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Parabolic obstacle problems, quasi-convexity and regularity

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Abstract. In a wide class of the so called Obstacle Problems of parabolic type it is shown how to improve the optimal regularity of the solution and as a consequence how to obtain space-time regularity of the corresponding free boundary.

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1. Introduction

Obstacle problems are characterized by the fact that the solution must satisfy unilateral constraint, *i.e.* must remain, on its domain of definition or part of it, above a given function the so called obstacle. Parabolic obstacle problems, *i.e.* when the involved operators are of parabolic type, can be formulated in various ways such as a system of inequalities, variational inequalities, Hamilton-Jacobi equation, etc. More precisely, as a system of inequalities, one seeks a solution $u(x,t)$ which satisfies

$$\begin{cases} u_t + Au \geq 0, \ u \geq \psi \\ (u_t + Au)(u-\psi) = 0 \end{cases} \quad \text{in } \Omega \times (0,T] \qquad u = \phi \quad \text{on } \partial_p(\Omega \times (0,T]) \tag{1.1}$$

or a solution $u(x,t)$ to

$$u_t + Bu = 0 \quad \text{in } \Omega \times (0,T]$$
$$\begin{cases} u \geq \psi, \quad \alpha u_t + u_\nu \geq 0 \\ (\alpha u_t + u_\nu)(u-\psi) = 0 \end{cases} \quad \text{on } \Gamma \times (0,T] \tag{1.2}$$
$$u = \phi \quad \text{on } \partial_p(\Omega \times (0,T]) \setminus (\Gamma \times (0,T]),$$

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where A and B are (non-negative) definite elliptic operators. Usually, (1.1) is referred as a thick obstacle problem and (1.2) with $\Gamma \subset \partial\Omega$ (when $\alpha = 0$) as a Signorini boundary obstacle problem (or thin obstacle problem if one takes Γ to be a $(n-1)-$ manifold in Ω). We shall refer to (1.2) as the dynamic thin obstacle problem if $\alpha > 0$ and to nondynamic thin obstacle or Signorini problem if $\alpha = 0$. Recently, there has been an intense interest, perhaps due to the connectivity to jump or anomalous diffusion, to study (1.1) in all of $\mathbb{R}^n$ when A is a non-local operator and especially the fractional Laplacian. Observe that when A is the $\frac{1}{2}-$Laplacian there is an obvious equivalence between (1.1) and (1.2) which is identified by the Neumann-Dirichlet map, provided that u_t term is absent and B is minus the Laplacian, $\Gamma \subset \mathbb{R}^{n-1}$. This equivalence remains true for any fractional Laplacian if B is replaced by an appropriate degenerate elliptic operator as it was introduced in [14].

Every problem of the above mentioned ones and their obvious generalizations is actually a minimum of linear monotone operators therefore second order incremental quotients are "supersolutions" and satisfy a minimum principle. That is "for $z = (x, t)$ with $x \in \Omega$ in (1.1) or $z = (x', t)$ with $x' \in \mathbb{R}^{n-1}$ in (1.2)

$$u(z + w) + u(z - w) - 2u(z)$$

has no interior minima". In particular, in the limit $D_{ww}u$ cannot attain a minimum in the interior of the domain of definition and on the hyperplane in the case (1.2). This means that minima must occur at the initial or lateral data (minus the hyperplane in case (1.2)). Therefore for an appropriate data we have an L^∞ bound from below. This is certainly true if the data is smooth enough or just when the data stays strictly above the obstacle (Section 2). In fact, we believe that an appropriate barrier would give interior quasi-convexity of solutions under general data; this is the content of a forthcoming paper where we discuss the limitations of the smooth fit principle, an important issue in mathematical finance and numerical analysis.

The purpose of this work is to show that the quasi-convexity property, absent in the literature so far, has strong implications in the study of the above problems. One such implication is the improvement of optimal time regularity, *i.e.* the positive time derivative is continuous, contrary to a long standing belief that the time derivative is only bounded. In Section 3 we prove this for a wide class of problems with no assumptions on the free boundary; let us mention that in the literature there are only three cases in which the positive time derivative is continuous and all three rely on the fact that the time derivative is a priori non negative. These are the one-phase Stefan problem [11], the (non-dynamic) thin obstacle problem ([3] only in $n = 2$) and, very recently, the parabolic fractional obstacle problem [10].

For further implications of the quasi-convexity assumption we concentrate on the (nondynamic) thin obstacle problem or (time-dependent) Signorini problem. The other cases, *i.e.* the dynamic parabolic obstacle problem, the nondynamic and dynamic fractional counterparts, as well as the one with parabolic nonlocal operators, is a long term project and they will be treated in forthcoming papers. Also, elsewhere we show how one can get with this approach free boundary regularity for the already known result [12] of the "thick" obstacle. Actually, in this case, *i.e.* the (time-dependent) Signorini problem, we prove the optimal regularity of the space derivative (Section 4.1), as a consequence of the parabolic monotonicity formula stated in the appendix of [4]. Secondly, we prove that the regularity of the time derivative (Section 4.2) near free boundary points of positive parabolic density with respect to the coincidence set is as "good" as that of the space derivative; let us point out that the results in Section 4.2 are, in fact, independent of the quasi-convexity. And finally, in Section 4.3, since Section 4.2 yields control of the speed of the free boundary, we prove (space and time) regularity of the free boundary near "non-degenerate" free boundary points.

The results of the present paper were presented by the first author in IMPA, Rio de Janeiro, August, 17- 21, 2015 during the "International Conference on Current Trends in Analysis and Partial Differential Equations". A video of the talk is available online at http://video.impa.br.

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2. Quasi-convexity

In this section we prove the quasi-convexity of the solution for a wide class of Parabolic Obstacle Problems. In order to avoid technicalities we shall concentrate on five prototypes of this class:

1*st prototype* (*Thick obstacle problem*). Given a bounded domain Ω in $\mathbb{R}^n$, a function $\psi(x,t)$ (the obstacle) where $\psi < 0$ on $\partial\Omega \times (0,T]$, $\max \psi(x,0) > 0$ and a function ϕ with $\phi = 0$ on $\partial\Omega \times (0,T]$, $\phi \geq \psi$ on $\Omega \times \{0\}$, find a function u such that

$$\begin{cases} u_t - \Delta u \geq 0, \quad u \geq \psi & \text{in } \Omega \times (0,T] \\ (u_t - \Delta u)(u - \psi) = 0 & \text{in } \Omega \times (0,T] \\ u = \phi & \text{on } \partial_p(\Omega \times (0,T]). \end{cases} \tag{2.1}$$

2*nd prototype* (*Nondynamic thin obstacle problem*). Given a bounded domain Ω in $\mathbb{R}^n$ with part of its boundary $\Gamma \subset \partial\Omega$ that lies on $\mathbb{R}^{n-1}$, a function $\psi(x,t)$ (the obstacle) where $\psi < 0$ on $(\partial\Omega \setminus \Gamma) \times (0,T]$, $\max \psi(x,0) > 0$ and a function ϕ

with $\phi = 0$ on $(\partial\Omega \setminus \Gamma) \times (0, T]$, $\phi \geq \psi$ on $\Gamma \times \{0\}$, find a function u such that

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times (0, T] \\ \partial_\nu u \geq 0, \quad u \geq \psi & \text{on } \Gamma \times (0, T] \\ (\partial_\nu u)(u - \psi) = 0 & \text{on } \Gamma \times (0, T] \\ u = \phi & \text{on } \partial_p(\Omega \setminus \Gamma \times (0, T]) \end{cases} \tag{2.2}$$

where ν is the outward normal on $\partial\Omega$.

3*nd prototype* (*Dynamic thin obstacle problem*). Given a bounded domain Ω in $\mathbb{R}^n$ with part of its boundary $\Gamma \subset \partial\Omega$ that lies on $\mathbb{R}^{n-1}$, a function $\psi(x, t)$ (the obstacle), $\psi < 0$ on $(\partial\Omega \setminus \Gamma) \times (0, T]$, $\max \psi(x, 0) > 0$ and a function ϕ with $\phi = 0$ on $(\partial\Omega \setminus \Gamma) \times (0, T]$, $\phi \geq \psi$ on $\Gamma \times \{0\}$, find a function u such that

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times (0, T] \\ \alpha\partial_t u + \partial_\nu u \geq 0, \quad u \geq \psi & \text{on } \Gamma \times (0, T] \\ (\alpha\partial_t u + \partial_\nu u)(u - \psi) = 0 & \text{on } \Gamma \times (0, T] \\ u = \phi & \text{on } \partial_p(\Omega \setminus \Gamma \times (0, T]) \end{cases} \tag{2.3}$$

where $\alpha \in (0, 1]$ and ν is the outward normal on $\partial\Omega$.

4*th prototype* (*Fractional obstacle problem*). Given a $\psi : \mathbb{R}^{n-1} \times [0, \infty) \to \mathbb{R}$ such that $\int_{\mathbb{R}^{n-1}} \frac{|\psi|}{(1+|x|)^{n-1+2s}} dx' < +\infty$ for all $t > 0$ and $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\int \frac{|\phi|}{(1+|x|)^{n-1+2s}} < +\infty$ for some $0 < s < 1$, find a function u such that

$$\begin{cases} \partial_t u + (-\Delta)^s u \geq 0, \quad u - \psi \geq 0 & \text{on } \mathbb{R}^{n-1} \times (0, T] \\ (\partial_t u + (-\Delta)^s u)(u - \psi) = 0 & \text{on } \mathbb{R}^{n-1} \times (0, T] \\ u(x, 0) = \phi(x) & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{2.4}$$

5*th prototype* (*General nonlocal operators*). Assume that $\psi : \mathbb{R}^{n-1} \times [0, \infty) \to \mathbb{R}$ is given and let

$$\mathcal{L}u := u_t - \int_{\mathbb{R}^{n-1}} g'(u(y, t) - u(x, t))K(y - x)dy,$$

where $g : \mathbb{R} \to [0, \infty)$ is a $C^2(\mathbb{R})$ function such that $g(0) = 0$ and $\Lambda^{-1/2} \leq g''(z) \leq \Lambda^{1/2}$, $z \in \mathbb{R}$ for a given constant $\Lambda > 1$. The kernel $K : \mathbb{R}^{n-1} \setminus \{0\} \to (0, \infty)$ satisfies

$$\begin{cases} K(-x) = K(x) & \text{for any } x \in \mathbb{R}^{n-1} \setminus \{0\} \\ \chi_{\{|x| \leq 3\}} \dfrac{\Lambda^{-1/2}}{|x|^{n-1+s}} \leq K(x) \leq \dfrac{\Lambda^{1/2}}{|x|^{n-1+s}} & \text{for any } x \in \mathbb{R}^{n-1} \setminus \{0\}. \end{cases} \tag{2.5}$$

Then find a function u such that

$$\begin{cases} \mathcal{L}u \geq 0, \quad u - \psi \geq 0 & \text{on } \mathbb{R}^{n-1} \times (0, T] \\ (u-\psi)\mathcal{L}u = 0 & \text{on } \mathbb{R}^{n-1} \times (0, T] \\ u(x,0) = \phi(x) & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{2.6}$$

In the following theorem we prove quasi-convexity for the first, the second, the third and the fourth prototype problems. The proof for the fifth prototype problem, although similar, will be treated in a forthcoming paper. The following theorem can be stated and proved using incremental quotients as it is mentioned in the introduction, for simplicity though, we prove it for the second t-derivative. Notice that the corresponding space quasi-convexity is well known from the outset of the problems.

Theorem 2.1. *Suppose that in the above problems ψ and ϕ are smooth. If $(\phi - \psi)\big|_{t=0} > 0$ then*

$$||(u_{tt})^{-}||_\infty \leq C,$$

where C depends on $||\psi_{tt}||_\infty$ and on the boundedness of the fourth space derivative of the initial data or of the second derivatives in (2.4), *and on the second t-derivative of the lateral data for the bounded domains case. If $(\phi - \psi)\big|_{t=0} \geq 0$ the same estimate holds provided in addition that $(\partial_t \psi - (-\Delta)^s \psi)\big|_{t=0} \geq M > 0$ for $s \in (0, 1]$ and M sufficiently large.*

Remark 2.2. Actually, if in the theorem above one assumes that the data stays above the obstacle, then the data can be taken to be very general since in its neighbourhood everything smooths out due to the classical theory of parabolic equations, hence interior quasiconvexity holds and C depends only on $||(\psi_{tt})||_\infty$.

Proof. In all four cases we use the penalization method, *i.e.* one obtains the solution u as a limit of u^ε as $\varepsilon \to 0$, where u^ε is a solution, in case (2.1) of the problem

$$\begin{cases} \Delta u^\varepsilon - \partial_t u^\varepsilon = \beta_\varepsilon(u^\varepsilon - \psi^\varepsilon) & \text{in } \Omega \times (0, T] \\ u^\varepsilon = \phi^\varepsilon + \varepsilon & \text{on } \partial_p(\Omega \times (0, T]), \end{cases} \tag{2.7}$$

or, in case (2.2), of the problem

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = 0, & \text{in } \Omega \times (0, T] \\ -\partial_\nu u^\varepsilon = \beta_\varepsilon(u^\varepsilon - \psi^\varepsilon) & \text{on } \Gamma \times (0, T] \\ u^\varepsilon = \phi^\varepsilon + \varepsilon & \text{on } \partial_p(\Omega \setminus \Gamma \times (0, T]), \end{cases} \tag{2.8}$$

or, in case (2.3), of the problem

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = 0, & \text{in } \Omega \times (0, T] \\ -\alpha \partial_t u^\varepsilon - \partial_\nu u^\varepsilon = \beta_\varepsilon(u^\varepsilon - \psi^\varepsilon) & \text{on } \Gamma \times (0, T] \\ u^\varepsilon = \phi^\varepsilon + \varepsilon & \text{on } \partial_p(\Omega \setminus \Gamma \times (0, T]), \end{cases} \tag{2.9}$$

or, in case (2.4), of the problem

$$\begin{cases} -(-\Delta)^s u^\varepsilon - \partial_t u^\varepsilon = \beta_\varepsilon(u^\varepsilon - \psi^\varepsilon) & \text{on } \mathbb{R}^{n-1} \times (0, T] \\ u^\varepsilon(x, 0) = \phi^\varepsilon(x) + \varepsilon & \text{on } \mathbb{R}^{n-1}, \end{cases} \tag{2.10}$$

where, for $\varepsilon > 0$, ϕ^ε, ψ^ε are smooth functions (with compact support in the case of the whole $\mathbb{R}^{n-1}$), $\beta_\varepsilon(s) = -Ne^{\frac{\varepsilon}{s-\varepsilon}}\chi_{s\leq\varepsilon}(s)$ with N large enough with $\psi^\varepsilon \to \psi$, $\phi^\varepsilon \to \phi$ (locally) uniformly as $\varepsilon \to 0$. If, now, $(\phi - \psi)\big|_{t=0} > 0$ then differentiating twice with respect to t we obtain

$$\begin{cases} \Delta u^\varepsilon_{tt} - \partial_t u^\varepsilon_{tt} \leq \beta'_\varepsilon(u^\varepsilon - \psi^\varepsilon)\big(u^\varepsilon_{tt} - \psi^\varepsilon_{tt}\big) & \text{in } \Omega \times (0, T] \\ u^\varepsilon_{tt} = \phi^\varepsilon_{tt} & \text{on } \partial\Omega \times (0, T] \\ u^\varepsilon_{tt}(x, 0) = \Delta^2 \phi^\varepsilon(x) & \text{in } \Omega \times \{0\}, \end{cases} \tag{2.11}$$

or

$$\begin{cases} \partial_t u^\varepsilon_{tt} - \Delta u^\varepsilon_{tt} = 0, & \text{in } \Omega \times (0, T] \\ -\partial_\nu u^\varepsilon_{tt} \geq \beta'_\varepsilon(u^\varepsilon - \psi)\big(u^\varepsilon_{tt} - \psi^\varepsilon_{tt}\big) & \text{on } \Gamma \times (0, T] \\ u^\varepsilon_{tt}(x, 0) = \Delta^2 \phi^\varepsilon(x) & \text{in } \Omega \times \{0\}, \end{cases} \tag{2.12}$$

or

$$\begin{cases} \partial_t u^\varepsilon_{tt} - \Delta u^\varepsilon_{tt} = 0, & \text{in } \Omega \times (0, T] \\ -\alpha\partial_t u^\varepsilon_{tt} - \partial_\nu u^\varepsilon_{tt} \geq \beta'_\varepsilon(u^\varepsilon - \psi^\varepsilon)\big(u^\varepsilon_{tt} - \psi^\varepsilon_{tt}\big) & \text{on } \Gamma \times (0, T] \\ u^\varepsilon_{tt}(x, 0) = \Delta^2 \phi^\varepsilon(x) & \text{in } \Omega \times \{0\}, \end{cases} \tag{2.13}$$

or

$$\begin{cases} -(-\Delta)^s u^\varepsilon_{tt} - \partial_t u^\varepsilon_{tt} = \beta'_\varepsilon(u^\varepsilon - \psi^\varepsilon)\big(u^\varepsilon_{tt} - \psi^\varepsilon_{tt}\big) & \text{on } \mathbb{R}^{n-1} \times (0, T] \\ u^\varepsilon_{tt}(x, 0) = \Delta^2 \phi^\varepsilon & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{2.14}$$

To finish the proof, apply the minimum principle to u^ε_{tt}.

