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A sufficient condition for the continuity of solutions to a logarithmic diffusion equation

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Abstract. This note gives a first sufficient condition that ensures a non-negative, locally bounded, local solution to a logarithmically singular parabolic equation is continuous at a vanishing point and an estimate of the modulus of continuity is given. Moreover, an estimate of the Hausdorff measure of the set of discontinuity is established.

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1. Introduction and main results

Let E be an open set in $\mathbb{R}^N$. For $T > 0$, let E_T denote the cylindrical domain $E \times (0, T]$. Consider the quasi-linear, parabolic differential equation

$$u_t - \Delta \ln u = 0 \quad \text{weakly in } E_T. \tag{1.1}$$

This equation is singular since its modulus of ellipticity $u^{-1} \to \infty$ as $u \to 0$. A non-negative function u satisfying

$$u \in C_{\text{loc}}\Big(0, T; L^2_{\text{loc}}(E)\Big), \quad \ln u \in L^2_{\text{loc}}\Big(0, T; W^{1,2}_{\text{loc}}(E)\Big)$$

is called a local, weak sub(super)-solution to (1.1) if for every compact set $K \subset E$ and every sub-interval $[t_1, t_2] \subset (0, T]$

$$\int_K u\varphi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big(-u\varphi_t + \frac{Du}{u}D\varphi\Big)dxdt \le (\ge)\, 0$$

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for all non-negative testing functions

$$\varphi \in W^{1,2}_{\mathrm{loc}}\Big(0,T;L^2(K)\Big) \cap L^2_{\mathrm{loc}}\Big(0,T;W^{1,2}_o(K)\Big).$$

A function u that is both a local, weak sub-solution and a local, weak super-solution is a local, weak solution.

For $\rho > 0$ we denote by $K_\rho(y)$ the cube centered at y with side length ρ. If $y = 0$ we use K_ρ. For $\theta > 0$ introduce the cylinder with "vertex" at (0, 0)

$$Q_\rho(\theta) = K_\rho \times (-\theta\rho^2, 0].$$

If $\theta = 1$ we use Q_ρ. Also a cylinder with "vertex" at (y, s) is

$$(y,s) + Q_\rho(\theta) = K_\rho(y) \times (s - \theta\rho^2, s].$$

Assume u is a locally bounded, local solution. Let us suppose $\rho > 0$ is so small that the cylinder $(y,s) + Q_\rho \subset E_T$. Up to a translation we may assume $(y,s) = (0,0)$ and let

$$\omega = \operatorname*{ess\,osc}_{Q_\rho} u.$$

Without loss of generality we assume $\omega \le 1$ such that

$$Q_\rho(\omega) \subset Q_\rho \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_\rho(\omega)} u \le \omega.$$

Suppose in addition to the notion of solution that

$$D\ln u \in L^p_{\mathrm{loc}}(E_T) \quad \text{for some } p > \frac{N+2}{2}. \tag{1.2}$$

Note that when $N = 1$ the integrability condition (1.2) is inherent in the notion of solution while in other cases it has to be imposed. Accordingly we define the quantity

$$I_{p,\rho}(y,s) = \rho\left(\fint\!\!\fint_{(y,s)+Q_\rho} |D\ln u|^p \, dxdt\right)^{\frac{1}{p}}$$

and $I_{p,\rho} = I_{p,\rho}(0,0)$. Then we have the following main theorem.

Theorem 1.1. *Let u be a non-negative, locally bounded, local solution to* (1.1) *and assume* (1.2) *is satisfied. Then there exist constants $\bar{C} > 1$ and $\alpha \in (0,1)$ depending only on N, such that for any $\mu \in (0,1)$ and $0 < r < \rho \le R_o$ we have*

$$\operatorname*{ess\,osc}_{Q_r(\omega)} u \le \bar{C}\left[\omega\left(\frac{r}{R_o}\right)^{(1-\mu)\alpha} + I_{p,R_o^{1-\mu}r^\mu}\right]$$

In particular, the solution u is continuous at the origin provided

$$\limsup_{r\to 0} I_{p,r} = 0. \tag{1.3}$$

Remark 1.2. Strictly speaking, we need the convention that the function $\rho \to I_{p,\rho}$ is non-decreasing. In order to realize that, we need only to replace $I_{p,\rho}$ by

$$\tilde{I}_{p,\rho} \stackrel{\text{def}}{=} \sup_{0<r<\rho} I_{p,r}$$

in Theorem 1.1

Remark 1.3. For $\lambda \geq 0, T > 0$ and $N \geq 3$, the explicit solution

$$u(x,t) = \frac{2(N-2)(T-t)^{\frac{N}{N-2}}}{\lambda + (T-t)^{\frac{2}{N-2}} |x|^2} \tag{1.4}$$

is continuous up to its extinction time T. One verifies that when $\lambda > 0$ and for any fixed x_o, there is a positive constant $C(x_o, \lambda, N, p)$ such that

$$I_{p,r}(x_o, T) \leq C r^{\frac{4}{(N-2)}+2} \to 0 \quad \text{as } r \to 0.$$

When $\lambda = 0$, (1.4) gives an unbounded solution which, in particular, is discontinuous at $x = 0$. Condition (1.3) is verified everywhere except for $x = 0$. A direct calculation shows that

$$D \ln u \in L^p_{\text{loc}}\left(\mathbb{R}^N \times (-\infty, T]\right) \quad \text{for any} \quad \frac{N+2}{2} < p < N.$$

Furthermore, there exists some positive constant $C(N,p)$ such that for every $t < T$

$$I_{p,r}(0,t) = \begin{cases} C(N,p) & \frac{N+2}{2} < p < N \\ \infty & p \geq N. \end{cases}$$

Hence the condition (1.2) alone is not sufficient to ensure continuity.

Now define the set $S \subset E_T$ to consist of all discontinuous points of a local solution u and

$$S_o = \left\{ (y,s) \in E_T : \limsup_{\rho \to 0} \frac{1}{\rho^{N+2-p}} \iint_{(y,s)+Q_\rho} |D \ln u|^p \, dx dt > 0 \right\}.$$

As a direct consequence of Theorem 1.1 it is straightforward to see that $S \subset S_o$. Moreover we are going to obtain an estimate of the Hausdorff measure of the set S_o.

The *parabolic Hausdorff measure* $\mathcal{P}_k$ is defined in a way similar to the usual Hausdorff measure $\mathcal{H}_k$ but using the parabolic metric on $\mathbb{R}^N \times \mathbb{R}$. For any set $U \subset \mathbb{R}^N \times \mathbb{R}$ and $k \geq 0$ we define

$$\mathcal{P}_k(U) = \lim_{\delta \to 0} \mathcal{P}^\delta_k(U),$$

where

$$\mathcal{P}_k^\delta(U) = \inf\left\{\sum_{i=1}^{\infty} r_i^k : U \subset \bigcup_i [(y_i, t_i) + Q_{r_i}],\ r_i < \delta\right\}.$$

$\mathcal{P}_k$ so defined is an outer measure whose σ-algebra contains all Borel sets of $\mathbb{R}^N \times \mathbb{R}$ (Chapter 2, [10]). Comparing with the Lebesgue measure $\mathcal{L}^{N+1}$ in $\mathbb{R}^{N+1}$, the parabolic Hausdorff measure is a more suitable measure to quantify the size of the discontinuity set S. Also, it should be pointed out that the parabolic Hausdorff measure dominates the usual Hausdorff measure in the sense that there is some constant $C(N, k)$ such that for any subset U of $\mathbb{R}^N \times \mathbb{R}$ one has

$$\mathcal{H}_k(U) \leq C\mathcal{P}_k(U).$$

Regarding the Hausdorff measure of the discontinuity set we have the following consequence of Theorem 1.1.

