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Del Pezzo elliptic varieties of degree $d \leq 4$

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Abstract. Let Y be a smooth del Pezzo variety of dimension $n \geq 3$, *i.e.*, a smooth complex projective variety endowed with an ample divisor H such that $-K_Y = (n-1)H$. Let d be the degree H^n of Y and assume that $d \leq 4$. Consider a linear subsystem of $|H|$ whose base locus is zero-dimensional of length d. The subsystem defines a rational map onto $\mathbb{P}^{n-1}$ and, under some mild extra hypothesis, the general pseudofibers are elliptic curves. We study the elliptic fibration $X \to \mathbb{P}^{n-1}$ obtained by resolving the indeterminacy and call the variety X a del Pezzo elliptic variety. Extending the results of [7] we mainly prove that the Mordell-Weil group of the fibration is finite if and only if the Cox ring of X is finitely generated.

Mathematics Subject Classification (2010): 14C20 (primary); 14Q15, 14E05, 14N25 (secondary).

1. Introduction

Let Y be a smooth del Pezzo variety of degree d at most 4 and dimension n at least 3, and let H be an ample divisor such that $-K_Y = (n-1)H$. A linear subsystem $V \subseteq |H|$ whose base locus is a zero-dimensional subscheme of length d defines a rational map onto $\mathbb{P}^{n-1}$ whose general pseudofibers are curves of arithmetic genus-one. We focus on the case when the general pseudofiber is smooth and denote by

$$\pi : X \to \mathbb{P}^{n-1}$$

the elliptic fibration obtained by resolving the indeterminacy. We call the variety X a *del Pezzo elliptic variety* and the fibration π a *del Pezzo elliptic fibration*. In [5] the case of general V is considered in relation with the Morrison-Kawamata cone conjecture. In this paper we extend the results of [7] to del Pezzo elliptic varieties of

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degree $d \leq 4$. Our first result is about the Mordell-Weil groups of the corresponding del Pezzo elliptic fibrations (the notation will be explained in Section 3).

Theorem 1.1. *The Mordell-Weil groups* $\mathrm{MW}(\pi)$ *of the del Pezzo elliptic fibrations* $\pi : X \to \mathbb{P}^{n-1}$ *of degree* $d \leq 4$ *and dimension* $n \geq 3$ *are the following:*

Table 1.1.

d	Type of X	$\mathrm{MW}(\pi)$	d	Type of X	$\mathrm{MW}(\pi)$
1	X_1	$\langle 0\rangle$	4	X_{40}	$\mathbb{Z}^3$
2	X_{11}	$\mathbb{Z}$		X_{41}, X_{30}	$\mathbb{Z}^2$
	X_{SS}	$\mathbb{Z}/2\mathbb{Z}$		X_{42}	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
	X_2	$\langle 0\rangle$		$X_{31}, X_{20}, X'_{20}, X'_{21}$	$\mathbb{Z}$
3	X_{111}	$\mathbb{Z}^2$		X_{43}	$(\mathbb{Z}/2\mathbb{Z})^2$
	X_{S11}, X_{12}	$\mathbb{Z}$		X_{21}, X_{22}	$\mathbb{Z}/2\mathbb{Z}$
	X_{SSS}	$\mathbb{Z}/3\mathbb{Z}$		X_{10}, X_{11}	$\langle 0\rangle$
	X_{S2}	$\mathbb{Z}/2\mathbb{Z}$			
	X_3, X_S	$\langle 0\rangle$			

Our second result relates the finite generation of the Cox ring of del Pezzo elliptic varieties with the finiteness of the Mordell-Weil group of the corresponding elliptic fibration. We recall that an important result along this line was obtained by Totaro [11] in dimension two and his work has been the starting point for further developments (see, *e.g.*, [5] and [7]).

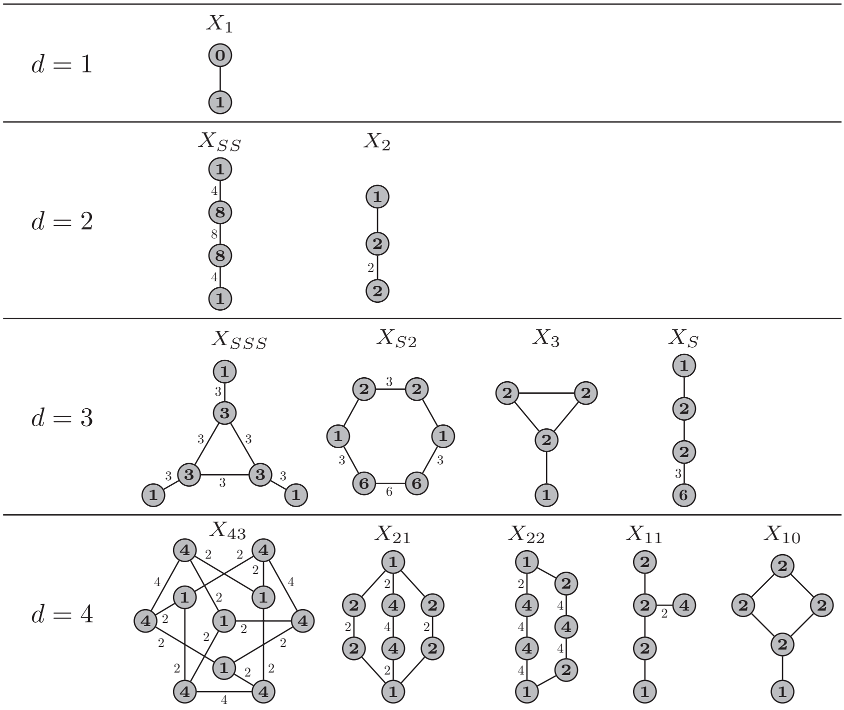
Theorem 1.2. *Let X be a del Pezzo elliptic variety of degree $d \leq 4$ and dimension $n \geq 3$. Then the following are equivalent:*

(1) *The Cox ring of X is finitely generated;*
(2) *The Mordell-Weil group of $\pi : X \to \mathbb{P}^{n-1}$ is finite.*

We prove Theorem 1.2 showing that any del Pezzo elliptic variety whose corresponding elliptic fibration has finite Mordell-Weil group, is a Mori dream space and vice versa. Then we conclude by Hu and Keel characterization of Mori dream spaces [8, Proposition 2.9]. The proof of our second theorem makes use of a detailed study of the structure of the moving and effective cones of del Pezzo elliptic varieties. In particular we prove the following result.

Theorem 1.3. *Let $\pi : X \to \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration of degree $d \leq 4$, with $n \geq 3$ and having finite Mordell-Weil group. Then the effective cone* $\mathrm{Eff}(X)$ *is generated by the vertical classes and the classes of sections and the moving cone* $\mathrm{Mov}(X)$ *is the dual of* $\mathrm{Eff}(X)$ *with respect to the bilinear form introduced in* (2.2). *The intersection graphs for the effective cones are given in the following table.*

Each vertex corresponds either to a section or a prime vertical class D explicitly given in Table 4.1 *of Proposition* 4.1*; the label in the vertex is $-(D, D)$ while the label on the edge connecting two vertices D and D' is (D, D') (we omit the label when it is equal to* 1).



The paper is structured as follows. In Section 2 we introduce del Pezzo elliptic fibrations and del Pezzo elliptic varieties and we define the bilinear form on the Picard group of such varieties. In Section 3 we study the geometry of these varieties and in the Section 4 we use these results in order to classify the Mordell-Weil groups of the del Pezzo elliptic fibrations, their vertical classes and sections. Section 5 contains the description of the nef, effective and moving cones of del Pezzo elliptic varieties and moreover in the same section we prove Theorem 1.2. In the last section we provide the Cox rings of the del Pezzo elliptic varieties whose fibration has finite Mordell-Weil group and having degree one an two and a lemma about the Cox ring of the blowup of the complete intersection of two quadrics at one point.

We work over an algebraically closed field of characteristic 0.

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2. Del Pezzo elliptic varieties

Let Y be a del Pezzo variety of dimension $n \geq 3$ such that $-K_Y = (n-1)H$, with H ample and $d := H^n \leq 4$. It is well known (see for instance [9]) that the Picard group of Y has rank one and it is generated by the class H. If $d = 1$ then Y is a smooth hypersurface of degree six of the weighted projective space $\mathbb{P}(3, 2, 1, \ldots, 1)$ and H is the restriction of a degree one class of the ambient space. If $d = 2$ then Y is a double cover of $\mathbb{P}^n$ branched along a smooth quartic hypersurface and H is the pull-back of a hyperplane of $\mathbb{P}^n$. If $d \in \{3, 4\}$ then Y is a projectively normal subvariety of $\mathbb{P}^{n+d-2}$ and H is the class of a hyperplane section.

Let us consider an $(n-1)$-dimensional linear subsystem of $|H|$, whose base locus Z has dimension zero and length d. In particular, if $d = 1$ we have $Z = V(x_3, \ldots, x_{n+2})$, if $d = 2$, Z is preserved by the covering involution and if $d \in \{3, 4\}$, Z spans a linear subspace $\Lambda \subseteq \mathbb{P}^{n+d-2}$ of dimension $d-2$. Let us denote by $\pi_Z : Y \dashrightarrow \mathbb{P}^{n-1}$ the rational map defined by the given subsystem and by $\pi : X \to \mathbb{P}^{n-1}$ the resolution of the indeterminacy of π_Z. The variety X comes with two morphisms:

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & \mathbb{P}^{n-1} \\ {\scriptstyle\sigma}\downarrow & \nearrow_{\pi_Z} & \\ Y & & \end{array}$$

where σ is the composition of d blowing-ups $\sigma_1, \ldots, \sigma_d$ at points $q_1, \ldots, q_d$, respectively. Moreover the general fiber of π is a smooth genus-one curve if $d \leq 3$ or if $d = 4$ and Λ is not contained in the tangent space of Y at any point of Z.

