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The homotopy sequence for regular singular stratified bundles

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Abstract. A separable proper morphism of varieties with geometrically connected fibers induces a homotopy exact sequence relating the étale fundamental groups of source, target and fiber. Extending work of dos Santos, we prove the existence of an analogous homotopy exact sequence for fundamental group schemes classifying regular singular stratified bundles, under the additional assumption that the morphism in question can be (partially) compactified to a log smooth morphism.

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1. Introduction

As proved in [7], if $f : Y \to X$ is a separable, proper morphism of schemes with geometrically connected fibers, and if $\bar{y}$ is a geometric point of Y, then the étale fundamental groups of Y, X and of the fiber $Y_{f(\bar{y})}$ are related via the exact sequence

$$\pi_1(Y_{f(\bar{y})}, \bar{y}) \to \pi_1(Y, \bar{y}) \to \pi_1(X, f(\bar{y})) \to 1. \tag{1.1}$$

In this article, we establish the existence of an exact sequence analogous to (1.1) for the affine group schemes classifying regular singular stratified bundles (Section 2) on X, Y and $Y_{f(\bar{y})}$.

Sequences similar to (1.1) have been studied for many different kinds of fundamental groups (see for example [3,4,8,19,24,25]). This article was inspired by the following two particular examples.

- In [8], it is proved that if $f : Y \to X$ is a log smooth morphism of fs log schemes with X connected and log regular, then there is an exact sequence (1.1) with the log fundamental group (which can be seen as a generalization of the tame

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fundamental group, see [9, Example 4.7, (c)]) instead of the étale fundamental group;

- In [4], it is proved that if f is smooth, projective and if Y, X are smooth connected varieties over an algebraically closed field k of arbitrary characteristic, then there is a homotopy exact sequence (1.1) for the affine k-group schemes which are obtained via Tannaka duality from the category of stratified bundles, that is from the category of $\mathcal{O}$-coherent $\mathscr{D}$-modules.

The objects studied in this article morally lie in the intersection of the above examples, as the notion of regular singularity for an $\mathcal{O}$-coherent $\mathcal{D}$-module naturally specializes to the notion of a tamely ramified covering (see Section 2 for details and definitions).

The main result is as follows. If k is an algebraically closed field of positive characteristic and if X is a smooth k-variety, let us denote by $\Pi^{\mathrm{rs}}(X, x)$ the affine k-group scheme corresponding to the category of regular singular stratified bundles on X with respect to the base point $x \in X(k)$ (see Definition 2.1).

Main Theorem (see Theorem 6.1). *Let $f : Y \to X$ be a smooth, projective morphism of smooth, connected k-varieties with geometrically connected fibers, and assume that X admits a good compactification $\overline{X}$ (Definition* 2.1, (b)). *Assume furthermore that:*

(i) *There is a good partial compactification $Y \subseteq \overline{Y}$ such that f extends to $\bar{f} : \overline{Y} \to \overline{X}$ and $\bar{f}(\overline{Y})$ contains every codimension* 1 *point of $\overline{X}$;*

(ii) *$\bar{f}$ is log smooth with respect to the natural fs log structure induced on $\overline{X}$ and $\overline{Y}$ by their divisors at infinity* (*see Remark* 3.1).

Then, for every $x \in X(k)$ and $y \in Y_x(k)$, there is an exact sequence of k-group schemes

$$\Pi^{\mathrm{rs}}(Y_x, y) \xrightarrow{j^*} \Pi^{\mathrm{rs}}(Y, y) \xrightarrow{f^*} \Pi^{\mathrm{rs}}(X, x) \to 1.$$

Moreover, the theorem admits a refinement dealing with the notion of regular singularity with respect some specific good partial compactification of X, therefore we obtain information which is not conditional on the existence of a good compactification of X. We would like to stress that the log smoothness assumption in (ii) cannot be dropped entirely. Namely (see Remark 6.2), we prove that the example described in Section 4, which is due to Raynaud, provides a counterexample to the exactness of the homotopy sequence in general.

The article is organized as follows. In Section 2 we recall the definitions of stratified bundles and of regular singularity; in Section 3 we recall the notion of log smoothness and study the pullback of logarithmic differential operators along such morphisms. In Section 4 we present the example of Raynaud. Section 5 establishes a criterion for a sequence of affine k-group schemes to be exact, and the proof of the main theorem is carried out in Section 6.

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2. Regular singular stratified bundles

Let k be an algebraically closed field of characteristic $p \geq 0$. We begin by recalling some basic facts about stratified bundles and regular singularity. For full details we refer to [5,16].

Definition 2.1. Let X be a smooth, separated, finite type k-scheme.

(a) A *stratified bundle on* X is a left-$\mathscr{D}_{X/k}$-module E which is coherent as an $\mathscr{D}^{\leq 0}_{X/k} = \mathcal{O}_X$-module. Here $\mathscr{D}_{X/k}$ is the sheaf of differential operators defined in [6, Section 16], and $\mathscr{D}^{\leq n}_{X/k}$, $n \geq 0$, the subsheaf of operators of order at most n. The category of stratified bundles with morphisms the morphisms of left-$\mathscr{D}_{X/k}$-modules, is denoted by $\mathrm{Strat}(X)$;

(b) If $\overline{X}$ is a smooth, separated, finite type k-scheme, such that there is an open immersion $X \hookrightarrow \overline{X}$, such that $\overline{X} \setminus X$ is the support of a strict normal crossings divisor, then the pair $(X, \overline{X})$ is called a *good partial compactification of* X. If $\overline{X}$ is proper, then $(X, \overline{X})$ is called a *good compactification of* X;

(c) If $(X, \overline{X})$ is a good partial compactification of X and if $D := (\overline{X} \setminus X)_{\mathrm{red}}$ is the associated strict normal crossings divisor, then an object $E \in \mathrm{Strat}(X)$ is said to be $(X, \overline{X})$*-regular singular* if there exists a $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{E}$, which is torsion-free and coherent with respect to the induced $\mathcal{O}_{\overline{X}}$-module structure, such that there is an isomorphism

$$E \cong \overline{E}|_X$$

in $\mathrm{Strat}(X)$. The sheaf $\mathscr{D}_{\overline{X}/k}(\log D)$ is the sheaf of subrings of $\mathscr{D}_{\overline{X}/k}$ generated by operators fixing all powers of the ideal of D. For more details see [1,5, 17] or [16, Paragraph 3].

We denote by $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$ the full subcategory of $\mathrm{Strat}(X)$ with objects the $(X, \overline{X})$-regular singular stratified bundles. If X is connected, then after fixing a base point $x \in X(k)$, $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$ is a neutral Tannakian category over k [16, Proposition 4.5] and we denote the associated group scheme by $\Pi_1^{\mathrm{rs}}((X, \overline{X}), x)$. If $(X, \overline{X})$ is a good compactification, then this group scheme is independent of the choice of $\overline{X}$ [5, Theorem 3.13], so we also write $\Pi_1^{\mathrm{rs}}(X, x)$. Finally, we denote by $\Pi_1(X, x)$ the associated group scheme to $\mathrm{Strat}(X)$.

It is proved in [16, Theorem 5.2] that $\overline{E}$ in Definition 2.1, (c) can be required to be a locally free $\mathcal{O}_X$-module without changing the definition. In this article, we also use the following simple criterion.

