Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XIX (2019), 889-911

Primitive equations with linearly growing initial data

AMRU HUSSEIN, MARTIN SAAL AND OKIHIRO SAWADA

Abstract. The primitive equations in a $3D$ infinite layer domain are considered with linearly growing initial data in the horizontal direction, which illustrates the global atmospheric rotating or straining flows. On the boundaries, Dirichlet, Neumann or mixed boundary conditions are imposed. The Ornstein-Uhlenbeck type operator appears in the linear parts, so the semigroup theory is established by Trotter's arguments due to decomposition of infinitesimal generators. To obtain smoothing properties of the semigroup, derivatives of the associated kernel are calculated. For proving time-local existence and uniqueness of mild solutions, the adapted Fujita-Kato scheme is used in certain Sobolev spaces.

Mathematics Subject Classification (2010): 35Q35 (primary); 76D03, 47D06, 35K65, 86A05 (secondary).

1. Introduction

The primitive equations for the ocean and atmosphere are considered to be a fundamental model for geophysical flows, which is derived from the Navier-Stokes or Boussinesq equations assuming a hydrostatic balance. The mathematical analysis of the primitive equations has been commenced by Lions, Teman and Wang [19–21]. For more information on previous works on the primitive equations, we refer to the books of Washington and Parkinson [31], Pedlosky [27], Majda [23] and Vallis [30]; see also the survey by Li and Titi [18] for recent results and further references.

The $3D$ primitive equations are derived from the Navier-Stokes equations in domains which are small in the vertical direction compared to the horizontal ones. This justifies the assumption of a hydrostatic balance in the vertical direction. Although the nonlinear structure of the primitive equations looks similar to that of the Navier-Stokes equations, it differs due to anisotropic features. As the Navier-Stokes equations, the primitive equations describe the velocity U of a fluid and the pressure

This work was partly supported by the DFG International Research Training Group IRTG 1529 and the JSPS Japanese-German Graduate Externship on Mathematical Fluid Dynamics. The first author has been supported by IRTG 1529 at TU Darmstadt.

Received January 25, 2017; accepted in revised form October 27, 2017.
Published online September 2019.

Π. Putting $U = (V, W)$, where $V = (V_1, V_2)$ denotes the horizontal components and W stands for the vertical one, it leads us to

$$\begin{cases} \partial_t V + V \cdot \nabla_H V + W \cdot \partial_z V - \Delta V + \nabla_H \Pi = f & \text{in } \Omega \times (0, T) \\ \Pi_z = 0 & \text{in } \Omega \times (0, T) \\ \operatorname{div}_H V + W_z = 0 & \text{in } \Omega \times (0, T) \\ V(0) = V_0 & \text{in } \Omega. \end{cases}$$

Here $\Omega := \mathbb{R}^2 \times (-h, h) \subset \mathbb{R}^3$ for $h > 0$ is an infinite layer domain; ∇_H and div_H denote the horizontal gradient and the horizontal divergence, respectively; Δ stands for the full Laplacian. It is remarkable that the time-global well-posedness for the $3D$ primitive equations has been proven by Cao and Titi [4] with initial data in H^1, while the question of time-global well-posedness for the Navier-Stokes equations still constitutes an open problem. So, in the study of the primitive equations – especially the study of admissible initial values – a legitimate first step would be to ask whether results known for the Navier-Stokes equations also hold for the primitive equations.

For the Navier-Stokes equations in the whole space $\mathbb{R}^d$ for $d \geq 2$, Hieber and the third author of this paper studied in [15] the particular case of linearly growing initial data. Concretely saying, the initial velocity V_0 is given by the form $V_0(x) = v_0(x) - Mx$ for $x \in \mathbb{R}^d$, where M is a constant and trace free matrix and $v_0 \in L^p_\sigma(\mathbb{R}^d)$ for $p \geq d$. For such an initial velocity the Navier-Stokes equations are indeed time-local well-posed in the mild sense. Such initial data occur in several applications. Typical examples of M with $d = 3$ in [14, 15] are $M = R, J, S$ and their respective sums with

$$R = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -b & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 2b \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. Here the matrix R describes a swirl by the Coriolis force; J models a drain along the horizontal axis and a jet flow in the vertical direction; S illustrates a model of a straining flow on the surface. For precise analysis, the reader can find for the case of pure rotation in, *e.g.*, [2, 16], for $M = J$ or S in, *e.g.*, [8, 22] and the references therein. In [3] the global existence of strong solutions in the case of a two dimensional exterior domain and a traceless matrix M was established for L^2-initial data of arbitrary size and in [11] a weaker version of that result has been shown recently in the L^p-setting ($p \geq 2$) considering only skew-symmetric matrices M.

Considering the primitive equations, the setting in the whole space does not make sense. However, a layer domain Ω imposing Dirichlet, Neumann or mixed boundary conditions constitutes an admissible setting for the hydrostatic balance assumption. The investigation in the L^p framework for the primitive equations in a cylinder $(0, 1)^2 \times (-h, 0) \subset \mathbb{R}^3$ with some boundary conditions has been started by Hieber and his collaborators in [9, 12, 13]. They treated initial data in $H^{2/p,p}$

for $p \in (1, \infty)$ which carries over to decaying initial data in $H^{2/p,p}$ on the layer domain. However, linearly growing data $V_0(x) = v_0(x) - Mx$ for $x \in \mathbb{R}^2$ have not been considered, so far. As an physical application one can think of

$$M = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

which produces a horizontal shear flow in the fluid layer.

Similar to the case of the Navier-Stokes equations in the whole space, with 2×2 matrix M the substitution

$$V = v - Mx_H, \quad v \in L^p(\Omega)^2, \quad x_H = (x, y) \in \mathbb{R}^2$$

one derives equations involving an Ornstein-Uhlenbeck type operator in the linear parts, whenever the boundary conditions are adopted in a certain sense. This operator contains linearly growing coefficients in the drift terms. The Ornstein-Uhlenbeck type operator generates a (C_0) semigroup in L^p for $p \in (1, \infty)$, however, this semigroup is not analytic, in general. So, it is *a priori* not clear whether it has suitable smoothing properties or not. The handling of the boundary conditions and the linearly growing coefficients in the drift terms are main technical issues that one has to overcome for proving eventually the existence of time-local unique mild solutions as well as the treatment of the non-linearity. Although the $3D$ primitive equations on bounded cylindrical domains are time-global well-posed, it is still open weather the same results hold with linearly growing data on layer domains. In this paper we aim to prove the local in time well-posedness of the $3D$ primitive equations. The question of time-global solutions will be part of a future work.

Comparing a situation of the Navier-Stokes equations, the main difficulties in our setting arise from a lack of smoothing property in the linearized problem. Although in the whole space the heat kernel is explicitly given, the hydrostatic Stokes operator is expressed as a perturbation of the Laplacian as shown in [9]. Hence, even though the structure of non-linearity in the primitive equations resembles that in the Navier-Stokes equations, one has to assume some additional regularity on initial data since for primitive equations the non-linearity which is actually bi-linear contains derivatives in both arguments. Therefore, it is also more convenient to apply the iteration scheme of Fujita-Kato [6] rather than that of Kato [17] for handling the non-linearity of the primitive equations.

This paper is organized as follows. In Section 2 we give preliminaries for basic setting of function spaces and reformulation for the problem with linearly growing initial data. In Section 3 we discuss theories of the Ornstein-Uhlenbeck type operator in an infinite layer domain, proving smoothing properties of the corresponding semigroup in Section 4. In Section 5 the existence of time-local unique mild solutions is proved.

ACKNOWLEDGEMENTS. The authors would like to thank Matthias Hieber for helpful discussions on linearly growing data for Navier Stokes equations and Robert Haller-Dintelmann for the valuable discussion of Ornstein-Uhlenbeck operators in various domains, in particular pointing out the works [10] and [32].

2. Preliminaries

We consider the $3D$ primitive equations in an infinite layer domain

$$\Omega := G \times (-h, h) \subset \mathbb{R}^3 \quad \text{with} \quad G := \mathbb{R}^2, \quad h > 0.$$

Here the horizontal coordinates are denoted by $x, y \in G$ and $x_H := (x, y)$, and the vertical one by $z \in (-h, h)$. For the sake of simplicity, only the velocity and the surface pressure are considered in this paper, omitting the temperature, salinity and further quantities which are incorporated into the full model discussed in [19–21]. The unknowns are the velocity U of the fluid described as $U = (V, W)$, where the vector $V = (V_1, V_2)$ denotes the horizontal components and the scalar W stands for of the vertical one, and the surface pressure written as Π_s. So, we consider the equations

$$\begin{cases} \partial_t V + V \cdot \nabla_H V + W \cdot \partial_z V - \Delta V + \nabla_H \Pi_s = f & \text{in } \Omega \times (0, T) \\ \operatorname{div}_H \overline{V} = 0 & \text{in } \Omega \times (0, T) \\ V(0) = V_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here Π_s can be regarded as a function defined in $\Omega \times (0, T)$ using $\Pi_s(x, y, z, t) = \Pi_s(x, y, t)$. We have used the notations

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad \nabla_H = (\partial_x, \partial_y), \quad V \cdot \nabla_H = V_1 \partial_x + V_2 \partial_y,$$
$$\operatorname{div}_H V = \partial_x V_1 + \partial_y V_2 \quad \text{and} \quad \overline{V} = \frac{1}{2h} \int_{-h}^{h} V(\cdot, \cdot, \xi) d\xi.$$

The system is supplemented by the mixed boundary conditions on

$$\Gamma_b := G \times \{-h\} \quad \text{and} \quad \Gamma_u := G \times \{h\}$$

denoting the bottom and the upper parts of the boundary $\partial\Omega$, respectively. Here and hereafter, for justification reducing equations of disturbance (2.4) below, we impose adopted boundary conditions on the Dirichlet parts of the boundary along with linearly growing initial data:

$$V = M x_H \text{ on } \Gamma_D \times (0,T), \quad \partial_z V = 0 \text{ on } \Gamma_N \times (0,T), \quad W = 0 \text{ on } \partial\Omega \times (0,T). \tag{2.2}$$

Here $M \in \mathbb{R}^{2\times 2}$ is a given constant matrix with $\operatorname{tr} M = 0$; Dirichlet, Neumann and mixed boundary conditions are comprised by the notation

$$\Gamma_D \in \{\emptyset, \Gamma_u, \Gamma_b, \Gamma_u \cup \Gamma_b\} \quad \text{and} \quad \Gamma_N := (\Gamma_u \cup \Gamma_b) \setminus \Gamma_D.$$

In the literature, several situation of boundary conditions have also been considered. For example, in [19, Equation (1.37) and (1.37)'] Dirichlet and mixed boundary conditions are treated respectively, while in [4] Neumann boundary conditions are assumed.