In what follows, by abuse of notation, we use the same letter H to denote the pull-back of H via σ while we denote by E_i the pull-back of the exceptional divisor of σ_i, for $i \in \{1, \ldots, d\}$. Observe that some of the points $q_2, \ldots, q_d$ can lie on the exceptional divisor of one of the σ_i's. Therefore E_i can be either a $\mathbb{P}^{n-1}$ or the union of a $\mathbb{P}^{n-1}$ with some other components isomorphic to the projectivization of the vector bundle $\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. In any case, we can write

$$\operatorname{Pic}(X) = \langle H, E_1, \ldots, E_d \rangle,$$

where, with abuse of notation, we are adopting the same symbols for the divisors and for their classes. We will also adopt the following notation

$$F := -\frac{1}{n-1} K_X. \tag{2.1}$$

Observe that F is the pull-back of a hyperplane section of $\mathbb{P}^{n-1}$ via π, so that $F = H - \sum_{i=1}^{d} E_i$.

Remark 2.1. The map σ is a composition of blowups at points and we claim that we blow up at most one point on each exceptional divisor. Indeed, since by assumption a general pseudofibre is smooth, when we blow up a point in the base locus the proper transform of a general pseudofibre only meets the exceptional divisor in one point.

2.1. A bilinear form on the Picard group

We are now going to introduce a bilinear form on $\mathrm{Pic}(X)$, where X is the blowup of Y at r general points. A similar form has been introduced for the blowup of products of projective spaces in [4] and [3].

Using the above notation for F, we define

$$(A, B) := F^{n-2} \cdot A \cdot B \tag{2.2}$$

for any two divisors A and B on X. Thus the quadratic form q induced by the above bilinear form is hyperbolic and the matrix with respect to the basis $(H, E_1, \dots, E_r)$ is diagonal with entries $d, -1, \dots, -1$. Since $(F, F) = d - r$, the sublattice $F^\perp$ is negative definite if $1 < r < d$ and it is negative semidefinite if $r = d$. In the first case, a basis consists of the classes $E_1 - E_2, \dots, E_{r-1} - E_r$, while in the second case it consists of the above classes plus F. These are roots lattices of type A_{r-1} and $\tilde{A}_{d-1}$, respectively.

When $r = d$ and the linear system $|F|$ on the blowup X induces the elliptic fibration $\pi\colon X \to \mathbb{P}^{n-1}$, we observe that F^{n-2} is rationally equivalent to a smooth rational elliptic surface S which is the preimage via π of a line. Thus we have $(A, B) = A|_S \cdot B|_S$, where the right hand side is the intersection product in $\mathrm{Pic}(S)$.

Proposition 2.2. *Let A and B be effective divisors of X with B a prime divisor. If $(A, B) < 0$ then B is contained in the stable base locus of $|A|$.*

Proof. Let ℓ be a general line of $\mathbb{P}^{n-1}$ and let S be the surface $\pi^{-1}(\ell)$. According to the definition of the bilinear form we have $A|_S \cdot B|_S < 0$. Being B prime and ℓ general, the divisor $B|_S$ of S is prime as well. Thus the linear series $|A|_S|$ contains $B|_S$ into its base locus and the same holds for the linear series $|A|$. Varying ℓ we get the claim. □

In particular the above proposition implies that the moving cone of X is contained in the dual of the effective cone with respect to the bilinear form. We will see in Section 5 that under an extra hypothesis on the fibration the two cones are indeed equal.

3. Types

In this section we are going to describe the possible types of del Pezzo elliptic varieties of degree $d \le 4$.

3.1. Degree one

In this case Y is a degree 6 hypersurface of the $(n+1)$-dimensional weighted projective space $\mathbb{P}(3,2,1,\dots,1)$. A defining equation for Y has the form

$$x_1^2 + x_1x_2f_1 + x_1f_3 + x_2^3 + x_2^2f_2 + x_2f_4 + f_6 = 0,$$

where f_t is a degree t homogeneous polynomial in $x_3,\dots,x_{n+2}$. After applying the change of coordinates $x_1 \mapsto x_1 - \frac{1}{2}x_2f_1 - \frac{1}{2}f_3$ the above equation takes the form

$$x_1^2 + x_2^3 + x_2^2f_2' + x_2f_4' + f_6' = 0.$$

By further applying the change of coordinates $x_2 \mapsto x_2 - \frac{1}{3}f_2'$ and exchanging the sign of x_3 the equation takes the form

$$x_1^2 - x_2^3 + x_2f_4 + f_6 = 0, \tag{3.1}$$

where with abuse of notation we are denoting by f_t the new coefficients after the last change of coordinates. The blowup $\sigma\colon X \to Y$ is centred at the point $q = (1,1,0,\dots,0) \in Y$ and the rational map $Y \dashrightarrow \mathbb{P}^{n-1}$ is defined by $(x_1,\dots,x_{n+2}) \mapsto (x_3,\dots,x_{n+2})$.

3.2. Degree two

In this case Y is a double cover $\varphi\colon Y \to \mathbb{P}^n$ branched along a smooth quartic hypersurface $\Sigma = V(f)$. A defining equation for Y has the form

$$x_{n+2}^2 = f. \tag{3.2}$$

In order to distinguish the different cases that can occur, we observe that the preimage of a line ℓ through a point $p := \varphi(q_1)$ is one of the following:

$$\varphi^{-1}(\ell) = \begin{cases} \text{elliptic curve} & \text{if } |\ell \cap \Sigma| = 4 \\ \text{rational nodal curve} & \text{if } |\ell \cap \Sigma| = 3 \\ \text{union of two smooth rational curves} & \text{if } \ell \text{ is bitangent to } \Sigma. \end{cases} \tag{3.3}$$

Therefore we distinguish three different cases depending on the position of p with respect to Σ and on the dimension of the variety $B \subseteq \mathbb{P}^n$ spanned by all the bitangent lines to Σ passing through p.

Case 1. The point p does not lie on Σ and B is not a hypersurface. In this case the preimage of p in the double covering $Y \to \mathbb{P}^n$ consists of two distinct points q_1 and q_2. We denote by X_{11} the variety that we obtain by blowing up these two points.

Case 2. The point p does not lie on Σ and B is a hypersurface. In this case after a linear change of coordinates we can assume that $p = (0,\dots,0,1)$. The defining polynomial for Σ in (3.2) is

$$f = x_{n+1}^4f_0 + x_{n+1}^3f_1 + x_{n+1}^2f_2 + x_{n+1}f_3 + f_4, \tag{3.4}$$

where $f_d \in \mathbb{C}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree d, with $f_0 \neq 0$. Let us consider a point $q \in \Sigma$ and the line parametrised by $\lambda p + q$. The line is tangent to Σ at q if and only if $f(\lambda p + q)$ has a double zero at $\lambda = 0$ or equivalently if the partial derivative ∂f vanishes at q, where $\partial := \partial/\partial x_{n+1}$, that is q lies on the polar $P_p(\Sigma)$ to p with respect to Σ. In order to be a bitangent line, the discriminant of the quadratic polynomial $f(\lambda p + q)/\lambda^2$ must vanish. An easy calculation shows that this is equivalent to q lying on the quadric $Q_p(\Sigma)$ defined by the polynomial $(\partial^3 f)^2 - 3(\partial^2 f)(\partial^4 f)$. Thus the locus of bitangency is

$$Z := (B \cap \Sigma)_{\mathrm{red}} \subseteq \Sigma \cap P_p(\Sigma) \cap Q_p(\Sigma). \tag{3.5}$$

Since we are in characteristic zero, a general generatrix ℓ of the cone B intersects Z transversally. Being ℓ bitangent to Σ, for any $z \in \ell \cap Z$ we have the inclusion $T_z\ell \subseteq T_z\Sigma$. Thus the tangent spaces T_zB and $T_z\Sigma$ are equal, being both equal to $T_zZ \oplus T_z\ell$. In particular we have an equality of divisors $B \cap \Sigma = mZ$ for some $m \in \{2, 4\}$. Being Σ a quartic hypersurface of dimension at least two, its Picard group is torsion free, so that by the previous argument the class of Z in $\mathrm{Pic}(\Sigma)$ is linearly equivalent to a multiple sL of the hyperplane section L (observe that L must be a primitive class since $L^{n-1} = 4$). In particular we have $\deg B = ms$. By Bertini's second theorem [2] the intersection of Σ with a general plane Π through p is a smooth plane quartic curve C. By (3.5) the degree of $Z \cap \Pi$ is at most 8 so that $B \cap \Pi$ is the union of at most 4 lines, which implies that $\deg B \leq 4$. Therefore the only possible values for (m, s) are $(2, 1)$, $(4, 1)$ and $(2, 2)$. The case $(2, 1)$ does not occur since L has degree 4 and can not be contained in the intersection of the quadric B with a hyperplane. We conclude that $\deg B = 4$ and Z is cut out either by a quadric or a hyperplane. The defining polynomial for Σ in (3.2) is therefore

$$f = g + h^2, \tag{3.6}$$

where $g \in \mathbb{C}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree four such that $B = V(g)$, while $h \in \mathbb{C}[x_1, \dots, x_{n+1}]$ is homogeneous of degree two, possibly a square. We denote by X_{SS} the variety obtained by blowing up the two distinct points q_1 and q_2 in the preimage of p.

