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Hyperbolic 3-manifolds groups are subgroup conjugacy separable

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Abstract. A group G is called subgroup conjugacy separable if for every pair of non-conjugate finitely generated subgroups of G, there exists a finite quotient of G where the images of these subgroups are not conjugate. It is proved that the fundamental group of a hyperbolic 3-manifold (closed or with cusps) is subgroup conjugacy separable.

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1. Introduction

O. Bogopolski and F. Grunewald [7] introduced an important notion of subgroup conjugacy separability for a group G. A group G is said to be subgroup conjugacy separable if for every pair of non-conjugate finitely generated subgroups H and K of G, there exists a finite quotient of G where the images of these subgroups are not conjugate. Thus the subgroup conjugacy separability is a residual property of groups, which logically continues the following classical residual properties of groups: the residual finiteness, the conjugacy separability, and the subgroup separability (LERF). A. I. Mal'cev was the first, who noticed, that finitely presented residually finite (respectively conjugacy separable) groups have solvable word problem (respectively conjugacy problem) [20]. Arguing in a similar way, one can show that finitely presented subgroup separable groups have solvable membership problem and that finitely presented subgroup conjugacy separable groups have solvable conjugacy problem for finitely generated subgroups. The last means, that there is an algorithm, which given a finitely presented subgroup conjugacy separable group $G = \langle X \mid R \rangle$ and two finite sets of elements Y and Z, decides whether the subgroups $\langle Y \rangle$ and $\langle Z \rangle$ are conjugate in G.

Bogopolski and Grunewald proved that free groups and the fundamental groups of finite trees of finite groups subject to a certain normalizer condition, are

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subgroup conjugacy separable. For finitely generated virtually free groups the result was proved by the authors in [9]. Also, O. Bogopolski and K-U. Bux in [5] proved that surface groups are subgroup conjugacy separable. In [10] the authors of the present paper extended this result to limit groups.

The objective of this paper is to show that the fundamental group of a hyperbolic 3-manifold is subgroup conjugacy separable. Bogopolski and Bux posed it as an open question in [6, page 3].

Theorem 1.1. *The fundamental group $\pi_1 M$ of a hyperbolic 3-manifold M (closed or with cusps) is subgroup conjugacy separable.*

Note that the fundamental group $\pi_1 M$ of a hyperbolic 3-manifold M is subgroup separable (see [3, Corollary 5.5 (1)]), a crucial property used in the proof. In fact, the proof is valid for hyperbolic subgroup separable virtually compact special groups.

Theorem 1.2. *A hyperbolic subgroup separable virtually compact special group G is subgroup conjugacy separable.*

These theorems were known before only for quasiconvex subgroups. In [10, Theorem 1.2], the authors of this paper proved that quasiconvex subgroups of hyperbolic virtually compact special groups are subgroup conjugacy separable (Bogopolsky and Bux gave an independent proof of this result under the complementary torsion freeness assumption).

Virtually compact special groups own its importance due to Daniel Wise who proved in [31] that 1-relator groups with torsion are virtually compact special, answering positively a question of Gilbert Baumslag who asked in [4] whether these groups are residually finite. In fact, many groups of geometric origin are virtually compact special: the fundamental group of a hyperbolic 3-manifold [1], small cancellation groups (a combination of [31] and [1]) and hyperbolic Coxeter groups [14].

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2. Preliminaries

In this section we introduce the reader to concepts and terminology of the profinite version of the Bass-Serre theory of groups acting on trees used in the paper.

We consider the following standard definitions. Our graphs are oriented graphs. A graph Γ is a set together with a distinguished subset of *vertices* $V = V(\Gamma)$ and together with two maps $d_0, d_1 : \Gamma \longrightarrow V$, which are the identity when restricted to V. This graph is called *profinite* if Γ is a profinite space (*i.e.*, a compact, Hausdorff and totally-disconnected topological space), V is a closed subset of Γ, and the mappings d_i are continuous. If $e \in \Gamma$, we say that $d_0(e)$ and $d_1(e)$ are the origin and terminal vertex of e, respectively. The complement $E = E(\Gamma) = \Gamma - V(\Gamma)$ of $V(\Gamma)$ in Γ is called the set (space) of *edges* of Γ. For basic concepts such as connectedness, or of when a graph is a tree, see [12, Chapter I], or [29, Part I], for abstract graphs. We assume that the reader is familiar with basic notions of Bass-Serre theory of groups acting on trees treated in these books.

We also assume that the reader knows basic facts about profinite groups, in particular the notion of the profinite topology on a group that can be found in [28, Chapter 3]. Following the tradition of combinatorial group theory a subgroup H of a group G will be called *separable* if it is closed in the profinite topology of G.