If, on the other hand, $(\phi - \psi)\big|_{t=0} \geq 0$, following the steps above, we notice that since $||\beta||_\infty < +\infty$ and $\beta' \geq 0$ it is enough to have $(\partial_t \psi - (-\Delta)^s \psi)\big|_{t=0} \geq M > 0$ for $s \in (0, 1]$ and M sufficiently large. □

3. A general implication on the optimality of the time derivative

In this section we show that the quasi-convexity property obtained in the previous section improves the time regularity for a wide class of problems. More precisely, we prove that the positive time derivative of the solution is always continuous for this class. Our approach will be as follows: we penalize the problems, we subtract the obstacle from the solution, then we differentiate with respect to time and we work with the derived equations. We shall obtain then a global uniform modulus of continuity independent of ε, which will yield in the limit the desired result.

In order to avoid having a lengthy paper, in the present section we concentrate on the first three prototype problems stated in (Section 2). The fourth and the fifth prototype problems will be treated in forthcoming papers.

3.1. The “thick” obstacle problem

In this situation the derived problem takes the form:

$$\begin{cases} \Delta v^\varepsilon - \partial_t v^\varepsilon = \beta'_\varepsilon(u^\varepsilon - \psi^\varepsilon)v^\varepsilon + f_t & \text{in } Q := \Omega \times (0,T] \\ v^\varepsilon = (\phi^\varepsilon - \psi^\varepsilon)_t & \text{on } \partial_p(\Omega \times (0,T]) \\ v^\varepsilon = \Delta(\phi^\varepsilon - \psi^\varepsilon) & \text{on } \Omega \times \{0\}, \end{cases} \tag{3.1}$$

where $v^\varepsilon = (u^\varepsilon - \psi^\varepsilon)_t$ and $f = -(\Delta\psi^\varepsilon - \partial_t \psi^\varepsilon)$.

Our method, which uses the approach of [9], is essentially that of De Giorgi, first appeared in his celebrated work [17]. To simplify matters we start with a normalized situation, *i.e.* we assume that our solution is between zero and one in the unit parabolic cylinder. We will prove (Proposition 3.5) that if at the top center v^ε is zero then in a concentric subcylinder into the future v^ε decreases. Then we rescale and repeat. But before that we need several lemmata. Our first lemma asserts that if v^ε is “most of the time” very near to its positive maximum in some cylinder, then in a smaller cylinder into the future v^ε is strictly positive.

Lemma 3.1. *Let* $Q_1(x_0,t_0) \subset Q$ *where* $Q_1(x_0,t_0) := B_1(x_0,0) \times (t_0 - 1, t_0]$ *with* $B_1 := \{x \in \mathbb{R}^n : |x - x_0| \le 1\}$. *Suppose that* $0 < v^\varepsilon < 1$ *in* $Q_1(x_0,t_0)$ *where* v^ε *is a solution to* (3.1)*; then there exists a constant* $\sigma > 0$*, independent of* ε*, such that*

$$\int_{Q_1(x_0,t_0)} (1 - v^\varepsilon)^2 dx < \sigma \tag{3.2}$$

implies that $v^\varepsilon \ge 1/2$ *in* $Q_{1/2}(x_0,t_0)$.

Proof. For simplicity we drop the ε, we shift (x_0,t_0) to $(0,0)$, and write Q_1 for $Q_1(0,0)$. First, we derive an energy inequality suited to our needs. Therefore we set $w = 1 - v$, so that the equation becomes

$$\Delta w - \partial_t w = \beta'(u - \psi)(w - 1) - f_t.$$

Choose a smooth cutoff function ζ vanishing near the parabolic boundary of Q_1 and $k \ge 0$. Multiply the above equation by $\zeta^2(w-k)^+$ and integrate by parts to obtain

$$\begin{aligned} &\frac{1}{2}\int_{Q_1} \partial_t \left[\left(\zeta(w-k)^+\right)^2 \right] dxdt + \int_{Q_1} \left|\nabla\left(\zeta(w-k)^+\right)\right|^2 dxdt \\ &= \int_{Q_1} \beta_t(u - \psi)\zeta^2(w-k)^+ dxdt \\ &\quad + \int_{Q_1} \left[(w-k)^+\right]^2 \left(|\nabla\zeta|^2 + \zeta\partial_t\zeta\right) dxdt + \int_{Q_1} f_t \zeta^2 (w-k)^+ dxdt. \end{aligned} \tag{3.3}$$

Integrating by parts in t the first term on the right we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{B_1}\big(\zeta(w-k)^+\big)^2(x,0)dx+\int_{Q_1}\big|\nabla\big(\zeta(w-k)^+\big)\big|^2dxdt\\
&=-\int_{Q_1}\beta(u-\psi)\partial_t\big(\zeta^2(w-k)^+\big)dxdt\\
&\quad+\int_{B_1}\beta(u-\psi)\zeta^2(w-k)^+(x,0)dx\\
&\quad+\int_{Q_1}\big[(w-k)^+\big]^2\big(|\nabla\zeta|^2+\zeta\partial_t\zeta\big)dxdt+\int_{Q_1}f_t\zeta^2(w-k)^+dxdt.
\end{aligned}\tag{3.4}$$

Now, since β is nonpositive and the upper limit of t-integration, $t=0$, could have been replaced by any $-1\le t\le 0$, our energy inequality takes the form

$$\begin{aligned}
&\max_{-1\le t\le 0}\int_{B_1}\big(\zeta(w-k)^+\big)^2dx+\int_{Q_1}\Big|\nabla\big(\zeta(w-k)^+\big)\Big|^2dxdt\\
&\le C\int_{Q_1}\Big(\big[(w-k)^+\big]^2\big(|\nabla\zeta|^2+|\partial_t\zeta|\big)+(w-k)^+(|\partial_t\zeta|+1)+\chi_{\{w>k\}}\Big)\,dxdt
\end{aligned}$$

with $C=2\max\{1,||\beta||_\infty(2+||(u_{tt})^-||_\infty),||f_t||_\infty\}$, where we have used the time quasiconvexity of the solution u.

Now, we want to obtain an iterative sequence of inequalities; thus we define, for $m=0,1,2,\dots$,

$$k_m:=\frac{1}{2}\left(1-\frac{1}{2^m}\right),\quad R_m:=\frac{1}{2}\left(1+\frac{1}{2^m}\right)$$
$$Q_m:=\Big\{(x,t):|x|\le R_m,\ -R_m^2\le t\le 0\Big\}$$

and the smooth cutoff functions

$$\chi_{Q_{m+1}}\le\zeta_m\le\chi_{Q_m}$$

with

$$|\nabla\zeta_m|\le C2^m,\quad|\partial_t\zeta_m|\le C4^m.$$

Substituting $\zeta=\zeta_m$ and setting $w_m=(w-k_m)^+$ we obtain, by the Sobolev inequality, that

$$\begin{aligned}
&\left(\int_{Q_m}(\zeta_m w_m)^{2\frac{n+2}{n}}dxdt\right)^{\frac{n}{n+2}}\\
&\le C\left(4^mC\int_{Q_m}w_m^2dxdt+C4^m\int_{Q_m}w_mdxdt+|Q_m\cap\{w_m\neq 0\}|\right)\\
&\le C\left(4^mC\int_{Q_m}w_m^2dxdt+\left(\frac{4^m}{2}+1\right)\big|Q_m\cap\{w_m\neq 0\}\big|\right).
\end{aligned}$$

Since

$$(k_m - k_{m-1})^2 |Q_m \cap \{w_m \neq 0\}| \leq \int_{Q_m} w_{m-1}^2 dxdt$$

we obtain

$$\begin{aligned}\int (\zeta_m w_m)^2 dxdt &\leq \left(\int (\zeta_m w_m)^{2\frac{n+2}{n}} dxdt \right)^{\frac{n}{n+2}} \left| Q_m \cap \{w_m \neq 0\} \right|^{\frac{2}{n+2}} \\ &\leq C 16^m \left(\int (\zeta_{m-1} w_{m-1})^2 dxdt \right)^{\frac{n+4}{n+2}}.\end{aligned} \tag{3.5}$$

Setting

$$I_m := \int (\zeta_m w_m)^2 dxdt$$

these quantities satisfy the recursive inequality

$$I_m \leq C 16^m I_{m-1}^{1+\frac{2}{n+2}}.$$

It is well known from De Giorgi's work (see for instance [19, Lemma II.5.6, page 95]) that $I_m \to 0$ as $m \to 0$ provided that

$$I_0 \leq \frac{1}{2^{(n+2)^2} C^{\frac{n+2}{n}}} =: \sigma.$$ □

Our second lemma asserts that if v^ε is very tiny "most of the time" in some cylinder, then in a smaller concentric cylinder, v^ε goes down to $1/2$. The fact that $\beta' > 0$ renders v^ε, more so any nonnegative solution to (3.1), a subsolution (subcaloric).

Lemma 3.2. *Let Q_1 be as in Lemma* 3.1. *Suppose that v^ε is a subsolution to* (3.1) *and that* $0 < v^\varepsilon < 1$ *in Q_1. Then there exists a constant $\bar{\sigma} > 0$, independent of ε, such that*

$$\int_{Q_1} (v^\varepsilon)^2 dxdt < \bar{\sigma}$$

implies that $v^\varepsilon \leq 1/2$ in $Q_{1/2}$.

Proof. It is identical to the proof of Lemma 3.1 except for the energy inequality which is in fact much simpler. As before we drop ε. We see that

$$\Delta v - \partial_t v \geq f_t \quad \text{in} \quad Q_1. \tag{3.6}$$

Therefore we multiply the equation by $\zeta^2 (v-k)^+$ where ζ and k are as in the proof of Lemma 3.1 and integrate by parts to obtain the energy inequality

$$\begin{aligned}&\max_{-1 \leq t \leq 0} \int_{B_1} \left(\zeta (v-k)^+\right)^2 dx + \int_{Q_1} \left|\nabla\left(\zeta (v-k)^+\right)\right|^2 dxdt \\ &\leq 2 \int_{Q_1} \left[(v-k)^+\right]^2 \left(|\nabla \zeta|^2 + |\partial_t \zeta|\right) dxdt.\end{aligned}$$

Again, we substitute $\zeta = \zeta_m$ and we set $v_m = (v - k_m)^+$ where ζ_m and k_m are as in Lemma 3.1. By the Sobolev inequality

$$\int (\zeta_m v_m)^{2\frac{n+2}{n}} dxdt \le C4^m \int v_m^2 dxdt$$

and since

$$(k_m - k_{m-1})^2 |Q_m \cap \{v_m \neq 0\}| \le \int v_{m-1}^2 dxdt$$

we obtain

$$\begin{aligned}\int (\zeta_m v_m)^2 dxdt &\le \left(\int (\zeta_m v_m)^{2\frac{n+2}{n}} dxdt \right)^{\frac{n}{n+2}} \big|Q_m \cap \{v_m \neq 0\}\big|^{\frac{2}{n+2}} \\ &\le C16^m \left(\int (\zeta_{m-1} v_{m-1})^2 dxdt \right)^{\frac{2}{n+2}}.\end{aligned} \tag{3.7}$$

Hence, if

$$I_m := \int (\zeta_m w_m)^2 dxdt$$

we have

$$I_m \le C16^m I_{m-1}^{1+\frac{2}{n+2}}$$

i.e. $I_m \to 0$ as $m \to 0$ provided that

$$I_0 \le \frac{1}{2^{(n+1)^2} C^{\frac{n+2}{2}}} =: \bar{\sigma}.$$ □

The next lemma is the parabolic version of De Giorgi's isoperimetric lemma. One version of this lemma is proved in [15] and with proper adjustments applies to our situation. We state it as our next lemma.

Lemma 3.3. *Given* $\epsilon_1 > 0$*, there exists a* $\delta_1 > 0$ *such that for every subsolution* v^ε *to* (3.1) *satisfying* $0 < v^\varepsilon < 1$ *in* Q_1 *and*

$$|\{(x,t) \in Q_1 : v^\varepsilon = 0\}| \ge \sigma_0 |Q_1|,$$

if

$$|\{(x,t) \in Q_1 : 0 < v^\varepsilon < 1/2\} < \delta_1 |Q_1|,$$

then

$$\int_{Q_{R'}} \left[\left(v^\varepsilon - \frac{1}{2} \right)^+ \right]^2 dxdt \le C\epsilon_1$$

where $R' = c\sigma_0$ *for* $\sigma_0 > 0$ *and some* $0 < c < 1$.

In order to achieve our decay estimate to zero we shall take a point $v^\varepsilon(0,0) = 0$ at the top center of Q_1 and show that in $Q_{R'}(0,0)$, for some $R' < 1$, v^ε is pointwise strictly less than one. This is the content of our next lemma.

Lemma 3.4. *Let Q_1 and σ be as in Lemma* 3.1. *Suppose that v^ε is a solution to* (3.1) *such that $v^\varepsilon(0,0) = 0$ and $0 \le v^\varepsilon \le 1$ in Q_1, then $v^\varepsilon \le 1 - C\sigma$ in $Q_{R'}(0,0)$ where C is independent of ε and $R' = \frac{\sigma}{8}$.*

Proof. Again, we drop ε. Since $v(0,0) = 0$, by Lemma 3.1

$$\int_{Q_1} (1-v)^2 dxdt \ge \sigma.$$

Then it follows that

$$\left|\left\{v < 1 - \frac{\sigma}{4}\right\} \cap Q_1\right| \ge \frac{1}{4}\sigma|Q_1|.$$

Therefore, we set

$$w := \frac{4}{\sigma}\left[v - \left(1 - \frac{\sigma}{4}\right)\right]^+$$

and we see that w is a subsolution to (3.1). Following De Giorgi's method we will consider a dyadic sequence of normalized truncations

$$w_k := 2^k\big[w - (1 - 2^{-k})\big]^+,$$

still subsolutions to (3.1). We will show that, in finite number of steps $k_0 = k_0(\delta_1)$ (where δ_1 is defined in Lemma 3.3 and $C\epsilon_1 \le \bar{\sigma}$, where $\bar{\sigma}$ is that of Lemma 3.2),

$$|\{w_{k_0} > 0\}| = 0.$$

Note that, for every k, one has $0 \le w_k \le 1$ and $|\{w_k = 0\} \cap Q_1| \ge \frac{\sigma}{4}|Q_1|$. Assume, now, that for every k $|\{0 < w_k < 1/2\} \cap Q_1| \ge \delta_1|Q_1|$. Then, for every k,

$$|\{w_k = 0\}| = |\{w_{k-1} = 0\}| + |\{0 < w_k < 1/2\}| \ge |\{w_{k-1} = 0\}| + \delta_1|Q_1|.$$

Therefore, after a finite number of steps, say $k_0 > 1/\delta_1$, we get $|\{w_{k_0} = 0\}| \ge |Q_1|$. Thus $w_{k_0} < 0$, *i.e.* $2^{k_0}[w - (1 - 2^{-k_0})]^+ = 0$ or $w < 1 - 2^{-k_0}$. Suppose, now, that there exists k' with $0 \le k' \le k_0$ such that

$$\left|\left\{0 < w_{k'} < \frac{1}{2}\right\}\right| < \delta_1|Q_1|.$$

By Lemma 3.3 applied to $w_{k'}$ with $\sigma_0 = \frac{\sigma}{4}$, and consequently by Lemma 3.2 applied to $w_{k'+1}$, we obtain $w_{k'+1} \le 1/2$ in $Q_{R'}$, where $R' = \frac{\sigma}{8}$, *i.e.* $w < 1 - 2^{-(k'+2)}$. Hence in both cases $w < 1 - 2^{-(k_0+2)}$ in $Q_{R'}$ or $v < 1 - 2^{-(k_0+4)}\sigma$. □

The estimates obtained above are all independent of ε. We would like to iterate the lemmata above to force the maximum of v^ε to decrease to zero in a dyadic sequence of decreasing parabolic cylinders in order to obtain the continuity of v^ε.

Proposition 3.5. *Let v^ε be a solution to* (3.1) *in Q then*

$$\left|(v^\varepsilon)^+(x,t) - (v^\varepsilon)^+(x_0,t_0)\right| \le C\omega\big(|x-x_0|^2 + |t-t_0|\big)$$

for any (x,t) and (x_0,t_0) in Q, where C is independent of ε and ω denotes the modulus of continuity.