Case 3. The point p lies on Σ. As before we can suppose that $p = (0, \dots, 0, 1)$ so that the defining polynomial for Σ in (3.2) is

$$f = x_{n+1}^3 f_1 + x_{n+1}^2 f_2 + x_{n+1} f_3 + f_4, \tag{3.7}$$

where $f_d \in \mathbb{C}[x_1, \dots, x_n]$ is homogeneous of degree d. In this case in order to get an elliptic fibration we need to blow up the point $q_1 := \varphi^{-1}(p)$ and the point on the exceptional divisor which is invariant with respect to the lifted involution. We denote by X_2 the variety that we obtain after the blowing-ups.

3.3. Degree four

Let us first collect some facts about smooth complete intersections of two hyperquadrics $Y := Q \cap Q' \subseteq \mathbb{P}^{n+2}$, for $n \geq 3$ (see also [10]). Observe that any quadric in the pencil Λ generated by Q and Q' has rank at least $n+2$, since otherwise Y would not be smooth, and there are $n+3$ singular quadrics in the pencil, counting muliplicities. We claim that there are exactly $n+3$ quadrics of rank $n+2$ and their vertices are in general position in $\mathbb{P}^{n+2}$. Indeed, let us suppose that either there are less than $n+3$ vertices or that they are not in general position.

In the former case the pencil of quadrics is tangent to the discriminant hypersurface at some point. Without loss of generality we can assume Q to be a cone of vertex $p = (1, 0, \dots, 0)$ in diagonal form g and the pencil $\Lambda : g + tg' = 0$ is tangent to the discriminant hypersurface at $t = 0$. If the Hessian matrix of g is $M = (m_{ij})$ and that of g' is $M' = (m'_{ij})$, the above tangency condition is equivalent to the vanishing of the following derivative

$$\frac{d}{dt}\mathrm{Det}\big(M + tM'\big)|_{t=0}.$$

Expanding the above derivative and using the fact that M is diagonal we see that the above is equivalent to $m'_{11} = 0$, that is $p \in Q'$. This is not possible since it contradicts the smoothness of Y.

In the latter case there exists a hyperplane $H \subseteq \mathbb{P}^{n+2}$ containing all the vertices and if we restrict Λ to H we obtain a pencil Λ_H of quadrics in $\mathbb{P}^{n+1}$, containing at least $n+3$ singular quadrics (counting multiplicities). Hence all the quadrics of Λ_H must be singular and by Bertini's theorem their vertices are contained in the base locus of Λ_H. This implies that all the vertices of these cones are in Y and this is a contradiction since they give singular points of Y.

Let us prove now the following result that will be useful in the next section.

Proposition 3.1. *Let $Y \subseteq \mathbb{P}^{n+2}$ be a smooth complete intersection of two quadrics defining a pencil Λ. Let q_1 and q_2 be two, possibly infinitely near, points of Y such that the line $\langle q_1, q_2\rangle$ is not contained in Y. Then the following are equivalent:*

(1) *The conics of Y through q_1 and q_2 span a hypersurface of Y;*
(2) *The line $\langle q_1, q_2\rangle$ passes through the vertex of a singular quadric of Λ.*

Proof. To prove (1) $\Rightarrow$ (2), let us suppose that the conics through q_1 and q_2 span a hypersurface S, *i.e.* there exists an $(n-2)$-dimensional family of such conics. If C is a conic contained in Y then its linear span $\langle C\rangle$ is contained in a quadric Q of the pencil Λ which passes through a point of $\langle C\rangle \setminus C$. There is only one such quadric Q in the pencil since otherwise we would have the inclusions $\langle q_1, q_2\rangle \subseteq \langle C\rangle \subseteq Y$, contradicting the hypothesis.

Moreover all such planes have to be contained in the same quadric Q since otherwise their common line $\langle q_1, q_2\rangle$ would be contained in Y, again contradicting the hypothesis. We claim that Q is a cone whose vertex lies on the line $\langle q_1, q_2\rangle$.

Indeed if we take a point $q \in \langle q_1, q_2 \rangle$, the tangent space $T_q Q$ intersects Q along an n-dimensional quadric that contains all the linear spaces through q. In particular this quadric must be the union of the $(n-2)$-dimensional family of planes containing the line $\langle q_1, q_2 \rangle$ and hence it is singular along the whole line. We conclude that Q must have a singular point on this line.

In order to prove (2) $\Rightarrow$ (1), let p be the vertex of a quadric $Q \in \Lambda$ lying on the line $\langle q_1, q_2 \rangle$, so that this line is a generatrix of the cone Q. We can write $Y = Q \cap Q'$, where Q' is any other quadric of the pencil. We conclude observing that there exists an $(n-2)$-dimensional family of planes of Q containing a generatrix and each of them intersects Q' along a conic through the two fixed points. □

In what follows we will denote by $Q_1, \ldots, Q_{n+3}$ the singular quadrics of the pencil Λ and by $p_1, \ldots, p_{n+3}$ the corresponding vertices. By the above discussion, we can assume that p_i is the i-th fundamental point of $\mathbb{P}^{n+2}$ for $i = 1, \ldots, n+3$, so that Q_1 and Q_2 are defined by diagonal forms. Moreover, after possibly rescaling the variables, we can assume the quadrics to be defined by the following polynomials

$$x_2^2 - x_3^2 + x_4^2 + \cdots + x_{n+3}^2 \qquad x_1^2 - x_3^2 + \alpha_4 x_4^2 + \cdots + \alpha_{n+3} x_{n+3}^2 \tag{3.8}$$

respectively, where the coefficients α_i are distinct and not in $\{0, 1\}$.

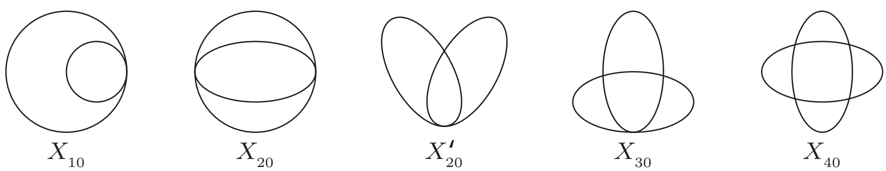
Let us fix now a plane $\Pi \subseteq \mathbb{P}^{n+2}$ and let us analyze the different types of del Pezzo elliptic varieties of degree four. By Proposition 3.1 the type depends not only on the number of points we blow up but also on the number of vertices p_i contained in the plane Π. Hence we are going to use the symbol X_{kl} to denote the variety that we obtain by choosing a plane Π intersecting Y in k distinct points and containing l vertices.

Let us spend few words about the geometry of this construction and about the possible values of k and l for X_{kl}. We remark that we can write

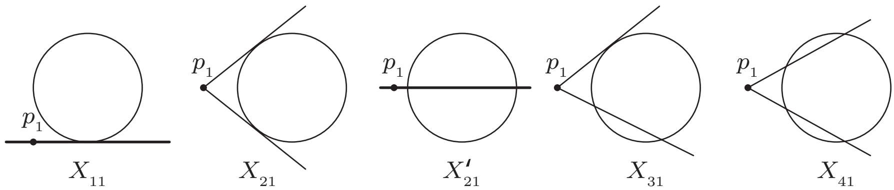
$$\Pi \cap Y = C \cap C'$$

where $C := Q \cap \Pi$ and $C' := Q' \cap \Pi$ are two plane conics. We discuss four cases.

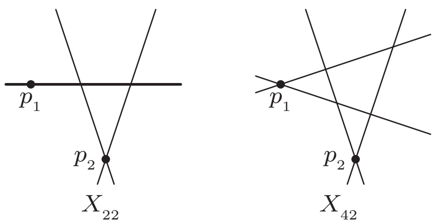
Case 1. If Π contains no vertices, then we have two smooth conics, whose intersection consists of k distinct points, for $k \in \{1, 2, 3, 4\}$ and hence we get the types $X_{10}, X_{20}, X'_{20}, X_{30}, X_{40}$. Observe that when $k = 2$ we have two possibilities: either C and C' are tangent at their two intersection points q_1 and q_2, or they intersect transversally at q_2 and with multiplicity three at q_1 (we denote this last case by X'_{20}).



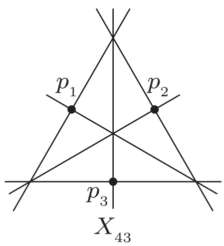
Case 2. If Π contains one vertex, say p_1, then we can suppose that C is a smooth conic while $C' := \Pi \cap Q_1$ has (at least) a singular point at the vertex $p_1 \in \Pi$. The intersection of C and C' consists of k points, for $k \in \{1, 2, 3, 4\}$ and we obtain the types X_{11}, X'_{21}, X_{31}, X_{41}. As before, when $k = 2$ we have two possibilities. Either C' is the union of two distinct lines and each of them is tangent to the conic C, or C' is a double line (which means that Π is tangent to Q_1) intersecting C in two distinct points (we denote this last case by X'_{21}).



Case 3. If Π contains two vertices, say p_1 and p_2, then we can suppose that both C and C' are singular and they can not intersect at the vertices, since Y is smooth, so that k can be either 1, 2 or 4. Moreover, when $k = 1$ we deduce that the plane Π is contained in the tangent space to Y at the only intersection point q_1. We are not going to consider this case since it does not give an elliptic fibration, being all the fibers singular rational curves. Hence we have only the two types X_{22} and X_{42}.



Case 4. Finally, observe that if Π contains three vertices, say p_1, p_2 and p_3, then it can intersect Y only at four distinct points (as it follows from (3.8)), giving case X_{43}.



