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A geometric second-order-rectifiable stratification for closed subsets of Euclidean space

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Abstract. Defining the m-th stratum of a closed subset of an n dimensional Euclidean space to consist of those points, where it can be touched by a ball from at least $n - m$ linearly independent directions, we establish that the m-th stratum is second-order rectifiable of dimension m and a Borel set. This was known for convex sets, but is new even for sets of positive reach. The result is based on a sufficient condition of parametric type for second-order rectifiability.

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1. Introduction

The main purpose of the present paper is to establish the following theorem; our notation is based on [5, pages 669–676], see the end of this introduction.

Structural theorem on the singularities of closed sets (see Theorem 4.12)

Suppose A is a closed subset of $\mathbf{R}^n$, for $a \in A$, $\operatorname{Dis}(A, a)$ is the set of $v \in \mathbf{R}^n$ satisfying $A \cap \{x : |x - (a + v)| < |v|\} = \varnothing$, m is an integer, $0 \le m \le n$, and

$$B = A \cap \{a : \operatorname{Dis}(A, a) \text{ contains at least } n - m \text{ linearly independent vectors}\}.$$

Then, B can be $\mathscr{H}^m$ almost covered by the union of a countable collection of m dimensional, twice continuously differentiable submanifolds of $\mathbf{R}^n$.

In the terminology of [11, page 1018] for $m \ge 1$, the conclusion asserts that B is countably $(\mathscr{H}^m, m)$ rectifiable of class 2. If A is convex, then B consists of the set of points, where the dimension of the normal cone of A is at least $n - m$, see Remark 4.14. Hence, our theorem contains the structural theorem on the singularities of convex sets, see [1, Theorem 3] or [18, Theorem 3]. We also prove,

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that B is a countably m rectifiable Borel set, see Theorem 4.12; in particular, if $m \geq 1$, then B can be covered (without exceptional set) by a countable family of images of Lipschitzian functions from $\mathbf{R}^m$ into $\mathbf{R}^n$, and, if $m = 0$, then B is countable.

Our approach rests on two pillars. The first may be stated as follows.

Sufficient condition of parametric type for second-order rectifiability (see Corollary 2.5)

Suppose W is an $\mathscr{L}^n$ measurable subset of $\mathbf{R}^n$, m is an integer, $1 \leq m \leq n$, $f : W \to \mathbf{R}^\nu$ is a locally Lipschitzian map,

$$Z = \mathbf{R}^\nu \cap \{z : \mathscr{H}^{n-m}(f^{-1}[\{z\}]) > 0\},$$

and, for $\mathscr{H}^m$ almost all $z \in Z$, there exists an m dimensional subspace U of $\mathbf{R}^\nu$ satisfying

$$\limsup_{y \to x} |y - x|^{-2} \operatorname{dist}(f(y) - f(x), U) < \infty \quad \textit{whenever } x \in f^{-1}[\{z\}].$$

Then, Z can be $\mathscr{H}^m$ almost covered by the union of a countable collection of m dimensional, twice continuously differentiable submanifolds of $\mathbf{R}^\nu$.

Note that $f^{-1}[\{z\}]$ abbreviates $\{x : f(x) = z\}$, see below. The key to reduce this sufficient condition to the nonparametric case is the construction (in Theorem 2.1) of a countable collection G of m rectifiable subsets P of W with $\mathscr{H}^m(Z \sim f[\bigcup G]) = 0$ such that, for each $P \in G$, the restriction $f|P$ is univalent and its inverse $(f|P)^{-1}$ is Lipschitzian. The nonparametric case was comprehensively studied in [11]; however, for the present purpose, also [14] would be sufficient (see Remark 2.6).

The second pillar of the proof of the structural theorem is the next result that we state here for the special case of a convex set A. It concerns the relation of the nearest point projection, $\boldsymbol{\xi}_A$, with the tangent and normal cones of A.

A geometric observation for convex sets (see Lemma 4.11 together with Lemma 3.9 (1) (3), and Remark 4.4)

If A is a nonempty closed convex subset of $\mathbf{R}^n$, m is an integer, $1 \leq m < n$, $x \in \mathbf{R}^n \sim A$, $a = \boldsymbol{\xi}_A(x)$, $\dim \operatorname{Nor}(A, a) \geq n - m$, U is an m dimensional subspace of $\mathbf{R}^n$, $U \subset \operatorname{Tan}(A, a)$, and $x - a$ belongs the relative interior of $\operatorname{Nor}(A, a)$, then

$$\limsup_{y \to x} |y - x|^{-2} \operatorname{dist}(\boldsymbol{\xi}_A(y) - a, U) < \infty.$$

This geometric observation and its generalisation to closed sets in Lemma 4.11 owe much to Federer's treatment of sets of positive reach (a concept that embraces convex sets and submanifolds of class 2) in [4]. Since it is elementary, that the set B in the structural theorem is countably m rectifiable, the sufficient condition of parametric type for second-order rectifiability then is readily applied with the function $f = \boldsymbol{\xi}_A|W$ for suitable W.

Connection to curvature measures Instead of using second-order rectifiability, curvature properties can also be studied via general Steiner formulae. This approach was taken, for sets of positive reach and various more general classes of sets, by Federer in [4], Stachó in [16], Zähle in [17], Rataj and Zähle in [10], and Hug, Last, and Weil in [6]; in fact, [16] and [6] treat arbitrary closed subsets of Euclidean space. The relation of both notions of curvature is characterised by the second author in the sequel [13].