Proof. It is enough to consider only the case when $(v^\varepsilon)^+(x_0,t_0) = 0$, since, otherwise, v^ε satisfies a nice equation with smooth data and with regular boundary. Therefore, for simplicity, we take $(x_0,t_0) = (0,0)$ and Q_1 as before. Again, we drop the ε and we set

$$Q_k := Q_{R_k}, \quad M_k := \sup_{Q_k} v$$

where $R_k := \frac{\sigma}{8} M_k$ and

$$\bar{v} := \frac{v_k}{M_k}$$

where $v_k(x,t) := v(R_k x, (R_k)^2 t)$. Then $\bar{v}$ satisfies

$$\Delta\bar{v} - \partial_t \bar{v} \ge \bar{f}_t \quad \text{in} \quad Q_1.$$

Therefore, by Lemma 3.4,

$$\sup_{Q_{R'}} \bar{v} \le 1 - C\sigma,$$

or, in our original setting,

$$\sup_{Q_{k+1}} v \le \mu_k \sup_{Q_k} v$$

where $\mu_k = 1 - C(\sup_{Q_k} v^+)^{1+\frac{n}{2}}$. So, even, if $\mu_k \to 1$ as $k \to \infty$, $M_k \to 0$.

To finish the proof, we use a standard barrier argument to get the continuity from the future. □

Theorem 3.6. *If u be a solution to* (2.1)*, then $(u-\psi)_t^+$ is continuous.*

Proof. It is well known that a subsequence of v^ε will converge uniformly to the unique solution of (2.1). □

3.2. The (nondynamic) thin obstacle problem or Signorini problem

Let us extend ψ^ε to all Ω, *i.e.* we take any function $\tilde{\psi}^\varepsilon(x', x_n, t)$ such that $\tilde{\psi}^\varepsilon(x', 0, t) = \psi^\varepsilon(x', t)$, $\tilde{\psi}^\varepsilon(x', x_n, t) < \phi$ on $\partial_p((\Omega \setminus \Gamma) \times (0, T])$, and $\frac{\partial \tilde{\psi}^\varepsilon}{\partial \nu}(x', 0, t) = 0$. Then our problem takes the form

$$\begin{cases} \Delta v^\varepsilon - \partial_t v^\varepsilon = f_t & \text{in } \Omega \times (0, T] \\ -\partial_\nu v^\varepsilon = \beta'_\varepsilon\big(u^\varepsilon - \tilde{\psi}^\varepsilon\big) v^\varepsilon & \text{on } \Gamma \times (0, T] \\ v^\varepsilon = \big(\phi^\varepsilon - \tilde{\psi}^\varepsilon\big)_t & \text{on } \partial_p(\overline{\Omega} \setminus \Gamma \times (0, T]) \\ v^\varepsilon = \Delta\big(\phi^\varepsilon - \tilde{\psi}^\varepsilon\big) & \text{on } \Omega \times \{0\}, \end{cases} \tag{3.8}$$

where $v^\varepsilon = (u^\varepsilon - \tilde{\psi}^\varepsilon)_t$ and $f = -(\Delta \tilde{\psi}^\varepsilon - \partial_t \tilde{\psi}^\varepsilon)$.

We shall repeat the approach of Section 3.1 but, instead of parabolic cylinders, we take parabolic rectangular cylinders with one of its sides lying on Γ. We normalize again, *i.e.* our solution is between zero and one, and we prove that, if v^ε is zero on the top center and on Γ in such a cylinder, then in a concentric subcylinder into the future v^ε is smaller than one. Then we rescale and repeat.

Our first lemma asserts that if v^ε is "most of the time" very near to its positive maximum in some cylinder sitting in $(\mathbb{R}^{n+1})$ against the hyperplane $x_n = 0$ and going backwards in time then in a smaller cylinder into the future, v^ε is strictly positive.

Lemma 3.7. *Let* $Q_1(x'_0, 0, t_0) \subset \Omega \times (0, T]$ *where* $Q_1(x'_0, 0, t_0) = B_1(x'_0, 0) \times (t_0 - 1, t_0]$, $B_1(x'_0, 0) = B'_1(x'_0) \times (0, 1)$, $B'_1(x_0) = \{x' : |x' - x'_0| < 1\}$ *and* $Q'_1(x'_0, t_0) = B'_1(x'_0) \times (t_0 - 1, t_0]$. *Suppose that* $0 < v^\varepsilon < 1$ *in* $Q_1(x_0, t_0)$ *where* v^ε *is a solution to* (3.8). *Then there exists a constant* $\sigma > 0$, *independent of* ε, *such that*

$$⨍_{Q'_1(x'_0, t_0)} \chi_{\{1 - v^\varepsilon > 0\}} dx' dt + ⨍_{Q_1(x_0, t_0)} (1 - v^\varepsilon)^2 dx dt < \sigma$$

implies that

$$v^\varepsilon \geq \frac{1}{2}$$

in $Q_{1/2}(x_0, t_0)$.

Proof. For simplicity we drop the superscript ε, shift $(x_0, 0, t_0)$ to $(0, 0, 0)$ and write Q_1 for $Q_1(0, 0, 0)$. We first derive an energy inequality associated to our problem. Setting $w = 1 - v$ the problem becomes

$$\begin{cases} \Delta w - \partial_t w = -f_t & \text{in } \Omega \times (-T, T] \\ \partial_\nu w = \beta_t\big(u - \tilde{\psi}\big) & \text{on } \Gamma \times (-T, T] \\ w = 1 - \big(\phi - \tilde{\psi}\big)_t & \text{on } \partial_p(\overline{\Omega} \setminus \Gamma \times (-T, T]) \\ w = 1 - \Delta\big(\phi - \tilde{\psi}\big) & \text{on } \Omega \times \{-T\}. \end{cases} \tag{3.9}$$

Choose a smooth cutoff function ζ vanishing near the parabolic boundary of Q_1, except on Q_1', and $k \geq 0$. Multiply the above equations by $\zeta^2(w-k)^+$ and integrate by parts to obtain

$$\int_{Q_1} \Big[\nabla\big(\zeta^2(w-k)^+\big)\nabla w + \zeta^2(w-k)^+\partial_t w\Big]\,dxdt$$
$$= \int_{Q_1'} \zeta^2(w-k)^+\partial_\nu w\,dx'dt + \int_{Q_1} f_t\zeta^2(w-k)^+dxdt$$

and

$$\begin{aligned}
&\int_{Q_1}\left[\frac{1}{2}\partial_t\Big[\big(\zeta(w-k)^+\big)^2\Big] + \Big|\nabla\big(\zeta(w-k)^+\big)\Big|^2\right]dxdt\\
&= \int_{Q_1'} \partial_t\Big[\zeta^2(w-k)^+\beta(u-\psi)\Big]dx'dt\\
&\quad -\int_{Q_1'} \zeta^2\partial_t(w-k)^+\beta(u-\psi)dx'dt + 2\int_{Q_1'} \zeta\,\partial_t\zeta\,(w-k)^+\beta(u-\psi)dx'dt\\
&\quad + \int_{Q_1} \big(|\nabla\zeta|^2 + \zeta\,\partial_t\zeta\big)\big[(w-k)^+\big]^2dxdt + \int_{Q_1} f_t\zeta^2(w-k)^+dxdt.
\end{aligned} \tag{3.10}$$

Now, using the fact that β is bounded and negative, $(u-\psi)_{tt}$ is bounded below, and since the upper limit of the t-integration $t=0$ can be replaced by any $-1 \leq t \leq 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\max_{-1\leq t\leq 0}\int_{B_1}\big[(w-k)^+\zeta\big]^2dx + \int_{Q_1}\big|\nabla\big((w-k)^+\zeta\big)\big|^2dxdt\\
&\leq \|\beta\|_\infty\big\|(u-\psi)_{tt}^-\big\|_\infty \int_{Q_1\cap\{w>k\}} \zeta^2dx'dt + \|\beta\|_\infty\int_{Q_1'}(w-k)^+\partial_t\zeta\,dx'dt\\
&\quad + \int_{Q_1}\big[(w-k)^+\big]^2\big(|\nabla\zeta|^2 + \partial_t\zeta\big)dxdt + \int_{Q_1} f_t\zeta^2(w-k)^+dxdt
\end{aligned}$$

and, a fortiori, we have the "energy inequality"

$$\begin{aligned}
&\max_{-1\leq t\leq 0}\int_{B_1}\big[(w-k)^+\zeta\big]^2dx + \int_{Q_1}\big|\nabla\big((w-k)^+\zeta\big)\big|^2dxdt\\
&\leq C\bigg(\int_{Q_1'}\big(\partial_t\zeta^2(w-k)^+ + \chi_{\{w>k\}}\big)dx'dt\\
&\quad + \int_{Q_1}\big[(w-k)^+\big]^2\big(|\nabla\zeta|^2 + \partial_t\zeta\big)dxdt + \int_{Q_1}\zeta^2(w-k)^+dxdt\bigg),
\end{aligned} \tag{3.11}$$

where $C = \overline{C}\max\{1, \|\beta\|_\infty(2 + \|(u_{tt})^-\|_\infty), \|f_t\|_\infty\}$.

Now, having our energy inequality, we shall obtain an iterative sequence of inequalities. We, therefore, define

$$k_m = \frac{1}{2}\left(1-2^{-m}\right), \quad R_m = \frac{1}{4}\left(1+\frac{1}{2^m}\right),$$
$$Q_m := B_m \times \left(R_m^2, 0\right], \quad B_m := B'_m \times [0, R_m], \quad B'_m := B'_{R_m} = \{|x'| < R_m\},$$
$$Q'_m := \{(x_1, \ldots, x_n, t) : -R_m \le x_i \le R_m, -R_m \le t \le 0\},$$

and we choose smooth cutoff functions ζ_m such that $\chi_{Q_{m+1}} \le \zeta_m \le \chi_{Q_m}$, $|\nabla\zeta_m| \le C2^m$ and $0 \le \partial_t\zeta_m \le C4^m$. We set $w_m = (w-k_m)^+$ and we denote by

$$I_m := \max_{-R^2 \le t \le 0} \int (\zeta_m w_m)^2 dx + \int |\nabla(\zeta_m w_m)|^2 dxdt.$$

We want to prove that for every $m \ge 0$, $I_m \le \alpha_0 M^{-m}$ with $\alpha_0 > 0$ and $M > 1$ to be chosen. The proof is by induction. For $1 \le m \le 2$ we choose σ such that $4C\sigma < M^{-2}$, and for $m \ge 3$ we have

$$\begin{aligned} I_m &\le C16^m \left(\int (w_{m-1}\zeta_{m-1})^2 dxdt + \int (w_{m-1}\zeta_{m-1})^2 dx'dt\right) \\ &= C16^m \left(\int (w_{m-1}\zeta_{m-1})^2 dxdt - 2\int (w_{m-1}\zeta_{m-1})(w_{m-1}\zeta_{m-1})_{x_n} dxdt\right) \\ &\le C16^m \Bigg[\int (w_{m-1}\zeta_{m-1})^2 dxdt \\ &\quad + \left(\int (w_{m-1}\zeta_{m-1})^2 dxdt\right)^{1/2} \left(\int |\nabla(\zeta_{m-1} w_{m-1})|^2 dxdt\right)^{1/2}\Bigg], \end{aligned} \tag{3.12}$$

where we used the divergence theorem and Hölder's inequality. Now, by Sobolev's inequality, we obtain

$$\begin{aligned} &\int_{Q_m} (w_{m-1}\zeta_{m-1})^2 dxdt \\ &\le \left(\int_{Q_{m-1}} (w_{m-1}\zeta_{m-1})^{2\frac{n+2}{n}} dxdt\right)^{\frac{n}{n+2}} \left(\int_{Q_{m-1}} \chi_{\{w_{m-1}\ne 0\}} dxdt\right)^{\frac{2}{n+2}} \\ &\le 2^{4m} I_{m-2}^{1+\frac{2}{n}}. \end{aligned}$$

Therefore, by substituting in the above we obtain

$$I_m \le C2^{8m}\left(I_{m-2}^{1+\frac{2}{n}} + I_{m-1}^{\frac{1}{2}} I_{m-2}^{\frac{1}{2}+\frac{1}{n}}\right).$$

Hence, if we choose $M = 2^{8n}$ and $\alpha_0 = C^{-\frac{n}{2}} 2^{-8n(n+2)}$, the claim is proved. $\square$

From this point on we observe that, since $\beta' > 0$, the boundary integral is nonnegative and can be omitted; then by reflecting about the hyperplane we are in the same situation as that of Section 3.1 with square cylinders. Therefore we have arrived at out main result:

Proposition 3.8. *Let v^ε be a solution to* (3.8) *in Q; then*

$$\left|(v^\varepsilon)^+(x,t) - (v^\varepsilon)^+(x_0,t_0)\right| \le C\big(|x-x_0|^2 + |t-t_0|\big)^\alpha$$

for any (x,t) and (x_0,t_0) in Q and some $0 < \alpha < 1$, where C and α are independent of ε.

Proof. It is enough to consider only the case when $(v^\varepsilon)^+(x_0,t_0) = 0$. For simplicity, we take $(x_0,t_0) = (0,0)$ and Q_1 as before. Again, we drop the ε and we set

$$Q_k := Q_{R_k}, \quad M_k := \sup_{Q_k} v$$

where $R_k := \frac{\sigma}{8} M_k$ and

$$\bar{v} := \frac{v_k}{M_k},$$

where $v_k(x,t) := v(R_k x, (R_k)^2 t)$. Then $\bar{v}$ satisfies

$$\Delta\bar{v} - \partial_t\bar{v} \ge \bar{f}_t \quad \text{in} \quad Q_1$$

and

$$\sup_{Q_{R'}} \bar{v} \le 1 - C\sigma,$$

or, in our original setting,

$$M_{k+1} \le \mu_k M_k,$$

where $\mu_k = 1 - C(\frac{M_k}{R_k})^{1+\frac{n}{2}}$. So, even, if $\mu_k \to 1$ as $k \to \infty$, $M_k \to 0$. As a matter of fact $M_k \sim 2^{-k}$ and $R_k \sim 2^{-k}$.

To finish the proof, we use a standard barrier argument to get the Hölder continuity. □

Theorem 3.9. *Let u be a solution to* (2.2)*; then $(u-\psi)_t^+$ is Hölder continuous.*

Proof. It is well known that a subsequence of v^ε will converge uniformly to the unique solution of (2.2). □

3.3. The dynamic thin obstacle problem

Let's extend ψ^ε to all Ω, *i.e.* we take any function $\tilde{\psi}^\varepsilon(x',x_n,t)$ such that $\tilde{\psi}^\varepsilon(x',0,)=\psi^\varepsilon(x',t)$, $\tilde{\psi}^\varepsilon(x',x_n,t)<\phi$ on $\partial_p((\Omega\backslash\Gamma)\times(0,T])$ and $\frac{\partial\tilde{\psi}^\varepsilon}{\partial\nu}(x',0,t)=0$. Subtracting $\tilde{\psi}^\varepsilon$ from the solution and differentiating with respect to t we obtain

$$\begin{cases} \Delta v^\varepsilon - \partial_t v^\varepsilon = -\big(\Delta\tilde{\psi}^\varepsilon - \partial_t\tilde{\psi}^\varepsilon\big)_t & \text{in } \Omega\times(0,T] \\ -\alpha\partial_t v^\varepsilon - \partial_\nu v^\varepsilon = \beta'_\varepsilon\big(u^\varepsilon - \tilde{\psi}^\varepsilon\big)v^\varepsilon + \alpha\partial_t\tilde{\psi}^\varepsilon_t & \text{on } \Gamma\times(0,T] \\ v^\varepsilon = \big(\phi^\varepsilon - \tilde{\psi}^\varepsilon\big)_t & \text{on } \partial_p(\overline{\Omega}\setminus\Gamma\times(0,T]) \\ v^\varepsilon = \Delta\big(\phi^\varepsilon - \tilde{\psi}^\varepsilon\big) & \text{on } \Omega\times\{0\}, \end{cases} \tag{3.13}$$

where $v^\varepsilon = (u^\varepsilon - \tilde{\psi}^\varepsilon)_t$. In order to avoid technicalities, in this more complex situation, and bring forth the main idea, we shall assume throughout this section that $(\Delta\tilde{\psi}^\varepsilon - \partial_t\tilde{\psi}^\varepsilon)_t = 0$.

We shall repeat the approach of Section 3.1 but, instead of parabolic cylinders, we take "hyperbolic" hypercubes with one of its sides lying on Γ. We normalize again, *i.e.* our solution is between zero and one, and we prove (Lemma 3.13) that, if v^ε is zero on the top center and on Γ in such a hypercube, then in a concentric subhypercube into the future v^ε is smaller than one. Then we rescale and repeat. The rescaling, of course, is hyperbolic appropriate for the boundary term on Γ but diminishes the time derivative in the heat equation; this though does not prevent us to obtain the continuity, as it was done in [5].

