6.5}$$

Here we have used (A.6r) and $\omega(e_1) = H = 0$ on Σ. The surfaces $\varphi_t(\Sigma \backslash S_\Sigma)$ are the level sets of a defining function ρ. Here $\dot{\varphi}_t = fe_2 + gT$. It follows that $(fe_2 + gT)(\rho) = 1$. On the other hand, $(\alpha e_2 + T)(\rho) = 0$ from the definition of α. So $T(\rho) = -\alpha e_2(\rho)$ and $e_2(\rho) = (f - \alpha g)^{-1}$ (where $f - \alpha g \neq 0$). Applying (A.6r) and (A.7r) to ρ, and using the above formulas, we obtain

$$\omega(e_2) = h^{-1}e_1(h) + 2\alpha\,, \tag{6.6a}$$

$$\omega(T) = e_1(\alpha) - \alpha h^{-1}e_1(h) - \mathrm{Im}A_{11} \tag{6.6b}$$

where $h = f - \alpha g$. Now substituting (6.6a), (6.6b) into (6.5), we get

$$e_2(H) = 2W - 2\mathrm{Im}A_{11} + 4e_1(\alpha) + 4\alpha^2 + h^{-1}e_1^2(h) + 2\alpha h^{-1}e_1(h)\,. \tag{6.7}$$

Observing that $e_1(e_1(h^2))\Theta\wedge e^1 = \Theta\wedge d(e_1(h^2)) = -d(e_1(h^2)\Theta)+2e_1(h^2)\alpha e^1\wedge\Theta$ on Σ by (A.1r) and (6.3), we integrate $\frac{1}{2}e_1(e_1(h^2)) = (e_1(h))^2+he_1^2(h)$ to obtain

$$-\int_\Sigma he_1^2(h)\Theta\wedge e^1 = \int_\Sigma [(e_1(h))^2 + 2\alpha h e_1(h)]\Theta\wedge e^1\,.$$

Substituting (6.7) into (6.4) and (6.4) into (6.1) and using the above formula, we finally reach the following second variation formula.

Proposition 6.1. *Suppose the surface Σ is p-minimal as defined in Section 2. Let f, g be functions with compact support away from the singular set and the boundary of Σ. Then*

$$\begin{aligned}&\delta^2_{fe_2+gT}\int_\Sigma \Theta\wedge e^1\\ &=\int_\Sigma \{(e_1(f-\alpha g))^2+(f-\alpha g)^2[-2W+2ImA_{11} - 4e_1(\alpha) - 4\alpha^2]\}\Theta\wedge e^1\,.\end{aligned} \tag{6.8}$$

Note that the Webster-Tanaka curvature W and the torsion A_{11} are geometric quantities of the ambient pseudohermitian 3-manifold M. When the torsion A_{11} vanishes and W is positive, we can easily discuss the stability of a p-minimal surface (see Example 2 in Section 7). If both W and A_{11} vanish, e.g. in the case of $M = H_1$ (see the Appendix), we can compute α, $e_1(\alpha)$ for a graph $z = u(x, y)$ as follows. First note that the defining function $\rho = (z-u(x, y))D^{-1}$ satisfies the condition $e_2(\rho) = 1$ (recall $e_2 = -[(u_x - y)\hat{e}_1 + (u_y + x)\hat{e}_2]D^{-1}$ in Section 2). So $\alpha = -T(\rho) = -\partial\rho/\partial z = -D^{-1}$ and a direct computation shows (recall $e_1 = [-(u_y + x)\hat{e}_1 + (u_x - y)\hat{e}_2]D^{-1}$ in Section 2) that

$$\begin{aligned}-4e_1(\alpha) - 4\alpha^2 = 4\{&(u_x - y)(u_y + x)(u_{xx} - u_{yy})\\ &+ [(u_y + x)^2 - (u_x - y)^2]u_{xy}\}D^{-4}\,.\end{aligned} \tag{6.9}$$

For example if $u = xy$, the right-hand side of (6.9) equals $1/x^2$. So away from the singular set, the second variation of the p-area is nonnegative according to (6.8) (it is easy to see that the second variation in the e_1 direction always vanishes). Note that $\{x = 0\}$ is the singular set in this example. From the p-area minimizing property shown below (Proposition 6.2), we know the second variation of the p-area for any p-minimal graph over the xy-plane with no singular points must be nonnegative.

If we consider only the variation in the T direction, i.e., $f = 0$, we should combine the term in (6.9) with terms involving $e_1(\alpha)$ in the expansion of $(e_1(-\alpha g))^2$ (to get a better expression of (6.8)). For instance, take a graph

$(x, y, u(x, y)) \in H_1$ over a domain Ω in the xy-plane. We denote the energy functional for the p-area by

$$E(u) = \int_\Omega D dx \wedge dy$$

in view of (2.5) and (2.11). A direct computation shows that $\frac{d^2}{d\varepsilon^2}|_{\varepsilon=0} E(u + \varepsilon v) = \int_\Omega D^{-1}(e_1(v))^2 dx \wedge dy$ for a variation $v = v(x, y)$. On the other hand, this should be obtained from (6.8) by letting $f = 0$ and $g = v$ (with compact support away from the singular set and $\partial\Omega$). It turns out to be equivalent to verifying the following integral formula (note that $\alpha = -D^{-1}$, $\Theta \wedge e^1 = Ddx \wedge dy$)

$$\int_\Omega \{e_1(v^2)D^{-1}e_1(D^{-1}) + v^2[(e_1(D^{-1}))^2 + 4D^{-2}e_1(D^{-1}) - 4D^{-4}]\} Ddx \wedge dy = 0 .$$

We leave this verification to the reader (Hint: we need an integration by parts formula $-\int e_1(\varphi)\psi\Theta \wedge e^1 = -\int[\varphi e_1(\psi) + 2\varphi\psi\alpha]\Theta \wedge e^1$. Express $D^2 = (e_1(\sigma))^2 + (e_2(\sigma))^2$ where $\sigma = z - u(x, y)$. The following formulas: $e_1^2(\sigma) = e_1(\sigma) = 0$, $e_2(\sigma) = D$, $[e_1, e_2] = -2\partial_z - e_2(\theta)e_2$, $e_1(D^2) = -4D - 2e_2(\theta)D^2$, and $e_1^2(D^2) = 2(e_1(D))^2 + 4e_2(\theta)D + 4(e_2(\theta)D)^2$ are useful).

