

Nef partitions for codimension 2 weighted complete intersections

VICTOR PRZYJALKOWSKI AND CONSTANTIN SHRAMOV

Abstract. We prove that a smooth well-formed Fano weighted complete intersection of codimension 2 has a nef partition. We discuss applications of this fact to Mirror Symmetry. In particular we list all nef partitions for smooth well-formed Fano weighted complete intersections of dimensions 4 and 5 and present weak Landau–Ginzburg models for them.

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1. Introduction

In [8] (see also [9]) Givental defined a Landau–Ginzburg model for a Fano complete intersection X in a smooth toric variety. This Landau–Ginzburg model is a precisely described quasi-projective family over \mathbb{A}^1 . Givental proved that an *I-series* for X , that is a generating series of genus 0 one-pointed Gromov–Witten invariants that count rational curves lying on X , provides a solution of Picard–Fuchs equation of the Landau–Ginzburg model. Givental’s construction may be used for smooth well-formed complete intersections in weighted projective spaces (as well as it is expected to work for complete intersections in varieties that admit “good” toric degenerations like Grassmannians, see [1, 3]) in the same way as for complete intersections in smooth toric varieties, see Section 2 below for details.

The key ingredient in Givental’s construction is a notion of *nef partition*. Let us describe it for the case we are mostly interested in, that is for complete intersections in weighted projective spaces (we refer the reader to [6] and [10] for the

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definitions and basic information about weighted projective spaces and complete intersections therein). Let X be a smooth well-formed Fano complete intersection of hypersurfaces of degrees d_1, \dots, d_c in $\mathbb{P}(a_0, \dots, a_n)$.

Definition 1.1. A *nef partition* for the complete intersection X is a splitting

$$\{0, \dots, n\} = S_0 \sqcup S_1 \sqcup \dots \sqcup S_c$$

such that $\sum_{j \in S_i} a_j = d_i$ for every $i = 1, \dots, c$. The nef partition is called *nice* if there exists an index $r \in S_0$ such that $a_r = 1$.

Given a nef partition, one can easily write down Givental's Landau–Ginzburg model. Moreover, if the nef partition is nice, one can birationally represent it as a complex torus with a function on it, that is just a Laurent polynomial, see Section 2, which we call f_X . Such Laurent polynomials are called *weak Landau–Ginzburg models*. This way of presenting Landau–Ginzburg models has many advantages. “Good” weak Landau–Ginzburg models are expected to have *Calabi–Yau compactifications*. As a result one gets Landau–Ginzburg models from Homological Mirror Symmetry point of view. Another expectation is that f_X can be related to a toric degeneration of X via its Newton polytope. If both expectations hold, f_X is called a *toric Landau–Ginzburg model*, see for instance [23] for more details.

It appears (see Section 2) that a crucial ingredient for the construction of Givental's Landau–Ginzburg model for a weighted Fano complete intersection is the existence of a nef partition, and a crucial ingredient for the construction of toric Landau–Ginzburg model is the existence of a nice nef partition. In [22] this was shown for complete intersections of Cartier divisors in weighted projective spaces. In particular, [22] implies the following.

Theorem 1.2. *Let X be a smooth well-formed Fano weighted hypersurface. Then there exists a nice nef partition for X , and X has a toric Landau–Ginzburg model.*

The main result of the present paper is the following.

Theorem 1.3. *Let X be a smooth well-formed Fano weighted complete intersection of codimension 2. Then there exists a nice nef partition for X .*

As it is discussed above, this result, together with [8, 21, 23], and [11] (see also Section 2 below), gives the following.

Corollary 1.4. *In the assumptions of Theorem 1.3 the complete intersection X has a toric Landau–Ginzburg model.*

Keeping in mind Theorems 1.2 and 1.3 (together with Corollary 1.4), we believe that the following is true.

Conjecture 1.5. *Let X be a smooth well-formed Fano weighted complete intersection. Then there exists a nice nef partition for X , and X has a toric Landau–Ginzburg model.*

The plan of the paper is as follows. In Section 2 we give definitions of nef partitions and Landau–Ginzburg models they correspond to. In Section 3 we introduce the combinatorial method to deal with nef partitions based on certain graphs with vertices labelled by non-trivial weights of the weighted projective space. In Section 4 we prove Theorem 1.3 and make some remarks on its possible generalizations. In Section 5 we write down nice nef partitions and weak Landau–Ginzburg models for four- and five-dimensional smooth well-formed Fano weighted complete intersections that are not intersections with linear cones to give additional evidence for Conjecture 1.5, and make a couple of concluding remarks.

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2. Nef partitions and Landau–Ginzburg models

Let X be a smooth well-formed Fano complete intersection of hypersurfaces of degrees d_1, \dots, d_c in $\mathbb{P}(a_0, \dots, a_n)$. Assume that X admits a nef partition

$$\{0, \dots, n\} = S_0 \sqcup S_1 \sqcup \dots \sqcup S_c.$$

Definition 2.1. Givental’s Landau–Ginzburg model is a quasi-projective variety in $(\mathbb{C}^*)^{n+1}$ with coordinates x_0, \dots, x_n given by equations

$$\begin{cases} x_0^{a_0} \cdots x_n^{a_n} = 1 \\ \sum_{j \in S_i} x_j = 1 & i = 1, \dots, c, \end{cases}$$

together with a function $\sum_{j \in S_0} x_j$ called *superpotential*.

If the nef partition is nice, then one can birationally represent Givental’s Landau–Ginzburg model by a complex torus with a function on it. This function is represented by the following Laurent polynomial. Let $s_{i,1}, \dots, s_{i,r_i}$, where $r_i = |S_i|$, be elements of S_i and let $x_{i,1}, \dots, x_{i,r_i}$ be formal variables of weights $a_{s_{i,1}}, \dots, a_{s_{i,r_i}}$. Since the nef partition is nice, we can assume that $a_{s_{0,r_0}} = 1$. Then Givental’s Landau–Ginzburg model for X is birational to $(\mathbb{C}^*)^{n-c}$ with coordinates $x_{i,j}$ with superpotential

$$f_X = \frac{\prod_{i=1}^c (x_{i,1} + \dots + x_{i,r_i-1} + 1)^{d_i}}{\prod_{\substack{i=0,\dots,c, \\ j=1,\dots,r_i-1}} x_{i,j}^{a_{i,j}}} + x_{0,1} + \dots + x_{0,r_0-1}, \quad (2.1)$$

see [23] and [30, Section 3]. Indices of variables in the factors in the numerator are $(i, 1), \dots, (i, s_i - 1)$. However one can choose any $s_i - 1$ indices among $(i, 1), \dots, (i, s_i)$ to distinguish such variables. The resulting family is relatively birational to the one presented above.

Remark 2.2. We see that the main difficulty to represent Givental's Landau–Ginzburg model for a weighted complete intersection by a Laurent polynomial is to find a nice nef partition; once it is found it is easy to get a birational isomorphism between Givental's Landau–Ginzburg model and a complex torus, so any nice nef partition gives a Laurent polynomial in this way. Givental's construction of Landau–Ginzburg models can be applied, besides complete intersections in smooth toric varieties or weighted projective spaces, to other related cases such as complete intersections in Grassmannians or partial flag varieties, see [3]. Unlike the case of weighted projective spaces, it is easy to describe nef partitions in the latter cases, and this can be done in a lot of ways. However the main problem for representing Landau–Ginzburg models by Laurent polynomials in this case is to find a “good” nef partition among all of them, and to construct the birational isomorphism with a complex torus, see [5, 7, 19, 26, 28, 30].

