

Poincaré inequalities for the maximal function

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Abstract. We study generalized Poincaré inequalities. We prove that if a function satisfies an inequality of Poincaré type, then the Hardy-Littlewood maximal function also obeys an estimate of similar form. As a by-product, we get a unified approach to proving that the maximal operator is bounded on Sobolev, Lipschitz and BMO spaces.

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1. Introduction

It was proved by Kinnunen in [22] that the Hardy-Littlewood maximal function gives rise to a bounded operator

$$M : W^{1,p}(\mathbb{R}^n) \rightarrow W^{1,p}(\mathbb{R}^n), \quad 1 < p \leq \infty.$$

Similar results have later been established for maximal functions restricted to domains [23], for fractional maximal functions [18, 19, 24], and for certain convolution type maximal functions [8, 10]. Continuity as well as action on some function spaces in the Triebel-Lizorkin scale have been studied in [25, 29, 30]. It is also well known that the maximal function behaves well on BMO and spaces of Hölder continuous functions. See [1, 5].

Many smoothness properties of functions can be characterized using generalized Poincaré inequalities. For instance, $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ has weak derivatives in $L^1_{\text{loc}}(\mathbb{R}^n)$ if and only if there is a non-negative function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$\int_Q |u - u_Q| \, dx \lesssim \text{diam}(Q) \int_Q g \, dx$$

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for all cubes Q [15]. See Section 2 for definitions. Replacing the functional

$$a(Q) = \text{diam}(Q) \oint_Q g \, dx$$

by more general *fractional averages*

$$a(Q) = \text{diam}(Q)^\alpha \frac{\mu(Q)}{|Q|}, \quad \alpha \in [0, 1],$$

we recover characterizations of other spaces of regular functions.

The present paper is devoted to studying the action of the Hardy-Littlewood maximal operator at the level of generalized Poincaré inequalities. Our main theorem states that certain general Poincaré type inequalities are preserved when we apply the maximal operator. See Theorem 3.1 for the precise statement. As a consequence, we can include many results about the regularity of the maximal functions in a single theorem. The result is also interesting from the point of view of more abstract Poincaré inequalities; see [2, 3, 13, 21, 33]. We do not know what are the most general functionals $a(Q)$ to which our approach applies, but a list of examples and corollaries demonstrates that the result for fractional averages is already interesting. Our proof is very geometric, and even though we restrict ourselves to the Euclidean space for the present paper, the approach is likely to extend to slightly more general metric spaces. In fact, the main technical tool we use, a self-improvement lemma by Franchi-Pérez-Wheeden [13], is valid in spaces of homogeneous type.

In addition to finding a unified approach to different function spaces, there is one more motivation for studying Poincaré inequalities that the maximal function satisfies. Namely, it is an open problem whether $u \mapsto |\nabla Mu|$ is a bounded operator from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for $n > 1$. This is Question 1 in [17], and it has attracted a lot of attention; see for instance [4, 7, 9] and the references therein. Progress on this problem has been restricted to dimension one, where the question was answered in positive by Tanaka [38] and Kurka [26], and to the radial case, where the problem was solved by Luiro [31].

Even if we do not know how to bound $|\nabla Mu|$ from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, a result from [17] ensures the local integrability of ∇Mu under the additional hypothesis $|\nabla u| \in L \log L$. This partial result can be included in the list of corollaries that follow from our main theorem. What prevents us from solving the question in full generality is the fact that our approach is based on dominating the gradient by the maximal function of the gradient, which need not be locally integrable a priori. Better results on this problem would require an more delicate analysis than what is possible to carry out in our general setting.

The structure of the present paper is as follows. Section 2 introduces the notation and preliminary results. Section 3 contains the main theorem which establishes that the Poincaré inequalities are preserved under the action of the maximal operator. Several corollaries are also discussed. Section 4 gives an application towards the study of the endpoint Sobolev space $W^{1,1}(\mathbb{R}^n)$. Section 5 contains a few remarks on the extension of the results to the fractional maximal function.

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2. Notation and preliminaries

We use standard notation in $\mathbb{R}^n$, $n > 1$. The letter C denotes a constant only depending on quantities we do not keep track of. Such quantities include the absolute constants and the dependency on the numerical value of the dimension $n \geq 1$. If $a \leq Cb$, we write $a \lesssim b$. For a measurable set $E \subset \mathbb{R}^n$ we denote by $|E|$ its Lebesgue measure. By a function we mean a measurable real valued function of n real variables. If u is a locally integrable function, then

$$u_E = \int_E u \, dx = \frac{1}{|E|} \int_E u \, dx.$$

The diameter of a bounded set E is denoted by

$$\text{diam}(E) = \sup\{|x - y| : x, y \in E\}.$$

Sometimes we only use positive parts of functions. Then we denote $u^+ = 1_{\{u>0\}}u$.

By a cube we always mean a cube with sides parallel to coordinate axes. In addition, our cubes are open even though sets of measure zero do not matter in our considerations. For a cube Q , we denote its side length by $\ell(Q)$ and its center by $c(Q)$. For a positive constant $\lambda > 0$, we understand λQ to be the cube with center $c(Q)$ and side length $\lambda\ell(Q)$.

We define the non-centred Hardy-Littlewood maximal function of a locally integrable function to be

$$Mu(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |u| \, dx$$

and of a locally finite Borel measure to be

$$M\mu(x) = \sup_{Q \ni x} \frac{\mu(Q)}{|Q|}$$

where the supremum is over all cubes with sides parallel to coordinate axes. The centred maximal function uses the supremum with the restriction that the cubes must be centred at x . We do not introduce separate notation for it since most of our results holds for both of them. We mention in our statements which variant we are using. We need the basic fact that $M : L^1(\mathbb{R}^n) \rightarrow L^{1,\infty}(\mathbb{R}^n)$ is bounded where

$$\|u\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |u(x)| > \lambda\}|^{1/p}, \quad 0 < p < \infty.$$

Also the boundedness $L^{p,\infty} \rightarrow L^{p,\infty}$ with $p > 1$ is needed. For the proof of that fact (by interpolation) and all other basic properties of $L^{p,\infty}$, we refer to the textbook [14]. For an exponent $p \in (1, \infty)$, we denote the conjugate exponent by $p' = p/(p-1)$.