Connection to varifold theory The original motivation of the first author for the present study was to create a deeper understanding of a relation proven by Almgren in his area-mean-curvature characterisation of the sphere in [2]. There, an equation relating the curvature measures (similar to those of [17]) of the convex hull of the support of a certain varifold to the perpendicular part of the mean curvature of the varifold is established in [2, Section 6 (2)]. The results of the present paper shall serve as tools for further investigations of both authors of the second-order rectifiability properties of classes of varifolds.

Notation Our notation and terminology is that of [5, pages 669–676], except that, as in [7, page 8], we denote the image of A under a relation r by

$$r[A] = \{y : (x, y) \in r \text{ for some } x \in A\}.$$

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2. Coarea formula

The purpose of the present section is to prove the sufficient condition for second-order rectifiability in Corollary 2.5. We begin by establishing a theorem that allows to construct univalent parametrisations from a Lipschitzian given one.

Theorem 2.1. *Suppose W is an $\mathscr{L}^n$ measurable subset of $\mathbf{R}^n$, m is an integer, $1 \le m \le n$, and $f : W \to \mathbf{R}^\nu$ is a locally Lipschitzian map. Then, there exists a countable collection G of compact subsets P of W, such that $f|P$ is univalent and $(f|P)^{-1}$ is Lipschitzian, satisfying*

$$\mathscr{H}^m\big(\mathbf{R}^\nu \cap \{z : \mathscr{H}^{n-m}(f^{-1}[\{z\}]) > 0\} \sim \bigcup\{f[P] : P \in G\}\big) = 0.$$

Moreover, each member of G is contained in some m dimensional affine plane.

Proof. We firstly consider *the special case that W is a compact subset of* $\mathbf{R}^n$. Choose $F : \mathbf{R}^n \to \mathbf{R}^\nu$ with $F|W = f$ and $\operatorname{Lip} F = \operatorname{Lip} f < \infty$ by Kirszbraun's theorem [5, 2.10.43]; in particular, $\mathrm{D}\,F$ is a Borel function whose domain is a Borel set by [5, 3.1.2]. Defining $Z_i = \mathbf{R}^\nu \cap \{z : \mathscr{H}^{n-m}(f^{-1}[\{z\}]) \geq 1/i\}$ whenever i is a positive integer, we note that

$$\mathbf{R}^\nu \cap \{z : \mathscr{H}^{n-m}(f^{-1}[\{z\}]) > 0\} = \textstyle\bigcup_{i=1}^{\infty} Z_i.$$

Moreover, the sets Z_i are Borel sets by [5, 2.10.26] and $(\mathscr{H}^m, m)$ rectifiable by [5, 3.2.31]. We define, for every positive integer i, the class Ω_i to consist of all families G of compact subsets P of $f^{-1}[Z_i]$ such that

$$f[P] \cap f[Q] \neq \varnothing \text{ if and only if } P = Q$$

whenever $P, Q \in G$, and such that

$$\mathscr{H}^m(P) > 0, \quad f|P \text{ is univalent}, \quad (f|P)^{-1} \text{ is Lipschitzian},$$
$$P \text{ is contained in some } m \text{ dimensional affine subspace of } \mathbf{R}^n$$

whenever $P \in G$. Clearly, each member of Ω_i is countable. Using Hausdorff's maximal principle (see [7, page 33]), we choose maximal elements G_i of Ω_i. The proof of the present case will be concluded by establishing

$$\mathscr{H}^m\big(Z_i \sim \textstyle\bigcup\{f[P] : P \in G_i\}\big) = 0 \quad \text{for every positive integer } i.$$

For this purpose, fix such i and define Borel sets $T = Z_i \sim \bigcup\{f[P] : P \in G_i\}$ and $S = f^{-1}[T]$. If T had positive $\mathscr{H}^m$ measure, then, noting [5, 2.10.35],

$$B = S \cap \big\{w : \| \textstyle\bigwedge_m \mathrm{D}\,F(w)\| > 0\big\}$$

would be a Borel set and have positive $\mathscr{L}^n$ measure by the coarea formula [5, 3.2.22 (3)] with W, Z, and f replaced by S, T, and $f|S$, since

$$(\mathscr{L}^n \llcorner S, n) \operatorname{ap} \mathrm{D}(f|S)(w) = \mathrm{D}\,F(w) \quad \text{for } \mathscr{L}^n \text{ almost all } w \in S$$

by [5, 2.10.19 (4)].

Consequently, identifying $\mathbf{R}^n \simeq \mathbf{R}^m \times \mathbf{R}^{n-m}$, there would exist a linear isometry $g : \mathbf{R}^n \to \mathbf{R}^n$ such that $\mathscr{L}^n(A) > 0$ with

$$\begin{aligned} A &= B \cap \big\{w : \textstyle\bigwedge_m(\mathrm{D}\,F(w)|g[\mathbf{R}^m \times \{0\}]) \neq 0\big\} \\ &= g\big[g^{-1}[B] \cap \{x : \textstyle\bigwedge_m(\mathrm{D}(F \circ g)(x)|\mathbf{R}^m \times \{0\}) \neq 0\}\big] \end{aligned}$$

and, as A would be a Borel set, $\eta \in \mathbf{R}^{n-m}$ so that $\mathscr{L}^m(R) > 0$ with

$$R = \mathbf{R}^m \cap \{\xi : (\xi, \eta) \in g^{-1}[A]\}$$

by Fubini's theorem, see [5, 2.6.2 (3)]. Since R would be a Borel set, we could apply [5, 3.2.2] to the function $h : \mathbf{R}^m \to \mathbf{R}^\nu$ defined by $h(\xi) = (F \circ g)(\xi, \eta)$ for $\xi \in \mathbf{R}^m$, and use the Borel regularity of $\mathscr{H}^m$ to construct a subset P of $g[R \times \{\eta\}]$ with $G_i \cup \{P\} \in \Omega_i$, contrary to the maximality of G_i.