Givental in [8] computed I -series of complete intersections in smooth toric varieties, that is a generating series of genus zero one-pointed Gromov–Witten invariants with descendants. He proved that this series gives a solution of Picard–Fuchs equation for the family of fibers of the superpotential. The I -series is described in terms of boundary divisors of the toric variety and the hypersurfaces that define the complete intersection. In [20] it was shown that Givental's recipe for I -series for complete intersections in singular toric varieties works in the same way provided that the complete intersection does not intersect the singular locus of the toric variety. The reason is that curves lying on the complete intersection (that is ones that we count) do not intersect the singular locus, so we can resolve singularities of the toric variety and apply Givental's recipe; the exceptional divisors do not contribute to the I -series. Thus one can explicitly write down an I -series for X . One can easily compute the main period for f_X , see, for instance, [21], and check that it coincides with the I -series for X . Moreover, if the Newton polytope of f_X is reflexive (which holds for complete intersections in usual projective spaces, see [24] and [27], but in fact it is not common for weighted complete intersections in weighted projective spaces with non-trivial weights), then f_X admits a Calabi–Yau compactification (see [25, Remark 9]). The Laurent polynomial f_X also corresponds to a certain toric degeneration of X , see [11]. In other words, in this case f_X is a *toric Landau–Ginzburg model* of X , see more details, say, in [23].

3. Weighted projective graphs

In this section we establish auxiliary combinatorial results that will be used in the proof of Theorem 1.3. Given a graph Γ , we will denote by $V(\Gamma)$ the set of its vertices.

Definition 3.1. A *weighted projective graph*, or a *WP-graph*, is a non-empty non-oriented graph Γ without loops and multiple edges together with a *weight function*

$$\alpha_\Gamma: V(\Gamma) \rightarrow \mathbb{Z}_{\geq 2}$$

such that the following properties hold

- For any two vertices $v_1, v_2 \in V(\Gamma)$ there exists an edge connecting v_1 and v_2 in Γ if and only if the numbers $\alpha_\Gamma(v_1)$ and $\alpha_\Gamma(v_2)$ are not coprime;
- For any three vertices $v_1, v_2, v_3 \in V(\Gamma)$ the numbers $\alpha_\Gamma(v_1)$, $\alpha_\Gamma(v_2)$, and $\alpha_\Gamma(v_3)$ are coprime.

The motivation for Definition 3.1 is as follows. If $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ is a well-formed weighted projective space such that every three numbers a_{i_1} , a_{i_2} , and a_{i_3} are coprime, we can produce a WP-graph whose vertices are labelled by the indices i such that $a_i > 1$, and whose weight function assigns the weight a_i to the corresponding vertex. We will use this graph to describe singularities of \mathbb{P} and complete intersections therein, see Section 4.

Definition 3.2. If Γ is a WP-graph, we define $\Sigma\Gamma$ to be the sum of $\alpha_\Gamma(v)$ over all vertices v of Γ , and $\text{lcm } \Gamma$ to be the least common multiple of $\alpha_\Gamma(v)$ over all vertices v of Γ .

Our current goal is to show that under certain assumptions on a WP-graph Γ one has $\text{lcm } \Gamma \geq \Sigma\Gamma$. However, this is not always the case for an arbitrary WP-graph.

Example 3.3. Let $\Delta = \Delta(6, 10, 15)$ be a graph with three vertices v_1, v_2 , and v_3 , and three edges connecting the pairs of the vertices. Put

$$\alpha_\Delta(v_1) = 6, \quad \alpha_\Delta(v_2) = 10, \quad \alpha_\Delta(v_3) = 15,$$

see Figure 3.1. Then Δ is a WP-graph with $\Sigma\Delta = 31$ and $\text{lcm } \Delta = 30$.

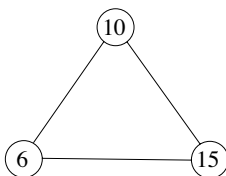


Figure 3.1. WP-graph $\Delta(6, 10, 15)$.

Remark 3.4. Suppose that a WP-graph Γ contains a WP-subgraph $\Delta(6, 10, 15)$. Then it is easy to see that $\Delta(6, 10, 15)$ is a connected component of Γ , and such subgraph is unique.

Definition 3.5. Let Γ be a WP-graph, and v be its vertex. We say that v is *weak* if there is an edge connecting v with another vertex v' of Γ such that $\alpha_\Gamma(v)$ divides $\alpha_\Gamma(v')$. If v is not weak, we say that it is *strong*.

Example 3.6. The graph on Figure 3.2 contains three weak vertices: one labelled by weight 7 and two labelled by weight 17.

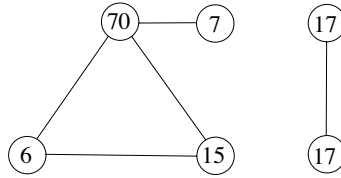


Figure 3.2. Weak and strong vertices.

It easily follows from the definitions that if v is a weak vertex of a WP-graph Γ , then there is only one edge in Γ containing v . We will see later that (surprisingly) the only WP-graph Γ without weak vertices such that $\text{lcm } \Gamma < \Sigma \Gamma$ is $\Delta(6, 10, 15)$.

To proceed we will need the following elementary computation.

Lemma 3.7. *Let N and M be positive integers such that $N \geq 4$ and $M \geq \lceil \frac{N-1}{2} \rceil$. Let a_1, \dots, a_M be integers such that all a_i are greater than 1, and a_i are pairwise coprime. Then $\prod a_i \geq N$.*

Proof. We can assume that $a_M \geq 2$ and $a_1 \geq \dots \geq a_{M-1} \geq 3$. This implies that

$$\prod a_i \geq 2 \cdot 3^{M-1} \geq 2 \cdot 3^{\lceil \frac{N-3}{2} \rceil}.$$

The latter value is not smaller than N for $N \geq 4$, which is easily checked by induction on N . \square

Lemma 3.8. *Let Γ be a connected WP-graph without weak vertices. The following assertions hold.*

- (i) *If Γ has at most two vertices, then $\text{lcm } \Gamma \geq \Sigma \Gamma$;*
- (ii) *If Γ has three vertices, then $\text{lcm } \Gamma \geq \Sigma \Gamma - 1$, and $\text{lcm } \Gamma \geq \Sigma \Gamma$ unless Σ is the WP-graph $\Delta(6, 10, 15)$.*

Proof. If Γ has only one vertex, then one clearly has $\text{lcm } \Gamma = \Sigma \Gamma$.