For later use, we deduce a different expression for $\Xi \equiv [-2W + 2\mathrm{Im}A_{11} - 4e_1(\alpha) - 4\alpha^2]\Theta \wedge e^1$ in (6.8). Noting that $e^2 \wedge \Theta = 0$ on Σ, we can easily get

$$d(\alpha\Theta) = [-e_1(\alpha) - 2\alpha^2]\Theta \wedge e^1 \tag{6.10}$$

by (A.1r) and (6.3). From (A.3r), (6.3), and (6.6a), we can relate $\mathrm{Im}A_{11}$ to $\omega(T)$ as follows:

$$(\mathrm{Im}A_{11})\Theta \wedge e^1 = -de^2 - [\omega(T) + \alpha h^{-1}e_1(h) + 2\alpha^2]\Theta \wedge e^1 . \tag{6.11}$$

In view of (6.10) and (6.11), we can express Ξ in the following form:

$$\Xi = -2[W + \omega(T) + \alpha h^{-1}e_1(h)]\Theta \wedge e^1 + d(4\alpha\Theta - 2e^2) . \tag{6.12}$$

In Euclidean 3-geometry, we take the interior product of the volume form with a vector field normal to a family of surfaces as a calibrating form ([HL]). This 2-form restricts to the area form on surfaces, and its exterior differentiation equals the mean curvature times the volume form along a surface. We have analogous results. Suppose M is foliated by a family of surfaces Σ_t, $-\varepsilon < t < \varepsilon$. Let e_1 be a vector field which is characteristic along any surface Σ_t. We are assuming Σ_t's have no singular points. Let $e_2 = Je_1$ denote the Legendrian normal along each Σ_t. Then the 2-form $\Phi = \frac{1}{2}i_{e_2}(\Theta \wedge d\Theta)$ satisfies the following properties. First, a direct computation shows that $\Phi = \Theta \wedge e^1$, our area 2-form from formula (A.1r). Secondly, $d\Phi = -H\Theta \wedge e^1 \wedge e^2$ by (6.2). So $\{\Sigma_t\}$ are p-minimal surfaces if and only if $d\Phi = 0$. Now suppose this is the case and Σ'

is a deformed surface with no singular points near a p-minimal surface $\Sigma = \Sigma_0$ having the same boundary. Also suppose the Poincaré lemma holds. That is to say, there is a 1-form Ψ such that $\Phi = d\Psi$. Then by Stokes' theorem, we have

$$\text{p-Area } (\Sigma) = \int_{\Sigma} \Phi = \int_{\partial\Sigma} \Psi = \int_{\partial\Sigma'} \Psi = \int_{\Sigma'} \Phi\,. \tag{6.13}$$

For Σ', we have corresponding $e_1', e_2', e^{1\prime}, e^{2\prime}$. There is a function α' such that $T + \alpha' e_2'$ is tangent to Σ'. Applying $\Phi = \Theta \wedge e^1$ to the basis $(T + \alpha' e_2', e_1')$ of $T\Sigma'$, we obtain $e^1(e_1')$. It follows that $\Phi = e^1(e_1')\Theta \wedge e^{1\prime}$ when restricted to Σ'. So we have

$$\begin{aligned} \int_{\Sigma'} \Phi &= \int_{\Sigma'} e^1(e_1')\Theta \wedge e^{1\prime} \\ &\le \int_{\Sigma'} \Theta \wedge e^{1\prime} = \text{ p-Area } (\Sigma') \text{ (since } e^1(e_1') \le 1)\,. \end{aligned} \tag{6.14}$$

From (6.13) and (6.14), we have shown that

$$\text{p-Area}(\Sigma) \le \text{ p-Area}(\Sigma')\,. \tag{6.15}$$

Let us summarize the above arguments in the following

Proposition 6.2. *Suppose we can foliate an open neighborhood of a p-minimal surface Σ by a family of p-minimal surfaces with no singular points, and in this neighborhood the Poincaré lemma holds (i.e., any closed 2-form is exact). Then Σ has the local p-area-minimizing property. That is to say, if Σ' is a deformed surface with no singular points near Σ having the same boundary, then* (6.15) *holds.*

We remark that a p-minimal surface in H_1 with no singular points, which is a graph over the xy-plane, satisfies the assumption in Proposition 6.2. Note that a translation of such a p-minimal graph in the z-axis is still p-minimal (quantitatively $u + c$ is again a solution if $u = u(x, y)$ is a solution to (pMGE)). Also a vertical (i.e. perpendicular to the xy-plane) plane in H_1 satisfies the assumption in Proposition 6.2. Note that a vertical plane is a p-minimal surface with no singular points, and a family of parallel such surfaces surely foliates an open neighborhood of a given one.

7. Closed p-minimal surfaces in the 3-sphere and proof of Theorem E

First let us describe the standard pseudohermitian 3-sphere $(S^3, \hat{J}, \hat{\Theta})$ (see the Appendix for the definition of basic notions). The unit 3-sphere S^3 in C^2 inherits a standard contact structure $\xi = TS^3 \cap J_{C^2}(TS^3)$ where J_{C^2} denotes the almost complex structure of C^2. The standard CR structure $\hat{J}$ compatible with ξ is nothing but the restriction of J_{C^2} on ξ. Let $r = |\zeta^1|^2 + |\zeta^2|^2 - 1$ where $(\zeta^1, \zeta^2) \in C^2$. The contact form $\hat{\Theta} \equiv -i\,\partial r = -i(\bar{\zeta}^1 d\zeta^1 + \bar{\zeta}^2 d\zeta^2)$ restricted to $S^3 \equiv \{r = 0\}$

gives rise to the Reeb vector field $\hat{T} = i\zeta^1\partial_{\zeta^1} + i\zeta^2\partial_{\zeta^2} - i\bar{\zeta}^1\partial_{\bar{\zeta}^1} - i\bar{\zeta}^2\partial_{\bar{\zeta}^2}$. Take the complex vector field $\hat{Z}_1 = \bar{\zeta}^2\partial_{\zeta^1} - \bar{\zeta}^1\partial_{\zeta^2}$ and the complex 1-form $\hat{\theta}^1 = \zeta^2 d\zeta^1 - \zeta^1 d\zeta^2$ such that $\{\hat{\Theta}, \hat{\theta}^1, \hat{\theta}^{\bar{1}}\}$ is dual to $\{\hat{T}, \hat{Z}_1, \hat{Z}_{\bar{1}}\}$ and $d\hat{\Theta} = i\hat{\theta}^1 \wedge \hat{\theta}^{\bar{1}}$. It follows that $\hat{\omega}_1^1 = -2i\hat{\Theta}$, $\hat{A}_{\bar{1}}^1 = 0$ in the corresponding (A.3) and (A.4) in the Appendix. Also in the corresponding (A.5), $\hat{W} = 2$. Write $\hat{Z}_1 = \frac{1}{2}(\hat{e}_1 - i\hat{e}_2)$ for real vector fields $\hat{e}_1$, $\hat{e}_2$. Let $\hat{\nabla}^{p.h.}$ denote the pseudohermitian connection of $(\hat{J}, \hat{\Theta})$. From $\hat{\nabla}^{p.h.}\hat{Z}_1 = \hat{\omega}_1^1 \otimes \hat{Z}_1$ (see (A.2)), we have

$$\hat{\nabla}^{p.h.}\hat{e}_1 = -2\hat{\Theta} \otimes \hat{e}_2, \hat{\nabla}^{p.h.}\hat{e}_2 = 2\hat{\Theta} \otimes \hat{e}_1 . \tag{7.1}$$

Recall that a Legendrian geodesic (with respect to $\hat{\nabla}^{p.h.}$) is a Legendrian curve γ such that $\hat{\nabla}^{p.h.}_{\dot{\gamma}}\dot{\gamma} = 0$. Here $\dot{\gamma} = \frac{d\gamma}{ds}$ is the unit tangent vector with respect to the Levi metric and s is a parameter of unit speed. A Legendrian great circle of $(S^3, \hat{J}, \hat{\Theta})$ is a great circle in the usual sense, whose tangents belong to the kernel of $\hat{\Theta}$.