A function u is said to belong to the local Sobolev space $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ for $p \in [1, \infty)$ if both u and its distributional gradient belong to $L_{\text{loc}}^p(\mathbb{R}^n)$, that is, for all compact K

$$\int_K (|u|^p + |\nabla u|^p) \, dx < \infty.$$

The global Sobolev space $W^{1,p}(\mathbb{R}^n)$ is defined similarly, but global integrability is required instead. We can norm this space by summing the L^p norms of the function and its gradient. Sobolev spaces can also be defined on domains; we mean open and connected subsets of $\mathbb{R}^n$. Weak differentiability of Sobolev functions can be characterized through Poincaré type inequalities.

Definition 2.1. Let $u \in L_{\text{loc}}^1(\mathbb{R}^n)$ and let μ be a locally finite positive Borel measure. We say that u satisfies (q, α, μ) -Poincaré inequality if for all cubes Q it holds

$$\left(\int_Q |u - u_Q|^q \, dx \right)^{1/q} \leq \text{diam}(Q)^\alpha \frac{\mu(Q)}{|Q|}.$$

Here $q \geq 1$ and $\alpha \geq 0$.

This general form of a Poincaré type inequality can be used to characterize various function spaces. As we show that it is preserved under the action of the Hardy-Littlewood maximal operator, we get a unified approach to studying the maximal function on such spaces. We list some examples.

- Let $p \in [1, \infty]$. Then $u \in W^{1,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and it satisfies the inequality of Definition 2.1 with $\alpha = 1$, $q = 1$ and $\mu \in L^p(\mathbb{R}^n)$. For sufficiency, see [15] and [27]. The necessity is the classical Poincaré inequality, see [12];
- $u \in BV$ if and only if $u \in L^1(\mathbb{R}^n)$ and it satisfies the inequality of Definition 2.1 with $\alpha = 1$, $q = 1$ and μ a Radon measure. See for instance [36] and [12];
- $u \in \Lambda(\alpha)$ (the space of α -Hölder continuous functions) if and only if $u \in L_{\text{loc}}^1(\mathbb{R}^n)$ and it satisfies the inequality of Definition 2.1 with $\alpha \in (0, 1)$, $q = 1$ and $\mu \in L^\infty(\mathbb{R}^n)$. See [6] and [35];
- $u \in BMO$ if and only if $u \in L_{\text{loc}}^1(\mathbb{R}^n)$ and it satisfies the inequality of Definition 2.1 with $\alpha = 0$, $q = 1$ and $\mu \in L^\infty(\mathbb{R}^n)$. This is just the definition of BMO.

We emphasize that Poincaré inequalities encode the local behaviour. For example, local Sobolev spaces are nested and decreasing with p . We can always study the validity of $(1, 1, \mu)$ -Poincaré inequality in order to find out whether or not the function has a weak derivative. It is then another task to find out if it belongs to the correct L^p space, locally or globally. Moreover, the modulus of the weak gradient of a Sobolev function u is, up to a dimensional constant, the minimal μ that can be

inserted to the Poincaré inequality. This is proved along with the characterization in [15].

All of the previous examples share the common feature of exhibiting a self-improving property. Namely, if the inequalities above hold with $q = 1$, then they also hold with some $q > 1$. The classical instances of this phenomenon are known as Sobolev-Poincaré and John-Nirenberg inequalities. However, similar phenomena also occur in the very general setting of μ being a locally finite Borel measure and beyond. This is sometimes called Franchi-Pérez-Wheeden self-improvement.

Lemma 2.2 (Franchi, Pérez, and Wheeden [13]). *Let $u \in L^1_{\text{loc}}$, μ be a locally finite positive Borel measure and $\alpha \in (0, 1]$. Assume that u satisfies $(1, \alpha, \mu)$ -Poincaré inequality of Definition 2.1. Then there exists $C > 0$ not depending on u such that for all cubes Q it holds*

$$\frac{\|1_Q(u - u_Q)\|_{L^{q,\infty}(\mathbb{R}^n)}}{|Q|^{1/q}} \leq C \operatorname{diam}(Q)^\alpha \frac{\mu(Q)}{|Q|}$$

where $q = n/(n - \alpha)$.

Proof. This is just a special case of [13, Theorem 2.3]. The explicit value of q is given in Example 2.2 of that paper. The fact that we are in $\mathbb{R}^n$ and use cubes and the Lebesgue measure saves us from dilating the cube in the right side, as pointed out in [13, Remark 2.6]. The constant C does not depend on u because the possible constants in Definition 2.1 are hidden in the measure μ . $\square$

Finally, we point out that in order to get the self-improvement, it would be enough to assume much less than the $(q = 1, \alpha, \mu)$ -Poincaré inequality, see [28], but for our purposes the case $q = 1$ is enough.

3. Poincaré inequality and maximal function

The fact that the maximal function preserves some Poincaré inequalities is a consequence of two phenomena. The first one is the improvement of local integrability. Concrete instances of this phenomenon are the John-Nirenberg inequality and the Sobolev-Poincaré inequality. Improvement of local integrability is important when working at small scales. So-called chaining arguments and the geometry of the underlying space become dominating at large scales. Similar ideas have appeared separately in the literature, but the way in which we interpret and combine them is new. See [1, 20, 32, 34].

It is possible to work with Sobolev spaces $W^{1,p}$ so that there is no need worry about what we call the local part if one uses the pointwise characterizations efficiently. See [17, 32]. Similarly, the role of what we call the non-local part is almost negligible in the case of BMO. Compare to [1] and the slightly more complicated setting of [37]. The following theorem is concerned with the general case involving complications from both of the extreme cases. The assumptions of the theorem are motivated by the list of examples given after Definition 2.1.