Our first lemma asserts that if v^ε is "most of the time" very near to its positive maximum in some hypercube sitting in $(\mathbb{R}^{n+1})$ against the hyperplane $x_n = 0$ and going backwards in time, then in a smaller hypercube into the future v^ε is strictly positive.

Lemma 3.10. *Let $Q_R(x'_0,0,t_0)\subset\Omega\times(0,T]$ where $Q_R(x'_0,0,t_0) = B_R(x'_0,0)\times(t_0-R,t_0]$, $B_R(x'_0,0) = B'_R(x_0)\times(0,1)$, $B'_R(x_0) = \{x' = (x_1,\dots,x_{n-1}) : |x'_i - x'_{0i}| < R, i = 1,\dots,n-1\}$ and $Q_R(x'_0,0) = B'_R(x'_0)\times(t_0-R,t_0]$. Suppose that $0 < v^\varepsilon < 1$ in $Q_R(x_0,t_0)$ where v^ε is a solution to (3.13). Then there exists a constant $\sigma > 0$, independent of ε, such that*

$$⨍_{Q'_R(x'_0,t_0)}(1-v^\varepsilon)^2dx'dt + ⨍_{Q_R(x_0,t_0)}(1-v^\varepsilon)^2dxdt < \sigma$$

implies that

$$v^\varepsilon \geq \frac{1}{8}$$

in $Q_{r/8}(x_0,t_0) = B'_R(x'_0)\times\big(0,\frac{1}{8}\big)\times\big(t_0-\frac{R}{8},t_0\big]$.

Proof. For simplicity we drop the superscript ε, shift $(x_0, 0, t_0)$ to $(0, 0, 0)$ and write Q_R for $Q_R(0, 0, 0)$. We first derive an energy inequality associated to our problem. Setting $w = 1 - v$, the problem becomes

$$\begin{cases} \Delta w - \partial_t w = 0 & \text{in } \Omega \times (-T, T] \\ \alpha \partial_t w + \partial_\nu w = \beta_t\big(u - \tilde{\psi}\big) + \alpha \partial_t \tilde{\psi}_t & \text{on } \Gamma \times (-T, T] \\ w = 1 - \big(\phi - \tilde{\psi}\big)_t & \text{on } \partial_p(\overline{\Omega} \setminus \Gamma \times (-T, T]) \\ w = 1 - \Delta\big(\phi - \tilde{\psi}\big) & \text{on } \Omega \times \{-T\}. \end{cases} \tag{3.14}$$

Choose a smooth cut-off function ζ vanishing near the parabolic boundary of Q_R, except on Q'_R, and $k \geq 0$. Multiply the above equations by $\zeta^2(w-k)^+$ and integrate by parts to obtain

$$\int_{Q_R} \Big[\nabla\big(\zeta^2(w-k)^+\big)\nabla w + \zeta^2(w-k)^+\partial_t w\Big]\,dxdt - \int_{Q'_R} \zeta^2(w-k)^+\partial_\nu w\,dx'dt = 0$$

and

$$\begin{aligned} &\int_{Q_R} \Big[\Big|\nabla\big(\zeta^2(w-k)^+\big)\Big|^2 + \frac{1}{2}\partial_t\big(\zeta^2(w-k)^+\big)^2\Big]\,dxdt \\ &\quad - \int_{Q'_R} \zeta^2(w-k)^+(\beta_t(u-\psi) + \alpha\psi_{tt} - \alpha\partial_t w)\,dx'dt \\ &= \int_{Q_R} \big((w-k)^+\big)^2\big(|\nabla\zeta|^2 + \zeta\zeta_t\big)\,dxdt \end{aligned}$$

and

$$\begin{aligned} &\frac{\alpha}{2}\int_{Q'_R} \partial_t\Big[\big(\zeta(w-k)^+\big)^2\Big]dx'dt \\ &+ \int_{Q_R} \Big[\frac{1}{2}\partial_t\Big[\big(\zeta(w-k)^+\big)^2\Big] + \Big|\nabla\big(\zeta(w-k)^+\big)\Big|^2\Big]dxdt \\ =& \int_{Q'_R} \partial_t\big[\zeta^2(w-k)^+\beta(u-\psi)\big]dx'dt - \int_{Q'_R} \zeta^2\partial_t(w-k)^+\beta(u-\psi)\,dx'dt \\ &+ 2\int_{Q'_R} \zeta\partial_t\zeta(w-k)^+\beta(u-\psi)\,dx'dt + \alpha\int_{Q'_R} \zeta\partial_t\zeta\big((w-k)^+\big)^2dx'dt \\ &+ \alpha\int_{Q'_R} \zeta^2(w-k)^+\psi_{tt}\,dx'dt + \int_{Q_R} \big(|\nabla\zeta|^2 + \zeta\partial_t\zeta\big)\big[(w-k)^+\big]^2dxdt. \end{aligned} \tag{3.15}$$

Now, using the fact that β is bounded and negative, $(u-\psi)_{tt}$ is bounded below, and since the upper limit of the t-integration $t = 0$ can be replaced by any $-R \leq t \leq 0$, we obtain

$$
\begin{aligned}
&\frac{\alpha}{2}\max_{-R\le t\le 0}\int_{B'_R}\big[(w-k)^+\zeta\big]^2dx' + \frac{1}{2}\max_{-R\le t\le 0}\int_{B_R}\big[(w-k)^+\zeta\big]^2dx \\
&\quad+\int_{Q_R}\big|\nabla\big((w-k)^+\zeta\big)\big|^2dxdt \\
&\le ||\beta||_\infty||(u-\psi)^-_{tt}||_\infty\int_{Q_R\cap\{w>k\}}\zeta^2dx'dt + 2||\beta||_\infty\int_{Q'_R}(w-k)^+|\partial_t\zeta|dx'dt \\
&\quad+\alpha\int_{Q'_R}\big[(w-k)^+\big]^2|\partial_t\zeta|dx'dt + \alpha||\psi_{tt}||_\infty\int_{Q'_R}(w-k)^+dx'dt \\
&\quad+\int_{Q_R}\big[(w-k)^+\big]^2\big(|\nabla\zeta|^2+|\partial_t\zeta|\big)dxdt
\end{aligned}
$$

or

$$
\begin{aligned}
&\alpha\max_{-R\le t\le 0}\int_{B'_R}\big[(w-k)^+\zeta\big]^2dx' + \max_{-R\le t\le 0}\int_{B_R}\big[(w-k)^+\zeta\big]^2dx \\
&\quad+\int_{Q_R}\big|\nabla\big((w-k)^+\zeta\big)\big|^2dxdt \\
&\le C\left(\int_{Q'_R}\big[(w-k)^+\big]^2|\partial_t\zeta| + (w-k)^+(1+|\partial_t\zeta|)dx'dt + \int_{Q'_R\cap\{w>k\}}\zeta^2dx'dt\right) \\
&\quad+C\int_{Q_R}\big[(w-k)^+\big]^2\big(|\nabla\zeta|^2+|\partial_t\zeta|\big)dxdt,
\end{aligned}
$$

where $C = 2\max\{||\beta||_\infty, ||(u-\psi)^-_{tt}||_\infty, 1, \alpha||\psi_{tt}||_\infty\}$ and, a fortiori, we have the "energy inequality"

$$
\begin{aligned}
&\alpha\max_{-R\le t\le 0}\int_{B'_R}\big[(w-k)^+\zeta\big]^2dx' + \int_{Q_R}|\nabla((w-k)^+\zeta)|^2dxdt \\
&\le C\Bigg(\int_{Q'_R}\big[(w-k)^+\big]^2|\partial_t\zeta| + (w-k)^+(1+|\partial_t\zeta|) + \zeta^2\chi_{\{w>k\}}dx'dt \qquad (3.16)\\
&\qquad+\int_{Q_R}\big[(w-k)^+\big]^2\big(|\nabla\zeta|^2+|\partial_t\zeta|\big)dxdt\Bigg).
\end{aligned}
$$

Now, having this energy inequality at disposal, we shall obtain an iterative sequence of inequalities. More precisely, the method consists in taking a sequence of decreasing cutoff functions in space and time ζ_m that converges to the indicator function of $Q_{R/4}$ and simultaneously a series of cutoff functions of the graph of u, u_m that converge to $(w-7/8)^+$ and prove by iteration that in the limit $(w-7/8)^+ = 0$ on

$Q_{r/4}$. Therefore we define

$$k_m = \frac{9}{16} + \frac{1}{16}\left(1 - 2^{-m}\right) \quad R_m = \frac{R}{4}\left(1 + \frac{1}{2^m}\right)$$
$$Q'_m = \{(x_1, \dots, x_n, t) : -R_m \le x_i \le R_m, -R_m \le t \le 0\}$$

and we choose cutoff functions ζ_m depending only on x' and t and such that $\chi_{Q'_{m+1}} \le \zeta_m \le \chi_{Q'_m}$, $|\nabla\zeta_m| \le C2^m$ and $|\partial_t \zeta_m| \le C2^m$. We set $u_m = (u - k_m)^+$ and we define

$$I_m = \int_{Q'_R} (\zeta_m u_m)^2 dx'dt + \int_0^{\delta^m/2} \int |\nabla(\zeta_m u_m)|^2 dxdt,$$

where $0 < \delta < 1$ is chosen such that $2^n 2^{-\frac{(n+6)2^{-m-1}}{\delta^m}} \le 2^{-m-4}$ holds. We also choose M as in [5] satisfying $2^{n+1} M^{-\frac{m}{2}} (\delta^n)^{-m-1} \le 2^{-m-6}$, $M^{-m} \ge C4^{m(1+\frac{1}{n-1})} M^{-(m-3)(1+\frac{1}{n-1})}$, $m \ge 14(n-1)$.

We want to prove simultaneously that for every $m \ge 0$, $I_m \le M^{-m}$ and that $u_m = 0$ on $Q'_m \times \{\frac{\delta^m}{2}\}$. The proof is by induction and is identical with Step 2a and Step 2b of [5, Lemma 2.2], except that

$$\left|\left|u\chi_{Q'_R} * H(x_n)\right|\right| \le ||H(y)||_{\infty(\{x_n \ge 1\})} \int_{Q'_R} u dx'dt$$
$$\le \frac{2^{n+2}}{\pi^{\frac{n}{2}}} \left(\frac{n+2}{2e}\right)^{n+2} \left|Q'_R\right|^{1/2} \sigma^{1/2} \le \frac{1}{64}$$

for σ small enough. So we concentrate on Step 2c, where we will show that

$$I_m \le C4^{m(1+\frac{1}{n-1})} I_{m-3}^{1+\frac{1}{n-1}}, \quad m \ge 14n - 13.$$

By the energy inequality,

$$\begin{aligned} I_m \le & \int (w_m \zeta_{m-1})^2 dx'dt \\ & + C\left[C2^m \int (w_m \zeta_{m-1})^2 dx'dt + (1 + C2^m)\frac{1}{2} \int (w_m \zeta_{m-1})^2 dx'dt\right] \\ & + C\left[(1 + C2^m)\frac{1}{2}|Q_{m-1} \cap \{w_m \ne 0\}| + \left|Q'_{m-1} \cap \{w_m \ne 0\}\right| \right. \\ & \qquad \left. +(C2^m)^2 \int (w_m \zeta_{m-1})^2 dxdt\right] \\ & + C\left[\frac{1}{2}\int (w_m \zeta_{m-1})^2 dxdt + \frac{1}{2}|Q_{m-1} \cap \{w_m \ne 0\}|\right], \end{aligned}$$

where we have used Young's inequality. Since $w_m < w_{m-1}$ and $\{w_m \neq 0\} = \{w_{m-1} > 2^{-m-4}\}$, we have

$$I_m \le C2^m \int (w_{m-1}\zeta_{m-1})^2 dx'dt + C4^m \int (w_{m-1}\zeta_{m-1})^2 dxdt.$$

Also, the integral of the second term, satisfies

$$\begin{aligned}\int (w_{m-1}\zeta_{m-1})^2 dxdt &\le \int |(w_{m-2}\zeta_{m-2}) * H(x_n)|^2 dxdt\\ &\le ||H||^2_{L^1(Q_R)} \int (w_{m-2}\zeta_{m-2})^2 dx'dt.\end{aligned}$$

Therefore one has

$$\begin{aligned}I_m &\le C4^m \int (w_{m-2}\zeta_{m-2})^2 dx'dt\\ &\le C4^m \Big(\int (w_{m-2}\zeta_{m-2})^{2\frac{n}{n-1}} dx'dt \Big)^{\frac{n-1}{n}} \left|\{w_{m-2} \neq 0\} \cap Q'_{m-2}\right|^{\frac{1}{n}} \qquad (3.17)\\ &\le C4^{m(1+\frac{1}{n-1})} \int (w_{m-3}\zeta_{m-3})^{2\frac{n}{n-1}} dx'dt.\end{aligned}$$

By Sobolev's inequality

$$I_m \le C4^{m(1+\frac{1}{n-1})} \left(\int (w_{m-3}\zeta_{m-3})^2 dx'dt + \int \left|\Lambda^{1/2}(w_{m-3}\zeta_{m-3})\right|^2 dx'dt \right)^{\frac{n}{n-1}}$$

where $\Lambda(w_{m-3}\zeta_{m-3}) = -\frac{\partial}{\partial x_n}(w_{m-3}\zeta_{m-3})$. Since

$$\int \left|\Lambda^{1/2}(w_{m-3}\zeta_{m-3})\right|^2 dx'dt \le \int |\nabla(w_{m-3}\zeta_{m-3})|^2 dxdt,$$

we have

$$I_m \le C4^{m(1+\frac{1}{n-1})} I_{m-3}^{1+\frac{1}{n-1}} \quad \text{for every } \; m \ge 14(n-1)+1$$

and so $I_m \to 0$ as $m \to \infty$ provided that

$$I_0 \le C^{-(n-1)} 4^{-n(n-1)} =: \sigma.$$

Hence, to complete the proof, consider the function $\bar{w}$ defined by

$$\begin{cases} \Delta \bar{w} - \partial_t \bar{w} = 0 & \text{in } \; Q_{R/4}\\ \bar{w} = 1 & \text{on } \; \partial_p(\overline{Q_{R/4}} \setminus \{x_n = 0\}\\ \bar{w} = \dfrac{5}{8} & \text{on } \; Q'_{R/4}. \end{cases} \qquad (3.18)$$

Then $\bar{w} < 7/8$ in $Q_{R/8}$ and by the maximum principle $w \le \bar{w}$. □

Our second lemma asserts that if v^ε is very tiny "most of the time" in some hypercube (as above) then, in a smaller concentric hypercube, v^ε goes down from 1 to $7/8$.

Lemma 3.11. *Let $Q_R(x_0', 0, t_0)$ be as in Lemma* 3.10. *Suppose that v^ε is a subsolution to* (3.13) *and that $0 < v^\varepsilon < 1$ in $Q_R(x_0', 0, t_0)$. Then there exists a constant $\bar{\sigma} > 0$, independent of ε, such that*

$$\fint_{Q_R'(x_0',t_0)} (v^\varepsilon)^2 dx'dt + \fint_{Q_R(x_0',0,t_0)} (v^\varepsilon)^2 dxdt < \bar{\sigma}$$

implies that

$$v^\varepsilon \le \frac{7}{8}$$

in $Q_{r/8}(x_0, 0, t_0)$.