Theorem 1.4. *Let u be a non-negative, locally bounded, local solution to* (1.1) *and assume* (1.2) *is satisfied. Then we have*

$$\begin{array}{ll} \mathcal{P}_{N+2-p}(S) = 0, & N > 1 \quad \textit{and} \\ \mathcal{P}_1(S) = 0, & N = 1. \end{array}$$

Remark 1.5. When $N = 1$ the possible discontinuous points of a non-negative, locally bounded, local solution to (1.1) cannot occupy a line in $\mathbb{R}^2$. Generally one gets less discontinuity as the L^p integrability of $D \ln u$ increases and eventually, the solution is continuous at every point if one has $p \geq N + 2$.

1.1. Novelty and significance

Equation (1.1) describes the evolution of the Ricci flow for complete $\mathbb{R}^2$ [18]. It also arises from modeling the thickness of a viscous liquid thin film that lies on a rigid plate under the influence of the van der Waals force [17].

Physical and geometric motivations of (1.1) make sense mainly for $N = 2$, but the problem is intriguing in the effort to shed light on the structural properties of singular diffusion equations.

Questions concerning both existence and non-existence of solutions to the Cauchy problem of (1.1) and its related elliptic equation are investigated in [1,2,4, 5,11,12,16] (just mention few).

The study of local behavior of local solutions to (1.1) has been initiated in [6,7]. Equation (1.1) can be viewed as a formal limit of the porous medium equation

$$u_t - \operatorname{div}(u^{m-1} Du) = 0 \quad \text{as } m \to 0.$$

A proof of Hölder continuity for non-negative, locally bounded, local solutions to the porous medium equation can be found in Appendix B of [9]. However, the local behavior of local solutions to (1.1) presents many striking differences from that of local solutions to the porous medium equation. See [14] for more detailed discussion.

It was shown in [6] that if one assumes that

$$u \in L^r_{\mathrm{loc}}(E_T) \quad \text{for some} \quad r > \max\left\{1, \frac{N}{2}\right\},$$

then u is locally bounded. If in addition one assumes that

$$\ln u \in L^\infty_{\mathrm{loc}}\left(0, T; L^p_{\mathrm{loc}}(E)\right) \quad \text{for some} \quad p > N + 2,$$

then a Harnack-type inequality is established and thus, if the solution does not vanish identically on a hyperplane normal to the time axis, then the equation (1.1) is neither degenerate nor singular in a backward cylinder with its vertex on the hyperplane. As a result u is a classical solution in such a cylinder by the classical parabolic theory (see [13]). In fact, it is shown in [8] that under such circumstances the solution is analytic in space variables while infinitely differentiable in time.

Nevertheless, these results do not explain why some explicit solutions, (1.4) for example, could be continuous up to their extinction time. Theorem 1.1 gives a first sufficient condition that ensures continuity at a vanishing point of u, and an explicit estimate of the modulus of continuity is given. Moreover, we establish in Theorem 1.4 an estimate on the Hausdorff measure of the set of discontinuity of u.

This effort being made, it is interesting to ask whether the higher integrability conditon (1.2) of $D\ln u$ for $N > 1$ can be obtained from the notion of solution and whether the condition (1.3) is necessary for a point to be a continuity point of u. Last but not least, can we construct an explicit bounded solution with discontinuity? When $N = 1$, a solution discontinuous on a line segment was constructed in [15]. However, the notion of solution used seems different from this note, since our results indicate such a phenomenon is not allowed for our solutions.

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2. Proof of Theorem 1.4 assuming Theorem 1.1

The proof of Theorem 1.4 is based on the following

Proposition 2.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$, suppose $0 \le d < N + 2$ and define*

$$\Lambda_d = \left\{(y, s) \in \mathbb{R}^{N+1} : \limsup_{\rho\to 0} \frac{1}{\rho^s} \iint_{(y,s)+Q_\rho} |f|\, dxdt > 0\right\}.$$

Then

$$\mathcal{P}_d(\Lambda_d) = 0.$$

Proof. This is a parabolic counterpart of a similar result shown in [10, Section 2.4.3]. We include the proof here for the reader's convenience. We may assume $f \in L^1(\mathbb{R}^{N+1})$. First, we use the Lebesgue Differentiation Theorem [10, Section 1.7.1] to obtain that

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \iint_{(y,s)+\mathcal{Q}_\rho} |f|\, dxdt = 0 \quad \text{with} \quad (y,s)+\mathcal{Q}_\rho = K_\rho(y)\times(s-\rho^2, s+\rho^2],$$

for $\mathcal{L}^{N+1}$ a.e. (y,s), since $0 \le d < N+2$. Hence

$$\mathcal{L}^{N+1}(\Lambda_d) = 0.$$

Next, by the absolute continuity of the Lebesgue integral, for any $\sigma > 0$ there exists $\eta > 0$, such that

$$\mathcal{L}^{N+1}(U) \le \eta \quad \text{implies} \quad \iint_U |f|\, dxdt < \sigma.$$

Define

$$\Lambda_d^\epsilon = \left\{ (y,s) \in \mathbb{R}^{N+1} : \limsup_{\rho\to 0} \frac{1}{\rho^d} \iint_{(y,s)+\mathcal{Q}_\rho} |f|\, dxdt > \epsilon \right\},$$

which has zero $\mathcal{L}^{N+1}$ measure by the preceding. Therefore there exists an open set U such that $\Lambda_d^\epsilon \subset U$, $\mathcal{L}^{N+1}(U) < \eta$. Now, fix $\delta > 0$ and set a family of parabolic cylinders

$$\mathcal{F} = \Big\{ (y,s) + \mathcal{Q}_\rho : (y,s) \in \Lambda_d^\epsilon, 0 < \rho < \delta, (y,s) + \mathcal{Q}_\rho \subset U,$$
$$\frac{1}{\rho^d} \iint_{(y,s)+\mathcal{Q}_\rho} |f|\, dxdt > \epsilon \Big\}.$$

By the Vitali covering theorem [10, Section 1.5.1], there exist countable disjoint cylinders $\{\mathcal{Q}_i = (y_i, s_i) + \mathcal{Q}_{\rho_i}\}_{i=1}^\infty$ in $\mathcal{F}$ such that

$$\Lambda_d^\epsilon \subset \bigcup_{i=1}^\infty \hat{\mathcal{Q}}_i, \quad \text{where} \quad \hat{\mathcal{Q}}_i = (y_i, s_i) + \mathcal{Q}_{5\rho_i}.$$

