

Nonlinear CR automorphisms of Levi degenerate hypersurfaces and a new gap phenomenon

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Abstract. We give a complete classification of polynomial models for smooth real hypersurfaces of finite Catlin multitype in $\mathbb{C}^3$, which admit nonlinear infinitesimal CR automorphisms. As a consequence, we obtain a sharp 1-jet determination result for any smooth hypersurface with such a model. The results also prove a conjecture of the first author about the origin of such nonlinear automorphisms (AIM list of problems, 2010). As another consequence, we describe all possible dimensions of the Lie algebra of infinitesimal CR automorphisms, which leads to a new “secondary” gap phenomenon.

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1. Introduction

This paper provides an important necessary step towards solving the local equivalence problem for hypersurfaces of finite Catlin multitype, by giving a full classification of their polynomial models with nonlinear infinitesimal CR automorphisms. Note that by the classical Chern-Moser theory, the only strongly pseudoconvex hypersurface which admits such automorphisms is the sphere (see [9, 27]).

The Levi degenerate case has recently attracted considerable attention and has led to the discovery of new types of nonlinear symmetries (see, *e.g.*, [12, 16, 18]). In contrast to the $\mathbb{C}^2$ case, nonlinear infinitesimal CR automorphisms with coefficients of arbitrarily high degree may arise in $\mathbb{C}^n$, for $n > 2$ [11, 22].

Our aim in this paper is to study systematically nonlinear infinitesimal CR automorphisms of Levi degenerate hypersurfaces in complex dimension three. The results provide a description of hypersurfaces of finite Catlin multitype in $\mathbb{C}^3$, whose models admit such automorphisms. Since the Lie algebra of infinitesimal CR automorphisms of a polynomial model is in one-to-one correspondence with the kernel

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of the generalized Chern-Moser operator [17], we can prove a sharp 1-jet determination result for the biholomorphisms of any such hypersurface.

Moreover, we identify the common source of such automorphisms. In all cases they arise from suitable holomorphic mappings into a quadric in $\mathbb{C}^K$, for $K \geq 3$, as a pull-back of an automorphism of the quadric.

As an application, we will determine all possible dimensions of the Lie algebra of infinitesimal CR automorphisms for such models, which reveals a “secondary” gap phenomenon at dimension eight.

Our starting point are the sharp results of [17] which give an effective bound for the weighted degree of the coefficients of an infinitesimal CR automorphism, and put restrictions on the possible form of such vector fields.

Our first result deals with a holomorphically nondegenerate model hypersurface M_H given by a homogeneous polynomial P of degree $d > 2$ without pluriharmonic terms,

$$M_H := \{\operatorname{Im} w = P(z, \bar{z})\}, \quad (z, w) \in \mathbb{C}^2 \times \mathbb{C}. \quad (1.1)$$

Notice that the case of $d = 2$ corresponds to Levi nondegenerate models, *i.e.*, hyperquadrics. We showed in [17] and in [18] that the Lie algebra $\mathfrak{g} = \operatorname{aut}(M_H, 0)$ of all germs of infinitesimal automorphisms of M_H at 0 admits the weighted grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/d} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\tau/d} \oplus \mathfrak{g}_{1-1/d} \oplus \mathfrak{g}_1, \quad (1.2)$$

for some integer τ , where $1 \leq \tau \leq d - 2$. The following theorem shows that if $\dim \mathfrak{g}_{1-1/d} \neq 0$, then there is a unique choice for M_H .

Theorem 1.1. *Let M_H be the holomorphically nondegenerate model hypersurface given by (1.1) with $d > 2$, and let $\mathfrak{g}_{1-1/d}$ in (1.2) satisfy*

$$\dim \mathfrak{g}_{1-1/d} > 0. \quad (1.3)$$

Then M_H is biholomorphically equivalent to

$$\operatorname{Im} w = \operatorname{Re} z_1 \bar{z}_2^{d-1}. \quad (1.4)$$

Further we consider the more general case of a holomorphically nondegenerate weighted homogeneous model of finite Catlin multitype. Let

$$M_H := \{\operatorname{Im} w = P_C(z, \bar{z})\}, \quad (z, w) \in \mathbb{C}^2 \times \mathbb{C}, \quad (1.5)$$

where P_C is a weighted homogeneous polynomial of degree one with respect to the multitype weights μ_1, μ_2 (see Section 2 for the needed definitions). Note that in order to distinguish the homogeneous and the weighted homogeneous case, we denote the corresponding polynomial by P in the former and by P_C in the later case.

As proved in [17], the Lie algebra of infinitesimal automorphisms $\mathfrak{g} = \text{aut}(M_H, 0)$ of M_H admits the weighted decomposition given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \bigoplus_{j=1}^2 \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_n \oplus \mathfrak{g}_1, \quad (1.6)$$

where $\mathfrak{g}_c$ contains vector fields commuting with $W = \partial_w$ and $\mathfrak{g}_n$ contains vector fields not commuting with W , whose weights in both cases lie in the interval $(0, 1)$. Note that by a result of [17], vector fields in $\mathfrak{g}_j$ with $j < 0$ are regular and vector fields in $\mathfrak{g}_0$ are linear.

Theorem 1.2. *Let $P_C(z, \bar{z})$ be a weighted homogeneous polynomial of degree 1 with respect to the multitype weights, such that the hypersurface*

$$M_H := \{\text{Im } w = P_C(z, \bar{z})\}, \quad (z, w) \in \mathbb{C}^2 \times \mathbb{C}, \quad (1.7)$$

is holomorphically nondegenerate. Let $\mathfrak{g}_n$ in (1.6) satisfy

$$\dim \mathfrak{g}_n > 0. \quad (1.8)$$

Then M_H is biholomorphically equivalent to

$$\text{Im } w = \text{Re } z_1 \bar{z}_2^l \quad (1.9)$$

or

$$\text{Im } w = |z_1|^2 \pm |z_2|^{2l}. \quad (1.10)$$

Note that the Levi nondegenerate case, corresponding to $l = 1$, is covered by Theorem 1.2. The following result, which deals with the component $\mathfrak{g}_c$ was obtained in [18].

Definition 1.3. Let Y be a weighted homogeneous vector field. A pair of finite sequences of holomorphic weighted homogeneous polynomials $\{U^1, \dots, U^n\}$ and $\{V^1, \dots, V^n\}$ is called a symmetric pair of Y -chains if

$$Y(U^n) = 0, \quad Y(U^j) = c_j U^{j+1}, \quad j = 1, \dots, n-1, \quad (1.11)$$

$$Y(V^n) = 0, \quad Y(V^j) = d_j V^{j+1}, \quad j = 1, \dots, n-1, \quad (1.12)$$

where c_j, d_j are non zero complex constants, which satisfy

$$c_j = -\bar{d}_{n-j}. \quad (1.13)$$

If the two sequences are identical we say that $\{U^1, \dots, U^n\}$ is a symmetric Y -chain.

The following theorem shows that in general the elements of $\mathfrak{g}_c$ arise from symmetric pairs of chains.

Theorem 1.4. *Let M_H be a holomorphically nondegenerate hypersurface given by (1.7), which admits a nontrivial $Y \in \mathfrak{g}_c$. Then P_C can be decomposed in the following way*

$$P_C = \sum_{j=1}^M T_j, \quad (1.14)$$

where each T_j is given by

$$T_j = \operatorname{Re} \left(\sum_{k=1}^{N_j} U_j^k \overline{V_j^{N_j-k+1}} \right), \quad (1.15)$$

where $\{U_j^1, \dots, U_j^{N_j}\}$ and $\{V_j^1, \dots, V_j^{N_j}\}$ are a symmetric pair of Y -chains.