Lemma 7.1. *In $(S^3, \hat{J}, \hat{\Theta})$ a Legendrian geodesic is a part of a Legendrian great circle, and vice versa.*

Proof. Suppose γ is a Legendrian geodesic. Write $\dot{\gamma} = a(s)\hat{e}_1 + b(s)\hat{e}_2$ (note that $\hat{e}_1$ and $\hat{e}_2 = \hat{J}\hat{e}_1$ belong to, and form a basis of, the kernel of $\hat{\Theta}$). Compute $0 = \hat{\nabla}^{p.h.}_{\dot{\gamma}}\dot{\gamma} = \dot{a}\hat{e}_1 + \dot{b}\hat{e}_2$ since $\hat{\nabla}^{p.h.}_{\dot{\gamma}}\hat{e}_1 = \hat{\nabla}^{p.h.}_{\dot{\gamma}}\hat{e}_2 = 0$ by (7.1) and $\hat{\Theta}(\dot{\gamma}) = 0$. So a (b, respectively) is a constant c_1 (c_2, respectively) along γ. Note that $c_1^2 + c_2^2 = 1$ since $a^2 + b^2 = 1$ by the unity of $\dot{\gamma}$. Now write $\zeta^1 = x^1 + iy^1$, $\zeta^2 = x^2 + iy^2$. From the definition we can express

$$\begin{aligned} \hat{e}_1 &= x^2\partial_{x^1} - y^2\partial_{y^1} - x^1\partial_{x^2} + y^1\partial_{y^2} , \\ \hat{e}_2 &= y^2\partial_{x^1} + x^2\partial_{y^1} - y^1\partial_{x^2} - x^1\partial_{y^2} . \end{aligned} \tag{7.2}$$

Writing $\gamma(s) = (x^1(s), y^1(s), x^2(s), y^2(s))$, we can express the equation $\dot{\gamma} = c_1\hat{e}_1 + c_2\hat{e}_2$ by (7.2) as

$$\begin{aligned} \dot{x}^1 &= c_1x^2 + c_2y^2, \dot{y}^1 = c_2x^2 - c_1y^2 , \\ \dot{x}^2 &= -c_1x^1 - c_2y^1, \dot{y}^2 = -c_2x^1 + c_1y^1 . \end{aligned} \tag{7.3}$$

It is easy to see from (7.3) that $\ddot{x}^1 = -x^1$, $\ddot{y}^1 = -y^1$, $\ddot{x}^2 = -x^2$, $\ddot{y}^2 = -y^2$. Therefore

$$(x^1(s), y^1(s), x^2(s), y^2(s)) = \cos(s)(\alpha_1, \alpha_2, \alpha_3, \alpha_4) + \sin(s)(\beta_1, \beta_2, \beta_3, \beta_4). \tag{7.4}$$

Here the constant vector $(\beta_1, \beta_2, \beta_3, \beta_4)$ is determined by the constant vector $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ as follows:

$$(\beta_1, \beta_2) = (\alpha_3, \alpha_4)\begin{pmatrix} c_1 & c_2 \\ c_2 & -c_1 \end{pmatrix}, (\beta_3, \beta_4) = (\alpha_1, \alpha_2)\begin{pmatrix} -c_1 & -c_2 \\ -c_2 & c_1 \end{pmatrix} . \tag{7.5}$$

Using $c_1^2 + c_2^2 = 1$, we have $\beta_1^2 + \beta_2^2 = \alpha_3^2 + \alpha_4^2$, $\beta_3^2 + \beta_4^2 = \alpha_1^2 + \alpha_2^2$ by (7.5). Denote $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ by $\vec{\alpha}$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$ by $\vec{\beta}$. We can write (7.4) as

$$\gamma(s) = \cos(s)\vec{\alpha} + \sin(s)\vec{\beta}\,. \tag{7.4$'$}$$

A direct computation using (7.5) and $c_1^2 + c_2^2 = 1$ shows that $\vec{\alpha}$ is perpendicular to $\vec{\beta}$ and $|\vec{\alpha}| = |\vec{\beta}| = 1$ in R^4 since $\gamma(s) \in S^3$. It is now clear from (7.4$'$) that the Legendrian geodesic $\gamma(s)$, $0 \le s \le 2\pi$, is a great circle. Moreover, $\vec{\beta}$ sits in the contact plane at the point $\vec{\alpha}$. Write $\hat{\Theta} = x^1dy^1 - y^1dx^1 + x^2dy^2 - y^2dx^2$. Define $(e, f)^{\perp} = (-f, e)$ for a plane vector (e, f). Write $\vec{\alpha} = (\vec{\alpha}_1, \vec{\alpha}_2)$ and $\vec{\beta} = (\vec{\beta}_1, \vec{\beta}_2)$ for plane vectors $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\beta}_1, \vec{\beta}_2$. Then $\vec{\beta} \in \ker \hat{\Theta}$ at the point $\vec{\alpha}$ if and only if

$$(\vec{\alpha}_1^{\perp}, \vec{\alpha}_2^{\perp}) \cdot (\vec{\beta}_1, \vec{\beta}_2) = \vec{\alpha}_1^{\perp} \cdot \vec{\beta}_1 + \vec{\alpha}_2^{\perp} \cdot \vec{\beta}_2 = 0 \tag{7.6}$$