As a result,

$$\begin{aligned}\mathcal{P}_d^{20\delta}(\Lambda_d^{\epsilon}) &\le \sum_{i=1}^{\infty}(10\rho_i)^d\\ &\le \frac{10^d}{\epsilon}\sum_{i=1}^{\infty}\iint_{Q_i}|f|\,dxdt\\ &\le \frac{10^d}{\epsilon}\iint_{U}|f|\,dxdt\\ &\le \frac{10^d}{\epsilon}\sigma.\end{aligned}$$

By letting $\delta \to 0$, and then $\sigma \to 0$, we obtain

$$\mathcal{P}_d(\Lambda_d^{\epsilon}) = 0. \qquad \square$$

Now we are ready to present

Proof of Theorem 1.4. When $N > 1$, since we assume

$$D\ln u \in L^p_{\mathrm{loc}}(E_T) \quad \text{for some } p > \frac{N+2}{2},$$

a straightforward application of Proposition 2.1 yields the desired conclusion.
When $N = 1$, the notion of solution gives

$$D\ln u \in L^2_{\mathrm{loc}}(E_T)$$

and by the Hölder inequality with $p < 2$

$$\left[\frac{1}{\rho^{3-p}}\iint_{Q_\rho}|D\ln u|^p\,dxdt\right]^{\frac{1}{p}} \le \left[\frac{1}{\rho}\iint_{Q_\rho}|D\ln u|^2\,dxdt\right]^{\frac{1}{2}}.$$

Thus

$$S_o \subset \left\{(y,s)\in E_T : \limsup_{\rho\to 0}\frac{1}{\rho}\iint_{(y,s)+Q_\rho}|D\ln u|^2\,dxdt > 0\right\}$$

and again by Proposition 2.1 we obtain

$$\mathcal{P}_1(S_o) = 0.$$

This finishes the proof. $\square$

The rest of the note is devoted to proving Theorem 1.1.

3. Some preliminary estimates

3.1. Energy estimates

Proposition 3.1. *Let u be a local, weak super-solution to* (1.1). *Then there is a positive constant γ depending only on N such that for every cylinder $(y,s)+Q_\rho(\theta)\subset E_T$, every $k\in\mathbb{R}_+$, and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_\rho(y)$,*

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^2<t<s}\frac12\int_{K_\rho(y)}(u-k)_-^2\zeta^2\,dx+k^{-1}\iint_{(y,s)+Q_\rho(\theta)}|D[(u-k)_-\zeta]|^2\,dxdt\\
&\le\frac12\int_{K_\rho(y)}(u-k)_-^2\zeta^2(x,s-\theta\rho^2)\,dx\\
&\quad+\iint_{(y,s)+Q_\rho(\theta)}(u-k)_-^2\zeta|\zeta_t|\,dxdt\\
&\quad+k^{-1}\iint_{(y,s)+Q_\rho(\theta)}(u-k)_-^2|D\zeta|^2\,dxdt\\
&\quad+2\iint_{(y,s)+Q_\rho(\theta)}|D\ln u|(u-k)_-|D\zeta|\zeta\,dxdt.
\end{aligned}
$$

Proof. We may assume $(y,s)=(0,0)$. In the weak formulation for super-solutions to (1.1), we take the test function

$$\varphi=-(u-k)_-\zeta^2$$

over the cylinder

$$Q_t=K_\rho\times(-\theta\rho^2,t]\quad\text{for}\quad t\in(-\theta\rho^2,0],$$

modulo a standard Steklov averaging process. This gives

$$
\begin{aligned}
&-\iint_{Q_t}u_\tau(u-k)_-\zeta^2\,dxd\tau-\iint_{Q_t}\zeta^2D\ln uD(u-k)_-\,dxd\tau\\
&\le\iint_{Q_t}2\zeta(u-k)_-D\ln uD\zeta\,dxd\tau.
\end{aligned}
$$

The first term on the left-hand side is estimated by

$$
\begin{aligned}
&-\iint_{Q_t}u_\tau(u-k)_-\zeta^2\,dxd\tau\\
&\ge\frac12\int_{K_\rho}(u-k)_-^2\zeta^2(x,t)\,dx-\frac12\int_{K_\rho}(u-k)_-^2\zeta^2(x,-\theta\rho^2)\,dx\\
&\quad-\iint_{Q_\rho(\theta)}(u-k)_-^2\zeta|\zeta_\tau|\,dxd\tau,
\end{aligned}
$$

while the second term is estimated by

$$-\iint_{Q_t} \zeta^2 D\ln u D(u-k)_- \,dxd\tau \ge k^{-1} \iint_{Q_\rho(\theta)} |D(u-k)_-|^2 \zeta^2 \,dxd\tau.$$

Next the term on the right side is

$$\iint_{Q_t} 2\zeta(u-k)_- D\ln u D\zeta \,dxd\tau \le 2\iint_{Q_\rho(\theta)} |D\ln u|(u-k)_-|D\zeta|\zeta \,dxd\tau$$

Combining all these estimates yields the conclusion. □

Proposition 3.2. *Let u be a local, weak sub-solution to* (1.1) *in E_T. There exists a positive constant $\gamma = \gamma(N)$, such that for every cylinder $(y,s) + Q_\rho(\theta) \subset E_T$, every $k \in \mathbb{R}_+$, and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_\rho(y)$,*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^2<t\le s} \int_{K_\rho(y)} (u-k)_+^2 \zeta^2(x,t)dx \\
&\quad - \int_{K_\rho(y)} (u-k)_+^2 \zeta^2(x, s-\theta\rho^2)dx \\
&\quad + \iint_{(y,s)+Q_\rho(\theta)} \frac{|D[(u-k)_+\zeta]|^2}{u} dxdt \\
&\le \gamma \iint_{(y,s)+Q_\rho(\theta)} (u-k)_+^2 \zeta|\zeta_t| dxdt \\
&\quad + \gamma \iint_{(y,s)+Q_\rho(\theta)} \frac{(u-k)_+^2}{u} |D\zeta|^2 dxdt.
\end{aligned} \tag{3.1}$$

Proof. After a translation may assume $(y,s) = (0,0)$. Take the test function $\varphi = (u-k)_+\zeta^2$ over Q_t modulo a standard Steklov averaging process, and perform standard calculations. The various integrals are extended over the set $[u > k]$ and since $k > 0$, they are all well defined. □

3.2. A logarithmic estimate for sub-solutions

Introduce the logarithmic function

$$\psi(u) = \ln^+ \left[\frac{H}{H - (u-k)_+ + c} \right] \tag{3.2}$$

where

$$H = \operatorname*{ess\,sup}_{(y,s)+Q_\rho(\theta)} (u-k)_+, \quad 0 < c < \min\{1; H\},$$

and for $s > 0$

$$\ln^+ s = \max\{\ln s; 0\}.$$

In the cylinder $(y, s)+Q_\rho(\theta)$ take a non-negative, piecewise smooth cutoff function ζ independent of t.