Suppose that Γ has two or three vertices, and denote them by v_i , $1 \leq i \leq t$, where t equals either 2 or 3. Put $r_i = \text{lcm } \Gamma / \alpha_\Gamma(v_i)$. Then r_i are pairwise coprime positive integers; moreover, one has $r_i \geq 2$, because Γ has no weak vertices. If $t = 2$, then

$$\Sigma \Gamma = \text{lcm } \Gamma \cdot \left(\frac{1}{r_1} + \frac{1}{r_2} \right) < \text{lcm } \Gamma.$$

If $t = 3$, write

$$\Sigma \Gamma = \text{lcm } \Gamma \cdot \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right).$$

Therefore, one has $\Sigma \Gamma > \text{lcm } \Gamma$ if and only if

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} > 1.$$

This easily implies that $r_1 = 2$, $r_2 = 3$, and $r_3 = 5$ up to permutation, which in turn means that $\alpha_\Gamma(v_1) = 15$, $\alpha_\Gamma(v_2) = 10$, and $\alpha_\Gamma(v_3) = 6$. Therefore, Γ is the WP-graph $\Delta(6, 10, 15)$, and one has

$$\text{lcm } \Gamma = 30 = \Sigma \Gamma - 1. \quad \square$$

Lemma 3.9. *Let Γ be a connected WP-graph. Suppose that every vertex of Γ is contained in at least two edges of Γ . Suppose also that the number N of vertices of Γ is at least 4. Then $\text{lcm } \Gamma \geq \Sigma \Gamma$.*

Proof. Let v_{\max} be the vertex of Γ where α_Γ attains its maximum. Let E be the set of all edges of Γ that do not contain the vertex v_{\max} . It is easy to see that

$$|E| \geq \left\lceil \frac{N-1}{2} \right\rceil.$$

For every edge e connecting the vertices v_1 and v_2 of Γ , let a_e denote the greatest common divisor of $\alpha_\Gamma(v_1)$ and $\alpha_\Gamma(v_2)$. Note that all a_e are pairwise coprime integers, and all of them are greater than 1. By Lemma 3.7(ii) we have

$$\text{lcm } \Gamma \geq \alpha_\Gamma(v_{\max}) \cdot \prod_{e \in E} a_e \geq N \alpha_\Gamma(v_{\max}) \geq \sum_{v \in V(\Gamma)} \alpha_\Gamma(v) = \Sigma \Gamma. \quad \square$$

Lemma 3.10. *Let Γ be a connected WP-graph without weak vertices. Suppose that there is a vertex v of Γ contained in only one edge of Γ . Let Γ' be the WP-graph that is obtained from Γ by throwing away the vertex v and the edge containing v , and restricting the weight function to the remaining vertices. Suppose that*

$$\text{lcm } \Gamma' \geq \Sigma \Gamma' - 1.$$

Then

$$\text{lcm } \Gamma \geq \Sigma \Gamma.$$

Proof. Let v' be the vertex of Γ connected with the vertex v . Write $\alpha_\Gamma(v) = ab$ and $\alpha_\Gamma(v') = ac$, where b and c are coprime positive integers, and $a \geq 2$, see Figure 3.3.

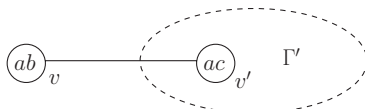


Figure 3.3. Strong vertex contained in a unique edge.

Note that $b \geq 2$ and $c \geq 2$, because v and v' are strong vertices. One has

$$\text{lcm } \Gamma = b \text{lcm } \Gamma', \quad \Sigma \Gamma = ab + \Sigma \Gamma'.$$

Note also that the graph Γ' is connected because the graph Γ is connected.

Suppose that $\Sigma\Gamma' \geq ab + 2$. Then

$$\text{lcm } \Gamma = b \text{lcm } \Gamma' \geq 2 \text{lcm } \Gamma' \geq 2\Sigma\Gamma' - 2 \geq \Sigma\Gamma' + ab = \Sigma\Gamma.$$

Now suppose that $\Sigma\Gamma' \leq ab + 1$. This is impossible if $c > b$, because $a \geq 2$. Thus we have $b \geq c$, which means $b > c$ since b and c are coprime. Hence

$$(b-1)c \geq 2b-2 \geq b+1.$$

Note that $\text{lcm } \Gamma' \geq ac$. This gives

$$\begin{aligned} \text{lcm } \Gamma &= b \text{lcm } \Gamma' = \text{lcm } \Gamma' + (b-1)\text{lcm } \Gamma' \geq \Sigma\Gamma' - 1 + (b-1)ac \\ &\geq \Sigma\Gamma' - 1 + (b+1)a > \Sigma\Gamma' + ab = \Sigma\Gamma. \end{aligned} \quad \square$$

Proposition 3.11. *Let Γ be a connected WP-graph without weak vertices. Then*

$$\text{lcm } \Gamma \geq \Sigma\Gamma - 1,$$

and moreover $\text{lcm } \Gamma \geq \Sigma\Gamma$ unless Γ is the WP-graph $\Delta(6, 10, 15)$.

Proof. We prove the assertion by induction on the number N of vertices of Γ . We know from Lemma 3.8 that the assertion holds for $N \leq 3$. If Γ has a vertex contained in only one edge of Γ , then the assertion follows by induction from Lemma 3.10. Therefore, we may assume that $N \geq 4$, and every vertex of Γ is contained in at least two edges of Γ . Now the assertion follows from Lemma 3.9. \square

Corollary 3.12. *Let Γ be a WP-graph without weak vertices. Suppose that Γ is not the WP-graph $\Delta(6, 10, 15)$. Then $\text{lcm } \Gamma \geq \Sigma\Gamma$.*

Proof. Let $\Gamma_1, \dots, \Gamma_r$ be connected components of Γ . Then

$$\text{lcm } \Gamma = \prod_{i=1}^r \text{lcm } \Gamma_i, \quad \Sigma\Gamma = \sum_{i=1}^r \Sigma\Gamma_i. \quad (3.1)$$

If a connected component Γ_i is not the WP-graph $\Delta(6, 10, 15)$, then $\text{lcm } \Gamma_i \geq \Sigma\Gamma_i$ by Proposition 3.11. Therefore, if none of Γ_i is $\Delta(6, 10, 15)$, then the assertion immediately follows from (3.1).

Suppose that some of Γ_i , say Γ_1 , is the WP-graph $\Delta(6, 10, 15)$. Then $r \geq 2$, and none of $\Gamma_2, \dots, \Gamma_r$ is $\Delta(6, 10, 15)$. Note that $\Sigma\Gamma_i \geq 2$ (and actually it is at least 7 for $2 \leq i \leq r$ because of coprimeness condition), so that

$$30 \prod_{i=2}^r \Sigma\Gamma_i \geq 31 + \prod_{i=2}^r \Sigma\Gamma_i \geq 31 + \sum_{i=2}^r \Sigma\Gamma_i.$$

Thus (3.1) implies

$$\text{lcm } \Gamma = 30 \prod_{i=1}^r \text{lcm } \Gamma_i \geq 30 \prod_{i=2}^r \Sigma\Gamma_i \geq 31 + \sum_{i=2}^r \Sigma\Gamma_i = \sum_{i=1}^r \Sigma\Gamma_i. \quad \square$$

Definition 3.13. Let d_1, \dots, d_c be positive integers. A *weighted complete intersection graph* (or a *WCI-graph*) of *multidegree* (d_1, \dots, d_c) is a WP-graph Γ such that the following condition holds: for every k and every choice of k vertices v_1, \dots, v_k of Γ for which the greatest common divisor δ of $\alpha_\Gamma(v_1), \dots, \alpha_\Gamma(v_k)$ is greater than 1, there exist k numbers d_{s_1}, \dots, d_{s_k} , $1 \leq s_1 < \dots < s_k \leq c$, whose greatest common divisor is divisible by δ . The number c is called the *codimension* of the WCI-graph Γ .