Conversely, if Y and P_C satisfy (1.11)-(1.15), then $Y \in \mathfrak{g}_c$.

Note that Y is uniquely and explicitly determined by P_C (see [18]). Hence for a given hypersurface this result also provides a constructive tool to determine $\mathfrak{g}_c$.

Definition 1.5. If P_C satisfies (1.11)-(1.15), the associated hypersurface M_H is called a chain hypersurface.

The description of the remaining component $\mathfrak{g}_1$ is a consequence of Theorem 4.7 in [17] (see section 2 for the notation).

Definition 1.6. Let $P_C(z, \bar{z})$ be a weighted homogeneous polynomial of degree 1 with respect to the multitype weights. We say that P_C is balanced if it can be written as

$$P_C(z, \bar{z}) = \sum_{|\alpha|_\Lambda = |\bar{\alpha}|_\Lambda = 1} A_{\alpha, \bar{\alpha}} z^\alpha \bar{z}^{\bar{\alpha}}, \quad (1.16)$$

for some pair of nonzero rational numbers $\Lambda = (\lambda_1, \lambda_2)$, where

$$|\alpha|_\Lambda := \lambda_1 \alpha_1 + \lambda_2 \alpha_2.$$

The associated hypersurface M_H is called a balanced hypersurface.

Note that P_C is balanced if and only if the linear vector field

$$Y = \lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2}$$

is a complex reproducing field in the terminology of [17], i.e., $Y(P_C) = P_C$.

Theorem 1.7. *The component $\mathfrak{g}_1$ satisfies $\dim \mathfrak{g}_1 > 0$ if and only if M_H is a balanced hypersurface.*

As a consequence, we obtain the following result for a general hypersurface of finite Catlin multitype.

Theorem 1.8. *Let M be a smooth hypersurface and $p \in M$ be a point of finite Catlin multitype with holomorphically nondegenerate model. If its model at p is neither a balanced hypersurface nor a chain hypersurface, then its automorphisms are determined by their 1-jets at p .*

Our results also confirm a conjecture about the origin of nonlinear automorphisms of Levi degenerate hypersurfaces formulated by the first author (see [18], [19]). Recall that two vector fields X_1 and X_2 are said to be f -related if $f_*(X_1) = X_2$.

Theorem 1.9. *Let M_H be a holomorphically nondegenerate hypersurface given by (1.7) and Y be a vector field of strictly positive weight. Then $Y \in \text{aut}(M_H, 0)$, if and only if there exists an integer $K \geq 3$ and a holomorphic mapping f from a neighbourhood of the origin in $\mathbb{C}^3$ into $\mathbb{C}^K$ which maps M_H into a Levi nondegenerate hyperquadric $Q \subseteq \mathbb{C}^K$ such that Y is f -related with an infinitesimal CR automorphism of Q .*

Let us remark that mappings of CR manifolds into hyperquadrics have been studied intensively in recent years (see, e.g., [1, 10] and the references therein). Here we ask in addition that the mapping be compatible with a symmetry of the hyperquadric.

As an application of our results, we will determine all possible dimensions of $\text{aut}(M_H, 0)$. Recall that the dimension of the symmetry group of a Levi nondegenerate hyperquadric in $\mathbb{C}^3$ is equal to 15. In complex dimension two, the possible dimensions of the symmetry group of a general hypersurface are known to be 1, 2, 3, 4, 5, 8 [23].

The problem of possible dimensions of the symmetry groups of a general hypersurface in $\mathbb{C}^3$ was considered by Beloshapka in [4]. He proved that for a germ of (an arbitrary) smooth real hypersurface, which is not equivalent to a hyperquadric, the dimension is at most 11. For related results on the gap phenomenon, see [25, 26].

The following result shows that for polynomial models of finite Catlin multitype the largest possible dimension is 10. Further, it demonstrates the existence of a secondary gap phenomenon in dimension 8.

Let us remark that the term gap phenomenon in this context was introduced by Kruglikov (see [14, 25])

Theorem 1.10. *Let M_H be a holomorphically nondegenerate hypersurface given by (1.7), such that 0 is a Levi degenerate point. Then*

$$\dim \text{aut}(M_H, 0) \leq 10.$$

If the dimension is equal to 10, then M_H is equivalent to (1.4). If the dimension is equal to 9, then M_H is equivalent to (1.10). Further, the dimension of $\text{aut}(M_H, 0)$ can attain any value from the set

$$\{2, 3, 4, 5, 6, 7, 9, 10\}.$$

In particular, there is no M_H with $\dim \text{aut}(M_H, 0) = 8$.

The paper is organized as follows. Section 2 contains the necessary definitions used in the rest of the paper. Section 3 deals with the $\mathfrak{g}_n$ component of the Lie algebra $\mathfrak{g}$. Section 4 contains the proofs of the results, up to Theorem 1.9. Section 5 deals with possible dimensions of $\text{aut}(M_H, 0)$ and contains the proof of Theorem 1.10.

2. Preliminaries

In this section we introduce notation and recall briefly some needed definitions (for more details, see, *e.g.*, [20]).

Consider a smooth real hypersurface $M \subseteq \mathbb{C}^3$ and let $p \in M$ be a point of finite type $m \geq 2$ (in the sense of Kohn and Bloom-Graham, [5, 15]).

Let (z, w) be local holomorphic coordinates centered at p , where $z = (z_1, z_2)$ and $z_j = x_j + iy_j$, $j = 1, 2$, and $w = u + iv$. We assume that the hyperplane $\{v = 0\}$ is tangent to M at p , so M is described near p as the graph of a uniquely determined real valued function

$$v = F(z_1, z_2, \bar{z}_1, \bar{z}_2, u), \quad (2.1)$$

with $dF(0) = 0$. We can assume that (see, *e.g.*, [5])

$$F(z_1, z_2, \bar{z}_1, \bar{z}_2, u) = P_m(z, \bar{z}) + o(u, |z|^m), \quad (2.2)$$

where $P_m(z, \bar{z})$ is a nonzero homogeneous polynomial of degree m without pluri-harmonic terms.

The definition of Catlin multitype involves rational weights associated to the variables w, z_1, z_2 . The variables w, u and v are given weight one, reflecting our choice of tangential and normal variables. The complex tangential variables (z_1, z_2) are treated as follows.

By a weight we understand a pair of nonnegative rational numbers $\Lambda = (\lambda_1, \lambda_2)$, where $0 \leq \lambda_j \leq \frac{1}{2}$, and $\lambda_1 \geq \lambda_2$. Let $\Lambda = (\lambda_1, \lambda_2)$ be a weight, and $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ be multiindices. The weighted degree κ of a monomial

$$q(z, \bar{z}, u) = c_{\alpha\beta l} z^\alpha \bar{z}^\beta u^l, \quad l \in \mathbb{N},$$

is then

$$\kappa := l + \sum_{i=1}^2 (\alpha_i + \beta_i) \lambda_i.$$

A polynomial $Q(z, \bar{z}, u)$ is weighted homogeneous of weighted degree κ if it is a sum of monomials of weighted degree κ .

Analogously, a holomorphic vector field with polynomial coefficients of the form

$$\sum_{j=1}^2 f^j(z, w) \partial_{z_j} + g(z, w) \partial_w, \quad (2.3)$$

is weighted homogeneous of weighted degree κ , provided $g(z, w)$ is weighted homogeneous of degree $\kappa + 1$, and each $f^j(z, w)$ is weighted homogeneous of degree $\kappa + \lambda_j$.

