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Exceptional zeros of L-series and Bernoulli-Carlitz numbers

BRUNO ANGLÈS, TUAN NGO DAC AND FLORIC TAVARES RIBEIRO

Abstract. Bernoulli-Carlitz numbers were introduced by L. Carlitz in 1935. They are analogues in positive characteristic of Bernoulli numbers. We prove a conjecture formulated by F. Pellarin and the first author on the non-vanishing modulo a given prime of families of Bernoulli-Carlitz numbers. The proof is intimately connected to the arithmetic properties of the L-series introduced by F. Pellarin. Finally, we formulate a conjecture concerning the exceptional zeros of these L-series and prove it in various cases.

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1. Introduction

There is a well-known analogy, started in the 1930s by Carlitz, between the arithmetic of number fields and global function fields. Some recent advances in the arithmetic of function fields [16, 17] and more specifically in the context of cyclotomic extensions [5, 18] have contributed to striking enrichments of this analogy. We briefly review some of these new results.

Let $\mathbb{F}_q$ be a finite field having q elements, q being a power of a prime number p. Let $A = \mathbb{F}_q[\theta]$ with θ an indeterminate over $\mathbb{F}_q$, $K = \mathbb{F}_q(\theta)$, $K_\infty = \mathbb{F}_q\left(\left(\frac{1}{\theta}\right)\right)$ and let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of K_∞. For $d \in \mathbb{N}$, $A_{+,d}$ denotes the set of monic elements in A of degree d. For $n \in \mathbb{Z}$, the value at n of the Carlitz-Goss zeta function is defined as follows

$$\zeta_A(n) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \in K_\infty.$$

One can show that $\zeta_A(n) \in A$ if $n \leq 0$ (this is a consequence of [11, Lemma 8.8.1]) and even $\zeta_A(n) = 0$ if $n < 0$ and $n \equiv 0 \pmod{q-1}$ [11, Example 8.13.9].

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Furthermore, for all $n \geq 1$, $\zeta_A(n) \in K_\infty^\times$ and can be written as a Euler product

$$\zeta_A(n) = \prod_{\substack{P \text{ monic} \\ P \text{ irreducible}}} \left(1 - \frac{1}{P^n}\right)^{-1}.$$

Choose $\lambda_\theta \in \mathbb{C}_\infty$ such that $\lambda_\theta^{q-1} = -\theta$. The Carlitz period, $\widetilde{\pi} \in \mathbb{C}_\infty^\times$, is defined by the following convergent product

$$\widetilde{\pi} = \lambda_\theta\, \theta \prod_{j \geq 1} \left(1 - \theta^{1-q^j}\right)^{-1}.$$

In the analogy developed by Carlitz, the A-module $\widetilde{\pi} A$ plays the role of the $\mathbb{Z}$-module $2i\pi\mathbb{Z}$. It is the kernel of the *Carlitz exponential*, $\exp_C$, defined by

$$\exp_C(z) = z \prod_{a \in A \setminus \{0\}} \left(1 - \frac{z}{\widetilde{\pi} a}\right) \qquad \text{for all } z \in \mathbb{C}_\infty.$$

One can show that [11, Chapter 3]

$$\exp_C(z) = \sum_{j \geq 0} \frac{z^{q^j}}{D_j} \qquad \text{for all } z \in \mathbb{C}_\infty,$$

where $D_0 = 1$, and for $j \geq 1$, $D_j = (\theta^{q^j} - \theta) D_{j-1}^q$. It satisfies the functional equation

$$\exp_C(\theta z) = \theta \exp_C(z) + \exp_C(z)^q \qquad \text{for all } z \in \mathbb{C}_\infty.$$

One can see that the Carlitz exponential induces a short exact sequence of A-modules

$$0 \to \widetilde{\pi} A \to \mathbb{C}_\infty \to C(\mathbb{C}_\infty) \to 0,$$

where $C(\mathbb{C}_\infty)$ denotes the $\mathbb{F}_q$-vector space $\mathbb{C}_\infty$ equipped with the A-module structure given by

$$\theta \cdot z = \theta z + z^q \qquad \text{for all } z \in \mathbb{C}_\infty.$$

Let $n \in \mathbb{N}$. We expand n in base $q : n = \sum_{j \geq 0} n_j q^j$ with $n_j \in \{0, \dots, q-1\}$. The Carlitz factorial, $\Pi(n) \in A$, is defined by

$$\Pi(n) = \prod_{j \geq 0} D_j^{n_j}.$$

In 1935, L. Carlitz has introduced analogues of the Bernoulli numbers for A [8]. For all $n \geq 0$, the *Bernoulli-Carlitz numbers*, $BC_n \in K$ are defined as follows. Let X be an indeterminate over K, then

$$\frac{X}{\exp_C(X)} = \sum_{n \geq 0} \frac{BC_n}{\Pi(n)} X^n.$$

It follows that $BC_0 = 1$, $BC_n = 0$ if $n \not\equiv 0 \pmod{q-1}$, and [11, Paragraph 9.2]

$$\frac{\zeta_A(n)}{\widetilde{\pi}^n} = \frac{BC_n}{\Pi(n)} \qquad \text{for all } n \geq 1, n \equiv 0 \pmod{q-1}.$$

Compare with the classical Euler formulas

$$\frac{\zeta(n)}{(2i\pi)^n} = -\frac{1}{2}\frac{B_n}{n!} \qquad \text{for all } n \geq 1, n \equiv 0 \pmod 2,$$

where $\zeta(.)$ denotes the Riemann zeta function, and B_k denotes the k-th Bernoulli number. The von-Staudt-Clausen Theorem [21, Theorem 5.10] asserts that for all $n \geq 1$, we have

$$B_{2n} + \sum_{\substack{p \text{ prime} \\ (p-1)|2n}} \frac{1}{p} \in \mathbb{Z}.$$

There exists an analogue of the above result for Bernoulli-Carlitz numbers [11, Paragraph 9.2]. In particular, if P denotes a monic irreducible polynomial in A of degree d, then BC_n is P-integral for $n \in \{0, \ldots, q^d - 2\}$.

Let p be a prime number and let ω_p be the p-adic Teichmüller character, let $1 < n < p-1$ be such that $n \equiv 1 \pmod 2$, then K. Ribet proved the following result [14]

$$\big(\mathrm{Cl}(\mathbb{Q}(\mu_p)) \otimes_{\mathbb{Z}} \mathbb{Z}_p\big)\big(\omega_p^n\big) \neq \{0\} \Leftrightarrow B_{p-n} \equiv 0 \pmod p,$$

where $\mathrm{Cl}(\mathbb{Q}(\mu_p))$ denotes the ideal class group of the ring of integers of $\mathbb{Q}(\mu_p)$, and $(\mathrm{Cl}(\mathbb{Q}(\mu_p)) \otimes_{\mathbb{Z}} \mathbb{Z}_p)(\omega_p^n)$ denotes the isotypic component attached to the character ω_p^n. We recall that the prime number p is said to be *regular* if $\mathrm{Cl}(\mathbb{Q}(\mu_p)) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \{0\}$, p is said to be *irregular* otherwise. We also recall that there exist infinitely many irregular primes [21, Theorem 5.17], and that the prime number p is regular if and only if $\prod_{n=1}^{\frac{p-3}{2}} B_{2n} \not\equiv 0 \pmod p$ [21]. In [21, page 63], a heuristic argument shows that 60, 65% of all primes should be regular. It is conjectured that there exist infinitely many regular primes. In particular, this latter conjecture implies the following conjecture.

Conjecture A. If $k \geq 3$ is an odd integer then there exist infinitely many primes p such that $B_{p-k} \not\equiv 0 \pmod p$.

Let us consider the function field case. Let P be a monic irreducible polynomial in A of degree d. Let $E = K(\exp_C(\frac{\widetilde{\pi}}{P}))$ be the P-th cyclotomic function field which is the analogue of $\mathbb{Q}(\mu_p)$ in our context. Let O_E be the integral closure of A in E. Quite recently, L. Taelman has introduced a class module $H(O_E)$ attached to O_E and the Carlitz exponential $\exp_C$ [16, 17]. This latter module can be viewed as an A-analogue in our context of the ideal class group of a number field. It is defined by

$$H(O_E) = \frac{E_\infty}{O_E + \exp_C(E_\infty)},$$

where $E_\infty = E \otimes_K K_\infty$. L. Taelman proved that $H(O_E)$ is a finite A-module [16]. Let $\widehat{A}_P$ be the P-adic completion of A and ω_P be the P-adic Teichmüller character. Let $1 < n < q^d - 1$ be such that $n \equiv 1 \pmod{q-1}$, we have an analogue of Ribet's theorem [5,18]

$$\big(H(O_E) \otimes_A \widehat{A}_P\big)\big(\omega_P^n\big) \neq \{0\} \Leftrightarrow BC_{q^d-n} \equiv 0 \pmod{P}.$$

Under the light of the above analogies, it is natural to ask if Conjecture A is valid in the carlitzian context. In this paper, we prove a stronger result which answers positively to this question and also to a Conjecture formulated in [2].

Theorem B. *Let $N \geq 2$ be an integer, $N \equiv 1 \pmod{q-1}$. Let $\ell_q(N)$ be the sum of the digits in base q of N. Let $P \in A$ be a monic irreducible polynomial of degree d such that $q^d > N$. If $d \geq \left(\frac{\ell_q(N)-1}{q-1}\right) N$, then*

$$BC_{q^d-N} \not\equiv 0 \pmod{P}.$$

Note that the above result is due to L. Carlitz when N is a power of q ([11, Lemma 8.22.4], see also [20, Theorem 4.16.1] and [12, Paragraph 3.4]). To prove our theorem, we use the deep connection between the several variable L-series introduced in [13] and the Bernoulli-Carlitz numbers, which will be explained in Sections 3 and 4.

Let $s \geq 1$ be an integer and let $\mathbb{T}_s$ be the Tate algebra in s variables $t_1, \dots, t_s$, with coefficients in $\mathbb{C}_\infty$. In 2012, F. Pellarin [13] introduced the following remarkable elements in $\mathbb{T}_s^\times$ called the *several variable L-series*

$$L_s(\underline{t}_s) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a}.$$

For $s = 1$, he proved the following identity [13, Theorem 1]

$$\frac{L_1(t_1)\omega(t_1)}{\widetilde{\pi}} = \frac{1}{\theta - t_1}$$

where $\omega(t_1)$ is the special function introduced in [1] and defined by

$$\omega(t_1) = \lambda_\theta \prod_{j \geq 0} \left(1 - \frac{t_1}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_1^\times.$$

For $s \geq 2$ and $s \equiv 1 \pmod{q-1}$, F. Pellarin and the first author ([2, Corollary 21], see also [4, Corollary 7.4]) showed that there exists a polynomial $\mathbb{B}_s \in \mathbb{F}_q[t_1, \dots, t_s][\theta]$ such that

$$\frac{L_s(\underline{t}_s)\, \omega(t_1) \cdots \omega(t_s)}{\widetilde{\pi}} = (-1)^{\frac{s-1}{q-1}} \mathbb{B}_s.$$

Further, these several variable Bernoulli-type objects $\mathbb{B}_s$ are linked by a class formula à la Taelman [17] to the several variable L-series $L_s(\underline{t}_s)$ [4, Theorem 5.11].

We expand N in base q

$$N = \sum_{i=1}^{s} q^{e_i},$$

with $e_i \in \mathbb{N}$, $e_1 \leq e_2 \leq \cdots \leq e_s$. We set

$$B_N(t,\theta) = \mathbb{B}_s \mid_{t_i = t^{q^{e_i}}} \in \mathbb{F}_q[t,\theta]$$

and

$$L_N(t) = L_s(\underline{t}_s) \mid_{t_i = t^{q^{e_i}}} \in K_\infty[[t]].$$

Using the combinatorial properties of these polynomials $\mathbb{B}_s$ (Proposition 3.7), we show that (see Section 4.3)

Theorem C. *Let $N \geq 2$ be an integer, $N \equiv 1 \pmod{q-1}$. Then for all $i \in \mathbb{N}$,*

$$B_N(t,\theta) \mid_{t=\theta^{q^i}} \neq 0.$$

Theorem B is a direct consequence of Theorem C, see Section 4.4.

In the remaining sections, we propose a refinement of Theorem C. In Section 5, we study the series $L_N(t)$. We show that this series is an entire function on $\mathbb{C}_\infty$. Further, we prove that the elements of $S_N := \{\theta^{q^i}, i \in \mathbb{Z}, Nq^i \geq 1\}$ are zeros of this entire series. We call them the *trivial zeros* of $L_N(t)$. The other zeros are called *exceptional zeros* of $L_N(t)$. It turns out that each exceptional zero of $L_N(t)$ is a root of $B_N(t,\theta)$, with the same multiplicity. Motivated by Theorem C, we formulate the following conjecture.

Conjecture D. Let $N \geq 2$ be an integer, $N \equiv 1 \pmod{q-1}$. Then the polynomial $B_N(t,\theta)$ (in the variable t) has no zeros in the set $\{\theta^{q^i}, i \in \mathbb{Z}\}$.

In Section 6, we will prove this conjecture when $q = p$ (Theorem 6.8). For the general case (Section 7), we are able to settle the conjecture when N is q-minimal, that is N satisfies certain combinatorial conditions (Theorem 7.7). Our proof uses combinatorial techniques introduced by F. Diaz-Vargas [10] and J. Sheats [15].

Theorem E. *Conjecture D holds in the following cases:*

1) *$q = p$;*
2) *$q > p$ and N is q-minimal.*

An analogue of Greenberg's pseudo-cyclicity Conjecture for the Iwasawa module associated to the p-part of the ideal class groups along the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}(\mu_p)$ (p is an odd prime number) was proved in [6, Theorem 4]. As an application of our methods, we deduce another proof of this latter result. Finally, we present some numerical evidence to support our conjecture when N is no longer q-minimal.