Lemma 2.2. *Let X be a smooth, separated, finite type k-scheme, let $(X, \overline{X})$ a good partial compactification, let $D := (\overline{X} \setminus X)_{\mathrm{red}}$ be the associated strict normal crossings divisor, and let $j : X \hookrightarrow \overline{X}$ be the associated open immersion. Let E be a stratified bundle on X, and let $\overline{E}$ be any locally free, coherent $\mathcal{O}_{\overline{X}}$-module extending E. Then the adjunction map $\overline{E} \to j_* j^* \overline{E} = j_* E$ is injective and we consider $\overline{E}$ as a sub-$\mathcal{O}_{\overline{X}}$-module of $j_* E$. Recall that $j_* E$ carries a natural $\mathscr{D}_{\overline{X}/k}(\log D)$-action. The following statements are equivalent.*

(a) *E is $(X, \overline{X})$-regular singular;*
(b) *The $\mathscr{D}_{\overline{X}/k}(\log D)$-submodule of $j_* E$ generated by $\overline{E}$ is $\mathcal{O}_{\overline{X}}$-coherent.*

Proof. Only (a) $\Rightarrow$ (b) is nontrivial. We may assume that $\overline{X} = \operatorname{Spec} A$ is affine and D regular, defined by $t \in A$. Assume that E is $(X, \overline{X})$-regular singular. According [16, Theorem 5.2] there exists a locally free, coherent $\mathcal{O}_{\overline{X}}$-module $\overline{E}'$ with an action of $\mathscr{D}_{\overline{X}/k}(\log D)$, such that $\overline{E}'|_X \cong E$ as $\mathscr{D}_{X/k}$-modules. Then there exists some $N \geq 0$, such that $\overline{E} \subseteq t^{-N} \overline{E}'$ as coherent $\mathcal{O}_{\overline{X}}$-submodules of $j_* E$. Note that $t^{-N} \overline{E}'$ is also a $\mathscr{D}_{\overline{X}/k}(\log D)$-submodule of $j_* E$. Thus the $\mathscr{D}_{X/k}(\log D)$-submodule of $j_* E$ generated by $\overline{E}$ is contained in $t^{-N} \overline{E}'$ and hence coherent, which is what we wanted to prove. $\square$

3. Logarithmic differential operators and logarithmic smoothness

For the details of the theory of logarithmic schemes we refer to [10], but we briefly recall the facts that we use in this article. More details can also be found in [13, 2.1].

Remark 3.1. The notion of an fs (fine and saturated) log structure can be understood as a generalization of the notion of a good partial compactification (Definition 2.1). Let k be a field. If X is a smooth, separated, finite type k-scheme and $(X, \overline{X})$ a good partial compactification of X, write $M_{\overline{X}} := M_{(X, \overline{X})}$ for the presheaf given by

$$M_{(X,\overline{X})}(U) := \left\{ g \in \mathcal{O}_{\overline{X}}(U) \mid g|_{U \cap X} \in \mathcal{O}_X(X \cap U)^{\times} \right\},$$

for $U \subseteq \overline{X}$ open. This is in fact a sheaf of monoids and the natural map $M_{\overline{X}} \to \mathcal{O}_{\overline{X}}$ makes $(\overline{X}, M_{\overline{X}})$ into an fs log scheme.

It is easy to write down local charts for this log scheme. If $U = \operatorname{Spec} A \subseteq \overline{X}$ is an affine open subset such that the boundary divisor is defined by $V(t_1 \cdot \cdots \cdot t_r)$, where $t_1, \ldots, t_r$ are part of a regular system of parameters for A, then we obtain a morphism of monoids $\mathbb{N}^r \to M_{\overline{X}}(U)$, $(m_1, \ldots, m_r) \mapsto \prod_{i=1}^r t_i^{m_i}$. This defines a chart $(U, M_{\overline{X}}|_U) \to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r]$.

More generally, given a finite set of units $u_1, \dots, u_n \in A^\times$, the morphism of monoids

$$\mathbb{Z}^n \oplus \mathbb{N}^r \to M_{\overline{X}}(U), (e_1, \dots, e_n, m_1, \dots, m_r) \mapsto \prod_{i=1}^n u_i^{e_i} \prod_{i=1}^r t_i^{m_i}$$

also induces a chart.

For a particular example of a chart for a morphism arising from good compactifications, and for what it means for such a morphism to be log smooth, see Example 3.7.

Definition 3.2. As in the non-log case, if (X, M_X) is an fs (S, M_S)-log scheme, for every $n \geq 0$, one defines *the sheaf* $\mathscr{P}^n_{(X,M_X)/(S,M_S)}$ *of logarithmic principal parts of order n* as the structure sheaf of the n-th order thickening of a certain log diagonal [10, Remark 5.8]. It is equipped with three ring homomorphisms

$$\mathcal{O}_X \underset{d_2}{\overset{d_1}{\rightrightarrows}} \mathscr{P}^n_{(X,M_X)/(S,M_S)} \xrightarrow{\Delta} \mathcal{O}_X$$

such that $\mathrm{id} = \Delta \circ d_1 = \Delta \circ d_2$.

The *sheaf of logarithmic differential operators of order* $\leq n$ is then defined as

$$\mathscr{D}^{\leq n}_{(X,M_X)/(S,M_S)} = \mathscr{H}om_{\mathcal{O}_X}\left(\mathscr{P}^n_{(X,M_X)/(S,M_S)}, \mathcal{O}_X\right),$$

and the *sheaf of logarithmic differential operators* as

$$\mathscr{D}_{(X,M_X)/(S,M_S)} = \bigcup_n \mathscr{D}^{\leq n}_{(X,M_X)/(S,M_S)}.$$

Proposition 3.3. *If $f : (X, M_X) \to (S, M_S)$ is a log smooth morphism of fs log schemes, then for every $n \geq 0$, the sheaf of logarithmic principal parts $\mathscr{P}^n_{(X,M_X)/(S,M_S)}$ is locally free of finite rank with respect to both its left and right $\mathcal{O}_X$-structures.*

More precisely, let $d_1, d_2 : \mathcal{O}_X \to \mathscr{P}^n_{(X,M_X)/(S,M_S)}$ denote the two structure maps. Let $\bar{x}$ be a geometric point of X and let $m_1, \dots, m_r \in M_{X,\bar{x}}$ be elements such that $\mathrm{dlog}(m_1), \dots, \mathrm{dlog}(m_r)$ *freely generate $\Omega^1_{(X,M_X)/(S,M_S),\bar{x}}$. Then $d_1(m_i) = d_2(m_i) \cdot u_i$ with $u_i \in (\mathscr{P}^n_{(X,M_X)/(S,M_S)})^\times_{\bar{x}}$. The assignment $m_i \mapsto u_i$ extends to a functorial morphism of monoids $\mu : M_X \to (\mathscr{P}^n_{(X,M_X)/(S,M_S)}, \cdot)$ and $\mathscr{P}^n_{(X,M_X)/(S,M_S),\bar{x}}$ is freely generated as either left- or right-$\mathcal{O}_{X,\bar{x}}$-module by monomials of degree $\leq n$ in $1 - \mu(m_1), \dots, 1 - \mu(m_r)$.*

Proof. The proof is completely analogous to the proof of [10, Proposition 6.5], forgetting about divided powers. □

Proposition 3.4. *Let (S, M_S) be a fs log scheme and let $f : (Y, M_Y) \to (X, M_X)$ be a log smooth (S, M_S)-morphism of fs log schemes which are log smooth over (S, M_S). Then the natural map*

$$\mathscr{D}_{(Y,M_Y)/(S,M_S)} \to f^*\mathscr{D}_{(X,M_X)/(S,M_S)}$$

is surjective.