Proposition 3.3. *Let u be a non-negative, locally bounded, local, weak sub-solution to equation* (1.1) *in E_T. There exists a constant γ, depending only on N, such that for every cylinder*

$$(y, s) + Q_\rho(\theta) \subset E_T$$

and for every level $k \geq 0$ we have

$$\begin{aligned}\operatorname*{ess\,sup}_{s-\theta\rho^2<t<s} &\int_{K_\rho(y)} \psi^2(u)(x,t)\zeta^2(x)dx \\ \leq \int_{K_\rho(y)} &\psi^2(u)(x, s-\theta\rho^2)\zeta^2(x)dx + \gamma \iint_{(y,s)+Q_\rho(\theta)} \frac{\psi(u)}{u}|D\zeta|^2 dxdt.\end{aligned} \tag{3.3}$$

Proof. Take $(y, s) = (0, 0)$ and work within the cylinder Q_t introduced before in the energy estimates. In the weak formulation of (1.1) take the testing function

$$\varphi = \frac{\partial}{\partial u}\big[\psi^2(u)\big]\zeta^2 = 2\psi\psi'\zeta^2.$$

By direct calculation

$$\big[\psi^2(u)\big]'' = 2(1+\psi)\psi'^2 \in L^\infty_{\mathrm{loc}}(E_T)$$

which implies that such φ is an admissible testing function, modulo a Steklov averaging process. Since $\psi(u)$ vanishes on the set where $(u-k)_+ = 0$, we have

$$\iint_{Q_t} u_\tau[\psi^2]'\zeta^2 dxd\tau = \int_{K_\rho} \psi^2(x,t)\zeta^2 dx - \int_{K_\rho} \psi^2(x, -\theta\rho^2)\zeta^2 dx.$$

As for the remaining term

$$\begin{aligned}&\iint_{Q_t} \frac{Du}{u}\cdot D\varphi dxd\tau \\ &\geq 2\iint_{Q_t}(1+\psi)\psi'^2\frac{|Du|^2}{u}\zeta^2 dxd\tau - 4\iint_{Q_t}\frac{|Du|}{u}\psi\psi'\zeta|D\zeta|dxd\tau \\ &\geq \iint_{Q_t}(1+\psi)\psi'^2\frac{|Du|^2}{u}\zeta^2 dxd\tau - \gamma\iint_{Q_t}\frac{\psi}{u}|D\zeta|^2 dxd\tau.\end{aligned}$$

Collecting these estimates establishes the proposition. □

4. De Giorgi-type lemmas

For a cylinder $(y,s)+Q_{2\rho}(\theta)\subset E_T$ denote by $\mu_\pm$ and ω, numbers satisfying

$$\mu_+ \geq \operatorname*{ess\,sup}_{(y,s)+Q_{2\rho}(\theta)} u, \qquad \mu_- \leq \operatorname*{ess\,inf}_{(y,s)+Q_{2\rho}(\theta)} u, \qquad \omega=\mu_+-\mu_-.$$

Denote by ξ and a fixed numbers in $(0,1)$.

Lemma 4.1. *Let u be a non-negative, locally bounded, local, weak super-solution to* (1.1). *Then there is a constant ν_- depending on the data and θ, ξ, ω, a, such that if*

$$|[u\leq \mu_-+\xi\omega]\cap[(y,s)+Q_{2\rho}(\theta)]|\leq \nu_-|Q_{2\rho}(\theta)|,$$

then either

$$\xi\omega\leq I_{p,\rho}(y,s),$$

or

$$u\geq \mu_-+a\xi\omega \quad \textit{a.e. in } (y,s)+Q_\rho(\theta).$$

Proof. We may take $(y,s)=(0,0)$. Set

$$\rho_n=\rho+\frac{\rho}{2^n}, \quad K_n=K_{\rho_n}, \quad Q_n=K_n\times(-\theta\rho_n^2,0].$$

Consider a non-negative, piecewise smooth cutoff function on Q_n of the form $\zeta(x,t)=\zeta_1(x)\zeta_2(t)$, where

$$\zeta_1=\begin{cases}1 & \text{in } K_{n+1}\\ 0 & \text{in } \mathbb{R}^N-K_n\end{cases} \qquad |D\zeta_1|\leq\frac{1}{\rho_n-\rho_{n+1}}=\frac{2^{n+1}}{\rho}$$

$$\zeta_2=\begin{cases}0 & \text{for } t<-\theta\rho_n^2\\ 1 & \text{for } t\geq-\theta\rho_{n+1}^2\end{cases} \qquad 0\leq\zeta_{2,t}\leq\frac{1}{\theta(\rho_n^2-\rho_{n+1}^2)}\leq\frac{2^{2(n+1)}}{\theta\rho^2}.$$

Now apply the energy estimate to $(u-k_n)_-$ in the cylinder Q_n with

$$k_n=\mu_-+a\xi\omega+\frac{1-a}{2^n}\xi\omega$$

to obtain

$$\begin{aligned}&\operatorname*{ess\,sup}_{-\theta\rho^2<t<0}\int_{K_n}(u-k_n)_-^2\zeta^2(x,t)\,dx+\frac{1}{\mu^-+\xi\omega}\iint_{Q_n}|D[(u-k_n)_-\zeta]|^2\,dxdt\\ &\leq\frac{4^n}{\theta\rho^2}\iint_{Q_n}(u-k_n)_-^2\,dxdt+\frac{4^n}{(\mu^-+a\xi\omega)\rho^2}\iint_{Q_n}(u-k_n)_-^2\,dxdt\\ &\quad+4^n\frac{M\xi\omega}{\rho}|[u<k_n]\cap Q_n|^{1-\frac{1}{p}}\end{aligned}$$

where

$$M = \left(\iint_{Q_\rho} |D \ln u|^p \, dxdt \right)^{\frac{1}{p}}.$$

Let $A_n = [u < k_n] \cap Q_n$. By the standard parabolic embedding theorem (Proposition 3.1, Chapter 1 of [3]), we obtain

$$\begin{aligned}
&\left(\frac{1-a}{2^{n+1}} \xi\omega \right)^2 |A_{n+1}| \\
&\le \iint_{Q_n} (u - k_n)_-^2 \zeta^2 \, dxdt \\
&\le \left(\iint_{Q_n} [(u - k_n)_- \zeta]^{2\frac{N+2}{N}} \, dxdt \right)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\
&\le \left(\iint_{Q_n} |D[(u - k_n)_- \zeta]|^2 \, dxdt \right)^{\frac{N}{N+2}} \\
&\quad \times \left(\operatorname*{ess\,sup}_{-\theta\rho^2 < t < 0} \int_{K_n} (u - k_n)_-^2 \zeta^2(x,t) \, dx \right)^{\frac{2}{N+2}} |A_n|^{\frac{2}{N+2}} \\
&\le 4^n |A_n|^{\frac{2}{N+2}} (\mu_- + \xi\omega)^{\frac{N}{N+2}} \\
&\quad \times \left[\left(\frac{1}{\theta} + \frac{1}{\mu_- + a\xi\omega} \right) \frac{(\xi\omega)^2}{\rho^2} |A_n| + \frac{M\xi\omega}{\rho} |A_n|^{1-\frac{1}{p}} \right].
\end{aligned}$$

Setting

$$Y_n = \frac{|A_n|}{|Q_n|},$$

we have

$$\begin{aligned}
Y_{n+1} \le \frac{\gamma 4^{2n}}{(1-a)^2} \Bigg[&\left(\left(\frac{\mu_- + \xi\omega}{\theta} \right)^{\frac{N}{N+2}} + \gamma_o \left(\frac{\theta}{\mu_- + a\xi\omega} \right)^{\frac{2}{N+2}} \right) Y_n^{1+\frac{2}{N+2}} \\
&+ \frac{I_{p,\rho}}{\xi\omega} \left(\frac{\mu_- + \xi\omega}{\theta} \right)^{\frac{N}{N+2}} Y_n^{1-\frac{1}{p}+\frac{2}{N+2}} \Bigg]
\end{aligned}$$

where

$$\gamma_o = \left(\frac{\mu_- + \xi\omega}{\mu_- + a\xi\omega} \right)^{\frac{N}{N+2}}.$$