For a profinite space X that is the inverse limit of finite discrete spaces X_j, $[[\widehat{\mathbb{Z}}X]]$ is the inverse limit of $[\widehat{\mathbb{Z}}X_j]$, where $[\widehat{\mathbb{Z}}X_j]$ is the free $\widehat{\mathbb{Z}}$-module with basis X_j. For a pointed profinite space $(X, *)$ that is the inverse limit of pointed finite discrete spaces $(X_j, *)$, $[[\widehat{\mathbb{Z}}(X, *)]]$ is the inverse limit of $[\widehat{\mathbb{Z}}(X_j, *)]$, where $[\widehat{\mathbb{Z}}(X_j, *)]$ is the free $\widehat{\mathbb{Z}}$-module with basis $X_j \setminus \{*\}$ [28, Chapter 5.2].

Given a profinite graph Γ define the pointed space $(E^*(\Gamma), *)$ as $\Gamma/V(\Gamma)$ with the image of $V(\Gamma)$ as a distinguished point $*$. By definition a profinite tree Γ is a profinite graph with a short exact sequence

$$0 \to [[\widehat{\mathbb{Z}}(E^*(\Gamma), *)]] \xrightarrow{\delta} [[\widehat{\mathbb{Z}}V(\Gamma)]] \xrightarrow{\epsilon} \widehat{\mathbb{Z}} \to 0,$$

where $\delta(\bar{e}) = d_1(e) - d_0(e)$ for every $e \in E(\Gamma)$, $\bar{e}$ the image of e in $E^*(\Gamma)$ and $\epsilon(v) = 1$ for every $v \in V(\Gamma)$.

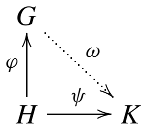
We refer the reader to [25] for further details of the profinite version of the Bass-Serre theory. If v and w are vertices of a tree (respectively, of a profinite tree) Γ, we denote by $[v, w]$ the smallest subtree (respectively, a profinite subtree) of Γ containing v and w.

A group H is said to act on a graph Γ if it acts on Γ as a set and if in addition $d_i(hm) = hd_i(m)$, for all $h \in H$ and $m \in \Gamma$ $(i = 0, 1)$; if Γ is a profinite graph and H a profinite group, we assume that the action is continuous. The quotient Γ/H inherits a natural graph structure (respectively, profinite graph structure).

Let G_1 and G_2 be profinite groups with a common closed subgroup H; the *profinite free amalgamated product* $G_1 \amalg_H G_2$ is the push-out G of G_1 and G_2 over H in the category of profinite groups; if the canonical homomorphisms $G_1 \longrightarrow G$ and $G_2 \longrightarrow G$ are embeddings, one says that G is proper (see [28, Chapter 9] for more details). Note that if G is residually finite then G is proper.

Let H be a profinite group and let $f : A \longrightarrow B$ be a continuous isomorphism between closed subgroups A, B of H. A *profinite HNN-extension* of H with associated subgroups A, B consists of a profinite group $G = \mathrm{HNN}(H, A, t)$, an element $t \in G$, and a continuous homomorphism $\varphi : H \longrightarrow G$ with $t(\varphi(a))t^{-1} = \varphi f(a)$ and satisfying the following universal property: for any profinite group K, any $k \in K$ and any continuous homomorphism $\psi : H \longrightarrow K$ satisfying $k(\psi(a))k^{-1} =$

$\psi f(a)$ for all $a \in A$, there is a unique continuous homomorphism $\omega : G \longrightarrow K$ with $\omega(t) = k$ such that the diagram



is commutative. We shall refer to ω as the homomorphism induced by ψ.

Observe that one needs to test the above universal property only for finite groups K, for then it holds automatically for any profinite group K, since K is an inverse limit of finite groups.

We define the standard tree $S(G)$ on which G acts (respectively, $S(\widehat{G})$ on which the profinite completion $\widehat{G}$ acts) for the cases of an amalgamated free product $G = G_1 *_H G_2$ (respectively, $\widehat{G} = \widehat{G}_1 \amalg_{\widehat{H}} \widehat{G}_2$) and an HNN-extension $G = HNN(G_1, H, t)$ (respectively, $\widehat{G} = HNN(\widehat{G}_1, \widehat{H}, t)$) since we shall use them frequently for these cases.