By the boundary conditions on W and $\operatorname{div} U = 0$ the vertical component of the velocity $W = W(V)$ is determined as

$$W(x, y, z, t) = -\int_{-h}^{z} \operatorname{div}_H V(x, y, \xi, t) d\xi$$

for each $(x, y, z, t) \in \Omega \times (0, T)$; see, *e.g.*, [12].

Similarly to the Navier-Stokes equations, one can consider the solenoidal subspace of $L^p(\Omega)^2$ for $p \in (1, \infty)$. The classical Helmholtz projection onto the solenoidal space $L^p_\sigma(\mathbb{R}^2)$ for $p \in (1, \infty)$ is denoted by Q_p here. In addition, benefits from the approach developed in [12, Section 3 and 4] carry over to the present situation. So, considering

$$L^p_{\overline{\sigma}}(\Omega) := \overline{\{v \in C^\infty(\Omega)^2 : \operatorname{div}_H \overline{v} = 0\}}^{\|\cdot\|_{L^p(\Omega)^2}}, \tag{2.3}$$

there exists a continuous projection P_p, called *hydrostatic Helmholtz projection*, from $L^p(\Omega)^2$ onto $L^p_{\overline{\sigma}}(\Omega)$ which can be represented by

$$P_p v = Q_p \overline{v} + \tilde{v}, \quad \tilde{v} := v - \overline{v}.$$

Note that $v = \overline{v}$ holds if and only if $\tilde{v} = 0$, compare also [9]. In particular P_p annihilates the gradient of the surface pressure term $\nabla_H \pi_s$. As in [12], we can define the *hydrostatic Stokes operator* A_p in $L^p_{\overline{\sigma}}(\Omega)$ by

$$A_p v := P_p \Delta v, \quad D(A_p) := \left\{ v \in H^{2,p}(\Omega)^2 : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0 \right\} \cap L^p_{\overline{\sigma}}(\Omega).$$

In what follows, we deal with the initial velocity V_0 of the form

$$V_0(x_H, z) := v_0(x_H, z) - M x_H \quad \text{for} \quad x_H \in G, \quad z \in (-h, h),$$

where $v_0 \in L^p_{\overline{\sigma}}(\Omega)$ and $M \in \mathbb{R}^{2\times 2}$ with $\operatorname{tr} M = 0$, that is,

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \quad \text{with} \quad m_{11} = -m_{22}.$$

The condition $\operatorname{tr} M = 0$ implies the compatibility condition:

$$\operatorname{div}_H \overline{V_0} = 0 \quad \text{in} \quad \Omega.$$

We now assume that V solves the primitive equations with initial data V_0. Substituting

$$V(x_H, z, t) = v(x_H, z, t) - M x_H \quad \text{for } v(t) \in L^p_{\overline{\sigma}}(\Omega)$$

yields for v

$$\partial_t V = \partial_t v, \quad \Delta V = \Delta v, \quad \operatorname{div}_H \overline{V} = \operatorname{div}_H \overline{v} \quad \text{and} \quad W(V)\partial_z V = W(v)\partial_z v.$$

For the last two identities, the assumption $\operatorname{tr} M = 0$ is essential. Moreover,

$$V \cdot \nabla_H V = v \cdot \nabla_H v - Mx_H \cdot \nabla_H v - v \cdot \nabla_H Mx_H + Mx_H \cdot \nabla_H Mx_H,$$

and the individual terms are more explicitly given by

$$\begin{aligned} v \cdot \nabla_H Mx_H &= v_1 \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} + v_2 \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = Mv, \\ Mx_H \cdot \nabla_H v &= (m_{11}x + m_{12}y)\partial_x v + (m_{21}x + m_{22}y)\partial_y v, \\ Mx_H \cdot \nabla_H Mx_H &= (m_{11}x + m_{12}y) \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} + (m_{21}x + m_{22}y) \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}. \end{aligned}$$

Using the assumption $\operatorname{tr} M = 0$, the last quadratic term simplifies to become

$$Mx_H \cdot \nabla_H Mx_H = \big(m_{11}^2 + m_{12}m_{21}\big)x_H = -(\det M)x_H.$$

Hence, to absorb this term by the modified pressure, we put $\tilde{\pi}_s = \Pi_s - g$ with $g(x, y) := -\frac{1}{2}\det(M)(x^2 + y^2)$. So, if (V, Π_s) solves (2.1), then $(v, \tilde{\pi}_s)$ solves the following equations in $\Omega \times (0, T)$

$$\begin{cases} \partial_t v + w\partial_z v - \Delta v + v \cdot \nabla_H v - Mv - Mx_H \cdot \nabla_H v + \nabla_H \tilde{\pi}_s = f \\ \operatorname{div}_H \overline{v} = 0, \end{cases} \tag{2.4}$$

with initial conditions $v(0) = v_0$ which belongs to a suitable subspace of $L^p_{\overline{\sigma}}(\Omega)$. Since $V(x_H, z) = v(x_H, z) - Mx_H$, where V satisfies the boundary conditions (2.2) one obtains using $\partial_z Mx_H \mid_{\Gamma_N} = 0$, $\partial_z V \mid_{\Gamma_N} = 0$ and $V \mid_{\Gamma_D} + Mx_H \mid_{\Gamma_D} = 0$ that v satisfies the boundary conditions

$$v = 0 \text{ on } \Gamma_D \times (0, T) \quad \text{and} \quad \partial_z v = 0 \text{ on } \Gamma_N \times (0, T). \tag{2.5}$$

So, solving system (2.4) for a real matrix M with $\operatorname{tr} M = 0$ and boundary conditions (2.5) for $v(\cdot, t) \in L^p_{\overline{\sigma}}(\Omega)$ and initial data $v_0 \in L^p_{\overline{\sigma}}(\Omega)$ is equivalent to solving the original equations (2.1) with boundary conditions (2.2) and linearly growing initial data $V_0 = v_0 - Mx_H$. The linear parts of (2.4) are associated with an operator of Ornstein-Uhlenbeck type which will be investigated in the next section for general real matrices M, that is not assuming necessarily $\operatorname{tr} M = 0$.

The spectrum and the structure of M are relevant for computing the matrix exponential appearing in the Kolmogorov kernel given in (3.2) below. In fact, the eigenvalues of M dominate the dynamics of solutions to (2.4), directly. The eigenvalues of $M \in \mathbb{R}^{2\times 2}$ with $\operatorname{tr} M = 0$ are explicitly given as

$$\lambda_\pm = \pm\sqrt{m_{11}^2 + m_{12} \cdot m_{21}}.$$

Therefore, one can distinguish the following cases:

(a) If $m_{11}^2 + m_{12} \cdot m_{21} > 0$, then there are two different real eigenvalues, and M is similar to a sign-indefinite symmetric matrix, and therefore e^{tM} is not exponentially stable;
(b) If $m_{11}^2 + m_{12} \cdot m_{21} < 0$, then there are two distinct and purely imaginary eigenvalues. Hence, M is similar to a skew-symmetric matrix making e^{tM} similar to a unitary group and $\|e^{tM}\| < C$ for all $t \in \mathbb{R}$, where $C > 0$ depends only on M;
(c) If $m_{11}^2 + m_{12} \cdot m_{21} = 0$, then zero is an eigenvalue of algebraic multiplicity two. If $M \neq 0$ then 0 is an eigenvalue of algebraic multiplicity one, and hence M is similar to a Jordan block with zero on the diagonal, and hence e^{tM} is similar to the matrix exponential $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$.

In particular, $\|e^{tM}\| \leq C$ holds with some $C > 0$ and all $t > 0$ if and only if $M = 0$ or case (b) holds. Furthermore, we can take $C = 1$ if M is anti-symmetric, that is, the pure rotation case.

Remark 2.1. To absorb the term $Mx_H \cdot \nabla_H Mx_H$ into the pressure, one needs that it is a gradient field. As described above this holds for traceless matrices. However, this is also true for instance for symmetric matrices, *i.e.*, $m_{12} = m_{21}$, where

$$\begin{aligned} Mx_H \cdot \nabla_H Mx_H &= \nabla\varphi, \\ \varphi(x, y) &= \frac{1}{2}\big(m_{11}^2 + m_{12}^2\big)x^2 + \frac{1}{2}\big(m_{22}^2 + m_{12}^2\big)y^2 + m_{12}\big(m_{11} + m_{22}\big)xy. \end{aligned}$$

3. Ornstein-Uhlenbeck operator in a layer

We define the Ornstein-Uhlenbeck type operator $\mathcal{L}_p$ in $L^p(\Omega)^2$ by

$$\begin{aligned} \mathcal{L}_p v &:= \Delta v - (\mathbb{1} - P_p)\,\mathrm{tr}_D(\partial_z v) + Mx_H \cdot \nabla_H v - Mv, \\ D(\mathcal{L}_p) &:= \big\{v \in H^{2,p}(\Omega)^2 : \partial_z u \mid_{\Gamma_N} = 0 \text{ and } u \mid_{\Gamma_D} = 0,\ Mx_H \cdot \nabla_H v \in L^p(\Omega)^2\big\}. \end{aligned}$$

Here we have used

$$\mathrm{tr}_D(\partial_z v) := \frac{1}{2h}\big(\alpha(b)\partial_z v|_{\Gamma_u} - \alpha(u)\partial_z v|_{\Gamma_b}\big)$$

with $\alpha(a) = 1$ if $\Gamma_a \subset \Gamma_D$ for $a \in \{u, b\}$, and $\alpha(a) = 0$ otherwise. So, one can consider the following time-evolutionary ordinary differential equation

$$\partial_t v - \mathcal{L}_p v = f \quad \text{for} \quad t > 0 \quad \text{and} \quad v(0) = v_0.$$

In fact, by Proposition 3.3 below, this problem is well-defined in $L^p_{\overline{\sigma}}(\Omega)$. In previous results on the bounded cylindrical domain case, the correction terms

$-(\mathbb{1} - P_p)\,\mathrm{tr}_D(\partial_z v)$ have also been discussed. In [9, Section 4] the key idea is to solve the surface pressure terms, firstly. This method carries over to the case in an infinite layer domain.