Proof. The proof is identical to the proof of Lemma 3.10 except from the energy inequality. For simplicity again we drop the ε and take $(x_0', 0, t_0) = (0, 0, 0)$ with $Q_R = Q_R(0, 0, 0)$. Since $\beta' \ge 0$, v satisfies

$$\begin{cases} \Delta v - \partial_t v = 0 & \text{in } Q_R \\ -\alpha \partial_t v - \partial_\nu v \ge \alpha \partial_t \psi_t & \text{on } Q_R'. \end{cases} \tag{3.19}$$

Choose again, a smooth cutoff function ζ vanishing near the parabolic boundary of Q_R except on Q_R' and $k \ge 0$. Multiply the above by $\zeta^2 (v-k)^+$ and integrate by parts to obtain

$$\int_{Q_R} \Big[\nabla\big(\zeta^2 (v-k)^+\big)\nabla v + \zeta^2 (v-k)^+ \partial_t v \Big] dxdt - \int_{Q_R'} \zeta^2 (v-k)^+ \partial_\nu v dx'dt \le 0$$

or

$$\begin{aligned} &\int_{Q_R} \Big[\Big|\nabla\big(\zeta (v-k)^+\big)\Big|^2 + \frac{1}{2}\partial_t\Big(\big(\zeta (v-k)^+\big)^2\Big)\Big] dxdt \\ &\quad + \int_{Q_R'} \zeta^2 (v-k)^+ \left[\alpha \psi_{tt} + \alpha \partial_t v\right] dx'dt \\ &\le \int_{Q_R} \Big(\big(v-k\big)^+\Big)^2 \Big[|\nabla \zeta|^2 + \zeta \partial_t \zeta \Big] dxdt \end{aligned}$$

and

$$\begin{aligned} &\frac{\alpha}{2} \int_{Q_R'} \partial_t \Big[\big(\zeta (v-k)^+\big)^2\Big] dx'dt + \int_{Q_R} \Big[\partial_t \big[(\zeta (v-k)^+]^2\big) + \big|\nabla\big(\zeta (v-k)^+\big)\big|^2 \Big] dxdt \\ &\le \alpha \int_{Q_R'} \left[(v-k)^+\right]^2 |\partial_t \zeta| dx'dt + \alpha ||\psi_{tt}||_{L^\infty} \int_{Q_R'} (v-k)^+ dx'dt \\ &\quad + \int_{Q_R} \left[(v-k)^+\right]^2 \big(|\nabla \zeta|^2 + |\partial_t \zeta|\big) dxdt. \end{aligned}$$

Taking again as upper limit any $-R \le t \le 0$, we obtain

$$\alpha \max_{-R\le t\le 0} \int_{B'_R} \left[\zeta(v-k)^+\right]^2 dx' + \int_{Q_R} \left|\nabla\big(\zeta(v-k)^+\big)\right|^2 dxdt$$
$$\le \bar{C}\left\{\int_{Q'_R} \left[\left[(v-k)^+\right]^2 |\partial_t\zeta| + (v-k)^+\right] dx'dt \right.$$
$$\left. + \int_{Q_R} \left[(v-k)^+\right]^2 \big(|\nabla\zeta|^2 + |\partial_t\zeta|\big) dxdt\right\}$$

where $\bar{C} = 2\max\{1, \alpha||\psi_{tt}||_{L^\infty}\}$.

Now, since we have our energy inequality, the rest is as that of Lemma 3.10 and we define

$$\bar{\sigma} := \bar{C}^{-(n-1)} 4^{-n(n-1)}. \qquad \square$$

We proceed, now, by using the parabolic version of De Giorgi's isoperimetric lemma. This lemma is proved in [15] and with proper adjustments applies to our situation. We state it as our next lemma.

Lemma 3.12. *Given $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that for every subsolution v^ε to* (3.13) *satisfying $0 < v^\varepsilon < 1$ in Q_R and*

$$|\{(x,t) \in Q_R : v^\varepsilon = 0\}| \ge \sigma_0 |Q_R|,$$

if

$$|\{(x,t) \in Q_R : 0 < v^\varepsilon < 1/2\} < \delta_1 |Q_R|$$

then

$$⨍_{Q'_{R'}} \left[\left(v^\varepsilon - \frac{1}{2}\right)^+\right]^2 dx'dt + ⨍_{Q_{R'}} \left[\left(v^\varepsilon - \frac{1}{2}\right)^+\right]^2 dxdt < C\sqrt{\epsilon_1},$$

where $R' = \frac{\sigma_0}{2} R$ for $\sigma_0 > 0$.

We are now ready to obtain our basic decay estimate to zero.

Lemma 3.13. *Let $Q_R(x'_0, 0, t_0)$ and σ be as in Lemma* 3.10. *Suppose that v^ε is a solution to* (3.13) *such that $v^\varepsilon(x'_0, 0, t_0) = 0$ and $0 \le v^\varepsilon \le 1$ in $Q_R(x'_0, 0, t_0)$. Then $v^\varepsilon \le 1 - C\sigma$ in $Q_{R'}(x'_0, 0, t_0)$ where C is independent of ε and $R' = \frac{\sigma}{16} R$.*

Proof. We drop the ε, take $(x'_0, 0, t_0)$ to be $(0,0,0)$ (by translation), and set $Q_R = Q_R(0,0,0)$. Since $v(0,0,0) = 0$, by Lemma 3.10

$$⨍_{Q'_R} (1-v)^2 dx'dt + ⨍_{Q_R} (1-v)^2 dxdt \ge \sigma.$$

Then it follows that there exists a constant $c_0 < 1$ such that

$$\left|\left\{v < 1 - \frac{\sigma}{4}\right\} \cap Q_R\right| \geq c_0\sigma|Q_R|.$$

Therefore set

$$w := \frac{4}{\sigma}\left(v - \left(1 - \frac{\sigma}{4}\right)\right)^+$$

and observe that w is a (nonnegative) subsolution to (3.13). Again by De Giorgi's lemma, the normalized truncations, *i.e.*

$$w_k := 2^k\Big(w - \big(1 - 2^{-k}\big)\Big)^+,$$

are still subsolutions to (3.13). We will show now that in a finite number of steps $k_0 = k_0(\delta_1)$ (δ_1 as in Lemma 3.12) there holds $|\{w_{k_0} > 0\}| = 0$. Note that for every k, $0 \leq w_k \leq 1$ and $|\{w_k = 0\} \cap Q_R| \geq \sigma_1|Q_R|$. Set $C\sqrt{\epsilon_1} \leq \bar{\sigma}$ where ϵ_1 is defined in Lemma 3.12 and $\bar{\sigma}$ in Lemma 3.11. Hence we assume that for every k, $|\{0 < w_k < \frac{1}{2}\} \cap Q_R| \geq \delta_1|Q_R|$. Then, for every k, one has

$$|\{w_k = 0\}| = |\{w_{k-1} = 0\}| + |\{0 < w_{k-1} < 1/2\}| \geq |\{w_{k-1} = 0\}| + \delta_1|Q_R|.$$

Hence after a finite number of steps, say $k_0 > 1/\delta_1$, we get $|\{w_{k_0} = 0\}| \geq |Q_R|$. Thus $w_{k_0} < 0$, *i.e.* $2^{k_0}[w - (1 - 2^{-k_0})]^+ = 0$ or $w < 1 - 2^{-k_0}$. Suppose, now, that there exists k', $0 \leq k' \leq k_0$, such that

$$\left|\left\{0 < w_{k'} < \frac{1}{2}\right\}\right| < \delta_1.$$

By Lemma 3.12 applied to $w_{k'}$, and consequently by Lemma 3.11 applied to $w_{k'+1}$, we conclude that $w_{k'+1} \leq 7/8$ in $Q_{R'}$, where $R' = \frac{\sigma}{16}R$, *i.e.* $w < 1 - \frac{1}{8}2^{-(k'+1)}$. A fortiori, in both cases we have $w < 1 - 2^{-(k_0+4)}$ in $Q_{R'}$, that is $v < 1 - 2^{-k_0-6}\sigma$. □

The estimates we obtained above are all independent of ε and remain invariant under hyperbolic scaling much the same way as in [5]. Although the time derivative term diminishes in the rescaling, we still obtain the continuity of the time derivative.

Proposition 3.14. *Let v^ε be a solution to* (3.13) *in Q_R. Suppose that* $0 \leq v^\varepsilon \leq M$, *where M is independent of ε. If* $v^\varepsilon(0,0,0) = 0$ *then*

$$v^\varepsilon\big(x', x_n, t\big) \leq \omega\big(|x'|, |x_n|, |t|\big),$$

where ω is a modulus of continuity (*required to be monotone and to satisfy* $\omega(0) = 0$) *independent of ε.*

Proof. We drop as usual the ε. Set

$$Q_k := Q_{R_k} = (-R_k, R_k)^{n-1} \times (0, r_k) \times (-R_k, 0] \text{ and } M_k := \sup_{Q_k} v$$

where $R_k := r_k R, r_k := \frac{\sigma}{16} M_k$. Define

$$\bar{v} := \frac{v_k}{M_k}, \quad \text{where } v_k(x,t) := v\big(r_k x', r_k x_n, r_k t\big).$$

Then $\bar{v}$ verifies

$$\begin{cases} \Delta\bar{v} - r_k \partial_t \bar{v} = 0 & \text{in } Q_R \\ -\alpha\partial_t\bar{v} - \partial_\nu\bar{v} = \beta'(u-\psi)\bar{v} + \bar{\psi}_t & \text{on } Q'_R \end{cases} \tag{3.20}$$

where $\bar{\psi}_t = \alpha\partial_t\psi_t / M_k$. We apply Lemma 3.13 to $\bar{v}$ to obtain

$$\sup_{Q_{R'}} \bar{v} \leq 1 - C\sigma.$$

Hence in our original setting

$$\sup_{Q_{k+1}} v \leq \mu_k \sup_{Q_k} v,$$

where $\mu_k = 1 - C(\sup_{Q_k} v)^{n-1}$. Therefore $\mu_k \to 1$ as $k \to \infty$ only if $\sup_{Q_k} u \to 0$ which yields a modulus of continuity. Finally, a standard barrier argument yields the continuity from the future, too. $\square$

Theorem 3.15. *Let u be a solution to* (2.3) *then* $(u-\psi)_t^+$ *is continuous with a uniform modulus of continuity.*

4. Further implications on the (nondynamic) thin obstacle problem or (time dependent) Signorini problem

In the present section we shall concentrate on the nondynamic parabolic "thin" obstacle or parabolic Signorini problem and we will show how the quasi-convexity yields the optimal regularity of the solution as well as free boundary regularity. The other cases, as it was mentioned in Section 1, will be treated in forthcoming papers. Since it is easier to work with the zero obstacle, we extend the obstacle as it was done in Section 3.2 in all of Ω and subtract it from the solution which we still denote by u.

More precisely, given $\Omega \subset \mathbb{R}^n$, an open bounded set with smooth boundary $\partial\Omega$ and $\Gamma \subset \partial\Omega$ lying in $\mathbb{R}^{n-1}$, we consider the following problem:

$$\begin{cases} \Delta u - \partial_t u = f, & \text{in } \Omega \times (-T, T] \\ \partial_\nu u \geq 0, \quad u \geq 0 & \text{on } \Gamma \times (-T, T] \\ u\partial_\nu u = 0 & \text{on } \Gamma \times (-T, T] \\ u = \phi - \psi & \text{on } \partial_p(\Omega \setminus \Gamma \times (-T, T]), \end{cases} \tag{4.1}$$

where ν is the unit outward normal, the functions $\psi(x', t)$ and $\phi(x, t)$ are smooth functions, satisfying the compatibility conditions of Section 2, and $f := -(\Delta\tilde{\psi} - \partial_t\tilde{\psi})$. Notice that the extended $\tilde{\psi}$ can be chosen, with no loss of generality, in such a way so that f is independent of x_n.

The methods to follow can be easily extended to cover a more general non-homogeneous term f. But, in order to avoid minor technicalities and set forth the ideas involved behind it, we work with (4.1).

4.1. Optimal regularity of the space derivative

The solution to the problem (4.1) is globally Lipschitz continuous in space and furthermore the space normal to the hyperplane derivative enjoys a C^α parabolic regularity, for $0 < \alpha \leq \frac{1}{2}$, up to the hyperplane (see [1,2]). We will prove in this subsection that, actually, $\alpha = \frac{1}{2}$. Recently, in [16], the optimal space derivative regularity was also obtained using the parabolic Almgren's frequency formula approach.

First, we want to complete what had started in [4], *i.e.* to prove a parabolic monotonicity formula analogous to the elliptic one for the global zero obstacle case. We thus take in (4.1) $f = 0$ and the domain Ω to be the half space $\mathbb{R}^n_+$. In this situation, it is clear, perhaps by appropriately blowing up the local solution, that the solution u is convex in the tangential and time directions. For simplicity we take the origin to be a free boundary point. The proof of the monotonicity result relies on the following eigenvalue problem (see the appendix of [4]):

Lemma 4.1. *Set*

$$\lambda_0 = \inf_{\substack{w \in H^1(\mathbb{R}^n_+) \\ w=0 \text{ on } \mathbb{R}^{n-1}_-}} \frac{\int_{\mathbb{R}^n_+} |\nabla w(y,-1)|^2 e^{-\frac{-|y|^2}{4}} dy}{\int_{\mathbb{R}^n_+} w^2(y,-1) e^{-\frac{-|y|^2}{4}} dy},$$

where

$$\mathbb{R}^n_+ := \left\{ x = (x', x_n) \in \mathbb{R}^n : x_n > 0 \right\}$$

and

$$\mathbb{R}^{n-1}_- := \left\{ (x', 0) : x' \in \mathbb{R}^{n-1},\ x_{n-1} < 0 \right\}.$$

Then $\lambda_0 = 1/4$.

Let w be any function in $\overline{\mathbb{R}^n_+} \times [-1, 0]$ that is caloric in $\mathbb{R}^n_+ \times [-1, 0]$, where $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. We assume that w has moderate growth at infinity,

$$\int_{B_R} w^2(x,-1)dx \leq C e^{\frac{R^2}{4+\varepsilon}}$$

for some positive constant C, $R > 0$ large and some $\varepsilon > 0$. We also set

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0 \\ 0 & t \le 0. \end{cases} \tag{4.2}$$

Lemma 4.2. *Set $w(x,t) = u_{x_n}(x,t)$ where u is a solution, with the above restrictions, to the problem* (4.1)*, and assume that $w(0,0) = 0$. If*

$$\varphi(t) := \frac{1}{t^{1/2}} \int_{-t}^{0} \int_{\mathbb{R}^n_+} |\nabla w|^2 G(x,-s)\,dx\,ds,$$

then $\varphi(t)$ is increasing in t.

Proof. Note that $\Delta w^2 = 2w\Delta w + 2|\nabla w|^2$. We compute $\varphi'(t)$, with a usual mollification argument, to obtain

$$\begin{aligned}\varphi'(t) &= -\frac{1}{2t^{3/2}} \int_{-t}^{0} \int_{\mathbb{R}^n_+} |\nabla w|^2 G(x,-s)\,dx\,ds + \frac{1}{t^{1/2}} \int_{\mathbb{R}^n_+} |\nabla w(x,-t)|^2 G(x,t)\,dx \\ &= -\frac{1}{2t^{3/2}} \int_{-t}^{0} \int_{\mathbb{R}^n_+} \left(\frac{1}{2}\Delta w^2 - w w_t\right) G(x,-s)\,dx\,ds \\ &\quad + \frac{1}{t^{1/2}} \int_{\mathbb{R}^n_+} |\nabla w(x,-t)|^2 G(x,t)\,dx.\end{aligned}$$

By integrating by parts and noticing that $\Delta G + G_t = \delta_{(0,0)}$, $w(0,0) = 0$ and $G(x,0) = 0$, we obtain,

$$\begin{aligned}\varphi'(t) = &-\frac{1}{4t^{3/2}} \int_{\mathbb{R}^n_+} w^2(x,-t) G(x,t)\,dx + \frac{1}{t^{1/2}} \int_{\mathbb{R}^n_+} |\nabla w(x,-t)|^2 G(x,t)\,dx \\ &- \frac{1}{4t^{3/2}} \int_{-t}^{0} \int_{\mathbb{R}^{n-1}} 2w w_\nu G(x',0,-s)\,dx'\,ds.\end{aligned} \tag{4.3}$$

Hence, by the eigenvalue problem of Lemma 4.1 and the complimentary conditions of the solution on $\mathbb{R}^{n-1}$, $\varphi'(t) \ge 0$. □

Theorem 4.3. *If u is a solution to the global convex case of* (4.1) *then $\nabla u \in C^{1/2,1/4}_{x,t}$ up to the coincidence set.*

Proof. It is enough to prove that u tends to zero in a parabolic C^1 fashion as (x,t), a point in the noncoincidence set, approaches a point (x_0, t_0) in the coincidence set which we take to be the origin. Set $w = u_{x_n}$, then w satisfies the hypothesis of Lemma 4.2. In particular, w vanishes at the origin, therefore

$$\frac{1}{t^{1/2}} \int_{-t}^{0} \int_{\mathbb{R}^n_+} |\nabla w(x,s)|^2 G(x,-s)\,dx\,ds \le C. \tag{4.4}$$

Since w vanishes on at least half of the space for all $t \leq 0$, the Poincaré inequality implies that

$$\int_{\mathbb{R}^n_+} w^2(y,-r^2)G(x-y, t+r^2)dy \leq 4r^2 \int_{\mathbb{R}^n_+} \left|\nabla w(y, -r^2)\right|^2 G(x-y, t+r^2)dy. \quad (4.5)$$

Since w^2 is a subsolution across $x_n = 0$ we have, for every $(x, t) \in Q^-_{r/2}$ and $s < r/2$,

$$w^2(x, t) \leq \int_{\mathbb{R}^n} w^2(y, s)G(x - y, t - s)dy. \quad (4.6)$$

Now integrate (4.6) with respect to s from $-r^2$ to $-r^2/2$ to obtain

$$r^2 w^2(x, t) \leq \int_{-r^2}^{-r^2/2} \int_{\mathbb{R}^n} w^2(y, s)G(x - y, t - s)dyds \quad (4.7)$$

and combining with Poincaré inequality we have

$$w^2(x, t) \leq 4 \int_{-r^2}^{-r^2/2} \int_{\mathbb{R}^n} |\nabla w|^2 G(x - y, t - s)dyds \quad (4.8)$$

for every $(x, t) \in Q^-_{r/2}$. Hence by (4.4), the proof is complete. □

Now, we remove the restrictions previously imposed and we show how to improve the $0 < \alpha < 1$ in the C^α regularity to get $C^{1/2}$ regularity. First we prove a lemma, which uses the normal semi-concavity, the tangential semi-convexity, and the time semi-convexity.