Remark 3.2. We provide here an example of defining equations for Π for each of the following five types:

$$X_{43}: \Pi = V(x_4, x_5, \dots, x_{n+3})$$
$$X_{22}: \Pi = V(x_3 - x_4, x_5, \dots, x_{n+3})$$
$$X_{21}: \Pi = V\left(\sqrt{\alpha_4+\alpha_5}\cdot x_2 - \sqrt{\alpha_4+\alpha_5-2}\cdot x_3, x_4 - x_5, \dots, x_{n+3}\right)$$
$$X_{11}: \Pi = V(x_1 - \alpha_4 x_2 + (\alpha_4 - 1)\, x_3, x_5, \dots, x_{n+3})$$
$$X_{10}: \Pi = V(2x_1 - (\alpha_4+\alpha_5)x_2 + (\alpha_4+\alpha_5-2)x_3, x_4 - x_5, x_6, \dots, x_{n+3}),$$

where $\alpha_4 + \alpha_5 \neq 0, 2$ in cases X_{21} and X_{10}.

Remark 3.3. We recall that if $Y = Q \cap Q' \subseteq \mathbb{P}^{n+2}$ and $n \geq 3$, then through any point of Y we have at least one line of Y. So let us fix a point $q_i \in Y$ and a line ℓ of Y, passing through this point, and let us describe the fiber of $\pi: X \to \mathbb{P}^{n-1}$ containing the strict transform of that line. The image of this fiber inside Y is the curve obtained by intersecting Y with the $\mathbb{P}^3$ spanned by the plane Π and the line ℓ. This can also be described as the base locus of the pencil of quadric surfaces obtained by restricting Λ to the $\mathbb{P}^3$ that we are considering. Observe that any time we have a vertex p_i in Π, the intersection of Q_i with the $\mathbb{P}^3$ is a quadric cone containing a line not passing through p_i. Hence it must be the union of two planes intersecting along a line passing through p_i. Therefore, in Case 1 the image of the fiber inside Y is obtained by intersecting two smooth quadric surfaces sharing a line and hence it is the union of that line and a rational normal cubic, intersecting in two points. In Case 2 the base locus is the intersection of a smooth quadric with a reducible one and then it is the union of two lines and a smooth conic. In Case 3 the base locus is the intersection of two reducible quadrics and hence it consists of four lines. Finally, in Case 4 we have the base locus of a pencil containing three reducible quadrics. Thus, after a possible renaming of the coordinates, the pencil has the form $(x_1^2 - x_2^2) + t(x_2^2 - x_3^2)$. All the quadrics in this pencil are singular at the point $p = (1, 1, 1, 1)$ and the base locus of the pencil consists of four lines intersecting at the point p. We remark that in this last case the corresponding fiber in X is the union of four rational components passing through one point and hence it is a type that does not appear in the Kodaira's list of singular fibers for elliptic surfaces. As a consequence this fibre is not contained in any nonsingular elliptic surface inside X.

4. Mordell-Weil groups

The main result of this section is the proof of Theorem 1.1 but we postpone it to the end of the section and we begin by studying the *prime vertical classes* of all the del Pezzo elliptic fibrations of degree $d \leq 4$, that is classes of prime divisors whose support does not dominate $\mathbb{P}^{n-1}$. If $d = 1$, then $\mathrm{Pic}(X)$ has rank two and is generated by the class E_1 of the exceptional divisor together with $F = H - E_1$.

When $d = 2$, recall that there is a double covering $\varphi\colon Y \to \mathbb{P}^n$ branched along a smooth quartic hypersurface Σ and $\sigma : X \to Y$ is the blowup of Y at two points q_1, q_2 exchanged by the covering involution. If D is a prime vertical class of X whose class is not a multiple of F, then either D is a prime component of the exceptional locus of σ, or by (3.3) $\varphi(D)$ is covered by bitangent lines to Σ. Therefore in case X_{11} there are no vertical classes other than multiples of F.

In case X_{SS} we have the two vertical classes $2H - 4E_1, 2H - 4E_2$, and assuming that Y has the equation (3.6), they are the classes of the strict transforms of $V(x_{n+2} - h)$ and $V(x_{n+2} + h)$, respectively.

Finally, in case X_2, $E_1 - E_2$ and $H - 2E_1$ are the only prime vertical classes.

The case $d = 3$ has already been studied in [7] and we refer to that paper for the classification of the prime vertical classes.

For $d = 4$, if D is a prime vertical class, then D is properly contained in the support of $\pi^*\pi(D)$. Let γ be a general fiber of π over a point $q \in \pi(D)$ and let us denote by C the image $\sigma(\gamma)$ in Y. Then either (j) C is an irreducible rational curve or (jj) it contains lines and/or conics.

In case (j), C is singular at one of the points $q_i \in \Pi \cap Y$ and the union of these curves gives a prime vertical class having class $H - 2E_i - E_j - E_k$. In order to obtain the class of a fiber we have to add some exceptional prime vertical classes of the form $E_i - E_{i+1}$ for some i.

In case (jj), observe that by [5, Section 2.2] through any point of Y there is only a $(n-3)$-dimensional family of lines and hence they can not fill up a divisor. Therefore the curve C must contain a conic through two points q_i and q_j of $\Pi \cap Y$, possibly infinitely near. By Proposition 3.1 the line $\langle q_i, q_j\rangle$ passes through one vertex p_k and hence $p_k \in \Pi$. In this case the class of one of the irreducible components of $\pi^*\pi(D)$ is of the form $H - 2E_i - 2E_j$. For instance, in case X_{31} we can write $Y = Q_1 \cap Q$, where Q_1 is a cone with vertex $p_1 \in \Pi$ and Q is a smooth quadric. Furthermore, Q intersects Π along a smooth conic C while $Q_1 \cap \Pi$ is the union of two generatrices and one of them is tangent to C at q_1 while the other one intersects C in q_3 and q_4. Therefore we have the prime vertical class $H - 2E_1 - 2E_2$ corresponding to the conics through q_1 and whose tangent line at q_1 is the line $\langle q_1, p_1\rangle$ and the prime vertical class $H - 2E_3 - 2E_4$ corresponding to the conics through q_3 and q_4. Observe that the sum of these classes gives twice a fiber. Moreover, we also have the prime vertical class $E_1 - E_2$ sitting inside the exceptional locus and the prime vertical class $H - 2E_1 - E_3 - E_4$ which is spanned by the union of the strict transforms of the singular rational quartic curves of Y obtained intersecting it with a hyperplane tangent to Y at q_1.

We summarise the above observations in the following:

Proposition 4.1. *Let $\pi : X \to \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration of degree $d \leq 4$ and dimension $n \geq 3$. Then for each type of X the sections and the vertical classes are as follows:*

Table 4.1. Sections and prime vertical classes of del Pezzo elliptic fibrations with $d \leq 4$.

d	Type	Sections	Prime vertical classes (omitting F)
1	X_1	E_1	
2	X_{11}	E_1, E_2	
	X_{SS}	E_1, E_2	$2H-4E_1, 2H-4E_2$
	X_2	E_2	$E_1-E_2, H-2E_1$
3	X_{111}	E_1, E_2, E_3	
	X_{S11}	E_1, E_2, E_3	$H-3E_1, 2H-3E_2-3E_3$
	X_{SSS}	E_1, E_2, E_3	$H-3E_1, H-3E_2, H-3E_3$
	X_{12}	E_1, E_3	$E_2-E_3, H-E_1-2E_2$
	X_{S2}	E_1, E_3	$H-3E_1, 2H-3E_2-3E_3, E_2-E_3, H-E_1-2E_2$
	X_3	E_3	$E_1-E_2, E_2-E_3, H-2E_1-E_2$
	X_S	E_3	$E_1-E_2, E_2-E_3, H-3E_1$
4	X_{40}	E_1, E_2, E_3, E_4	
	X_{41}	E_1, E_2, E_3, E_4	$H-2E_1-2E_2, H-2E_3-2E_4$
	X_{42}	E_1, E_2, E_3, E_4	$H-2E_1-2E_2, H-2E_3-2E_4$ $H-2E_1-2E_3, H-2E_2-2E_4$
	X_{43}	E_1, E_2, E_3, E_4	$H-2E_i-2E_j,\ 1 \leq i < j \leq 4$
	X_{30}	E_2, E_3, E_4	$E_1-E_2, H-2E_1-E_3-E_4$
	X_{31}	E_2, E_3, E_4	$H-2E_1-2E_2, H-2E_3-2E_4$ $E_1-E_2, H-2E_1-E_3-E_4$
	X_{20}	E_3, E_4	$E_1-E_3, H-2E_1-E_2-E_4, E_2-E_4, H-E_1-2E_2-E_3$
	X'_{20}	E_3, E_4	$E_1-E_3, E_3-E_4, H-2E_1-E_2-E_3$
	X_{21}	E_3, E_4	$E_1-E_3, E_2-E_4, H-2E_1-2E_3, H-2E_2-2E_4$ $H-2E_1-E_2-E_4, H-E_1-2E_2-E_3$
	X'_{21}	E_3, E_4	$E_1-E_3, E_2-E_4, H-2E_1-E_2-E_4, H-E_1-2E_2-E_3$
	X_{22}	E_3, E_4	$H-2E_1-2E_3, H-2E_2-2E_4$ $E_1-E_3, E_2-E_4, H-2E_1-2E_2$
	X_{10}	E_4	$E_1-E_2, E_2-E_3, E_3-E_4, H-2E_1-E_2-E_3$
	X_{11}	E_4	$E_1-E_2, E_2-E_3, E_3-E_4, H-2E_1-2E_2$

Proof of Theorem 1.1. Recall that the Mordell-Weil group of the elliptic fibration $\pi : X \to \mathbb{P}^{n-1}$ is the group of rational sections of π or, equivalently, the group of $K = \mathbb{C}(\mathbb{P}^{n-1})$-rational points $X_\eta(K)$ of the generic fiber X_η of π once we choose one of such points O as an origin for the group law. Let $\mathscr{T}$ be the subgroup of $\mathrm{Pic}(X)$ generated by the prime vertical classes and by the class of the section O. There is an exact sequence [12, Section 3.3]:

$$0 \longrightarrow \mathscr{T} \longrightarrow \mathrm{Pic}(X) \longrightarrow X_\eta(K) \longrightarrow 0. \tag{4.1}$$

In degree d, the Picard group of X is free of rank $d+1$, generated by the classes $H, E_1, \ldots, E_d$. Observe that if F is defined as in (2.1), then $\langle F, E_d \rangle \subseteq \mathscr{T}$ holds and by Proposition 4.1 and the sequence (4.1) we get the statement. □

5. Cones

The aim of this section is to provide a description of the nef, effective and moving cone of a del Pezzo elliptic variety X in order to prove Theorem 1.2. We will show that the moving cone $\mathrm{Mov}(X)$ is a finite union of polyhedral chambers, each of which is the pullback of the semiample cone of a small modification of X. According to Hu and Keel theorem [8, Proposition 2.9] this decomposition will provide a proof for Theorem 1.2.