The motivation for Definition 3.13 comes from the fact that a smooth weighted complete intersection of codimension 1 or 2 produces a WCI-graph of codimension 1 or 2, respectively, and some important properties of the weighted complete intersection are controlled by this WCI-graph, see Section 4 for details. Therefore, in this paper we will be mostly interested in WCI-graphs of codimension 1 and 2.

Remark 3.14. It would be more precise to say that a WCI-graph is not just a WP-graph Γ but rather a collection that consists of Γ and the multidegree (d_1, \dots, d_c) . In particular, one may have several different WP-graphs with the same Γ and different multidegrees, and even different codimensions. However, in this paper we are going to deal only with WCI-graphs of codimension 1 and 2, and in any case we want to avoid this complication of notation and hope that no confusion will arise.

Lemma 3.15. Let Γ be a WCI-graph of codimension 2 and bidegree (d_1, d_2) . Then the set of vertices $V(\Gamma)$ is a disjoint union

$$V(\Gamma) = V_1 \sqcup V_2,$$

such that the complete subgraphs Γ_1 and Γ_2 of Γ with vertices V_1 and V_2 are WP-graphs without weak vertices, none of Γ_1 and Γ_2 contains a connected component $\Delta(6, 10, 15)$, and $\text{lcm } \Gamma_i$ divides d_i .

Proof. Let $V' \subset V(\Gamma)$ be the set of strong vertices of Γ , and $V'' = V(\Gamma) \setminus V'$ be the set of weak vertices. If Γ does not contain a subgraph $\Delta(6, 10, 15)$, put

$$V'_1 = \{v \in V' \mid \alpha_\Gamma(v) \text{ divides } d_1\}.$$

If Γ contains a subgraph $\Delta(6, 10, 15)$, then it is easy to see that both d_1 and d_2 are divisible by $\text{lcm } \Delta(6, 10, 15) = 30$. In this case we put

$$V'_1 = \{v \in V' \setminus V(\Delta(6, 10, 15)) \mid \alpha_\Gamma(v) \text{ divides } d_1\} \cup \{v_1\},$$

where v_1 is an arbitrarily chosen vertex of $\Delta(6, 10, 15)$. We also put $V'_2 = V' \setminus V'_1$. It follows from the definition of a WCI-graph that for every $v \in V'_2$ the number $\alpha_\Gamma(v)$ divides d_2 .

For every weak vertex v of Γ denote by $\tau(v)$ the unique vertex of Γ connected to v by an edge. It follows from the definition of a WP-graph that either $\alpha_\Gamma(\tau(v)) > \alpha_\Gamma(v)$, so that $\tau(v)$ is a strong vertex of Γ , or $\alpha_\Gamma(\tau(v)) = \alpha_\Gamma(v)$, so that v and $\tau(v)$ are both weak vertices. In the latter case the vertices v and $\tau(v)$ together with the

edge connecting them form a connected component of Γ (note however that v and $\tau(v)$ together with the corresponding edge may form a connected component of Γ if $\tau(v)$ is a strong vertex as well). Let us refer to the former vertices as *weak vertices of the first type*, and to the latter vertices as *weak vertices of the second type*. In both cases it follows from the definition of a WCI-graph that the degrees d_1 and d_2 are divisible by $\alpha_\Gamma(v)$. Let V_1'' be the set of all weak vertices of the first type such that $\tau(v) \in V_2'$, and V_2'' be the set of all weak vertices of the first type such that $\tau(v) \in V_1'$. Finally, let \tilde{V}_1'' and \tilde{V}_2'' be sets of weak vertices of the second type each containing one and only one vertex from each pair connected by an edge.

Put

$$V_1 = V_1' \cup V_1'' \cup \tilde{V}_1'', \quad V_2 = V_2' \cup V_2'' \cup \tilde{V}_2'.$$

Then for every $v \in V_1$ the number $\alpha_\Gamma(v)$ divides d_1 , and for every $v \in V_2$ the number $\alpha_\Gamma(v)$ divides d_2 . The graphs Γ_1 and Γ_2 are WP-graphs since they are complete subgraphs of a WP-graph. None of them contains a subgraph $\Delta(6, 10, 15)$; indeed, if one of them does, then $\Delta(6, 10, 15)$ is also a subgraph of Γ , and all three vertices of $\Delta(6, 10, 15)$ cannot simultaneously appear as vertices of any of Γ_i by construction. We also see that $\text{lcm } \Gamma_i$ divides d_i . Moreover, if $v \in V_1$ (respectively, $v \in V_2$) is a weak vertex of Γ , then $\tau(v) \in V_2$ (respectively, $\tau(v) \in V_1$). This means that the graphs Γ_1 and Γ_2 do not have weak vertices themselves, because any weak vertex of Γ_i would also be a weak vertex of Γ . \square

Example 3.16. Let Γ be a WP-graph from Figure 3.2. The vertex of Γ labelled by weight 7 is a weak vertex of the first type, while the two vertices labelled by weight 17 are weak vertices of the second type. All other vertices are strong. The WP-graph Γ can be considered as a WCI-graph of codimension 2 and bidegree (d, d) , where

$$d = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 17 = 3570.$$

Following the proof of Lemma 3.15, one forms the set V_1' that consists of the vertices labelled by 70, 15, and 6, the set V_2'' that consists of the vertex labelled by 7, the sets \tilde{V}_1'' and \tilde{V}_2'' each consisting of one vertex labelled by 17, and puts $V_2' = V_1'' = \emptyset$.

Corollary 3.17. *Let Γ be a WCI-graph of codimension 2 and bidegree (d_1, d_2) . Then the set of vertices $V(\Gamma)$ is a disjoint union $V(\Gamma) = V_1 \sqcup V_2$ such that*

$$\sum_{v \in V_i} \alpha_\Gamma(v) \leq d_i.$$

Proof. Choose V_1 and V_2 as in Lemma 3.15, and let Γ_1 and Γ_2 be the complete subgraphs of Γ with vertices V_1 and V_2 . We know that d_i is divisible by $\text{lcm } \Gamma_i$. By Corollary 3.12 one has

$$d_i \geq \text{lcm } \Gamma_i \geq \Sigma \Gamma_i = \sum_{v \in V_i} \alpha_\Gamma(v). \quad \square$$

Example 3.18. Let Γ be a WP-graph from Figure 3.2 considered as a WCI-graph of codimension 2 and bidegree $(3570, 3570)$, see Example 3.16. Then one can take Γ_1 to be the graph with two connected components, one of them a triangle with vertices labelled by 70, 15, and 6 together with the edges connecting them, and the other a single point labelled by 17, while Γ_2 will be a graph with two connected components, each of them just a single point, one labelled by 7 and the other by 17.

4. Proof of the main theorem

In this section we prove Theorem 1.3 and make some remarks about its possible generalizations.

Proof of Theorem 1.3. Let X be a weighted complete intersection of hypersurfaces of degrees d_1 and d_2 in $\mathbb{P}(a_0, \dots, a_n)$. Since X is smooth and well-formed, by [29, Lemma 2.15] for every k and every choice of k weights

$$a_{i_1}, \dots, a_{i_k}, 0 \leq i_1 < \dots < i_k \leq n,$$

whose greatest common divisor δ is greater than 1, there exist k degrees

$$d_{s_1}, \dots, d_{s_k}, 1 \leq s_1 < \dots < s_k \leq 2,$$

whose greatest common divisor is divisible by δ . In particular, any three weights $a_{i_1}, a_{i_2}, a_{i_3}$ are coprime.