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2. Notation

In this paper, we will use the following notation:

- $\mathbb{N}$: the set of non-negative integers;
- $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$: the set of positive integers;
- $\mathbb{Z}$: the set of integers;
- $\mathbb{F}_q$: a finite field having q elements;
- p: the characteristic of $\mathbb{F}_q$;
- θ: an indeterminate over $\mathbb{F}_q$;
- A: the polynomial ring $\mathbb{F}_q[\theta]$;
- A_+: the set of monic elements in A;
- For $d \in \mathbb{N}$, $A_{+,d}$ denotes the set of monic elements in A of degree d;
- $K = \mathbb{F}_q(\theta)$: the fraction field of A;
- ∞ : the unique place of K which is a pole of θ;
- v_∞ : the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$;
- $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$: the completion of K at ∞;
- $\mathbb{C}_\infty$: the completion of a fixed algebraic closure of K_∞. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞;
- λ_θ : a fixed $(q-1)$th-root of $-\theta$ in $\mathbb{C}_\infty$;
- For $s \in \mathbb{N}$, $\{t_1, t_2, \ldots, t_s\}$ denotes a set of s variables and we will also denote this set by $\underline{t}_s$.

3. The several variable polynomials $\mathbb{B}_s$

3.1. Tate algebras

Let L be an extension of K_∞ in $\mathbb{C}_\infty$ such that L is complete with respect to $v_\infty \mid_L$. The absolute value of L is defined by

$$|a| = q^{-v_\infty(a)} \quad \text{for } a \in L.$$

Let $s \in \mathbb{N}$ be a non-negative integer. The polynomial ring $L[\underline{t}_s] = L[t_1, \ldots, t_s]$ is equipped with the Gauss valuation: for a polynomial $f \in L[\underline{t}_s]$, if we write

$$f = \sum_{i_1,\ldots,i_s \in \mathbb{N}} a_{i_1,\ldots,i_s} t_1^{i_1} \ldots t_s^{i_s}, \quad a_{i_1,\ldots,i_s} \in L,$$

then the Gauss valuation of f is defined by

$$v_\infty(f) := \inf\left\{v_\infty(a_{i_1,\ldots,i_s}), i_1, \ldots, i_s \in \mathbb{N}\right\}.$$

By definition, the Tate algebra $\mathbb{T}_s(L)$ in the variables $t_1, \ldots, t_s$ with coefficients in L is the completion of $L[\underline{t}_s]$ with respect to the Gauss valuation. Explicitly, $\mathbb{T}_s(L)$ is the set of formal series

$$f = \sum_{i_1,\ldots,i_s \in \mathbb{N}} a_{i_1,\ldots,i_s} t_1^{i_1} \ldots t_s^{i_s}, \quad a_{i_1,\ldots,i_s} \in L,$$

such that

$$\lim_{i_1+\ldots+i_s \to +\infty} v_\infty(a_{i_1,\ldots,i_s}) = +\infty.$$

When $L = \mathbb{C}_\infty$, we will write $\mathbb{T}_s$ instead of $\mathbb{T}_s(\mathbb{C}_\infty)$. Let $\tau : \mathbb{T}_s \to \mathbb{T}_s$ be the continuous homomorphism of $\mathbb{F}_q[\underline{t}_s]$-algebras such that

$$\tau(c) = c^q, \quad \forall c \in \mathbb{C}_\infty.$$

Explicitly, for a formal series $f \in \mathbb{T}_s$, we write

$$f = \sum_{i_1,\ldots,i_s \in \mathbb{N}} a_{i_1,\ldots,i_s} t_1^{i_1} \ldots t_s^{i_s}, \quad a_{i_1,\ldots,i_s} \in \mathbb{C}_\infty,$$

then

$$\tau(f) = \sum_{i_1,\ldots,i_s \in \mathbb{N}} a_{i_1,\ldots,i_s}^q t_1^{i_1} \ldots t_s^{i_s}.$$

3.2. The several variable polynomials $\mathbb{B}_s$

For the rest of this section, we will always suppose that $s \geq 2$ and $s \equiv 1 \pmod{q-1}$.

We set

$$r = \frac{s-q}{q-1} \in \mathbb{N}.$$

Recall that λ_θ is a fixed $(q-1)$th-root of $-\theta$ in $\mathbb{C}_\infty$. We have set

$$\widetilde{\pi} = \lambda_\theta \theta \prod_{j \geq 1} \left(1 - \theta^{1-q^j}\right)^{-1} \in \mathbb{C}_\infty^\times,$$

and for $i = 1, \dots, s$, we set

$$\omega(t_i) = \lambda_\theta \prod_{j\geq 0} \left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_s^\times.$$

Furthermore, we introduce

$$L_s(\underline{t}_s) = \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(t_1)\cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

We observe that it can be written as a Euler product

$$L_s(\underline{t}_s) = \prod_{\substack{P\in A_+ \\ P \text{ irreducible}}} \left(1 - \frac{P(t_1)\cdots P(t_s)}{P}\right)^{-1}.$$

We define

$$\mathbb{B}_s = (-1)^{\frac{s-1}{q-1}} \frac{L_s(\underline{t}_s)\omega(t_1)\dots\omega(t_s)}{\widetilde{\pi}} \in \mathbb{T}_s, \tag{3.1}$$

then by [4, Lemma 7.6] (see also [2, Corollary 21]), we know that:

Proposition 3.1. *The element $\mathbb{B}_s$ is a polynomial in $\mathbb{F}_q[\underline{t}_s, \theta]$. Moreover, it is a monic polynomial in θ of degree $r = \frac{s-q}{q-1}$ and a symmetric polynomial in $\underline{t}_s$.*

The polynomial $\mathbb{B}_s$ is intimately connected to the class module H_ϕ of a certain Drinfeld $A[\underline{t}_s]$-module ϕ of rank one as follows (we refer the interested reader to [4, Section 7] for more details). Let $\phi : A[\underline{t}_s] \to \mathbb{T}_s[\underline{t}_s]\{\tau\}$ be the Drinfeld $A[\underline{t}_s]$-module over $\mathbb{T}_s$ given by a homomorphism of $\mathbb{F}_q[\underline{t}_s]$-algebras such that

$$\phi_\theta = (t_1 - \theta)\cdots(t_s - \theta)\tau + \theta.$$

There exists a unique formal series $\exp_\phi \in \mathbb{T}_s\{\{\tau\}\}$ called the *exponential series* attached to ϕ such that

$$\exp_\phi \equiv 1 \pmod{\tau}$$

and

$$\phi_a \exp_\phi = \exp_\phi a, \quad \forall a \in A[\underline{t}_s].$$

One can show that the exponential series induces a natural $\mathbb{F}_q[\underline{t}_s]$-linear map

$$\exp_\phi : \mathbb{T}_s \to \mathbb{T}_s.$$

Following Taelman [17, 18], we define the class module H_ϕ by

$$H_\phi := \frac{\phi(\mathbb{T}_s(K_\infty))}{\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A[\underline{t}_s])}$$

where $\phi(A[\underline{t}_s])$ is the $\mathbb{F}_q[\underline{t}_s]$-module $A[\underline{t}_s]$ equipped with the $A[\underline{t}_s]$-module structure induced by ϕ. Then by [4, Proposition 7.2], the class module H_ϕ is a finitely generated $\mathbb{F}_q[\underline{t}_s]$-module of rank $r = \frac{s-q}{q-1}$. The importance of the polynomials $\mathbb{B}_s$ is dictated by [4, Theorem 7.7].

Theorem 3.2. *We denote by* $\mathrm{Fitt}_{A[\underline{t}_s]}(H_\phi)$ *the Fitting ideal of the torsion* $A[\underline{t}_s]$-*module of finite type* H_ϕ. *Then*

$$\mathrm{Fitt}_{A[\underline{t}_s]}(H_\phi) = \mathbb{B}_s A[\underline{t}_s].$$

In particular,

$$\mathbb{B}_s = \det_{\mathbb{F}_q[\underline{t}_s][Z]} \Big(Z \cdot \mathrm{Id} - \phi_\theta \mid_{H_\phi \otimes_{\mathbb{F}_q[\underline{t}_s]} \mathbb{F}_q[\underline{t}_s][Z]} \Big) \mid_{Z=\theta} . \tag{3.2}$$

We are now ready to give a few explicit examples of the polynomials $\mathbb{B}_s$. We need to introduce some more notation. Denote by S the set of all (finite) sequences of non-negative integers $\underline{m}$. For any sequence $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ of non-negative integers, we set

$$m_0 = s - (m_1 + \cdots + m_d) \in \mathbb{Z}$$

and

$$\sigma_s(\underline{m}) = \sigma_s(m_1, \dots, m_d) = \sum \prod_{u=1}^{d} \prod_{i \in J_u} t_i^u,$$

where the sum runs through the disjoint unions $J_1 \bigsqcup \cdots \bigsqcup J_d \subset \{1, \dots, s\}$ such that $\mid J_u \mid = m_u$, for $1 \le u \le d$. In particular, $\sigma_s(\underline{m}) = 0$ if $m_1 + \cdots + m_d > s$, which is equivalent to $m_0 < 0$. The reader should keep in mind that m_i may be 0. For example,

$$\sigma_s(0, 0, 1) = \sum_{i=1}^{s} t_i^3.$$

Here are some more explicit examples that will appear in Lemma 3.4.

$$\sigma_{2q-1}(q) = \sum_{1 \le i_1 < \cdots < i_q \le 2q-1} t_{i_1} \cdots t_{i_q},,$$

$$\sigma_{3q-2}(q) = \sum_{1 \le i_1 < \cdots < i_q \le 3q-2} \prod_{j=1}^{q} t_{i_j},$$

$$\sigma_{3q-2}(2q-1) = \sum_{1 \le i_1 < \cdots < i_{2q-1} \le 3q-2} \prod_{j=1}^{2q-1} t_{i_j},$$

$$\sigma_{3q-2}(2q) = \sum_{1 \le i_1 < \cdots < i_{2q} \le 3q-2} \prod_{j=1}^{2q} t_{i_j},$$

$$\sigma_{3q-2}(q-1, q) = \sum_{1 \le i_1 < \cdots < i_{q-1} \le 3q-2} \sum_{\substack{1 \le k_1 < \cdots < k_q \le 3q-2 \\ k_{j'} \ne i_j}} \prod_{j=1}^{q-1} t_{i_j} \prod_{j'=1}^{q} t_{k_{j'}}^2.$$

We remark that any symmetric polynomial f in $\mathbb{F}_q[\underline{t}_s]$ can be uniquely written as a combination

$$f = \sum_{\substack{\underline{m} \in S \\ 0 \leq m_0 \leq s}} a_{\underline{m}} \sigma_s(\underline{m}) \quad \text{with } a_{\underline{m}} \in \mathbb{F}_q.$$

Lemma 3.3. *Let $\underline{m} \in S$. We have*

$$\sigma_s(\underline{m}) \mid_{t_{s-(q-2)}=\ldots=t_s=0} = \sigma_{s-(q-1)}(\underline{m}).$$

In particular, if $m_0 < q-1$, we have

$$\sigma_s(\underline{m}) \mid_{t_{s-(q-2)}=\ldots=t_s=0} = 0.$$

Proof. This is a straight computation. □

Lemma 3.4. *With the previous notation, we have*

1) $\mathbb{B}_q = 1$;
2) $\mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q)$;
3) $\mathbb{B}_{3q-2} = \theta^2 - \theta[\sigma_{3q-2}(q) + \sigma_{3q-2}(2q-1)] + [\sigma_{3q-2}(q-1, q) + \sigma_{3q-2}(2q)]$.

Proof. We follow closely [4, Proposition 7.2] and its proof.

1) See [4, Corollary 7.3];

2) Remark that the formula for $\mathbb{B}_{2q-1}$ is already given in [4, Section 1.2]. We provide a proof for the convenience of the reader. Recall that $\mathbb{T}_{2q-1}(K_\infty)$ is the Tate algebra in the variables $t_1, \ldots, t_{2q-1}$ with coefficients in K_∞. We set

$$N = \left\{ f \in \mathbb{T}_{2q-1}(K_\infty), v_\infty(f) \geq 2 \right\}.$$

By [4, Proposition 7.2] and its proof, we obtain

(i) $\mathbb{T}_{2q-1}(K_\infty) = \frac{1}{\theta} A[t_1, \ldots, t_{2q-1}] \oplus N$;
(ii) $N = \exp_\phi(\mathbb{T}_{2q-1}(K_\infty))$;
(iii) H_ϕ is a free $\mathbb{F}_q[t_1, \ldots, t_{2q-1}]$-module of rank one generated by $\frac{1}{\theta}$.