Suppose $I_{p,\rho} \leq \xi\omega$; we have

$$Y_{n+1} \leq \frac{\gamma 4^{2n}}{(1-a)^2}\left[\left(\frac{\mu_- + \xi\omega}{\theta}\right)^{\frac{N}{N+2}} + \gamma_o\left(\frac{\theta}{\mu_- + a\xi\omega}\right)^{\frac{2}{N+2}}\right] Y_n^{1+\beta}$$

where

$$\beta = \frac{2}{N+2} - \frac{1}{p} > 0.$$

It follows from [3, Lemma 4.1, Chapter 1] that Y_n tend to 0 provided

$$Y_o \leq \nu_- \overset{\text{def}}{=} A^{-\frac{1}{\beta}} 16^{-\frac{1}{\beta^2}}, \tag{4.1}$$

where

$$A = \frac{\gamma}{(1-a)^2}\left[\left(\frac{\mu_- + \xi\omega}{\theta}\right)^{\frac{N}{N+2}} + \gamma_o\left(\frac{\theta}{\mu_- + a\xi\omega}\right)^{\frac{2}{N+2}}\right].$$

This finishes the proof. □

Some remarks are in order.

Remark 4.2. Without loss of generality, we may assume that $\mu_- < \frac{1}{2}\xi\omega$. In such a case the quantity A above reduces to

$$A = \frac{\gamma}{(1-a)^2}\left[\left(\frac{\xi\omega}{\theta}\right)^{\frac{N}{N+2}} + \left(\frac{\theta}{a\xi\omega}\right)^{\frac{2}{N+2}}\right].$$

Remark 4.3. The either-or conclusion is necessary. Without $\xi\omega > I_{p,\rho}(y,s)$, in general one cannot obtain

$$u \geq \mu_- + a\xi\omega \quad \text{a.e. in } (y,s) + Q_\rho(\theta).$$

See [14, Remark C.1 in Appendix C].

Lemma 4.4. *Let u be a non-negative, locally bounded, local, weak sub-solution to equation* (1.1)*, in E_T. Assume that*

$$\omega \geq \frac{1}{b+1}\mu_+, \tag{4.2}$$

for some positive parameter b to be chosen later. There exists a positive number ν_+, depending upon ω, θ, ξ, a and N, such that if

$$|[u \geq \mu_+ - \xi\omega] \cap [(y,s) + Q_{2\rho}(\theta)]| \leq \nu_+ |Q_{2\rho}(\theta)|$$

then

$$u \leq \mu_+ - a\xi\omega \qquad \text{a.e. in } (y,s) + Q_\rho(\theta).$$

Proof. Assume $(y, s) = (0, 0)$ and for $n = 0, 1, \ldots$ set

$$\rho_n = \rho + \frac{\rho}{2^n}, \qquad K_n = K_{\rho_n}, \qquad Q_n = K_n \times (-\theta\rho_n^2].$$

Let ζ be a non-negative, piecewise smooth cutoff function on Q_n defined as in the previous lemma. Introduce the sequence of truncating levels

$$k_n = \mu_+ - \xi_n\omega \qquad \text{with} \qquad \xi_n = a\xi + \frac{1-a}{2^n}\xi,$$

and write down the energy estimates (3.1) over the cylinder Q_n, for the truncated function $(u - k_n)_+$. Taking also into account (4.2), this gives

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta\rho_n^2<t\le 0} \int_{K_n} (u-k_n)_+^2 \zeta^2(x,t)dx + \iint_{Q_n} \frac{|D[(u-k_n)_+\zeta]|^2}{u} dx d\tau \\
&\le \gamma \frac{2^{2n}}{\rho^2}(\xi\omega)^2 \iint_{Q_n} \Big(\frac{1}{(1-\xi)\omega} + \frac{1}{\theta}\Big)\chi_{[u>k_n]} dx d\tau \\
&\le \gamma \frac{2^{2n}}{\rho^2}(\xi\omega)^2 \frac{1}{(1-\xi)\omega}\Big(1 + \frac{\omega}{\theta}\Big)|[u > k_n] \cap Q_n|.
\end{aligned}$$

To estimate below the second integral on the left-hand side, take into account that $u \le \mu_+$ and (4.2). This gives

$$\iint_{Q_n} \frac{|D[(u-k_n)_+\zeta]|^2}{u} dx d\tau \ge \frac{1}{(b+1)\omega} \iint_{Q_n} |D[(u-k_n)_+\zeta]|^2 dx d\tau.$$

Setting

$$A_n = [u > k_n] \cap Q_n \quad \text{and} \quad Y_n = \frac{|A_n|}{|Q_n|},$$

and combining these estimates gives

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta\rho_n^2<t\le 0} \int_{K_n} (u-k_n)_+^2 \zeta^2(x,t)dx \\
&\quad + \frac{1}{(b+1)\omega} \iint_{Q_n} |D[(u-k_n)_+\zeta]|^2 dx d\tau \qquad (4.3)\\
&\le \gamma \frac{2^{2n}}{\rho^2} \frac{(\xi\omega)^2}{(1-\xi)\omega}\Big(1 + \frac{\omega}{\theta}\Big)|A_n|.
\end{aligned}$$

Apply Hölder's inequality and the embedding [3, Proposition 3.1 in Chapter 1], and recall that $\zeta = 1$ on Q_{n+1}, to get

$$\begin{aligned}\Big(\frac{1-a}{2^{n+1}}\Big)^2(\xi\omega)^2|A_{n+1}| &\le \iint_{Q_{n+1}} (u-k_n)_+^2\,dxd\tau \\ &\le \Big(\iint_{Q_n} [(u-k_n)_+\zeta]^{2\frac{N+2}{N}}\,dxd\tau\Big)^{\frac{N}{N+2}}|A_n|^{\frac{2}{N+2}} \\ &\le \gamma\Big(\iint_{Q_n} |D[(u-k_n)_+\zeta]|^2 dxd\tau\Big)^{\frac{N}{N+2}} \\ &\quad\times\Big(\operatorname*{ess\,sup}_{-\theta\rho_n^2<t\le 0}\int_{K_n}[(u-k_n)_+\zeta]^2(x,t)dx\Big)^{\frac{2}{N+2}}|A_n|^{\frac{2}{N+2}}\end{aligned}$$

for a constant γ depending only upon N. Combine this with (4.3) to get

$$|A_{n+1}| \le \frac{\gamma\, 2^{4n}}{(1-a)^2\rho^2}\frac{(b+1)^{\frac{N}{N+2}}}{(1-\xi)}\frac{1}{\omega^{\frac{2}{N+2}}}\Big(1+\frac{\omega}{\theta}\Big)|A_n|^{1+\frac{2}{N+2}}.$$