- Let $G = G_1 *_H G_2$. Then the vertex set is $V(S(G)) = G/G_1 \cup G/G_2$, the edge set is $E(S(G)) = G/H$, and the initial and terminal vertices of an edge gH are respectively gG_1 and gG_2;
- Similarly, let $\widehat{G} = \widehat{G}_1 \amalg_{\widehat{H}} \widehat{G}_2$. Then the vertex set is $V(S(\widehat{G})) = \widehat{G}/\widehat{G}_1 \cup \widehat{G}/\widehat{G}_2$, the edge set is $E(S(\widehat{G})) = \widehat{G}/\widehat{H}$, and the initial and terminal vertices of an edge $g\widehat{H}$ are respectively $g\widehat{G}_1$ and $g\widehat{G}_2$;
- Let $G = HNN(G_1, H, t)$. Then the vertex set is $V(S(G)) = G/G_1$, the edge set is $E(S(G)) = G/H$, and the initial and terminal vertices of an edge gH are respectively gG_1 and gtG_1;
- Similarly let $\widehat{G} = HNN(\widehat{G}_1, \widehat{H}, t)$. Then the vertex set is $V(S(\widehat{G})) = \widehat{G}/\widehat{G}_1$, the edge set is $E(S(\widehat{G})) = \widehat{G}/\widehat{H}$, and the initial and terminal vertices of an edge $g\widehat{H}$ are respectively $g\widehat{G}_1$ and $gt\widehat{G}_1$.

The tree $S(G)$ naturally embeds in $S(\widehat{G})$ if and only if the subgroups H, G_1 and G_2 are separable in G, or equivalently H is closed in G_1 (and in G_2 in the case of amalgamation) with respect to the topology induced by the profinite topology on G (see [11, Proposition 2.5]).

These constructions are particular cases of the general construction of the profinite fundamental group of a finite graph of profinite groups.

When we say that $\mathcal{G}$ is a finite graph of profinite groups we mean that it contains the data of the underlying finite graph, the edge profinite groups, the vertex profinite groups and the attaching continuous maps. More precisely, let Δ be a connected finite graph. A graph of profinite groups $(\mathcal{G}, \Delta)$ over Δ consists of a specifying profinite group $\mathcal{G}(m)$ for each $m \in \Delta$, and continuous monomorphisms $\partial_i : \mathcal{G}(e) \longrightarrow \mathcal{G}(d_i(e))$ for each edge $e \in E(\Delta)$. The fundamental group

$$\Pi = \Pi_1(\mathcal{G}, \Delta)$$

of the graph of profinite groups $(\mathcal{G}, \Delta)$ is defined by means of a universal property: Π is a profinite group together with the following data and conditions:

(i) A maximal subtree T of Δ;
(ii) A collection of continuous homomorphisms

$$\nu_m : \mathcal{G}(m) \longrightarrow \Pi \quad (m \in \Delta),$$

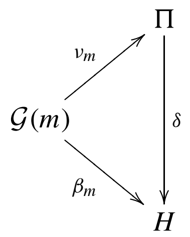
and a continuous map $E(\Delta) \longrightarrow \Pi$, denoted $e \mapsto t_e$ $(e \in E(\Delta))$, such that $t_e = 1$, if $e \in E(T)$, and

$$(\nu_{d_0(e)}\partial_0)(x) = t_e(\nu_{d_1(e)}\partial_1)(x)t_e^{-1}, \quad \forall x \in \mathcal{G}(e),\ e \in E(\Delta);$$

(iii) The following universal property is satisfied whenever one has the following data:

- H is a profinite group;
- $\beta_m : \mathcal{G}(m) \longrightarrow \Pi \quad (m \in \Delta)$ a collection of continuous homomorphisms;
- a map $e \mapsto s_e$ $(e \in E(\Delta))$ with $s_e = 1$, if $e \in E(T)$;
- $(\beta_{d_0(e)}\partial_0)(x) = s_e(\beta_{d_1(e)}\partial_1)(x)s_e^{-1}$, $\forall x \in \mathcal{G}(e),\ e \in E(\Delta)$,

then there exists a unique continuous homomorphism $\delta : \Pi \longrightarrow H$ such that $\delta(t_e) = s_e$ $(e \in E(\Delta))$, and for each $m \in \Delta$ the diagram



commutes.

In [34, Paragraph 3.3], the fundamental group Π is defined explicitly in terms of generators and relations. It is also proved there that the definition given above is independent of the choice of the maximal subtree T. We use the notation $\Pi(m) = \mathrm{Im}(\nu_m)$.

Associated with the graph of groups $(\mathcal{G}, \Delta)$ there is a corresponding *standard profinite graph* (or universal covering graph) $S = S(\Pi) = \dot\bigcup \Pi/\Pi(m)$. The vertices of S are those cosets of the form $g\Pi(v)$, with $v \in V(\Delta)$ and $g \in \Pi$; the incidence maps of S are given by the formulas:

$$d_0(g\Pi(e)) = g\Pi(d_0(e)); \quad d_1(g\Pi(e)) = gt_e\Pi(d_1(e))\ (e \in E(\Delta)).$$

In fact S is a profinite tree (*cf.* [34, Theorem 3.8]. There is a natural action of Π on S, and clearly $S/\Pi = \Delta$.