We observe that

$$\phi_\theta\left(\frac{1}{\theta}\right) \equiv \frac{\sum_{1 \leq i_1 < \ldots < i_q \leq 2q-1} t_{i_1} \cdots t_{i_q}}{\theta} \quad (\text{mod } A[t_1, \ldots, t_{2q-1}] \oplus N).$$

By Theorem 3.2, we have

$$\begin{aligned} \mathbb{B}_{2q-1} &= \det_{\mathbb{F}_q[t_1,\ldots,t_{2q-1}][Z]} \left(Z\mathrm{Id} - \phi_\theta \mid_{H_\phi \otimes_{\mathbb{F}_q[t_1,\ldots,t_{2q-1}]} \mathbb{F}_q[t_1,\ldots,t_{2q-1}][Z]} \right) \mid_{Z=\theta} \\ &= \theta - \sum_{1 \leq i_1 < \ldots < i_q \leq 2q-1} t_{i_1} \cdots t_{i_q} \\ &= \theta - \sigma_{2q-1}(q). \end{aligned}$$

3) The calculation is similar, but more involved and left to the reader. □

Lemma 3.5. *For $s \geq 2q-1$, $s \equiv 1 \pmod{q-1}$, we have*

$$\mathbb{B}_s\big(t_1, \ldots, t_{s-(q-1)}, 0, \ldots, 0\big) = \big(\theta - t_1 \cdots t_{s-(q-1)}\big)\mathbb{B}_{s-(q-1)}\big(t_1, \ldots, t_{s-(q-1)}\big).$$

Proof. By (3.1), we obtain

$$\begin{aligned}
&\mathbb{B}_s(t_1, \ldots, t_{s-(q-1)}, 0, \ldots, 0) \\
&= (-\theta)(-1)^{\frac{s-1}{q-1}} \frac{\omega(t_1)\ldots\omega(t_{s-(q-1)})}{\widetilde{\pi}} \prod_{\substack{P \in A_+ \\ P \text{ irreducible} \\ P \neq \theta}} \left(1 - \frac{P(t_1)\cdots P(t_{s-(q-1)})}{P}\right)^{-1} \\
&= \big(\theta - t_1 \cdots t_{s-(q-1)}\big)(-1)^{\frac{s-q}{q-1}} \frac{\omega(t_1)\ldots\omega(t_{s-(q-1)})}{\widetilde{\pi}} \\
&\quad \prod_{\substack{P \in A_+ \\ P \text{ irreducible}}} \left(1 - \frac{P(t_1)\cdots P(t_{s-(q-1)})}{P}\right)^{-1} \\
&= \big(\theta - t_1 \cdots t_{s-(q-1)}\big)\mathbb{B}_{s-(q-1)}\big(t_1, \ldots, t_{s-(q-1)}\big).
\end{aligned}$$

The proof is finished. □

Let $\rho : \mathbb{F}_q[\underline{t}_s] \to \mathbb{N} \cup \{+\infty\}$ be the valuation given by

(i) If $f = 0$, then

$$\rho(f) = +\infty;$$

(ii) Otherwise, we write $f = \sum a_{i_1,\ldots,i_s} t_1^{i_1} \cdots t_s^{i_s}$ with $a_{i_1,\ldots,i_s} \in \mathbb{F}_q$, then

$$\rho(f) = \operatorname{Inf}\big\{i_1 + \ldots + i_s,\ a_{i_1,\ldots,i_s} \neq 0\big\}.$$

Remark 3.6. Observe that for any sequence $\underline{m} = (m_1, \ldots, m_d) \in S$, we have

$$\rho(\sigma_s(\underline{m})) = \begin{cases} m_1 + 2m_2 + \ldots + dm_d & \text{if } m_1 + \cdots + m_d \leq s, \\ +\infty & \text{otherwise.} \end{cases}$$

By Proposition 3.1, $\mathbb{B}_s \in \mathbb{F}_q[\underline{t}_s, \theta]$ is a monic polynomial in θ of degree $r = \frac{s-q}{q-1}$ and a symmetric polynomial in $\underline{t}_s$. We write

$$\mathbb{B}_s = \sum_{i=0}^{r} B_{i,s}\theta^{r-i}, \tag{3.3}$$

where $B_{i,s} \in \mathbb{F}_q[\underline{t}_s]$ are symmetric polynomials, and $B_{0,s} = 1$.

We are now ready to prove the key result of this section.

Proposition 3.7. *For $i = 1, \dots, r$, we have*

$$\rho(B_{i,s}) \geq i(q-1)+1.$$

Proof. The proof is by induction on r. If $r = 1$, then $s = 2q-1$. By Lemma 3.4 2), we know that

$$\mathbb{B}_{2q-1} = \theta - \sigma_{2q-1}(q),$$

and the proposition is verified.

We suppose that $r \geq 2$ and that the proposition is true for $r-1$, *i.e.*, for $i = 1, \dots, r-1$, we have

$$\rho\big(B_{i,s-(q-1)}\big) \geq i(q-1)+1.$$

We set $B_{r,s-(q-1)} = 0$.

We will now prove that for $i = 1, \dots, r$, we have

$$\rho(B_{i,s}) \geq i(q-1)+1.$$

Let i be an integer with $1 \leq i \leq r$. Since $B_{i,s} \in \mathbb{F}_q[\underline{t}_s]$ is symmetric, we write

$$B_{i,s} = \sum_{\substack{\underline{m} \in S \\ 0 \leq m_0 \leq s}} a_{i,\underline{m}} \sigma_s(\underline{m}) \qquad \text{with } a_{i,\underline{m}} \in \mathbb{F}_q.$$

We put

$$\widehat{B}_{i,s} = \sum_{\substack{\underline{m} \in S \\ 0 \leq m_0 < q-1}} a_{i,\underline{m}} \sigma_s(\underline{m}),$$

$$\widetilde{B}_{i,s} = \sum_{\substack{\underline{m} \in S \\ q-1 \leq m_0 \leq s}} a_{i,\underline{m}} \sigma_s(\underline{m}).$$

Then

$$B_{i,s} = \widehat{B}_{i,s} + \widetilde{B}_{i,s}.$$

Lemma 3.8. *We have*

$$\rho\big(\widehat{B}_{i,s}\big) \geq r(q-1)+2.$$

Proof of Lemma 3.8. If $\widehat{B}_{i,s} = 0$ then $\rho(\widehat{B}_{i,s}) = +\infty$ and we are done. Otherwise, $\widehat{B}_{i,s} \neq 0$. By definition of the valuation ρ and Remark 3.6, we have

$$\begin{aligned}
\rho\big(\widehat{B}_{i,s}\big) &= \rho\left(\sum_{\substack{\underline{m}\in S\\ 0\le m_0<q-1}} a_{i,\underline{m}}\sigma_s(\underline{m})\right)\\
&= \mathrm{Inf}\big\{m_1 + 2m_2 + \ldots + dm_d \mid 0 \le m_0 < q-1 \text{ and } a_{i,\underline{m}} \neq 0\big\}\\
&\ge \mathrm{Inf}\big\{m_1 + m_2 + \ldots + m_d \mid 0 \le m_0 < q-1 \text{ and } a_{i,\underline{m}} \neq 0\big\}\\
&\ge \mathrm{Inf}\big\{s - m_0 \mid 0 \le m_0 < q-1 \text{ and } a_{i,\underline{m}} \neq 0\big\}\\
&\ge s - (q-2) = r(q-1) + 2.
\end{aligned}$$

The proof of Lemma 3.8 is finished. □

Lemma 3.9. *We have*

$$\rho\big(\widetilde{B}_{i,s}\big) \ge i(q-1) + 1.$$

Proof of Lemma 3.9. By Lemma 3.3, we have

$$\begin{aligned}
\widehat{B}_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0} &= 0,\\
\widetilde{B}_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0} &= \sum_{\substack{\underline{m}\in S\\ q-1\le m_0\le s}} a_{i,\underline{m}}\sigma_{s-(q-1)}(\underline{m}).
\end{aligned}$$

Hence,

$$\begin{aligned}
B_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0} &= \widehat{B}_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0} + \widetilde{B}_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0}\\
&= \widetilde{B}_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0}\\
&= \sum_{\substack{\underline{m}\in S\\ q-1\le m_0\le s}} a_{i,\underline{m}}\sigma_{s-(q-1)}(\underline{m}).
\end{aligned}$$

Since the polynomials $B_{i,s}$ are symmetric, it follows that

$$\rho(B_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0}) = \rho\big(\widetilde{B}_{i,s} \mid_{t_{s-(q-2)}=\ldots=t_s=0}\big) = \rho\big(\widetilde{B}_{i,s}\big). \tag{3.4}$$

By Lemma 3.5, we know that

$$\mathbb{B}_s \mid_{t_{s-(q-2)}=\ldots=t_s=0} = \big(\theta - t_1 \cdots t_{s-(q-1)}\big)\mathbb{B}_{s-(q-1)}.$$

Therefore, we get

$$B_{i,s} \mid_{t_s=\ldots=t_{s-(q-2)}=0} = B_{i,s-(q-1)} - t_1 \cdots t_{s-(q-1)} B_{i-1,s-(q-1)}. \tag{3.5}$$

Now, by the induction hypothesis, we have already known

$$\rho\big(B_{i,s-(q-1)} - t_1 \cdots t_{s-(q-1)} B_{i-1,s-(q-1)}\big) \geq i(q-1)+1.$$

Combined with (3.4) and (3.5), we obtain

$$\rho\big(\widetilde{B}_{i,s}\big) \geq i(q-1)+1.$$

The proof of Lemma 3.9 is finished. □

We are back to the proof of Proposition 3.7. Note that $B_{i,s} = \widehat{B}_{i,s} + \widetilde{B}_{i,s}$. Hence, the proposition is an immediate consequence of Lemma 3.8 and Lemma 3.9. □

4. The two variable polynomials $B_N(t,\theta)$

Let $N \geq 2$ be an integer such that $N \equiv 1 \pmod{q-1}$. We denote by $\ell_q(N)$ the sum of the digits of the expansion in base q of N. Let us assume that $\ell_q(N) \geq 2$. We set

$$s = \ell_q(N),$$

then $s \geq 2$ and $s \equiv 1 \pmod{q-1}$. We put

$$r = \frac{s-q}{q-1} \in \mathbb{N}.$$

Let us expand N in base q

$$N = \sum_{i=1}^{s} q^{e_i},$$

with $e_i \in \mathbb{N}$, $e_1 \leq e_2 \leq \cdots \leq e_s$. We remark that each identical exponent appears at most $q-1$ times in the above expansion of N.

4.1. The two variable polynomials $B_N(t,\theta)$

In the previous section, we have defined the several variable polynomial $\mathbb{B}_s \in \mathbb{F}_q[\underline{t}_s, \theta]$. We define

$$B_N(t,\theta) = \mathbb{B}_s \,|_{t_i = t^{q^{e_i}}} \in \mathbb{F}_q[t,\theta].$$

By Proposition 3.1, $B_N(t,\theta)$ is a monic polynomial in θ of degree r.

The following lemma summarizes some properties of the polynomial $B_N(t,\theta)$.

Lemma 4.1. *We keep the previous notation. Then*

1) $B_N(t^p, \theta^p) = B_N(t,\theta)^p$;
2) $B_{qN}(t,\theta) = B_N(t^q, \theta)$;

3) *If* $N \equiv 0 \pmod{p}$, *then* $B_N(t,\theta) \in \mathbb{F}_p[t^p,\theta]$*;*
4) $B_N(t,\theta) \equiv (\theta-t)^r - r(\theta-t)^{r-1}(t^q-t) \pmod{(t^q-t)^2\mathbb{F}_p[t,\theta]}$.

Proof. 1) By (3.1), we deduce the following formula

$$B_N(t,\theta) = \frac{(-1)^{\frac{s-1}{q-1}}}{\widetilde{\pi}} \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(t)^N}{a} \prod_{i=1}^{s} \omega\big(t^{q^{e_i}}\big). \tag{4.1}$$

Note that

$$\widetilde{\pi}^{-1} \prod_{i=1}^{s} \omega\big(t^{q^{e_i}}\big) \in \mathbb{F}_p\left(\left(\frac{1}{\theta}\right)\right)[[t]]$$

and

$$\sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(t)^N}{a} \in \mathbb{F}_p[t]\left[\left[\frac{1}{\theta}\right]\right].$$

Therefore, we obtain $B_N(t,\theta) \in \mathbb{F}_p[t,\theta]$. Thus the assertion 1) follows immediately.

2) This is a consequence of the definition of $B_N(t,\theta)$.

3) Since $N \equiv 0 \pmod{p}$, from the formula (4.1), we deduce

$$B_N(t,\theta) \in \mathbb{F}_q[t^p]\left(\left(\frac{1}{\theta}\right)\right).$$

By the assertion 1), we know that $B_N(t,\theta) \in \mathbb{F}_p[t,\theta]$. Hence we get

$$B_N(t,\theta) \in \mathbb{F}_q[t^p]\left(\left(\frac{1}{\theta}\right)\right) \cap \mathbb{F}_p[t,\theta] = \mathbb{F}_p[t^p,\theta].$$

4) Let $\zeta \in \mathbb{F}_q$. By [3, Theorem 2.9], we have

$$\omega(t)\mid_{t=\zeta} = \exp_C\left(\frac{\widetilde{\pi}}{\theta-\zeta}\right),$$

where $\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is the Carlitz exponential introduced in the Introduction (see [11, Chapter 3, Paragraph 3.2]). Now, by [13, Theorem 1], we get

$$\sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(t)^N}{a}\mid_{t=\zeta} = \sum_{d\geq 0} \sum_{a\in A_{+,d}} \frac{a(\zeta)}{a} = \frac{\widetilde{\pi}}{(\theta-\zeta)\exp_C\left(\frac{\widetilde{\pi}}{\theta-\zeta}\right)}.$$

Note that

$$\exp_C\left(\frac{\widetilde{\pi}}{\theta-\zeta}\right)^{q-1} = -(\theta-\zeta).$$

Thus we obtain

$$B_N(t,\theta)\mid_{t=\zeta}=(\theta-\zeta)^r. \tag{4.2}$$

Now, we will calculate $\frac{d}{dt}B_N(t,\theta)\mid_{t=\zeta}$. We have expanded $N=\sum_{i=1}^s q^{e_i}$ in base q. We set

$$n_0=\mathrm{Card}\{1\le i\le s, e_i=0\}.$$

Note that $0\le n_0\le q-1$ and $n_0\equiv N \pmod q$. By (4.1), we get

$$\frac{d}{dt}B_N(t,\theta)=\frac{n_0(-1)^{\frac{s-1}{q-1}}}{\widetilde{\pi}}\prod_{i=1}^s\omega\big(t^{q^{e_i}}\big)\sum_{d\ge0}\sum_{a\in A_{+,d}}\left(\frac{\frac{d}{dt}(a(t))a(t)^{N-1}}{a}-\frac{a(t)^N}{a}\frac{\frac{d}{dt}(\omega(t))}{\omega(t)}\right).$$