5.1. The nef cones

Given a subset I of $\{1, \dots, d\}$, in what follows we denote by F_I the divisor $H - \sum_{i \in I} E_i$. Moreover we denote by e_i the class of a line in the exceptional divisor E_i and by h the class of the preimage of a degree one curve of Y.

Theorem 5.1. *Let $\pi : X \to \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration with $n \geq 3$. Then the extremal rays of the nef cone* $\mathrm{Nef}(X)$ *are all the F_I such that $I \subseteq \{1, \dots, d\}$ and $(F_I, V) \geq 0$ for each exceptional vertical class V.*

Proof. Let us consider the subcone $\mathcal{C}$ of the Mori cone of X generated by the following classes:

- e_i such that E_i is a section;
- $e_i - e_j$ such that $E_i - E_j$ is a prime vertical divisor;
- $h - e_i$ for each $q_i \in Y$.

Let $D := \alpha H - \sum m_i E_i$ be a class in the dual $\mathcal{C}^*$. Then we have the following inequalities: $m_i \geq 0$ $\forall i$, $m_i \geq m_j$ if the point q_j lies on the exceptional divisor of the blowing-up at q_i and finally $\alpha \geq m_i$ $\forall i$. Let us write $\{m_1, \dots, m_d\} = \{\mu_1, \dots, \mu_r\}$, where $r \leq d$ and $0 = \mu_0 \leq \mu_1 < \cdots < \mu_r$, and let us denote by $I_i := \{j \mid m_j \geq \mu_i\}$, for each $i = 1, \dots, r$. Then we can write

$$D = (\alpha - \mu_r)H + \sum_{i=1}^{r} (\mu_i - \mu_{i-1}) F_{I_i},$$

where the F_{I_i} are nef and their product with any effective $E_j - E_k$ is non negative. In order to conclude the proof we need to show that these F_I are extremal rays of the nef cone.

Let us first suppose that X is obtained by blowing up d distinct points on Y. In this case, we have to consider all the F_I as I varies in the subsets of $\{1, \dots, d\}$, and by induction on d it can be proved that they are vertices of a d-dimensional hyper-cube. In particular, they are extremal rays of the cone they generate.

In addition, we can also infer that no F_I lies in the convex hull of the remaining and hence the general case follows. ☐

5.2. The effective and moving cones

We now restrict our attention to del Pezzo elliptic fibrations of degree $d \leq 4$ and having finite Mordell-Weil group, proving Theorem 1.3.

Proof of Theorem 1.3. Let us consider, for each del Pezzo elliptic variety X the cone $\mathcal{M}$ of $\mathrm{Pic}_{\mathbb{Q}}(X)$ generated by the vertical classes and the sections of π. Let $\varrho_1, \ldots, \varrho_n$ be the extremal rays of $\mathcal{M}$. We claim that the following inclusions hold

$$\bigcap_{i=1}^{n} \mathrm{cone}(\varrho_1, \ldots, \overset{\vee}{\varrho_i}, \ldots, \varrho_n) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Eff}(X)^\vee \subseteq \mathcal{M}^\vee. \tag{5.1}$$

In order to prove the first inclusion let us denote by D_i a prime divisor whose class generates the ray ρ_i and let D be a divisor whose class is in the above intersection. Observe that the support of D is contained in the union of $D_1, \ldots, D_n$ and moreover $[D] \in \mathrm{cone}(\varrho_1, \ldots, \overset{\vee}{\varrho_i}, \ldots, \varrho_n)$ implies that, up to multiples, D is linearly equivalent to an effective divisor that do not contain D_i in its support. Since the above condition is true for any $i = 1, \ldots, n$, we conclude that the stable base locus of $|D|$ does not contain any D_i, for $i = 1, \ldots, n$, and hence it can not be divisorial. In particular $[D]$ is a movable class. The second inclusion is a consequence of Proposition 2.2 while the last one follows from $\mathcal{M} \subseteq \mathrm{Eff}(X)$, so that the claim holds. Now the proof goes as follows. If the degree d is at most three, then the Cox ring is known (Theorem 6.1 for degree one or two and [7] for degree three) and a direct computation shows that the rays of $\mathcal{M}^\vee$ are movable. When $d = 4$ and X is of type X_{43}, X_{22}, X_{21}, the Test function presented in the appendix shows that the first cone and the last one in (5.1) are equal and hence the two assertions of the theorem follow. In the remaining cases, we are going to check that the rays of $\mathcal{M}^\vee$ are movable.

If X is of type X_{11}, looking at the Magma session in the appendix we see that the only class we have to check is $H - 2E_1$ (all the other rays of $\mathcal{M}^\vee$ being nef classes). We are going to see that the base locus of the linear system $|H - 2E_1|$ has codimension two. Indeed, this linear system corresponds on Y to the linear system of hyperplane sections containing the tangent space at q_1, whose base locus is the union of the lines passing through q_1. When we blow up q_1, the strict transforms of these lines intersect E_1 along a subvariety of codimension two. Observe that the second point q_2 that we blow up do not lie on this subvariety, since otherwise the plane Π would intersect Y along a line. Then the base locus of $|H - 2E_1|$ can not be divisorial.

In case X_{10}, looking again at the Magma session in the appendix we see that the only classes we have to check are $H - 2E_1$ and $3H - 4E_1 - 4E_2$. The first one can be done as in case X_{11} while the second one can be obtained as the image of H via the Geiser involution described in Subsection 5.3, and hence it is movable. □

As a consequence of Theorem 1.3, if X is a del Pezzo elliptic variety of degree $d \leq 4$ such that the Mordell-Weil group of $\pi : X \to \mathbb{P}^{n-1}$ is finite, then the effective cone $\mathrm{Eff}(X)$ can be read from Table 4.1. The graphs of the quadratic form on the primitive generators of the extremal rays of $\mathrm{Eff}(X)$ are listed in Theorem 1.3.

Let us consider an example in which the Mordell-Weil group of the fibration is not finite and the moving cone is the union of infinitely many chambers.

When the elliptic fibration has degree $d = 2$ and type X_{11}, we have seen that the Mordell-Weil group is $\langle\sigma\rangle \cong \mathbb{Z}$. The action of σ on the Picard group of X, with respect to the basis $B := (H - E_1 - E_2,\ E_2 - E_1,\ E_1)$, is given by the following matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

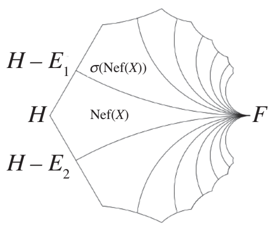
The cone $\sigma^k(\mathrm{Nef}(X))$ is generated by the classes corresponding to the columns of the matrix

$$\begin{pmatrix} 1 & k^2+k+1 & k^2-k+1 & 2k^2+1 \\ 0 & k+1 & k & 2k+1 \\ 0 & 1 & 1 & 2 \end{pmatrix},$$

with respect to the basis B. We claim that the classes $\sigma^k(H)$ are extremal rays of the moving cone and generate it, so that the following equality holds

$$\mathrm{Mov}(X) = \bigcup_{k\in\mathbb{Z}} \sigma^k(\mathrm{Nef}(X)).$$

First of all, observe that for each $k \in \mathbb{Z}$ the cones $\sigma^k(\mathrm{Nef}(X))$ and $\sigma^{k+1}(\mathrm{Nef}(X))$ share the two-dimensional face generated by F and $\sigma^k(H - E_1) = \sigma^{k+1}(H - E_2)$. Moreover $\sigma^k(H) + \sigma^{k+1}(H) = 4\sigma^k(H - E_1)$, so that the union of the cones $\sigma^k(\mathrm{Nef}(X))$ is a convex cone and the classes $\sigma^k(H - E_i)$, $i = 1, 2$, are on its boundary but they are not extremal rays. Now observe that the right hand side cone is contained in $\mathrm{Mov}(X)$.



Finally, since the property of lying on the boundary of $\mathrm{Mov}(X)$ is preserved by σ^k, we only have to prove that the two faces $\langle H, H - E_i\rangle$, for $i = 1, 2$, are on the boundary of the moving cone $\mathrm{Mov}(X)$. We conclude observing that if we move outside from $\mathrm{Nef}(X)$ along a direction orthogonal to the face $\langle H, H - E_1\rangle$ (respectively $\langle H, H - E_2\rangle$) we obtain classes containing E_2 (respectively E_1) in the stable base locus.

5.3. Generalized Bertini and Geiser involutions

We consider here a generalization of the classical Bertini and Geiser involutions to blowups of del Pezzo varieties. Let Y be a degree $d \geq 3$ del Pezzo variety and let $Z \subseteq Y$ be a zero-dimensional subscheme such that $\dim\langle Z\rangle = l(Z) - 1$, where $l(Z)$ is the length of Z, and the intersection of $n-1$ general elements of $\mathcal{L}_Z := |\mathcal{O}_Y(1) \otimes \mathcal{I}_Z|$ is a smooth curve of genus-one. We denote by $\sigma : Y_Z \to Y$ the blowup of Y along Z as in Section 2.