To treat the *general case*, we pick an increasing sequence of compact subsets K_i of $\mathbf{R}^n$ with $\mathscr{L}^n\big(W \sim \bigcup_{i=1}^\infty K_i\big) = 0$. Since, in conjunction with [5, 2.4.5], [5, 2.10.25] applied with A replaced by $W \sim \bigcup_{i=1}^\infty K_i$ implies that

$$\mathscr{H}^{n-m}\big(f^{-1}[\{z\}] \sim \textstyle\bigcup_{i=1}^\infty K_i\big) = 0 \quad \text{for } \mathscr{H}^m \text{ almost all } z \in \mathbf{R}^\nu$$

and $\lim_{i\to\infty} \mathscr{H}^{n-m}(f^{-1}[\{z\}] \cap K_i) = \mathscr{H}^{n-m}(f^{-1}[\{z\}])$ for such z, we readily infer the conclusion. □

Remark 2.2. The contradiction argument is inspired by [5, 3.2.21].

Remark 2.3. For the nearest point projection onto a set of positive reach, the idea of exhaustion by means of images from lower dimensional parts of the domain of f is employed in [4, 4.15 (3)]. The important additional feature of members P in our collection G is the Lipschitz continuity of $(f|P)^{-1}$.

Remark 2.4. One readily verifies that Theorem 2.1 also holds with $m = 0$, but this will not be needed in the present paper.

The sufficient condition of parametric type for second-order rectifiability now reads as follows.

Corollary 2.5. *Under the hypotheses of Theorem* 2.1, *if*

$$Z = \mathbf{R}^\nu \cap \{z : \mathscr{H}^{n-m}(f^{-1}[\{z\}]) > 0\},$$

and, for $\mathscr{H}^m$ *almost all* $z \in Z$, *there exists an* m *dimensional subspace* U *of* $\mathbf{R}^\nu$ *satisfying*

$$\limsup_{y\to x} |y-x|^{-2} \operatorname{dist}(f(y) - f(x), U) < \infty \quad \textit{whenever } x \in f^{-1}[\{z\}],$$

then Z *can be* $\mathscr{H}^m$ *almost covered by the union of a countable collection of* m *dimensional submanifolds of* $\mathbf{R}^\nu$ *of class* 2.

Proof. Whenever $P \in G$, as $(f|P)^{-1}$ is Lipschitzian, we note, for $\mathscr{H}^m$ almost all $z \in Z \cap f[P]$, there exists an m dimensional subspace U of $\mathbf{R}^\nu$ such that

$$\limsup_{f[P]\ni\zeta\to z} |\zeta - z|^{-2} \operatorname{dist}(\zeta - z, U) < \infty.$$

Therefore, the conclusion follows from [11, 5.4] and [5, 3.1.15]. □

Remark 2.6. With little additional effort, the final argument could have been based on [14, A.1] instead of [11, 5.4] and [5, 3.1.15].

Remark 2.7. In conjunction with the preceding corollary, the following observation will be useful. *If B is a countably $(\mathscr{H}^m, m)$ rectifiable subset of $\mathbf{R}^\nu$, then, for $\mathscr{H}^m$ almost all $b \in B$, there exists an m dimensional subspace U of $\mathbf{R}^\nu$ such that $U \subset \operatorname{Tan}(B, b)$;* in fact, [5, 2.1.4, 3.1.21] reduce the problem to Borel sets B, in which case [5, 2.10.19 (4), 3.2.17, 3.2.18] apply.

3. Convex sets

In the present section, we mainly collect some basic properties of convex sets and related definitions in Definition 3.1–Remark 3.11 for convenient reference. Additionally, we note an observation concerning convex cones in Definition 3.12–Corollary 3.14.

Definition 3.1. Suppose $A \subset \mathbf{R}^n$ and $x \in \mathbf{R}^n$. Then, the *distance of x to A* is denoted by $\operatorname{dist}(x, A) = \inf\{|x - a| : a \in A\}$.

Remark 3.2. If $A \neq \varnothing$, then $\operatorname{dist}(\cdot, A)$ is real valued and $\operatorname{Lip} \operatorname{dist}(\cdot, A) \leq 1$.

Remark 3.3. If $R = (\mathbf{R}^n \times A) \cap \{(x, a) : |x - a| = \operatorname{dist}(x, A)\}$, then, using Remark 3.2, one verifies that $\{a : (x, a) \in R \text{ for some } x \in B\}$ is bounded whenever B is a bounded subset of $\mathbf{R}^n$. Moreover, if A is closed, so is R.

Definition 3.4 (see [4, 4.1]). Suppose $A \subset \mathbf{R}^n$ and U is the set of all $x \in \mathbf{R}^n$ such that there exists a unique $a \in A$ with $|x - a| = \operatorname{dist}(x, A)$. Then, the *nearest point projection onto A* is the map $\boldsymbol{\xi}_A : U \to A$ characterised by the requirement $|x - \boldsymbol{\xi}_A(x)| = \operatorname{dist}(x, A)$ for $x \in U$.