We may assume that

$$1 = a_0 = \dots = a_p < a_{p+1} \leq \dots \leq a_n.$$

Let Γ be a WP-graph defined as follows. The vertices of Γ are v_{p+1}, \dots, v_n , and two vertices v_i and v_j are connected by an edge if and only if the weights a_i and a_j are not coprime. Furthermore, we put $\alpha_\Gamma(v_i) = a_i$. It is easy to see that Γ is a WP-graph. Moreover, Γ is a WCI-graph of codimension 2 and bidegree (d_1, d_2) . By Corollary 3.17 there are two disjoint sets V_1 and V_2 such that

$$V_1 \sqcup V_2 = \{p+1, \dots, n\}$$

and $\sum_{j \in V_i} a_j \leq d_i$ for $i = 1, 2$. Since X is Fano, we have

$$\sum_{i=0}^n a_i > d_1 + d_2, \quad (4.1)$$

see [6, Theorem 3.3.4] or [10, 6.14]. This implies that one can add the indices of several unit weights, i.e. some indices from $\{0, \dots, p\}$, to the sets V_1 and V_2 to form two disjoint subsets $S_1 \supset V_1$ and $S_2 \supset V_2$ of $\{0, \dots, n\}$ such that $\sum_{j \in S_i} a_j = d_i$

for $i = 1, 2$. Moreover, since the inequality in (4.1) is strict, we conclude that the set

$$S_0 = \{0, \dots, n\} \setminus (S_1 \cup S_2)$$

is not empty. All weights a_i with indices $i \in S_0$ equal 1, so that the nef partition

$$\{0, \dots, n\} = S_0 \sqcup S_1 \sqcup S_2$$

is nice. □

Example 4.1. Let X be a complete intersection of two hypersurfaces of degree 3570 in $\mathbb{P}(1^k, 6, 15, 70, 7, 17, 17)$, where 1^k stands for 1 repeated k times. This is a well-formed Fano weighted complete intersections if k is large enough (and X is general). Example 3.18 provides a nice nef partition for X . Of course, there are many more nice nef partitions in this case. Note that X is smooth if it is general enough.

If $X \subset \mathbb{P}(a_0, \dots, a_n)$ is a smooth well-formed Fano weighted hypersurface, then the corresponding WP-graph Γ has no edges at all. Thus the inequality $\text{lcm } \Gamma \geq \Sigma \Gamma$ is obvious in this case, and similarly to the proof of Theorem 1.3 we immediately obtain a nice nef partition for X . This recovers the result of Theorem 1.2. Also, the proof of Theorem 1.3 gives the following by-product (cf. [29, Lemma 3.3]).

Corollary 4.2. *Let X be a smooth well-formed Fano weighted complete intersection of hypersurfaces d_1, \dots, d_c in the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$. Suppose that $c \leq 2$. Then the number of indices $i \in \{0, \dots, n\}$ such that $a_i = 1$ is at least $I(X) = \sum a_i - \sum d_j$.*

Actually, the assertion of Corollary 4.2 holds in the case of arbitrary codimension, see [17, Corollary 5.11].

If X is a smooth well-formed Calabi–Yau weighted complete intersection of codimension 1 or 2, we can argue in the same way as in the proof of Theorem 1.3 to show that there exists a nef partition for X , for which we necessarily have $S_0 = \emptyset$ in the notation of Definition 1.1. Constructing the dual nef partition we obtain a Calabi–Yau variety Y that is mirror dual to X , see [2]. In the same paper it is proved that the Hodge-theoretic mirror symmetry holds for X and Y . That is, for a given variety V one can define *string Hodge numbers* $h_{st}^{p,q}(V)$ as Hodge numbers of a crepant resolution of V if such resolution exists. Then, for $n = \dim X = \dim Y$, one has $h_{st}^{p,q}(X) = h_{st}^{n-p,q}(Y)$ provided that the ambient toric variety (weighted projective space in our case) is Gorenstein.

Finally, we would like to point out a possible approach to a proof of Conjecture 1.5 along the lines of the current paper. If X is a smooth well-formed Fano weighted complete intersection of codimension 3 or higher in a weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$, it is possible that some three weights a_{i_1}, a_{i_2} , and a_{i_3} are not coprime. Thus a WP-graph constructed in the proof of Theorem 1.3 does not provide an adequate description of singularities of the weighted projective space \mathbb{P} . An obvious way to (try to) cope with this is to replace a graph by a simplicial

complex that would remember the greatest common divisors of arbitrary subsets of weights a_i in Definition 3.1. However, this leads to combinatorial difficulties that we cannot overcome at the moment. Except for the most straightforward ones, like the effects on weak vertices (which would be not that easy to control) and possibly larger number of exceptions analogous to our WP-graph $\Delta(6, 10, 15)$, there is also a less obvious one (which is in fact easy to deal with). Namely, we need a finer information about weights and degrees than that provided by [29, Lemma 2.15].

Example 4.3. Let X be a weighted complete intersection of hypersurfaces of degrees 2, 3, 5, and 30 in $\mathbb{P}(1^k, 6, 10, 15)$, where 1^k stands for 1 repeated k times. Then X is a well-formed Fano weighted complete intersection provided that k is large and X is general. Note that the conclusion of [29, Lemma 2.15] holds for X . However, it is easy to see that X is not smooth. Moreover, there is no nef partition for X .

In any case, it is easy to see that the actual information one can deduce from the fact that a weighted complete intersection is smooth is much stronger than that provided by [29, Lemma 2.15]. We also expect that combinatorial difficulties that one has to face on the way to the proof of Conjecture 1.5 proposed above are possible to overcome.

5. Fano four- and fivefolds

Smooth well-formed Fano weighted complete intersections of dimensions 2 and 3 are known and well studied (see, for instance, [12]), as well as their toric Landau–Ginzburg models (see, for instance, [16] and [23]). In this section we write down nef partitions and weak Landau–Ginzburg models for four- and five-dimensional smooth well-formed Fano weighted complete intersections. Some of them have codimension greater than 2, which gives additional evidence for Conjecture 1.5. Providing such list is possible due to classification of smooth Fano weighted complete intersections obtained in [29, Section 5], because finding all nef partitions for a given complete intersection requires just a simple (though a bit lengthy) computation.

In Tables 5.1 and 5.2 below we list nef partitions and corresponding weak Landau–Ginzburg models of four- and five-dimensional smooth well-formed Fano weighted complete intersections that are not intersections with linear cones, see [29, Section 2] for definitions. These weighted complete intersections were classified in [29, Section 5], see also [14, Proposition 2.2.1], where the case of dimension 4 was originally established. In the first column of Tables 5.1 and 5.2 we put the number of the family according to tables in [29, Section 5]. The second column describes the weighted projective spaces where the weighted complete intersections live. Here we use the abbreviation

$$(a_0^{k_0}, \dots, a_m^{k_m}) = (\underbrace{a_0, \dots, a_0}_{k_0 \text{ times}}, \dots, \underbrace{a_m, \dots, a_m}_{k_m \text{ times}}),$$

Table 5.1. Fourfold Fano weighted complete intersections.