Proof. The argument is almost the same as in the non-log case and essentially in [18]. We recall it for completeness.

By definition we have

$$\mathscr{D}^{\leq n}_{(X,M_X)/(S,M_S)} = \mathscr{H}om_{\mathcal{O}_X}\left(\mathscr{P}^n_{(X,M_X)/(S,M_S)}, \mathcal{O}_X\right),$$

where $n \geq 0$ and $\mathscr{P}^n_{(X,M_X)/(S,M)}$ is the sheaf of the n-th log principal parts. As (X, M_X) is log smooth over (S, M_S), the $\mathcal{O}_X$-modules $\mathscr{P}^n_{(X,M_X)/(S,M_S)}$ are locally free with respect to both their left- and right-$\mathcal{O}_X$-structure. It follows that it is enough to show that for every $n \geq 0$ the canonical morphism

$$f^*\mathscr{P}^n_{(X,M_X)/(S,M_S)} \to \mathscr{P}^n_{(Y,M_Y)/(S,M_S)} \tag{3.1}$$

is injective. This follows from the fact [10, 3.12] that the natural morphism

$$f^*\Omega^1_{(X,M_X)/(S,M_S)} \to \Omega^1_{(Y,M_Y)/(S,M_S)} \tag{3.2}$$

is injective and that its image is locally a direct summand.

Indeed, étale locally on X, there exist sections $m_1, \ldots, m_r$ of M_X, such that $\mathrm{dlog}(m_1) \ldots, \mathrm{dlog}(m_r)$ freely generate $\Omega^1_{(X,M_X)/(S,M_S)}$. As (3.2) is locally split, this means that locally there are sections $m'_{r+1}, \ldots, m'_d$ of M_Y, such that $\Omega^1_{(Y,M_Y)/(S,M_S)}$ is freely generated by

$$\mathrm{dlog}(f^*m_1), \ldots, \mathrm{dlog}(f^*m_r), \mathrm{dlog}\, m'_{r+1}, \ldots, \mathrm{dlog}\, m'_d.$$

Finally, by Proposition 3.3 there are natural maps

$$\mu_X : M_X \to \mathscr{P}^n_{(X,M_X)/(S,M_S)}, \quad \mu_Y : M_Y \to \mathscr{P}^n_{(Y,M_X)/(S,M_S)},$$

such that

(i) $\mathscr{P}^n_{(X,M_X)/(S,M_S)}$ is freely generated by the set of monomials of degree $\leq n$ in

$$1 - \mu_X(m_1), \ldots, 1 - \mu_X(m_r);$$

(ii) $\mathscr{P}^n_{(Y,M_Y)/(S,M_S)}$ is freely generated by the set of monomials of degree $\leq n$ in

$$1-\mu_Y(f^*m_1),\ldots,1-\mu_Y(f^*m_r),1-\mu_Y(m'_{r+1}),\ldots,1-\mu_Y\big(m'_d\big);$$

(iii)
$$f^*\mathscr{P}^n_{(X,M_X)/(S,M_S)}\to\mathscr{P}^n_{(Y,M_Y)/(S,M_S)}$$
is given by $1-\mu_X(m_i)\mapsto 1-\mu_Y(f^*m_i)$.

This shows that (3.1) is injective and that its image locally is a direct summand of $\mathscr{P}^n_{(Y,M_Y)/(S,M_S)}$. □

Corollary 3.5. *Let k be an algebraically closed field, let X, Y be smooth, separated, finite type k-schemes with good partial compactifications $(X,\overline{X})$, $(Y,\overline{Y})$, and write $D_X:=(\overline{X}\setminus X)_{\mathrm{red}}$ and $D_Y:=(\overline{Y}\setminus Y)_{\mathrm{red}}$. Let $f:Y\to X$ be a morphism which extends to a morphism $\bar{f}:\overline{Y}\to\overline{X}$ with the following properties:*

(a) *$\bar{f}(\overline{Y})$ contains all generic points of D_X;*
(b) *$\bar{f}$ is log smooth with respect to the log structures on $\overline{Y}$ and $\overline{X}$ defined by D_Y, respectively D_X.*

If E is a stratified bundle on X such that f^*E is $(Y,\overline{Y})$-regular singular, then E is $(X,\overline{X})$-regular singular.

Remark 3.6. We keep the notations from Corollary 3.5. One of the main results of [16] is that a stratified bundle E on X with finite monodromy is $(X,\overline{X})$-regular singular if and only if the associated Picard-Vessiot torsor on X (which is just a Galois covering) is tamely ramified along D_X. Thus, for stratified bundles with finite monodromy, Corollary 3.5 is a special case of [1, Proposition 7.7], as first lines of the proof of the theorem show that we can assume that $\bar{f}$ is faithfully flat.

Proof of Corollary 3.5. Without loss of generality we can assume that X and Y are connected. Let $\overline{Y}'\subseteq\overline{Y}$ be the largest open subset on which $\bar{f}$ is flat [6, Theorem 11.1.1]. Then $Y\subseteq\overline{Y}'$, as f is smooth. Moreover, $\bar{f}(\overline{Y}')\subseteq\overline{X}$ is open, $X\subseteq\bar{f}(\overline{Y}')$, and according to assumption (a), $\bar{f}(\overline{Y}')$ contains all generic points of D_X. Indeed, if $\eta\in\overline{X}$ is a generic point of D_X, then $\mathcal{O}_{\overline{X},\eta}$ is a discrete valuation ring, and thus the dominant morphism $\bar{f}$ is flat in a neighborhood of $\bar{f}^{-1}(\eta)$. If E is a stratified bundle on X, then it is $(X,\overline{X})$-regular singular if and only if it is $(X,\bar{f}(\overline{Y}'))$-regular singular ([16, Proposition 4.3]). Similarly, if f^*E is $(Y,\overline{Y})$-regular singular, then it is also $(Y,\overline{Y}')$-regular singular. Replacing $\overline{Y}$ by $\overline{Y}'$, and $\overline{X}$ by $\bar{f}(\overline{Y}')$, we may assume that $\bar{f}$ is faithfully flat.

Now the argument is almost identical to the proof of [14, Proposition 6.1], using Proposition 3.4 instead of the analogous surjectivity statement for the relative Frobenius morphism. We fix notations as in the following diagram:

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d, "f"] & \overline{Y} \arrow[d, "\bar{f}"] \\ X \arrow[r, hook, "j"] & \overline{X}. \end{tikzcd}$$

Let E be a stratified bundle on X and assume that f^*E is regular singular. Fix a locally free extension $E' \subseteq j_*E$ of E to $\overline{X}$. Denote by $\overline{E} \subseteq j_*E$ the $\mathscr{D}_{\overline{X}/k}(\log D_X)$-submodule of j_*E generated by E'; in other words $\overline{E}$ is the image of the evaluation map

$$\mathscr{D}_{\overline{X}/k}(\log D_X) \otimes_{\mathcal{O}_{\overline{X}}} E' \to j_*E. \tag{3.3}$$

The $\mathcal{O}_{\overline{X}}$-module $\overline{E}$ is quasi-coherent, and to show that E is regular singular, it suffices to show that $\overline{E}$ is coherent. As $\bar{f}$ is faithfully flat by assumption, it is enough to show that $\bar{f}^*\overline{E}$ is coherent. Note that $\bar{f}^*\overline{E}$ is the image of the pullback of (3.3) along $\bar{f}$.