Remark 3.5. Using Remark 3.3, we obtain that the function $\boldsymbol{\xi}_A$ is continuous. Moreover, if A is closed, then $\operatorname{dmn} \boldsymbol{\xi}_A$ is a Borel set; in fact, one verifies, by means of Remark 3.3, that the function mapping $x \in \mathbf{R}^n$ onto

$$d(x) = \operatorname{diam}\{a : (x, a) \in R\} \in \overline{\mathbf{R}}$$

is upper semicontinuous, and $\operatorname{dmn} \boldsymbol{\xi}_A = \{x : d(x) = 0\}$.

Definition 3.6 (see [15, page xix]). If $A \subset \mathbf{R}^n$, then $\operatorname{aff} A$ denotes the *affine hull* of A.

Definition 3.7 (see [15, page 7, page xx]). Suppose C is a convex subset of $\mathbf{R}^n$. Then, the *dimension* of C, denoted by $\dim C$, is defined to be the dimension of $\operatorname{aff} C$, and the *relative boundary* [*interior*] of C is defined to be the boundary [interior] of C relative to $\operatorname{aff} C$.

Remark 3.8. If V is the relative interior of C, then V is convex, $\dim V = \dim C$, and

$$c + t(v - c) \in V \quad \text{whenever } v \in V, c \in C, \text{ and } 0 < t \leq 1;$$

in fact, reducing to the case $\operatorname{aff} C = \mathbf{R}^n$, this is [15, 1.1.9, 1.1.10, 1.1.13].

Lemma 3.9. *Suppose C is a nonempty closed convex subset of $\mathbf{R}^n$.*

Then, the following four statements hold.

(1) *There holds* $\operatorname{dmn} \boldsymbol{\xi}_C = \mathbf{R}^n$ *and* $\operatorname{Lip} \boldsymbol{\xi}_C \le 1$*;*
(2) *If* $c \in C$*, then* $\operatorname{Tan}(C, c) = \mathbf{R}^n \cap \{u : u \bullet v \le 0 \text{ for } v \in \operatorname{Nor}(C, c)\}$ *and*

$$C \subset \{c + u : u \in \operatorname{Tan}(C, c)\} \subset \operatorname{aff} C;$$

in particular, $\dim C = \dim \operatorname{Tan}(C, c)$*;*
(3) *If* $c \in C$*, then*

$$\operatorname{Nor}(C, c) = \{v : \boldsymbol{\xi}_C(c + v) = c\} = \mathbf{R}^n \cap \{v : v \bullet (x - c) \le 0 \text{ for } x \in C\};$$

(4) *If* B *is the relative boundary of* C*, then*

$$B = C \cap \{c : c + v \in \operatorname{aff} C \text{ for some } v \in \mathbf{S}^{n-1} \cap \operatorname{Nor}(C, c)\}.$$

Proof. (1) is asserted in [5, 4.1.16]. In view of (1), the first equation and the first inclusion in (2) are contained in [4, 4.8 (12)] and [4, 4.18], respectively; the remaining items of (2) then follow. The first equation in (3) follows from (1) and [4, 4.8 (12)]. The second equation in (3) follows from [9, I.2.3]. Finally, (4) is implied by [15, 1.3.2]. □

Theorem 3.10. *Suppose* $X = \mathbf{R}^n \cap \mathbf{B}(0, 1)$*,* F *is the family of nonempty closed subsets of* X *endowed with the Hausdorff metric, and* $G = F \cap \{C : C \text{ is convex}\}$*.*
Then, the following four statements hold.

(1) *The families* F *and* G *are compact;*
(2) *The function mapping* $(x, B) \in X \times F$ *onto* $\operatorname{dist}(x, B) \in \mathbf{R}$ *is continuous;*
(3) *The function mapping* $C \in G$ *onto* $\dim C \in \mathbf{Z}$ *is lower semicontinuous;*
(4) *If* $\Phi = (G \times F) \cap \{(C, B) : B \text{ is the relative boundary of } C\}$*, then* Φ *is a Borel function whose domain equals the Borel set* $G \cap \{C : \dim C \ge 1\}$*.*

Proof. (1) is contained in [5, 2.10.21]. (2) follows from Remark 3.2. We observe that, in order to prove (3) and (4), it sufficient to establish the following assertion. *If* k *is an integer,* C_i *is a sequence in* G *with* $\dim C_i = k$*,* $C \in G$*, and* $C_i \to C$ *as* $i \to \infty$*, then* $\dim C \le k$ *and, in case of equality with* $k \ge 1$*, we additionally have* $\Phi(C) = \lim_{i \to \infty} \Phi(C_i)$. For this purpose, we assume, possibly passing to a subsequence, that for some affine subspace Q of $\mathbf{R}^n$

$$\operatorname{dist}(v, \operatorname{aff} C_i) \to \operatorname{dist}(v, Q) \quad \text{as } i \to \infty \text{ for } v \in \mathbf{R}^n,$$

and, if $k \ge 1$, that for some $B \in F$, we have $\Phi(C_i) \to B$ as $i \to \infty$. It follows $C \subset Q$, whence we infer $\dim C \le \dim Q \le k$. Therefore, if $\dim C = k \ge 1$, then $Q = \operatorname{aff} C$ and we could assume, possibly replacing C_i by $g_i^{-1}[C_i]$ for a sequence of isometries g_i of $\mathbf{R}^n$ with $\lim_{i \to \infty} g_i(x) = x$ for $x \in \mathbf{R}^n$ and using Remark 3.2, that $C_i \subset Q$ for each index i; in which case $\Phi(C) = B$ follows readily from Lemma 3.9 (4). □