No.	\mathbb{P}	Degrees	Nef partitions	Weak Landau–Ginzburg models
1	$\mathbb{P}(1^3, 2^2, 3^2)$	6,6	$\{0\} \sqcup \{1, 2, 3, 4\} \sqcup \{5, 6\}$ $\{0\} \sqcup \{1, 3, 5\} \sqcup \{2, 4, 6\}$	$\frac{(x_1+x_2+x_3+1)^6(y_1+1)^6}{x_1x_2x_3^2y_1^3}$ $\frac{(x_1+x_2+1)^6(y_1+y_2+1)^6}{x_1x_2^2y_1y_2^2}$
2	$\mathbb{P}(1^4, 2, 5)$	10	$\{0\} \sqcup \{1, 2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+x_4+1)^{10}}{x_1x_2x_3x_4^2}$
3	$\mathbb{P}(1^4, 2^2, 3)$	4,6	$\{0\} \sqcup \{1, 2, 4\} \sqcup \{3, 5, 6\}$ $\{0\} \sqcup \{4, 5\} \sqcup \{1, 2, 3, 6\}$	$\frac{(x_1+x_2+1)^4(y_1+y_2+1)^6}{x_1x_2y_1y_2^2}$ $\frac{(x_1+1)^4(y_1+y_2+y_3+1)^6}{x_1^2y_1y_2y_3}$
4	$\mathbb{P}(1^5, 4)$	8	$\{0\} \sqcup \{1, 2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+x_4+1)^8}{x_1x_2x_3x_4}$
5	$\mathbb{P}(1^5, 2)$	6	$\{0\} \sqcup \{1, 2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+x_4+1)^6}{x_1x_2x_3x_4}$
6	$\mathbb{P}(1^5, 2^2)$	4,4	$\{0\} \sqcup \{1, 2, 3, 4\} \sqcup \{5, 6\}$ $\{0\} \sqcup \{1, 2, 5\} \sqcup \{3, 4, 6\}$	$\frac{(x_1+x_2+x_3+1)^4(y_1+1)^4}{x_1x_2x_3y_1^2}$ $\frac{(x_1+x_2+1)^4(y_1+y_2+1)^4}{x_1x_2y_1y_2}$
7	$\mathbb{P}(1^6, 3)$	2,6	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4, 5, 6\}$	$\frac{(x_1+1)^2(y_1+y_2+y_3+1)^6}{x_1y_1y_2y_3}$
8	\mathbb{P}^5	5	$\{0\} \sqcup \{1, 2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+x_4+1)^4}{x_1x_2x_3x_4}$
9	$\mathbb{P}(1^6, 2)$	3,4	$\{0\} \sqcup \{1, 2, 3\} \sqcup \{4, 5, 6\}$ $\{0\} \sqcup \{1, 6\} \sqcup \{2, 3, 4, 5\}$	$\frac{(x_1+x_2+1)^3(y_1+y_2+1)^4}{x_1x_2y_1y_1}$ $\frac{(x_1+1)^3(y_1+y_2+y_3+1)^4}{x_1y_1y_2y_3}$
10	\mathbb{P}^6	2,4	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4, 5, 6\}$	$\frac{(x_1+1)^2(y_1+y_2+y_3+1)^4}{x_1y_1y_2y_3}$
11	\mathbb{P}^6	3,3	$\{0\} \sqcup \{1, 2, 3\} \sqcup \{4, 5, 6\}$	$\frac{(x_1+x_2+1)^3(y_1+y_2+1)^3}{x_1x_2y_1y_2}$
12	\mathbb{P}^7	2,2,3	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6, 7\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+z_2+1)^3}{x_1y_1z_1z_2}$
13	\mathbb{P}^8	2,2,2,2	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6\} \sqcup \{7, 8\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+1)^2(w_1+1)^2}{x_1y_1z_1w_1}$
14	$\mathbb{P}(1^5, 3)$	6	$\{0, 1\} \sqcup \{2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+1)^6}{x_1x_2x_3t_1} + t_1$
15	\mathbb{P}^5	4	$\{0, 1\} \sqcup \{2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+1)^4}{x_1x_2x_3t_1} + t_1$
16	\mathbb{P}^6	2,3	$\{0, 1\} \sqcup \{2, 3\} \sqcup \{4, 5, 6\}$	$\frac{(x_1+1)^2(y_1+y_2+1)^3}{x_1y_1y_2t_1} + t_1$
17	\mathbb{P}^7	2,2,2	$\{0, 1\} \sqcup \{2, 3\} \sqcup \{4, 5\} \sqcup \{6, 7\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+1)^2}{x_1y_1z_1t_1} + t_1$
18	$\mathbb{P}(1^4, 2, 3)$	6	$\{0, 1, 2\} \sqcup \{3, 4, 5\}$ $\{0, 4\} \sqcup \{1, 2, 3, 5\}$	$\frac{(x_1+x_2+1)^6}{x_1x_2^2t_1t_2} + t_1 + t_2$ $\frac{(x_1+x_2+x_3+1)^6}{x_1x_2x_3t_1^2} + t_1$
19	$\mathbb{P}(1^5, 2)$	4	$\{0, 1, 2\} \sqcup \{3, 4, 5\}$ $\{0, 5\} \sqcup \{1, 2, 3, 4\}$	$\frac{(x_1+x_2+1)^4}{x_1x_2t_1t_2} + t_1 + t_2$ $\frac{(x_1+x_2+x_3+1)^4}{x_1x_2x_3t_1^2} + t_1$
20	\mathbb{P}^5	3	$\{0, 1, 2\} \sqcup \{3, 4, 5\}$	$\frac{(x_1+x_2+1)^3}{x_1x_2t_1t_2} + t_1 + t_2$
21	\mathbb{P}^6	2,2	$\{0, 1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6\}$	$\frac{(x_1+1)^2(y_1+1)^2}{x_1y_1t_1t_2} + t_1 + t_2$
22	\mathbb{P}^5	2	$\{0, 1, 2, 3\} \sqcup \{4, 5\}$	$\frac{(x_1+1)^2}{x_1t_1t_2t_3} + t_1 + t_2 + t_3$

Table 5.2. Fivefold Fano weighted complete intersections.