To apply known results for Ornstein-Uhlenbeck operators in the whole space, we decompose $\mathcal{L}_p$ into the horizontal parts $\mathcal{L}_H$ and the vertical one $\mathcal{L}_z$ as

$$\mathcal{L}_H v := \Delta_H v + M x_H \cdot \nabla_H v - M v \quad \text{and} \quad \mathcal{L}_z v := \partial_z^2 v - (\mathbb{1} - P_p)\,\mathrm{tr}_D(\partial_z v)$$

using anisotropic Sobolev spaces as the domains

$$\begin{aligned} D(\mathcal{L}_H) &:= \{ v \in L^p_z H^{2,p}_{xy} : M x_H \cdot \nabla_H v \in L^p(\Omega) \}, \\ D(\mathcal{L}_z) &:= \{ v \in H^{2,p}_z L^p_{xy} : \partial_z v|_{\Gamma_N} = 0,\ v|_{\Gamma_D} = 0 \}. \end{aligned}$$

Here, for $r, s \geq 0$ and $p, q \in (1, \infty)$ we have used the spaces

$$H^{r,q}_z H^{s,p}_{xy} := H^{r,q}((-h,h); H^{s,p}(G))$$

equipped with the norm $\|v\|_{H^{r,q}_z H^{s,p}_{xy}} := \| \|v(\cdot,z)\|_{H^{s,p}(G)} \|_{H^{r,q}(-h,h)}$ setting for brevity $H^{0,p} = L^p$. Note that

$$H^{r+s,p}(\Omega) \subset H^{r,p}_z H^{s,p}_{xy}. \tag{3.1}$$

Note that the boundary conditions of $\mathcal{L}_z$ are well-defined since the trace on Γ_u and Γ_b is well-defined in the anisotropic spaces with regularity in the vertical direction. Indeed, by Sobolev's embedding we see that

$$|v(x,y,\pm h)| \leq C \|v(x,y,\cdot)\|_{H^1_z} \quad \text{and} \quad |\partial_z v(x,y,\pm h)| \leq C \|v(x,y,\cdot)\|_{H^2_z}$$

for almost every $x, y \in G$. Taking L^p_{xy} norms into above, we thus have

$$\|v(\cdot,\cdot,\pm h)\|_{L^p_{xy}} \leq C \|v\|_{H^{1,p}_z L^p_{xy}} \quad \text{and} \quad \|\partial_z v(\cdot,\cdot,\pm h)\|_{L^p_{xy}} \leq C \|v\|_{H^{2,p}_z L^p_{xy}}.$$

The operator $\mathcal{L}_H$ in $L^p(\mathbb{R}^2)^2$ has been studied in [15] drawing back its main properties to the classical Ornstein-Uhlenbeck operator defined by $(\mathcal{L}_H + M)v$ in $L^p(\mathbb{R}^2)^2$ studied extensively in, *e.g.*, [10, 24, 25, 32]. Consider as in [15, Lemma 3.3] for $L^p(\mathbb{R}^2)^2$, the semigroup $T_H(t) = e^{t\mathcal{L}_H}$ in $L^p(\Omega)^2$ for $p \in (1, \infty)$ defined by

$$\begin{aligned} &(T_H(t)\psi)(x_H, z) \\ &:= \frac{1}{4\pi(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^2} \psi\left(e^{tM} x_H - x'_H, z\right) e^{-\frac{1}{4}\langle Q_t^{-1} x'_H, x'_H \rangle} dx'_H \end{aligned} \tag{3.2}$$

for $t > 0$ and $(x_H, z) \in \Omega$, where $\psi \in L^p(\Omega)^2$ and $Q_t := \int_0^t e^{sM} e^{sM^T} ds$. Define the associated kernel k_t by $k_t(x_H) := \frac{1}{4\pi(\det Q_t)^{1/2}} e^{-tM} e^{-\frac{1}{4}\langle Q_t^{-1} x_H, x_H \rangle}$, so we can write (3.2) as

$$(T_H(t)\psi)(x_H, z) = (k_t *_H \psi)\left(e^{tM} x_H, z\right) \quad \text{for} \quad t > 0,$$

where $*_H$ denotes by the convolution with respect to the x_H variables.

Note that there is a constant $C > 0$ independent of t such that

$$t^2 \leq \det Q_t \quad \text{and} \quad \left\| Q_t^{-1} \right\| \leq C t^{-1} \quad \text{for} \quad t > 0,$$

see [15, Equation (3.5)]. Let us consider the more general d-dimensional case. Since the matrix $M + M^T$ is symmetric, we can write it as ADA^{-1} with a diagonal matrix D and a regular matrix A. Hence, we obtain

$$\begin{aligned}\det Q_t &= \det \int_0^t e^{sADA^{-1}} ds = \det \left(A \int_0^t e^{sD} ds A^{-1} \right) \\ &= \det \int_0^t e^{sD} ds = t^k \prod_{i=1}^{d-k} \frac{e^{\lambda_i t} - 1}{\lambda_i}.\end{aligned}$$

Here k is the multiplicity of the eigenvalue 0, and the $\lambda_i \in \mathbb{R} \setminus \{0\}$ are non-zero eigenvalues. With $e^{\lambda_i t} - 1 \geq \lambda_i t$ for $t > 0$, we obtain $t^d \leq \det Q_t$. Similarly,

$$Q_t^{-1} = \left(A \int_0^t e^{sD} ds A^{-1} \right)^{-1} = A \begin{pmatrix} t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{e^{t\lambda_{d-k}} - 1}{\lambda_{d-k}} \end{pmatrix}^{-1} A^{-1}.$$

It follows from $e^{\lambda_i t} - 1 \geq \lambda_i t$ again that

$$\| Q_t^{-1} \| \leq \|A\| \| A^{-1} \| \max_{1 \leq i \leq d-k} \left\{ t^{-1}, \frac{\lambda_i}{e^{t\lambda_i} - 1} \right\} \leq \|A\| \| A^{-1} \| \cdot t^{-1}.$$

The operator $\mathcal{L}_z^0$ defined by

$$\mathcal{L}_z^0 \psi = \partial_z^2 \psi, \quad D\big(\mathcal{L}_z^0\big) = D(\mathcal{L}_z)$$

is certainly the generator of a bounded analytic semigroup in $L^p(\Omega)^2$. In $\mathbb{R}^2 \times S^1$ with $S^1 := \{z \in [-h, h] : -h \sim h\}$, a semigroup can be defined explicitly by means of Fourier series

$$\left(T_{z,h}^0(t)\psi \right)(x_H, z) := \sum_{k \in \mathbb{Z}} e^{-tk^2} \hat{\psi}_k(x_H) e^{ik\frac{\pi}{h}z} \tag{3.3}$$

for $t > 0$ and $(x_H, z) \in \Omega$, where

$$\hat{\psi}_k(x_H) := \frac{1}{2h} \int_{-h}^{h} e^{-ik\frac{\pi}{h}z} \psi(x_H, z) dz \tag{3.4}$$

for $t > 0$ and $(x_H, z) \in \Omega$. We may arrive at the Dirichlet, Neumann and mixed boundary conditions, taking odd parts of $T_{z,2h}^0$, even parts of $T_{z,2h}^0$, and odd and even part of $T_{z,4h}^0$, respectively.

We treat $\mathcal{L}_z = \mathcal{L}_z^0 + B_p$, which can be regarded as a relatively bounded perturbation from $\mathcal{L}_z^0$ by

$$B_p \colon H_z^{1+\frac{1}{p},p} L_{xy}^p(\Omega)^2 \to L^p(\Omega)^2, \quad B_p v := (\mathbb{1} - P_p)(\mathrm{tr}_D \, \partial_z v).$$

One sees that the operator B_p is bounded by the trace theorem applied to the vertical direction; see, *e.g.*, [28, Theorem 2.7.2] for half spaces, which carries over to the situation considered here localizing functions around Γ_u and Γ_b by cut-off functions, and taking L_{xy}^p norm. In particular, $C^1(-h,h) \hookrightarrow H_z^{1+\frac{1}{p},p}$ for $p \in (1,\infty)$. Also, $D(\mathcal{L}_z^0) \subset D(B_p)$ holds. Therefore, by interpolation inequality and $H_z^{1+\frac{1}{p},p} = [L_z^p, H_z^{2,p}]_{\frac{1}{2}+\frac{1}{2p}}$, where $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor, and by Young's inequality

$$\begin{aligned} \|B_p v\|_{L^p(\Omega)^2} &\le C\|v\|_{H_z^{1+\frac{1}{p},p} L_{xy}^p} \\ &\le C\|v\|_{L_z^p L_{xy}^p}^{\frac{1}{2}-\frac{1}{2p}} \|v\|_{H_z^{2,p} L_{xy}^p}^{\frac{1}{2}+\frac{1}{2p}} \\ &\le \varepsilon C\|v\|_{H_z^{2,p} L_{xy}^p} + \frac{C}{\varepsilon}\|v\|_{L^p(\Omega)^2} \end{aligned}$$

for $v \in D(\mathcal{L}_z^0)$ and $\varepsilon > 0$, where $C > 0$ is some constant depending only on p and h. Therefore, $\mathcal{L}_z$ with $D(\mathcal{L}_z) = D(\mathcal{L}_z^0)$ is the generator of an analytic semigroup $T_z(t) = e^{t\mathcal{L}_z}$ in $L^p(\Omega)$ for $p \in (1,\infty)$ by, *e.g.*, [26, Theorem 3.2.1]. Note that for the case of $\Gamma_D = \emptyset$, *i.e.* $\Gamma_N = \Gamma_b \cup \Gamma_u$, we see that $\mathcal{L}_z = \mathcal{L}_z^0$ and the semigroup can be given explicitly as even parts of $T_{z,2h}^0$.