Observe that

$$\begin{aligned}&\sum_{d\ge0}\sum_{a\in A_{+,d}}\left(\frac{\frac{d}{dt}(a(t))a(t)^{N-1}}{a}-\frac{a(t)^N}{a}\frac{\frac{d}{dt}(\omega(t))}{\omega(t)}\right)\mid_{t=\zeta}\\ =&\sum_{d\ge0}\sum_{\substack{a\in A_{+,d}\\ a(\zeta)\neq0}}\left(\frac{\frac{d}{dt}(a(t))}{a}-\frac{a(t)}{a}\frac{\frac{d}{dt}(\omega(t))}{\omega(t)}\right)\mid_{t=\zeta}.\end{aligned}$$

In particular, the latter expression does not depend on N. We claim that it vanishes. In fact, take $N_0=1+(q-1)q$. Then $\ell_q(N_0)=q$. By Lemma 3.4, we have $B_{N_0}(t,\theta)=1$. It implies

$$\frac{d}{dt}B_{N_0}(t,\theta)\mid_{t=\zeta}=0.$$

As a consequence of the previous discussion, we obtain

$$\sum_{d\ge0}\sum_{\substack{a\in A_{+,d}\\ a(\zeta)\neq0}}\left(\frac{\frac{d}{dt}(a(t))}{a}-\frac{a(t)}{a}\frac{\frac{d}{dt}(\omega(t))}{\omega(t)}\right)\mid_{t=\zeta}=0.$$

It follows that we always have

$$\frac{d}{dt}B_N(t,\theta)\mid_{t=\zeta}=0 \quad \text{for all } \zeta\in\mathbb{F}_q. \tag{4.3}$$

We are now ready to prove the assertion (4). From (4.2), we deduce that

$$B_N(t,\theta)\equiv(\theta-t)^r \pmod{t^q-t}.$$

We write

$$B_N(t,\theta)=(\theta-t)^r+W(t^q-t)$$

for some polynomial $W\in\mathbb{F}_q[t,\theta]$. From (4.3), we deduce that

$$\frac{d}{dt}B_N(t,\theta)\equiv0 \pmod{t^q-t}.$$

It implies $-r(\theta-t)^{r-1}-W\equiv 0 \pmod{t^q-t}$ and

$$W\equiv -r(\theta-t)^{r-1} \pmod{t^q-t}.$$

Thus we get the assertion (4), *i.e.*

$$B_N(t,\theta)\equiv(\theta-t)^r-r(\theta-t)^{r-1}(t^q-t) \pmod{(t^q-t)^2\mathbb{F}_p[t,\theta]}.$$

The proof is finished. □

Lemma 4.2. *We keep the previous notation. Then the total degree in* t,θ *of* $B_N(t,\theta)$ *is smaller than* $\mathrm{Max}\{rN+r-1,0\}$. *Furthermore,* $B_N(t,\theta)$ *(as a polynomial in* t*) is a primitive polynomial.*

Proof. Recall that if $r=0$, then $B_N(t,\theta)=1$ and we are done. We can assume that $r\geq 1$. By Lemma 4.1 (4), we have

$$B_N(t,0)\equiv -(-t)^{r-1}(t+r(t^q-t)) \pmod{(t^q-t)^2\mathbb{F}_p[t]}.$$

In particular, $\deg_t B_N(t,\theta)\geq p$.

Let $x\in\mathbb{C}_\infty$ be such that $v_\infty(x)>-\frac{1}{N}$. then for any $a\in A_{+,d}$, we get

$$v_\infty\left(a(x)^N/a\right)=Nv_\infty(a(x))+d=d(Nv_\infty(x)+1).$$

Since $v_\infty(x)>-\frac{1}{N}$, it follows

$$\lim_{\deg a\to+\infty} v_\infty\left(\frac{a(x)^N}{a}\right)=+\infty.$$

Hence, the sum $\sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(x)^N}{a}$ converges to an element of $\mathbb{C}_\infty^\times$. It follows that

$$\sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(x)^N}{a}=\prod_{\substack{P\in A_+\\ P\text{ irreducible}}}\left(1-\frac{P(x)^N}{P}\right)^{-1}\in\mathbb{C}_\infty^\times.$$

By (4.1), we have

$$B_N(t,\theta)\,|_{t=x}=\frac{(-1)^{\frac{s-1}{q-1}}}{\widetilde{\pi}}\sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(x)^N}{a}\prod_{i=1}^{s}\omega\big(t^{q^{e_i}}\big)\,|_{t=x}\,.$$

Since

$$\omega\big(t^{q^{e_i}}\big)\,|_{t=x}\in\mathbb{C}_\infty^\times \quad \text{for all } 1\leq i\leq s,$$

we obtain

$$B_N(t,\theta)\,|_{t=x}\neq 0.$$

Hence, we have shown that if $x\in\mathbb{C}_\infty$ is a root of $B_N(t,\theta)$, then $v_\infty(x)\leq -\frac{1}{N}<0$.

We set $m = \deg_t B_N(t,\theta)$. Recall that

$$B_N(t,\theta) = \theta^r + \sum_{i=1}^{m} a_i t^i$$

with $a_i \in \mathbb{F}_p[\theta]$ and $\deg_\theta a_i < r$. It implies the following expression in $\mathbb{C}_\infty[t]$

$$B_N(t,\theta) = \lambda \prod_{j=1}^{m} (t - x_j)$$

with $\lambda \in \mathbb{F}_p[\theta] \setminus \{0\}$ such that $\deg_\theta \lambda \leq r-1$ and $x_1, \dots, x_m \in \mathbb{C}_\infty$. Thus we get

$$\theta^r = (-1)^m \lambda \prod_{j=1}^{m} x_j.$$

We have proved that for all $1 \leq j \leq m$, $v_\infty(x_j) \leq -\frac{1}{N}$. Therefore

$$\deg_\theta \lambda - r \leq -\frac{m}{N}.$$

We finally obtain

$$m = \deg_t B_N(t,\theta) \leq (r - \deg_\theta \lambda)N \leq rN.$$

Since $B_N(t,\theta)$ is a monic polynomial in θ, the total degree in t, θ of $B_N(t,\theta)$ is smaller than its degree in t plus $r-1$ which is less than or equal to $rN + r - 1$.

To conclude, observe by Lemma 4.1 (4) that $B_N(1,\theta) = (\theta-1)^r$. Now write $B_N(t,\theta) = \alpha F$ with $\alpha \in \mathbb{F}_p[\theta]$ non zero and $F \in \mathbb{F}_p[\theta][t]$ primitive. Then it follows immediately that $\alpha \mid \theta^r$ and $\alpha \mid (\theta-1)^r$. Since $r \geq 1$, $\alpha \in \mathbb{F}_p^\times$. The proof is finished. □

4.2. Relations with Bernoulli-Carlitz numbers

Recall that in the Introduction, for $n \in \mathbb{N}$, we have introduced the coefficients D_n, the Carlitz factorials $\Pi(n)$ and the Bernoulli-Carlitz numbers BC_n. We refer the interested reader to [11, Chapter 9] for more details.

Proposition 4.3. *We keep the previous notation.*

1) *Let $d \geq 1$ be such that $q^d > N$, then we have the following equality in $\mathbb{C}_\infty$*

$$\frac{B_N(\theta, \theta^{q^d})}{\prod_{i=1}^{s} \prod_{n=0, n \neq e_i}^{d-1} (\theta^{q^{e_i}} - \theta^{q^n})} = (-1)^{\frac{s-q}{q-1}} \frac{BC_{q^d - N}}{\Pi(N)\Pi(q^d - N)};$$

2) *Let P be a monic irreducible polynomial in A of degree $d \geq 1$ such that $q^d > N$. Then*

$$BC_{q^d-N} \equiv 0 \pmod{P} \Longleftrightarrow B_N(\theta,\theta) \equiv 0 \pmod{P}.$$

Proof. 1) This assertion is a consequence of the proof of [2, Theorem 2]. For the convenience of the reader, we give a proof of this result. We will use the notation of Section 3. Recall that $\tau : \mathbb{T}_s \to \mathbb{T}_s$ is the homomorphism of $\mathbb{F}_q[\underline{t}_s]$-algebras such that $\tau(x) = x^q$ for all $x \in \mathbb{C}_\infty$. Since $\lambda_\theta^q = -\theta\lambda_\theta$, we have

$$\tau(\omega(t_i)) = (t_i - \theta)\omega(t_i) \quad \text{for all } i = 1, \ldots, s.$$

Applying τ^d to the equation (3.1), we obtain

$$\frac{\tau^d(\mathbb{B}_s)}{\prod_{n=0}^{d-1}(t_1 - \theta^{q^n}) \cdots (t_s - \theta^{q^n})} = (-1)^{\frac{s-1}{q-1}} \frac{\tau^d(L_s(\underline{t}_s))}{\widetilde{\pi}^{q^d}} \omega(t_1) \ldots \omega(t_s)$$

or equivalently,

$$\frac{\tau^d(\mathbb{B}_s)(t_1 - \theta^{q^{e_1}}) \cdots (t_s - \theta^{q^{e_s}})}{\prod_{n=0}^{d-1}(t_1 - \theta^{q^n}) \cdots (t_s - \theta^{q^n})}$$
$$= (-1)^{\frac{s-1}{q-1}} \frac{\tau^d(L_s(\underline{t}_s))}{\widetilde{\pi}^{q^d}} (t_1 - \theta^{q^{e_1}})\omega(t_1) \ldots \left(t_s - \theta^{q^{e_s}}\right)\omega(t_s).$$

Now, we specialize to $t_i = t^{q^{e_i}}$ for $1 \leq i \leq s$. Recall that, by (4.1), we have

$$\mathbb{B}_s \mid_{t_i = t^{q^{e_i}}} = B_N(t,\theta).$$

Moreover, by [13], formula (24), for $1 \leq i \leq s$, we have

$$\left(t_i - \theta^{q^{e_i}}\right)\omega(t_i) \mid_{t_i = \theta^{q^{e_i}}} = -\frac{\widetilde{\pi}^{q^{e_i}}}{D_{e_i}}.$$

Putting altogether, we finally obtain

$$\frac{B_N(t, \theta^{q^d})}{\prod_{i=1}^{s} \prod_{n=0, n \neq e_i}^{d-1} (t^{q^{e_i}} - \theta^{q^n})} \mid_{t=\theta} = (-1)^{\frac{s-q}{q-1}} \frac{BC_{q^d-N}}{\Pi(N)\Pi(q^d - N)}.$$

2) The assertion follows from the fact that

$$B_N(\theta,\theta) \equiv B_N\left(\theta, \theta^{q^d}\right) \pmod{P}.$$

□

4.3. Proof of Theorem C

We prove a slightly stronger version of Theorem C.

Theorem 4.4. *For all* $a \in \overline{\mathbb{F}}_q[\theta]$, *we have*

$$B_N(t,\theta)\mid_{t=a} \neq 0.$$

Proof. By Proposition 3.7, we have

$$B_N(t,\theta) - \theta^r \in t(t,\theta)^r$$

where (t,θ) denotes the ideal of $\overline{\mathbb{F}}_q[t,\theta]$ generated by t and θ. The theorem follows immediately. □

4.4. Proof of Theorem B

For $s = 1$, *i.e.*, $\ell_q(N) = 1$, the result is well-known and is a consequence of [11, Lemma 8.22.4]. Thus, we can assume that $s = \ell_q(N) \geq q$. Recall that $r = \frac{s-q}{q-1}$. Then $r \geq 0$ and $r < s \leq N$. By Lemma 4.2, the total degree in t, θ of $B_N(t,\theta)$ is at most $\mathrm{Max}\{rN + r - 1, 0\}$, hence strictly less than $(r+1)N$. In particular,

$$\deg_\theta B_N(\theta,\theta) < (r+1)N.$$

Now, by Theorem 4.4,

$$B_N(\theta,\theta) \neq 0.$$

Since P is a monic irreducible polynomial of degree d in A such that

$$d = \deg_\theta P \geq \frac{\ell_q(N) - 1}{q-1} N = (r+1)N,$$

we get

$$B_N(\theta,\theta) \not\equiv 0 \pmod{P}.$$

Hence, the theorem follows from Proposition 4.3 2) since $q^d > N$. The proof is finished.

5. The L-series $L_N(t)$

5.1. The L-series $L_N(t)$

Let $N \geq 1$ be an integer. We set

$$S_{d,N}(t) = \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \in K_\infty[t], \quad d \geq 0,$$

and

$$L_N(t) = \sum_{d \geq 0} S_{d,N}(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \in \mathbb{T}_t^\times.$$

Lemma 5.1. *For* $d \geq [\frac{\ell_q(N)}{q-1}] + 2$, *we have*

$$v_\infty(S_{d,N}(t)) \geq d + q^{d-[\frac{\ell_q(N)}{q-1}]-1}.$$

Proof. Let d be an integer such that $d \geq [\frac{\ell_q(N)}{q-1}] + 2$. Following D. Goss, for a monic polynomial a in A, we set

$$\langle a \rangle_\infty = \frac{a}{\theta^{\deg_\theta a}} \in 1 + \frac{1}{\theta}\mathbb{F}_q\left[\left[\frac{1}{\theta}\right]\right].$$

It follows

$$S_{d,N}(t) = \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} = \frac{1}{\theta^d} \sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{-1}.$$

We write

$$m = d - 2 - \left[\frac{\ell_q(N)}{q-1}\right] \in \mathbb{N},$$

and we set

$$y_m = \sum_{n=0}^{m} (q-1)q^n.$$

Then

$$y_m \equiv -1 \pmod{q^{m+1}} \quad \text{and} \quad \ell_q(y_m) = (m+1)(q-1).$$

Therefore, we obtain

$$v_\infty\left(\sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{-1} - \sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{y_m}\right) \geq q^{m+1}. \tag{5.1}$$

Now, we will use the following elementary fact ([2, Lemma 4], see also [11, Lemma 8.8.1]).