(in which "$\cdot$" denotes the inner product). Now it is easy to see that (7.5) implies (7.6) for $\vec{\alpha}_1 = (\alpha_1, \alpha_2)$, $\vec{\alpha}_2 = (\alpha_3, \alpha_4)$, $\vec{\beta}_1 = (\beta_1, \beta_2)$, $\vec{\beta}_2 = (\beta_3, \beta_4)$. Conversely, given an arbitrary point $\vec{\alpha} \in S^3$ and a unit tangent vector $\vec{\beta}$ in the contact plane at $\vec{\alpha}$, we claim that the great circle $\gamma(s)$ defined by (7.4$'$) is Legendrian and a Legendrian geodesic. From $\dot{\gamma}(s) = -\sin(s)\vec{\alpha} + \cos(s)\vec{\beta}$, we compute

$$\begin{aligned}
&\{\cos(s)(\vec{\alpha}_1^{\perp}, \vec{\alpha}_2^{\perp}) + \sin(s)(\vec{\beta}_1^{\perp}, \vec{\beta}_2^{\perp})\} \cdot \{-\sin(s)(\vec{\alpha}_1, \vec{\alpha}_2) + \cos(s)(\vec{\beta}_1, \vec{\beta}_2)\} \\
&= -\sin^2(s)(\vec{\beta}_1^{\perp} \cdot \vec{\alpha}_1 + \vec{\beta}_2^{\perp} \cdot \vec{\alpha}_2) + \cos^2(s)(\vec{\alpha}_1^{\perp} \cdot \vec{\beta}_1 + \vec{\alpha}_2^{\perp} \cdot \vec{\beta}_2) \\
&= \vec{\alpha}_1^{\perp} \cdot \vec{\beta}_1 + \vec{\alpha}_2^{\perp} \cdot \vec{\beta}_2 \quad (\text{since } \vec{\eta} \cdot \vec{\zeta} = \vec{\eta}^{\perp} \cdot \vec{\zeta}^{\perp} \quad \text{and } (\vec{\eta}^{\perp})^{\perp} = -\vec{\eta}) \\
&= 0 \quad (\text{by (7.6)})\,.
\end{aligned}$$

So $\gamma(s)$ is Legendrian by (7.6) again. From (7.2) we can express $\hat{e}_1, \hat{e}_2$ at $\gamma(s)$ as follows:

$$\hat{e}_1(\gamma(s)) = \cos(s)(\alpha_3, -\alpha_4, -\alpha_1, \alpha_2) + \sin(s)(\beta_3, -\beta_4, -\beta_1, \beta_2)\,,$$

$$\hat{e}_2(\gamma(s)) = \cos(s)(\alpha_4, \alpha_3, -\alpha_2, -\alpha_1) + \sin(s)(\beta_4, \beta_3, -\beta_2, -\beta_1)\,.$$

Recall that we write $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $\vec{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)$. Equating $(\dot{\gamma}(s) =) -\sin(s)\vec{\alpha} + \cos(s)\vec{\beta} = c_1\hat{e}_1(\gamma(s)) + c_2\hat{e}_2(\gamma(s))$ with $c_1^2 + c_2^2 = 1$ gives the equations (7.5) (requiring $c_1^2 + c_2^2 = 1$ gets rid of other equivalent equations). Solving (7.5) for c_1, c_2, we obtain $c_1 = (\beta_1\alpha_3 - \beta_2\alpha_4)(\alpha_3^2 + \alpha_4^2)^{-1}$, $c_2 = (\beta_1\alpha_4 + \beta_2\alpha_3)(\alpha_3^2 + \alpha_4^2)^{-1}$ if $\alpha_3^2 + \alpha_4^2 \neq 0$ or $c_1 = (-\alpha_1\beta_3 + \alpha_2\beta_4)(\alpha_1^2 + \alpha_2^2)^{-1}$, $c_2 = (-\alpha_1\beta_4 - \alpha_2\beta_3)(\alpha_1^2 + \alpha_2^2)^{-1}$ if $\alpha_1^2 + \alpha_2^2 \neq 0$. Note that two expressions for c_1 (c_2, respectively) are equal where $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\alpha_3^2 + \alpha_4^2 \neq 0$ by the condition (7.6) and $\vec{\alpha} \cdot \vec{\beta} = 0$. Now $\hat{\nabla}_{\dot{\gamma}}^{p.h.}\dot{\gamma} = c_1\hat{\nabla}_{\dot{\gamma}}^{p.h.}\hat{e}_1(\gamma(s)) + c_2\hat{\nabla}_{\dot{\gamma}}^{p.h.}\hat{e}_2(\gamma(s)) = 0$ by (7.1). We have proved that $\gamma(s)$ is a Legendrian geodesic. □

Recall (see Section 2) that the p-mean curvature H of a surface Σ in a pseudohermitian 3-manifold (M, J, Θ) depends on (J, Θ). Let $\tilde{H}$ denote the p-mean curvature associated to another contact form $\tilde{\Theta} = \lambda^2\Theta$, $\lambda > 0$ with J fixed. Let e_2 denote the Legendrian normal to Σ. Then we have the following transformation law.

Lemma 7.2. *Suppose* $\tilde{\Theta} = \lambda^2\Theta$, $\lambda > 0$. *Then* $\tilde{H} = \lambda^{-2}(\lambda H - 3e_2(\lambda))$.

Proof. Let $\tilde{e}_1$ denote the characteristic field with respect to $\tilde{\Theta}$. Then it follows from the definition that $\tilde{e}_1 = \lambda^{-1}e_1$. Applying (5.7) in [Lee] to e_1 (in our case, $n = 1$, $Z_1 = \frac{1}{2}(e_1 - ie_2)$), we obtain

$$\lambda\tilde{\omega}_1^1(\tilde{e}_1) = \omega_1{}^1(e_1) - 3i\lambda^{-1}e_2(\lambda)\,. \tag{7.7}$$

Note that $H = \omega(e_1) = -i\omega_1{}^1(e_1)$ (see the remark after (2.8)). Rewriting (7.7) in terms of H and $\tilde{H}$ gives what we want. □

We define the Cayley transform $F\colon S^3\backslash\{(0,-1)\} \to H_1$ by

$$x = \mathrm{Re}\left(\frac{\zeta^1}{1+\zeta^2}\right),\ y = \mathrm{Im}\left(\frac{\zeta^1}{1+\zeta^2}\right),\ z = \frac{1}{2}\mathrm{Re}\left[i\left(\frac{1-\zeta^2}{1+\zeta^2}\right)\right]$$

where $(\zeta^1, \zeta^2) \in S^3 \subset C^2$ satisfies $|\zeta^1|^2 + |\zeta^2|^2 = 1$ (see, e.g., [JL1]). A direct computation shows that

$$\hat{\Theta} = F^*(\lambda^2\Theta_0) \tag{7.8}$$

where $\lambda^2 = 4[4z^2 + (x^2+y^2+1)^2]^{-1}$ (recall that $\Theta_0 = dz + xdy - ydx$ is the standard contact form for H_1).

Lemma 7.3. *Let* Σ *be a* C^2 *smoothly embedded p-minimal surface in* $(S^3, \hat{J}, \hat{\Theta})$. *Suppose* $p \in \Sigma$ *is an isolated singular point. Then there exists a neighborhood* V *of* p *in* Σ *such that* V *is contained in the union of all Legendrian great circles through* p.