Definition 2.1. For a weight Λ , the weighted length of a multiindex $\alpha = (\alpha_1, \alpha_2)$ is defined by

$$|\alpha|_\Lambda := \lambda_1 \alpha_1 + \lambda_2 \alpha_2.$$

Similarly, if $\alpha = (\alpha_1, \alpha_2)$ and $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2)$ are two multiindices, the weighted length of the pair $(\alpha, \hat{\alpha})$ is

$$|(\alpha, \hat{\alpha})|_\Lambda := \lambda_1(\alpha_1 + \hat{\alpha}_1) + \lambda_2(\alpha_2 + \hat{\alpha}_2).$$

Definition 2.2. A weight Λ will be called distinguished for M if there exist local holomorphic coordinates (z, w) in which the defining equation of M takes form

$$v = P_C(z, \bar{z}) + o_\Lambda(1), \quad (2.4)$$

where $P_C(z, \bar{z})$ is a nonzero Λ -homogeneous polynomial of weighted degree 1 without pluriharmonic terms, and $o_\Lambda(1)$ denotes a smooth function whose derivatives of weighted order less than or equal to one vanish.

It follows from (2.2) that one can always find distinguished weights. When (2.2) is valid, we can take, e.g., $\Lambda = (\frac{1}{m}, \frac{1}{m})$.

In the following we shall consider the standard lexicographic order on the set of pairs. We recall the following definition (see [7]):

Definition 2.3. Let $\Lambda_M = (\mu_1, \mu_2)$ be the infimum of all possible distinguished weights Λ with respect to the lexicographic order. The multitype of M at p is defined to be the pair

$$(m_1, m_2),$$

where

$$m_j = \begin{cases} \frac{1}{\mu_j} & \text{if } \mu_j \neq 0 \\ \infty & \text{if } \mu_j = 0. \end{cases}$$

If no m_j is infinity, we say that M is of finite multitype at p . Clearly, since the definition of multitype includes all distinguished weights, the infimum is a biholomorphic invariant.

Coordinates corresponding to the multitype weight Λ_M , in which the local description of M has form (2.4), with P_C being Λ_M -homogeneous, are called multitype coordinates.

Definition 2.4. Let M be given by (2.4). We define a model hypersurface M_H associated to M at p by

$$M_H = \{(z, w) \in \mathbb{C}^{n+1} \mid v = P_C(z, \bar{z})\}. \quad (2.5)$$

Next let us recall the following definitions:

Definition 2.5. Let X be a holomorphic vector field of the form

$$X = \sum_{j=1}^2 f^j(z, w) \partial_{z_j} + g(z, w) \partial_w. \quad (2.6)$$

We say that X is rigid if f^1, f^2, g are all independent of the variable w .

We can divide homogeneous rigid vector fields into three types, and introduce the following terminology:

Definition 2.6. Let $X \in \text{aut}(M_H, 0)$ be a rigid weighted homogeneous vector field. X is called

- (1) A shift if the weighted degree of X is less than zero;
- (2) A rotation if the weighted degree of X is equal to zero;
- (3) A generalized rotation if the weighted degree of X is bigger than zero and less than one.

3. Computing $\mathfrak{g}_n$

In this section we describe all holomorphically nondegenerate model hypersurfaces M_H which have a nontrivial $\mathfrak{g}_n$.

For easier notation, we will write P instead of P_C in (1.7). If M_H has nontrivial $\mathfrak{g}_{-\mu_j}$ and $X \in \mathfrak{g}_{-\mu_j}$, then by [17, Lemma 6.1] there exist local holomorphic coordinates preserving the multitype (with pluriharmonic terms allowed), such that

$$X = i \partial_{z_j}. \quad (3.1)$$

By a permutation of indices, if needed, we can assume that $j = 1$ (hence we allow here $\mu_1 < \mu_2$).

By assumption, we may write P as

$$P(z, \bar{z}) = \sum_{j=0}^m x_1^j P_j(z_2, \bar{z}_2), \quad (3.2)$$

where $P_j(z_2, \bar{z}_2)$ are real valued homogeneous polynomials and $P_m \neq 0$.

Lemma 3.1. *Let $X = i \partial_{z_1}$ be in $\text{aut}(M_H, 0)$ and P be of the form (3.2). If there is a vector field Y in $\text{aut}(M_H, 0)$ such that $[Y, W] = X$, then $m \leq 2$, i.e., P has the form*

$$P(z, \bar{z}) = x_1^2 P_2(z_2, \bar{z}_2) + x_1 P_1(z_2, \bar{z}_2) + P_0(z_2, \bar{z}_2). \quad (3.3)$$

Proof. Suppose, by contradiction, that $m > 2$. We split Y according to the powers of z_1 , writing

$$Y = i w \partial_{z_1} + \sum_{j=-m}^k Y_j, \quad (3.4)$$

where each Y_j has the form

$$Y_j = \varphi_1^j(z_2)z_1^{j+1}\partial_{z_1} + \varphi_2^j(z_2)z_1^j\partial_{z_2} + \psi^j(z_2)z_1^{j+m}\partial_w, \quad (3.5)$$

with $\varphi_1^{j-1}(z_2) = \varphi_2^j(z_2) = 0$ for $j < 0$ and $Y_k \neq 0$. Note that by weighted homogeneity of Y , each coefficient is homogeneous in z_2 , hence a monomial.

We claim that $2m - 1 \leq m + k$. Indeed, if not, while applying $\operatorname{Re} Y$ to $P - v$, the first term in the right-hand side of (3.4) gives

$$-\frac{m}{2}P_m^2x_1^{2m-1}$$

while all other terms give polynomials in which z_1 has maximal exponent $m + k$, yielding a contradiction that proves our claim.

We will next show that $\psi^k(z_2) = 0$. Indeed, consider the leading term with respect to the variable z_1 in the tangency equation $\operatorname{Re} Y(P - v) = 0$. We obtain

$$\begin{aligned} \delta_{k,m-1}mx_1^{2m-1}P_m^2 &= mx_1^{m-1}P_m\operatorname{Re}\varphi_1^kz_1^{k+1} \\ &\quad + 2x_1^m\operatorname{Re}\varphi_2^kz_1^k\frac{\partial P_m}{\partial z_2} - \operatorname{Im}\psi^kz_1^{k+m}, \end{aligned} \quad (3.6)$$

where δ is the Kronecker symbol. We observe that, since $m > 2$, $\operatorname{Im}\psi^k(z_2)z_1^{k+m}$ cannot contain terms in $x_1y_1^{m+k-1}$ and in y_1^{m+k} . Hence $\psi^k(z_2) = 0$.

Next we distinguish two cases.

Case 1. Let P_m be nonconstant. We claim that $k = m - 1$. Indeed, if $k > m - 1$, then we have

$$0 = mP_m\operatorname{Re}\varphi_1^kz_1^{k+1} + 2x_1\operatorname{Re}\varphi_2^kz_1^k\frac{\partial P_m}{\partial z_2}. \quad (3.7)$$

Since $m > 2$ and $k > 1$, the first, but not the second summand in the right-hand side, is harmonic in z_1 . That gives a contradiction. Now, setting $k = m - 1$, we compare degrees in z_2 in (3.6). By homogeneity it follows that φ_1^k and P_m have the same degree. Dividing by x_1^{m-1} and looking at coefficients of y_1^{k+1} we obtain that φ_1^k is a constant, hence P_m is a constant, which gives a contradiction.