Theorem 3.1. Denote by M the Hardy-Littlewood maximal function, centred or non-centred. Let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a positive function such that $Mu \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let either

- $\alpha = 0$ and μ equal the Lebesgue measure, or
- $\alpha \in (0, 1]$ and μ be a locally finite positive Borel measure.

Suppose that there is a constant $C > 0$ such that for all cubes Q

$$\oint_Q |u - u_Q| dx \leq C \text{diam}(Q)^\alpha \frac{\mu(Q)}{|Q|}. \quad (3.1)$$

Then there is a constant C comparable to the constant appearing in (3.1) such that

$$\oint_Q |Mu - (Mu)_Q| dx \leq C \text{diam}(Q)^\alpha \inf_{z \in Q} M\mu(z)$$

for all Q .

Proof. Take any cube $Q_0 = Q(x_0, r_0) \subset \mathbb{R}^n$ with side length r_0 . Denote

$$U_1(x) = \sup\{u_Q : Q \in \mathcal{Q}(x), \ell(Q) \leq r_0\}$$

$$U_2(x) = \sup\{u_Q : Q \in \mathcal{Q}(x), \ell(Q) > r_0\}$$

$$U(x) = \max(U_1(x), U_2(x)).$$

If M is the centred maximal operator, then $\mathcal{Q}(x)$ is the collection of cubes centred at x . If M is the non-centred maximal operator, then $Q \in \mathcal{Q}(x)$ whenever $x \in Q$. Our aim is to show that $U = Mu$ satisfies a Poincaré inequality. We reduce the task to controlling the quantity

$$\begin{aligned} \oint_{Q_0} |U - U_{Q_0}| dx &= 2 \oint_{Q_0} (U - U_{Q_0})^+ dx \\ &= \frac{2}{|Q_0|} \int_{Q_0 \cap \{U_2 \leq U_1\}} (U_1 - U_{Q_0})^+ dx \\ &\quad + \frac{2}{|Q_0|} \int_{Q_0 \cap \{U_2 > U_1\}} (U_2 - U_{Q_0})^+ dx \\ &= \frac{2}{|Q_0|} (I + II) \end{aligned} \quad (3.2)$$

in two parts. The first term corresponds to local behaviour.

3.1. Local part

Take the cube $3Q_0$ that is centred at x_0 and that has side length $3r_0$. Every cube Q with $Q \cap Q_0 \neq \emptyset$ and $\ell(Q) \leq r_0$ is contained in $3Q_0$. Consequently, we can regard U_1 as a maximal function localized in $3Q_0$, that is,

$$U_1(x) - u_{3Q_0} \leq M[1_{3Q_0}(u - u_{3Q_0})](x)$$

for every $x \in Q_0$. Remember that $u \geq 0$. Here M is the non-centred Hardy-Littlewood maximal function. It is known to be bounded on $L^{q,\infty}$ (this can be deduced from the off-diagonal Marcinkiewicz interpolation theorem, see [14]) so

$$\begin{aligned} \|1_{Q_0}(U_1 - u_{3Q_0})\|_{L^{q,\infty}(\mathbb{R}^n)} &\leq \|M(1_{3Q_0}(u - u_{3Q_0}))\|_{L^{q,\infty}(\mathbb{R}^n)} \\ &\lesssim \|1_{3Q_0}(u - u_{3Q_0})\|_{L^{q,\infty}(\mathbb{R}^n)} \\ &\lesssim \text{diam}(Q_0)^\alpha \frac{\mu(3Q_0)}{|3Q_0|^{1/q'}}. \end{aligned} \quad (3.3)$$

where $q = \frac{n}{n-\alpha} > 1$ if $\alpha > 0$. Inequality (3.3) comes from an application of Lemma 2.2. If $\alpha = 0$ and $\mu = 1$, then we have the same bound by the John-Nirenberg lemma. Hence

$$\begin{aligned} \frac{I}{|Q_0|} &\leq \int_{Q_0} (U_1 - U_{Q_0})^+ dx \leq \int_{Q_0} (U_1 - u_{3Q_0})^+ dx + (u_{3Q_0} - u_{Q_0})^+ \\ &\lesssim |Q_0|^{-1/q} \|1_{Q_0}(U_1 - u_{3Q_0})\|_{L^{q,\infty}(\mathbb{R}^n)} + \int_{3Q_0} |u - u_{3Q_0}| dx \\ &\lesssim \text{diam}(Q_0)^\alpha \frac{\mu(3Q_0)}{|3Q_0|} \end{aligned} \quad (3.4)$$

where we used the simple fact that

$$\frac{1}{|E|^{1/q'}} \int_E |f| dx \lesssim_q \|f\|_{L^{q,\infty}(\mathbb{R}^n)}$$

for all measurable sets E that have finite measure.

Inequality (3.4) is the desired bound for the local part. Note that the proof is very abstract. It works for both centred and non-centred maximal functions as such. Transition to a maximal function using balls does not make any difference either.

3.2. Preparation for the non-local part

In the non-local part, we have to implement a construction that differs a bit for the centred and non-centred maximal functions. This is the only part of the proof where the two cases diverge, and it is packed into this subsection whose output is uniform for both cases. Our aim is to establish a pointwise bound for

$$(U_2(x) - U(y))^+, \quad x, y \in Q_0.$$

We proceed by proving a uniform bound in the case where $U_2(x)$ is replaced by an average over any admissible cube and $U(y)$ is estimated from below by a suitable cube. We start with the non-centred case. The construction is illustrated in Figure 3.1.

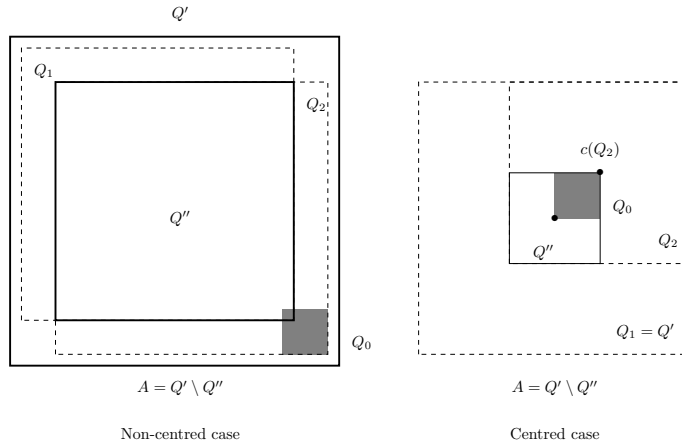


Figure 3.1. Construction of the annulus A for both the centred and non-centred cases.