Next, write G for the $\mathscr{D}_{\overline{Y}/k}(\log D_Y)$-submodule of $i_*f^*E = \bar{f}^*j_*E$ spanned by $\bar{f}^*E'$. In other words, G is the image of the evaluation map

$$\mathscr{D}_{\overline{Y}/k}(\log D_Y) \otimes_{\mathcal{O}_{\overline{Y}}} \bar{f}^*E' \to \bar{f}^*j_*E. \tag{3.4}$$

Note that $(\bar{f}^*E')|_Y \cong f^*E$. Thus, as f^*E is $(X, \overline{X})$-regular singular by assumption, Lemma 2.2 shows that G is coherent. We show that $G \cong \bar{f}^*\overline{E}$.

By the definition of the $\mathscr{D}_{\overline{Y}/k}(\log D_Y)$-action on $\bar{f}^*j_*E = i_*f^*E$, the maps (3.3) and (3.4) fit into the following commutative diagram

$$\begin{tikzcd} \bar{f}^*\left(\mathscr{D}_{\overline{X}/k}(\log D_X) \otimes_{\mathcal{O}_{\overline{X}}} E'\right) \arrow[r, "\bar{f}^*((4))"] \arrow[d, equal] & \bar{f}^*j_*E \arrow[ddd, equal] \\ \bar{f}^*\mathscr{D}_{\overline{X}/k}(\log D_X) \otimes_{\bar{f}^{-1}\mathcal{O}_{\overline{X}}} \bar{f}^{-1}E' & \\ \mathscr{D}_{\overline{Y}/k}(\log D_Y) \otimes_{\bar{f}^{-1}\mathcal{O}_{\overline{X}}} \bar{f}^{-1}E' \arrow[u, "\gamma\otimes\mathrm{id}"] \arrow[d, equal] & \\ \mathscr{D}_{\overline{Y}/k}(\log D_Y) \otimes_{\mathcal{O}_{\overline{Y}}} \bar{f}^*E' \arrow[r, "(5)"] & i_*f^*E, \end{tikzcd}$$

where

$$\gamma : \mathscr{D}_{\overline{Y}/k}(\log D_Y) \to \bar{f}^*\mathscr{D}_{\overline{X}/k}(\log D_X)$$

is the functoriality morphism for logarithmic differential operators. According to Proposition 3.4, this map is surjective, so the images of $\bar{f}^*(3.3)$ and (3.4) agree, which shows that $G = \bar{f}^*\overline{E}$. □

Example 3.7. Here are two fundamental (equicharacteristic) examples for log smoothness. Let k be a field of characterstic $p > 0$, $R = \operatorname{Spec} k[[t]]$, $S = \operatorname{Spec} R$, $K = k((t))$.

(a) Define
$$X_1 := \operatorname{Spec} R\Big[x_1, \dots, x_n\Big]\Big/\Big(x_1^{m_1} \cdot \ldots \cdot x_n^{m_n} - t\Big)$$
with $m_1, \dots, m_n \geq 0$ and at least one m_i prime to p;

(b) Define
$$X_2 := \operatorname{Spec} R\Big[u^{\pm 1}, x_1, \dots, x_n\Big]\Big/\Big(ux_1^{\ell_1} \cdot \ldots \cdot x_n^{\ell_n} - t\Big)$$
with $\ell_1, \dots, \ell_n \in \mathbb{Z}$. Note that if there exists one ℓ_i which is prime to p, then we can replace X_2 by a Kummer covering to arrive in case (a). Thus we assume that $\ell_1, \dots, \ell_n \in p\mathbb{Z}$.

Equip S, X_1 and X_2 with the logarithmic structure induced by the reduced special fiber. Then $X_1 \to S$ and $X_2 \to S$ are both log smooth. Indeed, S admits the chart $\mathbb{N} \to R$, $m \mapsto t^m$, and X_1 admits the chart $\mathbb{N}^n \to H^0(X_1, \mathcal{O}_{X_1})$, $(a_1, \dots, a_n) \mapsto \prod_{i=1}^n x_i^{a_i}$. The morphism $\varphi_1 : \mathbb{N} \to \mathbb{N}^n, a \mapsto (am_1, \dots, am_n)$ induces a commutative diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n] \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi_1} \\ S & \longrightarrow & \operatorname{Spec} \mathbb{Z}[\mathbb{N}], \end{array}$$

which is easily checked to be a chart for the morphism $X_1 \to S$.

Let $\varphi_1^{\mathrm{gp}} : \mathbb{Z} \to \mathbb{Z}^n$ be the homomorphism of abelian groups associated with φ_1. By [10, Theorem 3.5], X_1 is log smooth over S if and only if $\operatorname{coker}(\varphi_1^{\mathrm{gp}})$ has no p-torsion, and if the induced morphism $X_1 \to S \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]$ is étale. These conditions are easily seen to be satisfied if one of the $m_1, \dots, m_n$ is coprime to $p = \operatorname{char}(k)$.

For X_2, we can choose the chart
$$\mathbb{Z} \oplus \mathbb{N}^n \to H^0\big(X_2, \mathcal{O}_{X_2}\big), (a_0, a_1, \dots, a_n) \mapsto u^{a_0} \prod_{i=1}^n x_i^{a_i},$$
and $\varphi_2 : \mathbb{N} \to \mathbb{Z} \oplus \mathbb{N}^n, \varphi_2(a) = (a, a\ell_1, \dots, a\ell_n)$, induces a chart for the morphism $X_2 \to S$. We see that $X_2 \to S$ is log smooth.

We conclude the example by verifying Proposition 3.4 in this situation for $X_2 = \operatorname{Spec} k[\![t]\!][u^{\pm 1}, x]/(ux^\ell - t)$. The (completed) ring of differential operators $\mathscr{D}_{S/k}(\log(t))$ is isomorphic to $\bigoplus_{m\geq 0} k[\![t]\!]\delta_t^{(m)}$, where $\delta_t^{(m)}$ is the differential operator such that $\delta_t^{(m)}(t^r) = \binom{r}{m} t^r$ (*i.e.*, the characteristic free version of $\frac{t^m}{m!}\frac{\partial^m}{\partial t^m}$). The functions x and u are a system of coordinates for X_2 relative to k, and the (completed) ring of differential operators of X_2 relative to k is $\bigoplus_{m,n\geq 0}(k[\![t]\!]\delta_u^{(m)} \oplus k[\![t]\!]\delta_x^{(n)})$. We compute

$$\begin{aligned}\delta_u^{(m)}(t^r) &= \delta_u^{(m)}(u^r x^{\ell r})\\ &= \delta_u^{(m)}(u^r)x^{\ell r}\\ &= \binom{r}{m}u^r x^{\ell r}\\ &= \binom{r}{m}t^r.\end{aligned}$$

This shows that the map

$$\bigoplus_{m,n\geq 0}\left(k[\![t]\!]\delta_u^{(m)} \oplus k[\![t]\!]\delta_x^{(n)}\right) \to \bigoplus_{m\geq 0} k[\![t]\!]\delta_t^{(m)}$$

induced by $X_2 \to S$ is surjective.