If $l(Z) = d-2$, then the general $(d-2)$-dimensional linear space containing Z intersects $Y \setminus Z$ at two distinct points. The birational involution obtained by exchanging these two points induces a birational involution σ_G on the blowup Y_Z of Y at Z. We call this σ_G a *generalized Geiser involution*. Observe that a general $(d-1)$-dimensional linear space Λ containing Z cuts on Y a genus-one curve C, preserved by the Geiser involution. The quotient of C by the induced involution is the $\mathbb{P}^1$ given by the pencil of hyperplanes in Λ containing Z. In particular σ_G restricts to an hyperelliptic involution on C.

When $l(Z) = d-1$, denote by F the divisor on Y_Z defined as before. The base locus of the linear system $|F|$ consists of one point q while $|2F|$ defines a morphism φ. Since $F^n = 1$, we have that F^{n-1} is rationally equivalent to an elliptic curve C passing through q and the restriction $\varphi_{|C}$ is a double covering of a line passing through the point $p := \varphi(q)$. Hence the image $\varphi(Y_Z)$ is a cone V. If we denote by E the exceptional divisor corresponding to the last blowup of σ, we have that the restriction $\varphi_{|E}$ is the 2-veronese embedding v_2 of $\mathbb{P}^{n-1}$. We conclude that V is a cone over $v_2(\mathbb{P}^{n-1})$ and φ induces a birational involution σ_B on Y_Z that we call a *generalized Bertini involution*. We remark that if X is the del Pezzo elliptic variety obtained by blowing up Y_Z at q, then σ_B induces on X the hyperelliptic involution with respect to the origin given by the exceptional divisor.

Remark 5.2. If Y has degree four and the line $\langle Z\rangle$ does not contain any vertex p_i, then the indeterminacy locus of the corresponding Geiser involution σ_G has codimension two. Moreover, it lifts to an isomorphism in codimension one for the elliptic varieties of type X_{21} and X_{10}. The action on the Picard group of X in each case is given by the following matrices respectively

$$\sigma_{21} = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ -4 & -2 & -1 & 0 & 0 \\ -4 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \sigma_{10} = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ -4 & -1 & -2 & 0 & 0 \\ -4 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To prove this, we first claim that the lifted birational map preserves the elliptic fibration π and thus it is a flop. Indeed, if f is a fibre of π whose image C in Y is cut out by a three-dimensional linear space L and we fix a point $y \in C$, then the plane spanned by y and $\langle Z\rangle$ is contained in L and thus it intersects C at a fourth point, so that $\phi(f) = f$, which proves the claim. Since ϕ preserves the fibration π, its pull-back ϕ^* must preserve the set of sections of π and the set of vertical classes of X. A direct calculation shows that the representative matrix for ϕ^* with respect to the basis $(H, E_1, \dots, E_4)$ is one of the above in each case.

As we remarked before, the Geiser involution induces an hyperelliptic involution on a general fiber and thus on the generic fiber as well. Moreover σ_{10} has a fixed point on the generic fiber, defined over the function field $\mathbb{C}(\mathbb{P}^{n-1})$, which corresponds to the only section E_4. On the other hand σ_{21} has no fixed points defined over $\mathbb{C}(\mathbb{P}^{n-1})$ since it exchanges the two sections E_3 and E_4.

5.4. Mori chambers

Let X be a del Pezzo elliptic variety of degree four with finite Mordell-Weil group. We provide here the Mori chamber decomposition of the moving cone $\mathrm{Mov}(X)$ of X. In the following proposition, we will denote by N the nef cone of X and by

$$N_i := \mathrm{cone}(\{F_I \,:\, i \in I \text{ and } F_I \in N\} \cup \{H - 2E_i\}).$$

Proposition 5.3. *Let X be a del Pezzo elliptic variety of degree four such that the corresponding elliptic fibration has finite Mordell-Weil group. Then the Mori chamber decomposition of* $\mathrm{Mov}(X)$ *is given in the following table.*

Table 5.1.

Type of X	Cones
X_{43}	N, N_1, N_2, N_3, N_4
X_{22}	N, N_1, N_2
X_{21}	$N, N_1, N_2, \sigma_{21}^*(N), \sigma_{21}^*(N_1), \sigma_{21}^*(N_2)$
X_{11}	N, N_1
X_{10}	$N, N_1, \sigma_{10}^*(N), \sigma_{10}^*(N_1)$

Proof. Let $X \to X_i$ be the flop of the class $h - e_i$ of the strict transform C of a line through the point $q_i \in Y$. Note that such a flop exists by [5]. We show that the nef cone of X_i is N_i and then observe that the union of the cones in the table in the statement is $\mathrm{Mov}(X)$ for each type. To prove the claim, we begin by showing that the primitive generators of the extremal rays of the cone N_i are nef in X_i. Observe that each $F_I \in N_i$ is nef in both X and X_i since $F_I \cdot (h - e_i) = 0$ by our definition of N_i. Hence we only have to check that also $H - 2E_i$ is nef in X_i. Since $H - 2E_i$ is the pull-back of a class on the blowup $\widetilde{Y}$ of Y at q_i, it is enough to prove the claim on $\widetilde{Y}$. By Lemma 6.2 the Cox ring of $\widetilde{Y}$ is finitely generated and the moving cone decomposes as follows:

$$\mathrm{Mov}(\widetilde{Y}) = \mathrm{cone}(H, H - E_i) \cup \mathrm{cone}(H - E_i, H - 2E_i).$$

Thus, after flopping $h - e_i$ the class $H - 2E_i$ becomes nef as claimed so that we have the inclusion $N_i \subseteq \mathrm{Nef}(X_i)$. To prove that this is indeed an equality, we show that the extremal rays of the dual cone of N_i are classes of effective curves of X_i. To this aim we make use of [5, Lemma 4.1] which asserts that if Γ is a curve of X which meets C transversally at k points, and no other effective curve of class $h - e_i$, then the flop image Γ' of Γ has class

$$[\Gamma'] = [\Gamma] + k[C]. \tag{5.2}$$

By a direct calculation we see that the extremal rays of the dual cone of N_i are the following (here we list only the case $i = 1$, being the remaining cases analogous):

Table 5.2.

Type	**Extremal rays of the Mori cone**	**Extremal rays of the dual cone of N_1**
X_{43}	$e_1,\ e_2,\ e_3,\ e_4$ $h - e_1,\ h - e_2,\ h - e_3,\ h - e_4$	$-h + e_1,\ e_2,\ e_3,\ e_4,$ $2h-e_1-e_2,\ 2h-e_1-e_3,\ 2h-e_1-e_4$
$X_{22},\ X_{21}$	$e_2,\ e_4$ $h - e_1,\ h - e_3$ $e_1 - e_2,\ e_3 - e_4$	$-h + e_1, e_2,\ e_4$ $2h - e_1 - e_2,\ 2h - e_1 - e_3$ $e_3 - e_4$
$X_{11},\ X_{10}$	$h - e_1,\ e_4,\ e_1 - e_2$ $e_2 - e_3,\ e_3 - e_4$	$-h + e_1,\ e_4, e_2 - e_3,$ $e_3 - e_4,\ 2h - e_1 - e_2$

We are going to show that any class on the right column is effective since it can be obtained, by means of (5.2), from a curve of X having the same class and not intersecting any curve with class $h - e_1$. The assertion is straightforward for the classes e_i and $e_i - e_{i+1}$, with $i > 1$ and hence in order to conclude we have to consider the case in which Γ is a curve such that $[\Gamma] = 2h - e_1 - e_i$, with $i > 1$. We can assume that Γ is the strict transform of a smooth conic $\mathcal{C}$ of Y passing through q_1 and q_i (possibly infinitely near to q_1). The tangent line to $\mathcal{C}$ at q_1 can not be contained in Y since otherwise the plane spanned by $\mathcal{C}$ and this line would be contained into each quadric of the pencil and thus in Y. This gives a contradiction, since the line through q_1 and q_i is not contained in Y by hypothesis. Therefore any line of Y through q_1 is not tangent to $\mathcal{C}$ at this point so that their strict transforms do not intersect.

We conclude that $N_1 = \mathrm{Nef}(X_1)$ and anagously we can obtain the same equality for $i > 1$, proving the assertion for X_{43}, X_{22} and X_{11} (since in these cases the union of the cones N_i is the whole moving cone).

In case $X = X_{21}$ the chamber $\sigma_{21}^*(N)$ is the pull-back of the nef cone $N = \mathrm{Nef}(X)$ via the flop σ_{21}. Since σ_{21} is the generator of the Mordell-Weil group of π, we deduce that $\sigma_{21}(X)$ is a del Pezzo elliptic variety of the same type. Thus each chamber $\sigma_{21}^*(N_i)$, for $i = 1, 2$, is a flop image of $\sigma_{21}^*(N)$ exactly as N_i is a flop image of N. In particular the chamber $\sigma_{21}^*(N_i)$ is generated by finitely many semiample classes of $\sigma_{21}(X_i)$.