Remark 3.11. We observe that (2)–(4) imply that, *if* A *is a Borel subset of* $\mathbf{R}^n$ *and* $\Gamma : A \to G$ *is a Borel function, then the set of* $(a, v) \in A \times \mathbf{R}^n$ *such that* v *belongs to the relative interior of* $\Gamma(a)$ *is a Borel subset of* $\mathbf{R}^n \times \mathbf{R}^n$*.*

The corollary to the next theorem on convex cones will be one of the ingredients to the geometric observation for convex sets described in the introduction.

Definition 3.12. A subset C of $\mathbf{R}^n$ is said to be a *cone* if and only if $\lambda c \in C$ whenever $0 < \lambda < \infty$ and $c \in C$.

Theorem 3.13. *Suppose C is a convex cone in $\mathbf{R}^n$,*

$$D = \mathbf{R}^n \cap \{d : d \bullet c \leq 0 \text{ for } c \in C\},$$

U is an m dimensional plane in $\mathbf{R}^n$, $U \subset D$, $\dim C \geq n - m$, and v belongs to the relative interior of C.

Then, $\dim C = n - m$ and there exists $0 \leq \gamma < \infty$ satisfying

$$\operatorname{dist}(d, U) \leq -\gamma d \bullet v \quad \textit{for } d \in D.$$

Proof. Defining $V = \mathbf{R}^n \cap \{v : u \bullet v = 0 \text{ for } u \in U\}$, we see $C \subset V$ from [4, 4.5], hence $\operatorname{aff} C = V$; in particular, $\dim C = n - m$. Since D is closed under addition and $U \subset D$, D is invariant under directions in U. Therefore, it is sufficient to prove the existence of $0 \leq \gamma < \infty$ such that the inequality holds for $d \in D \cap V \cap \mathbf{S}^{n-1}$. If there were no such γ, then, by compactness, there would exist $d \in D \cap V \cap \mathbf{S}^{n-1}$ with $d \bullet v = 0$ which would imply that v belongs to the relative boundary of C, as $d \in \operatorname{aff} C$. □

Corollary 3.14. *Under the hypotheses of Theorem* 3.13*, there holds*

$$\operatorname{dist}(b, U) \leq -\gamma b \bullet v + (1 + \gamma|v|) \operatorname{dist}(b, D) \quad \textit{for } b \in \mathbf{R}^n.$$

Proof. In view of Remark 3.2 and Lemma 3.9 (1), one may apply Theorem 3.13 to $d = \boldsymbol{\xi}_D(b)$. □

4. Distance bundle

In this section, we introduce the distance bundle in Definition 4.1–Remark 4.6; its nonzero directions correspond to the *normal bundle* employed by Hug, Last, and Weil in [6], see Remark 4.6. Then, we extend (in Theorem 4.7) some basic estimates from Federer's treatment of sets of positive reach in [4] which lead to an important one-sided estimate for the nearest point projection in Corollary 4.9. Finally, we derive the geometric observation, described for convex sets in the introduction, in Lemma 4.11, and the main structural theorem on the singularities of closed sets in Theorem 4.12.

Definition 4.1. Suppose $A \subset \mathbf{R}^n$. Then, the *distance bundle of A* is defined by

$$\operatorname{Dis}(A) = (\mathbf{R}^n \times \mathbf{R}^n) \cap \{(a, v) : a \in \operatorname{Clos} A \text{ and } |v| = \operatorname{dist}(a + v, A)\}.$$

Moreover, we let $\operatorname{Dis}(A, a) = \{v : (a, v) \in \operatorname{Dis}(A)\}$ for $a \in \mathbf{R}^n$.

Remark 4.2. Clearly, $\operatorname{Dis}(A) = \operatorname{Dis}(\operatorname{Clos}(A))$, $\operatorname{Dis}(A)$ is closed, and $0 \in \operatorname{Dis}(A, a)$ if and only if $a \in \operatorname{Clos} A$. Moreover, $\operatorname{Dis}(A, a)$ is a convex subset of $\operatorname{Nor}(A, a)$ for $a \in \mathbf{R}^n$ by [4, 4.8 (2)].

Remark 4.3. Whenever X and G are as in Theorem 3.10, the function, that maps $a \in \operatorname{Clos} A$ onto $\operatorname{Dis}(A, a) \cap X \in G$, is a Borel function; in fact, Remark 4.2 implies that, in the terminology of [3, II.20], the function in question is an upper semicontinuous multifunction, whence the assertion follows by [3, III.3].

Remark 4.4. If $a \in A$, $v \in \operatorname{Dis}(A, a)$, and $0 \le t < 1$, then $\boldsymbol{\xi}_A(a + tv) = a$. In particular, $\boldsymbol{\xi}_A(a + v) = a$ whenever v belongs to the relative interior of $\operatorname{Dis}(A, a)$, and $\operatorname{Dis}(A, a)$ is the closure of $\{v : \boldsymbol{\xi}_A(a + v) = a\}$.