Lemma 4.4. *Let u be a solution of* (4.1) *in Q^+_1 with $\nabla u, u^+_t \in C^{\alpha,\frac{\alpha}{2}}_{x,t}$ Then there exists a $\delta = \delta(\alpha) > 0$ such that*

$$(0, 0, t) \notin \Gamma\Big(\big\{u_{x_n} < -r^{\alpha+\delta}\big\} \cap Q'_r\Big)$$

for every $t \in [-r^2, 0]$ and $0 < r < 1$, where $\Gamma(A)$ denotes the convex hull of the set A.

Proof. If

$$\big(x', 0, -r^2\big) \in \big\{u_{x_n} < -r^{\alpha+\delta}\big\}$$

then

$$u\big(x', h, -r^2\big) \leq -r^{\alpha+\delta}h + \frac{M}{2}h^2$$

since $u_{x_n x_n} < M$. Take $h = \frac{r^{\alpha+m\delta}}{M}$ for some $m > 1$; in this case

$$u\big(x', h, -r^2\big) \leq -\frac{r^{2\alpha+(m+1)\delta}}{2M}.$$

Moreover, if we restrict the considerations to $|x'| \leq \frac{r}{2M}$, then

$$u\big(x', h, -r^2\big) + M|x'|^2 \leq -\frac{r^{2\alpha+(m+1)\delta}}{4M} \tag{4.9}$$

provided that $\delta < \frac{2(1-\alpha)}{m+1}$. On the other hand, since $u_{tt} > -M_1$ and u_t^+ is Hölder continuous whose exponent, with no loss of generality, can be taken to be the same α as above, we have

$$\begin{aligned} u(0, h, -r^2) &\geq u(0, h, 0) - \max\big\{0, c_1 h^\alpha r^2\big\} - \frac{M_1}{2} r^4 \\ &\geq -c_0 h^{1+\alpha} - \max\big\{0, c_1 h^\alpha r^2\big\} - \frac{M_1}{2} r^4 \\ &> -\bar{c} r^{(\alpha+m\delta)(1+\alpha)}. \end{aligned} \tag{4.10}$$

Finally, if we choose $\delta > \frac{\alpha(1-\alpha)}{\alpha m-1}$ and $m > 1 + \frac{2}{\alpha}$ we get a contradiction to (4.9) above. Note that the same argument applies for any $t \in [-r^2, 0]$. □

We provide now our monotonicity formula for solutions to the local situation.

Lemma 4.5. *Let $\delta > 0$ and u be a solution to the Signorini problem* (4.1). *Set $w = u_{x_n}$ and*

$$\varphi(r) = \frac{1}{r} \int_{-r^2}^0 \int_{\mathbb{R}^n_+} |\nabla(\eta w)(x, s)|^2 G(x, -s) dx ds$$

for $r < 1$ where $\eta \in C_0^\infty(B_r)$ with $\eta \equiv 1$ and $\eta_{x_n}|_{B_r \cap \mathbb{R}^{n-1}} = 0$. There exists a universal constant $C > 0$ such that

(i) *If $2\alpha + \delta > 1$ then $\varphi(r) \leq C$;*
(ii) *If $2\alpha + \delta < 1$ then $\varphi(r) \leq C r^{2\alpha+\delta-1}$.*

Proof. We compute

$$|\nabla(\eta w)|^2 = \frac{1}{2}\big(\Delta(\eta w)^2 - \partial_t(\eta w)^2\big) - 2\eta w \nabla\eta \nabla w - \eta w^2 \Delta\eta \tag{4.11}$$

and

$$\begin{aligned} \varphi'(r) = &-\frac{1}{2r^2} \int_{-r^2}^0 \int_{\mathbb{R}^n_+} \big(\Delta(\eta w)^2 - \partial_t(\eta w)^2\big) G(x, -s) dx ds \\ &+ 2 \int_{\mathbb{R}^n_+} \big|\nabla(\eta w)\big(x, -r^2\big)\big|^2 G\big(x, r^2\big) dx \\ &+ \frac{1}{r^2} \int_{-r^2}^0 \int_{\mathbb{R}^n_+} \big(2\eta w \nabla\eta \nabla w + \eta w^2 \Delta\eta\big) dx dt. \end{aligned} \tag{4.12}$$

We integrate by parts to obtain

$$\begin{aligned}\varphi'(r) = \frac{1}{2r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^n_+} \big(\nabla(\eta w)^2\nabla G + \partial_t(\eta w)^2 G\big)dxds \\ - \frac{1}{2r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^{n-1}} \big[(\eta w)^2\big]_\nu(x',0,s)G(x',0,-s)dx'ds \\ + \frac{1}{r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^n_+} \big(2\eta\nabla w\nabla\eta + \eta w^2\Delta\eta\big)G(x,-s)dxds \\ + 2\int_{\mathbb{R}^n_+} \big|\nabla(\eta w)\big(x,-r^2\big)\big|^2 G\big(x,r^2\big)dx.\end{aligned} \tag{4.13}$$

Integrating again by parts, we obtain

$$\begin{aligned}\varphi'(r) = -\frac{1}{2r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^n_+} (\eta w)^2(\Delta G + \partial_t G)dxds \\ - \frac{1}{2r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^{n-1}} (\eta w)^2_\nu G(x,-s)dx'ds \\ - \frac{1}{2r^2}\int_{\mathbb{R}^n_+} (\eta w)^2\big(x,-r^2\big)G\big(x,r^2\big)dx \\ + \frac{1}{r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^n_+} \big(\eta\Delta\eta w^2 + 2\eta w\nabla\eta\nabla w\big)G(x,-s)dxds \\ + 2\int_{\mathbb{R}^n_+} \big|\nabla(\eta w)\big(x,-r^2\big)\big|^2 G(x,r^2)dx.\end{aligned} \tag{4.14}$$

Since $w(0,0) = 0$, we have

$$\begin{aligned}\varphi'(r) = -\frac{1}{2r^2}\int_{\mathbb{R}^n_+} (\eta w)^2\big(x,-r^2\big)G\big(x,r^2\big)dx \\ + 2\int_{\mathbb{R}^n_+} \big|\nabla(\eta w)\big(x,-r^2\big)\big|^2 G\big(x,r^2\big)dx \\ - \frac{1}{2r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^{n-1}} 2\eta w\eta w_\nu G(x,-s)dx'ds \\ + \frac{1}{r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^n_+} \eta\Delta\eta w^2 G(x,-s)dxds \\ + \frac{2}{r^2}\int_{-r^2}^{0}\int_{\mathbb{R}^n_+} \eta w\nabla\eta\nabla w G(x,-s)dxds\end{aligned} \tag{4.15}$$

or

$$
\begin{aligned}
\varphi'(r) = &-\frac{1}{2r^2}\int_{\mathbb{R}^n_+}(\eta w)^2\big(x,-r^2\big)G\big(x,r^2\big)dx \\
&+2\int_{\mathbb{R}^n_+}\big|\nabla(\eta w)\big(x,-r^2\big)\big|^2G\big(x,r^2\big)dx \\
&+\frac{1}{2r^2}\int_{-r^2}^0\int_{\mathbb{R}^n_+}\nabla\eta^2\nabla w^2 G(x,-s)dxds \qquad (4.16)\\
&+\frac{1}{r^2}\int_{-r^2}^0\int_{\mathbb{R}^n_+}\eta\Delta\eta w^2 G(x,-s)dxds \\
&+\frac{1}{r^2}\int_{-r^2}^0\int_{\mathbb{R}^{n-1}}\eta^2 wf\ G(x,-s)dx'ds
\end{aligned}
$$

and finally

$$
\begin{aligned}
\varphi'(r) \ge &-\frac{1}{2r^2}\int_{\mathbb{R}^n_+}(\eta w)^2\big(x,-r^2\big)G\big(x,r^2\big)dx \\
&+2\int_{\mathbb{R}^n_+}\big|\nabla(\eta w)\big(x,-r^2\big)\big|^2G\big(x,r^2\big)dx - Cr^\alpha.
\end{aligned}
$$

Now, consider the truncated function $\overline{w} = -(w + r^{\alpha+\delta})^-$ and note that

$$
\int_{\mathbb{R}^n_+}\big|\nabla(\eta\overline{w})\big(x,-r^2\big)\big|^2G\big(x,r^2\big)dx \le \int_{\mathbb{R}^n_+}\big|\nabla(\eta w)\big(x,-r^2\big)\big|^2G\big(x,r^2\big)dx.
$$

Hence we have

$$
\begin{aligned}
\varphi'(r) \ge &-\frac{1}{2r^2}\int_{\mathbb{R}^n_+}[\eta(w-\overline{w})+\eta\overline{w}]^2\big(x,-r^2\big)G\big(x,r^2\big)dx \\
&+2\int_{\mathbb{R}^n_+}\big|\nabla(\eta\overline{w})\big(x,-r^2\big)\big|^2G\big(x,r^2\big)dx - Cr^\alpha
\end{aligned}
$$

and

$$
\varphi'(r) \ge -\frac{1}{2r^2}\int_{\mathbb{R}^n_+}\eta^2\big[(w-\overline{w})^2+2\overline{w}(w-\overline{w})\big]G\big(x,r^2\big)dx - Cr^\alpha
$$

or

$$
\varphi'(r) \ge -\frac{3}{2}r^{2\alpha+2\delta-2} - Cr^\alpha \ge -\frac{3}{2}r^{2\alpha+\delta-2}.
$$

Therefore

$$
\varphi(1)-\varphi(r) \ge -\frac{3}{2}\left(\frac{1}{2\alpha+\delta-1}\right)+\frac{3}{2}\left(\frac{1}{2\alpha+\delta-1}\right)r^{2\alpha+\delta-1}.
$$

Since $\varphi(1)$ is universally bounded the proof is complete. □

Next, we state our main result of this subsection:

Theorem 4.6. *Let u the solution of* (4.1), *then ∇u is $C^{\frac{1}{2},\frac{1}{4}}_{x,t}$ up to the hyperplane* $\mathbb{R}^{n-1}$.

Proof. Let $w = u_{x_n}$ and $\overline{w}$ be as in the proof of Lemma 4.5. Fix $s > 0$, choose $R > 0$ large enough and $\varepsilon < s$. We define a cutoff function $\eta = \eta(x)$ so that $\operatorname{supp}\eta \in B_{R+1}(0)$, $\eta \equiv 1$ on $B_R(0)$ and $|\nabla\eta| \leq C$.

Then

$$(\Delta - \partial_\xi)(\eta\overline{w})^2 = 2\eta^2|\nabla\overline{w}|^2 + 8\overline{w}\eta\nabla\overline{w}\nabla\eta + 2\big(\eta\Delta\eta + |\nabla\eta|^2\big)\overline{w}^2 + 2\eta^2\overline{w}(\Delta\overline{w} - \partial_\xi\overline{w}). \tag{4.17}$$

Recall that $(\Delta + \partial_\xi)G(x, -\xi) = \delta_{(0,0)}$; therefore using (4.17), an integration by parts, along with the fact that η is compactly supported, we obtain

$$\begin{aligned} &2\int_{-s}^{-\varepsilon}\int_{\mathbb{R}^n} \eta^2|\nabla\overline{w}|^2 G(x, -\xi)dxd\xi \\ &= -\int_{\mathbb{R}^n} \eta^2\overline{w}^2 G(x, \varepsilon)dx + \int_{\mathbb{R}^n} \eta^2\overline{w}^2 G(x, s)dx \\ &\quad - 8\int_{-s}^{-\varepsilon}\int_{\mathbb{R}^n} \overline{w}\eta\nabla\eta\nabla\overline{w}G(x, -\xi)dxd\xi \\ &\quad - 2\int_{-s}^{-\varepsilon}\int_{\mathbb{R}^n} \big(\eta\Delta\eta + |\nabla\eta|^2\big)\overline{w}^2 G(x, -\xi)dxd\xi \\ &\quad - 2\int_{-s}^{-\varepsilon}\int_{\mathbb{R}^n} \eta^2\overline{w}(\Delta\overline{w} - \partial_\xi\overline{w})G(x, -\xi)dxd\xi. \end{aligned} \tag{4.18}$$

Observe that

$$\begin{aligned} \int_{-s}^{-\varepsilon}\int_{\mathbb{R}^n} \overline{w}\eta|\nabla\eta||\nabla\overline{w}|G(x, -\xi)dxd\xi &\leq C\int_{-s}^{-\varepsilon}\int_{B_{R+1}\setminus B_R} |\overline{w}||\nabla\overline{w}|\frac{e^{-R^2/4|\xi|}}{|\xi|^{n/2}}dxd\xi \\ &\leq Ce^{-R^2/4+\varepsilon_0}\int_{-s}^{0}\int_{B_{R+1}\setminus B_R} |\overline{w}||\nabla\overline{w}|dxd\xi. \end{aligned}$$

Using the Cauchy-Schwartz inequality, we conclude that the last three terms on the right hand side of (4.18) behave in the same way, in particular they decay to zero as $R \to \infty$. Therefore we conclude that

$$(\eta\overline{w})^2(0, 0) \leq \int_{\mathbb{R}^n} (\eta\overline{w})^2 G(x, s)dx$$

or, after rescaling,

$$(\eta\overline{w})^2(x, t) \leq \int_{\mathbb{R}^n} (\eta\overline{w})^2(y, s)G(x - y, t - s)dy \tag{4.19}$$

for every $(x,t) \in Q^+_{r/2}$ and $-r^2 < s < -\frac{r^2}{2}$. By Poincaré inequality for Gaussian measures (see [7]) we have that

$$\begin{aligned}&\int_{\mathbb{R}^n} (\eta\overline{w})^2(y,s)G(x-y,t-s)dy\\ &\le 2|s|\int_{\mathbb{R}^n} |\nabla(\eta\overline{w})(y,s)|^2 G(x-y,t-s)dy\end{aligned} \tag{4.20}$$

for $(x,t) \in Q^+_{r/2}$ and $-r^2 < s < -\frac{r^2}{2}$. Combine (4.19) and (4.20) to obtain

$$(\eta\overline{w})^2(x,t) \le C|s|\int_{\mathbb{R}^n_+} |\nabla(\eta\overline{w})(y,s)|^2 G(x-y,t-s)dy \tag{4.21}$$

for every $(x,t) \in Q^+_{r/2}$ and $-r^2 < s < -\frac{r^2}{2}$. An integration with respect to s in (4.21) shows that

$$(\eta\overline{w})^2(x,t) \le C\int_{-r^2}^{-r^2/2}\int_{\mathbb{R}^n} |\nabla(\eta\overline{w})(y,s)|^2 G(x-y,t-s)dyds$$

for every $(x,t) \in Q^+_{r/2}$. Now the dichotomy for $\varphi(r)$ in Lemma 4.5 provides a $C^{1/2}$ modulus of continuity for w, as in [4, proof of Theorem 5]. □

4.2. Hölder continuity of the time derivative near a free boundary point of positive parabolic density

Although the positive time derivative is always Hölder continuous (see Section 3.2), one does not expect to obtain continuity of the full time derivative without further restrictions. The purpose of this section is to show that, indeed, Hölder continuity of the full time derivative can be achieved near free boundary points of positive parabolic density with respect to the coincidence set. In order to achieve this desired result we employ the well known "hole filling" method of Widman (see [22]) adapted for parabolics by Struwe (see [21]). As it was mentioned in the introduction, the results of the present section are independent of the quasi-convexity.

Definition 4.7. A free boundary point $(x'_0, 0, t_0)$ is of positive parabolic density with respect to the coincidence set if there exist positive constants $c > 0$ and $r_0 > 0$ such that

$$\left|Q'_r(x'_0,0,t_0) \cap \{u=0\}\right| \ge c\left|Q'_r(x'_0,0,t_0)\right|$$

for all $r < r_0$.

So the main result of this subsection is stated as follows:

Theorem 4.8. *Let* (x_0, t_0) *be a free boundary point of positive parabolic density with respect to the coincidence set to problem* (4.1). *Then* u_t *is Hölder continuous in a neighborhood of* (x_0, t_0).