Finally, in case $X = X_{10}$ we proceed as we did for X_{21}, considering σ_{10} instead of σ_{21}. □

Proof of Theorem 1.2. By [7, Lemma 3.5] (1) implies (2), so let us suppose that the Mordell-Weil group of π is finite. If $d = 1$ or 2, then we conclude by means of Theorem 6.1, while the case $d = 3$ has been proved in [7, Theorem 3.6]. Finally, when $d = 4$, we observe that by Proposition 5.3, if the Mordell-Weil group of the fibration is finite, then the moving cone $\mathrm{Mov}(X)$ satisfies all the hypotheses of Hu and Keel theorem [8, Proposition 2.9]. □

6. Cox rings

In this section we provide a presentation for the Cox rings of the del Pezzo elliptic varieties of degree $d \leq 2$. We recall that given a normal projective variety X with finitely generated Picard group, its Cox ring $\mathcal{R}(X)$ can be defined as (see [1])

$$\mathcal{R}(X) = \bigoplus_{[D]\in \mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

We apply [6, Algorithm 5.4] and we will explain all the steps in the algorithm for the convenience of the reader. Let Y_1 be a smooth projective variety with finitely generated Cox ring R_1, which admits a presentation $R_1 = \mathbb{C}[T_1, \dots, T_{r_1}]/I_1$. Note that R_1 is K_1-graded, where $K_1 = \mathrm{Cl}(Y_1)$. Define $\overline{Y}_1 = \mathrm{Spec}(R_1)$ and let $\widehat{Y}_1 \subseteq \overline{Y}_1$ be the characteristic space of Y_1 with characteristic map $p\colon \widehat{Y}_1 \to Y_1$. Let Y_2 be the blowup of Y_1 at a point $q \in Y_1$. Let $I \subseteq R_1$ be the ideal of $p^{-1}(q)$ in $\overline{Y}_1$, and let $J \subseteq R_1$ be the irrelevant ideal, *i.e.* the ideal of $\overline{Y}_1 \setminus \widehat{Y}_1$. Then the Cox ring of Y_2 is isomorphic to the extended saturated Rees algebra:

$$R_1[I]^{\mathrm{sat}} = \sum_{m\in\mathbb{Z}} (I^m : J^\infty)t^{-m},$$

where $I^m = R_1$ if $m \leq 0$. An element of $I^m : J^\infty$ defines a hypersurface of Y_1 which has multiplicity at least m at q. We choose a finite set of elements $g_i \in I^{m_i} : J^\infty$, with $1 \leq i \leq k$ and form the subalgebra $A = R_1[g_1t^{-m_1}, \dots, g_kt^{-m_k}, t]$ of $R_1[I]^{\mathrm{sat}}$, which admits a presentation $\mathbb{C}[T_1, \dots, T_{r_1+k}, S]/I_2$, where

$$I_2 = \left\langle T_{r_1+1}S^{m_1} - g_1, \dots, T_{r_1+k}S^{m_k} - g_k \right\rangle + I_1 : \langle S\rangle^\infty. \tag{6.1}$$

The test to verify the equality $A = R_1[I]^{\mathrm{sat}}$ is to check whether the inequality

$$\dim I_2 + \langle S\rangle > \dim I_2 + \langle S, T_\nu\rangle, \tag{6.2}$$

holds, where T_ν is the product of all the T_i's, for $1 \leq i \leq r_1$, such that T_i does not vanish identically at $p^{-1}(q)$.

6.1. Degree one and two

In this subsection we provide a presentation for the Cox rings of the del Pezzo elliptic varieties of degree at most two with finite Mordell-Weil group. Our main result is the following.

Theorem 6.1. *Let $\pi : X \to \mathbb{P}^{n-1}$ be a del Pezzo elliptic fibration of degree $d \leq 2$ having finite Mordell-Weil group. Then the Cox ring of X and its grading matrix are listed in the following table:*

Table 6.1.

Type	Cox ring	Grading matrix
X_1	$\dfrac{\mathbb{C}[T_1, \ldots, T_{n+2}, S]}{\langle T_1^2 - T_2^3 + T_2 \tilde{f}_4 S^4 + \tilde{f}_6 S^6 \rangle}$ $\tilde{f}_d := f_d(T_1, T_2, T_3 S, \ldots, T_{n+2} S)$	$\begin{bmatrix} 3 & 2 & 1 & \ldots & 1 & 0 \\ 0 & 0 & -1 & \ldots & -1 & 1 \end{bmatrix}$
X_{SS}	$\dfrac{\mathbb{C}[T_1, \ldots, T_{n+3}, S_1, S_2]}{\langle T_{n+2} S_1^4 - T_{n+3} S_2^4 + 2\tilde{h},\ T_{n+2} T_{n+3} - \tilde{g} \rangle}$ $\tilde{h} := h(T_1 S_1 S_2, \ldots, T_n S_1 S_2, T_{n+1}),$ $\tilde{g} := g(T_1, \ldots, T_n)$	$\begin{bmatrix} 1 & \ldots & 1 & 1 & 2 & 2 & 0 & 0 \\ -1 & \ldots & -1 & 0 & -4 & 0 & 1 & 0 \\ -1 & \ldots & -1 & 0 & 0 & -4 & 0 & 1 \end{bmatrix}$
X_2	$\dfrac{\mathbb{C}[T_1, \ldots, T_{n+2}, S_1, S_2]}{\langle T_{n+2}^2 - S_2^2 \tilde{f} - T_n T_{n+1}^3 \rangle}$ $\tilde{f} := \dfrac{f(T_1 S_1 S_2^2, \ldots, T_{n-1} S_1 S_2^2, T_n S_1^2 S_2^2, T_{n+1})}{S_1^2 S_2^2}$	$\begin{bmatrix} 1 & \ldots & 1 & 1 & 1 & 2 & 0 & 0 \\ -1 & \ldots & -1 & -2 & 0 & -1 & 1 & 0 \\ -1 & \ldots & -1 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$

Proof. In order to prove the case X_1, let Y_1 be the del Pezzo variety given by the polynomial (3.1) and let $q \in Y_1$ be the point of coordinates $(1, 1, 0, \ldots, 0)$. The ring R_1 equals $\mathbb{C}[T_1, \ldots, T_{n+2}]/I_1$, where I_1 is the principal ideal generated by the polynomial (3.1). We take $I, J \subseteq R_1$ as before and we choose the following homogenous elements $g_1, \ldots, g_n$:

$$T_3, \ldots, T_{n+2} \in I$$

i.e. for all of them we have $m_i = 1$. Observe that the saturated ideal (6.1) is

$$I_2 = \langle T_{n+2+i} S - g_i \, : \, 1 \leq i \leq n \rangle + I_1$$

since, after applying the substitution $T_{2+i} = T_{n+2+i} S$ for each $i = 1, \ldots, n$, the resulting polynomial $T_1^2 - T_2^3 + T_2 \tilde{f}_4 S^4 + \tilde{f}_6 S^6$ is not divisible by S. Finally, according to (6.2), we need to show that

$$\dim I_2 + \langle S \rangle > \dim I_2 + \langle S, T_1 T_2 \rangle,$$

and this is easily checked, being I_1 a principal ideal. We conclude that the ring $\mathbb{C}[T_1, \ldots, T_{2n+2}, S]/I_2$ is isomorphic to the the Cox ring of the blowup X_1 of Y at q. After eliminating the fake linear relations and renaming the variables, we get the claimed presentation for the Cox ring.

We now prove the case X_{SS}. Let Y_1 be the del Pezzo variety given by the polynomial (3.6) and let $q \in Y_1$ be the point of coordinates $(0, \dots, 0, 1, 1)$. The ring R_1 equals $\mathbb{C}[T_1, \dots, T_{n+2}]/I_1$, where I_1 is the principal ideal generated by the polynomial (3.6). We take $I, J \subseteq R_1$ as before and choose the following homogeneous elements $g_1, \dots, g_{n+1}$:

$$T_1, \dots, T_n \in I, \qquad T_{n+2} - h \in \bigl(I^4 \colon J^\infty\bigr),$$

that is the first n sections have $m_i = 1$, while $m_{n+1} = 4$. Observe that the ideal in (6.1) is

$$I_2 = \bigl\langle T_{n+2+i} S_1^{m_i} - g_i \,:\, 1 \le i \le n+1 \bigr\rangle + \bigl\langle T_{2n+3}^2 S_1^4 + 2\,h' T_{2n+3} - g' \bigr\rangle,$$

where $h' = h(T_{n+3}S_1, \dots, T_{2n+2}S_1, T_{n+1})$ and $g' = g(T_{n+3}, \dots, T_{2n+2})$. According to (6.2), we can easily check that the following inequality holds:

$$\dim I_2 + \langle S_1 \rangle > \dim I_2 + \langle S_1, T_{n+1} T_{n+2} \rangle.$$

Thus, after eliminating the fake linear relations from I_2 and renaming the variables, we can conclude that the Cox ring and the grading matrix of the blowup Y_2 of Y_1 at q are the following

$$R_2 = \frac{\mathbb{C}[T_1, \dots, T_{n+2}, S_1]}{\bigl\langle T_{n+2}^2 S_1^4 + 2h'' T_{n+2} - g'' \bigr\rangle} \qquad \begin{bmatrix} 1 & \dots & 1 & 1 & 2 & 0 \\ -1 & \dots & -1 & 0 & -4 & 1 \end{bmatrix}$$

where $h'' = h(T_1 S_1, \dots, T_n S_1, T_{n+1})$ and $g'' = g(T_1, \dots, T_n)$. The irrelevant ideal is $J_2 = \langle T_1, \dots, T_n, T_{n+2} \rangle \cap \langle T_{n+1}, S_1 \rangle$. We now repeat the procedure blowing-up Y_2 at the point q_2' which lies over $q_2 = (0, \dots, 0, 1, -1) \in Y$. Recall that there is a $\mathbb{C}^*$-equivariant embedding of total coordinate spaces

$$\overline{Y}_1 \to \overline{Y_2} \qquad (T_1, \dots, T_{n+2}) \to (T_1, \dots, T_{n+1}, T_{n+2} - h, 1)$$

which induces the birational map $Y_1 \dashrightarrow Y_2$. The image of q_2 is the point of homogeneous coordinates $q_2' = (0, \dots, 0, 1, -2, 1)$. We choose the following homogenous elements $g_1, \dots, g_{n+2}$:

$$T_1, \dots, T_n, 2T_{n+1}^2 + T_{n+2} S_1^4 \in I, \qquad T_{n+2} S_1^4 + 2h'' \in \bigl(I^4 \colon J^\infty\bigr),$$

that is the first $n+1$ sections have $m_i = 1$, while $m_{n+2} = 4$. The ideal in (6.1) is

$$I_3 = \bigl\langle T_{n+3+i} S_2^{m_i} - g_i \,:\, 1 \le i \le n+2 \bigr\rangle + \bigl\langle T_{n+2}^2 S_1^4 + 2 T_{n+2} h''' - S_2^4 g''' \bigr\rangle$$

where $h''' = h(T_{n+4} S_1 S_2, \dots, T_{2n+3} S_1 S_2, T_{n+1})$ and $g''' = g(T_{n+4}, \dots, T_{2n+3})$. After eliminating the fake linear relations from the above ideal and renaming the variables, we get the statement for X_{SS}.