No.	\mathbb{P}	Degrees	Nef partitions	Weak Landau–Ginzburg models
1	$\mathbb{P}(1^5, 2, 3, 3)$	6, 6	$\{0\} \sqcup \{1, 2, 3, 4, 5\} \sqcup \{6, 7\}$ $\{0\} \sqcup \{1, 2, 3, 6\} \sqcup \{4, 5, 7\}$	$\frac{(x_1+x_2+x_3+x_4+1)^6(y_1+1)^6}{x_1x_2x_3x_4y_1^3}$ $\frac{(x_1+x_2+x_3+1)^6(y_1+y_2+1)^6}{x_1x_2x_3y_1y_2^2}$
2	$\mathbb{P}(1^6, 5)$	10	$\{0\} \sqcup \{1, 2, 3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+x_4+x_5+1)^{10}}{x_1x_2x_3x_4x_5}$
3	$\mathbb{P}(1^6, 2, 3)$	4, 6	$\{0\} \sqcup \{1, 2, 3, 4\} \sqcup \{5, 6, 7\}$ $\{0\} \sqcup \{1, 7\} \sqcup \{2, 3, 4, 5, 6\}$ $\{0\} \sqcup \{1, 2, 6\} \sqcup \{3, 4, 5, 7\}$	$\frac{(x_1+x_2+x_3+1)^4(y_1+y_2+1)^6}{x_1x_2x_3y_1y_2^2}$ $\frac{(x_1+1)^4(y_1+y_2+y_3+y_4+1)^6}{x_1y_1y_2y_3y_4}$ $\frac{(x_1+x_2+1)^4(y_1+y_2+y_3+1)^6}{x_1x_2y_1y_2y_3}$
4	$\mathbb{P}(1^7, 4)$	2, 8	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4, 5, 6, 7\}$	$\frac{(x_1+1)^2(y_1+y_2+y_3+y_4+1)^8}{x_1y_1y_2y_3y_4}$
5	\mathbb{P}^6	6	$\{0\} \sqcup \{1, 2, 3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+x_4+x_5+1)^6}{x_1x_2x_3x_4x_5}$
6	$\mathbb{P}(1^7, 2)$	4, 4	$\{0\} \sqcup \{1, 2, 3, 4\} \sqcup \{5, 6, 7\}$	$\frac{(x_1+x_2+x_3+1)^4(y_1+y_2+1)^4}{x_1x_2x_3y_1y_2}$
7	$\mathbb{P}(1^8, 3)$	2, 2, 6	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6, 7, 8\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+z_2+z_3+1)^6}{x_1y_1z_1z_2z_3}$
8	\mathbb{P}^7	2, 5	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4, 5, 6, 7\}$	$\frac{(x_1+1)^2(y_1+y_2+y_3+y_4+1)^5}{x_1y_1y_2y_3y_4}$
9	\mathbb{P}^7	3, 4	$\{0\} \sqcup \{1, 2, 3\} \sqcup \{4, 5, 6, 7\}$	$\frac{(x_1+x_2+1)^3(y_1+y_2+y_3+1)^4}{x_1x_2y_1y_2y_3}$
10	\mathbb{P}^8	2, 2, 4	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6, 7, 8\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+z_2+z_3+1)^4}{x_1y_1z_1z_2z_3}$
11	\mathbb{P}^8	2, 3, 3	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4, 5\} \sqcup \{6, 7, 8\}$	$\frac{(x_1+1)^2(y_1+y_2+1)^3(z_1+z_2+1)^3}{x_1y_1y_2z_1z_2}$
12	\mathbb{P}^9	2, 2, 2, 3	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6\} \sqcup \{7, 8, 9\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+1)^2(u_1+u_2+1)^3}{x_1y_1z_1u_1u_2}$
13	\mathbb{P}^{10}	2, 2, 2, 2, 2	$\{0\} \sqcup \{1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6\} \sqcup \{7, 8\} \sqcup \{9, 10\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+1)^2(u_1+1)^2(w_1+1)^2}{x_1y_1z_1u_1v_1}$
14	$\mathbb{P}(1^4, 2, 2, 3, 3)$	6, 6	$\{0, 1\} \sqcup \{2, 3, 4, 5\} \sqcup \{6, 7\}$ $\{0, 1\} \sqcup \{2, 4, 6\} \sqcup \{3, 5, 7\}$	$\frac{(x_1+x_2+x_3+1)^6(y_1+1)^6}{x_1x_2x_3^2y_1^3t_1} + t_1$ $\frac{(x_1+x_2+1)^6(y_1+y_2+1)^6}{x_1x_2^2y_1y_2^2t_1} + t_1$
15	$\mathbb{P}(1^5, 2, 5)$	10	$\{0, 1\} \sqcup \{2, 3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+x_4+1)^{10}}{x_1x_2x_3x_4^2t_1} + t_1$
16	$\mathbb{P}(1^5, 2, 2, 3)$	4, 6	$\{0, 1\} \sqcup \{2, 3, 5\} \sqcup \{4, 6, 7\}$ $\{0, 1\} \sqcup \{2, 7\} \sqcup \{3, 4, 5, 6\}$ $\{0, 1\} \sqcup \{5, 6\} \sqcup \{2, 3, 4, 7\}$	$\frac{(x_1+x_2+1)^4(y_1+y_2+1)^6}{x_1x_2y_1y_2^2t_1} + t_1$ $\frac{(x_1+1)^4(y_1+y_2+y_3+1)^6}{x_1y_1y_2y_3^2t_1} + t_1$ $\frac{(x_1+1)^4(y_1+y_2+y_3+1)^6}{x_1^2y_1y_2y_3t_1} + t_1$
17	$\mathbb{P}(1^6, 4)$	8	$\{0, 1\} \sqcup \{2, 3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+x_4+1)^8}{x_1x_2x_3x_4t_1} + t_1$
18	$\mathbb{P}(1^6, 2)$	6	$\{0, 1\} \sqcup \{2, 3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+x_4+1)^6}{x_1x_2x_3x_4t_1} + t_1$
19	$\mathbb{P}(1^6, 2, 2)$	4, 4	$\{0, 1\} \sqcup \{2, 3, 4, 5\} \sqcup \{6, 7\}$ $\{0, 1\} \sqcup \{2, 3, 6\} \sqcup \{4, 5, 7\}$	$\frac{(x_1+x_2+x_3+1)^4(y_1+1)^4}{x_1x_2x_3y_1^2t_1} + t_1$ $\frac{(x_1+x_2+1)^4(y_1+y_2+1)^4}{x_1x_2y_1y_2t_1} + t_1$
20	$\mathbb{P}(1^7, 3)$	2, 6	$\{0, 1\} \sqcup \{2, 3\} \sqcup \{4, 5, 6, 7\}$	$\frac{(x_1+1)^2(y_1+y_2+y_3+1)^6}{x_1y_1y_2y_3t_1} + t_1$
21	\mathbb{P}^6	5	$\{0, 1\} \sqcup \{2, 3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+x_4+1)^5}{x_1x_2x_3x_4t_1} + t_1$