4. An example of Raynaud

The log smoothness condition in Corollary 3.5 cannot be dropped, as the following example shows. It is a variant of [21, Remark 9.4.3 (c)] and is contained in a letter from M. Raynaud to H. Esnault dated May 15, 2009.

We will show: there is a smooth affine curve S over an algebraically closed field k of characteristic $p > 0$, a closed point $s \in S$, a morphism $f : X \to S$, and a finite map $g : S' \to S$, such that

(a) X is regular, hence smooth over k;
(b) f is faithfully flat;
(c) If $D = f^{-1}(s)_{\mathrm{red}}$, then $X \setminus D \to S \setminus \{s\}$ is smooth, projective, with geometrically connected fibers;
(d) $g : S' \to S$ is étale over $S \setminus \{s\}$;
(e) $g : S' \to S$ is wildly ramified over s;
(f) If X' is the normalization of $X \times_S S'$, then the finite map $g_X : X' \to X$ is étale.

The wildly ramified covering g pulls back to an étale covering of X, and a fortiori to a tamely ramified covering. Translated into the language of stratified bundles, this means that if $\Sigma \subseteq D$ is a 0-dimensional subset such that $D \setminus \Sigma$ is regular, then the stratified bundle $(g_*\mathcal{O}_{S'})|_{S\setminus\{s\}}$, which is irregular over s, pulls back to a stratified bundle on $X \setminus D$ which extends to X. In particular, this pullback is $(X \setminus \Sigma, D \setminus \Sigma)$-regular singular, so the conclusion of Corollary 3.5 fails.

Let S be an affine smooth curve over an algebraically closed field of characteristic $p > 0$ and let $E \to S$ be a family of ordinary elliptic curves. In particular, E is an abelian S-scheme (the Néron model of its generic fiber). After possibly replacing S by a étale open, we may assume that E contains an S-subgroup scheme G which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})_S$. Indeed, according to [12, 12.3], if E_p is the kernel of $E \xrightarrow{\cdot p} E$, then E_p sits in a short exact sequence $0 \to \ker(F) \to E_p \to \ker(V) \to 0$, where F and V denote the absolute Frobenius and Verschiebung. As E is ordinary, V is étale and $\ker(V)$ is a cyclic finite étale S-group scheme of order p. This means that there is a finite separable extension L of $K(S)$ over which $(E_p)_L = \ker(F)_L \times_L (\mathbb{Z}/p\mathbb{Z})_L$. We replace S by the integral closure of a suitable open subset in L, and from now on assume that there is a S-subgroup scheme $G \subseteq E$ with $G \cong (\mathbb{Z}/p\mathbb{Z})_S$.

Pick a closed point $s \in S$, write $K = K(S)$, $K_s := \operatorname{Frac}(\widehat{\mathcal{O}_{S,s}})$, fix separable closures $K^{\text{sep}} \subseteq (K_s)^{\text{sep}}$. Consider the following commutative diagram of Galois cohomology groups:

$$\begin{tikzcd}
H^1(K, G(K^{\text{sep}})) \arrow[r] \arrow[d] \arrow[dr, dashed, "\varphi"] & H^1(K, E(K^{\text{sep}})) \arrow[d] \\
H^1(K_s, G(K_s^{\text{sep}})) \arrow[r] & H^1(K_s, E_{K_s}(K_s^{\text{sep}})).
\end{tikzcd}$$

Claim 4.1. There exists an element $\alpha \in H^1(K, G(K^{\text{sep}}))$ of order p, such that $\varphi(\alpha) \neq 0$.

Proof. Write $R = \widehat{\mathcal{O}_{S,s}}$ and let R^{sep} be the integral closure of R in K_s^{sep}. If E_R is the elliptic curve on Spec R obtained by restricting E, then

$$E_R\left(K_s^{\text{sep}}\right) = E_R\left(R^{\text{sep}}\right) \twoheadrightarrow E_R(k(s)),$$

where $k(s) = k$ is the algebraically closed residue field of R. Moreover, if $E_R(k(s))$ is equipped with the trivial $\operatorname{Gal}(K_s^{\text{sep}}/K_s)$-action, we obtain a commutative diagram of continuous $\operatorname{Gal}(K_s^{\text{sep}}/K_s)$-modules

$$\begin{tikzcd}
E_{K_s}(K_s^{\text{sep}}) \arrow[rr, two heads] & & E(k(s)) \\
G(K_s^{\text{sep}}) \arrow[u, hook] \arrow[r, equal] & G(K_s) \arrow[r, equal] & G(k(s)) \arrow[u, hook]
\end{tikzcd}$$

and consequently

$$\begin{tikzcd}
H^1(K_s, E(K_s^{\mathrm{sep}})) \arrow[r] & \mathrm{Hom}^{\mathrm{cont}}(\mathrm{Gal}(K_s^{\mathrm{sep}}/K_s), E(k(s))) \\
H^1(K_s, G(K_s^{\mathrm{sep}})) \arrow[u] \arrow[r, equal] & \mathrm{Hom}^{\mathrm{cont}}(\mathrm{Gal}(K_s^{\mathrm{sep}}/K_s), G(k(s))). \arrow[u, hook]
\end{tikzcd}$$

This shows we can find $\alpha_s \in H^1(K_s, G(K_s^{\mathrm{sep}}))$ of order p with nontrivial image in $H^1(K_s, E(K_s^{\mathrm{sep}}))$.

Finally, note that the theorem of Katz-Gabber [11, Theorem 1.7.2] shows that there is a surjective morphism $\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(K_s^{\mathrm{sep}}/K_s)$, so we can lift α_s to $\alpha \in H^1(K, G(K^{\mathrm{sep}})) \cong \mathrm{Hom}^{\mathrm{cont}}(\mathrm{Gal}(K^{\mathrm{sep}}/K), \mathbb{Z}/p\mathbb{Z})$ of order p, such that $\varphi(\alpha) \neq 0$. ☐

The element α corresponds to an Artin-Schreier extension of $K(S)$, totally wildly ramified in s. Let S' be the integral closure of S in this Artin-Schreier extension. After possibly shrinking S around s, we can assume that $g : S' \to S$ is étale away from s.

The image of α in $H^1(K, E_K(K^{\mathrm{sep}}))$ corresponds to an E_K-torsor $X_K \to \operatorname{Spec} K$ which becomes trivial on $K(S')$. Let $f : X \to S$ be the minimal regular model of X_K. In [20, pages 82–83] it is shown that X can be constructed as a quotient of $E_{S'}$ by a finite flat equivalence relation. After perhaps shrinking S around s, we may assume that $f : X \to S$ is smooth away from s. Let $X' \to S'$ be the normalization of $X_{S'}$. Then $X' \cong E_{S'}$. Indeed, the quotient map $E_{S'} \to X$ induces a birational, integral map $E_{S'} \to X_{S'}$. As $E_{S'}$ is normal (even regular), this means that $E_{S'} \cong X'$.