Finally, let us prove the statement for X_2. Let Y_1 be the del Pezzo variety given by $x_{n+2}^2 - f$, where f is the polynomial appearing in (3.7), and let $q \in Y_1$ be the point of coordinates $(0, \dots, 0, 1, 0)$. After a linear change of coordinates we can suppose that $f_1 = x_n$. The ring R_1 equals $\mathbb{C}[T_1, \dots, T_{n+2}]/I_1$, where I_1 is the principal ideal generated by the polynomial of Y_1. We take $I, J \subseteq R_1$ as before and choose the following homogenous elements $g_1, \dots, g_{n+1}$:

$$T_1, \dots, T_{n-1}, T_{n+2} \in I, \qquad T_n \in \left(I^2 \colon J^\infty\right),$$

that is the first n sections have $m_i = 1$, while $m_{n+1} = 2$. Observe that the ideal in (6.1) is

$$I_2 = \left\langle T_{n+2+i} S^{d_i} - g_i \,:\, 1 \leq i \leq n+1 \right\rangle + \left\langle T_{2n+2}^2 - f' - T_{2n+3} T_{n+1}^3 \right\rangle$$

with $f' = S^{-2} f(T_{n+3} S_1, \dots, T_{2n+1} S, T_n S^2, T_{n+1})$. According to (6.2) it can be easily checked that

$$\dim I_2 + \langle S \rangle > \dim I_2 + \langle S, T_{n+1} \rangle.$$

Thus, after eliminating the fake linear relations from I_2 and renaming the variables, we conclude that the Cox ring and the grading matrix of the blowup Y_2 of Y_1 at q are the following

$$R_2 = \frac{\mathbb{C}[T_1, \dots, T_{n+2}, S]}{\left\langle T_{n+2}^2 - f'' - T_n T_{n+1}^3 \right\rangle} \qquad \begin{bmatrix} 1 & \dots & 1 & 1 & 1 & 2 & 0 \\ -1 & \dots & -1 & -2 & 0 & -1 & 1 \end{bmatrix}$$

with $f'' = S^{-2} f(T_1 S_1, \dots, T_{n-1} S, T_n S^2, T_{n+1})$. The irrelevant ideal of R_2 is $J_2 = \langle T_1, \dots, T_{n-1}, T_{n+2} \rangle \cap \langle T_n, S \rangle$. Now repeat the procedure by blowing up Y_2 at the point $q_2' = (0, \dots, 0, 1, 1, 1, 0)$ which is the invariant point with respect to the lifted involution $(T_1, \dots, T_{n+2}, S) \mapsto (T_1, \dots, T_{n+1}, -T_{n+2}, S)$, and it corresponds to the generator of the kernel of the differential $d\varphi_q$. We choose the following homogenous elements $g_1, \dots, g_n$:

$$T_1, \dots, T_{n-1}, S \in I,$$

i.e. $m_i = 1$ for all the sections. The ideal in (6.1) is

$$I_3 = \left\langle T_{n+3+i} S_2 - g_i \,:\, 1 \leq i \leq n \right\rangle + \left\langle T_{n+2}^2 - \tilde{f} - T_n T_{n+1}^3 \right\rangle$$

where $\tilde{f} = T_{2n+3}^{-2} S_2^{-2} f''(T_{n+4} T_{2n+3} S_2^2, \dots, T_n T_{2n+3}^2 S_2^2, T_{n+1})$. After eliminating the fake linear relations from the above ideal and renaming the variables, we obtain the statement for X_2. □

6.2. Degree four

In this last subsection we provide the following presentation for the Cox ring of the blowup of a del Pezzo variety of degree four at a point. This is used in the proof of Proposition 5.3.

Lemma 6.2. *Let Y be a smooth complete intersection of two quadrics of $\mathbb{P}^{n+2}$. After applying a linear change of coordinates, the ideal of Y is generated by $x_2x_3 - x_1x_2 + f(x_4, \dots, x_{n+3})$ and $x_2x_3 - x_1x_3 + g(x_4, \dots, x_{n+3})$. The blowup $\widetilde{Y}$ of Y at the point $q = (1, 0, \dots, 0) \in Y$ has the following Cox ring and grading matrix*

$$\frac{\mathbb{C}[T_1, \dots, T_{n+3}, S]}{\left\langle T_2 T_3 S^2 - T_1 T_2 + f, T_2 T_3 S^2 - T_1 T_3 + g \right\rangle} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ 0 & -2 & -2 & -1 & \dots & -1 & 1 \end{bmatrix}$$

respectively, where $f = f(T_4, \dots, T_{n+3})$ and $g = g(T_4, \dots, T_{n+3})$.

Proof. After applying a linear change of coordinates, we can assume that q is a point of $Y = Q \cap Q'$, where Q is singular at $(1, 1, 0, 0, \dots, 0)$ and Q' is singular at $(1, 0, 1, 0, \dots, 0)$, and that the tangent hyperplanes to Q and Q' at q are $V(T_2)$ and $V(T_3)$, respectively. This proves the first claim.

To prove the second statement, we take R_1 to be $\mathbb{C}[T_1, \dots, T_{n+3}]/I_1$, where I_1 is the ideal of Y, and we apply [6, Algorithm 5.4]. We take $I, J \subseteq R_1$ as before and choose the following homogenous elements $g_1, \dots, g_{n+2}$:

$$T_4, \dots, T_{n+3} \in I, \qquad T_2, T_3 \in \left(I^2 \colon J^\infty\right),$$

that is the first n sections have $m_i = 1$, while $m_{n+1} = m_{n+2} = 2$. The ideal in (6.1) is

$$\begin{aligned} I_2 = {} & \left\langle T_{n+3+i} S^{m_i} - g_i \,:\, 1 \le i \le n+2 \right\rangle + \\ & + \left\langle T_{2n+4} T_{2n+5} S^2 - T_1 T_{2n+4} + \tilde{f}, T_{2n+4} T_{2n+5} S^2 - T_1 T_{2n+5} + \tilde{g} \right\rangle, \end{aligned}$$

where $\tilde{f} := f(T_{n+4}, \dots, T_{2n+3})$ and $\tilde{g} := g(T_{n+4}, \dots, T_{2n+3})$. According to (6.2) it can be easily checked that

$$\dim I_2 + \langle S \rangle > \dim I_2 + \langle S, T_1 \rangle.$$

After eliminating the fake linear relations from I_2 and renaming the variables, we get the second statement. □

Appendix

In this appendix we provide some Magma programs used in the proof of Theorem 1.3. First of all we describe the function `Test` that verifies if the inclusions in (5.1) are in fact equalities.

```
L := ToricLattice(5);
H := Basis(L)[1];
E := Basis(L)[2..5];
D := DiagonalMatrix([4,–1, −1, −1, −1]);

Test := function(C)
  DualC := Cone([L!r : r in Rays(Dual(C*D))]);
  CapC := &meet[Cone(Remove(Rays(C),i)) : i in [1..#Rays(C)]];
   return DualC eq CapC;
end function;
```

We now proceed to use the above function `Test` to check that (5.1) are equalities for X_{43}, X_{22}, X_{21}.

```
C43 := Cone(E[1..4] cat [H-2*E[i]-2*E[j] : i,j in [1..4] — i lt j]);
C22 := Cone(E[3..4] cat [E[1]-E[3], E[2]-E[4]] cat
  [H-2*E[1]-2*E[2], H-2*E[1]-2*E[3], H-2*E[2]-2*E[4]]);
C21 := Cone(E[3..4] cat [E[1]-E[3], E[2]-E[4]] cat
  [H-2*E[1]-2*E[3], H-2*E[2]-2*E[4], H-2*E[1]-E[2]-E[4], H-E[1]-    2*E[2]-
E[3]]);

for C in [C43,C22,C21] do Test(C); end for;
```

Finally we compute the primitive generators of the extremal rays of the dual cone $\mathcal{M}^\vee$ for X_{11} and X_{10}.

```
> C11 := Cone([E[4],H-2*E[1]-2*E[2]] cat [E[i]-E[i+1] : i in [1..3]]);
> C10 := Cone([E[4],H-2*E[1]-E[2]-E[3]]cat [E[i]-E[i+1]: i in [1..3]]);
> for C in [C11,C10] do
    Rays(Cone([L!r : r in Rays(Dual(C*D))]));
end for;
[
    (1, -2, 0, 0, 0),
   (1, -1, -1, -1, -1),
   (1, -1, -1, -1, 0),
   (1, -1, -1, 0, 0),
   (1, 0, 0, 0, 0)
]
[
   (1, -2, 0, 0, 0),
   (1, -1, -1, -1, -1),
   (1, -1, -1, -1, 0),
   (1, 0, 0, 0, 0),
   (3, -4, -4, 0, 0)
]
```

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