Lemma 3.1.

(a) *The operators $T_H(t)$ and $T_z(t)$ define (C_0)-semigroups in $L^p(\Omega)^2$ for $p \in (1,\infty)$ the infinitesimal generators of which are $\mathcal{L}_H$ and $\mathcal{L}_z$, respectively;*
(b) *They commute, that is,*

$$T_z(s)T_H(t) = T_H(t)T_z(s) \quad \textit{for} \quad s,t > 0;$$

(c) *$T_H(t)$ and $T_z(t)$ restrict to (C_0)-semigroups in the complementary spaces $L_{\overline{\sigma}}^p(\Omega)$ and $\nabla_H(W^{1,p}(\mathbb{R}^2))$.*

Proof. The operator $\mathcal{L}_z$ is by construction the infinitesimal generator of $T_z(t)$. For $\mathcal{L}_H$, one can verify that the proof of the corresponding statement on $L^p(\mathbb{R}^2)$ for the generator; see [24, Proposition 3.2] as well as the characterization of the domain [25, Theorem 4.1] which carry over one-to-one to the present situation, and one therefore concludes that $\mathcal{L}_H$ is indeed the generator of T_H.

In the case of $\Gamma_D = \emptyset$, both semigroups $T_H(t)$ and $T_z(t)$ are given explicitly by (3.2) and (3.3), respectively. So, it is straight forward to verify directly that they commute interchanging the order of the integration with respect to dx_H and dz.

Considering the case $\Gamma_D \neq \emptyset$, it suffices to show that the resolvents of $\mathcal{L}_H$ and $\mathcal{L}_z$ commute by Trotter's approximation formula [26, Theorem 3.4.4]. We now prove that their resolvents commute or, equivalently $\rho(\mathcal{L}_H) \cap \rho(\mathcal{L}_z) \neq \emptyset$, that is, for $\lambda \in \rho(\mathcal{L}_H)$

$$(\mathcal{L}_H - \lambda)^{-1} D(\mathcal{L}_z) \subset D(\mathcal{L}_z) \quad \text{and} \quad (\mathcal{L}_H - \lambda)^{-1} \mathcal{L}_z v = \mathcal{L}_z (\mathcal{L}_H - \lambda)^{-1} v \tag{3.5}$$

for $v \in D(\mathcal{L}_z)$; see, *e.g.*, [1, Section 4.2]. Here, the resolvent of $\mathcal{L}_H$ can be given explicitly by the Laplace transform

$$(\mathcal{L}_H - \lambda)^{-1} v = \int_0^\infty e^{-\lambda t} T_H(t) v \, dt \quad \text{for} \quad \lambda > 0, \tag{3.6}$$

since $\mathcal{L}_H$ generates a quasi-contractive (C_0)-semigroup in $L^p(\Omega)^2$. In [24, Lemma 3.1] $\mathcal{L}'_H v = \mathcal{L}_H v + M x_H$ is considered, which is a given as integral kernel only in x_H direction, we thus have

$$T_H(t) \partial_z^2 v = \partial_z^2 T_H(t) v, \quad v \in D(\mathcal{L}_z), \quad t \geq 0,$$

since by the arguments of a relatively bounded perturbation one has $D(\mathcal{L}_z) = D(\mathcal{L}_z^0)$. Using (3.6), we see

$$(\mathcal{L}_H - \lambda)^{-1} \partial_z^2 v = \partial_z^2 (\mathcal{L}_H - \lambda)^{-1} v \in L^p(\Omega)^2 \quad \text{for} \quad v \in D(\mathcal{L}_z).$$

Therefore, it holds that

$$(\mathcal{L}_H - \lambda)^{-1} D(\mathcal{L}_z) \subset D(\mathcal{L}_z).$$

We calculate further the horizontal derivatives of $v(e^{tM} x_H)$. To shorten the notation, we often omit z. By the chain rule we get

$$\partial_x \left[v_1\big(e^{tM} x_H\big) \right] = \Big\langle \partial_x e^{tM} x_H, (\nabla v_1)(e^{tM} x_H) \Big\rangle = \left\langle e^{tM} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (\nabla v_1)\big(e^{tM} x_H\big) \right\rangle \tag{3.7}$$

and

$$\partial_y \left[v_1\big(e^{tM} x_H\big) \right] = \left\langle e^{tM} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, (\nabla v_1)\big(e^{tM} x_H\big) \right\rangle \tag{3.8}$$

for a scalar function v_1, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$. So, we see that

$$\nabla_H \left[v_1\big(e^{tM} x_H\big) \right] = \big(e^{tM}\big)^T (\nabla_H v_1)\big(e^{tM} x_H\big). \tag{3.9}$$

The same holds for v_2. Let $\nabla_H v := (\nabla_H v_1 \;\; \nabla_H v_2)^T$ for $v = (v_1, v_2)$, so we write

$$\nabla_H \left[v\big(e^{tM} x_H\big) \right] = \left[\big(e^{tM}\big)^T \big(\nabla v_1 \;\; \nabla v_2\big)\big(e^{tM} x_H\big) \right]^T = (\nabla_H v)(e^{tM} x_H) e^{tM}.$$

For checking that $T_H(\cdot)$ restricts to a semigroup on $L^p_{\overline{\sigma}}(\Omega)$, we now compute

$$\begin{aligned}
&\operatorname{div}_H \overline{(T_H(t)v)}(x_H)\\
&= \frac{1}{4\pi(\det Q_t)^{\frac12}} \operatorname{div}_H e^{-tM} \int_{\mathbb{R}^2} \overline{v}\big(e^{tM}x_H - x'_H\big) e^{-\frac14 \langle Q_t^{-1}x'_H, x'_H\rangle} dx'_H\\
&= \frac{1}{4\pi(\det Q_t)^{\frac12}} \operatorname{tr} \nabla_H \left(e^{-tM} \int_{\mathbb{R}^2} \overline{v}\big(e^{tM}x_H - x'_H\big) e^{-\frac14 \langle Q_t^{-1}x'_H, x'_H\rangle} dx'_H \right)\\
&= \frac{1}{4\pi(\det Q_t)^{\frac12}} \operatorname{tr} \left(e^{-tM} \int_{\mathbb{R}^2} (\nabla_H \overline{v})\big(e^{tM}x_H - x'_H\big) e^{-\frac14 \langle Q_t^{-1}x'_H, x'_H\rangle} dx'_H e^{tM} \right)\\
&= \frac{1}{4\pi(\det Q_t)^{\frac12}} \int_{\mathbb{R}^2} (\operatorname{div}_H \overline{v})\big(e^{tM}x_H - x'_H\big) e^{-\frac14 \langle Q_t^{-1}x'_H, x'_H\rangle} dx'_H\\
&= e^{tM} (T_H(t) \operatorname{div}_H \overline{v})(x_H).
\end{aligned}$$

Here we have used the representation $\operatorname{div}_H \overline{v} = \operatorname{tr} \nabla_H \overline{v}$ and with a slightly abusive notation $\operatorname{div}_H \overline{(T_H(t)v)}$ involves the actual Ornstein-Uhlenbeck semigroup in $L^p(\Omega)^2$, while $T_H(t)\operatorname{div}_H \overline{v}$ uses the corresponding semigroup in $L^p(\mathbb{R}^2)^2$. Since e^{tM} and $T_H(t)$ are boundedly invertible, one can conclude form the above identity that $\operatorname{div}_H \overline{(T_H(t)v)} = 0$ if and only if $\operatorname{div}_H \overline{v} = 0$. Hence, $T_H(t)$ maps onto $L^p_{\overline{\sigma}}(\Omega)$ or, equivalently expressed in terms of projection $P_p T_H(t) = T_H(t) P_p$. Therefore, the same holds also for the complementary space $(\mathbb{1} - P_p) T_H(t) = T_H(t)(\mathbb{1} - P_p)$.

Concerning $T_z(\cdot)$, we note that $\mathcal{L}_z^0$ is boundedly invertible if $\Gamma_D \neq \emptyset$. Since $\mathcal{L}_z$ is a bounded perturbation, we take $\lambda > 0$ sufficiently large such that the operator $(\lambda - \mathcal{L}_z)$ is boundedly invertible as well as

$$\begin{aligned}
\operatorname{div}_H \overline{(\lambda - \mathcal{L}_z)v} &= \lambda \operatorname{div}_H \overline{v} - \operatorname{div}_H \left(\frac{1}{2h} \int_{-h}^{h} \partial_z^2 v + (\mathbb{1} - P_p) \operatorname{tr}_D(\partial_z v) \right)\\
&= \lambda \operatorname{div}_H \overline{v} - \operatorname{div}_H P_p \operatorname{tr}_D(\partial_z v) = \lambda \operatorname{div}_H \overline{v}.
\end{aligned}$$

This gives that $\operatorname{div}_H \overline{(\lambda - \mathcal{L}_z)v} = 0$ if and only if $\operatorname{div}_H \overline{v} = 0$. Similarly, we have

$$(\lambda - \mathcal{L}_z)\nabla_H g = \lambda \nabla_H g$$

for $g \in H^{1,p}(\mathbb{R}^2)$. Hence, the operator $(\lambda - \mathcal{L}_z)$ is mapping divergence free fields as well as gradient fields onto. Therefore, the semigroup T_z generated by $\mathcal{L}_z$ restricts to (C_0)-semigroups on these invariant subspaces; see, *e.g.*, [26, Theorem 4.5.5].

