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Isogenies of Abelian Anderson A-modules and A-motives

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Abstract. As a generalization of Drinfeld modules, Greg Anderson introduced Abelian t-modules and t-motives over a perfect base field. In this article we study relative versions of these defined over base rings. We investigate isogenies among them. Our main results state that every isogeny possesses a dual isogeny in the opposite direction, and that a morphism between Abelian t-modules is an isogeny if and only if the corresponding morphism between their associated t-motives is an isogeny. We also study torsion submodules of Abelian t-modules which in general are non-reduced group schemes. They can be obtained from the associated t-motive via the finite shtuka correspondence of Drinfeld and Abrashkin. The inductive limits of torsion submodules are the function field analogs of p-divisible groups. These limits correspond to the local shtukas attached to the t-motives associated with the Abelian t-modules. In this sense the theory of Abelian t-modules is captured by the theory of t-motives.

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1. Introduction

In 1974 Drinfeld [12] defined "elliptic modules" as function field analogs of elliptic curves. These are today called *Drinfeld modules*. As higher dimensional generalizations of Drinfeld modules and function field analogs of Abelian varieties, Greg Anderson [2] introduced *Abelian t-modules* and *t-motives* over a perfect base field. In this article we study families, that is, relative versions of these defined over base rings, and we generalize them to *Abelian Anderson A-modules* and *A-motives*. The upshot of our results is that the entire theory of Abelian Anderson A-modules is contained in the theory of A-motives. More precisely, let $\mathbb{F}_q$ be a finite field with q elements, let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ and let $Q = \mathbb{F}_q(C)$ be its function field. Let $\infty \in C$

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be a closed point and let $A = \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$ be the ring of functions which are regular outside ∞. Let (R, γ) be an *A-ring*, that is a commutative unitary ring together with a ring homomorphism $\gamma\colon A \to R$. We consider the ideal $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a)\colon a \in A) = \ker(\gamma \otimes \mathrm{id}_R\colon A_R \to R) \subset A_R := A \otimes_{\mathbb{F}_q} R$ and the endomorphism $\sigma := \mathrm{id}_A \otimes \mathrm{Frob}_{q,R}\colon a \otimes b \mapsto a \otimes b^q$ of A_R. For an A_R-module M we set $\sigma^*M := M \otimes_{A_R, \sigma} A_R = M \otimes_{R, \mathrm{Frob}_{q,R}} R$, and for an element $m \in M$ we write $\sigma_M^* m := m \otimes 1 \in \sigma^*M$. Also we let $\mathbb{N}_0$ be the set of non-negative integers.

Definition 1.1. An *effective A-motive of rank r over an A-ring R* is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_R-module M of rank r and an A_R-homomorphism $\tau_M\colon \sigma^*M \to M$ whose cokernel is annihilated by $\mathcal{J}^n$ for some positive integer n. We say that $\underline{M}$ has *dimension d* if $\operatorname{coker}\tau_M$ is a locally free R-module of rank d and annihilated by $\mathcal{J}^d$. We write $\operatorname{rk}\underline{M} = r$ and $\dim \underline{M} = d$ for the rank and the dimension of $\underline{M}$.

A *morphism* $f\colon (M, \tau_M) \to (N, \tau_N)$ between effective A-motives is an A_R-homomorphism $f\colon M \to N$ which satisfies $f \circ \tau_M = \tau_N \circ \sigma^* f$.

Note that τ_M is always injective and $\operatorname{coker}(\tau_M)$ is always a finite locally free R-module by Proposition 2.3 below. We give some explanations for this definition in Section 2 and also define non-effective A-motives. If R is a perfect field, $q = p$ is a prime, $A = \mathbb{F}_p[t]$ and in addition, M is finitely generated over the non-commutative polynomial ring $R\{\tau\} := \left\{ \sum_{i=0}^n b_i \tau^i : n \in \mathbb{N}_0, b_i \in R \right\}$ with $\tau b = b^p \tau$, which acts on $m \in M$ via $\tau\colon m \mapsto \tau_M(\sigma_M^* m)$, then (M, τ_M) is a t-motive in the sense of Anderson [2, Section 1.2].