Case 2. Let P_m be constant. Without loss of generality we will assume $P_m = 1$. If $k \geq m$, the leading equation (3.6) must be trivial and we have necessarily

$$Y_k = \beta z_1^k \partial_{z_2} \quad (3.8)$$

for some constant $\beta \in \mathbb{C}$. Further, the weight of z_1 is $\frac{1}{m}$ and the weighted degree of Y is $1 - \frac{1}{m}$. It follows that $\frac{k}{m} - \mu_2 = 1 - \frac{1}{m}$ and, since $\mu_2 \leq \frac{1}{2}$, we have

$$k \leq \frac{3}{2}m - 1.$$

Further we have $Y_l = 0$ for all $m \leq l \leq k - 1$, hence the first nontrivial terms have degree $2m - 1$ in z_1 and we obtain

$$mx_1^{2m-1} = mx_1^{m-1} \operatorname{Re} \varphi_1^{m-1} z_1^m + 2x_1^{2m-1-k} \operatorname{Re} \beta z_1^k \frac{\partial P_{2m-1-k}}{\partial z_2} - \operatorname{Im} \psi^{m-1} z_1^{2m-1}. \quad (3.9)$$

By the same argument as above, we obtain $\psi^{m-1}(z_2) = 0$. Indeed, we have $2m - 1 - k \geq 2m - 1 - (\frac{3}{2}m - 1) = \frac{m}{2}$. Since $2m - 1 - k$ is an integer, we obtain $2m - 1 - k \geq 2$. Hence $\psi^{m-1}(z_2) z_1^{2m-1}$ contains no terms in x_1^{2m-1} or $x_1^{2m-2} y_1$, which gives $\psi^{m-1}(z_2) = 0$. Next, dividing by x_1^{2m-1-k} we obtain

$$mx_1^k = mx_1^{k-m} \operatorname{Re} \varphi_1^{m-1} z_1^m + 2 \operatorname{Re} \beta z_1^k \frac{\partial P_{2m-1-k}}{\partial z_2}. \quad (3.10)$$

The left, but not the right-hand side, contains a nonzero multiple of $z_1^{k-m+1} \bar{z}_1^{m-1}$. That gives a contradiction, which settles the case $k \geq m$. It remains to consider $k = m - 1$. We obtain

$$mx_1^{2m-1} = mx_1^{m-1} \operatorname{Re} \varphi_1^{m-1} z_1^m - \operatorname{Im} \psi^{m-1} z_1^{2m-1}. \quad (3.11)$$

Using the same type of argument, we obtain immediately a contradiction.

This completes the proof of the lemma. $\square$

Lemma 3.2. *Let $X = i\partial_{z_1}$ be in $\operatorname{aut}(M_H, 0)$ and P be of the form (3.2), with $m = 2$. If there is a vector field Y in $\operatorname{aut}(M_H, 0)$ such that $[Y, W] = X$, then P has the form*

$$P(z, \bar{z}) = Cx_1^2 + P_1(z_2, \bar{z}_2)x_1 + P_0(z_2, \bar{z}_2) \quad (3.12)$$

for some nonzero real constant C .

Proof. Let k be as in the definition of Y in (3.4). First assume $k = 1$. From coefficients of degree three with respect to z_1 , we obtain,

$$2x_1^3 P_2^2 = 2x_1 P_2 \operatorname{Re} \varphi_1^1 z_1^2 + 2x_1^2 \operatorname{Re} \varphi_2^1 z_1 \frac{\partial P_2}{\partial z_2} - \operatorname{Im} \psi^1 z_1^3. \quad (3.13)$$

If $\psi^1(z_2) \neq 0$ in (3.13), then it is a constant, by comparing terms in y_1^3 . Hence, by homogeneity, P_2 is constant.

Next, assume that $\psi^1(z_2) = 0$. Comparing degrees in z_2 , we see that φ_1^1 and P_2 have the same degree, or $\varphi_1^1 = 0$. If $\varphi_1^1 \neq 0$, then from the coefficients of $x_1 y_1^2$ we obtain that φ_1^1 is a constant, hence P_2 is a constant. On the other hand, $\varphi_1^1 = 0$ implies

$$x_1^3 P_2^2 = x_1^2 \operatorname{Re} \varphi_2^1 z_1 \frac{\partial P_2}{\partial z_2} \quad (3.14)$$

which is impossible, since by positivity, the left hand side contains a nonzero balanced term in z_2 , while the right-hand side has no such terms, since $\deg \varphi_2^1 = \deg P_2 + 1$.

Now assume that $k \neq 1$. For terms of degree $k + 2$ we get

$$0 = 2x_1 P_2 \operatorname{Re} \varphi_1^k z_1^{k+1} + 2x_1^2 \operatorname{Re} \varphi_2^k z_1^k \frac{\partial P_2}{\partial z_2} - \operatorname{Im} \psi^k z_1^{k+2}. \quad (3.15)$$

From the coefficients of y_1^{2+k} , we see that $\psi^k(z_2)$ is a constant. If $\psi^k(z_2) \neq 0$, then by homogeneity P_2 is a constant. Next, let $\psi^k(z_2) = 0$. After dividing by x_1 , the first term is harmonic in z_1 , while the second one is not, unless $k = 0$. But we know from the proof of the previous lemma that $k \geq m - 1$, hence $k = 0$ implies $P_2 = 0$. This completes the proof of the lemma. $\square$

In the following lemma we consider the case $m = 1$. Notice that for $m = 1$ the definition of mutitype implies $\mu_1 = \mu_2$.

Lemma 3.3. *Let $X = i\partial_{z_1}$ be in $\operatorname{aut}(M_H, 0)$ and P be of the form (3.2) with $m = 1$ and assume that P_0 does not contain any harmonic terms. Then there is a vector field Y in $\operatorname{aut}(M_H, 0)$ such that $[Y, W] = X$ if and only if either*

$$P(z, \bar{z}) = x_1 \operatorname{Re} \alpha z_2^{l+1} \quad (3.16)$$

for some $l \in \mathbb{N}$, $\alpha \in \mathbb{C}$, or

$$P(z, \bar{z}) = x_1 \operatorname{Re} \alpha z_2 + \varepsilon |z_2|^2, \quad (3.17)$$

where $\varepsilon \in \mathbb{R}$ and $\alpha \in \mathbb{C}$.

Proof. The vector field Y has the form

$$Y = iw\partial_{z_1} + \sum_{j=1}^2 \varphi_j \partial_{z_j} + \psi \partial_w. \quad (3.18)$$

From $\operatorname{Re} Y(P - v) = 0$, using $\operatorname{Re} X(P) = 0$, we obtain

$$P_0 P_1 + x_1 P_1^2 = 2x_1 \operatorname{Re} \varphi_2 \frac{\partial P_1}{\partial z_2} + \operatorname{Re} \varphi_1 P_1 + 2\operatorname{Re} \varphi_2 \frac{\partial P_0}{\partial z_2} - \operatorname{Im} \psi. \quad (3.19)$$

Hence for the constant and linear terms in z_1 we have

$$P_0 P_1 = 2\operatorname{Re} \varphi_2^0 \frac{\partial P_0}{\partial z_2} + \operatorname{Re} \varphi_1^{-1} P_1 - \operatorname{Im} \psi^{-1} \quad (3.20)$$

and

$$x_1 P_1^2 = 2x_1 \operatorname{Re} \varphi_2^0 \frac{\partial P_1}{\partial z_2} + \operatorname{Re} \varphi_1^0 z_1 P_1 + 2\operatorname{Re} \varphi_2^1 z_1 \frac{\partial P_0}{\partial z_2} - \operatorname{Im} \psi^0 z_1. \quad (3.21)$$

Let k be as in (3.4) and let first assume that $k > 0$. We get

$$0 = \operatorname{Re} \varphi_1^k z_1^{k+1} P_1 + 2x_1 \operatorname{Re} \varphi_2^k z_1^k \frac{\partial P_1}{\partial z_2} - \operatorname{Im} \psi^k z_1^{k+1}. \quad (3.22)$$

Being the only one factoring the monomial $\bar{z}_1 z_1^k$, the middle summand in the right-hand side is zero. It follows that $\operatorname{Re} \varphi_1^k z_1^{k+1} P_1$ is pluriharmonic, hence P_1 is constant. This yields a contradiction, showing that $k \leq 0$.