Remark 2.1. If $\pi_1(\mathcal{G}, \Gamma)$ is the fundamental group of a finite graph of groups then one has the induced graph of profinite completions of edge and vertex groups $(\widehat{\mathcal{G}}, \Gamma)$ and a natural homomorphism $\pi = \pi_1(\mathcal{G}, \Gamma) \longrightarrow \Pi_1(\widehat{\mathcal{G}}, \Gamma)$. It is an embedding if $\pi_1(\mathcal{G}, \Gamma)$ is residually finite. In this case $\Pi_1(\widehat{\mathcal{G}}, \Gamma) = \widehat{\pi_1(\mathcal{G}, \Gamma)}$ is simply the profinite completion. Moreover,

(i) The tree $S(\pi)$ naturally embeds in $S(\widehat{\pi})$ if and only if the edge and vertex groups $\mathcal{G}(e), \mathcal{G}(v)$ are separable in $\pi_1(\mathcal{G}, \Gamma)$, or equivalently $\mathcal{G}(e)$ are closed in $\mathcal{G}(d_0(e))$, $\mathcal{G}(d_1(v))$ with respect to the topology induced by the profinite topology on π (see [11, Proposition 2.5]);

(ii) If H is an infinite finitely generated subgroup of π then by combination of Theorem 4.12 and Proposition 4.13 of Chapter 1 in [12] there exists a minimal H-invariant subtree T_H of $S(\pi)$ and it is unique. Moreover, T_H/H is finite;

(iii) If $S(\pi)$ naturally embeds in $S(\widehat{\pi})$, the closure $\overline{T}_H$ in $S(\widehat{\pi})$ is a $\overline{H}$-invariant profinite subtree and by [28, Lemma 1.5] contains a unique (in $S(\widehat{\pi})$) minimal $\overline{H}$-invariant subtree $\widehat{T_{\overline{H}}}$. Moreover, $\widehat{T_{\overline{H}}}/\overline{H}$ is finite since it is a subgraph of a quotient graph $\overline{T}_H/\overline{H}$ of the finite graph T_H/H.

Lemma 2.2. *Within the hypotheses of Remark* 2.1 (iii) *suppose π is subgroup separable and H acts freely on $S(\pi)$. Then $\overline{T}_H = \widehat{T}_H$ and T_H is a connected component of $\overline{T}_H$ (considered as a usual graph).*

Proof. For a graph Δ denote by D_Δ a maximal subtree of Δ. Since H acts freely on $S(\pi)$ it is free of rank $(T_H/H) \setminus D_{T_H/H}$. Since G is subgroup separable $\overline{H} \cong \widehat{H}$ is a free profinite group of the same rank as H. By [26, Lemma 2.8] $\overline{H}$ acts on $S(\widehat{\pi})$ freely as well. By [33, Proposition 2.11] $\overline{H}$ is a free profinite group of rank $(\overline{T}_H/\overline{H}) \setminus D_{\overline{T}_H/\overline{H}}$ and since T_H/H is a covering of $\overline{T}_H/\overline{H}$ (because of the free action of $\overline{H}$) we deduce that $T_H/H = \overline{T}_H/\overline{H}$. To see that $\overline{T}_H = \widehat{T}_H$ let Σ be a connected transversal of $\widehat{T}_H/\overline{H}$ in $S(\pi)$ with $d_0(\Sigma) \subseteq \Sigma$ and Ω its maximal connected subtree. Put $K = \langle k_e \in \pi \mid k_e d_1(e) \in \Sigma \setminus \Omega\rangle$. Then K is a free group freely generated by $\{k_e \in \pi \mid k_e d_1(e) \in \Sigma\}$ and also $\overline{H}$ is freely generated (as a profinite group) by $\{k_e \in \pi \mid k_e d_1(e) \in \Sigma\}$ (see [33, Lemma 2.3]). It follows that $\widehat{H} \cong \overline{H} = \overline{K} \cong \widehat{K}$. Since $\overline{T}_H/\overline{H}$ contains $\widehat{T}_H/\overline{H}$ by Remark 2.1 (iii), $H = K * L$ is a free product for some L and so $\widehat{H} = \widehat{K} \amalg \widehat{L}$ the free profinite product. It follows from $\widehat{H} \cong \widehat{K}$ that $\widehat{L} = 1 = L$ so that $K = H$. Then by the minimality of H-invariant subtree T_H, we have $T_H = H\Sigma$ and so $\overline{T}_H = \overline{H\Sigma} = \widehat{T}_H$.