Table 5.2. *continued*

No.	\mathbb{P}	Degrees	Nef partitions	Weak Landau–Ginzburg models
22	$\mathbb{P}(1^7, 2)$	3, 4	$\{0, 1\} \sqcup \{2, 3, 4\} \sqcup \{5, 6, 7\}$	$\frac{(x_1+x_2+1)^3(y_1+y_2+1)^4}{x_1x_2y_1y_2t_1} + t_1$
			$\{0, 1\} \sqcup \{2, 7\} \sqcup \{3, 4, 5, 6\}$	$\frac{(x_1+1)^3(y_1+y_2+y_3+1)^4}{x_1y_1y_2y_3t_1} + t_1$
23	\mathbb{P}^7	2, 4	$\{0, 1\} \sqcup \{2, 3\} \sqcup \{4, 5, 6, 7\}$	$\frac{(x_1+1)^2(y_1+y_2+y_3+1)^4}{x_1y_1y_2y_3t_1} + t_1$
24	\mathbb{P}^7	3, 3	$\{0, 1\} \sqcup \{2, 3, 4\} \sqcup \{5, 6, 7\}$	$\frac{(x_1+x_2+1)^3(y_1+y_2+1)^3}{x_1x_2y_1y_2t_1} + t_1$
25	\mathbb{P}^8	2, 2, 3	$\{0, 1\} \sqcup \{2, 3\} \sqcup \{4, 5\} \sqcup \{6, 7, 8\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+z_2+1)^3}{x_1y_1z_1z_2t_1} + t_1$
26	\mathbb{P}^9	2, 2, 2, 2	$\{0, 1\} \sqcup \{2, 3\} \sqcup \{4, 5\} \sqcup \{6, 7\} \sqcup \{8, 9\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+1)^2(u_1+1)^2}{x_1y_1z_1u_1t_1} + t_1$
27	$\mathbb{P}(1^6, 3)$	6	$\{0, 1, 2\} \sqcup \{3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+1)^6}{x_1x_2x_3t_1t_2} + t_1 + t_2$
28	\mathbb{P}^6	4	$\{0, 1, 2\} \sqcup \{3, 4, 5, 6\}$	$\frac{(x_1+x_2+x_3+1)^4}{x_1x_2x_3t_1t_2} + t_1 + t_2$
29	\mathbb{P}^7	2, 3	$\{0, 1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6, 7\}$	$\frac{(x_1+1)^2(y_1+y_2+1)^3}{x_1y_1y_2t_1t_2} + t_1 + t_2$
30	\mathbb{P}^8	2, 2, 2	$\{0, 1, 2\} \sqcup \{3, 4\} \sqcup \{5, 6\} \sqcup \{7, 8\}$	$\frac{(x_1+1)^2(y_1+1)^2(z_1+1)^2}{x_1y_1z_1t_1t_2} + t_1 + t_2$
31	$\mathbb{P}(1^5, 2, 3)$	6	$\{0, 1, 2, 3\} \sqcup \{4, 5, 6\}$	$\frac{(x_1+x_2+1)^6}{x_1x_2^2t_1t_2t_3} + t_1 + t_2 + t_3$
			$\{0, 1, 5\} \sqcup \{2, 3, 4, 6\}$	$\frac{(x_1+x_2+x_3+1)^6}{x_1x_2x_3t_1t_2^2} + t_1 + t_2$
			$\{0, 6\} \sqcup \{1, 2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+x_4+1)^6}{x_1x_2x_3x_4t_1^3} + t_1$
32	$\mathbb{P}(1^6, 2)$	4	$\{0, 1, 2, 3\} \sqcup \{4, 5, 6\}$	$\frac{(x_1+x_2+1)^4}{x_1x_2t_1t_2t_3} + t_1 + t_2 + t_3$
			$\{0, 1, 6\} \sqcup \{2, 3, 4, 5\}$	$\frac{(x_1+x_2+x_3+1)^4}{x_1x_2x_3t_1t_2^2} + t_1 + t_2$
33	\mathbb{P}^6	3	$\{0, 1, 2, 3\} \sqcup \{4, 5, 6\}$	$\frac{(x_1+x_2+1)^3}{x_1x_2t_1t_2t_3} + t_1 + t_2 + t_3$
34	\mathbb{P}^7	2, 2	$\{0, 1, 2, 3\} \sqcup \{4, 5\} \sqcup \{6, 7\}$	$\frac{(x_1+1)^2(y_1+1)^2}{x_1y_1t_1t_2t_3} + t_1 + t_2 + t_3$
35	\mathbb{P}^6	2	$\{0, 1, 2, 3, 4\} \sqcup \{5, 6\}$	$\frac{(x_1+1)^2}{x_1t_1t_2t_3t_4} + t_1 + t_2 + t_3 + t_4$

where k_0, \dots, k_m are any positive integers. If some of k_i is equal to 1 we drop it for simplicity. In the third column we put the degrees of weighted hypersurfaces that cut out our complete intersections. The forth column describes nice nef partitions; note that in general there are many of them in every case, but we do not distinguish between nef partitions obtained by permuting indices corresponding to equal weights. In the fifth column we write down the corresponding Landau–Ginzburg models. The latter are obtained using formula (2.1), where instead of variables $x_{0,j}$, $x_{1,j}$, $x_{2,j}$, \dots , we use variables t_j , x_j , y_j , \dots , respectively, to simplify notation. We exclude four- and five-dimensional projective spaces (which are complete intersections of codimension 0 in themselves) from the tables to unify them with tables from [29, Section 5].

Remark 5.1. The set S_0 in nef partitions obtained as in the proof of Theorem 1.3 consists of indices only of such variables that have weight 1. However some smooth

well-formed complete intersections may admit other nef partitions, having non-trivial weights in S_0 , see for instance No. 18 and 19 in Table 5.1, and No. 31 and 32 in Table 5.2.

Question 5.2. One sees that varieties No. 1, 3, 6, 9, 18, 19 from Table 5.1 and No. 1, 14, 19, 22, 32 from Table 5.2 have two different nice nef partitions, while varieties No. 3, 16, and 31 from Table 5.2 have three different nice nef partitions. Thus they have two or three weak Landau–Ginzburg models given by these nef partitions. In [4, 15] (see also [18]) it is proved (under mild additional assumptions) that for complete intersections in Gorenstein toric varieties Landau–Ginzburg models provided by different nef partitions are birational. Does this hold for complete intersections in weighted projective spaces?

Remark 5.3. Varieties listed in Tables 5.1 and 5.2 admit degenerations to toric varieties whose fan polytopes coincide with Newton polytopes of their weak Landau–Ginzburg models, see [11]. Most of them are complete intersections in usual projective spaces. Thus one can prove the existence of (log) Calabi–Yau compactifications for them, see [23, 27], and [25]. Moreover, their existence can be proved for some other varieties: for variety No. 18 from Table 5.2 using a method from [27] and for varieties No. 18, 19 (for the second nef partition), 22 (for the first nef partition), 27, 32 (for both nef partitions) from Table 5.2 using a method from [25]. Thus one can prove that these varieties have toric Landau–Ginzburg models (listed in the last column of the tables).

Question 5.4. In [13] Landau–Ginzburg Hodge numbers are defined, see [16] for some discussion on this definition. Using this definition in [13] the authors formulated Hodge-theoretic Mirror Symmetry conjecture for Fano varieties by an analogy with the conjecture for smooth Calabi–Yau varieties. This conjecture was proved for del Pezzo surfaces in [16]. One of Hodge numbers can be conjecturally interpreted via number of components of reducible fibers, see [23, 27]. In [27] this conjecture was checked for complete intersections in usual projective spaces. Does the Hodge-theoretic Mirror Symmetry conjecture hold for varieties listed in Tables 5.1 and 5.2? Does one have an interpretation via the number of irreducible components of reducible fibers in this case? Does it hold for all Fano complete intersections in weighted projective spaces having nice nef partitions?

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Steklov Mathematical Institute of RAS

8 Gubkina street

Moscow, 119991, Russia

and

National Research University Higher School of Economics

Russian Federation

Laboratory of Mirror Symmetry, NRU HSE

6 Usacheva street

Moscow, 119048, Russia

victorprz@mi.ras.ru

victorprz@gmail.com

Steklov Mathematical Institute of RAS

8 Gubkina street

Moscow 119991, Russia

and

National Research University Higher School of Economics

Laboratory of Algebraic Geometry

6 Usacheva street

Moscow, 119048, Russia

costya.shramov@gmail.com