Assuming that $k = 0$, we get

$$x_1 P_1^2 = 2x_1 \operatorname{Re} \varphi_2^0 \frac{\partial P_1}{\partial z_2} + \operatorname{Re} \varphi_1^0 z_1 P_1 - \operatorname{Im} \psi^0 z_1. \quad (3.23)$$

By equating to zero the summands factoring y_1 , we get

$$-\operatorname{Im} \varphi_1^0 P_1 - \operatorname{Re} \psi^0 = 0. \quad (3.24)$$

This implies that P_1 is harmonic, namely $P_1 = c \operatorname{Re} \varphi_1^0$ for some $c \in \mathbb{R}$. Notice that $\varphi_1^0 = 0$ leads to contradiction. Indeed, if $\varphi_1^0 = 0$, then $\psi^0 = 0$, since P_1 cannot be constant. It follows that

$$P_1^2 = 2 \operatorname{Re} \varphi_2^0 \frac{\partial P_1}{\partial z_2}, \quad (3.25)$$

which is not allowed, since the left hand side contains a nonzero balanced term in z_2 , while the right-hand side has no such terms, since $\deg \varphi_2^0 = \deg P_1 + 1$. This gives the contradiction. Next consider the equation for the coefficients of x_1 in (3.23),

$$P_1^2 = 2 \operatorname{Re} \varphi_2^0 \frac{\partial P_1}{\partial z_2} + \operatorname{Re} \varphi_1^0 P_1 - \operatorname{Im} \psi^0. \quad (3.26)$$

By substituting $P_1 = c \operatorname{Re} \varphi_1^0$, from the mixed terms we obtain $c = 1$. Set $P_1 = \operatorname{Re} \varphi_1^0 = \operatorname{Re} \alpha z_2^l$. Notice that the degree of P_0 is $l + 1$, by weighted homogeneity, since by the definition of the Catlin multitype $\mu_1 = \mu_2$. For terms of order zero we obtain

$$P_0 P_1 = \operatorname{Re} \varphi_1^{-1} P_1 + 2 \operatorname{Re} \varphi_2^0 \frac{\partial P_0}{\partial z_2} - \operatorname{Im} \psi^{-1}. \quad (3.27)$$

Using the form of P_1 , we obtain

$$P_0 \operatorname{Re} (\alpha z_2^l) = \operatorname{Re} (\delta z_2^{l+1}) \operatorname{Re} (\alpha z_2^l) + \operatorname{Re} \left(\beta z_2^{l+1} \frac{\partial P_0}{\partial z_2} \right) + \operatorname{Im} \gamma z_2^{2l+1} \quad (3.28)$$

for some $\beta, \gamma, \delta \in \mathbb{C}$. In particular, $2\varphi_2^0 = \beta z_2^{l+1}$. We write P_0 as

$$P_0(z_2, \bar{z}_2) = \sum_{j=j_0}^{l+1-j_0} A_j z_2^j \bar{z}_2^{l+1-j} \quad (3.29)$$

with $A_{j_0} \neq 0$. Recall that $j_0 \neq 0$ by assumption. Substituting into (3.28) and comparing coefficients of $z_2^{j_0} \bar{z}_2^{2l+1-j_0}$ and $z_2^{l+1-j_0} \bar{z}_2^{l+j_0}$ we obtain

$$\alpha = j_0 \beta, \quad \alpha = (l+1-j_0) \beta. \quad (3.30)$$

This gives

$$2j_0 = l+1 \quad (3.31)$$

and $\beta = \frac{2}{l+1} \alpha$. It follows that l is odd and $P_0 = d|z|^{l+1}$ for some $d \in \mathbb{R}$. If $d \neq 0$, we use the explicit forms of P_0 and P_1 along with (3.23) and (3.27). By (3.24) we have

$$-\frac{1}{2} \operatorname{Im} (\varphi_1^0)^2 = \operatorname{Re} \psi^0,$$

which gives

$$\psi^0 = \frac{i}{2} (\varphi_1^0)^2.$$

From (3.26) we obtain from the equation for the coefficient of z_2^{2l}

$$\frac{1}{2} \alpha^2 = \frac{l}{l+1} \alpha^2,$$

which gives $l = 1$. The vice versa is immediate to verify, using the above calculations (see also Section 4). That finishes the proof. $\square$

Next we turn to the second case, where $P_2 \neq 0$. Using rescaling in the z_1 variable, we may assume $P_2 = 1$.

Lemma 3.4. *Let $X = i \partial_{z_1}$ be in $\operatorname{aut}(M_H, 0)$ and P be of the form*

$$P(z, \bar{z}) = x_1^2 + x_1 P_1(z_2, \bar{z}_2) + P_0(z_2, \bar{z}_2). \quad (3.32)$$

Then there is a vector field Y in $\operatorname{aut}(M_H, 0)$ such that $[Y, W] = X$, if and only if P is biholomorphically equivalent, by a change of multitype coordinates, to

$$P(z, \bar{z}) = x_1^2 + c|z_2|^{2l} \quad (3.33)$$

for some $c \in \mathbb{R} \setminus \{0\}$ and $l \in \mathbb{N}$.

Proof. Since $X = i \partial_{z_1}$, we have again

$$Y = iw \partial_{z_1} + \varphi_1 \partial_{z_1} + \varphi_2 \partial_{z_2} + \psi \partial_w. \quad (3.34)$$

Without any loss of generality, we can assume that both P_1 and P_0 contain no harmonic terms. Note that harmonic terms in P_1 can be eliminated by a change of variables $z'_1 = z_1 + \alpha z_2^l$, where l is the degree of P_1 and α is a suitable constant.

In the first part of the proof, we will show that under this assumption, $P_1 = 0$. Applying $\operatorname{Re} Y$ to $P - v$ gives

$$-(2x_1 + P_1)(x_1^2 + x_1 P_1 + P_0) + 2\operatorname{Re} \varphi_1 \frac{\partial P}{\partial z_1} + 2\operatorname{Re} \varphi_2 \frac{\partial P}{\partial z_2} - \operatorname{Im} \psi = 0. \quad (3.35)$$

Let k be as in (3.4). Assume first that $k > 1$. If $\varphi_1^k \neq 0$ we get

$$x_1 \operatorname{Re} \varphi_1^k z_1^{k+1} - \frac{1}{2} \operatorname{Im} \psi^k z_1^{k+2} = 0,$$

which gives a contradiction, since the second term is harmonic in z_1 , while the first one is not. Hence $\varphi_1^k = 0$ and we have, for terms of degree $k + 1$ in z_1 ,

$$2\delta_{k,2} x_1^3 = 2x_1 \operatorname{Re} \varphi_1^{k-1} z_1^k + 2x_1 \operatorname{Re} \varphi_2^k z_1^k \frac{\partial P_1}{\partial z_2} - \operatorname{Im} \psi^{k-1} z_1^{k+1}. \quad (3.36)$$

Looking at the coefficients of y_1^{k+1} we see that ψ^{k-1} is a real or imaginary constant, depending on the parity of k . If $\psi^{k-1} \neq 0$, then by homogeneity P_1 has degree at most one, hence it must be zero. If $\psi^{k-1} = 0$, after dividing by x_1 , the right-hand side is harmonic in z_1 . It follows that $\delta_{k,2} = 0$, hence $k > 2$. But then $\varphi_2^k \frac{\partial P_1}{\partial z_2}$ cannot contain any mixed terms, hence $P_1 = 0$, as claimed.