Remark 4.5. In view of Remark 4.4, we could have alternatively formulated our main theorem (see Theorem 4.12), for closed sets, in terms of the bundle $\{(a, v) : \boldsymbol{\xi}_A(a + v) = a\}$ which would be more in line with Stachó's definition of *prenormals* in [16, page 192]. Our choice of bundle is motivated by the fact that $\operatorname{Dis}(A)$ is closed.

Remark 4.6. If A is closed, then Remark 4.4 yields that

$$\left\{(a, |v|^{-1} v) : (a, v) \in \mathbf{R}^n \times \mathbf{R}^n \text{ and } 0 \neq v \in \operatorname{Dis}(A, a)\right\}$$

equals the *normal bundle* of A defined in [6, page 239].

Basic estimates for the distance bundle are collected in the following theorem.

Theorem 4.7. *Suppose $A \subset \mathbf{R}^n$. Then, the following three statements hold.*

(1) *If $0 < q < \infty$, $a \in \operatorname{Clos} A$, $b \in \operatorname{Clos} A$, $v \in \mathbf{R}^n$, and*

$$\textit{either} \quad v = 0 \quad \textit{or} \quad q|v|^{-1} v \in \operatorname{Dis}(A, a),$$

then $(b - a) \bullet v \le (2q)^{-1} |b - a|^2 |v|$;

(2) *If $0 < r < q < \infty$, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^n$, $a \in A$, $b \in A$, and*

$$|x - a| = \operatorname{dist}(x, A) \le r, \quad |y - b| = \operatorname{dist}(y, A) \le r,$$
$$\textit{either} \quad x = a \quad \textit{or} \quad q|x - a|^{-1}(x - a) \in \operatorname{Dis}(A, a),$$
$$\textit{either} \quad y = b \quad \textit{or} \quad q|y - b|^{-1}(y - b) \in \operatorname{Dis}(A, b),$$

then $\boldsymbol{\xi}_A(x) = a$, $\boldsymbol{\xi}_A(y) = b$, and

$$|b - a| \le q(q - r)^{-1} |y - x|;$$

(3) *If $0 < q < \infty$, $a \in \operatorname{Clos} A$, $b \in \operatorname{Clos} A$, C is a convex cone in $\mathbf{R}^n$,*

$$qv \in \operatorname{Dis}(A, a) \quad \textit{whenever } v \in C \cap \mathbf{S}^{n-1},$$

and $D = \mathbf{R}^n \cap \{u : u \bullet v \le 0 \text{ for } v \in C\}$, then

$$\operatorname{dist}(b - a, D) \le (2q)^{-1} |b - a|^2.$$

Proof. To prove (1), we assume $v \neq 0$, let $w = |v|^{-1} v$, and compute

$$|a + qw - b| \geq \operatorname{dist}(a + qw, A) = q, \quad |a - b|^2 + 2qw \bullet (a - b) + q^2 \geq q^2,$$
$$2qw \bullet (b - a) \leq |b - a|^2, \quad v \bullet (b - a) \leq (2q)^{-1} |b - a|^2 |v|.$$

To prove (2), we notice that $a = \boldsymbol{\xi}_A(x)$ and $b = \boldsymbol{\xi}_A(y)$ by Remark 4.4 and infer

$$(b - a) \bullet (x - a) \leq |b - a|^2 r/(2q), \quad (a - b) \bullet (y - b) \leq |b - a|^2 r/(2q)$$

from applying (1) twice; once with v replaced by $x - a$ and once with a, b, and v replaced by b, a, and $y - b$. Therefore, we obtain

$$\begin{aligned} |b - a||y - x| &\geq (b - a) \bullet (y - x) \\ &= (b - a) \bullet ((b - a) + (a - x) + (y - b)) \geq |b - a|^2 (1 - r/q), \end{aligned}$$

whence we infer $|x - y| \geq |a - b|(q - r)/q$.

To prove (3), we suppose $a = 0$. Whenever $v \in C$, we notice that

$$v \bullet b \leq (2q)^{-1} |b|^2 |v|$$

by (1), and estimate

$$|b - v|^2 = |b|^2 + |v|^2 - 2b \bullet v \geq |b|^2 + |v|^2 - |b|^2 |v|/q \geq |b|^2 - |b|^4/(4q^2).$$

Consequently, $\operatorname{dist}(b, C)^2 \geq |b|^2 - |b|^4/(4q^2)$ and (3) is implied by [4, 4.16]. □

Remark 4.8. The proof is almost verbatim taken from [4, 4.8 (7) (8), 4.18 (2)], where slightly stronger hypotheses were made.

Next, we derive a crucial one-sided estimate for the nearest point projection.

Corollary 4.9. *Suppose* $A \subset \mathbf{R}^n$, $0 < s < r < q < \infty$, *and*

$$x \in \operatorname{dmn} \boldsymbol{\xi}_A, \quad s \leq \operatorname{dist}(x, A) \leq r, \quad v = \frac{x - \boldsymbol{\xi}_A(x)}{|x - \boldsymbol{\xi}_A(x)|}, \quad qv \in \operatorname{Dis}(A, \boldsymbol{\xi}_A(x)),$$
$$y \in \operatorname{dmn} \boldsymbol{\xi}_A, \quad s \leq \operatorname{dist}(y, A) \leq r, \quad w = \frac{y - \boldsymbol{\xi}_A(y)}{|y - \boldsymbol{\xi}_A(y)|}, \quad qw \in \operatorname{Dis}(A, \boldsymbol{\xi}_A(y)).$$

Then, there holds

$$(\boldsymbol{\xi}_A(x) - \boldsymbol{\xi}_A(y)) \bullet v \leq \kappa |y - x|^2,$$

where $\kappa = (2s)^{-1}(1 + 2q/(q - r))^2$.