Non-centred case. Take any Q_1 such that $\ell(Q_1) \geq r_0$ and $Q_1 \cap Q_0 \neq \emptyset$. Let $Q_2 = Q_1 + h$ where $h \in \mathbb{R}^n$ is such that $Q_2 \supset Q_0$ and $|Q_1 \cap Q_2|$ is maximized. Now

$$\begin{aligned} (u_{Q_1} - u_{Q_2})^+ &= \frac{1}{|Q_1|} \left(\int_{Q_1 \setminus Q_2} u \, dx - \int_{Q_2 \setminus Q_1} u \, dx \right)^+ \\ &\leq \frac{1}{|Q_1|} \left| \int_A \varphi u \, dx \right| \end{aligned}$$

where $\varphi = 1_{Q_2 \setminus Q_1} - 1_{Q_1 \setminus Q_2}$ obviously has mean zero and A is an annulus

$$A = Q(x', \ell') \setminus Q(x'', \ell'') = Q' \setminus Q'' \supset (Q_2 \setminus Q_1) \cup (Q_1 \setminus Q_2).$$

We can choose the annulus such that $\ell' - \ell'' = 2r_0$ and $\ell' \leq 10\ell(Q_1)$. Consequently,

$$\frac{|A|}{|Q_1|} \leq C \frac{r_0}{\ell(Q_1)}.$$

Note also that by the fact that φ has mean zero in A , we have that

$$\left| \int_A \varphi u \, dx \right| \leq \int_A |u - u_{Q_*}| \, dx \quad (3.5)$$

for any constant u_{Q_*} . We choose Q_* to be the central cube obtained by dividing each side of Q' into three equal parts. The data A, Q_*, Q_1 will be needed in completing the proof. Before proceeding to that, we construct these objects in the centred setting.

Centred case. We modify the previous argument for the centred maximal operator. We also do it so that the argument carries over to the domain setting to be discussed later. We take again Q_1 that is centred at a point $y \in Q_0$. If $\ell(Q_1) < 10r_0$, we can estimate

$$(u_{Q_1} - u_{Q_2})^+ \lesssim r_0^\alpha \frac{\mu(Q_1)}{|Q_1|} \lesssim r_0^\alpha \inf_{z \in Q_0} M\mu(z)$$

for any Q_2 centred at a point $x \in Q_0$ and $\ell(Q_2) = r_0$. Taking the supremum over Q_1 we get the correct bound pointwise for

$$(U_2(y) - U_{Q_0})^+ \leq (U_2(y) - \inf_{z \in Q_0} U(z))^+,$$

and we are done.

Assume then that $\ell(Q_1) \geq 10r_0$. Note that $Q_0 \subset Q_1$. Take $z \in Q_0$. We choose the maximal cube Q_2 centred at z such that $Q_2 \subset Q_1$. Then

$$\begin{aligned} (u_{Q_1} - u_{Q_2})^+ &= \frac{1}{|Q_1|} \left(\int_{Q_1} u \, dx - \int_{Q_2} u \, dx + (|Q_2| - |Q_1|)u_{Q_2} \right)^+ \\ &= \frac{1}{|Q_1|} \left(\int_{Q_1 \setminus Q_2} u \, dx - (|Q_1| - |Q_2|)u_{Q_2} \right)^+ \\ &\leq \frac{1}{|Q_1|} \int_{Q_1 \setminus Q_2} (u - u_{Q_2})^+ \, dx. \end{aligned}$$

We can compute further

$$\begin{aligned} \frac{1}{|Q_1|} \int_{Q_1 \setminus Q_2} (u - u_{Q_2})^+ \, dx &\leq \frac{1}{|Q_1|} \int_{Q_1 \setminus Q_2} (u - u_{Q_*})^+ \, dx \\ &\quad + \frac{|Q_1 \setminus Q_2|}{|Q_1|} (u_{Q_*} - u_{Q_2})^+. \end{aligned} \tag{3.6}$$

The choice of Q_* is done as follows. In the first term, note that $Q_1 \setminus Q_2$ is contained in an annulus $A = Q' \setminus Q''$ such that $\ell(Q') - \ell(Q'') = 2r_0$. We can choose $Q_1 = Q'$ in this case. Later in connection Corollary 4.3, it will be crucial that we can choose $A \subset Q_1$. We let Q_* be the central triadic subcube of Q' . The second term is then clear by the fact that

$$\frac{|Q_1 \setminus Q_2|}{|Q_1|} (u_{Q_*} - u_{Q_2})^+ \lesssim \frac{r_0}{\ell(Q_1)} \text{diam}(Q_2)^\alpha \frac{\mu(Q_2)}{|Q_2|} \lesssim r_0^\alpha M\mu(x_0)$$

where we used that $r_0 \lesssim \text{diam}(Q_2) \approx \ell(Q_1)$ and $\alpha \in [0, 1]$. Hence we have reduced the proof to estimating (3.5) or equivalently the first term in (3.6):

$$\frac{1}{|Q_1|} \int_A (u - u_{Q_*})^+.$$

3.3. Whitney decomposition

We form a Whitney decomposition of Q' . Namely, we first divide each side by 3. We choose the middle cube to $\mathcal{W}_1$. This was previously called Q_* . We subdivide the remaining cubes by halving each side length and the resulting cubes that touch the element of $\mathcal{W}_1$ form $\mathcal{W}_2$. Continuing inductively by dividing always the side length by 2, we get a collection of cubes $\mathcal{W} = \cup_{i=1}^{\infty} \mathcal{W}_i$ with the properties

$$\begin{aligned} Q \cap P &\in \{\emptyset, Q, P\}, \quad Q, P \in \mathcal{W}, \\ \ell(Q) &= d(Q, \partial Q'), \\ \cup_{Q \in \mathcal{W}} Q &= Q' \quad \text{up to a set of measure zero.} \end{aligned}$$

Let $\mathcal{A} = \{Q \in \mathcal{W} : Q \cap A \neq \emptyset\}$.