We restrict this situation to a local setup: Let $s' \in S'$ be a point over s, and write $R := \widehat{\mathcal{O}_{S,s}}$, $R' := \widehat{\mathcal{O}_{S',s'}}$. We obtain the following commutative diagram:

$$\begin{tikzcd}
X_R \arrow[d] & X_{R'} \arrow[l] \arrow[d] & X'_{R'} \arrow[l] \arrow[dl] \\
\mathrm{Spec}\ R & \mathrm{Spec}\ R'. \arrow[l] &
\end{tikzcd}$$

As S is excellent, $X'_{R'} \to X_{R'}$ is the normalization of $X_{R'}$. Indeed, we can proceed as in [22, Tag 07TD]: $X'_{R'}$ is integral over $X_{R'}$, $X'_{R'}$ is normal by [6, Proposition 6.14.1] , and as $X'_{R'} \to X'$ is flat, $X'_{R'} \to X_{R'}$ is birational. This means we have a factorization as in the following diagram:

$$\begin{tikzcd}
(X_{R'})^{\mathrm{norm}} \arrow[r, dashed, "\cong"] \arrow[dr, "\text{normalization}"'] & X'_{R'} \arrow[r, "\text{flat}"] \arrow[d, "\text{birat.}"] & X' \arrow[d, "\text{normalization}"] \\
& X_{R'} \arrow[r] & X_{S'}.
\end{tikzcd}$$

It now suffices to prove that the finite map

$$E_{R'} \cong X'_{R'} \to X_R \tag{4.1}$$

is étale.

Note that X_R is the minimal regular model of X_{K_s} [17, Proposition 9.3.28] over R, and that X_{K_s} is the E_{K_s}-torsor associated with the cohomology class α_s from above. By [20, pages 82–83], the E_{K_s}-action on X_{K_s} extends to an E_R-action on X_R. If X_s and E_s are the special fibers of X and E, then the induced action of the abelian variety E_s on X_s is set-theoretically transitive. This means that $(X_s)_{\mathrm{red}}$ is isomorphic to E_s/H where H is some subgroup scheme of E_s.

Recall that we assume the existence of a finite étale group scheme $G \subseteq E$, $G \cong (\mathbb{Z}/p\mathbb{Z})_S$. Write $F := E/G$. By construction α_s is the image of a G_{K_s}-torsor, so there is a canonical E_{K_s}-equivariant morphism $u_{K_s} : X_{K_s} \to F_{K_s}$. By functoriality of the minimal regular model, it extends to an E_R-equivariant morphism $u_R : X_R \to F_R$. We obtain an E_s-equivariant morphism

$$(u_s)_{\mathrm{red}} : E_s/H \cong (X_s)_{\mathrm{red}} \to (F_s)_{\mathrm{red}} \cong E_s/G_s.$$

This shows that $H \subseteq G_s$ and that H is finite étale over $k(s)$. On the other hand, G_s acts trivially on $(X_s)_{\mathrm{red}}$ (thus implying that $H = G_s$), as the residue field extension of $R \subseteq R'$ is trivial and $X_R \to \operatorname{Spec} R$ factors through $\operatorname{Spec} R' \to \operatorname{Spec} R$: this is clear if for example one construct the correspondence between first cohomology classes and torsors using Weil's descent, remembering that K' corresponds to the class α in $H^1(K, G(K^{\mathrm{sep}}))$ and X_K corresponds to its image in $H^1(K, E_K(K^{\mathrm{sep}}))$.

Finally, the map (4.1) is finite of degree p, and $X'_{R'} \cong E_{R'}$. We have seen that on reduced special fibers, this map induces an étale map of degree p:

$$E_{s'} = E_s \to (X_s)_{\mathrm{red}} = E_s/G_s.$$

This implies that (4.1) is étale.

5. An exactness criterion for sequences of affine group schemes

In this section we establish an additional technical result, which characterizes the exactness of a sequence of affine k-group schemes in terms of the exactness of the induced sequences of their algebraic quotients.

Definition 5.1. Let $(\mathcal{T}, \omega)$ be a neutral Tannakian category over a field k and let $\mathcal{S}$ be a subset of its objects.

(a) The $\mathcal{T}$*-span* $\langle \mathcal{S} \rangle_{\otimes}^{\mathcal{T}}$ *of* $\mathcal{S}$ is the smallest full sub-Tannakian category of $\mathcal{T}$ containing $\mathcal{S}$ which is closed under subquotients and isomorphisms. When the ambient category is clear, we also drop the superscript $\mathcal{T}$ and write $\langle \mathcal{S} \rangle_{\otimes}$;

(b) We write $\Pi^{\mathcal{T}}_{\mathcal{S}}$, or simply $\Pi_{\mathcal{S}}$ when no confusion is possible, for the affine k-group scheme associated with $(\langle \mathcal{S}\rangle_{\otimes}, \omega)$ via the Tannaka formalism. If $\mathcal{S} = \{S\}$ is a singleton, we simply write Π_S instead of $\Pi_{\{S\}}$;

(c) A sub-Tannakian category $\mathcal{T}'$ of $\mathcal{T}$ is called *replete* if it is equal to the $\mathcal{T}$-span of its objects. By [2, Proposition 2.21] replete sub-Tannakian categories of $\mathcal{T}$ correspond to quotients of $\Pi_{\mathcal{T}} := \underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Recall that every affine group scheme over a field is the inverse limit of its finite type quotients (see for example [23, 3.3, corollary]). From the Tannakian perspective, this can be seen as follows: if $(\mathcal{T}, \omega)$ is a neutral Tannakian category over a field k, and if S_1, S_2 are two objects of $\mathcal{T}$, then for $i = 1, 2$, there are full embeddings $\langle S_i\rangle_{\otimes} \hookrightarrow \langle S_1 \oplus S_2\rangle_{\otimes}$,which induce quotient maps $\Pi_{S_1\oplus S_2} \twoheadrightarrow \Pi_{S_i}$. These maps are the transition maps in a projective system $\{\Pi_S | S \in \mathcal{T}\}$, and

$$\Pi_{\mathcal{T}} \cong \varprojlim_{S\in\mathcal{T}} \Pi_S.$$

Lemma 5.2. *Let*

$$\left(\mathcal{T}'', \omega''\right) \xrightarrow{B} (\mathcal{T}, \omega) \xrightarrow{A} \left(\mathcal{T}', \omega'\right)$$

be a sequence of additive, exact, tensor functors between neutral Tannakian categories over a field k, and let

$$\Pi_{\mathcal{T}'} \xrightarrow{a} \Pi_{\mathcal{T}} \xrightarrow{b} \Pi_{\mathcal{T}''} \tag{5.1}$$

be the associated sequence of affine k-group schemes.

The following statements are equivalent:

(a) *The sequence* (5.1) *is exact in the middle;*

(b) *For every object $S \in \mathcal{T}$ the sequence*

$$1 \to \Pi_{A(S)} \xrightarrow{\bar{a}} \Pi_S \xrightarrow{\bar{b}} \Pi_{\mathcal{T}_S} \to 1 \tag{5.2}$$

of affine k-group schemes is fpqc-exact, where $\mathcal{T}_S$ is the set of objects in $\langle S\rangle_{\otimes}$ which are in the span of the essential image of B.

Proof. Let $\mathcal{I}$ be the essential image of A and let $\mathcal{J}$ be the essential image of B. Then, by [2, Proposition 2.21] the sequence (5.1) factors as

$$\begin{tikzcd} \Pi_{\mathcal{T}'} \arrow[r, two heads] \arrow[rr, bend left, "a"] & \Pi^{\mathcal{T}'}_{\mathcal{I}} \arrow[r, hook] & \Pi_{\mathcal{T}} \arrow[r, two heads] \arrow[rr, bend left, "b"] & \Pi^{\mathcal{T}}_{\mathcal{J}} \arrow[r, hook] & \Pi_{\mathcal{T}''}. \end{tikzcd}$$

By replacing $\mathcal{T}'$ with $\langle\mathcal{I}\rangle^{\mathcal{T}'}_{\otimes}$ and $\mathcal{T}''$ with $\langle\mathcal{J}\rangle^{\mathcal{T}}_{\otimes}$, we can and will assume that a is injective (or, equivalently a closed immersion, see [23, Theorem 15.3]) and that b is faithfully flat.