Proof. We let $a = \boldsymbol{\xi}_A(x)$ and $b = \boldsymbol{\xi}_A(y)$, hence we have $a = x - |x-a|v$ and $b = y - |y-b|w$. Next, we estimate

$$(a-b) \bullet v \le (2s)^{-1}|y-x|^2$$

in case that $\operatorname{dist}(x, A) = \operatorname{dist}(y, A) = s$; in fact, noting $\operatorname{dist}(y, A) \le |y-a|$ and $|v| = |w| = 1$, we obtain

$$s^2 \le |y-(x-sv)|^2, \quad (x-y) \bullet v \le (2s)^{-1}|y-x|^2, \quad (w-v) \bullet v \le 0,$$
$$(a-b) \bullet v = (x-y) \bullet v + s(w-v) \bullet v \le (2s)^{-1}|y-x|^2.$$

In the general case, we let (see Lemma 3.9 (1))

$$\alpha = a + \boldsymbol{\xi}_{\mathbf{B}(0,s)}(x-a), \quad \beta = b + \boldsymbol{\xi}_{\mathbf{B}(0,s)}(y-b),$$

notice $\alpha = a + sv$ and $\beta = b + sw$, and infer

$$\alpha \in \operatorname{dmn} \boldsymbol{\xi}_A, \quad \boldsymbol{\xi}_A(\alpha) = a, \quad \beta \in \operatorname{dmn} \boldsymbol{\xi}_A, \quad \boldsymbol{\xi}_A(\beta) = b,$$
$$|\beta - \alpha| \le |y-x| + 2|b-a| \le (1 + 2q/(q-r))|y-x|$$

from Remark 4.4, Lemma 3.9 (1), and Theorem 4.7 (2). Therefore, we may apply the previous case with x and y replaced by α and β to deduce the conclusion. □

Remark 4.10. One could also derive a two-sided estimate; in fact, this is done in [12, 6.6 (1)].

We now have all ingredients at our disposal to derive the geometric observation, formulated in the introduction for convex sets, in full generality.

Lemma 4.11. *Suppose $A \subset \mathbf{R}^n$, $0 < q < \infty$, m is an integer, $1 \le m < n$, W is the set of $y \in \operatorname{dmn} \boldsymbol{\xi}_A$ satisfying*

$$0 < \operatorname{dist}(y, A) < q \quad \textit{and} \quad q|y - \boldsymbol{\xi}_A(y)|^{-1}(y - \boldsymbol{\xi}_A(y)) \in \operatorname{Dis}(A, \boldsymbol{\xi}_A(y)),$$

$x \in W$, $a = \boldsymbol{\xi}_A(x)$, $\dim \operatorname{Dis}(A, a) \ge n - m$, U is an m dimensional subspace of $\mathbf{R}^n$, $U \subset \operatorname{Tan}(A, a)$, and

$$q|x-a|^{-1}(x-a) \textit{ belongs to the relative interior of } \operatorname{Dis}(A, a).$$

Then,

$$\limsup_{W \ni y \to x} |y-x|^{-2} \operatorname{dist}(\boldsymbol{\xi}_A(y) - a, U) < \infty.$$

Proof. Assume $a = 0$, choose s and r such that $0 < s < |x| < r < q$, and let $Q = \operatorname{aff} \operatorname{Dis}(A, 0)$. Then, the set X of all $v \in Q \sim \{0\}$, such that $q|v|^{-1}v$ belongs to the relative interior of $\operatorname{Dis}(A, 0)$, is relatively open in Q and $x \in X$. This implies the existence of $\varepsilon > 0$ such that the convex cone

$$C = Q \cap \{v : |rv - x| < \varepsilon \text{ for some } 0 < r < \infty\}$$

satisfies $C \cap \{v : |v| = |x|\} \subset X$, hence

$$qv \in \operatorname{Dis}(A, 0) \quad \text{whenever } v \in C \cap \mathbf{S}^{n-1};$$

in particular, $C \subset \operatorname{Nor}(A, 0)$ by Remark 4.2. We note that $\dim C = \dim Q \geq n - m$ and that x belongs to the relative interior of C, as $Q \cap \mathbf{U}(x, \varepsilon) \subset C$. Abbreviating $D = \mathbf{R}^n \cap \{d : d \bullet c \leq 0 \text{ for } c \in C\}$, we observe $U \subset D$ from [4, 4.5], and employing $0 \leq \gamma < \infty$ from Theorem 3.13 with $v = x$, we estimate

$$\begin{aligned}\operatorname{dist}(\boldsymbol{\xi}_A(y), U) &\leq -\gamma \boldsymbol{\xi}_A(y) \bullet x + (1 + \gamma|x|) \operatorname{dist}(\boldsymbol{\xi}_A(y), D)\\ &\leq \gamma\kappa|x||y - x|^2 + (1 + \gamma|x|)(2q)^{-1}|\boldsymbol{\xi}_A(y)|^2 \leq \lambda|y - x|^2\end{aligned}$$

whenever $y \in W$ and $s \leq \operatorname{dist}(y, A) \leq r$ by Corollary 3.14, Corollary 4.9, Theorem 4.7 (3), and Theorem 4.7 (2), where

$$\kappa = (2s)^{-1}(1 + 2q/(q - r))^2, \quad \lambda = \gamma\kappa|x| + (1 + \gamma|x|)2^{-1}q(q - r)^{-2}.$$

Finally, x belongs to the interior of $W \cap \{y : s \leq \operatorname{dist}(y, A) \leq r\}$ relative to W by Remark 3.2. □

Finally, we establish the structural theorem on the singularities of closed sets; in fact, we may formulate it for arbitrary subsets of Euclidean space.