Fact. Let $s \geq 1$ be an integer and let $t_1, \ldots, t_s$ be s variables over $\mathbb{F}_q$. If d is an integer such that $d(q-1) > s$, then we have

$$\sum_{a \in A_{+,d}} a(t_1) \cdots a(t_s) = 0.$$

Since $\ell_q(N) + \ell_q(y_m) = \ell_q(N) + (m+1)(q-1) < d(q-1)$, the previous claim implies

$$\sum_{a \in A_{+,d}} a(t)^N a^{y_m} = 0. \tag{5.2}$$

From (5.1) and (5.2), we deduce that

$$v_\infty\left(\sum_{a\in A_{+,d}} a(t)^N\langle a\rangle_\infty^{-1}\right)\geq q^{m+1}.$$

We finally obtain

$$v_\infty(S_{d,N}(t))=v_\infty\left(\frac{1}{\theta^d}\sum_{a\in A_{+,d}} a(t)^N\langle a\rangle_\infty^{-1}\right)\geq d+q^{m+1}=d+q^{d-[\frac{\ell_q(N)}{q-1}]-1}$$

as required. □

Corollary 5.2. *The function $L_N(t)$ is an entire function in $\mathbb{C}_\infty$.*

Proof. Since $\deg_t(S_{d,N}(t))\leq Nd$, this corollary is an immediate consequence of the above lemma. □

Lemma 5.3. *Let $j\in\mathbb{Z}$. Then $L_N(t)\mid_{t=\theta^{q^j}}=0$ if and only if $N\equiv 1 \pmod{q-1}$, and $q^jN>1$.*

Proof. Because of Lemma 5.1, we have

$$L_N(t)\mid_{t=\theta^{q^j}}=\sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{a(t)^N\mid_{t=\theta^{q^j}}}{a}=\sum_{d\geq 0}\sum_{a\in A_{+,d}} a^{Nq^j-1}.$$

The lemma follows immediately from the following facts:

1) For $n<0$, we always have $\sum_{d\geq 0}\sum_{a\in A_{+,d}}\frac{1}{a^n}\neq 0$;
2) For $n\geq 0$, $\sum_{d\geq 0}\sum_{a\in A_{+,d}} a^n=0$ if and only if $n\geq 1, n\equiv 0 \pmod{q-1}$. □

5.2. Basic sums

Let $d\geq 1$ be an integer. For a tuple $\mathbf{k}=(k_0,\dots,k_{d-1})\in\mathbb{N}^d$, we set

$$\begin{aligned}\ell(\mathbf{k})&=d\in\mathbb{N}^*,\\ |\mathbf{k}|&=k_0+\cdots+k_{d-1}\in\mathbb{N},\\ w(\mathbf{k})&=k_{d-1}+\cdots+(d-1)k_1+dk_0\in\mathbb{N},\\ C_{\mathbf{k}}&=\frac{|\mathbf{k}|!}{k_0!\cdots k_{d-1}!}\in\mathbb{F}_p.\end{aligned}$$

If $a=a_0+a_1\theta+\cdots+a_{d-1}\theta^{d-1}+\theta^d$ with $a_i\in\mathbb{F}_q$, then we write

$$a^{\mathbf{k}}=\prod_{i=0}^{d-1}a_i^{k_i}$$

with $0^0 := 1$. Finally, we will attach to the above tuple $\mathbf{k}$ another d-tuple

$$\overline{\mathbf{k}} = (\overline{k}_0, \dots, \overline{k}_{d-1}) \in \{0, \dots, q-1\}^d$$

defined as follows: for $0 \leq i \leq d-1$, $\overline{k}_i \in \{0, \dots, q-1\}$ is the least integer such that $k_i + \overline{k}_i \in (q-1)\mathbb{N}^*$.

Now we write

$$L_N(t) = \sum_{i \geq 0} \alpha_{i,N}(t)\theta^{-i}, \qquad \text{with } \alpha_{i,N}(t) \in \mathbb{F}_q[t]. \tag{5.3}$$

It is immediate that $\alpha_{0,N}(t) = 1$. We give an explicit expression for the polynomials $\alpha_{i,N}(t)$ in the following lemma.

Lemma 5.4. *For $i \geq 0$, we have*

$$\alpha_{i,N}(t) = \sum_{\ell(\mathbf{k})+w(\mathbf{k})=i} (-1)^{|\mathbf{k}|} C_{\mathbf{k}} \sum_{a \in A_{+,\ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}.$$

Proof. Let $a \in A_{+,d}$. We expand

$$\frac{1}{a} = \frac{1}{\theta^d} \sum_{\mathbf{k} \in \mathbb{N}^d} (-1)^{|\mathbf{k}|} C_{\mathbf{k}} a^{\mathbf{k}} \frac{1}{\theta^{w(\mathbf{k})}}.$$

It follows that

$$\sum_{a \in A_{+,d}} \frac{a(t)^N}{a} = \frac{1}{\theta^d} \sum_{\mathbf{k} \in \mathbb{N}^d} (-1)^{|\mathbf{k}|} C_{\mathbf{k}} \frac{1}{\theta^{w(\mathbf{k})}} \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}}.$$

We finally get

$$\alpha_{i,N}(t) = \sum_{\ell(\mathbf{k})+w(\mathbf{k})=i} (-1)^{|\mathbf{k}|} C_{\mathbf{k}} \sum_{a \in A_{+,\ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}.$$

The proof is finished. □

We analyze the *basic sums*

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \tag{5.4}$$

which appear in the previous expressions $\alpha_{i,n}(t)$. We see that

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = \sum_{a \in A_{+,d}} \sum_{\substack{\mathbf{m} \in \mathbb{N}^{d+1} \\ |\mathbf{m}|=N}} C_{\mathbf{m}} a^{\mathbf{k}} a^{\mathbf{m}} t^{m_1+2m_2+\cdots+dm_d}.$$

Recall that by Lucas' theorem, $C_{\mathbf{m}} \neq 0$ if and only if the sum $N = m_0 + \cdots + m_d$ has no carryover in base p. Furthermore, for $n \in \mathbb{N}$, we see immediately that

$$\sum_{\lambda \in \mathbb{F}_q} \lambda^n \neq 0 \Leftrightarrow n \in (q-1)\mathbb{N}^*.$$

Thus for $\mathbf{m} \in \mathbb{N}^{d+1}$, we have

$$\sum_{a \in A_{+,d}} a^{\mathbf{m}} = \begin{cases} (-1)^d & \text{if } (m_0, \dots, m_{d-1}) \in ((q-1)\mathbb{N}^*)^d \\ 0 & \text{otherwise.} \end{cases}$$

Following J. Sheats [15, Section 1], for $d \geq 1$, $\mathbf{k} \in \{0, \dots, q-1\}^d$, we denote by $U_d(N, \mathbf{k})$ the set of tuples $\mathbf{m} \in \mathbb{N}^{d+1}$ such that:

1) There is no carryover of p-digits in the sum $N = m_0 + \cdots + m_d$;
2) For $n = 0, \dots, d-1$, we have $m_n - k_n \in (q-1)\mathbb{N}$.

For $\mathbf{m} \in U_d(N, \mathbf{k})$, we set

$$\deg \mathbf{m} = m_1 + 2m_2 + \cdots + dm_d.$$

An element $\mathbf{m} \in U_d(N, \mathbf{k})$ is called *optimal* if

$$\deg \mathbf{m} = \mathrm{Max}\{\deg \mathbf{n}, \mathbf{n} \in U_d(N, \mathbf{k})\}.$$

If $U_d(N, \mathbf{k}) \neq \emptyset$, the *greedy element* of $U_d(N, \mathbf{k})$ is the element $\mathbf{m} = (m_0, \dots, m_d) \in U_d(N, \mathbf{k})$ such that $(m_d, \dots, m_1)$ is largest lexicographically.

To summarize, we have shown that:

Lemma 5.5. *With the previous notation, we have*

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = (-1)^d \sum_{\mathbf{m} \in U_d(N, \overline{\mathbf{k}})} C_{\mathbf{m}} t^{\deg \mathbf{m}}$$

where $\overline{\mathbf{k}}$ is the d-tuple attached to $\mathbf{k}$ *defined at the beginning of this section.*

5.3. An example: the series $L_1(t)$

For the convenience of the reader, we treat a basic example: $N = 1$. We keep the previous notation. We will write $S_d(t)$ instead of $S_{d,1}(t)$. Thus we get

$$L_1(t) = \sum_{d \geq 0} S_d(t)$$

where

$$S_d(t) = \sum_{a \in A_{+,d}} \frac{a(t)}{a} \in K[t].$$

We set

$$\ell_0 = 1,$$
$$\ell_d = (\theta - \theta^q)\cdots\left(\theta - \theta^{q^d}\right), \quad d \geq 1.$$

Lemma 5.6. *For $d \geq 0$, we have*

$$\sum_{a \in A_{+,d}} \frac{1}{a} = \frac{1}{\ell_d}.$$

Proof. This is a well-known consequence of a result of Carlitz [11, Theorem 3.1.5]. The equality is immediate for $d = 0$. We can assume that $d \geq 1$. We put

$$e_d(X) = \prod_{\substack{a \in A \\ \deg_\theta a < d}} (X - a) \in A[X].$$

By [11, Theorem 3.1.5], we have

$$e_d(X) = \sum_{i=0}^{d} \frac{D_d}{D_i \ell_{d-i}^{q^i}} X^{q^i},$$

where $D_0 = 1$, and for $i \geq 1$, $D_i = (\theta^{q^i} - \theta) D_{i-1}^q$. Now observe that

$$\frac{\frac{d}{dX}\left(e_d\left(X - \theta^d\right)\right)}{e_d\left(X - \theta^d\right)} \mid_{X=0} = -\sum_{a \in A_{+,d}} \frac{1}{a}.$$

Since $e_d(\theta^d) = D_d$ [11, Corollary 3.1.7], we get the desired result. □

Lemma 5.7. *For $d \geq 0$, we have*

$$S_d(t) = \frac{(t - \theta)\cdots\left(t - \theta^{q^{d-1}}\right)}{\ell_d}.$$

Proof. The equality is immediate for $d = 0$. We can assume that $d \geq 1$. Recall that

$$S_d(t) = \sum_{a \in A_{+,d}} \frac{a(t)}{a} \in K[t].$$

Then for $i = 0, \ldots, d-1$, it is immediate that

$$S_d(t) \mid_{t=\theta^{q^i}} = 0.$$

Furthermore, by Lemma 5.6, $S_d(t)$ has degree d in t and the coefficient of t^d is $\frac{1}{\ell_d}$. The lemma follows. □

We write

$$L_1(t) = \sum_{d \geq 0} \alpha_d t^d \in K_\infty[[t]] \quad \text{with } \alpha_d \in K_\infty.$$

Recall that the Newton polygon for $L_1(t)$ is the lower convex hull in $\mathbb{R}^2$ of the points

$$(d, v_\infty(\alpha_d))_{d \geq 0}.$$

Its sides determine the valuation of the entire series $L_1(t)$: if the Newton polygon of $L_1(t)$ has a side of slope α whose projection onto the horizontal axis has length k, then $L_1(t)$ has precisely k zeros (counted with multiplicity) with valuation $-\alpha$.

Lemma 5.8. *The edge points of the Newton polygon of $L_1(t)$ are $(d, q\frac{q^d-1}{q-1})$ with $d \in \mathbb{N}$.*

Proof. Let $d \in \mathbb{N}$. We claim that

$$v_\infty(\alpha_d) = q\frac{q^d - 1}{q - 1}.$$

In fact, for any $d' \geq 0$, we denote by $\alpha_d(d') \in K_\infty$ the coefficient of t^d in the polynomial $S_{d'}(t) \in K_\infty[t]$. It follows that

$$\alpha_d = \sum_{d' \geq 0} \alpha_d(d').$$

By Lemma 5.7, we get

1) For $d' < d$, we get $\alpha_d(d') = 0$;
2) For $d' = d$, we get $\alpha_d(d) = \frac{1}{l_d}$ and the

$$v_\infty(\alpha_d(d)) = v_\infty\left(\frac{1}{l_d}\right) = q\frac{q^d - 1}{q - 1};$$

3) For $d' > d$, we get the following inequality

$$v_\infty(\alpha_d(d')) \geq -v_\infty(\ell_{d'}) - q^d - \cdots - q^{d'-1} > q\frac{q^d - 1}{q - 1}.$$

Our claim is proved and the lemma follows immediately. □

Recall that $\lambda_\theta \in \mathbb{C}_\infty^\times$ is a fixed $(q-1)$-th root of $-\theta$. As a consequence of Lemma 5.8, we obtain the following formula due to F. Pellarin [13, Theorem 1].

Proposition 5.9. *Recall that*

$$\widetilde{\pi} = \lambda_\theta \theta \prod_{j\geq 1} \left(1 - \theta^{1-q^j}\right)^{-1} \in \mathbb{C}_\infty^\times.$$

Then we have

$$(\theta - t)L_1(t) = \frac{\widetilde{\pi}}{\lambda_\theta} \prod_{j\geq 0} \left(1 - \frac{t}{\theta^{q^j}}\right).$$

Proof. By Lemma 5.8, the entire function $(t-\theta)L_1(t)$ has simple zeros in K_∞ whose valuations lie in the set $\{-q^j, j \in \mathbb{N}\}$. By Lemma 5.3,

$$L_1(t) \mid_{t=\theta^{q^j}} = 0 \quad \text{for all } j \geq 1.$$

Thus there exists $\alpha \in \mathbb{C}_\infty^\times$ such that

$$(t-\theta)L_1(t) = \alpha \prod_{j\geq 0} \left(1 - \frac{t}{\theta^{q^j}}\right).$$

We observe that

$$\frac{\widetilde{\pi}}{\lambda_\theta} \prod_{j\geq 1} \left(1 - \frac{t}{\theta^{q^j}}\right) \mid_{t=\theta} = \theta$$

and

$$L_1(\theta) = 1.$$

Hence we conclude

$$\alpha = -\frac{\widetilde{\pi}}{\lambda_\theta}.$$

The proof is finished. □

5.4. Exceptional zeros of the series $L_N(t)$

For the rest of this section, we will suppose that $N \geq 2$ and $N \equiv 1 \pmod{q-1}$.