Now let us assume that $k = 1$. For the third order terms in z_1 we obtain

$$2x_1^3 = 2x_1 \operatorname{Re} \varphi_1^1 z_1^2 - \operatorname{Im} \psi^1 z_1^3. \quad (3.37)$$

Note that this determines φ_1^1 and ψ^1 . Looking at terms of second order in z_1 we obtain from (3.35) and (3.32)

$$-3x_1^2 P_1 + 2x_1 \operatorname{Re} \varphi_1^0 z_1 + 2x_1 \operatorname{Re} \varphi_2^1 z_1 \frac{\partial P_1}{\partial z_2} + 2\operatorname{Re} \varphi_1^1 z_1^2 P_1 - \operatorname{Im} \psi^0 z_1^2 = 0. \quad (3.38)$$

Looking at coefficients of y_1^2 , we obtain that P_1 is harmonic, hence $P_1 = 0$.

We will further assume that $P_1 = 0$. We obtain for terms linear in z_1 ,

$$-2x_1 P_0 + 2\operatorname{Re} z_1 \varphi_2^1 \frac{\partial P_0}{\partial z_2} + 2x_1 \operatorname{Re} \varphi_1^{-1} - \operatorname{Im} \psi^{-1} z_1 = 0 \quad (3.39)$$

which gives equations for coefficients of x_1 and y_1 . Namely

$$2P_0 + 2\operatorname{Re} \varphi_2^1 \frac{\partial P_0}{\partial z_2} + 2\operatorname{Re} \varphi_1^{-1} - \operatorname{Im} \psi^{-1} = 0 \quad (3.40)$$

and

$$-2\operatorname{Im} \varphi_2^1 \frac{\partial P_0}{\partial z_2} - \operatorname{Re} \psi^{-1} = 0. \quad (3.41)$$

Using $\varphi_2^1 = \alpha z_2$ for some $\alpha \in \mathbb{C}$ and the fact that P_0 contains no harmonic terms, it follows that

$$\operatorname{Im} \varphi_2^1 \frac{\partial P_0}{\partial z_2} = 0, \quad (3.42)$$

hence

$$P_0 = -\varphi_2^1 \frac{\partial P_0}{\partial z_2}. \quad (3.43)$$

It follows that P_0 has a complex reproducing field, hence $P_0 = c|z_2|^{2l}$, as claimed. That finishes the proof. $\square$

4. Proofs of Theorems 1.1-1.9

In this section we complete the proofs of the results stated in the introduction, up to Theorem 1.9. Theorem 1.1 is an immediate consequence of Theorem 1.2.

Proof of Theorem 1.2. We apply Lemma 3.1-3.4. Note that by Lemma 3.1 and Lemma 3.2, we obtain either $\mu_1 = \frac{1}{2}$, or $\mu_1 = \mu_2$. Hence we can assume without any loss of generality that $i\partial_{z_1} \in \operatorname{aut}(M_H, 0)$. Using suitable scaling, rotation and adding harmonic terms leads to the form given in the statement of the theorem. $\square$

Theorem 1.7 follows immediately from [17, Theorem 4.7]. Combining Theorem 1.2, 1.4 and 1.7 with the results of [17] leads to Theorem 1.8.

Proof of Theorem 1.9. If $\mathfrak{g}_1 \neq 0$, then by [17, Theorem 4.7] we have

$$P(z, \bar{z}) = \sum_{|\alpha|_\Lambda = |\bar{\alpha}|_\Lambda = 1} A_{\alpha, \bar{\alpha}} z^\alpha \bar{z}^{\bar{\alpha}} \quad (4.1)$$

for some pair $\Lambda = (\lambda_1, \lambda_2)$ (not necessarily equal to the multitype weight). Let K be the number of nonzero terms in the sum. We order the multiindices and write P as

$$P(z, \bar{z}) = \sum_{j=1}^K A_j z^{\alpha_j} \bar{z}^{\alpha_{K+j}}. \quad (4.2)$$

Consider the hyperquadric in $\mathbb{C}^{2K+1}$ defined by

$$\operatorname{Im} \eta = \sum_{j=1}^K A_j \zeta_j \overline{\zeta_{K+j}}, \quad (4.3)$$

and the mapping $f : \mathbb{C}^3 \rightarrow \mathbb{C}^{2K+1}$ given by $\eta = w$ and $\zeta_j = z^{\alpha_j}$ for $j = 1, \dots, 2K$.

It is immediate to verify that the vector field in $\text{aut}(M_H, 0)$

$$Y = (\lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2}) w + \frac{1}{2} w^2 \partial_w,$$

is f -related to the infinitesimal automorphism of the above hyperquadric given by

$$Z = \frac{1}{2} \eta \sum_{j=1}^{2K} \zeta_j \partial_{\zeta_j} + \frac{1}{2} \eta^2 \partial_\eta.$$

Next, if $\mathfrak{g}_n \neq 0$ we consider the two cases from Theorem 1.2. We have $K = 3$ in both cases. In the first case, we define f by $\eta = w$, $\zeta_1 = z_1$, $\zeta_2 = z_2^l$. We verify that the vector field

$$Y_1 = aw \partial_{z_1} - i\bar{a} z_1 z_2^l \partial_{z_1} - i\bar{a} \frac{1}{l} z_2^{l+1} \partial_{z_2} + 2i\bar{a} z_2^l w \partial_w, \quad a \in \mathbb{C},$$

in $\text{aut}(M_H, 0)$ is f -related to the infinitesimal automorphism of the hyperquadric,

$$\text{Im } \eta = \text{Re } \zeta_1 \bar{\zeta}_2, \quad (4.4)$$

given by

$$Z_1 = a\eta \partial_{\zeta_1} - i\bar{a} \zeta_1 \zeta_2 \partial_{\zeta_1} - i\bar{a} \zeta_2^2 \partial_{\zeta_2} + 2i\bar{a} \zeta_2 \eta \partial_\eta, \quad a \in \mathbb{C}.$$

The second case is completely analogous. The case of $\mathfrak{g}_c$ follows from [18, Theorem 1.2]. This finishes the proof. $\square$

5. Dimension of $\text{aut}(M_H, 0)$

In this section we will again assume that M_H is a holomorphically nondegenerate model given by (1.7).

We will first prove two auxiliary lemmata. Let us denote $\mathfrak{g}_t = \mathfrak{g}_{-\mu_1} \oplus \mathfrak{g}_{-\mu_2}$, the part of $\text{aut}(M_H, 0)$ containing complex tangential shifts.

Lemma 5.1. *Let there exist two regular vector fields in $\mathfrak{g}_t$ whose values at 0 are linearly independent over $\mathbb{R}$, but dependent over $\mathbb{C}$. Then M_H is biholomorphic to*

$$\text{Im } w = Cx_1^2 + x_1 \text{Re } \alpha z_2^l + Q(z_2, \bar{z}_2) \quad (5.1)$$

for some $C \in \mathbb{R}$, $\alpha \in \mathbb{C}$ and homogeneous polynomial Q without harmonic terms.