Proof. Since, by Subsection 3.2, u_t^+ is Hölder continuous, it suffices to prove the theorem for u_t^-. Actually, we will show that u_t^- decays to zero in parabolic cylinders shrinking to the free boundary point (x_0, t_0). We consider the penalized solution u^ε of (4.1) in $Q_r^+(x_0, t_0)$ with $r < r_0$, where r_0 is as in Definition 4.7. For simplicity we take $(x_0, t_0) = (0, 0)$ and $r = 1$. Differentiate with respect to t to have as in (3.8)

$$\begin{cases} \Delta v^\varepsilon - \partial_t v^\varepsilon = f_t^\varepsilon, & \text{in } Q_1^+ \\ -\partial_\nu v^\varepsilon = \beta_\varepsilon'(u^\varepsilon) v^\varepsilon & \text{on } Q_1', \end{cases} \tag{4.22}$$

where $v^\varepsilon := (u^\varepsilon)_t$. For any $(\xi, \tau) \in Q^+_{\frac{1}{5}}$ we want to multiply the equation by an appropriate test function and integrate by parts over the set $Q^+_{\frac{3}{5}}(\xi, \tau) := Q_{\frac{3}{5}}(\xi, \tau) \cap \{x_n \geq 0\} \subset Q_1^+$. This will lead us to an estimate which will be iterated to yield the desired result.

The aforesaid appropriate test function will be the product of following three functions:

- The first one is the square of a smooth function $\zeta(x, t)$ supported in $Q^+_{\frac{3}{5}}(\xi, \tau)$ such that $\zeta \equiv 1$ for every $(x, t) \in Q^+_{\frac{2}{5}}(\xi, \tau)$, $|\nabla \zeta| \leq c$ with $supp(\nabla\zeta) \subset (B^+_{\frac{3}{5}}(\xi, \tau) \setminus B^+_{\frac{2}{5}}(\xi, \tau)) \times (\tau - \frac{9}{25}, \tau]$, $0 \leq \zeta_t \leq c$ with $supp(\zeta_t) \subset B_{\frac{3}{5}}(\xi, \tau) \times (\tau - \frac{9}{25}, \tau - \frac{4}{25})$;
- The second one is a smoothing of the fundamental solution $G(x, t)$ of the heat equation (see (4.2)), *i.e*.

$$G_\delta^{(\xi,\tau)}(x, t) := \Big(G(x - \xi, \tau - t)\chi(x, t)_{E^c_\delta(\xi,\tau)} + p(x - \xi, t - \tau)\Big)\chi(x, t)_{E_\delta(\xi,\tau)})\chi(x, t)_{\{t<\tau\}}$$

  where $E_\delta(\xi, \tau) := \{(x, t) \in R^{n+1} : t \leq \tau,\ G(x-\xi, \tau-t) \geq \frac{1}{\delta^n}\}$, the "heat" ball of "radius" δ about (ξ, τ), and $p(x, t) := \frac{1}{\delta^n}(\log \frac{e(4\pi|t|)^{n/2}}{\delta^n} - \frac{|x|^2}{4t})\chi(x, t)_{\{t<0\}}$. Notice that $G_\delta^{(\xi,\tau)}$ is a C^1 function everywhere in $\mathbb{R}^{n+1}$ except at (ξ, τ). In order to deal with this problem we just translate the singularity outside of our domain by a small amount $\varepsilon' > 0$ and then we let ε' to tend to zero, for simplicity we omit this technicality;
- Finally the third function is $(v^\varepsilon)^-$ which can be smoothed out by the standard way; again we omit it for the sake of simplicity.

Therefore we multiply the equation in (4.22) by $\zeta^2 G_\delta^{(\xi,\tau)}(v^\varepsilon)^-$ and integrate by parts over $Q^+_{\frac{3}{5}}(\xi,\tau)$ to obtain

$$\begin{aligned}
&\int_{Q^+_{\frac{3}{5}}(\xi,\tau)} \Big(\nabla\Big(\zeta^2 G_\delta^{(\xi,\tau)}(v^\varepsilon)^-\Big)\nabla v^\varepsilon + \Big(\zeta^2 G_\delta^{(\xi,\tau)}(v^\varepsilon)^-\Big)\partial_t v^\varepsilon\Big)dxdt \\
&= -\int_{Q'_{\frac{3}{5}}(\xi,\tau)} \Big(\zeta^2 G_\delta^{(\xi,\tau)}(v^\varepsilon)^-\Big)\beta'(u^\varepsilon)v^\varepsilon dx'dt \qquad (4.23)\\
&\quad -\int_{Q^+_{\frac{3}{5}}(\xi,\tau)} \zeta^2 G_\delta^{(\xi,\tau)}(v^\varepsilon)^- f_t dxdt.
\end{aligned}$$

By calculating appropriately and by noticing that due to the non negativity of β'_ε the boundary integral term has the right sign, so it can be omitted, we obtain

$$\begin{aligned}
&\int_{Q^+_{\frac{3}{5}}(\xi,\tau)}\Big(G_\delta^{(\xi,\tau)}\Big|\nabla\big(\zeta(v^\varepsilon)^-\big)\Big|^2 + \frac{1}{2}\Big[\nabla G_\delta^{(\xi,\tau)}\nabla\big(\zeta(v^\varepsilon)^-\big)^2 + G_\delta^{(\xi,\tau)}\partial_t\big(\zeta(v^\varepsilon)^-\big)^2\Big]\Big)dxdt \\
&\le \int_{Q^+_{\frac{3}{5}}(\xi,\tau)} G_\delta^{(\xi,\tau)}\Big(|\nabla\zeta|^2 + \zeta\zeta_t\Big)\Big(\big(v^\varepsilon\big)^-\Big)^2 dxdt \\
&\quad + \frac{1}{2}\int_{Q^+_{\frac{3}{5}}(\xi,\tau)} \nabla G_\delta^{(\xi,\tau)}\nabla\zeta^2\Big(\big(v^\varepsilon\big)^-\Big)^2 dxdt + \int_{Q^+_{\frac{3}{5}}(\xi,\tau)} \zeta^2 G_\delta^{(\xi,\tau)}(v^\varepsilon)^- f_t dxdt.
\end{aligned}$$

Using the fact that $\mathrm{supp}(\mu) = E_\delta(\xi,\tau)$ where $\mu := -(\Delta+\partial_t)G_\delta^{(\xi,\tau)}$ with $d\mu = \frac{1}{4\delta^n}dE_\delta(\xi,\tau)$ and $|E_\delta(\xi,\tau)| = 4\delta^n$ (see [18]) and that, for δ small enough, one has the inequalities $0 \le G_\delta^{(\xi,\tau)} \le C(n)$ in $(B^+_{\frac{4}{5}} \setminus B^+_{\frac{1}{5}}) \times (-\frac{2}{5}, 0)$, and $c(n) \le G_\delta^{(\xi,\tau)} \le C(n)$ in $B^+_{\frac{4}{5}} \times (-\frac{2}{5}, -\frac{4}{25})$, we have

$$\begin{aligned}
&\int_{Q^+_{\frac{2}{5}}(\xi,\tau)} G_\delta^{(\xi,\tau)}|\nabla(v^\varepsilon)^-|^2 dxdt + \fint_{E^+_\delta(\xi,\tau)} (v^\epsilon)^- dE_\delta(\xi,\tau) \\
&\le C(n)\int_{-\frac{2}{5}}^{-\frac{4}{25}}\int_{B^+_{\frac{4}{5}}} ((v^\varepsilon)^-)^2 dxdt \qquad (4.24)\\
&\quad + C(n)\int_{-\frac{2}{5}}^{0}\int_{B^+_{\frac{4}{5}}\setminus B^+_{\frac{1}{5}}} ((v^\varepsilon)^-)^2 dxdt + C(n)M
\end{aligned}$$

where $M := ||v^\varepsilon||_\infty ||f_t||_\infty$.

Now, we first let ε tend to 0 in order to obtain (4.24) for v^-, then we let δ to go to 0, and, finally, we take the supremum over $(\xi,\tau)\in Q^+_{\frac{1}{4}}$ to obtain, a fortiori,

$$\begin{aligned}\int_{Q^+_{\frac{1}{5}}} G(x,-t)|\nabla v^-|^2 dxdt + \sup_{Q^+_{\frac{1}{5}}}(v^-)^2 \le C(n)\int_{-\frac{2}{5}}^{-\frac{4}{25}}\int_{B^+_{\frac{4}{5}}}(v^-)^2 dxdt \\ + C \sup_{Q^+_1\setminus Q^+_{\frac{1}{5}}}(v^-)^2 + CM.\end{aligned}\tag{4.25}$$

Next we want to control the first integral of the right hand side of (4.25) by one similar to the first integral of the left hand side of (4.25). To do this first we multiply the equation in (4.22) by $\zeta^2(v^\varepsilon)^-$, where ζ is a smooth cutoff function supported in $B_1\times(-1,t)$, for any $t\le -\frac{4}{25}$, $\zeta\equiv 1$ on $B_{\frac{4}{5}}\times(-\frac{2}{5},t)$, and vanishing near its parabolic boundary with $|\nabla\zeta|\le c$ and $0\le \zeta_t\le c$, then we integrate by parts over this set intersected by $\mathbb{R}^n_+$ to have

$$\begin{aligned}&\int_{-1}^{t}\int_{B_1^+}\Big(\nabla\big(\zeta^2(v^\varepsilon)^-\big)\nabla v^\varepsilon + \big(\zeta^2(v^\varepsilon)^-\big)\partial_t v^\varepsilon\Big)\,dxdt\\ &= -\int_{-1}^{t}\int_{B_1'}\big(\zeta^2(v^\varepsilon)^-\big)\beta'(u^\varepsilon)v^\varepsilon dx'dt\\ &\quad -\int_{-1}^{t}\int_{B_1^+}\zeta^2(v^\varepsilon)^- f_t dxdt.\end{aligned}$$

Again, exploiting the positivity of β' and letting ε go to zero, we arrive, as above but in a much simpler way, at the following inequality

$$\int_{B^+_{\frac{4}{5}}}(v^-)^2(x,t)dx + \int_{-\frac{2}{5}}^{t}\int_{B^+_{\frac{4}{5}}}|\nabla v^-|^2 dxdt \le c\int_{Q_1^+}(v^-)^2 dxdt + C(n)Mr^{n+2}$$

$\forall\, t\in(-\frac{2}{5},-\frac{4}{25})$. Observe that a sufficient portion of the coincidence set is present in Q_1 so that the parabolic Poincaré inequality can be applied to dominate the integral on the right hand side of the above inequality. Therefore, since the second term on the left hand side is non negative, we have, for every $-\frac{2}{5}\le t\le -\frac{4}{25}$,

$$\int_{B^+_{\frac{4}{5}}}(v^-)^2(x,t)dx \le C(n)\int_{Q_1^+}|\nabla v^-|^2 dxdt + C(n)M.$$

We then integrate the above inequality with respect to t from $-\frac{2}{5}$ to $-\frac{4}{25}$ to get

$$\int_{-\frac{2}{5}}^{-\frac{4}{25}}\int_{B^+_{\frac{4}{5}}}(v^-)^2 dxdt \le C(n)\int_{Q_1^+}|\nabla(v^-)|^2 dxdt + C(n)M.$$

Insert this in (4.25) above and, using the fact that $G(x, -t) \geq c(n)$ for $-\frac{2}{5} \leq t \leq -\frac{4}{25}$, to have

$$\begin{aligned} &\int_{Q^+_{\frac{1}{5}}} G(x,-t)|\nabla v^-|^2 dxdt + \sup_{Q^+_{\frac{1}{5}}}(v^-)^2 \\ &\leq C(n)\left(\int_{Q^+_1\setminus Q^+_{\frac{1}{5}}} G(x,-t)|\nabla v^-|^2 dxdt + \sup_{Q^+_1\setminus Q^+_{\frac{1}{5}}}(v^-)^2)\right) + C'(n)M. \end{aligned} \tag{4.26}$$

Set $\omega(\rho) := \int_{Q^+_\rho} G|\nabla v^-|^2 dxdt + \sup_{Q^+_\rho}(v^-)^2$, then add $C(n)\omega(\frac{1}{5})$ to both sides of (4.26) and divide the new inequality by $1 + C(n)$ to have

$$\omega\left(\frac{1}{5}\right) \leq \lambda\omega(1) + c, \tag{4.27}$$

where $\lambda = \frac{C(n)}{1+C(n)}$. Iteration of (4.27) implies that there exists an $\alpha = \alpha(\lambda) \in (0, 1)$ and a constant $C = C(n, ||u_t||_\infty, ||f_t||_\infty)$ such that

$$\omega(\rho) \leq C\rho^\alpha$$

for every $0 < \rho \leq \frac{r_0}{5}$. This concludes the Hölder continuity from the past. The continuity from the future follows, now, by standard methods. □

Remark. After the presentation of the present paper in IMPA (August 2015), we have been informed by A. Petrosyan that a similar result to our Theorem 4.8 has also appeared in [20].

4.3. Free boundary regularity

In the study of free boundary regularity it turns out that in order to achieve smoothness of the free boundary one has to focus his attention in a neighborhood of certain free boundary points, which we shall call them non-degenerate, (see Definition 4.9 below). A good candidate for a non-degenerate free boundary point must include one of positive parabolic density of the coincidence set. The fact, that u_t is Hölder continuous at such a point (see Section 4.2), yields a control of the speed of the interphase, a crucial step for our further analysis of the regularity of the free boundary. Since it is more convenient to work with the zero obstacle and with the right hand side of the equation to vanish at the point, which, for simplicity, we take it to be the origin, we set $\tilde{u}(x', x_n, t) = u(x', x_n, t) - \psi(x', t) + \frac{1}{2}H\psi(0, 0)x_n^2$ $(H := \Delta - \partial_t)$. Observe that $\{\tilde{u}(x', x_n, t) = 0\} = \{u(x', x_n, t) = \psi(x', t)\}$, and upon reflection in

$B_1^* := \{(x,t) \in \mathbb{R}^{n+1} : |x|^2 + t^2 < 1\}$ $\tilde{u}$ satisfies:

$$\begin{cases} \tilde{u}(x',0,t) \geq 0 & \text{in } B_1^* \cap \{x_n = 0\} \\ \tilde{u}(x',x_n,t) = \tilde{u}(x',-x_n,t) & \text{in } B_1^* \\ \Delta\tilde{u}(x',x_n,t) - \partial_t\tilde{u}(x',x_n,t) = H\psi(0,0) - H\psi(x',t) & \text{in } B_1^* \setminus \{\tilde{u} = 0\} \\ \Delta\tilde{u}(x',x_n,t) - \partial_t\tilde{u}(x',x_n,t) \leq H\psi(0,0) - H\psi(x',t) & \text{in } B_1^*. \end{cases} \tag{4.28}$$

For simplicity of notation we replace u with $\tilde{u}$ for the rest of this section.

Now we pass the u_t term to the right hand side of the equation and if we assume that $H\psi$ is at least C^α we can apply the elliptic theory developed in [6,13] and extended in [8] at the t-level of the point. In the Appendix we show how we apply the elliptic theory in our case. Consequently, if the origin is regular then at $t = 0$ the blow up limit v_0 of the solution u (up to sub-sequences) exists, and, in appropriate coordinates,

$$v_0(x) = \frac{2}{3}\rho^{\frac{3}{2}} \cos\left(\frac{3}{2}\theta\right),$$

where $\rho = \sqrt{x_1^2 + x_n^2}$ and $\theta = \arctan(\frac{x_n}{x_1})$ (unique up to rotations).

Now, we are ready to state the "hyperbolic" definition of our non-degenerate free boundary point.

Definition 4.9. Let (x_0, t_0) be a free boundary point and $B_r^*(x_0,t_0) := \{(x,t) \in R^{n+1} : (x-x_0)^2 + (t-t_0)^2 < r^2\}$, and set

$$l := \limsup_{r \to 0^+} \frac{||u||_{L^\infty(B_r^*(x_0,t_0))}}{r^{3/2}}.$$

A point (x_0, t_0) is called a non-degenerate free boundary point if it is of positive parabolic density of the coincidence set and $0 < l < \infty$; otherwise the point is called degenerate.

With this definition at our hands we state the main result of this section:

Theorem 4.10. *Let u be a solution to* (4.28). *Assume the origin to be a non-degenerate free boundary point. Then the free boundary is a $C^{1,\alpha}$ n-dimensional surface about the origin.*

The following "hyperbolic" blow up sequence will be very useful for our analysis since, at a point, it preserves the geometry of the free boundary:

$$u_r(x,t) := \frac{u(rx,rt)}{r^{3/2}}.$$

Lemma 4.11. *Let u be a solution to* (4.28). *If* $(0,0)$ *is a non-degenerate free boundary point then there exists a sequence u_{r_j} of blow ups which converges uniformly on compact subsets to a function u_0 such that* (*in appropriate coordinates*)

$$u_0(x,t) = \frac{2}{3}\rho(t)^{\frac{3}{2}} \cos\left(\frac{3}{2}\theta(t)\right)$$

where $\rho(t) := \sqrt{(x_1 + \omega t)^2 + x_n^2}$ and $\theta(t) :=$ $\arctan(\frac{x_n}{x_1+\omega t})$ for some $\omega \in \mathbb{R}$.