If T_H is not a connected component of $\overline{T}_H$ then there exists an edge $e \in \overline{T}_H \setminus T_H$ with an incident vertex $v \in T_H$. Since $\overline{T}_H/\overline{H} = T_H/H$, $\bar{h}e \in T_H$ for some $\bar{h} \in \overline{H}$ and so $\bar{h}v \in T_H$. Hence there exists $h \in H$ with $hv = \bar{h}v$ and since the action of $\overline{H}$ on $S(\widehat{\pi})$ is free we have $\bar{h} = h$ implying $e \in S(H)$, a contradiction. □

The following term will be important in the following section to perform an induction on hierarchy.

Definition 2.3. We say that a residually finite group G is adjustable if for any pair of finitely generated subgroups A and B of G such that $\overline{A}^\gamma = \overline{B}$ for some $\gamma \in \widehat{G}$ there exists $\beta \in \overline{B}$, such that $A^{\gamma\beta} \cap B \neq 1$.

Remark 2.4. Note that adjustability is preserved by commensurability. Indeed, a finite index subgroup of an adjustable group is clearly adjustable.

If a group H has a finite index adjustable subgroup G, then H is adjustable. Indeed, passing to the core we may assume that G is normal. Suppose A, B are finitely generated subgroups of G such that $\overline{A}^{\gamma} = \overline{B}$ for some $\gamma \in \widehat{G}$. If A, B are finite then they coincide with their closures and there is nothing to show. If on the other hand they are infinite then $(\overline{A} \cap \widehat{G})^{\gamma} = \overline{B} \cap \widehat{G}$ and since G is adjustable, there exists $\beta \in \overline{B} \cap \widehat{G}$, such that $A^{\gamma\beta} \cap B \neq 1$ as needed.

Proposition 2.5. *Let $G = \pi_1(\mathcal{G}, \Gamma)$ be the fundamental group of a finite graph of finitely generated adjustable groups. Suppose G is subgroup separable. Then G is adjustable.*

Proof. Let A, B be infinite finitely generated subgroups of G such that $\overline{A}^{\gamma} = \overline{B}$ for some $\gamma \in \widehat{G}$ (if A and B are finite there is nothing to prove). Let S be a standard tree on which G acts and let T_A and T_B be the minimal A-invariant and B-invariant subtrees of the standard tree S on which G acts (see Remark 2.1 (ii)). Denote by $\widehat{T_{\overline{A}}}$ and $\widehat{T_{\overline{B}}}$ the (unique) minimal $\overline{H}_1$ and $\overline{H}_2$-invariant profinite subtrees in $\widehat{S}$ respectively (see Remark 2.1 (iii)). Then, $\gamma^{-1}\widehat{T_{\overline{A}}} = \widehat{T_{\overline{B}}}$, by the uniqueness of the minimal $\overline{B}$-invariant subtree $\widehat{T_{\overline{B}}}$ in $\widehat{S}$.

(i) If the action of A on S is free, then by [26, Lemma 2.8] $\overline{A}$, $\overline{B}$ and B have trivial edge and vertex stabilizers as well. Hence by [27, Proposition 1.6] $\overline{T}_A = \widehat{T_{\overline{A}}}$, $\overline{T}_B = \widehat{T_{\overline{B}}}$ and by Lemma 2.2 T_A and T_B are the (usual) connected component of $\widehat{T}_A$ and $\widehat{T}_B$ respectively. Since $\overline{T}_B = \overline{B}T_B$ this means that $\beta^{-1}\gamma^{-1}T_A = T_B$ for some $\beta \in \overline{B}$. It follows that $A^{\gamma\beta} = B$, since $T_B/A^{\gamma\beta} = \overline{T_B}/\overline{A^{\gamma\beta}} = \overline{T_B}/\overline{B} = T_B/B$ by [27, Proposition 1.6].