We already remarked that

$$\Pi_{\mathcal{T}} \cong \varprojlim_{S \in \mathcal{T}} \Pi_S.$$

Moreover, as a is a closed immersion, [2, Proposition 2.21] shows that for every $S' \in \mathcal{T}'$ there exists $S \in \mathcal{T}$ such that $S' \in \langle A(S)\rangle_{\otimes}$, in particular the algebraic groups $\Pi_{A(S)}$ are cofinal in the projective system $\{\Pi_{S'}\}_{S' \in \mathcal{T}'}$ and thus

$$\Pi_{\mathcal{T}'} \cong \varprojlim_{S \in \mathcal{T}} \Pi_{A(S)}.$$

As b is faithfully flat, B is fully faithful and the essential image of B is replete in $\mathcal{T}$. In particular for every object $S'' \in \mathcal{T}''$ one has that $\Pi_{S''} = \Pi_{B(S'')}$ and thus

$$\Pi_{\mathcal{T}''} \cong \varprojlim_{S \in \mathcal{T}} \Pi_{\mathcal{T}_S}, \tag{5.3}$$

where $\mathcal{T}_S$ consists of all objects in $\langle S\rangle_{\otimes}$ that are in the essential image of B.

All together, we proved that the sequence (5.1) is the inverse limit over the sequences (5.2), indexed by the objects of $\mathcal{T}$.

If (5.2) is exact for every $S \in \mathcal{T}$, then (5.1) is exact in the middle because for a projective system $\{G_i\}_i$ of k-group schemes and for any commutative ring R, we have $(\varprojlim G_i)(R) = \varprojlim(G_i(R))$ and the inverse limit is a (left) exact functor. This proves (b) $\Rightarrow$ (a).

To prove (a) $\Rightarrow$ (b), assume that (5.1) is exact. Recall that we may assume that a is a closed immersion and that b is faithfully flat. We get the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{\mathcal{T}'} & \xrightarrow{a} & \Pi_{\mathcal{T}} & \xrightarrow{b} & \Pi_{\mathcal{T}''} & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \mathrm{pr}'} & & \big\downarrow{\scriptstyle \mathrm{pr}} & & \big\downarrow{\scriptstyle \mathrm{pr}''} & & \\
1 & \longrightarrow & \Pi_{A(S)} & \xrightarrow{\bar{a}} & \Pi_S & \xrightarrow{\bar{b}} & \Pi_{\mathcal{T}_S} & \longrightarrow & 1
\end{array} \tag{5.4}$$

where all vertical arrows are faithfully flat. We want to prove that the bottom row is exact. It follows directly from [2, Proposition 2.21] that the morphism $\bar{a}$ is a closed immersion and that $\bar{b}$ is faithfully flat. It is also clear that $\bar{b} \cdot \bar{a}$ is the trivial morphism. It remains to prove that (5.2) is exact in the middle.

We can write Π_S as $\Pi_{\mathcal{T}}/H$ and $\Pi_{A(S)}$ as $\Pi_{\mathcal{T}'}/H'$, for some normal subgroup schemes H, H'. As a is injective, we see that $\ker(\mathrm{pr} \circ a) = \Pi_{\mathcal{T}'} \cap H$, and as $\bar{a}$ is injective, $H' = \ker(\bar{a} \circ \mathrm{pr}')$. As (5.4) commutes, the two kernels must agree and thus $H' = \Pi_{\mathcal{T}'} \cap H$.

In particular, this shows that $\Pi_{A(S)}$ is normal in Π_S. Thus to prove that (5.2) is exact, it is enough to prove that $\Pi_{\mathcal{T}_S}$ is the cokernel of $\bar{a}$. By the universal property of the cokernel, together with [2, Corollary 2.9, Proposition 2.21], the cokernel of $\bar{a}$ corresponds uniquely to a replete sub-Tannakian category $\mathcal{C}$ of $\langle S\rangle_\otimes$, characterized by the property that for every $S' \in \langle S\rangle_\otimes$, $A(S')$ is trivial if and only if $S' \in \mathcal{C}$, where by *trivial* we mean that $A(S')$ is the direct sum of copies of the unit object of $\mathcal{T}''$. Our assumptions imply that $\Pi_{\mathcal{T}''}$ is the cokernel of a, hence if $S' \in \mathcal{T}$, $A(S')$ is trivial if and only if $S' \in \mathcal{T}''$, hence by definition $\mathcal{T}_S$ is the maximal replete sub-Tannakian category of $\langle S\rangle_\otimes$ whose objects are trivialized by A, which finishes the proof. □

6. The homotopy exact sequence for regular singular stratified bundles

We return to the notations that were introduced in Section 2. In particular, k is an algebraically closed field of characteristic $p > 0$ and X is a smooth, connected, separated scheme of finite type over k. We recall that given $x \in X(k)$, we have associated k-group schemes $\Pi(X, x)$, $\Pi^{\mathrm{rs}}(X, x)$ and $\Pi^{\mathrm{rs}}((X, \overline{X}), x)$, where $(X, \overline{X})$ is a good partial compactification of X (Definition 2.1).

Note that according to [16, Proposition 4.5], $\mathrm{Strat}^{\mathrm{rs}}((X, \overline{X}))$ is a replete sub-Tannakian category of $\mathrm{Strat}(X)$ and hence there is a quotient map $\Pi(X, x) \twoheadrightarrow \Pi^{\mathrm{rs}}((X, \overline{X}), x)$.

Theorem 6.1. *Let X and Y be smooth, connected, separated k-schemes of finite type and let $f : Y \to X$ be a smooth projective morphism with geometrically connected fibers. Fix good partial compactifications $(X, \overline{X})$ and $(Y, \overline{Y})$ and write $D_X := (\overline{X} \setminus X)_{\mathrm{red}}$, $D_Y := (\overline{Y} \setminus Y)_{\mathrm{red}}$. Let $\bar{f} : \overline{Y} \to \overline{X}$ be an extension of f satisfying:*

- (a) *$\bar{f}(\overline{Y})$ contains all generic points of D_X;*
- (b) *$\bar{f}$ is log smooth when $\overline{Y}$ and $\overline{X}$ are equipped with the log structures defined by D_Y, respectively D_X (Remark 3.1).*

If $y \in Y$ is a closed point, define $x := f(y)$ and let $j : Y_x \hookrightarrow \overline{Y}$ be the fiber over x. Then the sequence

$$\Pi(Y_x, y) \xrightarrow{j^*} \Pi^{\mathrm{rs}}((Y, \overline{Y}), y) \xrightarrow{f^*} \Pi^{\mathrm{rs}}((X, \overline{X}), x) \to 1 \tag{6.1}$$

of affine k-group schemes is fpqc exact.