In terms of $Y_n = \frac{|A_n|}{|Q_n|}$ this can be rewritten as

$$Y_{n+1} \le \frac{\gamma 2^{4n}(b+1)^{\frac{N}{N+2}}}{(1-a)^2(1-\xi)}\Big(\frac{\theta}{\omega}\Big)^{\frac{2}{N+2}}\Big(1+\frac{\omega}{\theta}\Big)Y_n^{1+\frac{2}{N+2}}.$$

By [3, Lemma 4.1 in Chapter 1], $\{Y_n\}\to 0$ as $n\to\infty$, provided

$$Y_o = \frac{|A_o|}{|Q_o|} \le \left[\frac{(1-a)^2(1-\xi)}{\gamma 4^{N+2}(b+1)^{\frac{N}{N+2}}}\right]^{\frac{N+2}{2}}\frac{\frac{\omega}{\theta}}{\big(1+\frac{\omega}{\theta}\big)^{\frac{N+2}{2}}} \stackrel{\text{def}}{=} \nu_+. \tag{4.4}$$

This finishes the proof. □

5. Proof of Theorem 1.1

Fix $(x_o,t_o)\in E_T$ and let $\rho > 0$ be so small that $(x_o,t_o)+Q_\rho \subset E_T$; we may assume that (x_o,t_o) coincides with the origin. Set

$$\mu_+ = \operatorname*{ess\,sup}_{Q_\rho} u, \quad \mu_- = \operatorname*{ess\,inf}_{Q_\rho} u, \quad \omega = \mu_+ - \mu_-.$$

Without loss of generality we may assume $\omega \le 1$, such that

$$Q_\rho(\omega) \subset Q_\rho \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_\rho(\omega)} u \le \omega.$$

The proof now unfolds along several cases.

5.1. Case I

First of all, let us suppose

$$\mu_- \geq \frac{1}{8}\omega \quad \Leftrightarrow \quad \mu_+ \leq 9\mu_-.$$

Without loss of generality we may assume $\mu_+ \leq 1$ such that

$$Q_\rho(\mu_+) \subset Q_\rho \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_\rho(\mu_+)} u \leq \omega.$$

Introduce the change of time variable and unknown function

$$\tau = \mu_+^{-1} t \quad \text{and} \quad v(\cdot, \tau) = \frac{u(\cdot, t)}{\mu_+}.$$

Then

$$v_\tau - \operatorname{div} \frac{Dv}{v} = 0 \quad \text{weakly in} \quad Q_\rho$$

with

$$\frac{1}{9} \leq v \leq 1.$$

Thus, by the classical parabolic theory [13], there exists $\eta \in (0, 1)$ depending only on N such that

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{2}}} v \leq (1 - \eta) \operatorname*{ess\,osc}_{Q_\rho} v.$$

Returning to the original coordinates we conclude that

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{2}}(\omega)} u \leq \operatorname*{ess\,osc}_{Q_{\frac{\rho}{2}}(\mu_+)} u \leq (1 - \eta) \operatorname*{ess\,osc}_{Q_\rho(\mu_+)} u \leq (1 - \eta)\omega.$$

5.2. Case II

Now suppose

$$\mu_+ > 9\mu_-,$$

which is equivalent to

$$\omega > \frac{8}{9}\mu_+.$$

Suppose in addition that

$$\left| \left[u \leq \mu_- + \frac{1}{2}\omega \right] \cap Q_\rho(\omega) \right| \leq \nu_- |Q_\rho(\omega)|$$

where ν_- is defined in (4.1) with $\xi = \frac{1}{2}$ and $\theta = \omega$. Then by Lemma 4.1 with $a = \frac{1}{2}$, we have either

$$\omega \leq 2I_{p,\rho}$$

or

$$u \geq \mu_- + \frac{1}{4}\omega \quad \text{a.e. in } Q_{\frac{\rho}{2}}(\omega).$$

The latter implies

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{2}}(\omega)} u \leq \frac{3}{4}\omega.$$

5.3. Case III

As in the previous case suppose that

$$\omega > \frac{8}{9}\mu_+,$$

but

$$\left| \left[u \leq \mu_- + \frac{1}{2}\omega \right] \cap Q_\rho(\omega) \right| > \nu_- |Q_\rho(\omega)|.$$

Then there exists some

$$-\omega\rho^2 \leq s \leq -\frac{1}{2}\nu_-\omega\rho^2$$

such that

$$\left| \left[u(\cdot, s) < \mu_- + \frac{1}{2}\omega \right] \cap K_\rho \right| > \frac{1}{2}\nu_- |K_\rho|. \tag{5.1}$$

Indeed, if the above inequality does not hold for any s in the given interval, then

$$\begin{aligned} \left| \left[u < \mu_- + \frac{1}{2}\omega \right] \cap Q_\rho(\omega) \right| &= \int_{-\omega\rho^2}^{-\frac{1}{2}\nu_-\omega\rho^2} \left| \left[u(\cdot, s) < \mu_- + \frac{1}{2}\omega \right] \cap K_\rho \right| ds \\ &\quad + \int_{-\frac{1}{2}\nu_-\omega\rho^2}^{0} \left| \left[u(\cdot, s) < \mu_- + \frac{1}{2}\omega \right] \cap K_\rho \right| ds \\ &\leq \nu_- |Q_\rho(\omega)|. \end{aligned}$$

Since $\mu_+ - \frac{1}{4}\omega > \mu_- + \frac{1}{2}\omega$ always holds, (5.1) implies

$$\left| \left[u(\cdot, s) > \mu_+ - \frac{1}{4}\omega \right] \cap K_\rho \right| \leq \left(1 - \frac{1}{2}\nu_- \right) |K_\rho|.$$

Based on this, we use the logarithmic estimate to show that such a measure theoretical information propagates in time.

Lemma 5.1. *There exists a positive integer n_* depending only on N such that*

$$\left| \left[u(\cdot, t) > \mu_+ - \frac{\omega}{2^{n_*}} \right] \cap K_\rho \right| \leq \left(1 - \frac{1}{4}\nu_-^2 \right) |K_\rho| \quad \text{for all } s < t < 0.$$

Proof. In the logarithmic estimate we take

$$k = \mu_+ - \frac{1}{4}\omega, \quad c = \frac{\omega}{2^{n+2}}.$$

This gives

$$\psi(u) = \ln^+ \left[\frac{H}{H - [u - (\mu_+ - \frac{1}{4}\omega)]_+ + \frac{\omega}{2^{n+2}}} \right]$$

where

$$H = \operatorname*{ess\,sup}_{K_\rho \times (s,0)} (u-k)_+.$$

Choose a cutoff function ζ which satisfies $\zeta = 1$ on $K_{(1-\sigma)\rho}$ and $\zeta = 0$ on ∂K_ρ, such that

$$|D\zeta| \le \frac{1}{\sigma\rho}.$$

Hence, for all $s < t < 0$,

$$\int_{K_{(1-\sigma)\rho}} \psi^2(u)(x,t)dx \le \int_{K_\rho} \psi^2(u)(x,s)dx + \frac{\gamma}{(\sigma\rho)^2}\int_s^0 \int_{K_\rho} \frac{\psi(u)}{u} dxdt.$$