In order to verify the commutator relation (3.5), we recall that $\mathcal{L}_z = \mathcal{L}_z^0 + (\mathbb{1} - P_p)(\operatorname{tr}_D \partial_z v)$ and the arguments above by the Fourier representation (3.4). Finally, we conclude that $\mathcal{L}_z^0$ as well as tr_D commute with $T_H(t)$ for $t > 0$, and $(\mathbb{1} - P_p)$ commutes with $(\mathcal{L}_H - \lambda)^{-1}$, since $T_H(\cdot)$ defines a (C_0)-semigroup on the two dimensional gradient fields $\operatorname{Ran}(\mathbb{1} - P_p)$. □

Consequently, we appeal to Trotter's results [29, Theorem 1], that is, if $T_A(t) = e^{tA}$ and $T_B(s) = e^{sB}$ commute for all $s, t > 0$, then the closure $\overline{A+B}$ generates the semigroup defined by $T(t) = e^{tA}e^{tB}$.

Lemma 3.2. *The sum $\mathcal{L}' = \mathcal{L}_H + \mathcal{L}_z$ with $D(\mathcal{L}') = D(\mathcal{L}_H) \cap D(\mathcal{L}_z)$ is closed and then $\mathcal{L}' = \mathcal{L}_p$. Especially, $\mathcal{L}_p$ is the generator of the semigroup $T(t) := T_H(t)T_z(t)$ in $L^p(\Omega)$ for $p \in (1, \infty)$.*

Proof. First one proves that $D(\mathcal{L}_H) \cap D(\mathcal{L}_z) = D(\mathcal{L}_p)$. Note that with Δ_H defined on $D(\Delta_H) = L^p_z H^{2,p}_{xy}$ we see

$$D(\mathcal{L}_H) = D(\Delta_H) \cap \left\{ v \in L^p(\Omega)^2 : Mx_H \cdot \nabla_H v \in L^p(\Omega)^2 \right\},$$

$$D(\mathcal{L}_p) = \left\{ v \in H^{2,p}(\Omega)^2 : \partial_z u|_{\Gamma_N} = 0,\ u|_{\Gamma_D} = 0 \right\} \cap \left\{ v \in L^p(\Omega)^2 : Mx_H \cdot \nabla_H v \in L^p(\Omega)^2 \right\}.$$

Hence, it is sufficient to prove that

$$L^p_z H^{2,p}_{xy} \cap \left\{ v \in H^{2,p}_z L^p_z : \partial_z u|_{\Gamma_N} = 0,\ u|_{\Gamma_D} = 0 \right\} = \left\{ v \in H^{2,p}(\Omega)^2 : \partial_z u|_{\Gamma_N} = 0,\ u|_{\Gamma_D} = 0 \right\}.$$

The inclusion "$\supset$" certainly holds by (3.1). Besides, for the converse "$\subset$" one has to prove for mixed derivatives $\nabla_H \partial_z v \in L^p(\Omega)^{2\times 2}$. Recall $D(\mathcal{L}_z) = D(\mathcal{L}^0_z)$ and the Fourier representation in $\mathbb{R}^2 \times S^1$ to see

$$v(x_H, z) := \int_{\mathbb{R}^2} \sum_{k\in\mathbb{N}_0} e^{i\xi_H \cdot x_H} \hat{v}_k(\xi_H) \cos\left(\frac{hkz}{\pi}\right) d\xi_H, \tag{3.10}$$

where

$$\hat{v}_k(\xi_H) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\xi_H \cdot x_H} \frac{1}{2h} \int_{-h}^{h} e^{-ik\frac{\pi}{h}z} v(x_H, z) dz dx_H.$$

Taking odd or even parts iteratively, one arrives at Dirichlet, Neumann and mixed boundary conditions. So, we can show that there exists a constant $C > 0$ such that

$$\|\nabla_H \partial_z v\|_{L^p(\Omega)} \leq C\|\Delta_H v\|_{L^p(\Omega)} + \left\|\partial_z^2 v\right\|_{L^p(\Omega)},$$

which proves the claim.

It is known that Ornstein-Uhlenbeck operators in the whole space are closed; see [24, Proposition 3.2] and [25, Theorem 4.1]. The proofs are based on the local elliptic regularity and the results by Dore and Venni [5] on closedness of commuting operators which uses bounded imaginary powers, respectively. To prove that the

Ornstein-Uhlenbeck operators $\mathcal{L}_p$ in the layer Ω is closed, we assume that there is a sequence $(v_n) \in D(\mathcal{L}_p)$ such that both (v_n) and $(w_n) := (\mathcal{L}_p v_n)$ are Cauchy sequences in $L^p(\Omega)^2$. Take a smooth partition of unity $\mathbb{1}_{[-h,h]}$ as

$$\begin{aligned} &\phi_u, \phi_b : [-h, h] \to [0, 1], && \phi_u + \phi_b \equiv 1, \\ &\operatorname{supp} \phi_u \subset (-h/2, h], && \operatorname{supp} \phi_b \subset [-h, h/2). \end{aligned}$$

Put $(v_n^u) := (\phi_u v_n)$, $(v_n^b) := (\phi_b v_n)$, $(w_n^u) := (\phi_u w_n)$ and $(w_n^b) := (\phi_b w_n)$ in $L^p(\Omega)^2$. We choose the alternative extension E by the odd and even reflection for Dirichlet and Neumann conditions, respectively, to extend these to sequences $((Ev)_n^u) = (E\phi_u v_n)$, $((Ev)_n^b) = (E\phi_b v_n)$, $((Ew)_n^u) = (E\phi_u w_n)$ and $((Ew)_n^b) = (E\phi_b w_n)$ in $L^p(\mathbb{R}^3)^2$ with supports in $\mathbb{R}^2 \times (-2h, 2h)$. Note that $E\phi_u w_n$ and $E\phi_b w_n$ are in the domain of the Ornstein-Uhlenbeck operator on $\mathbb{R}^3$ since the extension by reflexion the extended sequences satisfy the original boundary conditions as well. Hence, the closedness can be drawn back to the case in the whole space. Using the fact that the traces are bounded with respect to the graph norm of $\mathcal{L}_p$ which is stronger than $H^{2,p}$ norm, we guarantee that boundary conditions are preserved. □

Combing Lemma 3.1 and Lemma 3.2, we state the following proposition.

Proposition 3.3. *The restriction of $\mathcal{L}_p$ to $L^p_{\overline{\sigma}}(\Omega)$ is the generator of the (C_0)-semigroup defined by $T(t) = T_H(t)T_z(t)$ in $L^p_{\overline{\sigma}}(\Omega)$.*

4. Smoothing properties

In order to apply a Fujita-Kato type iteration as in [12], we need some smoothing properties of $T(t)$.

Proposition 4.1. *Let $p \in (1, \infty)$, $T > 0$ and $\theta_1, \theta_2 \in [0, 1]$ with $\theta_1 + \theta_2 \le 1$. There exists a constant $C > 0$ depending only on T and p such that*

$$\left\| e^{t\mathcal{L}_p} f \right\|_{H^{2\theta_1+2\theta_2,p}(\Omega)} \le Ct^{-\theta_1} \| f \|_{H^{2\theta_2,p}(\Omega)} \quad \textit{for} \quad t \in (0, T) \tag{4.1}$$

and $f \in H^{2\theta_2,p}(\Omega)^2$. If in addition $\theta_1 > 0$, then

$$\lim_{t \to 0+} t^{\theta_1} \left\| e^{t\mathcal{L}} f \right\|_{H^{2\theta_1+2\theta_2,p}(\Omega)} = 0.$$

Proof. For the first inequality with $\theta_1 = 1$ and $\theta_2 = 0$ we compute

$$\begin{aligned} \left\| e^{t\mathcal{L}_p} f \right\|_{H^{2,p}(\Omega)} &\le C \left\| \Delta e^{t\mathcal{L}_p} f \right\|_{H^{2,p}(\Omega)} \\ &\le C \left\| \Delta_H T_H(t) T_z(t) f \right\|_{L^p(\Omega)} + C \left\| \Delta_z T_z(t) T_H(t) f \right\|_{L^p(\Omega)}, \end{aligned}$$

for some $C > 0$ and with $\Delta = \Delta_H + \Delta_z$. By analyticity of $T_z(t) = e^{t\mathcal{L}_z}$, one already has

$$\|\Delta_z T_z(t) T_H(t) f\|_{L^p(\Omega)} \leq C t^{-1} \|T_H(t) f\|_{L^p(\Omega)}$$

for $t \in (0, T)$ with some $C > 0$. Considering $\Delta_H T_H(t) T_z(t) f$, we use the explicit representation of the Kolmogrov kernel (3.2) to follow the line of [15, Proposition 3.4] in the case $n = 2$ for $g := T_z(t) f$. From (3.7), (3.8) and (3.9) we deduce for any scalar valued function ψ

$$\begin{aligned}\Delta_H \left[\psi\left(e^{tM} x_H\right)\right] &= \operatorname{div}_H \nabla_H \left[\psi\left(e^{tM} x_H\right)\right] \\ &= \operatorname{div}_H \left[\left(e^{tM}\right)^T \nabla_H \psi\right]\left(e^{tM} x_H\right) \\ &= \left[\operatorname{div}_H e^{t(M^T+M)} \nabla_H \psi\right]\left(e^{tM} x_H\right).\end{aligned}$$