(ii) If $A_w \neq 1$ for some $w \in S$, then $A_v \neq 1$ for some $v \in T_A$. Since $\gamma^{-1}\widehat{T_{\overline{A}}} = \widehat{T_{\overline{B}}}$ and $\widehat{T_{\overline{B}}}/\overline{B}$ is a subgraph of a quotient graph of T_B/B (see Remark 2.1(iii)), we have $\beta^{-1}\gamma^{-1}v \in T_B$ for some $\beta \in \overline{B}$. Since $S(G)/G = S(\widehat{G})/\widehat{G}$, the vertices v, $\beta\gamma v$ are in the same G-orbit and so there exists $g \in G$ with $gv = \beta^{-1}\gamma^{-1}v$ so that $\gamma\beta g \in \widehat{G}_v$. Therefore $\overline{A}_v^{\gamma\beta g} = (\overline{A} \cap \widehat{G}_v)^{\gamma\beta g} = \overline{A}^{\gamma\beta g} \cap \widehat{G}_v = \overline{B}^g \cap \widehat{G}_v = (\overline{B}^g)_v$. Since G_v is adjustable there exists $\beta_v^g \in (\overline{B}^g)_v$ such that $A_v^{\gamma\beta g\beta_v^g} = A_v^{\gamma\beta\beta_v g} = (B^g)_v \leq B^g$. Then $A_v^{\gamma\beta\beta_v} \leq B$ and since $\beta\beta_v \in \overline{B}$ the result is proved. □

Remark 2.6. The case (i) of the proof of Proposition 2.5 shows that if the action of A on S is free then in addition we have $\beta^{-1}\gamma^{-1}T_A = T_B$ and $A^{\gamma\beta} = B$.

On the other hand if the action of A on S is not free the case (ii) of the proof shows that the element $a \neq 1$ with $a^{\gamma\beta} \in B$ exists in every non-trivial vertex stabilizer A_v.

3. General results

A subgroup H of a group G is called a virtual retract if H is a semidirect factor (retract) of some finite index subgroup of G. A group G is called hereditarily conjugacy separable if every finite index subgroup of G is conjugacy separable.

We begin this section with the key:

Lemma 3.1. *Let G be an adjustable conjugacy separable group and A, B be finitely generated separable subgroups of G. Suppose there exists an element $a \in A$ such that $C_{\widehat{G}}(a)G = C_{\overline{A}}(a)G$. Then the conjugacy of $\overline{A}$ and $\overline{B}$ in $\widehat{G}$ implies the conjugacy of A and B in G. In particular, the statement holds if $[C_G(a) : \langle a\rangle]$ is finite and $\overline{C_G(a)} = C_{\widehat{G}}(a)$; the latter equality holds for every $1 \neq a \in G$ if G is hereditary conjugacy separable.*

Proof. Suppose $\overline{A}^{\gamma} = \overline{B}$ for some γ in $\widehat{G}$. Since G is adjustable $a^{\gamma\beta} \in B$ for some $1 \neq a \in A$, $\beta \in \overline{B}$. Since G is conjugacy separable $a^g = a^{\gamma\beta}$ for some $g \in G$, so replacing B with $B^{g^{-1}}$ and γ with $\gamma\beta g^{-1}$ we may assume $\gamma \in C_{\widehat{G}}(a)$. By hypothesis, $C_{\widehat{G}}(a)G = C_{\overline{A}}(a))G$, so $\gamma = a'g$ for some $a' \in C_{\overline{A}}(a)$, $g \in G$ and therefore once more replacing B with $B^{g^{-1}}$ and γ with γg^{-1} we may assume that $\gamma \in C_{\overline{A}}(a)$. It follows then that $\overline{A} = \overline{B}$ and since A and B are separable we deduce that $A = B$.

To prove the last statement note that since $[C_G(a) : \langle a\rangle]$ is finite and $C_G(a)$ is dense in $C_{\widehat{G}}(a)$, then $C_{\widehat{G}}(a) = \overline{\langle a\rangle}C_G(a)$ and so $C_{\widehat{G}}(a)G = C_{\overline{A}}(a))G$ clearly holds. We conclude the proof observing that by [21, Lemma 12.3] hereditarily conjugacy separability of G implies $C_{\widehat{G}}(a) = \overline{C_G(a)}$ for every $a \in G$. □

Theorem 3.2. *Let G be an adjustable, subgroup separable and hereditarily conjugacy separable group. Suppose that for any element $1 \neq g \in G$ the index $[C_G(g) : \langle g\rangle]$ is finite. Then G is subgroup conjugacy separable.*

Proof. Let A, B be finitely generated subgroups of G such that $\overline{A}^{\gamma} = \overline{B}$ for some γ in $\widehat{G}$. Then all the premises of Lemma 3.1 are satisfies and so applying it we deduce that A and B are conjugate in G as required. □

Since the centralizer of a non-trivial element in a torsion free hyperbolic group is cyclic (see [2, Proposition 3.5]) we deduce the following:

Corollary 3.3. *A torsion free adjustable, subgroup separable and hereditarily conjugacy separable hyperbolic group is subgroup conjugacy separable.*

Definition 3.4. The class of groups with a *hierarchy* is the smallest class of groups, closed under isomorphism, that contains the trivial group, and such that, if

(1) $G = A *_C B$ and A, B each have a hierarchy;
(2) $G = A*_C$ and A has a hierarchy;

then G also has a hierarchy.