We recall that $\ell_q(N)$ denotes the sum of the digits of the expansion in base q of N. We assume that $\ell_q(N) \geq 2$. We set

$$s = \ell_q(N),$$

then $s \geq 2$ and $s \equiv 1 \pmod{q-1}$. We put

$$r = \frac{s-q}{q-1} \in \mathbb{N}.$$

We write N in base q

$$N = \sum_{i=0}^{k} n_i q^i \quad \text{with } 0 \leq n_i \leq q-1 \text{ and } n_k \neq 0. \tag{5.5}$$

Observe that

$$\sum_{i=0}^{k} n_i = s$$

and

$$q^k < N < q^{k+1}.$$

We will often use the second expression of this expansion

$$N = \sum_{i=1}^{s} q^{e_i}, \tag{5.6}$$

with $e_i \in \mathbb{N}$, $e_1 \leq e_2 \leq \cdots \leq e_s = k$. We remark that each identical exponent appears at most $q-1$ times in the above expansion of N.

By Lemma 5.3, the elements of the set

$$S_N = \left\{\theta^{q^j},\, j \geq -k\right\}$$

are zeros of the entire series $L_N(t)$. We call them the *trivial zeros* of $L_N(t)$. The other zeros are called *exceptional zeros* of $L_N(t)$. It turns out (Proposition 5.10) that they are intimately connected to several variable polynomials $\mathbb{B}_s$ and their specializations $B_N(t,\theta)$ which are introduced in [2] (see also [4]).

By (3.1), we get

$$B_N(t,\theta)\frac{(-1)^{r+1}\widetilde{\pi}}{\prod_{i=1}^{s}\omega(t^{q^{e_i}})} = L_N(t). \tag{5.7}$$

From this relation, we get immediately:

Proposition 5.10. *Each exceptional zero of $L_N(t)$ is a root of $B_N(t,\theta)$, with the same multiplicity. In particular, the set of exceptional zeros counted with multiplicity is finite.*

We recall (Section 4.4) that Theorem B is a direct consequence of the fact that $t = \theta$ is not a root of $B_N(t,\theta)$. Further, Theorem 4.4 implies that for $i \in \mathbb{N}$, $t = \theta^{q^i}$ is not a root of $B_N(t,\theta)$. Hence it is natural to ask whether all the roots of $B_N(t,\theta)$ (as a polynomial in t) are exceptional zeros of $L_N(t)$. It is tempting to make a slightly stronger conjecture which is Conjecture D in the Introduction.

Conjecture 5.11. The polynomial $B_N(t,\theta)$ in the variable t has no zeros in the set $\{\theta^{q^i}, i \in \mathbb{Z}\}$.

Remark 5.12. When $l_q(N) = q$, Lemma 3.4 1) implies that $B_N(t,\theta) = 1$ and Conjecture 5.11 holds.

We will prove this conjecture when $q = p$ (Theorem 6.8). The case $q > p$ is more subtle to handle. However, we are able to settle the conjecture when N is q-minimal, that is N satisfies certain combinatorial conditions (Theorem 7.7). Further, we present some numerical evidence to support our conjecture when N is no longer q-minimal.

6. Exceptional zeros of L-series: the case $q = p$

In this section we suppose that $q = p$. We will fix an integer $N \geq 2$ such that $N \equiv 1 \pmod{p-1}$ and $l_p(N) > p$ (see Remark 5.12).

We will use the notation of Section 5. For the convenience of the reader, we try to keep the text of this section as self-contained as possible. We refer the interested reader to [20, Paragraph 5.8], for a proof of the Riemann hypothesis for the Carlitz-Goss zeta function in the case $q = p$.

6.1. Preliminaries

We will use the second expression (5.6) of the expansion of N in base $q = p$

$$N = \sum_{i=1}^{s} p^{e_i}, \tag{6.1}$$

with $e_i \in \mathbb{N}$, $e_1 \leq e_2 \leq \cdots \leq e_s = k$. We will generalize several results in [10, Lemma 6.1 and Proposition 6.2].

Lemma 6.1. *Let $d \geq 1$ and $\overline{\mathbf{k}} = (\overline{k}_0, \dots, \overline{k}_{d-1}) \in \{0, \dots, p-1\}^d$. We assume that $|\overline{\mathbf{k}}| \leq \ell_p(N)$. We set*

$$\sigma_i = \begin{cases} 0 & \text{if } i = 0 \\ \sum_{n=0}^{i-1} \overline{k}_n & \text{if } 1 \leq i \leq d \\ \ell_p(N) & \text{if } i = d+1. \end{cases}$$

Denote by $\mathbf{m}(\overline{\mathbf{k}}) = (m_0, \dots, m_d) \in \mathbb{N}^{d+1}$ the element defined by

$$n = 0, \dots, d, \quad m_n = \begin{cases} \sum_{i=\sigma_n+1}^{\sigma_{n+1}} p^{e_i} & \text{if } \sigma_n < \sigma_{n+1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{m}(\overline{\mathbf{k}}) \in U_d(N, \overline{\mathbf{k}})$. Furthermore, $\mathbf{m}(\overline{\mathbf{k}})$ is the greedy element of $U_d(N, \overline{\mathbf{k}})$. In particular,

$$U_d(N, \overline{\mathbf{k}}) \neq \emptyset \text{ if and only if } |\overline{\mathbf{k}}| \leq \ell_p(N).$$

Proof. Observe that $\sigma_d = |\overline{\mathbf{k}}| \leq \ell_p(N)$. Thus $\mathbf{m}(\overline{\mathbf{k}})$ is well-defined. It is then straightforward to verify that $\mathbf{m}(\overline{\mathbf{k}}) \in U_d(N, \overline{\mathbf{k}})$ and that $\mathbf{m}(\overline{\mathbf{k}})$ is the greedy element of $U_d(N, \overline{\mathbf{k}})$.

To conclude, we have to show that if $U_d(N, \overline{\mathbf{k}}) \neq \emptyset$, then $|\overline{\mathbf{k}}| \leq \ell_p(N)$. Assume that $U_d(N, \overline{\mathbf{k}}) \neq \emptyset$. Let $\mathbf{m}' = (m'_0, \dots, m'_d) \in U_d(N, \overline{\mathbf{k}})$. Then

$$n = 0, \dots, d-1, \quad \ell_p\big(m'_n\big) \equiv \overline{k}_n \pmod{p-1}.$$

Since $0 \le \overline{k}_n \le p-1$, we deduce that

$$n = 0, \dots, d-1, \quad \ell_p\big(m'_n\big) \ge \overline{k}_n.$$

Thus

$$\ell_p(N) \ge \sum_{n=0}^{d-1} \ell_p\big(m'_n\big) \ge |\overline{\mathbf{k}}|.$$

The proof is finished. □

Proposition 6.2. *Let $d \ge 1$ and $\mathbf{k} \in \mathbb{N}^d$. We consider the tuple $\overline{\mathbf{k}}$ attached to $\mathbf{k}$ (see Section 5.2) and assume that $|\overline{\mathbf{k}}| \le \ell_p(N)$. Then $U_d(N, \overline{\mathbf{k}})$ contains a unique optimal element which is equal to the greedy element of $U_d(N, \overline{\mathbf{k}})$.*

In particular,

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \ne 0 \quad \textit{if and only if} \quad |\overline{\mathbf{k}}| \le \ell_p(N).$$

Proof. Let $\mathbf{m}(\overline{\mathbf{k}}) = (m_0, \dots, m_d) \in \mathbb{N}^{d+1}$ be the greedy element of $U_d(N, \overline{\mathbf{k}})$ defined in Lemma 6.1. Let $\mathbf{m}' = (m'_0, \dots, m'_d) \in U_d(N, \overline{\mathbf{k}})$ such that $\mathbf{m}' \ne \mathbf{m}(\overline{\mathbf{k}})$. We will show that $\mathbf{m}'$ is not optimal.

Write $c_n = \ell_p(m'_n)$ for $n = 0, \dots d-1$. Then for $n = 0, \dots, d-1$, we get

$$c_n \ge \overline{k}_n \text{ and } c_n \equiv \overline{k}_n \pmod{p-1}.$$

For $n = 0, \dots, d-1$, there exist $e_{n,1} \le \cdots \le e_{n,c_n}$ such that we can write in a unique way

$$m'_n = \sum_{i=1}^{c_n} p^{e_{n,i}}.$$

We distinguish two cases.

Case 1: There exists an integer j, $0 \le j \le d-1$, such that $c_j > \overline{k}_j$.

Let $\mathbf{m}'' = (m''_0, \dots, m''_d) \in \mathbb{N}^{d+1}$ be defined by

$$m''_n = \begin{cases} m'_n & \text{if } 0 \le n \le d-1, n \ne j \\ \sum_{i=1}^{\overline{k}_j} p^{e_{j,i}} & \text{if } n = j \\ N - m''_0 - \dots - m''_{d-1} & \text{if } n = d. \end{cases}$$

Then $\mathbf{m}'' \in U_d(N, \mathbf{k})$ and

$$\deg \mathbf{m}'' = \deg \mathbf{m}' + (d-j)(m'_j - m''_j) > \deg \mathbf{m}'.$$

Thus $\mathbf{m}'$ is not optimal.

Case 2: For $n = 0, \dots, d-1$, we have $c_n = \overline{k}_n$.

Let $j \in \{0, \dots d-1\}$ be the smallest integer such that $m'_j \neq m_j$. Then by the construction of $\mathbf{m}(\overline{\mathbf{k}})$, we have

$$m'_j > m_j.$$

Thus there exists an integer $l \in \mathbb{N}$ such that the number of times p^l appears in the sum of m'_j as $\overline{k}_j$ powers of p is strictly greater than the number of times it appears in the sum of m_j as $\overline{k}_j$ powers of p. Also, there exists an integer $v \in \mathbb{N}$ such that the number of times p^v appears in the sum of m_j as $\overline{k}_j$ powers of p is strictly greater than the number of times it appears in the sum of m'_j as $\overline{k}_j$ powers of p. Therefore, there exists an integer $t > j$ such that p^v appears in the sum of m'_t as $\ell_p(m'_t)$ powers of p. By the construction of $\mathbf{m}(\overline{\mathbf{k}})$, we can choose v and l such that $v < l$. Let $\mathbf{m}'' = (m''_0, \dots, m''_d) \in \mathbb{N}^{d+1}$ be defined by

$$m''_n = \begin{cases} m'_j - p^l + p^v & \text{if } n = j \\ m'_t - p^v + p^l & \text{if } n = t \\ m'_n & \text{otherwise.} \end{cases}$$

Then $\mathbf{m}'' \in U_d(N, \overline{\mathbf{k}})$ and

$$\deg \mathbf{m}'' = \sum_{n=0}^{d} n m''_n = \deg \mathbf{m}' + (t-j)\big(p^l - p^v\big) > \deg \mathbf{m}'.$$

Thus $\mathbf{m}'$ is not optimal. □

We have the following key result.

Proposition 6.3. *Let $d \geq 1$ such that $d(p-1) \leq \ell_p(N) - p$ and $\mathbf{k} \in \mathbb{N}^d$. Then*

$$N(d-1) < \deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \leq Nd.$$

Proof. It is clear that

$$\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \leq Nd.$$

Next, since $\mathbf{k} \in \mathbb{N}^d$, we get $|\overline{\mathbf{k}}| \leq d(p-1)$. Denote by $\mathbf{m}(\overline{\mathbf{k}})$ the greedy element of $U_d(N, \overline{\mathbf{k}})$ defined by Lemma 6.1. By Proposition 6.2, we have

$$\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = \sum_{n=0}^{d} n m_n = dN - \sum_{n=1}^{d} n m_{d-n}.$$

Recall that $s = \ell_p(N)$ and

$$N = \sum_{n=1}^{s} p^{e_n}$$

is the expansion of N in base q.

We claim that for $1 \leq n \leq d$,

$$m_{d-n} \leq (p-1)p^{e_{s-1}-n}.$$

In fact, fix $1 \leq n \leq d$. Since $d(p-1) \leq \ell_p(N) - p = s - p$, we get

$$\overline{k}_0 + \cdots + \overline{k}_{d-n} \leq (p-1)(d-n+1) \leq s - p - (p-1)(n-1).$$

By the above discussion, we deduce that

$$e_{\overline{k}_0+\cdots+\overline{k}_n} \leq e_{s-p-(p-1)(n-1)} \leq e_{s-1} - n.$$

It implies

$$m_{d-n} \leq \overline{k}_{d-n} p^{e_{\overline{k}_0+\cdots+\overline{k}_{d-n}}} \leq (p-1)p^{e_{s-1}-n}.$$

If $x \in \mathbb{R} \setminus \{1\}$, we have

$$\sum_{n=1}^{d} n x^{n-1} = \frac{1 - x^{d+1} + (d+1)(x-1)x^d}{(x-1)^2}.$$

It follows that

$$(p-1)\sum_{n=1}^{d} n p^{-n} = \frac{p - p^{-d} - (d+1)(p-1)p^{-d}}{p-1} < \frac{p}{p-1}.$$

Putting all together, we obtain

$$\begin{aligned}
\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} &= dN - \sum_{n=1}^{d} n m_{d-n} \\
&\geq dN - (p-1)p^{e_{s-1}} \sum_{n=1}^{d} n p^{-n} \\
&> dN - \frac{p}{p-1} p^{e_{s-1}} \\
&> (d-1)N.
\end{aligned}$$

The proof is finished. □

6.2. Newton polygon of truncated L-series

Recall that

$$N = \sum_{i=0}^{k} n_i p^i \quad \text{with } 0 \le n_i \le p-1 \text{ and } n_k \neq 0,$$

$$s = \ell_p(N) \quad \text{and} \quad r = \frac{s-p}{p-1} \in \mathbb{N}.$$

We consider the truncated L-series (truncation of (5.3)):

$$\Lambda_{r,N}(t) = \sum_{i=0}^{r} \alpha_{i,N}(t)\theta^{-i} \in K_\infty[t].$$

For $i \ge 0$, we set

$$S_i(N) = \sum_{a \in A_{+,i}} a(t)^N \in \mathbb{F}_p[t].$$

This is a particular case of the basic sums (5.4) with $\mathbf{k} = (\underbrace{0, \dots, 0}_{i \text{ times}}) \in \mathbb{N}^i$. By Proposition 6.2, we have

$$U_r\left(N, \underbrace{(p-1, \dots, p-1)}_{r \text{ times}}\right) \neq \emptyset.$$

Therefore, we have $S_i(N) \neq 0$ for $i = 0, \dots, r$.