Proof. Notice that the assumptions of the theorem imply that Y_x is proper, hence all stratified bundles on Y_x are trivially regular singular. We will deduce the theorem from [4, Theorem 1], which states that under our hypotheses the sequence

$$\Pi(Y_x, y) \to \Pi(Y, y) \to \Pi(X, x) \to 1$$

is exact. In particular, the top horizontal arrow in the commutative diagram

$$\begin{tikzcd} \Pi(Y,y) \arrow[r, "f^*"] \arrow[d, "\pi_Y"] & \Pi(X,x) \arrow[d, "\pi_X"] \\ \Pi^{\mathrm{rs}}((Y,\overline{Y}),y) \arrow[r, "f^*"] & \Pi^{\mathrm{rs}}((X,\overline{X}),x) \end{tikzcd}$$

is faithfully flat, and so are the vertical arrows. This shows that the bottom horizontal arrow is faithfully flat as well.

If $E \in \mathrm{Strat}^{\mathrm{rs}}((Y,\overline{Y}))$, then [16, Proposition 4.5] shows that the span $\langle E\rangle_{\otimes}$ is independent of whether we compute it in $\mathrm{Strat}^{\mathrm{rs}}((Y,\overline{Y}))$ or the ambient category $\mathrm{Strat}(Y)$.

Similarly, if $E \in \mathrm{Strat}^{\mathrm{rs}}((Y,\overline{Y}))$, then we can consider the span of j^*E in $\mathrm{Strat}(Y_x)$ and we write $\Pi_{j^*E} = \Pi_{j^*E}^{\mathrm{Strat}(Y_x)}$ for the affine k-group scheme attached to it. By Lemma 5.2, to prove that (6.1) is exact, it is enough to prove that for every $(Y,\overline{Y})$-regular singular stratified bundle E, the sequence

$$1 \to \Pi_{j^*E} \to \Pi_E \to \Pi_{\mathcal{T}_E^{\mathrm{rs}}} \to 1 \tag{6.2}$$

is exact, where $\mathcal{T}_E^{\mathrm{rs}}$ are the objects of $\langle E\rangle_{\otimes}$ that are also in $\mathrm{Strat}^{\mathrm{rs}}(X,\overline{X})$ (seen as a full replete sub-Tannakian category of $\mathrm{Strat}^{\mathrm{rs}}(Y,\overline{Y})$ via f^*).

By [4, Theorem 1] the sequence

$$\Pi(Y_x,y) \xrightarrow{j^*} \Pi(Y,y) \xrightarrow{f^*} \Pi(X,x) \to 1$$

is exact. Hence, the sequence

$$1 \to \Pi_{j^*E} \to \Pi_E \to \Pi_{\mathcal{T}_E} \to 1 \tag{6.3}$$

is exact for every $E \in \mathrm{Strat}^{\mathrm{rs}}((Y,\overline{Y}))$ (Lemma 5.2), where $\mathcal{T}_E$ are the objects of $\langle E\rangle_{\otimes}$ that are also in $\mathrm{Strat}(X)$. By Corollary 3.5, we have $\mathcal{T}_E = \mathcal{T}_E^{\mathrm{rs}}$, so the sequence (6.3) is isomorphic to the sequence (6.2), which is then exact. Using Lemma 5.2 again, this concludes the proof. □

Remark 6.2. The proof above shows a little more: namely, if there exists a stratified bundle E on X which is not $(X,\overline{X})$-regular singular but such that f^*E is $(Y,\overline{Y})$-regular singular (*e.g.*, as in the example of Section 4), then the sequence (6.1) cannot be exact.

To see this, note that under the conditions of Theorem 6.1 the surjectivity of

$$f^* : \Pi^{\mathrm{rs}}((Y,\overline{Y}),y) \to \Pi^{\mathrm{rs}}((X,\overline{X}),x)$$

holds even without the log smoothness hypothesis on $\bar{f}$. This allows us to see $\mathrm{Strat}(X)$ and $\mathrm{Strat}^{\mathrm{rs}}(X, \overline{X})$ as replete sub-Tannakian categories of $\mathrm{Strat}(Y)$. As in the proof of Theorem 6.1, consider the full subcategories

$$\mathcal{T}_{f^*E} := \langle f^*E \rangle_{\otimes} \cap \mathrm{Strat}(X) \subseteq \mathrm{Strat}(Y),$$

and

$$\mathcal{T}^{\mathrm{rs}}_{f^*E} := \langle f^*E \rangle_{\otimes} \cap \mathrm{Strat}^{\mathrm{rs}}((X, \overline{X})) \subseteq \mathrm{Strat}(Y).$$

The exactness of (6.3) for f^*E reduces to the isomorphism $\Pi_{f^*E} \cong \Pi_{\mathcal{T}^{\mathrm{rs}}_{f^*E}}$. Note that indeed by [4, Theorem1] and arguing as in the proof of Theorem 6.1 one has that $\Pi_{f^*E} \cong \Pi_{\mathcal{T}_{f^*E}}$. On the other hand, if E is not $(X, \overline{X})$-regular singular, then $f^*E \notin \mathcal{T}^{\mathrm{rs}}_{f^*E}$, so the inclusion $\mathcal{T}^{\mathrm{rs}}_{f^*E} \subseteq \mathcal{T}_{f^*E}$ is strict. This means that the induced morphism $\Pi_{f^*E} \to \Pi_{\mathcal{T}^{\mathrm{rs}}_{f^*E}}$ is not an isomorphism, so it follows from Lemma 5.2 the sequence (6.1) cannot be exact in the middle. This shows that the log smoothness assumption in Theorem 6.1 cannot be dropped entirely.

A stratified bundle E on X is called *regular singular* if it is $(X, \overline{X})$-regular singular for every good partial compactification $\overline{X}$. We denote by $\Pi^{\mathrm{rs}}(X, x)$ the Tannakian fundamental group associated with the full subcategory of $\mathrm{Strat}(X)$ given by all regular singular stratified bundles [16, Proposition 7.4]. By [16, Proposition 7.5] if X admits a good compactification $\overline{X}$, then a stratified bundle E is regular singular if and only if it is $(X, \overline{X})$-regular singular. Together with [15, Theorem 1.3] this implies the following.

Corollary 6.3. *Retain the notations and assumptions of Theorem* 6.1*, and assume furthermore that $\overline{X}$ is proper. Then the sequence*

$$\Pi(Y_x, y) \xrightarrow{j^*} \Pi^{\mathrm{rs}}(Y, y) \xrightarrow{f^*} \Pi^{\mathrm{rs}}(X, x) \to 1$$

is fpqc exact.

Corollary 6.4 (Künneth formula). *Let X and Y be smooth, connected k-varieties with Y projective, let $x \in X(k)$, $y \in Y(k)$ and let $z \in (Y \times_k X)(k)$ be the point induced by x and y. Then, the natural morphism induced by the projections*

$$\Pi^{\mathrm{rs}}(X \times_k Y, z) \to \Pi^{\mathrm{rs}}(X, x) \times \Pi^{\mathrm{rs}}(Y, y)$$

is an isomorphism.

Proof. Note that Y is proper, hence $\Pi(Y, y) = \Pi^{\mathrm{rs}}(Y, y)$.

The strategy of the proof is exactly the same as in [7, X, Corollary 1.7]. In order to make use of Theorem 6.1, we only need to remark that for every partial good compactification $\overline{X}$ of X there exists a good partial compactification of $X \times Y$ (log) smooth over $\overline{X}$, namely $\overline{X} \times Y$. $\square$

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