Note that

$$\psi \le n \ln 2.$$

The first term on the right-hand side is estimated by

$$\int_{K_\rho(y)} \psi^2(u)(x,s)dx \le n^2 \ln^2 2 \left(1 - \frac{1}{2}\nu_-\right)|K_\rho|.$$

The second term is estimated by

$$\frac{\gamma}{(\sigma\rho)^2}\int_s^0 \int_{K_\rho} \frac{\psi(u)}{u} dxdt \le \frac{\gamma n}{(\sigma\rho)^2}(\omega\rho^2)\omega^{-1}|K_\rho| \le \frac{\gamma n}{\sigma^2}|K_\rho|.$$

The left-hand side is estimated below by integrating over the smaller set

$$\left[u(\cdot,t) > \mu_+ - \frac{\omega}{2^{n+2}}\right] \cap K_{(1-\sigma)\rho}.$$

On such a set

$$\psi^2 \ge \ln^2\left(\frac{\frac{\omega}{4}}{\frac{\omega}{2^{n+1}}}\right) = (n-1)^2 \ln^2 2.$$

Thus, combining all above estimates yields

$$\left|\left[u(\cdot,t) > \mu_+ - \frac{\omega}{2^{n+2}}\right] \cap K_{(1-\sigma)\rho}\right| \le \left(\frac{n}{n-1}\right)^2 \left(1 - \frac{1}{2}\nu_-\right)|K_\rho| + \frac{\gamma}{n\sigma^2}|K_\rho|$$

for all $s < t < 0$. On the other hand,

$$\begin{aligned}
&\left|\left[u(\cdot,t) > \mu_+ - \frac{\omega}{2^{n+2}}\right] \cap K_\rho\right| \\
&\le \left|\left[u(\cdot,t) > \mu_+ - \frac{\omega}{2^{n+2}}\right] \cap K_{(1-\sigma)\rho}\right| + |K_\rho - K_{(1-\sigma)\rho}| \\
&\le \left|\left[u(\cdot,t) > \mu_+ - \frac{\omega}{2^{n+2}}\right] \cap K_{(1-\sigma)\rho}\right| + N\sigma|K_\rho|.
\end{aligned}$$

Therefore,

$$\left|\left[u(\cdot,t) > \mu_+ - \frac{\omega}{2^{n+2}}\right] \cap K_\rho\right| \le \left[\left(\frac{n}{n-1}\right)^2\left(1 - \frac{1}{2}\nu_-\right) + \frac{\gamma}{n\sigma^2} + N\sigma\right]|K_\rho|.$$

The claim is proved by choosing σ so small, and then n large enough. □

Using the measure theoretical information obtained for every time level of the cylinder

$$Q_\rho\left(\frac{1}{2}\nu_-\omega\right) = K_\rho \times \left(-\frac{1}{2}\nu_-\omega\rho^2, 0\right]$$

in Lemma 5.1, we are able to show

Lemma 5.2. *For any $\nu_* \in (0,1)$ there exists a positive integer l such that*

$$\left|\left[u > \mu_+ - \frac{\omega}{2^{n_*+l}}\right] \cap Q_\rho\left(\frac{1}{2}\nu_-\omega\right)\right| \le \nu_*\left|Q_\rho\left(\frac{1}{2}\nu_-\omega\right)\right|.$$

Proof. Let $Q = Q_\rho(\frac{1}{2}\nu_-\omega)$ and $Q' = Q_{2\rho}(\frac{1}{2}\nu_-\omega)$. Write down the energy estimate over Q' for

$$(u-k_j)_+ \quad \text{where } k_j = \mu_+ - \frac{\omega}{2^j} \quad \text{for } j = n_*, \cdots, n_*+l.$$

Choose a cutoff function ζ which satisfies $\zeta = 1$ on Q, and vanishes on the parabolic boundary of Q', such that

$$|D\zeta| \le \frac{1}{\rho}, \quad |\zeta_t| \le \frac{1}{\nu_-\omega\rho^2}.$$

Then keeping in mind we assumed at the beginning that $\omega > 8\mu_+/9$, the energy estimate gives

$$\omega^{-1}\iint_Q |D(u-k_j)_+|^2\,dxdt \le \frac{\gamma}{\nu_-\omega\rho^2}\left(\frac{\omega}{2^j}\right)^2|Q|. \tag{5.2}$$

Next, we apply the discrete isoperimetric inequality on page 15 of [3] to $u(\cdot,t)$ for $-\frac{1}{2}\nu_-\omega\rho^2 < t < 0$, over the cube K_ρ, for levels $k_j < k_{j+1}$. Taking into account the measure theoretical information from Lemma 5.1, this gives

$$\begin{aligned}
&\frac{\omega}{2^{j+1}}|[u(\cdot,t)>k_{j+1}]\cap K_\rho| \\
&\le \frac{\gamma\rho^{N+1}}{|[u(\cdot,t)<k_j]\cap K_\rho|}\int_{[k_j<u(\cdot,t)<k_{j+1}]\cap K_\rho}|Du|\,dx \\
&\le \frac{\gamma}{\nu_-^2}\rho\left(\int_{[k_j<u(\cdot,t)<k_{j+1}]\cap K_\rho}|Du|^2\,dx\right)^{\frac{1}{2}} \\
&\quad\times|([u(\cdot,t)>k_j]-[u(\cdot,t)>k_{j+1}])\cap K_\rho|^{\frac{1}{2}}.
\end{aligned}$$

Set

$$A_j = [u>k_j]\cap Q$$

and integrate the above estimate in dt over $-\frac{1}{2}\nu_-\omega\rho^2 < t < 0$; we obtain

$$\frac{\omega}{2^j}|A_{j+1}| \le \frac{\gamma}{\nu_-^2}\rho\left(\iint_Q |D(u-k_j)_+|^2\,dxdt\right)^{\frac{1}{2}}(|A_j|-|A_{j+1}|)^{\frac{1}{2}}.$$

Now square both sides of the above inequality and use (5.2) to estimate the term containing $D(u-k_j)_+$, to obtain

$$|A_{j+1}|^2 \le \frac{\gamma}{\nu_-^5}|Q|(|A_j|-|A_{j+1}|).$$

Add these inequalities from n_* to n_*+l-1 to obtain

$$l|A_{n_*+l}|^2 \le \sum_{j=n_*}^{n_*+l-1}|A_{j+1}|^2 \le \frac{\gamma}{\nu_-^5}|Q|^2.$$

From this

$$|A_{n_*+l}| \le \frac{1}{\sqrt{l}}\sqrt{\frac{\gamma}{\nu_-^5}}|Q|.$$

Given a number $\nu_* \in (0,1)$, we can choose l large enough to guarantee

$$\frac{1}{\sqrt{l}}\sqrt{\frac{\gamma}{\nu_-^5}} < \nu_*.$$

This finishes the proof. $\square$

Now we are ready to finish **Case III**. Choosing $a = \frac{1}{2}$, $b = \frac{1}{8}$ and $\theta = \frac{1}{2}\nu_-\omega$, the constant ν_+ from (4.4) becomes

$$\nu_+ = \frac{1}{\gamma}\nu_-^{\frac{N}{2}}(1-\xi)^{\frac{N+2}{2}}$$

with

$$\xi = \frac{1}{2^{n_*+l}}, \qquad n_* \text{ and } l \text{ to be determined.}$$