Lemma 6.4.

1) *For* $i = 0, \dots, r-1$, *we have*

$$p^{k+1} > \deg_t S_{i+1}(N) - \deg_t S_i(N) > p^k;$$

2) *For* $i = 1, \dots, r-1$, *we have*

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) > \deg_t S_{i+1}(N) - \deg_t S_i(N).$$

Proof. We can assume that $r \ge 1$. Let $\mathbf{m} \in \mathbb{N}^{r+1}$ be the optimal element of $U_r(N, \underbrace{(p-1, \dots, p-1)}_{r \text{ times}})$ given by Lemma 6.1 and Proposition 6.2. For $i = 0, \dots, r$, let $\mathbf{m}(i) = (m_0, \dots, m_{i-1}, N - \sum_{n=0}^{i-1} m_n) \in \mathbb{N}^{i+1}$. Then again by Lemma 6.1 and Proposition 6.2, $\mathbf{m}(i)$ is the optimal element of $U_i(N, \underbrace{(p-1, \dots, p-1)}_{i \text{ times}})$.

Therefore,

$$\deg_t S_i(N) = \deg \mathbf{m}(i)$$

for $i = 0, \dots, r$. For $i = 0, \dots, r$, we have:

$$p^{k+1} > N - \sum_{n=0}^{i-1} m_n > n_k p^k.$$

Now, let $i \in \{0, \dots, r-1\}$, we have

$$p^{k+1} > \deg \mathbf{m}(i+1) - \deg \mathbf{m}(i) = N - \sum_{n=0}^{i} m_n > p^k. \tag{6.2}$$

Thus, we get the assertion 1).

Furthermore, we observe that for $i \in \{1, \dots, r-1\}$, we have:

$$\deg \mathbf{m}(i) - \deg \mathbf{m}(i-1) > \deg \mathbf{m}(i+1) - \deg \mathbf{m}(i).$$

We get the assertion 2) and the proof is finished. □

Proposition 6.5. *For $i = 0, \dots, r$, we have*

$$\deg_t \alpha_{i,N}(t) = \deg_t S_i(N).$$

Proof. We can assume that $r \geq 1$. By Proposition 6.3, for $i = 0, \dots, r$,

$$\operatorname{Max} \left\{ \deg_t \sum_{a \in A_{+,\ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}, w(\mathbf{k}) + \ell(\mathbf{k}) = i \right\}$$

is attained for a unique tuple $\mathbf{k}$ which is $(0, \dots, 0) \in \mathbb{N}^i$. It remains to apply Lemma 5.4 to finish the proof. □

Proposition 6.6. *The truncated L-series $\Lambda_{r,N}(t)$ is a polynomial in t of degree $\deg_t S_r(N)$. The edge points of its Newton polygon are*

$$(\deg_t S_i(N), i)$$

for $i = 0, \dots, r$. Furthermore, all the roots of $\Lambda_{r,N}(t)$ are of valuation strictly greater than $-p^{-k}$.

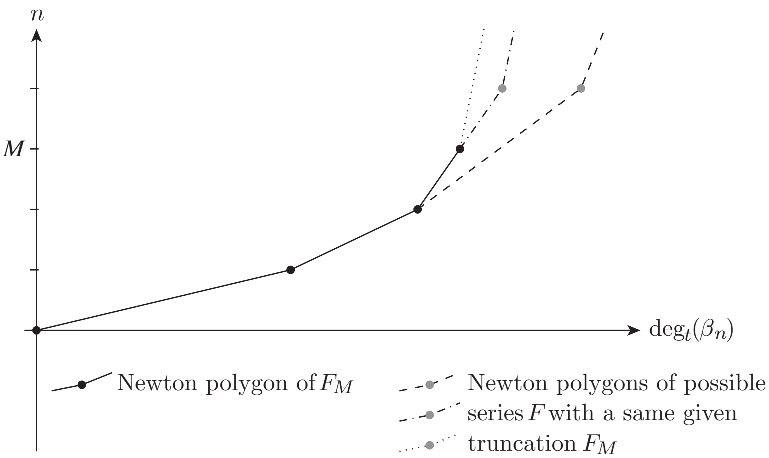
Proof. This is a consequence of Lemma 6.4 and Proposition 6.5. □

6.3. Newton polygon of the polynomial $B_N(t,\theta)$

In the following lemma, we do not suppose that $q = p$. We compare the Newton polygon of a truncated series with the truncation of the Newton polygon of this series.

Lemma 6.7. *Let* $F(t) = 1 + \sum_{n\geq 1} \beta_n(t)\frac{1}{\theta^n} \in \mathbb{F}_q[t][[\frac{1}{\theta}]]$, $\beta_n(t) \in \mathbb{F}_q[t]$, *and suppose that* $F(t)$ *converges on* $\mathbb{C}_\infty$. *Let* $M \geq 1$, *set* $F_M(t) = 1 + \sum_{n=1}^{M} \beta_n(t)\frac{1}{\theta^n} \in K[t]$, *let* $\rho \in \mathbb{R}$ *be the last slope of the Newton polygon of* $F_M(t)$ *and let* I_ρ *denote the multiset of the roots* ζ *of* F *of valuation* $v_\infty(\zeta) \geq -\rho$, *counted with multiplicity. If* $-\sum_{\zeta\in I_\rho} v_\infty(\zeta) \leq M$, *then all the edge points of the Newton polygon of* $F_M(t)$ *are edge points of the Newton polygon of* $F(t)$.

Proof. First, we remark that the Newton polygon of $F(t)$ and $F_M(t)$ is the lower convex hull of the points $(0,0)$ and $(\deg_t(\beta_n(t)), n)$, with $n \leq M$ for F_M.



If (w, n) is the end point of a side of slope α of the Newton polygon of F, then $-\sum v_\infty(\zeta) = n$ where the sum runs over all roots ζ of F of valuation $v_\infty(\zeta) \geq -\alpha$, counted with multiplicity. Thus, the hypothesis on the roots of F asserts that the end point of the last side of slope $\leq \rho$ of the Newton polygon of F is of the form $(\deg_t(\beta_{n_0}(t)), n_0)$, with $n_0 \leq M$.

Since the Newton polygon of F can only lie under the one of F_M,$(\deg_t(\beta_{n_0}(t)), n_0)$ is an edge point of the Newton polygon of F_M. Finally, if $n_0 < M$, then the next side of the Newton polygon of F is not a side of the one of F_M, its end point is then of the form $(\deg_t(\beta_{n_1}(t)), n_1)$ with $n_1 > M$, and of slope strictly less than ρ, which is a contradiction. □

We are now ready to prove the main theorem of this section.

Theorem 6.8.

1) *The polynomial* $B_N(t,\theta)$ *is of total degree* $\deg_t S_r(N)$ *in* t,θ *and has only one monomial of total degree* $\deg_t S_r(N)$, *which is* $t^{\deg_t S_r(N)}$;
2) *The edge points of the Newton polygon of* $\theta^{-r}B_N(t,\theta)$ *are*

$$(\deg_t S_i(N), i)$$

for $i=0,\dots,r$. *In other words, the polynomial* $\theta^{-r}B_N(t,\theta)$ *has the same Newton polygon as that of* $\Lambda_{r,N}(t)$;
3) *The polynomial* $B_N(t,\theta)$ *in the variable* t *has no zeros in* $\{\theta^{p^i}, i\in\mathbb{Z}\}$.

Remark 6.9. The assertion 3) proves Conjecture D when $q=p$.

Proof. First, we prove the assertion 2). By the formula (5.7), the roots of $L_N(t)$, are the roots of $B_N(t,\theta)$ and the roots of $\omega^{-1}(t^{p^{e_i}})$. The latter ones are all of the form $\theta^{p^{j-e_i}}$ with $j\geq 0$ and $e_i\leq k$, thus of valuation less than or equal to $-p^{-k}$. Thus, the roots of $L_N(t)$ of valuaton strictly greater than $-p^{-k}$ are all roots of $B_N(t,\theta)$. Therefore, we have $-\sum v_\infty(\zeta)\leq r$ where the sum runs over all roots ζ of $L_N(t)$ of valuation $v_\infty(\zeta)>-p^{-k}$, counted with multiplicity.

Combining Lemma 6.7 and Proposition 6.6, we get that the Newton polygon of $\Lambda_{r,N}(t)$ is a truncation of the Newton polygon of $L_N(t)$. As it describes the roots of $B_N(t,\theta)$, we obtain that the polynomial $\theta^{-r}B_N(t,\theta)$ has the same Newton polygon as that of $\Lambda_{r,N}(t)$. We get the assertion 2).

Second, the assertion 3) is then a consequence of Lemma 6.4: the roots of $B_N(t,\theta)$ have valuation w with $-p^{-k}<w<-p^{-(k+1)}$.

Finally, we prove the assertion 1). The total degree of $\beta_n(t)\theta^{r-n}$ is d where (d,r) is the end point of the segment starting at $(\deg_t\beta_n(t), n)$ and of slope 1. Since the slopes of the Newton polygon of $\theta^{-r}B_N(t,\theta)$ are all less than p^{-k}, the total degree of $B_N(t,\theta)$ is obtained for $n=r$. □

Corollary 6.10. *The polynomial* $B_N(t,\theta)$ *(viewed as a polynomial in* θ*) has* r *simple roots and all its roots are contained in* $\mathbb{F}_p((\frac{1}{t}))\setminus\{t^{p^i}, i\in\mathbb{Z}\}$.

Proof. As before, we write

$$\theta^{-r}B_N(t,\theta)=\sum_{i=0}^{r}\beta_i(t)\theta^{-i}\qquad\text{with } \beta_i(t)\in\mathbb{F}_p[t].$$

Observe that $\beta_0(t)=1$ and by Theorem 6.8,

$$\deg_t\beta_i(t)=\deg_t S_i(N),\qquad i=0,\dots,r.$$

By Lemma 6.4, we deduce that, for $i=1,\dots,r-1$,

$$\deg_t\beta_{i+1}(t)-\deg_t\beta_i(t)<\deg_t\beta_i(t)-\deg_t\beta_{i-1}(t).$$

Thus the edge points s of the Newton polygon of $\theta^{-r} B_N(t, \theta)$ viewed as a polynomial in $\frac{1}{\theta}$ are

$$(i, -\deg_t S_i(N)), \quad i = 0, \ldots, r.$$

We obtain the corollary. □

7. Exceptional zeros of L-series: toward the general case

In this section, q is no longer assumed to be equal to p.

7.1. The work of J. Sheats

Let $N \geq 1$ be an integer. For $d \geq 1$, we set

$$U_d(N) = U_d\left(N, \underbrace{(q-1, \ldots, q-1)}_{d \text{ times}}\right).$$

Recall (Section 5.2, [15], Section 1) that $U_d(N)$ is the set of tuples $\mathbf{m} \in \mathbb{N}^{d+1}$ such that

1) There is no carryover of p-digits in the sum $N = m_0 + \cdots + m_d$;
2) For $n = 0, \ldots, d-1$, we have $m_n \in (q-1)\mathbb{N}^*$.

Thus, by Lemma 5.5,

$$S_d(N) := \sum_{a \in A_{+,d}} a(t)^N = (-1)^d \sum_{\mathbf{m} \in U_d(N)} C_{\mathbf{m}} t^{\deg \mathbf{m}}.$$

J. Sheats proved ([15], Theorem 1.2 and Lemma 1.3) that if $U_d(N) \neq \emptyset$, $U_d(N)$ has a unique optimal element. Further, the optimal element is the greedy element of $U_d(N)$. In particular,

$$U_d(N) \neq \emptyset \Leftrightarrow S_d(N) \neq 0.$$

Observe that if $\mathbf{m} = (m_0, \ldots, m_d) \in U_d(N)$, then $(m_0, \ldots, m_{d-2}, m_{d-1} + m_d) \in U_{d-1}(N)$. In particular, $U_d(N) \neq \emptyset$ implies $U_{d-1}(N) \neq \emptyset$.

Proposition 7.1. *Let $d \geq 1$ such that $U_d(N) \neq \emptyset$. Then*

1) *For $i = 1, \ldots, d-1$,*

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) > \deg_t S_{i+1}(N) - \deg_t S_i(N);$$

2) *Further, if $N \equiv 0 \pmod{q-1}$ and if there is an element $(m_0, \ldots, m_d) \in U_d(N)$ such that $m_d \neq 0$. Then*

$$\deg_t S_d(N) > N(d-1);$$

3) *For* $i = 1, \dots, d$,

$$\deg_t S_d(N) \leq Nd.$$

Proof. 1) This assertion is a consequence of the proof of [15, Theorem 1.1], (see [15, pages 127 and 128]).

2) This assertion is proved in [15, Proposition 4.6].