Putting $A_H := \operatorname{div}_H e^{t(M^T+M)} \nabla_H$, we thus get

$$\begin{aligned}&\Delta_H T_H(t) g(x_H, z) \\ &= \frac{1}{4\pi(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^2} (\Delta_H)_{x_H} g\left(e^{tM} x_H - x'_H, z\right) e^{-\frac{1}{4}\langle Q_t^{-1} x'_H, x'_H\rangle} dx'_H \\ &= \frac{1}{4\pi(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^2} (A_H)_{x_H} g\left(e^{tM} x_H - x'_H, z\right) e^{-\frac{1}{4}\langle Q_t^{-1} x'_H, x'_H\rangle} dx'_H \\ &= \frac{1}{4\pi(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^2} (A_H)_{x'_H} g\left(e^{tM} x_H - x'_H, z\right) e^{-\frac{1}{4}\langle Q_t^{-1} x'_H, x'_H\rangle} dx'_H \\ &= \frac{1}{4\pi(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^2} g\left(e^{tM} x_H - x'_H, z\right) (A_H)_{x'_H} e^{-\frac{1}{4}\langle Q_t^{-1} x'_H, x'_H\rangle} dx'_H \\ &= \frac{1}{4\pi(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^2} g\left(e^{tM} x_H - x'_H, z\right) e^{-\frac{1}{4}\langle Q_t^{-1} x'_H, x'_H\rangle} \\ &\quad \cdot \frac{-1}{2}\left[\operatorname{tr}\left(e^{t(M^T+M)} Q_t^{-1}\right) - \frac{1}{2}\left\langle e^{t(M^T+M)} Q_t^{-1} x'_H, Q_t^{-1} x'_H\right\rangle\right] dx'_H.\end{aligned} \tag{4.2}$$

We substitute $x'_H = Q_t^{1/2} y'_H$ to give

$$\begin{aligned}\Delta_H T_H(t) g(x_H, z) &= \frac{-1}{8\pi} e^{-tM} \int_{\mathbb{R}^2} g\left(e^{tM} x_H - Q_t^{-\frac{1}{2}} y'_H, z\right) e^{-\frac{1}{4}\langle y'_H, y'_H\rangle} \\ &\quad \cdot \left[\operatorname{tr}\left(e^{t(M^T+M)} Q_t^{-1}\right) - \frac{1}{2}\left\langle e^{t(M^T+M)} Q_t^{-1} y'_H, y'_H\right\rangle\right] dy'_H \\ &= \frac{-1}{8\pi} e^{-tM} \int_{\mathbb{R}^2} g\left(Q_t^{-\frac{1}{2}}\left(Q_t^{\frac{1}{2}} e^{tM} x_H - y'_H\right), z\right) e^{-\frac{1}{4}\langle y'_H, y'_H\rangle} \\ &\quad \cdot \left[\operatorname{tr}\left(e^{t(M^T+M)} Q_t^{-1}\right) - \frac{1}{2}\left\langle e^{t(M^T+M)} Q_t^{-1} y'_H, y'_H\right\rangle\right] dy'_H \\ &= \left(g\left(Q_t^{-\frac{1}{2}} \cdot, z\right) *_H \Psi\right)\left(Q_t^{1/2} e^{tM} x_H\right).\end{aligned}$$

Here

$$\Psi(x_H) := \frac{-1}{8\pi} e^{-tM} e^{-\frac{1}{4}\langle x_H, x_H\rangle} \left[\operatorname{tr}\left(e^{t(M^T+M)} Q_t^{-1}\right) - \frac{1}{2}\left\langle e^{t(M^T+M)} Q_t^{-1} x_H, x_H \right\rangle \right].$$

By direct calculation we see

$$|\Psi(x_H)| \le \frac{1}{16\pi} \left\| e^{t(M^T+M)} Q_t^{-1} \right\| \cdot \left\| e^{-tM} \right\| \cdot \left(2 + |x_H|^2\right) e^{-\frac{1}{4}|x_H|^2},$$

and by Young's inequality,

$$\begin{aligned}
&\|\Delta_H T_H(t) g(x_H, z)\|_{L^p(\Omega)} \\
&= \left\| \left(g\left(Q_t^{-1/2}\cdot, \cdot\right) *_H \Psi \right) \left(Q_t^{1/2} e^{tM} \cdot \right) \right\|_{L^p(\Omega)} \\
&= \det\left(Q_t^{1/2} e^{tM}\right)^{1/p} \left\| g\left(Q_t^{-1/2}\cdot, \cdot\right) *_H \Psi \right\|_{L^p(\Omega)} \\
&\le \det\left(Q_t^{1/2} e^{tM}\right)^{1/p} \left\| g\left(Q_t^{-1/2}\cdot, \cdot\right) \right\|_{L^q(\Omega)} \|\Psi\|_{L^r(\Omega)} \\
&= \det\left(Q_t^{1/2} e^{tM}\right)^{1/p} \left(\det Q_t^{-1/2}\right)^{1/q} \|g\|_{L^q(\Omega)} \|\Psi\|_{L^r(\Omega)} \\
&\le C_r e^{t\cdot \operatorname{tr} M/p} (\det Q_t)^{\frac{1}{2p} - \frac{1}{2q}} \|e^{t(M^T+M)} Q_t^{-1}\| \cdot \|e^{-tM}\| \cdot \|g\|_{L^q(\Omega)} \\
&\le C_r e^{t\cdot \operatorname{tr} M/p} (\det Q_t)^{\frac{1}{2p} - \frac{1}{2q}} \|e^{tM}\|^2 \cdot \|e^{-tM}\| \cdot \|Q_t^{-1}\| \cdot \|g\|_{L^q(\Omega)}
\end{aligned}$$

for $1/p + 1 = 1/q + 1/r$, where $C_r := \frac{1}{8\pi} \|(2 + |\cdot|^2) e^{-\frac{1}{4}|\cdot|^2}\|_{L^r(\Omega)}$. Since $\|Q_t\|^{-1} \le \frac{C}{t}$ for $t > 0$ with some C, we have

$$\|\Delta_H T_H(t) g(x_H, z)\|_{L^p(\Omega)} \le \frac{C}{t} \|g\|_{L^p(\Omega)}$$

for $t \in (0, T)$ by choosing $q = p$ and $r = 1$.

For the case $\theta_1 = 0, \theta_2 = 1$ we obtain as above by the analyticity of $T_z(t)$

$$\left\| e^{t\mathcal{L}_p} f \right\|_{H^{2,p}(\Omega)} \le \|\Delta_H T_H(t) T_z(t) f\|_{L^p(\Omega)} + C \|T_H(t) f\|_{H^{2,p}(\Omega)}. \tag{4.3}$$

Putting $g := T_z(t) f$, we have from (4.2)

$$\begin{aligned}
&\Delta_H T_H(t) g(x_H, z) \\
&= \frac{1}{4\pi (\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^2} [(A_H)_{x_H} g]\left(e^{tM} x_H - x'_H, z\right) e^{-\frac{1}{4}\langle Q_t^{-1} x'_H, x'_H\rangle} dx'_H \\
&= (A_H g *_H k_t)\left(e^{tM} x_H, z\right).
\end{aligned}$$

Thus, it follows that

$$
\begin{aligned}
\|\Delta_H T_H(t) T_z(t) f\|_{L^p(\Omega)}^p &= \int_{-h}^{h} \big\|\big(A_H g *_H k_t\big)\big(e^{tM}\cdot, z\big)\big\|_{L^p(\mathbb{R}^2)}^p dz \\
&= e^{t\cdot \operatorname{tr} M} \int_{-h}^{h} \|(A_H g *_H k_t)(\cdot, z)\|_{L^p(\mathbb{R}^2)}^p dz \\
&\le e^{t\cdot \operatorname{tr} M} \int_{-h}^{h} \|A_H g(\cdot, z)\|_{L^p(\mathbb{R}^2)}^p \|k_t\|_{L^1(\mathbb{R}^2)}^p dz \\
&= e^{t\cdot \operatorname{tr} M} \big\|e^{-tM}\big\| \int_{-h}^{h} \|A_H g(\cdot, z)\|_{L^p(\mathbb{R}^2)}^p dz \\
&= e^{t\cdot \operatorname{tr} M} \big\|e^{-tM}\big\| \cdot \|A_H g\|_{L^p(\Omega)}^p \\
&\le e^{t\cdot \operatorname{tr} M} \big\|e^{-tM}\big\| \cdot \big\|e^{t(M^T+M)}\big\| \cdot \|g\|_{H^{2,p}(\Omega)}^p \\
&\le C\|f\|_{H^{2,p}(\Omega)}
\end{aligned}
$$

by the analyticity of $T_z(t)$. The second term in (4.3) can be handled similarly as

$$
\begin{aligned}
\|T_H(t) f\|_{H^{2,p}(\Omega)} &\le \|\Delta_H T_H(t) f\|_{L^p(\Omega)} + \|T_H(t)\partial_{zz} f\|_{L^p(\Omega)} \\
&\le C\|f\|_{H^{2,p}(\Omega)} + \|\partial_{zz} f\|_{L^p(\Omega)} \\
&\le C\|f\|_{H^{2,p}(\Omega)}.
\end{aligned}
$$

The statement for $\theta_1, \theta_2 \in (0, 1)$ follows from the interpolation of the inequalities

$$
\big\|e^{t\mathcal{L}_p} f\big\|_{H^{2,p}(\Omega)} \le Ct^{-1}\|f\|_{L^p(\Omega)} \quad \text{and} \quad \big\|e^{t\mathcal{L}_p} f\big\|_{H^{2,p}(\Omega)} \le C\|f\|_{H^{2,p}(\Omega)}.
$$

For the limit $t \to 0+$, we apply to an approximation argument as $g_n \in C_c(\Omega)$ for $n \in \mathbb{N}$ such that $g_n \to g$ in $H^{2\theta_2,p}(\Omega)$. So, we see

$$
\begin{aligned}
&t^{\theta_1}\|T_H(t) g(x_H, z)\|_{H^{2\theta_1+2\theta_2,p}(\Omega)} \\
\le& t^{\theta_1}\|T_H(t)(g(x_H, z) - g_n(x_H, z))\|_{H^{2\theta_1+2\theta_2,p}(\Omega)} + t^{\theta_1}\|T_H(t) g_n(x_H, z)\|_{H^{2\theta_1+2\theta_2,p}(\Omega)} \\
\le& C\|g(x_H, z) - g_n(x_H, z)\|_{H^{2\theta_2,p}(\Omega)} + t^{\theta_1}\|T_H(t) g_n(x_H, z)\|_{H^{2,p}(\Omega)} \\
\le& C\|g(x_H, z) - g_n(x_H, z)\|_{H^{2\theta_2,p}(\Omega)} + t^{\theta_1}\|g_n(x_H, z)\|_{H^{2,p}(\Omega)}.
\end{aligned}
$$

We firstly choose n sufficiently large so that the first term in the right hand side of the last inequality becomes small, and secondly take the limit $t \to 0+$. This completes the proof. □

Remark 4.2. If $\operatorname{tr} M = 0$ and M has two purely imaginary eigenvalues, then the constant C in (4.1) is independent of T. Moreover, if M is anti-symmetric, then $C = 1$.