Proof. Without loss of generality, we may assume $(0,-1) \notin \Sigma$. Consider $\Sigma_0 = F(\Sigma) \subset H_1$. Let H_0 denote the p-mean curvature of Σ_0 in H_1. By (7.8) and Lemma 7.2, we obtain $H_0 = 3\lambda^{-1}e_2(\lambda)$. Here e_2 is the Legendrian normal of Σ_0. Recall (Section 2) that $e_2 = -(\cos\theta)\hat{e}_1 - (\sin\theta)\hat{e}_2$. So $e_2(\lambda) = -(\cos\theta)\hat{e}_1(\lambda) - (\sin\theta)\hat{e}_2(\lambda)$ is bounded near the isolated singular point $F(p)$ since $\hat{e}_1(\lambda)$ and $\hat{e}_2(\lambda)$ are global C^∞ smooth functions. Thus H_0 is bounded near $F(p)$. (Note that near a singular point, Σ_0 is a graph over the xy-plane.) Observing that characteristic curves and the singular set are preserved under the contact diffeomorphism F, we conclude the proof of the lemma by Theorem 3.10 and Lemma 7.1. □

A direct proof of Corollary F (for the nonexistence of hyperbolic p-minimal surfaces). Let Σ be a closed, connected, C^2 smoothly embedded p-minimal surface in $(S^3, \hat{J}, \hat{\Theta})$. Without loss of generality, we may assume $(0,-1) \notin \Sigma$. Consider $\Sigma_0 = F(\Sigma) \subset H_1$. As argued in the proof of Lemma 7.3, H_0, the p-mean curvature of Σ_0, is bounded. According to Theorem B any singular point of Σ_0 is either isolated or contained in a C^1 smooth singular curve with no other singular points near it (note that near a singular point, Σ_0 is a graph over the xy-plane). So back to Σ through F^{-1}, we can only have isolated singular points or closed singular curves on Σ. Similarly via the Cayley transform we

can have an extension theorem analogous to Corollary 3.6 with the characteristic curve replaced by a characteristic (Legendrian) great circle (arc) in the case of $(S^3, \hat{J}, \hat{\Theta})$.

Now suppose Σ has an isolated singular point p. By Lemma 7.3 there exists a neighborhood V of p in Σ such that V is contained in the union of all Legendrian great circles through p. But this union simply forms a p-minimal 2-sphere. This 2-sphere must be the whole Σ since Σ is connected. Next suppose Σ does not have any isolated singular point. Then in view of the extension theorem, the space of leaves (Legendrian great circles) of the characteristic foliation (including touching points on singular curves) forms a closed, connected, 1-dimensional manifold. So it must be homeomorphic to S^1. In this case, Σ is topologically a torus. □

Example 1. Every coordinate sphere (defined by x^1, y^1, x^2 or $y^2 = 0$) is a closed, connected, embedded p-minimal surface of genus 0 in $(S^3, \hat{J}, \hat{\Theta})$. For instance, we can write $\{y^2 = 0\}$ as the union of Legendrian great circles: $\gamma_t(s) = \cos(s)(0, 0, 1, 0) + \sin(s)(\cos(t), \sin(t), 0, 0)$ parametrized by t (it is a simple exercise to verify (7.6)).

Example 2. Write $\zeta^1 = \rho_1 e^{i\varphi_1}$, $\zeta^2 = \rho_2 e^{i\varphi_2}$ in polar coordinates with $\rho_1^2 + \rho_2^2 = 1$ on S^3. We consider the surface Σ_c defined by $\rho_1 = c$, a constant between 0 and 1. Note that $\hat{\Theta} = \rho_1^2 d\varphi_1 + \rho_2^2 d\varphi_2$, $\hat{J}\partial_{\varphi_j} = -\rho_j \partial_{\rho_j}$ and $\hat{J}\partial_{\rho_j} = \rho_j^{-1}\partial_{\varphi_j}$, $j = 1, 2$ in polar coordinates. Here we have used $\partial_{\varphi_j} = i\zeta^j \partial_{\zeta^j} - i\bar{\zeta}^j \partial_{\bar{\zeta}^j}$, $\partial_{\rho_j} = \rho_j^{-1}(\zeta^j \partial_{\zeta^j} + \bar{\zeta}^j \partial_{\bar{\zeta}^j})$ for $j = 1, 2$ (no summation convention here). Next we compute the Reeb vector field T, the characteristic field e_1, and the Legendrian normal e_2 as follows:

$$T = \partial_{\varphi_1} + \partial_{\varphi_2}, e_1 = \frac{\rho_2}{\rho_1}\partial_{\varphi_1} - \frac{\rho_1}{\rho_2}\partial_{\varphi_2}, e_2 = \hat{J}e_1 = -\rho_2 \partial_{\rho_1} + \rho_1 \partial_{\rho_2}.$$

We then have $e^1 = \rho_1\rho_2(d\varphi_1 - d\varphi_2)$ and $e^2 = -\rho_2 d\rho_1 + \rho_1 d\rho_2$. So we can compute the p-area 2-form $\hat{\Theta} \wedge e^1 = -\rho_1\rho_2 d\varphi_1 \wedge d\varphi_2$, the volume form $\hat{\Theta} \wedge e^1 \wedge e^2 = \rho_1 d\rho_1 \wedge d\varphi_1 \wedge d\varphi_2$ and $d(\hat{\Theta} \wedge e^1) = -(\rho_2^2 - \rho_1^2)\rho_2^{-1} d\rho_1 \wedge d\varphi_1 \wedge d\varphi_2$ (noting that $\rho_1 d\rho_1 + \rho_2 d\rho_2 = 0$). By (6.2) we obtain the p-mean curvature $H = (\rho_2^2 - \rho_1^2)(\rho_1\rho_2)^{-1} = \rho_2\rho_1^{-1} - \rho_1\rho_2^{-1}$ for Σ_c where $0 < \rho_1 = c < 1$ and $\rho_2 = \sqrt{1 - c^2}$. Thus for $c = \sqrt{2}/2$ ($\rho_1 = \rho_2 = c$), Σ_c is a closed, connected, embedded p-minimal torus with no singular points (observing that T is tangent at every point of Σ_c) and the union of Legendrian great circles defined by $\varphi_1 + \varphi_2 = a$, $0 \le a < 2\pi$ with 0 identified with 2π. In any case, Σ_c is a torus of constant p-mean curvature. Also note that the p-minimal torus $\Sigma_{\sqrt{2}/2}$ is not stable. This can be seen from the second variation formula (6.8) in which $\alpha = 0$, $A_{11} = 0$, and $W = 2$.