3) This assertion is immediate. □

Lemma 7.2. *Let* $d \geq 1$ *such that* $U_{d+1}(N) \neq \emptyset$. *Let* $\mathbf{m} = (m_0, \dots, m_{d+1})$ *be the greedy element of* $U_{d+1}(N)$. *Then*

$$\deg_t S_d(N) > N(d-1),$$

and

$$\deg_t S_d(N) - \deg_t S_{d-1}(N) > m_{d+1}.$$

Proof. We set $\mathbf{m}' = (m_0, \dots, m_d) \in U_d(N - m_{d+1})$. Then

$$m_d \equiv 0 \pmod{q-1}, \quad m_d \geq q-1.$$

Furthermore, observe that $\mathbf{m}'$ is the greedy element of $U_d(N - m_{d+1})$. By Proposition 7.1, assertion 2), we get

$$\deg_t S_d(N - m_{d+1}) > (N - m_{d+1})(d-1).$$

Let $\mathbf{m}'' = (m_0, \dots, m_{d-1}, m_d + m_{d+1}) \in U_d(N)$. We have

$$\deg_t S_d(N) \geq \deg \mathbf{m}'' = m_1 + \cdots + (d-1)(m_{d-1}) + d(m_d + m_{d+1}).$$

Thus

$$\begin{aligned}\deg_t S_d(N) &\geq \deg_t S_d(N - m_{d+1}) + dm_{d+1} \\ &> (N - m_{d+1})(d-1) + dm_{d+1} = (d-1)N + m_{d+1}.\end{aligned}$$

It implies that

$$\deg_t S_d(N) > N(d-1),$$
$$\deg_t S_d(N) - \deg_t S_{d-1}(N) \geq \deg_t S_d(N) - (d-1)N > m_{d+1}.$$ □

Proposition 7.3. *Let* $d \geq 1$ *such that* $U_{d+1}(N) \neq \emptyset$. *We consider the truncated L-series*

$$\Lambda_{d,N}(t) = \sum_{i=0}^{d} \alpha_{i,N}(t)\theta^{-i} \in K_\infty[t].$$

Then

$$\deg_t \Lambda_{d,N}(t) = \deg_t S_d(N).$$

Further, the edge points of the Newton polygon of $\Lambda_{d,N}(t)$ *are*

$$(\deg_t S_i(N), i), \quad i = 0, \dots, d.$$

Proof. The proof uses similar arguments as that used in the proof of Proposition 6.6. Let $i \geq 0$. By Lemma 5.4, we have

$$\alpha_{i,N}(t) = \sum_{\ell(\mathbf{k})+w(\mathbf{k})=i} (-1)^{|\mathbf{k}|} C_{\mathbf{k}} \sum_{a \in A_{+,\ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}.$$

Observe that

$$\deg_t \sum_{a \in A_{+,\ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}} \leq \ell(\mathbf{k})N.$$

Thus, for $i = 0, \ldots, d$, by Proposition 7.1, assertion 2), we get

$$\deg_t \alpha_{i,N}(t) = \deg_t S_i(N).$$

In particular, again by Proposition 7.1, assertion 2),

$$\deg_t \Lambda_{d,N}(t) = \deg_t S_d(N).$$

Finally, by Proposition 7.1, assertion 1), $(\deg_t S_i(N), i), i = 0, \ldots, d$, are the edge points of the Newton polygon of $\Lambda_{d,N}(t)$. □

To conclude this paragraph, we recall the following crucial result which is implicit in [15] after G. Böckle [7, Theorem 1.2].

Theorem 7.4. *We keep the previous notation. Then*

$$U_d(N) \neq \emptyset \quad \Leftrightarrow \quad S_d(N) \neq 0 \quad \Leftrightarrow \quad d(q-1) \leq \mathrm{Min}\left\{\ell_q\big(p^i N\big), i \in \mathbb{N}\right\}.$$

7.2. The q-minimal case

An integer $N \geq 1$ will be called *q-minimal* if

$$\left[\frac{\ell_q(N)}{q-1}\right] = \mathrm{Min}\left\{\left[\frac{\ell_q(p^i N)}{q-1}\right], i \in \mathbb{N}\right\}.$$

Remark 7.5. Observe that if $q = p$, then every integer $N \geq 1$ is q-minimal.

For the rest of this section, we always suppose that $N \geq 2$ is q-minimal such that $N \equiv 1 \pmod{q-1}$ and $l_q(N) > q$ (see Remark 5.12).

We will use the first expansion (5.5) of N in base q

$$N = \sum_{i=0}^{k} n_i q^i$$

with $n_0, \ldots, n_k \in \{0, \ldots, q-1\}$ and $n_k \neq 0$. By Theorem 7.4, we know that

$$U_d(N) \neq \emptyset \quad \Leftrightarrow \quad S_d(N) \neq 0 \quad \Leftrightarrow \quad d \leq r+1 = \mathrm{Min}\left\{\left[\frac{\ell_q(p^i N)}{q-1}\right], i \in \mathbb{N}\right\}.$$

Proposition 7.6. *We keep the previous notation. Then for $i \in \{1, \dots, r\}$, we have*

$$q^k < \deg_t S_i(N) - \deg_t S_{i-1}(N) < q^{k+1}.$$

Proof. By Proposition 7.1, for $i \in \{1, \dots, r\}$,

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) \le \deg_t S_1(N) \le N < q^{k+1}.$$

Let $\mathbf{m} = (m_0, \dots, m_{r+1})$ be the greedy element of $U_{r+1}(N)$. Since $\ell_q(N) = (r+1)(q-1)+1$, we must have $\ell_q(m_{r+1}) = 1$.

If $m_{d+1} = q^k$, then by Lemma 7.2 and Proposition 7.1, for $i \in \{1, \dots, r\}$, we have

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) > q^k.$$

Suppose now that $m_{d+1} \neq q^k$. Since $\mathbf{m}$ is the greedy element of $U_{r+1}(N)$, p divides n_k. Otherwise, we would have $m_{d+1} = q^k$. In particular, $n_k \ge 2$, and it follows that $n_k q^k \ge \frac{2}{3}N$.

For all $i \in \mathbb{N}$, we have

$$\left[\frac{\ell_q(p^i(N - n_k q^k))}{q-1}\right] \ge \left[\frac{\ell_q(p^i N)}{q-1}\right] - 1 \ge r.$$

By Theorem 7.4, we deduce that $U_r(N - n_k q^k) \neq 0$. Let $\mathbf{u} = (u_0, \dots, u_r)$ be the greedy element of $U_r(N - n_k q^k)$. Then $\mathbf{u}' = (u_0, \dots, u_r + n_k q^k)$ is an element of $U_r(N)$, and $\mathbf{u}'' = (u_0, \dots, u_{r-1})$ is the greedy element of $U_{r-1}(N - n_k q^k - u_k)$. By Proposition 7.1, assertion 2), we have $\deg \mathbf{u}'' \ge (r-2)(N - n_k q^k - u_k)$. Thus we get

$$\begin{aligned}
\deg_t S_r(N) &\ge \deg \mathbf{u}' \\
&= \deg \mathbf{u}'' + r\big(n_k q^k + u_r\big) \\
&> (r-2)\big(N - n_k q^k - u_k\big) + r\big(n_k q^k + u_r\big) \\
&> (r-2)N + 2n_k q^k = (r-2)N + \frac{3}{2}n_k q^k + \frac{1}{2}n_k q^k \\
&> (r-1)N + q^k
\end{aligned}$$

where the last inequality comes from both estimations $n_k \ge 2$, and $n_k q^k \ge \frac{2}{3}N$.

By Proposition 7.1, assertion (3), we know that

$$\deg_t S_{r-1}(N) \le N(r-1).$$

We obtain

$$\deg_t S_r(N) - \deg_t S_{r-1}(N) > N(r-1) + q^k - N(r-1) = q^k.$$

Now, by Proposition 7.1, assertion 1), for $i \in \{1, \ldots, r\}$, we have

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) > \deg_t S_r(N) - \deg_t S_{r-1}(N).$$

Therefore,

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) > q^k.$$

The proof is complete. □

Theorem 7.7.

1) *The polynomial $B_N(t, \theta)$ is of total degree $\deg_t S_r(N)$ in t, θ and has only one monomial of total degree $\deg_t S_r(N)$, which is $t^{\deg_t S_r(N)}$;*
2) *The edge points of the Newton polygon of $\theta^{-r} B_N(t, \theta)$ are*

$$\big(\deg_t S_i(N), i\big) \quad \text{for } i = 0, \ldots, r;$$

3) *The polynomial $B_N(t, \theta)$ in the variable t has no zeros in $\{\theta^{q^i}, i \in \mathbb{Z}\}$.*

Remark 7.8. The assertion 3) proves Conjecture D when N is q-minimal. In fact, we prove a stronger assertion, which says that all the roots of $B_N(t, \theta)$ in the variable t are of valuation w with $-q^{-k} < w < -q^{-(k+1)}$.

Proof. The proof is similar to that of Theorem 6.8. Indeed, Proposition 7.3 combined with Proposition 7.6 show the equivalent of Proposition 6.6: the truncated L-series

$$\Lambda_{r,N}(t) = \sum_{i=0}^{r} \alpha_{i,N}(t)\theta^{-i} \in K_\infty[t]$$

is a polynomial in t of degree $\deg_t S_r(N)$. The edge points of its Newton polygon are $(\deg_t S_i(N), i)$ for $i = 0, \ldots, r$. Furthermore, all the roots of $\Lambda_{r,N}(t)$ are of valuation w with $-q^{-k} < w < -q^{-(k+1)}$. The rest of the proof is identical. □

This theorem implies immediately the following:

Corollary 7.9. *We assume that $r \geq 1$. Then:*

1) *The zeros of $B_N(t, \theta)$ are algebraic integers* (i.e., *they are integral over A);*
2) *$B_N(t, \theta)$ (viewed as a polynomial in θ) has only simple roots and its roots belong to $\mathbb{F}_p((\frac{1}{t})) \setminus \{t^{q^i}, i \in \mathbb{Z}\}$.*

7.3. Final remarks

As an application, we will deduce some properties on the several variable polynomials $\mathbb{B}_s$.

Proposition 7.10. *Let s be an integer such that $s \geq 2$ and $s \equiv 1 \pmod{q-1}$. Then*

1) *The polynomial $\mathbb{B}_s$ is square-free,* i.e., *$\mathbb{B}_s$ is not divisible by the square of a non-trivial polynomial in $\mathbb{F}_q[t_1, \dots, t_s, \theta]$;*
2) *For all $i, n \in \mathbb{N}$, $\mathbb{B}_s$ is relatively prime to $(t_1^{q^i} - \theta^{q^n})$;*
3) *For all monic irreducible primes P of A, $\mathbb{B}_s$ is relatively prime to $P(t_1) \cdots P(t_s) - P$.*

Proof. We choose $N = q^{e_1} + \dots + q^{e_s}$ with $0 \leq e_1 < e_2 < \dots < e_s$. Then

$$B_N(t, \theta) = \mathbb{B}_s \mid_{t_i = t^{q^{e_i}}}.$$

We see immediately that N is q-minimal. Corollary 7.9, assertion 2) implies that $B_N(t, \theta)$ is square-free and has no roots in $\{\theta^{q^i}, i \in \mathbb{Z}\}$. This proves 1) and 2).

We will prove the assertion 3). Let P be a monic irreducible polynomial in A. Suppose that that $P(t_1) \cdots P(t_s) - P$ and $\mathbb{B}_s$ are not relatively prime. Then $P(t)^N - P$ and $B_N(t, \theta)$ are not relatively prime. By Remark 7.8, if $\alpha \in \mathbb{C}_\infty$ is a root of $B_N(t, \theta)$, then

$$v_\infty(\alpha) > -q^{-k} > -\frac{1}{N}.$$

Now, observe that if $\beta \in \mathbb{C}_\infty$ is a root of $P(t)^N - P$, then $v_\infty(\beta) = -\frac{1}{N}$. This leads to a contradiction. □

The assertion 1) of the above proposition implies immediately the cyclicity result of [6, Theorem 4] by a completely different method.

Finally, we present an example of an integer N which is not q-minimal, so that our method does not apply. We choose $q = 4$ and

$$N = 682 = 2 + 2 \times 4 + 2 \times 4^2 + 2 \times 4^3 + 2 \times 4^4.$$

We get $l_q(N) = 10 = 3q - 2$ so that $\deg_\theta(B_N(t, \theta)) = 2$. Moreover, $l_q(pN) = 5$ so that N is not q-minimal. Since

$$q^4 = 256 < N < q^5 = 1024,$$

we get $k = 4$ and the set of trivial zeros of the series $L_N(t)$ is $\{\theta^{q^j}, j \geq -4\}$. By using the explicit examples given in Section 3, we get

$$B_N(t, \theta) = \theta^2 + \theta \left(t^{10} + t^{34} + t^{40} + t^{130} + t^{136} + t^{160} + t^{514} + t^{520} + t^{544} + t^{640}\right) + \left(t^{170} + t^{554} + t^{650} + t^{674} + t^{680}\right).$$

The Newton polygon of $B_N(t,\theta)$ has then the end points $(0,-2)$, $(640,-1)$, $(680,0)$. We deduce that $B_N(t,\theta)$ has 640 distinct zeroes of valuation $w_1=-\frac{1}{640}$ and 40 distinct zeros of valuation $w_2=-\frac{1}{40}$. The explicit bounds in Remark 7.8 do not hold: $w_2=-\frac{1}{40}<-q^{-k}=-\frac{1}{256}$. However, Conjecture D still holds for $N=682$, *i.e.*, $B_N(t,\theta)$ has no zero of the form θ^{q^i}, $i\in\mathbb{Z}$.

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Université de Caen Normandie
Laboratoire de Mathématiques Nicolas Oresme
CNRS UMR 6139
Campus II, Boulevard Maréchal Juin
B.P. 5186, 14032 Caen Cedex, France
bruno.angles@unicaen.fr
tuan.ngodac@unicaen.fr
floric.tavares-ribeiro@unicaen.fr