5. Mild solutions

On the primitive equations in the L^p framework with decaying initial data, time-local unique mild solutions have been constructed in [12] adapting the Fujita-Kato scheme in the spaces

$$V_{\theta,p} := \big[L^p_{\overline{\sigma}}(\Omega), D(A_p)\big]_\theta \subset H^{2\theta,p}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)$$

for $p \in (1,\infty)$ with some $\theta \in (0,1)$, where $[\cdot,\cdot]_\theta$ denotes by the complex interpolation; see [12, Equation (4.10)]. In what follows, the initial disturbance v_0 is taken from

$$V_{\frac{1}{p},p} := \Big\{v \in H^{\frac{2}{p},p}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v|_{\Gamma_D} = 0\Big\}$$

as in [13, Section 4]; where the reader can find the explicit characterization of the interpolation space. Besides, we deal with mild solutions v in

$$V_{\frac{1}{2}+\frac{1}{2p},p} \subset H^{1+\frac{1}{p},p}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) =: H^{1+\frac{1}{p},p}_{\overline{\sigma}}(\Omega)$$

for each t, whence the mild solution exists. Here, we appeal to the Fujita-Kato scheme directly in Sobolev spaces rather than in interpolation spaces, because the semigroup generated by $\mathcal{L}_p$ is neither analytic nor enjoying L^p-L^q smoothing properties onto the operator domain of $\mathcal{L}_p$. So, it is of benefit to argue in the Sobolev spaces $H^{1+\frac{1}{p},p}_{\overline{\sigma}}(\Omega)$; see Proposition 4.1.

We consider the non-linear and remainder terms

$$F_p(v) := -P_p\,(v\cdot\nabla_H v + w\partial_z v + 2Mv) \tag{5.1}$$

rewritten as $F_p(v) = F_p^*(v) - 2P_pMv$ with $F_p^*(v) := -P_p(v\cdot\nabla_H v + w\partial_z v)$. In [12, Lemma 5.1 (a)] it has been derived the estimate

$$\|F_p^*(v)\|_{H^{1+\frac{1}{p},p}} \le C\|v\|^2_{H^{1+\frac{1}{p},p}} \tag{5.2}$$

with some $C > 0$ for the case of a bounded cylindrical domain. The estimate (5.2) is still valid in an infinite layer domain, since by anisotropic Hölder estimates

$$\begin{aligned}\|w\partial_z v\|_{L^p(\Omega)} &\le \|w\|_{L^\infty_z L^{2p}_{xy}}\|\partial_z v\|_{L^p_z L^{2p}_{xy}} \le C\|w\|_{H^{1,p}_z L^{2p}_{xy}}\|v\|_{H^{1,p}_z L^{2p}_{xy}}\\ &\le C\|\mathrm{div}_H\, v\|_{L^p_z L^{2p}_{xy}}\|v\|_{H^{1,p}_z H^{\frac{1}{p},p}_{xy}} \le C\|v\|_{L^p_z H^{1,2p}_{xy}}\|v\|_{H^{1,p}_z H^{\frac{1}{p},p}_{xy}}\\ &\le C\|v\|^2_{H^{1+\frac{1}{p},p}}\end{aligned}$$

with $C > 0$ being a universal constant; recall that

$$w = w(v) = -\int_{-h}^{z} \mathrm{div}_H\, v(x,y,\xi,t)\,d\xi.$$

Here we have used the Sobolev embeddings

$$\begin{aligned} &H^{1,p}(-h,h) \hookrightarrow L^\infty(-h,h), \quad H^{1+\frac{1}{p},p}\big(\mathbb{R}^2\big) \hookrightarrow H^{1,2p}\big(\mathbb{R}^2\big), \\ &H^{\frac{1}{p},p}\big(\mathbb{R}^2\big) \hookrightarrow L^{2p}\big(\mathbb{R}^2\big) \end{aligned}$$

by, *e.g.*, [28, Theorem 3.3.1 and Theorem 2.7.1] and the Poincaré inequality $\|w\|_{H_z^{1,p}} \le C\|\partial_z w\|_{L_z^p}$ applied to w and (3.1). Similarly, the estimate for the term is derived as

$$\begin{aligned} \|v\cdot\nabla_H v\|_{L^p(\Omega)} &\le \|v\|_{L_z^\infty L_{xy}^{2p}} \|\nabla_H v\|_{L_z^p L_{xy}^{2p}} \le C\|v\|_{H_z^{1,p} L_{xy}^{2p}} \|v\|_{L_z^p H_{xy}^{1,2p}} \\ &\le C\|v\|_{H_z^{1,p} H_{xy}^{\frac{1}{p},p}} \|v\|_{L_z^p H_{xy}^{1+\frac{1}{p},p}} \le C\|v\|^2_{H^{1+\frac{1}{p},p}}. \end{aligned}$$

Since $-2P_p Mv$ is a linear term, analogously to [12, Lemma 5.1] we can state the following lemma.

Lemma 5.1. *For $p \in (1,\infty)$, the operator F_p maps from $H^{1+\frac{1}{p}}(\Omega)^2$ into $L^p_{\overline{\sigma}}(\Omega)$, and there exists a constant $C > 0$ such that the following two estimates hold:*

(a) *For $v \in H^{1+\frac{1}{p}}(\Omega)^2$*

$$\|F_p(v)\|_{L^p_{\overline{\sigma}}(\Omega)} \le C\left(\|v\|^2_{H^{1+\frac{1}{p}}(\Omega)^2} + \|v\|_{H^{1+\frac{1}{p}}(\Omega)^2}\right);$$

(b) *For $v, v_\flat \in H^{1+\frac{1}{p}}(\Omega)^2$*

$$\|F_p(v)-F_p(v_\flat)\|_{L^p_{\overline{\sigma}}(\Omega)} \le C\left(\|v\|_{H^{1+\frac{1}{p}}(\Omega)} + \|v_\flat\|_{H^{1+\frac{1}{p}}(\Omega)} + 1\right)\|v - v_\flat\|_{H^{1+\frac{1}{p}}(\Omega)}.$$

Let $T > 0$, and define the space

$$\begin{aligned} S_T := \Big\{ &v \in C^0\left([0,T]; V_{\frac{1}{p},p}\right) \cap C^0\left((0,T]; H_{\overline{\sigma}}^{1+\frac{1}{p},p}(\Omega)^2\right): \\ &\|v(t)\|_{H^{1+\frac{1}{p},p}} = o\left(t^{\frac{1}{2p}-\frac{1}{2}}\right) \text{ as } t \to 0 \Big\}. \end{aligned}$$

This becomes a Banach space equipped with the norm

$$\|v\|_{S_T} := \sup_{0\le t\le T} \|v(t)\|_{H^{\frac{2}{p},p}} + \sup_{0\le t\le T} t^{\frac{1}{2}-\frac{1}{2p}} \|v(t)\|_{H^{1+\frac{1}{p}}}.$$

The function $v \in C([0,T]; V_{\frac{1}{p},p})$ is called a *mild solution* to the primitive equations with linearly growing data, if v satisfies

$$v(t) = e^{t\mathcal{L}_p} v_0 + \int_0^t e^{(t-s)\mathcal{L}_p}\left(P_p f(s) + F_p(v(s))\right) ds \quad \text{for} \quad t \in [0,T].$$

Theorem 5.2. *Let $p \in (1, \infty)$ and $T > 0$. Assume that $v_0 \in V_{\frac{1}{p},p}$ and $P_p f \in C^0((0, T]; L^p_\sigma(\Omega))$ satisfying*

$$\|P_p f(t)\|_{L^p_\sigma(\Omega)} = o\left(t^{\frac{1}{p}-1}\right) \quad as\, t \to 0.$$

Then there exists $T_\sharp \in (0, T)$ and a unique mild solution $v \in S_{T_\sharp}$. If in addition $v_0 \in V_{\frac{1}{p}+\varepsilon,p}$ for some $\varepsilon \in (0, \frac{1}{2} - \frac{1}{2p}]$, then

$$v \in C^0\left([0, T_\sharp]; V_{\frac{1}{p}+\varepsilon,p}\right) \cap C^0\left((0, T_\sharp]; H^{1+\frac{1}{p},p}_{\overline{\sigma}}(\Omega)\right).$$

Proof. The proof is essentially based on that of [12, Proposition 5.2], using Proposition 4.1 and Lemma 5.1. We begin with the recursive sequence $(v_m)_{m\in\mathbb{N}}$ defined as

$$v_1(t) := e^{t\mathcal{L}_p} v_0 + \int_0^t e^{(t-s)\mathcal{L}_p} P_p f(s) ds \quad \text{and}$$
$$v_{m+1}(t) := v_1(t) + \int_0^t e^{(t-s)\mathcal{L}_p} F_p(v_m(s)) ds.$$

To shorten the notation, put $\gamma := \frac{1}{2} + \frac{1}{2p} \in (0, 1)$ and $V_\gamma := V_{\gamma,p}$. As in [12, Proposition 5.2] we may inductively prove that this sequence is well-defined in S_T, and that it converges in this space to prove the following two properties:

(a) There exists $T_\sharp \in (0, T]$ such that $\|v_m\|_{S_{T_\sharp}}$ is bounded uniformly in $m \in \mathbb{N}$;
(b) Let $d_m := v_{m+1} - v_m$, then there exists a constant $C_\sharp \in (0, 1)$ such that

$$\sup_{0\le t\le T_\sharp} t^{1-\gamma} \|d_{m+1}(t)\|_{V_\gamma} \le C_\sharp \sup_{0\le t\le T_\sharp} t^{1-\gamma} \|d_m(t)\|_{V_\gamma} \quad \text{for} \quad m \in \mathbb{N}.$$