To generalize Corollary F, we need to extend the theory of singular set in Section 3 to the general situation. Let Σ be a surface in an arbitrary 3-dimensional pseudohermitian manifold. Let p_0 be a singular point of Σ. Near

p_0 let $e_1,\ e_2 \in \xi$ denote an orthonormal basis with respect to the Levi metric G such that $\frac{1}{2}(e_1 - ie_2)$ forms a special frame in the sense of [JL2]. Then in pseudohermitian normal coordinates x, y, z associated to this special frame, we can describe Σ by a graph $z = u(x, y)$ with $p_0 = (0, 0, 0)$. Furthermore, we can write

$$\begin{aligned} e_1 &= \hat{e}_1 + \mathcal{O}_1 \\ e_2 &= \hat{e}_2 + \mathcal{O}_2 \end{aligned} \tag{7.9}$$

where $\hat{e}_1 \equiv \partial_x + y\partial_z$, $\hat{e}_2 \equiv \partial_y - x\partial_z$, and $\mathcal{O}_1, \mathcal{O}_2$ are real vector fields of degree ≥ 1 ([JL2]). Let $\psi = u(x, y) - z$ be a defining function for Σ. Recall that $\nabla_b\psi = (e_1\psi)e_1 + (e_2\psi)e_2$ and $\mathcal{D} \equiv |\nabla_b\psi|_G \equiv \sqrt{(e_1\psi)^2 + (e_2\psi)^2}$ (see Section 2). We compute the p-mean curvature (see (pMCE). Note that we take the positive sign below to be consistent with the Heisenberg group case in which we use $z - u(x, y)$ as a defining function):

$$\begin{aligned} H &= \operatorname{div}_b\left(\frac{\nabla_b\psi}{|\nabla_b\psi|_G}\right) \\ &= G\left(\nabla^{p.h.}_{e_1}\frac{\nabla_b\psi}{|\nabla_b\psi|_G}, e_1\right) + G\left(\nabla^{p.h.}_{e_2}\frac{\nabla_b\psi}{|\nabla_b\psi|_G}, e_2\right) \\ &= \left[e_1\left(\frac{e_1\psi}{\mathcal{D}}\right) + e_2\left(\frac{e_2\psi}{\mathcal{D}}\right)\right] - \frac{e_2\psi}{\mathcal{D}}B_1 - \frac{e_1\psi}{\mathcal{D}}B_2 \end{aligned} \tag{7.10}$$

where B_1 and B_2 are determined by $\nabla^{p.h.}_{e_1}e_1 = B_1e_2$ and $\nabla^{p.h.}_{e_2}e_2 = B_2e_1$, respectively. Note that both $\frac{e_1\psi}{\mathcal{D}}$ and $\frac{e_2\psi}{\mathcal{D}}$ are bounded by 1. Also since e_1, e_2 are local C^∞ smooth vector fields, B_1 and B_2 are bounded in a neighborhood of p_0. It follows from (7.10) that near p_0 where $\mathcal{D} = 0$ (i.e., $e_1\psi = e_2\psi = 0$)

$$H = \left[e_1\left(\frac{e_1\psi}{\mathcal{D}}\right) + e_2\left(\frac{e_2\psi}{\mathcal{D}}\right)\right] + \text{a bounded function.} \tag{7.11}$$

To generalize the results in Section 3, we replace $u_x - y$ and $u_y + x$ by $e_1\psi$ and $e_2\psi$, respectively in all the places. Thus the singular set $S(u)$ consists of the points where $e_1\psi = e_2\psi = 0$ while $F_{a,b}$ and U are defined to be $a(e_1\psi) + b(e_2\psi)$ and

$$\begin{bmatrix} (e_1\psi)_x & (e_1\psi)_y \\ (e_2\psi)_x & (e_2\psi)_y \end{bmatrix},$$

respectively. Note that the above U is reduced to (U) in the beginning of Section 3 at p_0 by (7.9). It is now easy to see that Lemma 3.1 and Lemma 3.2 continue to hold in the general situation.

To generalize Theorem 3.3, we need to modify the argument used to show that (2) $\Rightarrow$ (3). Let $N(u) \equiv \mathcal{D}^{-1}\ (e_1\psi,\ e_2\psi)$. Let $\eta_1 = \mathcal{O}_1(x)dy - \mathcal{O}_1(y)dx$, $\eta_2 = \mathcal{O}_2(x)dy - \mathcal{O}_2(y)dx$ (note that $e_1 \lrcorner dx\wedge dy = dy + \eta_1$ and $e_2 \lrcorner dx\wedge dy = -dx + \eta_2$). Making use of the formula: $\iint v(f)dx\wedge dy = \oint f(v\lrcorner dx\wedge dy) - \iint fd(v\lrcorner dx\wedge dy)$ for a vector field v and a function f, we compute

$$\iint_{\Omega_i}\left[e_1\left(\frac{e_1\psi}{\mathcal{D}}\right) + e_2\left(\frac{e_2\psi}{\mathcal{D}}\right)\right]dx\wedge dy = \oint_{\partial\Omega_i} N(u)\cdot v ds + E_1 + E_2 \tag{7.12}$$

where the unit outward normal $\nu = (\frac{dy}{ds}, -\frac{dx}{ds})$ and two error terms E_1, E_2 are given by

$$\begin{aligned} E_1 &= -\iint_{\Omega_i} \left(\frac{e_1\psi}{\mathcal{D}} d\eta_1 + \frac{e_2\psi}{\mathcal{D}} d\eta_2 \right) \\ E_2 &= \oint_{\partial\Omega_i} \frac{e_1\psi}{\mathcal{D}} \eta_1 + \frac{e_2\psi}{\mathcal{D}} \eta_2 \,. \end{aligned} \tag{7.13}$$

Observe that $\frac{e_1\psi}{\mathcal{D}}$ and $\frac{e_2\psi}{\mathcal{D}}$ are bounded by 1, $\eta_j = o(dr)$ (little "o"), and $d\eta_j = o(dx \wedge dy)$ (little "o"). By (7.12) we can then estimate

$$\begin{aligned} &\left| \iint_{\Omega_i} [H - \text{ (a bounded function)}] dx \wedge dy \right| \\ &= \left| \oint_{\partial\Omega_i} N(u) \cdot \nu ds + E_1 + E_2 \right| \\ &\geq \frac{|c_1 - c_2|}{2} |r(\tilde{e}_i) - r(e_i)| \end{aligned}$$

for large i. On the other hand, (3.6) still holds for H replaced by H-(a bounded function). So we get a contradiction, and hence validate Theorem 3.3 in the general situation. To generalize other results except Lemma 3.4 in Section 3, we observe that the error terms do not affect the major estimates in the proofs (Lemma 3.4 is used to let us assume that the x-axis through p_0 is transverse to the singular curve in the proof of Proposition 3.5. We may instead assume the y-axis through p_0 is transverse to the singular curve if the x-axis is not without using Lemma 3.4. Note that Lemma 3.4 is mainly used in the proof of Theorem A in Section 4). Therefore they all continue to hold in the general situation.