Proof. Let Z_1, Z_2 be such vector fields. Without loss of generality, we may assume that $Z_1 = i \partial_{z_1}$ and

$$Z_2 = \partial_{z_1} + \psi(z_1, z_2) \partial_w.$$

Note that this form is attained using transformations preserving multitype coordinates (with pluriharmonic terms allowed). The commutator of Z_1, Z_2 either vanishes, or lies in $\mathfrak{g}_{-1}$, *i.e.*, it is a real multiple of ∂_w . This leads to $\psi(z_1, z_2) = Cz_1 + \alpha z_2^l$ for some $\alpha \in \mathbb{C}, C \in \mathbb{R}$ and $l \in \mathbb{N}$. From $\operatorname{Re} Z_1(P) = 0$ it follows that

$$P(z_1, z_2, \bar{z}_1, \bar{z}_2) = \sum_{j=0}^m x_1^j P_j(z_2, \bar{z}_2). \quad (5.2)$$

From $\operatorname{Re} Z_2(P) = 0$ we obtain that $\operatorname{Re} \frac{\partial P}{\partial z_1}$ is pluriharmonic. It follows that

$$P(z_1, z_2, \bar{z}_1, \bar{z}_2) = Cx_1^2 + x_1 \operatorname{Re} \alpha z_2^l + P_0(z_2, \bar{z}_2), \quad (5.3)$$

which finishes the proof. $\square$

Recall that by the results of [17] and [18], $\dim \mathfrak{g}_c \leq 1$ and $\dim \mathfrak{g}_1 \leq 1$. The following lemma considers the case when both components are nontrivial.

Lemma 5.2. *Assume that $\dim \mathfrak{g}_c = \dim \mathfrak{g}_1 = 1$. Then $\dim \mathfrak{g}_0 = 3$.*

Proof. Let $Z \in \mathfrak{g}_0$ be a rotation and $Y \in \mathfrak{g}_1$ be nonzero vector field. By [17, Theorem 4.7], Y has the form

$$Y = \sum_{j=1}^2 \varphi_j(z) w \partial_{z_j} + \frac{1}{2} w^2 \partial_w \quad (5.4)$$

where the φ_j have the complex reproducing field property

$$2 \sum_{j=1}^2 \varphi_j(z) P_{z_j} = P(z, \bar{z}). \quad (5.5)$$

It is immediate to verify that in Jordan normal form the linear vector field $\sum_{j=1}^2 \varphi_j(z) \partial_{z_j}$ is diagonal with real coefficients. We can thus consider multitype coordinates in which Y is a real multiple of

$$(\lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2}) w + \frac{1}{2} w^2 \partial_w.$$

We claim that in such coordinates, Z is also diagonal. Indeed, let $X \in \mathfrak{g}_c$ be a nonzero vector field. The commutator of X and Y is of weight bigger than one, so $[X, Y] = 0$, by the results of [17]. It follows that the pair (λ_1, λ_2) is linearly independent with the multitype weights (μ_1, μ_2) . If $\mu_1 \neq \mu_2$, any rotation is diagonal. So we may assume $\mu_1 = \mu_2$, which implies $\lambda_1 \neq \lambda_2$. The commutator of Z with Y has to be a real multiple of Y . Computing the commutator, it follows immediately

that Z is diagonal. Next, the rotations with real coefficients have dimension one, the coefficients being given by $1 - \mu_1$ and $1 - \mu_2$. Let us write P in the form

$$P(z_1, z_2, \bar{z}_1, \bar{z}_2) = \sum_{|\alpha, \hat{\alpha}|_{\Lambda_M} = 1} A_{\alpha, \hat{\alpha}} z^\alpha \bar{z}^{\hat{\alpha}}.$$

If there exist in addition two linearly independent imaginary rotations, we have $\alpha_1 = \hat{\alpha}_1, \alpha_2 = \hat{\alpha}_2$ whenever $A_{\alpha, \hat{\alpha}} \neq 0$. From the real rotation and weighted homogeneity we obtain a unique solution for $\alpha_1, \alpha_2, \hat{\alpha}_1, \hat{\alpha}_2$. That contradicts holomorphic nondegeneracy of M_H . On the other hand, there is an imaginary rotation with coefficients $i\lambda_1, i\lambda_2$. Hence there are two linearly independent rotations, and therefore $\dim \mathfrak{g}_0 = 3$. $\square$

Lemma 5.3. *There exist multitype coordinates in which every rotation is linear. Moreover,*

$$\dim \mathfrak{g}_0 \leq 5.$$

Proof. The first part of the statement is proved in [17, Proposition 3.9]. In the normalization of [17, Proposition 2.6], it is immediate to check that the nonzero vector fields given by

$$Z = (az_1 + \beta z_2)\partial_{z_1} + cz_2\partial_{z_2}, \quad (5.6)$$

where $a, c \in \mathbb{R}, \beta \in \mathbb{C}$ do not belong to $\mathfrak{g}_0$. Hence the space of rotations is at most four dimensional, and $\dim \mathfrak{g}_0 \leq 5$. $\square$

Proof of Theorem 1.10. First we consider the case when $\dim \mathfrak{g}_n > 0$. By Theorem 1.2 M_H is given by (1.4) or (1.10). The vector fields in $\text{aut}(M_H, 0)$ of (1.4) are described explicitly in [22], which shows that the dimension is equal to 10. A completely analogous computation gives all vector fields in $\text{aut}(M_H, 0)$ for (1.10), and shows that in this case, $\dim \text{aut}(M_H, 0) = 9$. Indeed, we have $\dim \mathfrak{g}_{-\frac{1}{2}} = 2$ and $\dim \mathfrak{g}_{\frac{1}{2}} = 2$. Further, since $\mu_1 \neq \mu_2$, all rotations have to be diagonal, which gives immediately $\dim \mathfrak{g}_0 = 3$. By the results of [18], we have $\dim \mathfrak{g}_c = 0$. Since $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_1 = 1$, we obtain $\dim \text{aut}(M_H, 0) = 9$.

Next we consider the case when $\dim \mathfrak{g}_n = 0$ and show that in this case

$$\dim \text{aut}(M_H, 0) \leq 7.$$

We will consider all possible dimensions of $\mathfrak{g}_t$, namely 0, 1, 2, 3. Note that if $\dim \mathfrak{g}_t = 4$, by Lemma 5.1 it follows that $\mu_1 = \mu_2 = \frac{1}{2}$, hence M_H is a Levi nondegenerate hyperquadric.

Let us first assume that $\dim \mathfrak{g}_t = 0$. Then by Theorem 1.2, Lemma 5.2 and 5.3 we obtain immediately $\dim \text{aut}(M_H, 0) \leq 7$.

Next assume that $\dim \mathfrak{g}_t = 1$. If $\dim \mathfrak{g}_c + \dim \mathfrak{g}_1 = 2$, then by Lemma 5.2 $\dim \mathfrak{g}_0 \leq 3$, and since $\dim \mathfrak{g}_n = 0$, this leads to $\dim \text{aut}(M_H, 0) \leq 7$. Now let $\dim \mathfrak{g}_1 + \dim \mathfrak{g}_c \leq 1$. Without any loss of generality, we can assume $Y = \partial_{z_1} \in \mathfrak{g}_t$. By

the partial normalization [17, Proposition 2.6], the five dimensional linear subspace of vector fields of the form

$$Z = (\alpha z_1 + \beta z_2)\partial_{z_1} + dz_2\partial_{z_2}, \quad (5.7)$$

where $\alpha, \beta \in \mathbb{C}$, $d \in \mathbb{R}$ has trivial intersection with $\mathfrak{g}_0$. This follows from Lemma 5.3 and the fact that if $Z \in \text{aut}(M_H, 0)$, the commutator of Z and Y has to be a real multiple of Y . It implies $\dim \mathfrak{g}_0 \leq 4$. Since $\dim \mathfrak{g}_n = 0$, we obtain $\dim \text{aut}(M_H, 0) \leq 7$.