For each $Q \in \mathcal{A}$, we join the center $c(Q)$ to the center of Q' by a straight line γ . In the center, there is the Whitney cube Q_* . Let

$$\mathcal{C}(Q) = \{Q \in \mathcal{W} : Q \cap \gamma \neq \emptyset\}.$$

For all $Q \in \mathcal{A}$, we write $\mathcal{C}(Q) = \{Q^i\}_{i=1}^{m_Q}$ where the cubes are ordered according to their distance from the point $c(Q)$. Note that

$$\begin{aligned} |(2Q^i) \cap (2Q^{i+1})| &\approx_n |Q^i|, \\ Q^1 &= Q \quad \text{and} \quad Q^{m_Q} = Q_*. \end{aligned}$$

The previous construction at hand, we are ready to estimate

$$\int_A |u - u_{Q_*}| \, dx \leq \sum_{Q \in \mathcal{A}} \int_Q |u - u_{Q_*}| \, dx.$$

By the property $|2Q^i \cap 2Q^{i+1}| \approx_n |Q^i|$ there is $R^i \subset 2Q^i \cap 2Q^{i+1}$ with $|R^i| \approx_n |Q^i|$ so that

$$\begin{aligned} \int_Q |u - u_{Q_*}| \, dx &\leq \int_Q |u - u_Q| \, dx + |Q| \sum_{j=1}^{m_Q-1} |u_{Q^j} - u_{Q^{j+1}}| \\ &\leq \int_Q |u - u_Q| \, dx + |Q| \sum_{j=1}^{m_Q-1} \int_{R^j} (|u_{Q^j} - u| + |u - u_{Q^{j+1}}|) \, dx \\ &\lesssim |Q| \sum_{P \in \mathcal{C}(Q)} \text{diam}(P)^\alpha \frac{\mu(2P)}{|P|} \\ &= |Q| \sum_{P \in \mathcal{C}(Q)} |P|^{\alpha/n-1} \mu(2P) \end{aligned}$$

Let $\mathcal{S}(\mathcal{P}) = \{Q \in \mathcal{A} : P \in \mathcal{C}(Q)\}$. Notice the following:

- (i) If $P \in \mathcal{A}$, then $Q \subset c_n P$ for a dimensional constant c_n whenever $Q \in \mathcal{S}(P)$. This follows from the fact that $\mathcal{C}(Q)$ cannot contain cubes of equal size more than a uniformly bounded amount.
- (ii) If $P \in \mathcal{W} \setminus \mathcal{A}$, then $Q \subset A_P = (c_n P) \cap A$ for a dimensional constant c_n whenever $Q \in \mathcal{S}(P)$.

We change the order of summation in the previous estimate to get

$$\begin{aligned}
 \sum_{Q \in \mathcal{A}} \int_Q |u - u_{Q^*}| dx &\lesssim \sum_{Q \in \mathcal{A}} |Q| \sum_{P \in \mathcal{C}(Q)} |P|^{\alpha/n-1} \mu(2P) \\
 &\leq \sum_{P \in \mathcal{W}} \sum_{Q \in \mathcal{S}(P)} |Q| |P|^{\alpha/n-1} \mu(2P) \\
 &= \sum_{P \in \mathcal{W} \cap \mathcal{A}} \sum_{Q \in \mathcal{S}(P)} |Q| |P|^{\alpha/n-1} \mu(2P) \\
 &\quad + \sum_{P \in \mathcal{W} \setminus \mathcal{A}} \sum_{Q \in \mathcal{S}(P)} |Q| |P|^{\alpha/n-1} \mu(2P) \\
 &= S_1 + S_2.
 \end{aligned}$$

To estimate the term S_1 , we note that by (i)

$$\sum_{P \in \mathcal{W} \cap \mathcal{A}} \sum_{Q \in \mathcal{S}(P)} |Q| |P|^{\alpha/n-1} \mu(2P) \lesssim \sum_{P \in \mathcal{W} \cap \mathcal{A}} |P|^{\alpha/n} \mu(2P).$$

In term S_2 we use the fact (ii) and an estimate for the volume of a cube intersected with an annulus. We obtain the bound

$$\begin{aligned}
 \sum_{P \in \mathcal{W} \setminus \mathcal{A}} \sum_{Q \in \mathcal{S}(P)} |Q| |P|^{\alpha/n-1} \mu(2P) &\leq \sum_{P \in \mathcal{W} \setminus \mathcal{A}} |A_P| |P|^{\alpha/n-1} \mu(2P) \\
 &\lesssim \sum_{P \in \mathcal{W} \setminus \mathcal{A}} \frac{r_0 |P|}{|P|^{1/n}} \cdot |P|^{\alpha/n-1} \mu(2P) \\
 &= r_0^\alpha \sum_{P \in \mathcal{W} \setminus \mathcal{A}} \left(\frac{|P|^{1/n}}{r_0} \right)^{\alpha-1} \mu(2P) \\
 &\leq r_0^\alpha \sum_{P \in \mathcal{W} \setminus \mathcal{A}} \mu(2P)
 \end{aligned}$$

since $\alpha \in [0, 1]$ and $r_0 \leq |P|^{1/n}$ for $P \in \mathcal{W} \setminus \mathcal{A}$. Hence

$$S_1 + S_2 \lesssim r_0^\alpha \mu(Q').$$

The constants are uniform in the cubes Q_1 and Q_2 so we can conclude that

$$\int_{Q_0} (U_2 - U_{Q_0})^+ dx \lesssim \text{diam}(Q_0)^\alpha \inf_{z \in Q_0} M\mu(z). \quad (3.7)$$

This completes the proof of the bound for term II , both in the centred and the non-centred case.