Next let us define Abelian Anderson A-modules by first agreeing that all group schemes in this article are assumed to be commutative. In Section 3 we give some explanations on the terminology in the following

Definition 1.2. Let d and r be positive integers. An *Abelian Anderson A-module of rank r and dimension d over R* is a pair $\underline{E} = (E, \varphi)$ consisting of a smooth affine group scheme E over $\operatorname{Spec} R$ of relative dimension d, and a ring homomorphism $\varphi\colon A \to \mathrm{End}_{R\text{-groups}}(E)$, $a \mapsto \varphi_a$ such that

(a) There is a faithfully flat ring homomorphism $R \to R'$ for which $E \times_R \operatorname{Spec} R' \cong \mathbb{G}_{a,R'}^d$ as $\mathbb{F}_q$-module schemes, where $\mathbb{F}_q$ acts on E via φ and $\mathbb{F}_q \subset A$;
(b) $(\operatorname{Lie}\varphi_a - \gamma(a))^d = 0$ on $\operatorname{Lie} E$ for all $a \in A$;
(c) The set $M := M(\underline{E}) := M_q(\underline{E}) := \mathrm{Hom}_{R\text{-groups}, \mathbb{F}_q\text{-lin}}(E, \mathbb{G}_{a,R})$ of $\mathbb{F}_q$-equivariant homomorphisms of R-group schemes is a locally free A_R-module of rank r under the action given on $m \in M$ by

$$A \ni a\colon\; M \longrightarrow M,\; m \mapsto m \circ \varphi_a$$
$$R \ni b\colon\; M \longrightarrow M,\; m \mapsto b \circ m.$$

A *morphism* $f\colon (E, \varphi) \to (E', \varphi')$ between Abelian Anderson A-modules is a homomorphism of group schemes $f\colon E \to E'$ over R which satisfies $\varphi'_a \circ f = f \circ \varphi_a$ for all $a \in A$.

Remark 1.3. In particular, if $R = k$ is a perfect field, $q = p$ is a prime and $A = \mathbb{F}_p[t]$, then an Abelian Anderson A-module is nothing else than an Abelian t-module in the sense of Anderson [2, Section 1.1]. Indeed, Anderson requires that E is isomorphic to $\mathbb{G}_{a,k}^d$ over k. This is implied by our condition (a) by [28, Chapter VII, Proposition 11] and [29, XVII, Lemme 2.3 bis]. Our definition is the natural generalization to arbitrary A-rings R. Likewise our condition (c) that $M(\underline{E})$ is a locally free A_R-module generalizes Anderson's condition that $M(\underline{E})$ is a finite free module over the principal ideal domain $A_k = k[t]$ when $R = k$ is a perfect field; see [2, Section 1.1 and Lemma 1.4.5]. This is a severe restriction on $\underline{E}$, but was intended already by Anderson. Namely, we will see that for general A and R, the Abelian Anderson A-modules of dimension 1 over R are precisely the Drinfeld A-modules over R; see Definition 3.7 and Theorem 3.9. It was Anderson's motivation to define and study higher dimensional generalizations of Drinfeld A-modules in the same spirit as Abelian varieties are higher dimensional generalizations of elliptic curves, the number field analogs of Drinfeld A-modules. Condition (c) is crucial for the intended analogy between Abelian Anderson A-modules and Abelian varieties, because it determines the structure of endomorphism rings and torsion points; see [2, Corollary 1.7.3 and Proposition 1.8.3] and our generalizations Corollary 3.6 and Theorem 6.4.

When q is not a prime and R is not a field, we do not know the answer to the following

Question 1.4. If we weaken Definition 1.2(a) and only require that there is an isomorphism of *group schemes* $E \times_{\operatorname{Spec} R} \operatorname{Spec} R' \cong \mathbb{G}_{a,R'}^d$, do we get an equivalent definition?

Anderson's anti-equivalence [2, Theorem 1] between Abelian t-modules and t-motives directly generalizes to the following:

Theorem 3.5. *Let R be a fixed A-ring. If $\underline{E} = (E, \varphi)$ is an Abelian Anderson A-module over R, then $\underline{M}(\underline{E}) = (M, \tau_M)$ with $\tau_M \colon \sigma^* M \to M$, $\sigma_M^* m \mapsto \operatorname{Frob}_{q,\mathbb{G}_{a,R}} \circ m$ is an effective A-motive over R of the same rank and dimension as $\underline{E}$. The contravariant functor $\underline{E} \mapsto \underline{M}(\underline{E})$ between Abelian Anderson A-modules over R and A-motives over R is fully faithful. Its essential image consists of all effective A-motives $\underline{M} = (M, \tau_M)$ over R for which there exists a faithfully flat ring homomorphism $R \to R'$ such that $M \otimes_R R'$ is a finite free left $R'\{\tau\}$-module under the map $\tau \colon M \to M$, $m \mapsto \tau_M(\sigma_M^* m)$.*

The main purpose of this article is to study isogenies and their (co-)kernels over arbitrary A-rings R. Here a morphism $f \colon \underline{E} \to \underline{E}'$ between Abelian Anderson A-modules over R is an *isogeny* if it is finite and surjective. On the other hand, a morphism $f \in \operatorname{Hom}_R(\underline{M}, \underline{N})$ between A-motives over R is an *isogeny* if f is injective and $\operatorname{coker} f$ is finite and locally free as R-module. We give equivalent characterizations in Propositions 5.2, 5.4 and 5.8. The following are our two main results.