Theorem 4.12. *Suppose $A \subset \mathbf{R}^n$, m is an integer, and $0 \leq m \leq n$. Then,*

$$\{a : \dim \operatorname{Dis}(A, a) \geq n - m\}$$

is a countably m rectifiable Borel set which can be $\mathscr{H}^m$ almost covered by the union of a countable family of m dimensional submanifolds of $\mathbf{R}^n$ of class 2.

Proof. Let $B = \{a : \dim \operatorname{Dis}(A, a) \geq n - m\}$. We assume A to be a nonempty closed set by Remark 4.2, and also $m < n$. As $0 \in \operatorname{Dis}(A, a)$ for $a \in A$ by Remark 4.2, we obtain

$$\dim \operatorname{Dis}(A, a) = \dim(\operatorname{Dis}(A, a) \cap \mathbf{B}(0, 1)) \quad \text{for } a \in A$$

from Lemma 3.9 (2); in particular, B is a Borel set by Theorem 3.10 (3) and Remark 4.3. We define N to be the set of all $(a, v) \in A \times \mathbf{R}^n$ such that v belongs to the relative interior of $\operatorname{Dis}(A, a) \cap \mathbf{B}(0, 1)$, hence N is a Borel set by Remark 3.11 and Remark 4.3. By Remark 4.4, we have

$$\boldsymbol{\xi}_A(x + v) = a \quad \text{whenever } (a, v) \in N.$$

Noting Remark 3.2 and Remark 3.5, we define W_i to be the Borel set consisting of all $x \in \boldsymbol{\xi}_A^{-1}[B]$ such that

$$0 < \operatorname{dist}(x, A) < i^{-1} \quad \text{and} \quad \big(\boldsymbol{\xi}_A(x), i^{-1}|x - \boldsymbol{\xi}_A(x)|^{-1}(x - \boldsymbol{\xi}_A(x))\big) \in N$$

for every positive integer i. Then, $\boldsymbol{\xi}_A|W_i$ is locally Lipschitzian by Theorem 4.7 (2) and Remark 3.2, and

$$(\boldsymbol{\xi}_A(x), x - \boldsymbol{\xi}_A(x)) \in N \quad \text{for } x \in W_i$$

by Remark 3.8. We observe that this implies that

$$\mathscr{H}^{n-m}\big((\boldsymbol{\xi}_A|W_i)^{-1}[\{\boldsymbol{\xi}_A(x)\}]\big) > 0 \quad \text{whenever } x \in W_i,$$

since $(\boldsymbol{\xi}_A|W_i)^{-1}[\{\boldsymbol{\xi}_A(x)\}]$ is relatively open in $\{\boldsymbol{\xi}_A(x)+v : v \in \operatorname{aff}\operatorname{Dis}(A, \boldsymbol{\xi}_A(x))\}$.

We choose a countable family F of m dimensional affine planes in $\mathbf{R}^n$ such that $Q \cap \bigcup F$ is dense in Q, whenever Q is an affine subspace of $\mathbf{R}^n$ satisfying $\dim Q \geq n - m$; in fact, one may take F to be a countable dense subset in the family of all m dimensional affine planes in $\mathbf{R}^n$. Thence, we deduce, employing Remark 3.8, that

$$B = \textstyle\bigcup_{i=1}^{\infty} \boldsymbol{\xi}_A\big[W_i \cap \bigcup F\big];$$

in fact, whenever $a \in B$, we take $Q = \{a + v : v \in \operatorname{aff}\operatorname{Dis}(A, a)\}$, pick a positive integer i such that, for some $x \in Q$ with $0 < |x - a| < i^{-1}$, we have that the vector $i^{-1}|x - a|^{-1}(x - a)$ belongs to the relative interior of $\operatorname{Dis}(A, a) \cap \mathbf{B}(0, 1)$, choose such x within $\bigcup F$, and conclude $x \in W_i$ with $\boldsymbol{\xi}_A(x) = a$, as $(a, x - a) \in N$. It follows that B is countably m rectifiable.

To prove the remaining property of B, we assume $m \geq 1$. Then, in view of Remark 2.7 and Lemma 4.11, we may apply Corollary 2.5 with $f = \boldsymbol{\xi}_A|W_i$ for every positive integer i to obtain the conclusion. □

Remark 4.13. Our proof of the countable m rectifiability follows [4, 4.15 (3)], where the case of sets of positive reach was treated.

Remark 4.14. If A is a closed convex set, this property was proven, by different methods, in [1, Theorem 3]; the agreement, in this case, of the normal bundle used there with our distance bundle follows from Lemma 3.9 (1) (3) and Remark 4.4.

Remark 4.15. For $1 \leq m < n$, the preceding theorem may not be strengthened by replacing the distance bundle by the normal bundle, as is evident from considering a closed m dimensional submanifold of $\mathbf{R}^n$ of class 1 that meets every m dimensional submanifold of $\mathbf{R}^n$ of class 2 in a set of $\mathscr{H}^m$ measure zero; the existence of such A follows from [8].

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