Proof. Since $0 < l < \infty$, it is clear that we can extract a subsequence u_{r_j} converging uniformly on compact subsets to a non trivial limit u_0. This u_0 is a harmonic function for every fixed t outside of the coincidence set; the coincidence set, due to the density assumption, is a convex cone in $\mathbb{R}^n$, or more precisely in (x', t) variables. Also, by the discussion above, at $t = 0$ $u_0 = \frac{2}{3}\rho^{3/2}\cos\frac{3}{2}\theta$ where $\rho = \sqrt{x_1^2 + x_n^2}$ and $\theta = \arctan(\frac{x_n}{x_1})$. Moreover the convex cone is composed by the two supporting hyperplanes $Ax_1 + at = 0$ for $t \geq 0$ and $Bx_1 + bt = 0$, for $t \leq 0$, with $A \geq 0$, $B \geq 0$ and $bA \leq aB$. We want to prove that this convex cone is actually a non-horizontal half space, *i.e.* $A > 0$, $B > 0$, and $bA = aB$, and u_0 admits the stated representation; we do this in several steps:

Step I: $A > 0$ and $B > 0$. For, if $A = 0$ then for every $t > 0$ $u_0(x, t)$ is harmonic in all of $\mathbb{R}^n$, and therefore of polynomial growth. But for $t = 0$ u_0 has $3/2$ degrees of growth, therefore, by continuity of u_0, we have a contradiction. Similarly $B > 0$.

Step II: For each fixed t, $u \sim |x|^{\frac{3}{2}}$ as $|x| \to \infty$ with $x \cdot e_1 \geq \varepsilon$ for some $\varepsilon > 0$. It is enough to show the bound by below. Therefore take a sequence $x^{(j)}$ such that $|x^{(j)}| \to \infty$ with $x^{(j)} \cdot e_1 \geq \varepsilon$ for every j then by convexity $u_0(x^{(j)}, t) \geq u_0(x^{(j)}, 0) + (u_0)_t(x^{(j)}, 0)t$, hence by the behavior of u_0 at $t = 0$ the result follows.

Step III: For each fixed t,

$$u_0(x, t) = \frac{2}{3}\rho(t)^{\frac{3}{2}}\cos\left(\frac{3}{2}\theta(t)\right),$$

where for $t > 0$, $\rho(t) = \sqrt{(x_1 + \frac{a}{A}t)^2 + x_n^2}$, $\theta(t) = \arctan\frac{x_n}{x_1 + \frac{a}{A}t}$ and for $t < 0$, $\rho(t) = \sqrt{(x_1 + \frac{b}{B}t)^2 + x_n^2}$, $\theta(t) = \arctan\frac{x_n}{x_1 + \frac{b}{B}t}$. Indeed, for each fixed $t > 0$, u_0 is a harmonic function which vanishes for $\{x_1 \leq -\frac{a}{A}t\} \cap \{x_n = 0\}$ and grows at infinity with $\frac{3}{2}$ exponent, therefore by Phragmén-Lindelöf theorem we obtain the representation. We can proceed analogously for $t < 0$.

Step IV: $bA = aB$. If this is not true we have

$$\partial_t u_0(0, 0^+) - \partial_t u_0(0, 0^-) = \left(\frac{a}{A} - \frac{b}{B}\right)\rho^{\frac{1}{2}}\cos\left(\frac{1}{2}\theta\right) \neq 0,$$

whence, by approximation, a contradiction to the continuity of $\partial_t u$ at the origin.
Set $\omega := \frac{a}{A}$ and the proof is complete. □

Finally we prove our theorem:

Proof. Obviously the existence of ω in Lemma 4.11 implies the differentiability of the free boundary at the origin. Also, due to the upper semi-continuity of the elliptic Almgren's frequency function, we have the differentiability of the free boundary for any nearby point $p = (x_p, t_p)$ at least when $t_p \leq 0$, since u_t is continuous there. Now, if $t_p > 0$ and $p = (x_p, t_p)$ still near the origin, we observe that the frequency function will converge to $\frac{3}{2}$, and this implies that the positive density will propagate to p. Consequently, the point $p = (x_p, t_p)$ will be a free boundary point of positive parabolic density with respect to zero set, which renders u_t continuous there. Hence we have the differentiability of the free boundary there, too. To prove the continuous differentiability of it consider two distinct free boundary points nearby, say p and 0. Assume, on the contrary, that it is not true, that is $\omega(p)$ does not converge to $\omega(0)$ as $p \to 0$. Consider the blow up sequences $u_{r_j}^{(p)}$ and $u_{r_i}^{(0)}$ around p and 0, respectively, where $u_{r_j}^{(p)}(x,t) := \frac{u(r_j((x,t)-p))}{r_j^3/2}$. These sequences converge uniformly to

$$u_0^{(p)}(x,t) := \frac{2}{3}\rho^{\frac{3}{2}}(p,t)\cos\frac{3}{2}\theta(p,t)$$

and

$$u_0^{(0)}(x,t) := \frac{2}{3}\rho^{\frac{3}{2}}(0,t)\cos\frac{3}{2}\theta(0,t)$$

respectively, where $\rho(p,t) := \sqrt{(x_1(p) + \omega(p)t(p))^2 + x_n^2)}$ and

$$\theta(p,t) := \arctan\frac{x_n}{x(p)) + \omega(p)t(p)}.$$

So, if $\omega(p)$ does not converge to $\omega(0)$ then $u_0^{(p)}$ does not converge to $u_0^{(0)}$, therefore a contradiction to the continuity of the solution u. Hence a C^α estimate of the free boundary normals follows easily. □

Appendix A.

The purpose of this Appendix is to show how to apply the elliptic theory to our case. First we show that the Almgren's frequency formula holds even with right hand side in L^p for $p > n$ and secondly we show how to deduce $C^{1,\alpha}$ free boundary regularity even with $W^{-1,p}$ right hand side.

Frequency formula

The main issue in the proof arises when one differentiates the expression

$$D(r) := \frac{r}{2}\frac{d}{dr}\log\varphi(r) := \frac{r}{2}\frac{d}{dr}\log⨍_{\partial B_r} u^2 dS = \frac{r\int_{\partial B_r} u u_\nu dS}{\int_{\partial B_r} u^2 dS}$$
$$= \frac{r\int_{B_r}(|\nabla u|^2 + u\Delta u)dx}{\int_{\partial B_r} u^2 dS} =: \frac{rV(r)}{S(r)}.$$

Indeed

$$\frac{d}{dr}\log D(r) = \frac{1}{r} + \frac{V'(r)}{V(r)} - \frac{S'(r)}{S(r)}$$

and working as in [6] we end up

$$\begin{aligned}
&= 2\left(\frac{\int_{\partial B_r} u_\nu^2 dS}{\int_{\partial B_r} uu_\nu dS} - \frac{\int_{\partial B_r} uu_\nu dS}{\int_{\partial B_r} u^2 dS}\right) \\
&\quad + \frac{\int_{\partial B_r} u\Delta u dS - \frac{n-1}{r}\int_{B_r} u\Delta u dx - \frac{2}{r}\int_{B_r}(x\cdot\nabla u)\Delta u dx}{\int_{\partial B_r} uu_\nu dS} \\
&=: R(r) + \mathcal{E}(r).
\end{aligned}$$

Notice that if the right hand side of the equation were zero, *i.e.* $\Delta u = 0$, then we would obtain as in [6] only the first term, which being non-negative produces the monotonicity in the frequency formula. Since $\mathcal{E}(r) \neq 0$ we have, as in [13], to estimate it, of course, for small r. The key point in controlling this term relies on the following fact obtained in [13] for any fractional power, which includes our $\frac{1}{2}$ exponent case (see Lemma 2.13 in [13]). Notice the difference in the power of the second term; this occurs because we are using instead only the L^p bound of the Laplacian.

$$\int_{\partial B_r} u^2 dS \le Cr\int_{B_r} |\nabla u|^2 dx + Cr^{n+3-\frac{2n}{p}} \tag{A.1}$$

and its integrated analogue

$$\int_{B_r} u^2 dx \le Cr^2\int_{B_r} |\nabla u|^2 dx + Cr^{n+4-\frac{2n}{p}}. \tag{A.2}$$

Now, writing (A.1) as

$$⨍_{\partial B_r} u^2 dS \le Cr^2 ⨍_{B_r} |\nabla u|^2 dx + Cr^{4-\frac{2n}{p}}, \tag{A.3}$$

we observe that for any number a_0, $0 < a_0 < 1$, with

$$⨍_{\partial B_r} u^2 dS \ge r^{4-\frac{2n}{p}-a_0} \tag{A.4}$$

for every $r \le r_0$, with r_0 small enough, it holds

$$⨍_{\partial B_r} u^2 dS \le \frac{Cr^2}{1-Cr_0^{a_0}} ⨍_{B_r} |\nabla u|^2 dx, \tag{A.5}$$

$$r^2 \le \frac{Cr^{\frac{2n}{p}+a_0}}{1-Cr_0^{a_0}} ⨍_{B_r} |\nabla u|^2 dx, \tag{A.6}$$

and

$$⨍_{B_r} u^2 dx \le Cr^2 ⨍_{B_r} |\nabla u|^2 dx. \tag{A.7}$$

Thus, with (A.5), (A.6), and (A.7) at hands, we can estimate the error $\mathcal{E}(r)$ term by term: the first one

$$\begin{aligned}\left|\int_{\partial B_r} u\Delta u dS\right| &\le ||\Delta u||_p\left(\int_{\partial B_r} |u|^q dS\right)^{\frac{1}{q}} \le C\left(\int_{\partial B_r} u^{rq} dS\right)^{\frac{1}{r}} |\partial B_r|^{\frac{1}{s}}\\ &= Cr^{\frac{n-1}{qs}+\frac{n-1}{2}}\left(⨍_{\partial B_r} u^2 dS\right)^{\frac{1}{2}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $rq=2$ with $\frac{1}{r}+\frac{1}{s}=1$. Using (A.5) and (A.6), we obtain

$$\begin{aligned}\left|\int_{\partial B_r} u\Delta u dS\right| &\le Cr^{\frac{n-1}{qs}+\frac{n-1}{2}}\left(Cr^2 ⨍_{B_r} |\nabla u|^2 dx\right)^{\frac{1}{2}}\\ &\le Cr^{\frac{n-1}{qs}+\frac{n-1}{2}+1}\left(⨍_{B_r} |\nabla u|^2 dx\right)^{\frac{1}{2}} \le Cr^{-1+\frac{a_0}{2}}\int_{B_r} |\nabla u|^2 dx.\end{aligned} \tag{A.8}$$

In the second one, using (A.6) and (A.7), we get

$$\left|\int_{B_r} u\Delta u dx\right| \le Cr^{\frac{n}{qs}}\left(\int_{B_r} u^2 dx\right)^{\frac{1}{2}} \le Cr^{\frac{n}{qs}+1} ⨍_{B_r} |\nabla u|^2 dx = Cr^{\frac{a_0}{2}}\int_{B_r} |\nabla u|^2 dx. \tag{A.9}$$

In the third one of the numerator, using (A.6),

$$\begin{aligned}\left|\int_{B_r} (x\cdot\nabla u)\Delta u dx\right| &\le Cr^{\frac{n}{qs}}\left(\int_{B_r} |x\cdot\nabla u|^2 dx\right)^{\frac{1}{2}}\\ &\le Cr^{\frac{n}{qs}+1}\left(\int_{B_r} |\nabla u|^2 dx\right)^{\frac{1}{2}} \le Cr^{\frac{a_0}{2}}\int_{B_r} |\nabla u|^2 dx.\end{aligned} \tag{A.10}$$

Finally, by (A.8) the denominator

$$\left|\int_{\partial B_r} uu_\nu dS\right| \le \int_{B_r} |\nabla u|^2 dx + \left|\int_{B_r} u\Delta u dx\right| \le \left(1+Cr^{\frac{a_0}{2}}\right)\int_{B_r} |\nabla u|^2 dx. \tag{A.11}$$

Therefore

$$\mathcal{E}(r) \ge -\frac{Cr^{-1+\frac{a_0}{2}}}{1+Cr^{\frac{a_0}{2}}}. \tag{A.12}$$

Hence, multiplying by an appropriate in order to absorb this error and "cutting" to accommodate the opposite case in (A.4), we have shown that

$$\Phi(r) := \frac{r\left(1 + C_0 r^{\frac{a_0}{2}}\right)}{2} \frac{d}{dr} \log \max\{\varphi(r), r^{4-a_0}\}$$

is monotone non-decreasing function in r if C_0 is chosen large enough.

$C^{1,\alpha}$ Regularity of the free boundary

First we obtain Lipschitz continuity of the free boundary by using the following approximation lemma which has been proved in [13].

Lemma A.1. *Let Λ be a subset of $\mathbb{R}^{n-1} \times \{0\}$. Assume that h is a continuous function such that:*

(i) $\Delta h \le C$ *in* $B_1 \setminus \Lambda$*;*
(ii) $h \ge 0$ *for* $|x_n| \ge \sigma > 0$, $h = 0$ *on* Λ*;*
(iii) $h \ge c_0 > 0$ *for* $|x_n| \ge \frac{1}{8(n-1)}$*;*
(iv) $h \ge -\omega(\sigma)$ *for* $|x_n| < \sigma$*, where* ω *is the modulus of continuity of* h*.*

Then there exists $\sigma_0 = \sigma_0(n, c_0, \omega)$ *and* $C_0 = C_0(n, c_0, \omega)$ *such that, if* $\sigma < \sigma_0$ *and* $C < C_0$*, then* $h \ge 0$ *in* $B_{1/2}$*.*

We will obtain Lipschitz continuity of the free boundary by applying Lemma A.1 to $h = (D_\tau u_{r_j})^m$, where $m \in \mathbb{N}$ is large and odd, to be chosen later, u_{r_j} is the blow-up family that converges to u_0 (see Lemma 4.11) and D_τ denotes a tangential derivative. We compute

$$\Delta h = m(m-1)\big(D_\tau u_{r_j}\big)^{m-2}\big|\nabla\big(D_\tau u_{r_j}\big)\big|^2 + m\big(D_\tau u_{r_j}\big)^{m-1}\Delta\big(D_\tau u_{r_j}\big). \quad \text{(A.13)}$$

Fix $(x,t) \in B^*_{r_j} \setminus \{u = 0\}$ and let d be the distance of (x,t) to the free boundary. By Theorem 4.8 we know that $\partial_t u_{r_j}$ is Hölder continuous, therefore $|\nabla(D_\tau u_{r_j})| \sim d^{-\beta_1}$ and $\Delta(D_\tau u_{r_j}) \sim d^{-\beta_2}$ for β_1, β_2 positive. If we choose m large enough, Δh in (A.13) remains bounded and Lemma A.1 applies. As a result $(D_\tau u_{r_j})^m \ge 0$ in $B^*_{1/2}$ and the same is true for $D_\tau u_{r_j}$ since m is odd. This shows that the free boundary is (locally) the graph of a Lipschitz function.

To prove $C^{1,\alpha}$ regularity we proceed as in [13, Section 7]. Firstly, we observe that since $(D_\tau u_{r_j})^m$ is positive in $B^*_{1/2}$ and $\partial_t u_{r_j}$ is Hölder continuous at $(0,0)$, of [13, Lemma 7.3] can be applied to an appropriate multiple of $(D_\tau u_{r_j})^m$, for m large and odd. Then using Harnack inequality we obtain a nondegeneracy condition

$$\big(D_\tau u_{r_j}(x,t)\big)^m \ge Cd,$$

where d is the distance of (x,t) to the coincidence set. The $C^{1,\alpha}$ regularity will be a consequence of the boundary Harnack inequality, for the Laplacian with bounded right hand side, applied to two positive directional derivatives. The first part of the boundary Harnack principle (the Carleson estimate) can be proved with the right hand side due to the nondegeneracy condition. In our case, we make use again of the fact that $\partial_t u_{r_j}$ is Hölder continuous and that m can be chosen large enough to ensure the validity of [13, Lemma 7.5] for $(D_\tau u_{r_j})^m$. The proof of the second part of the boundary Harnack inequality follows the corresponding proof of [13, Lemma 7.6]. Finally, the $C^{1,\alpha}$ regularity of the free boundary is obtained by applying a standard iterative argument to the ratio $(D_\tau u)^m/(D_{\bar{\tau}} u)^m$ of two directional derivatives of u, see for instance the [6, proof of Theorem 6].

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