Proof of Theorem E. Locally near a singular point p_0 of Σ, we may assume that Σ is a C^2 smooth graph over the xy-plane in pseudohermitian normal coordinates x, y, z with $p_0 = (0, 0, 0)$. So the singular set S_Σ (depending only on Σ and the contact structure) consists of finitely many isolated points and C^1 smooth closed curves in view of generalized Theorem B (see the paragraph above). Also the (generalized) extension theorems (Corollary 3.6 and Theorem 3.10) hold in this situation. Now the configuration of characteristic foliation on Σ is clear. The associated line field (extended to include those defined on points of singular curves) has only isolated singular points of index 1 in view of (generalized) Lemma 3.8. Therefore the total index sum of this line field is nonnegative. This index sum is equal to the Euler characteristic number of the surface Σ according to the Hopf index theorem for a line field (e.g., [Sp]). On the other hand, the Euler characteristic number of Σ equals $2 - 2g(\Sigma)$ where $g(\Sigma)$ denotes the genus of Σ. It follows that $g(\Sigma) \leq 1$. □

Appendix. Basic facts in pseudohermitian geometry

Let M be a smooth (paracompact) 3-manifold. A contact structure or bundle ξ on M is a completely nonintegrable plane distribution. A contact form is a 1-form annihilating ξ. Let (M, ξ) be a contact 3-manifold with an oriented contact structure ξ. We say a contact form Θ is oriented if $d\Theta(u, v) > 0$ for (u, v) being an oriented basis of ξ. There always exists a global oriented contact form Θ, obtained by patching together local ones with a partition of unity. The Reeb vector field of Θ is the unique vector field T such that $\Theta(T) = 1$ and $\mathcal{L}_T\Theta = 0$ or $d\Theta(T, \cdot) = 0$. A CR-structure compatible with ξ is a smooth endomorphism $J : \xi \to \xi$ such that $J^2 = -$ Identity. We say J is oriented if (X, JX) is an oriented basis of ξ for any nonzero $X \in \xi$. A pseudohermitian structure compatible with ξ is a CR-structure J compatible with ξ together with a global contact form Θ.

Given a pseudohermitian structure (J, Θ), we can choose a complex vector field Z_1, an eigenvector of J with eigenvalue i, and a complex 1-form θ^1 such that $\{\Theta, \theta^1, \theta^{\bar{1}}\}$ is dual to $\{T, Z_1, Z_{\bar{1}}\}$ ($\theta^{\bar{1}} = (\theta^1)$,$Z_{\bar{1}} = (Z_1)$). It follows that $d\Theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$ for some nonzero real function $h_{1\bar{1}}$. If both J and Θ are oriented, then $h_{1\bar{1}}$ is positive. In this case we call such a pseudohermitian structure (J, Θ) oriented, and we can choose a Z_1 (hence θ^1) such that $h_{1\bar{1}} = 1$. That is to say

$$d\Theta = i\theta^1 \wedge \theta^{\bar{1}}\,. \tag{A.1}$$

We will always assume our pseudohermitian structure is oriented and $h_{1\bar{1}} = 1$. The pseudohermitian connection of (J, Θ) is the connection $\nabla^{p.h.}$ on $TM \otimes C$ (and extended to tensors) given by

$$\nabla^{p.h.} Z_1 = \omega_1{}^1 \otimes Z_1,\ \nabla^{p.h.} Z_{\bar{1}} = \omega_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}},\ \nabla^{p.h.} T = 0 \tag{A.2}$$

in which the 1-form $\omega_1{}^1$ is uniquely determined by the following equation with a normalization condition ([Ta, We]):

$$d\theta^1 = \theta^1 \wedge \omega_1{}^1 + A^1{}_{\bar{1}}\Theta \wedge \theta^{\bar{1}}\,, \tag{A.3}$$

$$\omega_1{}^1 + \omega_{\bar{1}}{}^{\bar{1}} = 0\,. \tag{A.4}$$

The coefficient $A^1{}_{\bar{1}}$ in (A.3) is called the (pseudohermitian) torsion. Since $h_{1\bar{1}} = 1$, $A_{\bar{1}\bar{1}} = h_{1\bar{1}}A^1{}_{\bar{1}} = A^1{}_{\bar{1}}$. And A_{11} is just the complex conjugate of $A_{\bar{1}\bar{1}}$. Differentiating $\omega_1{}^1$ gives

$$d\omega_1{}^1 = W\theta^1 \wedge \theta^{\bar{1}} + 2i\ \mathrm{Im}(A_{11,\bar{1}}\theta^1 \wedge \Theta) \tag{A.5}$$

where W is the Tanaka-Webster curvature. Write $\omega_1{}^1 = i\omega$ for some real 1-form ω by (A.4). This ω is just the one used in previous sections. Write $Z_1 = \frac{1}{2}(e_1 - ie_2)$ for real vectors e_1, e_2. It follows that $e_2 = Je_1$. Let $e^1 = \mathrm{Re}(\theta^1)$, $e^2 = \mathrm{Im}(\theta^1)$. Then $\{e^0 = \Theta, e^1, e^2\}$ is dual to $\{e_0 = T, e_1, e_2\}$. Now in

view of (A.1), (A.2) and (A.3), we have the following real version of structure equations:

$$d\Theta = 2e^1 \wedge e^2\,, \tag{A.1r}$$

$$\nabla^{p.h.} e_1 = \omega \otimes e_2,\ \nabla^{p.h.} e_2 = -\omega \otimes e_1\,, \tag{A.2r}$$

$$de^1 = -e^2 \wedge \omega \bmod \Theta;\quad de^2 = e^1 \wedge \omega \bmod \Theta\,. \tag{A.3r}$$

Similarly, from (A.5), we have the following equation for W:

$$d\omega(e_1, e_2) = -2W\,. \tag{A.5r}$$

Also by (A.1), (A.3) we can deduce

$$[Z_{\bar 1}, Z_1] = iT + \omega_1{}^1(Z_{\bar 1})Z_1 - \omega_{\bar 1}{}^{\bar 1}(Z_1)Z_{\bar 1}\,, \tag{A.6}$$

$$[Z_{\bar 1}, T] = A^1{}_{\bar 1} Z_1 - \omega_{\bar 1}{}^{\bar 1}(T)Z_{\bar 1}\,. \tag{A.7}$$

The real version of (A.6), (A.7) reads

$$[e_1, e_2] = -2T - \omega(e_1)e_1 - \omega(e_2)e_2\,, \tag{A.6r}$$

$$[e_1, T] = (\mathrm{Re}A_{11})e_1 - ((\ \mathrm{Im}A_{11}) + \omega(T))e_2\,, \tag{A.7r}$$

$$[e_2, T] = -((\mathrm{Im}A_{\bar 1\bar 1}) + \omega(T))e_1 + (\mathrm{Re}A_{\bar 1\bar 1})e_2\,.$$