Then after choosing n_* from Lemma 5.1, we can choose l so large that

$$\frac{1}{\sqrt{l}}\sqrt{\frac{\gamma}{\nu_-^5}} < \frac{1}{\gamma}\nu_-^{\frac{N}{2}}\left(1-\frac{1}{2^{n_*+l}}\right)^{\frac{N+2}{2}}.$$

By Lemma 4.4 and Lemma 5.2, we obtain that

$$u \le \mu_+ - \frac{\omega}{2^{n_*+l+1}} \quad \text{in} \quad Q_{\frac{\rho}{2}(\frac{1}{2}\nu_-\omega)}.$$

This in turn implies

$$\operatorname*{ess\,osc}_{Q_{\frac{\rho}{2}(\frac{1}{2}\nu_-\omega)}} u \le (1-\delta)\omega \quad \text{where} \quad \delta = \frac{1}{2^{n_*+l+1}}.$$

Proof of Theorem 1.1. Combining all these cases above, we have proved that once we have

$$\operatorname*{ess\,osc}_{Q_\rho(\omega)} u \le \omega,$$

we can find positive constants

$$c = \min\left\{\frac{1}{2}, \sqrt{\frac{\nu_-}{8}}\right\} \quad \text{and} \quad \lambda = \min\left\{\frac{3}{4},\ 1-\eta,\ 1-\delta\right\}$$

such that

$$\operatorname*{ess\,osc}_{Q_{c\rho}(\lambda\omega)} u \le \max\{\lambda\omega,\ 2I_{p,\rho}\}.$$

Relabel the quantities ρ and ω chosen above as ρ_o and ω_o. Now let

$$\omega_1 = \max\{\lambda\omega_o,\ 2I_{p,\rho_o}\} \quad \text{and} \quad \rho_1 = c\rho_o$$

such that

$$Q_{\rho_1}(\omega_1) \subset Q_{c\rho_o}(\lambda\omega_o) \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_{\rho_1}(\omega_1)} u \le \omega_1.$$

The above set inclusion is verified if

$$\omega_1\rho_1^2 = \max\{\lambda\omega_o,\ 2I_{q,\rho_o}\}(c\rho_o)^2 \le \lambda\omega_o(c\rho_o)^2.$$

This holds naturally unless

$$\lambda\omega_o \le 2I_{p,\rho_o},$$

but then there is nothing to prove since

$$\operatorname*{ess\,osc}_{Q_{\rho_o}(\omega_o)} u \le \frac{2}{\lambda} I_{p,\rho_o}.$$

Hence, with such a choice of c we now have

$$\operatorname*{ess\,osc}_{Q_{\rho_1}(\omega_1)} u \le \omega_1.$$

According to what we have shown, one has

$$\operatorname*{ess\,osc}_{Q_{c\rho_1}(\lambda\omega_1)} u \le \max\{\lambda\omega_1,\ 2I_{p,\rho_1}\}.$$

Now define

$$\omega_2 = \max\{\lambda\omega_1,\ 2I_{p,\rho_1}\} \quad \text{and} \quad \rho_2 = c\rho_1,$$

and we want to show that

$$Q_{\rho_2}(\omega_2) \subset Q_{c\rho_1}(\lambda\omega_1) \quad \text{and hence} \quad \operatorname*{ess\,osc}_{Q_{\rho_2}(\omega_2)} u \le \omega_2.$$

The above set inclusion is verified if

$$\omega_2\rho_2^2 = \max\{\lambda\omega_1,\ 2I_{p,\rho_1}\}(c\rho_1)^2 \le \lambda\omega_1(c\rho_1)^2.$$

This holds naturally unless

$$\lambda\omega_1 \le 2I_{p,\rho_1},$$

but then there is nothing to prove since

$$\operatorname*{ess\,osc}_{Q_{\rho_1}(\omega_1)} u \le \frac{2}{\lambda} I_{p,\rho_1}.$$

Iterating in this fashion, one concludes that there are positive numbers $c,\ \lambda \in (0, 1)$ such that constructing

$$\rho_n = c^n\rho_o \quad \text{and} \quad \omega_n = \max\{\lambda\omega_{n-1},\ 2I_{p,\rho_{n-1}}\},$$

one obtains

$$\operatorname*{ess\,osc}_{Q_{\rho_n}(\omega_n)} u \le \max\left\{\omega_n,\ \frac{2}{\lambda} I_{p,\rho_{n-1}}\right\}.$$

Let $0 < r < \rho_o \le R_o$ be fixed. Since the sequence $\{\omega_n\rho_n^2\}$ is strictly decreasing and gives a partition of the interval $(0,\ \omega_o\rho_o^2)$, there must be some positive integer n such that

$$\rho_{n+1}^2\omega_{n+1} \le r^2\omega_o < \rho_n^2\omega_n.$$

Noting that

$$\omega_n \le \omega_o \quad \text{and} \quad \omega_{n+1} \ge \lambda^{n+1}\omega_o,$$

this implies that $r < \rho_n$ and

$$n+1 \ge \frac{\ln\left(\frac{r}{\rho_o}\right)}{\ln(\sqrt{\lambda}c)}.$$

Then it is not hard to see that

$$Q_r(\omega_o) \subset Q_{\rho_n}(\omega_n),$$

and either

$$\operatorname*{ess\,osc}_{Q_r(\omega_o)} u \le \omega_n \quad \text{or} \quad \omega_{n-1} \le \frac{2}{\lambda} I_{p,\rho_{n-1}}.$$

Note that

$$\begin{aligned}\omega_n &= \max\left\{\lambda^n\omega_o, 2\lambda^{n-1}I_{p,\rho_o}, \cdots, 2\lambda I_{p,\rho_{n-2}}, 2I_{p,\rho_{n-1}}\right\}\\ &\le \lambda^n\omega_o + \frac{2}{1-\lambda}I_{p,\rho_o}.\end{aligned}$$

Here we made the convention that the function $\rho \to I_{p,\rho}$ is non-decreasing. Otherwise, one could use

$$\tilde{I}_{p,\rho} = \sup_{0<\tau<\rho} I_{p,\tau}.$$

Thus, there is some $\alpha \in (0, 1)$ depending only on N such that

$$\operatorname*{ess\,osc}_{Q_r(\omega_o)} u \le \lambda^n\omega_o + CI_{p,\rho_o} \le \bar{C}\left[\omega_o\left(\frac{r}{\rho_o}\right)^\alpha + I_{p,\rho_o}\right]$$

where

$$\alpha = \frac{\ln\lambda}{\ln(\sqrt{\lambda}c)} \quad \text{and} \quad \bar{C} = \max\left\{\frac{2}{1-\lambda}, \frac{2}{\lambda}\right\}.$$

One verifies easily that ρ_o can be replaced by any $\tilde{\rho} \in (r, \rho_o)$ in the above oscillation estimate. Then choose $\tilde{\rho} = R_o^{1-\mu}r^\mu$, and conclude we have

$$\operatorname*{ess\,osc}_{Q_r(\omega_o)} u \le \bar{C}\left[\omega_o\left(\frac{r}{R_o}\right)^{(1-\mu)\alpha} + I_{p,R_o^{1-\mu}r^\mu}\right]. \qquad \square$$

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