Groups with hierarchy allow to use induction on their hierarchy. Thus we can deduce from Proposition 2.5 the following

Proposition 3.5. *A subgroup separable group with hierarchy is adjustable.*

Theorem 3.6. *A hyperbolic hereditarily conjugacy separable group H having a finite index subgroup separable subgroup G with hierarchy is infinite subgroup conjugacy separable.*

Proof. By Proposition 3.5 and Remark 2.4 H is adjustable, and since subgroup separability passes to overgroups of finite index, is subgroup separable.

Let H_1, H_2 be infinite finitely generated subgroups of H such that $\overline{H}_1^{\gamma} = \overline{H}_2$ for some $\gamma \in \widehat{H}$.

Since a residually finite hyperbolic group is virtually torsion free (see [19, Theorem 5.1]) H contains a torsion free finite index subgroup K so replacing G by $G \cap K$ we may assume that G is torsion free. Then H_1 possesses an element of infinite order in G. The centralizer of an element h of infinite order in a hyperbolic group is virtually cyclic (see [2, Proposition 3.5]) and so h generates the subgroup of finite index in its centralizer. Thus by Lemma 3.1 H_1 and H_2 are conjugate in H. □

A group G is called virtually compact special if there exists a special compact cube complex X having a finite index subgroup of G as its fundamental group (see [31] for definition of special cube complex). Since the hyperbolic fundamental group of such a complex admits a hierarchy [17] we deduce from Proposition 3.5 and Remark 2.4 the following:

Corollary 3.7. *A hyperbolic virtually compact special group is adjustable.*

Lemma 3.8. *Let A be a finite subgroup of a hyperbolic virtually compact special group G. Then:*

(i) *$C_G(A)$ is a virtual retract of G and is virtually compact special;*
(ii) *$C_G(A)$ is dense in $C_{\widehat{G}}(A)$.*

Proof. (i) Since the group G is hyperbolic, it is well-known that centralizers of elements in G are quasiconvex (see, for example, [8, Chapter III.Γ, Proposition 4.14]) and are also hyperbolic (*cf.* [2, Lemma 3.8]). Then inductively on the number of elements using $C_G(a) \cap C_G(b) = C_{C_G(a)}(b)$ we deduce that the centralizer of any finite subgroup of a hyperbolic group is quasiconvex and hyperbolic. In [15, Corollary 7.8] Haglund and Wise proved that any quasiconvex subgroup of G is virtually compact special and in [14] that it is virtual retract of G. Thus the centralizer of a finite subgroup in G is a virtual retract of G.

(ii) Using (i) we prove (ii) by induction on $|A|$. By [23, Theorem 1.1] a virtually compact special group is hereditarily conjugacy separable and so by [21, Lemma 12.3] $C_G(a)$ is dense in $C_{\widehat{G}}(a)$ for any element $a \in G$. This gives the base of induction.

Let K be a maximal subgroup of A and $a \in A \setminus K$. Then $C_G(A) = C_{C_G(K)}(a)$ and $C_{\widehat{G}}(A) = C_{C_{\widehat{G}}(K)}(a)$. By induction hypothesis $C_G(K)$ is dense in $C_{\widehat{G}}(K)$ and $C_{C_G(K)}(a)$ is dense in $C_{\widehat{C_G(K)}}(a)$. Then using that the profinite topology of G induces the full profinite topology on virtual retracts we have $\overline{C_G(A)} = \overline{C_{C_G(K)}(a)} = C_{\widehat{C_G(K)}}(a) = C_{\overline{C_G(K)}}(a) = C_{C_{\widehat{G}}(K)}(a) = C_{\widehat{G}}(A)$ as required. □

The proof of the next proposition completes the proof of [10, Theorem 2.6] where the case of finite subgroups was left out. It was suggested by Ashot Minasyan.

Proposition 3.9. *A hyperbolic virtually compact special group G is finite subgroup conjugacy separable.*

Proof. Let H_1, H_2 be finite subgroups of G with $H_1^{\gamma} = H_2$ for some $\gamma \in \widehat{G}$. We shall use induction on the order $|H_1| = |H_2|$. The conjugacy separability of G proved in [23, Theorem 1.1] implies the result for H_1 cyclic of order p.