Note that $A_{\bar 1\bar 1} = A^1{}_{\bar 1}$ since $h_{1\bar 1} = 1$. We define the subgradient operator ∇_b acting on a smooth function f by

$$\nabla_b f = 2\{(Z_{\bar 1} f)Z_1 + (Z_1 f)Z_{\bar 1}\}\,. \tag{A.8}$$

It is easy to see that the definition of ∇_b is independent of the choice of unitary ($h_{1\bar 1} = 1$) frame Z_1. The real version of (A.8) reads

$$\nabla_b f = (e_1 f)e_1 + (e_2 f)e_2\,. \tag{A.8r}$$

Next we will introduce the 3-dimensional Heisenberg group H_1 (see, e.g., [FS]). For two points (x, y, z), $(x', y', z') \in \mathbb{R}^3$, we define the multiplication as follows: $(x, y, z) \circ (x', y', z') = (x + x',\ y + y',\ z + z' + yx' - xy')$. $\mathbb{R}^3$ endowed with this multiplication "$\circ$" forms a Lie group, called the (3-dimensional) Heisenberg group and denoted as H_1. It is a simple exercise to verify that:

$$\hat e_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ \hat e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ T_0 = \frac{\partial}{\partial z}$$

form a basis for the left-invariant vector fields on H_1. We can endow H_1 with a standard pseudohermitian structure. The plane distribution spanned by $\hat e_1, \hat e_2$ forms a contact structure ξ_0 (so that $\hat e_1, \hat e_2$ are Legendrian, i.e., sitting in the

contact plane). The CR structure J_0 compatible with ξ_0 is defined by $J_0(\hat{e}_1) = \hat{e}_2$, $J_0(\hat{e}_2) = -\hat{e}_1$. The contact form $\Theta_0 = dz + xdy - ydx$ gives rise to the Reeb vector field $T_0 = \frac{\partial}{\partial z}$. Observe that $\{dx, dy, \Theta_0\}$ is dual to $\{\hat{e}_1, \hat{e}_2, T_0\}$ and $d(dx + idy) = 0$. It follows from the structural equations (A.3), (A.4) and (A.5) that the connection form associated to the coframe $dx + idy$, the torsion and the Tanaka-Webster curvature are all zero.

References

[B] Z. BALOGH, *Size of characteristic sets and functions with prescribed gradient*, J. Reine Angew. Math. **564** (2003), 63–83.

[CDG] L. CAPOGNA, D. DANIELLI and N. GAROFALO, *The geometric Sobolev embedding fir vector fields and the isoperimetric inequality*, Comm. Anal. Geom. **2** (1994), 203–215.

[CF] P. CONCUS and R. FINN, *On capillary free surfaces in the absence of gravity*, Acta Math. **132** (1974), 177–198.

[CH] J.-H. CHENG and J.-F. HWANG, *Properly embedded and immersed minimal surfaces in the Heisenberg group*, Bull. Austral. Math. Soc. **70** (2004), 507–520.

[CK] P. COLLIN and R. KRUST, *Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non bornès*, Bull. Soc. Math. France **119** (1991), 443–462.

[DGN] D. DANIELLI, N. GAROFALO and D-M. NHIEU, *Minimal surfaces, surfaces of constant mean curvature and isoperimetry in Carnot groups*, Preprint, 2001.

[FS] G. B. FOLLAND and E. M. STEIN, *Estimates for the $\bar{\partial}_b$-complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[FSS] B. FRANCHI, R. SERAPIONI and F. SERRA CASSANO, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), 479–531.

[GN] N. GAROFALO and D.-M. NHIEU, *Isoperimetric and Sobolev inequalities for Carnot-Caratheodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081–1144.

[GP1] N. GAROFALO and S. PAULS, *The Bernstein problem in the Heisenberg group*, arXiv: math. DG/0209065.

[GP2] N. GAROFALO and S. PAULS, *The Bernstein problem in the Heisenberg group*, preprint, 2004.

[HL] R. HARVEY and H. B. LAWSON, *Calibrated geometries*, Acta Math. **148** (1982), 47-157.

[Hw1] J. F. HWANG, *Comparison principles and Liouville theorems for prescribed mean curvature equation in unbounded domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **15** (1988), 341–355.

[Hw2] J. F. HWANG, *Structural inequalities method for uniqueness theorems for the minimal surface equation*, Proc. of CMA (Joint Australia-Taiwan Workshop on Analysis and Application, Brisbane, March 1997), Australian National University, Vol. 37, 1999, 47–52.

[JL1] D. JERISON and J. M. LEE, *The Yamabe problem on CR manifolds*, J. Differential Geom. **25** (1987), 167–197.

[JL2] D. JERISON and J. M. LEE, *Intrinsic CR normal coordinates and the CR Yamabe problem*, J. Differential Geom. **29** (1989), 303–343.

[Jo] F. JOHN, "Partial Differential Equations", Springer-Verlag, 4th ed., 1982.

[La] H. B. LAWSON JR., *Complete minimal surfaces in S^3*, Ann. of Math. **92** (1970), 335-374.

[Lee] J. M. LEE, *The Fefferman metric and pseudohermitian invariants*, Trans. Amer. Math. Soc. **296** (1986), 411–429.

[LM] G. LEONARDI and S. MASNOU, *On the isoperimetric problem in the Heisenberg group H^n*, to appear on Annali Mat. Pura e Appl., 2002.

[LR] G. LEONARDI and S. RIGOT, *Isoperimetric sets on Carnot groups*, Houston J. Math. (2003).

[Mik] V. M. MIKLYUKOV, *On a new approach to Bernstein's theorem and related questions for equations of minimal surface type*, Mat. Sb. **108 (150)** (1979), 268–289; English transl. in Math. USSR Sb. **36** (1980), 251–271.

[Mil] J. MILNOR, "Topology from the Differentiable Viewpoint", University of Virginia Press, 1965.

[Mo] G. MONGE, "Application de l'Analyse à la Géométrie", Paris, Bachelier, 1850.

[Os] R. OSSERMAN, "A Survey of Minimal Surfaces", Dover Publications, Inc., New York, 1986.

[Pan] P. PANSU, *Une inegalite isoperimetrique sur le groupe de Heisenberg*, C.R. Acad. Sci. Paris I **295** (1982), 127–130.

[Pau] S. D. PAULS, *Minimal surfaces in the Heisenberg group*, Geom. Dedicata **104** (2004), 201–231.

[Sp] M. SPIVAK, "A Comprehensive Introduction to Differential Geometry", Vol. 3, Publish or Perish Inc., Boston, 1975.

[St] S. STERNBERG, "Lectures on Differential Geometry", 2nd ed., Chelsea Publishing Company, New York, 1983.

[SY] R. SCHOEN and S.-T. YAU, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), 45–76.

[Ta] N. TANAKA, "A Differential Geometric Study on Strongly Pseudo-Convex Manifolds", Kinokuniya Co. Ltd., Tokyo, 1975.

[We] S. M. WEBSTER, *Pseudohermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), 25–41.

Institute of Mathematics
Academia Sinica
Nankang, Taipei, Taiwan, 11529, R.O.C.
cheng@math.sinica.edu.tw

Institute of Mathematics
Academia Sinica
Nankang, Taipei, Taiwan, 11529, R.O.C.
majfh@math.sinica.edu.tw

SISSA
Via Beirut 2-4
34014 Trieste, Italy
malchiod@sissa.it

Department of Mathematics
Princeton University
Princeton, NJ 08544, U.S.A.
yang@Math.Princeton.EDU