3.4. Final estimate

Completing the estimate (3.2) by using (3.4) and (3.7) we reach the desired Poincaré inequality

$$\int_Q |Mu - (Mu)_Q| dx \leq C \operatorname{diam}(Q)^\alpha \inf_{z \in Q} M\mu(z).$$

Note that the maximal function on the right hand side is whichever we prefer to use since the centred and the non-centred maximal functions are pointwise comparable. $\square$

Remark 3.2. It is also possible to run the previous proof for the maximal function defined using balls. Even if the proof seems to heavily rely on the properties of cubes, most cubes appearing in the proof are not related to the maximal function. In the case of the maximal function using balls, the local step would be almost identical. In the non-local step, we have to use the Whitney decomposition of a ball instead of that of a cube, but are no essential differences. The argument for the centred maximal function also works in the setting where the centred maximal function is restricted to a proper subdomain of $\mathbb{R}^n$. Consequences of this extension are discussed in Corollary 4.3. The argument above does not work as such if we try to bound non-centred maximal function in a domain.

Remark 3.3. We can also write the inequality resulting from Theorem 3.1 as

$$\int_Q |Mu - (Mu)_Q| dx \leq C \operatorname{diam}(Q)^\alpha \cdot \sup_{Q' \supset Q} \frac{\mu(Q')}{|Q'|},$$

and this can be verified using the same proof with a more careful bookkeeping. It is also possible to interpret the quantities appearing in the proof in the spirit of the representation formulas of [29].

Next we point out how several classical results can be deduced from Theorem 3.1. Consider functions $u \in W^{1,p}(\mathbb{R}^n)$ where $p \in (0, 1)$. Since

$$\inf_{z \in Q} M|\nabla u|(z) \leq \left(\int_Q M(|\nabla u|)^p dx \right)^{1/p}$$

for all $p \in (1, \infty)$, our Theorem 3.1 and in Hajlasz [15, Lemma 6] give a new (though not very simple) proof for the boundedness result $M : W^{1,p}(\mathbb{R}^n) \rightarrow W^{1,p}(\mathbb{R}^n)$. This was originally proved by Kinnunen in [22]. See also [17].

Corollary 3.4 (Kinnunen [22]). *Let $p > 1$. Then the Hardy-Littlewood maximal function is bounded $M : W^{1,p}(\mathbb{R}^n) \rightarrow W^{1,p}(\mathbb{R}^n)$. In addition,*

$$|\nabla Mu| \lesssim M|\nabla u|.$$

Proof. Note that $|\nabla u| = |\nabla|u||$. Then $|u|$ and $|\nabla u|$ satisfy a Poincaré inequality with $\alpha = 1$. Since $u \in L^p(\mathbb{R}^n)$, Mu is locally integrable. By Theorem 3.1, Mu and $M|\nabla u|$ satisfy a Poincaré inequality with $\alpha = 1$. By Lemma 6 in Hajlasz [15] the claim follows. $\square$

Specializing to $\mu \in L^\infty(\mathbb{R}^n)$ and lowering the value of α in the Poincaré inequality, we end up studying spaces of Hölder continuous functions. We get the following corollary. See also [11] for a related result.

Corollary 3.5 (Buckley [5]). *Let $u \in \Lambda(\alpha)$ with $\alpha \in (0, 1]$ be such that $Mu \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then $Mu \in \Lambda(\alpha)$ with the same α .*

Proof. The space $\Lambda(\alpha)$ consist of the measurable functions that have an α -Hölder continuous representative. It is characterized by a Poincaré inequality of u and $\mu = 1$ with exponent α for the diameter (see [35]). By Theorem 3.1, the maximal function preserves this Poincaré inequality provided that Mu is locally integrable. Hence the claim follows. $\square$

Finally, looking at $\alpha = 0$ and $\mu \in L^\infty(\mathbb{R}^n)$, we can include BMO. It is interesting that we can cover both BMO and Sobolev spaces with a single proof. Finding such a proof was our original motivation for writing this note about Sobolev spaces, Poincaré inequalities, and the maximal function.

Corollary 3.6 (Bennett, DeVore, and Sharpley [1]). *Let $u \in \text{BMO}$ with $Mu \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then*

$$\|Mu\|_{\text{BMO}} \lesssim \|u\|_{\text{BMO}}.$$

The claim follows directly from Theorem 3.1. On the other hand, the proof of Theorem 3.1 also contains a simplification of the original argument of [1]. Since this might be of independent interest, we write down another simple proof of this corollary. We need not use the A_p weights like for instance the proof in [11].

Another simple proof. Let M be the non-centred maximal function. Since $|u| \in \text{BMO}$, we can assume that $u \geq 0$. Take a cube Q . Let $E = Q \cap \{Mu = M(1_3Qu)\}$. Then

$$\begin{aligned} \int_Q |Mu - (Mu)_Q| \, dx &= 2 \int_Q (Mu - (Mu)_Q)^+ \, dx \\ &= 2 \int_E (M(1_3Qu) - (Mu)_Q)^+ \, dx + 2 \int_{Q \setminus E} (Mu - (Mu)_Q)^+ \, dx \\ &\leq 2 \int_Q |M(1_3Q(u - u_{3Q}))| \, dx + 2 \int_{Q \setminus E} \sup_{\substack{x \in P \\ |P| > |Q|}} |u_P - u_{2P}| \, dx \\ &= I + II. \end{aligned}$$

P denotes a cube in the supremum. Clearly $II \lesssim |Q| \|u\|_{\text{BMO}}$. By Hölder's inequality, L^2 boundedness of M and the John-Nirenberg inequality

$$\begin{aligned} I &\lesssim |Q|^{1/2} \|M(1_Q(u - u_{3Q}))\|_{L^2} \lesssim |Q|^{1/2} \|1_Q(u - u_{3Q})\|_{L^2} \\ &\lesssim |Q| \|u\|_{\text{BMO}} \end{aligned}$$

so the claim follows. $\square$

4. The endpoint Sobolev space

Next we use Theorem 3.1 in the endpoint space $W^{1,1}(\mathbb{R}^n)$ to prove that the distributional partial derivatives of Mu acting on test functions vanishing at a singularity set (whose size we cannot bound) are actually functions in $L^{1,\infty}(\mathbb{R}^n)$. Moreover, if we assume that $|\nabla u| \in L \log L(\mathbb{R}^n)$, we recover local integrability of ∇Mu . This was first proved by Hajlasz and Onninen in [17]. Of course, we do not know if any of the results is optimal. Recall that no counterexample for $u \in W^{1,1}(\mathbb{R}^n)$ implying $|\nabla Mu| \in L^1_{\text{loc}}$ is known at the moment. The validity of the boundedness of $u \mapsto |\nabla Mu|$ from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ is an interesting open question.