Next, let $\dim \mathfrak{g}_t = 2$ and the values of two generators of $\mathfrak{g}_t$ at 0 are complex independent. We claim that $\dim \mathfrak{g}_0 \leq 2$. Indeed, let $Z_1 \in \mathfrak{g}_{-\mu_1}$, $Z_2 \in \mathfrak{g}_{-\mu_2}$ be two vector fields, whose values at 0 are complex independent. Since their commutator is of weight $-\mu_1 - \mu_2 > -1$, it must vanish. We may therefore assume $Z_1 = i\partial_{z_1}$, $Z_2 = i\partial_{z_2}$. Hence P is a function of x_1, x_2 . Again, this form is attained by a transformation preserving multitype coordinates (with pluriharmonic terms allowed). Computing commutators of Z_1 and Z_2 with a rigid element of $\mathfrak{g}_0$ of the form

$$Y = (az_1 + bz_2)\partial_{z_1} + (cz_1 + dz_2)\partial_{z_2} + \psi(z_1, z_2)\partial_w, \quad (5.8)$$

we see that the coefficients a, b, c, d must be real. It follows that

$$\text{Re } Y(P - v) = (ax_1 + bx_2)\frac{\partial P}{\partial x_1} + (cx_1 + dx_2)\frac{\partial P}{\partial x_2} - \frac{1}{2}\text{Im } \psi(z_1, z_2) = 0.$$

The last term is pluriharmonic and the remaining part is a function of x_1, x_2 . It follows that the pluriharmonic term vanishes, and we obtain

$$\frac{\frac{\partial P}{\partial x_2}}{\frac{\partial P}{\partial x_1}} = -\frac{ax_1 + bx_2}{cx_1 + dx_2}.$$

It follows that a, b, c, d are determined uniquely by P , up to a real multiple. Note that the left hand side is nonconstant, otherwise M_H is holomorphically degenerate. Hence $\dim \mathfrak{g}_0 \leq 2$. It follows that $\dim \mathfrak{g} \leq 7$.

If $\mathfrak{g}_t$ contains two complex dependent at 0 vector fields, we use Lemma 5.1 and consider a rotation

$$Y = (az_1 + bz_2)\partial_{z_1} + (cz_1 + dz_2)\partial_{z_2} + \psi(z_1, z_2)\partial_w. \quad (5.9)$$

Let first $C \neq 0$. After scaling in z_1 and absorbing the mixed term into x_1^2 , we may assume

$$P(z_1, z_2, \bar{z}_1, \bar{z}_2) = x_1^2 + Q(z_2, \bar{z}_2),$$

where Q is different from $|z_2|^l$, since $\dim \mathfrak{g}_n = 0$. We obtain

$$2x_1 \text{Re } (az_1 + bz_2) + \text{Re } (cz_1 + dz_2)\frac{\partial Q}{\partial z_2} - \frac{1}{2}\text{Im } \psi(z_1, z_2) = 0. \quad (5.10)$$

From the coefficient of y_1 we obtain $\operatorname{Im}(c \frac{\partial Q}{\partial z_2}) = 0$, which implies $ic\partial_{z_2} \in \mathfrak{g}_t$. It follows that $c = 0$. Hence

$$2x_1 \operatorname{Re}(az_1) + 2x_1 \operatorname{Re}(bz_2) + \operatorname{Re} d z_2 \frac{\partial Q}{\partial z_2} - \frac{1}{2} \operatorname{Im} \psi(z_1, z_2) = 0. \quad (5.11)$$

The first three terms have different powers of z_1 , so they must all be pluriharmonic. It follows that a is purely imaginary and $b = d = 0$. Hence $\dim \mathfrak{g}_0 \leq 2$ which implies $\dim \mathfrak{g} \leq 6$.

Let now $C = 0$, *i.e.*,

$$P(z_1, z_2, \bar{z}_1, \bar{z}_2) = x_1 \operatorname{Re} \alpha z_2^l + Q(z_2, \bar{z}_2).$$

Without any loss of generality, we will assume that Q contains no harmonic terms, and no $(1, l)$ terms, which can be absorbed into the first term by a change of variables $z_1^* = z_1 + \gamma z_2$. We have

$$\begin{aligned} & 2\operatorname{Re}(az_1 + bz_2) \operatorname{Re} \alpha z_2^l + 2lx_1 \operatorname{Re}(\alpha z_2^{l-1}(cz_1 + dz_2)) \\ & + 2\operatorname{Re} \left[(cz_1 + dz_2) \frac{\partial Q}{\partial z_2} \right] - \operatorname{Im} \psi(z_1, z_2) = 0. \end{aligned}$$

From the equation for the coefficient of $z_1 \bar{z}_1 z_2^{l-1}$ we obtain that $c = 0$. From terms of order zero in z_1 , looking at the coefficient of $\bar{z}_2 z_2^l$ we obtain $b = 0$, since Q contains no $(1, l)$ terms. From the coefficient of $z_1 \bar{z}_2^l$ we obtain $a + l\bar{d} = 0$. This gives $\dim \mathfrak{g}_0 \leq 3$ and $\dim \mathfrak{g} \leq 7$.

If $\dim \mathfrak{g}_t = 3$, we use Lemma 5.1 to conclude that P must be biholomorphic to

$$\operatorname{Im} w = |z_1|^2 + (\operatorname{Re} z_2)^l, \quad (5.12)$$

where $l > 2$. Since $\mu_1 \neq \mu_2$, the rotations have to be diagonal. It is immediate to verify that the only rotations are real multiples of $iz_1 \partial_{z_1}$. Hence $\dim \mathfrak{g}_0 = 2$ and $\dim \mathfrak{g} \leq 7$.

To finish the proof, we give examples of models with $\operatorname{aut}(M_H, 0) = 7, 6, 5, 4, 3, 2$. The hypersurface given by

$$\operatorname{Im} w = \left(\sum_{j=1}^2 |z_j|^2 \right)^2 \quad (5.13)$$

admits the same rotations as the sphere, hence $\dim \mathfrak{g}_0 = 5$ and $\dim \operatorname{aut}(M_H, 0) = 7$.

Six-dimensional $\operatorname{aut}(M_H, 0)$ occurs for

$$\operatorname{Im} w = |z_1|^2 + (\operatorname{Re} z_2)^3, \quad (5.14)$$

which has $\dim \mathfrak{g}_t = 3$ and $\dim \mathfrak{g}_0 = 2$.

Dimension five occurs for

$$\operatorname{Im} w = \sum_{j=1}^2 |z_j|^4,$$

where $\dim \mathfrak{g}_0 = 3$.

Dimension four occurs for

$$\operatorname{Im} w = (\operatorname{Re} z_1)^3 + (\operatorname{Re} z_2)^4, \quad (5.15)$$

where $\dim \mathfrak{g}_t = 2$.

Dimension three occurs for

$$\operatorname{Im} w = \operatorname{Re} z_1 \bar{z}_1^3 + (\operatorname{Re} z_2)^3, \quad (5.16)$$

with $\dim \mathfrak{g}_t = 1$.

Dimension two occurs for a generic model, e.g.,

$$\operatorname{Im} w = \operatorname{Re} \left(\sum_{j=1}^2 z_j \bar{z}_j^3 \right). \quad (5.17)$$

This finishes the proof of the theorem. $\square$

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