As usual, we conclude that the limit of this successive approximation is in fact a unique mild solution. So, our main task is to prove that

$$k_m(t) := \sup_{0\le s\le t} s^{1-\gamma} \|v_m\|_{H^{1+\frac{1}{p},p}}$$

is a bounded sequence uniformly in $m \in \mathbb{N}$ for t sufficiently small. By definition of v_1, we see that

$$\begin{aligned}
&\|v_1(t)\|_{H^{1+\frac{1}{p},p}} \\
&\le \left\|e^{t\mathcal{L}_p} v_0\right\|_{H^{1+\frac{1}{p},p}} + \int_0^t \left\|e^{(t-s)\mathcal{L}_p}\right\|_{\mathcal{L}(L^p_\sigma(\Omega), H^{1+\frac{1}{p},p}_{\overline{\sigma}})} \|P_p f(s)\|_{L^p_\sigma(\Omega)} ds \\
&\le C t^{\gamma-1} \|v_0\|_{H^{\frac{2}{p},p}} + C \sup_{0\le s\le t} s^{1-\frac{1}{p}} \|P_p f(s)\|_{L^p_\sigma(\Omega)} \int_0^t (t-s)^{-\gamma} s^{\frac{1}{p}-1} ds \\
&= C t^{\gamma-1} \|v_0\|_{H^{\frac{2}{p},p}} + C \sup_{0\le s\le t} s^{1-1/p} \|P_p f(s)\|_{L^p_\sigma(\Omega)} \cdot t^{1-\gamma} \int_0^1 (1-s)^{-\gamma} s^{1/p-1} ds
\end{aligned}$$

and we thus multiply $t^{1-\gamma}$ in both side to derive

$$k_1(t) \le C\|v_0\|_{H^{\frac{2}{p},p}} + CB\left(\frac{1}{2}-\frac{1}{2p},\frac{1}{p}\right)\sup_{0\le s\le t}\left(s^{1-\frac{1}{p}}\|P_p f(s)\|_{L^p_{\overline{\sigma}}(\Omega)}\right)$$

where $B(x,y)$ denotes by Euler's beta function. Here we have used Proposition 4.1 with $\theta_1 = \frac{1}{2}-\frac{1}{2p}, \theta_2 = \frac{1}{p}$ to estimate $\|e^{t\mathcal{L}_p}v_0\|_{H^{1+\frac{1}{p},p}}$ and $\theta_1 = \frac{1}{2}+\frac{1}{2p}, \theta_2 = 0$ to estimate $\|e^{(t-s)\mathcal{L}_p}\|_{\mathcal{L}\left(L^p_{\overline{\sigma}}(\Omega), H^{1+\frac{1}{p},p}_{\overline{\sigma}}\right)}$. By assumption one can confirm that $k_1(t) \le k_\flat$ for any small $k_\flat > 0$, if t is taken sufficiently small. Similarly, for $m \ge 2$ we have

$$k_{m+1}(t) \le k_1(t) + C\sup_{0\le s\le t}\left(s^{1-\gamma}\|v_m(s)\|_{H^{1+\frac{1}{p},p}}\right)^2 + Ct^{1-\gamma}\sup_{0\le s\le t}s^{1-\gamma}\|v_m(s)\|_{H^{1+\frac{1}{p},p}}$$

with some constant $C > 0$. We now obtain that

$$k_{m+1} \le k_\flat + C\left(k_m(t)^2 + t^{1-\gamma}k_m(t)\right).$$

Therefore, $k_m(t) \le 2k_\flat$ for all $m \in \mathbb{N}$, if we choose t small enough so that $Ct^{1-\gamma} \le \frac{1}{4}$ and $k_\flat \le \frac{1}{8C}$. Inductively we may check $\lim_{t\to 0^+} k_m(t) = 0$. The other properties such as uniqueness can be shown by minor adjustments as in [12, Section 5]. □

Remark 5.3. It is known that the Ornstein-Uhlenbeck semigroup is not analytic. So, it does not map after short time into its generators domain, but only into a Sobolev space. Therefore, it is not expected that the mild solution is a strong one. However, once we guarantee more smoothing on the semigroup, it might be possible to show that the mild solution v satisfies (2.4) in the classical sense as well as $V = v + Mx_h$ to (2.1).

References

[1] W. ARENDT, *Semigroups and evolution equations: functional calculus, regularity and kernel estimates*, In: "Evolutionary Equations", Vol. I, Handb. Differ. Equ., 2004, 1–85.

[2] A. BABIN, A. MAHALOV and B. NICOLAENKO, *3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity*, Indiana Univ. Math. J. **50** (2001), 1–35.

[3] M. CAMPITI, G. P. GALDI and M. HIEBER, *Global existence of strong solutions for 2-dimensional Navier-Stokes equations on exterior domains with growing data at infinity*, Comm. Pure Appl. Anal. **13** (2014), 1613–1627.

[4] CH. CAO and E. TITI, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math. **166** (2007), 245–267.

[5] G. DORE and A. VENNI, *On the closedness of the sum of two closed operators*, Math. Z. **196** (1987), 189–201.

[6] H. FUJITA and T. KATO, *On the Navier-Stokes initial value problem, I*, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[7] G. P. GALDI, M. HIEBER and T. KASHIWABARA, *Strong time-periodic solutions to the 3D primitive equations subject to arbitrary large forces*, Nonlinearity **30** (2017), 3979–3992.

[8] TH. GALLAY and Y. MAEKAWA, *Three-dimensional stability of Burgers vortices*, Comm. Math. Phys. **302** (2011), 477–511.

[9] Y. GIGA, M. GRIES, A. HUSSEIN, M. HIEBER and T. KASHIWABARA, *Bounded H^∞-calculus for the hydrostatic Stokes operator on L^p-spaces and applications*, Proc. Amer. Math. Soc. **145** (2017), 3865–3876.

[10] R. HALLER-DINTELMANN and J. WIEDL, *Kolmogorov kernel estimates for the Ornstein-Uhlenbeck operator*, Ann. Scuola Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 729–748,

[11] B. HAN, S. SHAO, S. WANG and W.-Q. XU, *Global existence for the 2D Navier-Stokes flow in the exterior of a moving or rotating obstacle*, Kinet. Relat. Models **9** (2016), 767–776.

[12] M. HIEBER and T. KASHIWABARA, *Global strong well-posedness of the three dimensional primitive equations in L^p-spaces*, Arch. Rational Mech. Anal. **221** (2016), 1077–1115.

[13] M. HIEBER, T. KASHIWABARA and A. HUSSEIN, *Global strong L^p well-posedness of the 3D primitive equations with heat and salinity diffusion*, J. Differential Equations **261** (2016), 6950–6981.

[14] M. HIEBER, A. RHANDI and O. SAWADA, *The Navier-Stokes flow for globally Lipschitz continuous initial data*, In: "RIMS Kôkyûroku Bessatsu", Vol. B1, 2007, 159–165.

[15] M. HIEBER and O. SAWADA, *The Navier-Stokes equations in $\mathbb{R}^n$ with linearly growing initial data*, Arch. Ration. Mech. Anal. **175** (2005), 269–285.

[16] T. HISHIDA, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Ration. Mech. Anal. **150** (1999), 307–348.

[17] T. KATO, *Strong L^p-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions*, Math. Z. **187** (1984), 471–480.

[18] J. LI and E. TITI, *Recent advances concerning certain class of geophysical flows*, In: "Handbook of Mathematical Analysis in Mechanics of Viscous Fluids", Springer International Publishing, 2016.

[19] J. L. LIONS, R. TEMAM and SH. H. WANG, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity **5** (1992), 237–288.

[20] J. L. LIONS, R. TEMAM and SH. H. WANG, *On the equations of the large-scale ocean*, Nonlinearity **5** (1992), 1007–1053.

[21] J. L. LIONS, R. TEMAM and SH. H. WANG, *Models for the coupled atmosphere and ocean*, (CAO I, II), Comput. Mech. Adv. **1** (1993), 3–119.

[22] A. MAJDA, *Vorticity and the mathematical theory of incompressible fluid flow*, Comm. Pure Appl. Math. **39** (1986) 187–220.

[23] A. MAJDA, "Introduction to PDEs and Waves for the Atmosphere and Ocean", Courant Lecture Notes in Mathematics Vol. 9, Providence, RI, American Mathematical Society, 2003.

[24] G. METAFUNE, *L^p-spectrum of Ornstein-Uhlenbeck operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), 97–124.

[25] G. METAFUNE, J. PRÜSS, A. RHANDI and R. SCHNAUBELT, *The domain of the Ornstein-Uhlenbeck operator on an L^p-space with invariant measure*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002), 471–485.

[26] A. PAZY, "Semigroups of Linear Operators and Applications to Partial Differential Equations", Springer, New York, 1983.

[27] J. PEDLOSKY, "Geophysical Fluid Dynamics", second edition, Springer, New York, 1987.

[28] H. TRIEBEL, " Theory of Function Spaces", (Reprint of 1983 edition) Springer AG, Basel, 2010.

[29] H. F. TROTTER, *On the product of semi-groups of operators*, Proc. Amer. Math. Soc. **10** (1959), 545–551.

[30] G. K. VALLIS, "Atmospheric and Oceanic Fluid Dynamics", second edition, Cambridge Univ. Press, 2006.
[31] W. M. WASHINGTON and C. L. PARKINSON, "An Introduction to Three Dimensional Climate Modeling", second edition, University Science Books, Oxford, 1986.
[32] J. WIEDL, "Analysis of Ornstein-Uhlenbeck Operators", PhD thesis, TU Darmstadt, 2007.

Departement of Mathematics
TU Darmstadt, Schlossgartenstr. 7
64289 Darmstadt, Germany
hussein@mathematik.tu-darmstadt.de
msaal@mathematik.tu-darmstadt.de

Applied Physics Course
Gifu University, Yanagido 1-1
501-1193 Gifu, Japan
okihiro@gifu-u.ac.jp