There are also other approaches to the maximal function of $u \in W^{1,1}(\mathbb{R}^n)$. Hajlasz and Malý [16] show that Mu is approximately differentiable almost everywhere, which implies that it coincides with a continuously differentiable function outside an open set of arbitrarily small measure. This property is weaker than weak differentiability. On the other hand, they prove the result in a more general setting under less stringent assumptions. See also the related paper by Luiro [31].

Corollary 4.1. *Let $u \in W^{1,1}(\mathbb{R}^n)$. Then the distributional partial derivatives $\partial_i Mu$ can be represented as functions $h_i \in L^{1,\infty}(\mathbb{R}^n)$ when they act on smooth functions with compact support not meeting*

$$\left\{ x \in \mathbb{R}^n : \liminf_{\delta \rightarrow 0} \int_{B(x,\delta)} M|\nabla u| \, dy = \infty \right\}.$$

Moreover,

$$|h_i| \lesssim M|\nabla u|, \quad i \in \{1, \dots, n\}.$$

Proof. First of all, note that $|u|$ satisfies the assumptions of Theorem 3.1. The property $Mu \in L^1_{\text{loc}}(\mathbb{R}^n)$ follows from the Sobolev embedding theorem. The function Mu gives rise to a bounded linear functional acting on smooth functions with compact support. We study the distributional derivative defined by

$$\partial_i Mu(\varphi) = - \int_{\mathbb{R}^n} Mu \cdot (\partial_i \varphi) \, dx$$

where φ belongs to the space of smooth and compactly supported functions $C_0^\infty(\mathbb{R}^n)$. The distributional derivative of Mu need not arise from a locally integrable function since $L^{1,\infty}(\mathbb{R}^n)$ is the best integrability that we can expect at the moment.

Let

$$A = \left\{ x \in \mathbb{R}^n : \liminf_{\delta \rightarrow 0} \int_{B(x, \delta)} M|\nabla u| \, dy = \infty \right\}.$$

Note that A is closed. Indeed, take any $x \in \overline{A}$. Take a sequence $A \ni x_i \rightarrow x$. Take $\epsilon > 0$. There is i such that $x_i \in B(x, \epsilon)$. Since $x_i \in A$, there is $\delta > 0$ such that $B(x_i, \delta) \subset B(x, \epsilon)$ and

$$\int_{B(x_i, \delta)} M|\nabla u| \, dy = \infty.$$

Consequently

$$\int_{B(x, \epsilon)} M|\nabla u| \, dy \geq \int_{B(x_i, \delta)} M|\nabla u| \, dy = \infty$$

and $x \in A$. So A is closed. We define $\mathcal{D}(A^c)$ to consist of all test functions $\varphi \in C_0^\infty(\mathbb{R}^n)$ that are supported outside A . When we talk about distributions, we refer to the dual of this test function class supported outside A . We prove that $\partial_i Mu$ on $\mathcal{D}(A^c)$ is given by integration against $h \in L^{1, \infty}(\mathbb{R}^n)$.

To prove that $\partial_i Mu$ is a measurable function, we use the argument of Hajlasz [15]. Let $\psi \in C_0^\infty(B(0, 1))$ be a positive function with $\int_{\mathbb{R}^n} \psi \, dx = 1$. For $\epsilon \in (0, 1)$ we define the dilations $\psi_\epsilon(x) = \frac{1}{\epsilon^n} \psi(\frac{x}{\epsilon})$. We have that

$$\partial_i Mu = \lim_{\epsilon \rightarrow 0} (Mu * \partial_i \psi_\epsilon)$$

in the sense of distributions. Since $\int \partial_i \psi_\epsilon = 0$, also

$$(Mu * \partial_i \psi_\epsilon)(y) = [(Mu - (Mu)_{B(y, \epsilon)}) * \partial_i \psi_\epsilon](y).$$

By Theorem 3.1

$$|(Mu * \partial_i \psi_\epsilon)(y)| \leq C \epsilon^{-n-1} \int_{B(y, \epsilon)} |Mu - (Mu)_{B(y, \epsilon)}| \, dx \leq C M|\nabla u|(y).$$

Let

$$g_\epsilon = (Mu * \partial_i \psi_\epsilon) \quad \text{and} \quad g = \lim_{\epsilon \rightarrow 0} g_\epsilon = \lim_{\epsilon \rightarrow 0} (Mu * \partial_i \psi_\epsilon),$$

where the limit is again in the sense of distributions.

Since

$$|g(\varphi)| = \lim_{\epsilon \rightarrow 0} |g_\epsilon(\varphi)| \leq C \|\varphi\|_{L^\infty} \int_{\text{supp } \varphi} M|\nabla u| \, dx,$$

g is a bounded linear functional on $C_0(\mathbb{R}^n \setminus A)$. By the Riesz representation theorem it is a signed Radon measure. It is also absolutely continuous with respect to the Lebesgue measure. By the Radon-Nikodym theorem it is represented by integration against a measurable function h . This proves the existence of the distributional derivative as a measurable function.