Suppose now $|H_1| > p$ and let A be a maximal subgroup of H_1. Since G is virtually torsion free there exist a finite index torsion free normal subgroup U in G. Since $\widehat{G} = \widehat{U}G$ replacing H_2 by its conjugate in G we may assume that $\gamma \in \widehat{U}$. Then $H_2 \leq H_1\overline{U} \cap G = H_1U$ and so we may assume that $G = UH_1 = U \rtimes H_1$. By induction hypothesis $A^{\gamma} = A^{g}$ for some $g \in G$. Since $g = h_1u$ for some $u \in U, h_1 \in H_1$ and since $AU/U = A\widehat{U}/\widehat{U} = A^{\gamma}\widehat{U}/\widehat{U} = A^{g}U/U = A^{h_1}U/U$, h_1 normalizes A and therefore $A^{u} = A^{\gamma}$. Then replacing H_2 by $H_2^{u^{-1}}$ and γ with γu^{-1} we may assume that $\gamma \in C_{\widehat{U}}(A) = N_{\widehat{U}}(A)$.

Pick $h \in H_1 \setminus A$. Then $h^{\gamma} \in H_2$ and so since G is conjugacy separable $h^{g} = h^{\gamma}$ for some $g \in G$, *i.e.* $g \in C_{\widehat{G}}(h)\gamma$. Then $g \in C_{\widehat{G}}(h)C_{\widehat{G}}(A)$ and since by [15, Theorem 1.5] every quasiconvex subgroup of G is separable and then by [22, Theorem 1.1] the product of quasiconvex subgroups of G is separable we deduce that $g \in C_G(h)C_G(A)$, *i.e.* $g = c_hc_A$ for some $c_h \in C_G(h), c_A \in C_G(A)$. Then $h^{c_A} = h^{\gamma}$ and so $H_1^{c_A} = H_2$. □

Now from Theorem 3.6 and Proposition 3.9 we deduce the main general result of this paper.

Theorem 3.10. *A hyperbolic subgroup separable virtually compact special group G is subgroup conjugacy separable.*

4. Manifolds

Here we apply the general result of the previous section to closed and cusp hyperbolic 3-manifolds. For closed 3-manifolds the result follows quickly.

Theorem 4.1. *The fundamental group π_1M of a closed hyperbolic 3-manifold M is subgroup conjugacy separable.*

Proof. In this case π_1M is hyperbolic. By result of Agol [1] π_1M is virtually compact special and subgroup separable. It is also hereditarily conjugacy separable (see [3, G8]). Thus the result follows from Theorem 3.10. □

We consider the cusped case now. Recall that a subgroup of π_1M is called *peripheral* if it is conjugate to the fundamental group of a cusp and so is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

It is well-known that $\pi_1 M$ is relatively hyperbolic to peripheral subgroups [13, Theorem 5.1]. We refer the reader to [18] for a survey of the various equivalent definitions of relative hyperbolicity.

Theorem 4.2. *The fundamental group $H = \pi_1 M$ of a cusped hyperbolic 3-manifold M is subgroup conjugacy separable.*

Proof. The group H is subgroup separable [3, Corollary 5.5] and hereditarily conjugacy separable (see in [3, Section 5.2 (H.8)]). Since by [32, Theorem 9.1] H admits a hierarchy by Proposition 3.5 combined with Remark 2.4 it is adjustable.

Let A, B be finitely generated subgroups of H such that $\overline{A}^{\gamma} = \overline{B}$ for some $\gamma \in \widehat{H}$. Note that by [21, Lemma 12.3] hereditary conjugacy separability of G implies $\overline{C_G(a)} = C_{\widehat{G}}(a)$ for any a.

If A is not contained in a peripheral subgroup then since H is relatively hyperbolic to peripheral subgroups [13, Theorem 5.1] there exists $a \in A$ such that $C_G(a)$ is infinite cyclic (*cf.* [24, Theorem 4.3]). Hence $[C_G(a) : \langle a \rangle]$ is finite. Therefore we deduce from Lemma 3.1 that A and B are conjugate.

If A is contained in a peripheral subgroup P then it is either free Abelian of rank 2 (and so is of finite index in P) or cyclic. In the first case the condition $C_{\widehat{G}}(a)G = C_{\overline{A}}(a)G$ is satisfied so by Lemma 3.1 A and B are conjugate. If A and B are cyclic, then since H is adjustable there exists $1 \neq a \in A$ and $\beta \in \overline{B}$ with $a^{\gamma\beta} \in B$ and since H is conjugacy separable $a = a^{\gamma\beta h}$ for some h in H. Then conjugating B by h^{-1} we may assume that $a = a^{\gamma\beta}$ and since peripheral subgroups pairwise intersect trivially (*cf.* [16, Lemma 4.7]) this implies that $B \leq P$. But P is free Abelian, so cyclic subgroups A, B intersecting non-trivially must coincide. $\square$

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