Since

$$\int \varphi \partial_i M u \, dx \lesssim \|\varphi\|_{L^\infty} \int_{\text{supp } \varphi} M |\nabla u| \, dx$$

holds for all test functions, we can conclude that $\partial_i M u \lesssim M |\nabla u|$ almost everywhere. $\square$

The proof was based on Theorem 3.1 which in turn holds whenever the modulus of the generalized gradient is a general Radon measure. It need not be locally integrable function. Hence we can extend the result to the class of BV functions. See [12] for the precise definitions.

Corollary 4.2. *Let $u \in \text{BV}$. Then the distributional partial derivatives $\partial_i M u$ can be represented as functions $h_i \in L^{1,\infty}(\mathbb{R}^n)$ when they act on smooth functions with compact support not meeting*

$$\left\{ x \in \mathbb{R}^n : \liminf_{\delta \rightarrow 0} \int_{B(x,\delta)} M |\nabla u| \, dy = \infty \right\}.$$

Moreover,

$$|h_i| \lesssim M |\nabla u|, \quad i \in \{1, 2, \dots, n\}.$$

The result concerning the endpoint Sobolev space can be localized. By minor modifications in the arguments, we can prove that Corollary 4.1 holds true for centred maximal operators in subdomains $\Omega \subset \mathbb{R}^n$. By a domain we mean an open and connected set. We define

$$M_\Omega u(x) = \sup_{Q_x \subset \Omega} \int_{Q_x} |u| \, dx$$

where the supremum is over all cubes centred at x and with sides parallel to the coordinate axes.

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $u \in W^{1,1}(\Omega)$. Then the distributional partial derivatives $\partial_i M_\Omega u$ can be represented as functions in $h_i \in L^{1,\infty}(\Omega)$ when they act on smooth functions with compact support not meeting*

$$\left\{ x \in \Omega : \liminf_{\delta \rightarrow 0} \int_{B(x,\delta)} M_\Omega |\nabla u| \, dy = \infty \right\}.$$

Moreover,

$$|h_i| \lesssim M_\Omega |\nabla u|, \quad i \in \{1, 2, \dots, n\}.$$

Proof. We first note that Theorem 3.1 carries over to this setting provided that we weaken the claimed inequality to only hold for Q such that $20Q \subset \Omega$. However, this is enough for the proof of Corollary 4.1. The claim follows from a verbatim repetition of its proof. $\square$

The same method also applies to the case $p > 1$. Namely, we get a new proof for the theorem of Kinnunen and Lindqvist [23]. Even if the approaches of the original proofs were different, the proof via Poincaré inequalities does not see any difference between the full space and a proper domain as far as centred maximal functions are concerned.

Corollary 4.4 (Kinnunen and Lindqvist [23]). *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $u \in W^{1,p}(\Omega)$. Then $M_\Omega u \in W^{1,p}(\Omega)$ and*

$$|\nabla M_\Omega u| \lesssim M_\Omega |\nabla u|.$$

Proof. This is essentially same as the proof given for Corollary 3.4 as the proof of Corollary 4.3 reveals. $\square$

Note that it is necessary to use the centred maximal function here if we talk about cubes. If we define

$$M_\Omega^* u(x) = \sup_{Q \subset \Omega} \frac{1_Q(x)}{|Q|} \int_Q |u| dy$$

we can consider a planar domain Ω that is a wide square at the end of a narrow corridor, like the interior of $[-1, 0] \times [0, 1] \cup [0, 2] \times [0, 2]$. A smooth bump φ supported in $(1, 2) \times (1, 2)$ is a Sobolev function. However, its non-centred maximal function necessarily has a jump on the line $\{(0, t) : t \in (0, 1)\}$. We have $M_\Omega^* \varphi = 0$ in $[-1, 0] \times [0, 1]$ but $M_\Omega^* \varphi > \varphi_{(0,2) \times (0,2)}$ in $(0, 2) \times (0, 2)$. Such a function cannot belong to any Sobolev space.

5. The fractional maximal function

In this concluding section, we make some remarks on how the method applies to the fractional maximal function. We define the non-centred fractional maximal function to be

$$M_\beta f = \sup_{Q \ni x} \ell(Q)^\beta \int_Q |f| dx, \quad \beta \in [0, n].$$

We will next inspect how the local part of the proof (Section 3.1) behaves with the fractional maximal function.

Take a cube Q_0 and a triple (u, μ, α) as in the premises of Theorem 3.1 except that now we assume that $M_\beta u \in L^1_{\text{loc}}(\mathbb{R}^n)$ in addition to $Mu \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $E = Q_0 \cap \{M_\beta u = M_\beta(1_{3Q_0}u)\}$. We see that for all $x \in E$ it is clear that

$$M_\beta u - (M_\beta u)_{Q_0} \leq M_\beta(1_{3Q_0}u) - \ell(3Q_0)^\beta u_{3Q_0} \lesssim \ell(Q_0)^\beta (M(1_{3Q_0}u) - u_{3Q_0})$$

where the rightmost quantity can be estimated as in the proof of Theorem 3.1. The outcome is that

$$\frac{1}{|Q_0|} \int_E (M_\beta u - (M_\beta u)_{Q_0})^+ dx \lesssim \text{diam}(Q_0)^\alpha \inf_{z \in Q_0} M_\beta \mu(z).$$

In the non-local part (Section 3.2) the changes that occur are even more marginal. Namely, the relevant quantity

$$(\ell(Q_1)^\beta u_{Q_1} - \ell(Q_2)^\beta u_{Q_2})^+ = \ell(Q_1)^\beta (u_{Q_1} - u_{Q_2})^+$$

reduces to the case already known modulo the factor $\ell(Q_1)^\beta$. This will be absorbed in the fractional maximal function appearing in the right side of the final estimate.

Hence the proof of Theorem 3.1 with the initial changes mentioned above gives the estimate

$$\oint_Q |M_\beta u - (M_\beta u)_Q| dx \leq C \operatorname{diam}(Q)^\alpha \inf_{z \in Q} M_\beta \mu(z)$$

for the fractional maximal function provided that the minimal assumptions to make the quantities above well defined are fulfilled. One can use this inequality to derive a bound for the gradient of the fractional maximal function of a Sobolev function. This will be a repetition of the argument in [15] so we omit it.

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