Theorem 5.9. *Let $f \in \mathrm{Hom}_R(\underline{E}, \underline{E}')$ be a morphism between Abelian Anderson A-modules over an A-ring R, and let $\underline{M}(f) \in \mathrm{Hom}_R(\underline{M}', \underline{M})$ be the associated morphism between the associated effective A-motives $\underline{M} = \underline{M}(\underline{E})$ and $\underline{M}' = \underline{M}(\underline{E}')$ over R. Then*

(a) *f is an isogeny if and only if $\underline{M}(f)$ is an isogeny;*
(b) *If f is an isogeny, then* $\ker f$ *and* $\mathrm{coker}\,\underline{M}(f)$ *correspond to each other under the finite shtuka equivalence which we review in Section* 4.

Corollary 5.15. *If $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ is an isogeny between A-motives over an A-ring R, then there is an element $0 \neq a \in A$ and an isogeny $g \in \mathrm{Hom}_R(\underline{N}, \underline{M})$ with $f \circ g = a \cdot \mathrm{id}_{\underline{N}}$ and $g \circ f = a \cdot \mathrm{id}_{\underline{M}}$. The same is true for Abelian Anderson A-modules.*

This leads to the following result about torsion points in Section 6. Let $(0) \neq \mathfrak{a} \subset A$ be an ideal and let $\underline{E} = (E, \varphi)$ be an Abelian Anderson A-module over R. The *$\mathfrak{a}$-torsion submodule $\underline{E}[\mathfrak{a}]$ of $\underline{E}$* is the closed subscheme of E defined by $\underline{E}[\mathfrak{a}](S) = \{\, P \in E(S) \colon \varphi_a(P) = 0 \text{ for all } a \in \mathfrak{a} \,\}$ on any R-algebra S.

Theorem 6.4. *$\underline{E}[\mathfrak{a}]$ is a finite locally free group scheme over R. It is étale over R if and only if $\mathfrak{a} + \mathcal{J} = A_R$. If $\underline{M} = \underline{M}(\underline{E})$ is the associated A-motive then $\underline{E}[\mathfrak{a}]$ and $\underline{M}/\mathfrak{a}\underline{M}$ correspond to each other under the finite shtuka equivalence reviewed in Section* 4.

If $\mathfrak{a} + \mathcal{J} = A_R$ and $\bar{s} = \mathrm{Spec}\,\Omega$ is a geometric base point of $\mathrm{Spec}\,R$, then we also prove in Theorem 6.6 that $\underline{E}[\mathfrak{a}](\Omega)$ is a free $A/\mathfrak{a}$-module of rank r which carries a continuous action of the étale fundamental group $\pi_1^{\text{ét}}(\mathrm{Spec}\,R, \bar{s})$.

In the final Section 7 we turn towards the case where $\mathfrak{p} \subset A$ is a maximal ideal and where all elements of $\gamma(\mathfrak{p}) \subset R$ are nilpotent. In this case, we can associate with an A-motive $\underline{M}$ over R a *local shtuka* $\hat{\underline{M}}_{\mathfrak{p}}(\underline{M})$; see Example 7.2 and with an Abelian Anderson A-module $\underline{E}$ a *divisible local Anderson module* $\underline{E}[\mathfrak{p}^\infty] := \varinjlim \underline{E}[\mathfrak{p}^n]$ in the sense of [19]; see Definition 7.3 and Theorem 7.6. If $\underline{M} = \underline{M}(\underline{E})$ then $\hat{\underline{M}}_{\mathfrak{p}}(\underline{M})$ and $\underline{E}[\mathfrak{p}^\infty]$ correspond to each other under the local shtuka equivalence from [19]; see Theorems 7.4 and 7.6.

Notation

Throughout this article we denote by

$\mathbb{N}_{>0}$ and $\mathbb{N}_0$	the positive, respectively the non-negative integers,
$\mathbb{F}_q$	a finite field with q elements and characteristic p,
C	a smooth projective geometrically irreducible curve over $\mathbb{F}_q$,
$Q := \mathbb{F}_q(C)$	the function field of C,
∞	a fixed closed point of C,

$\mathbb{F}_\infty$	the residue field of the point $\infty \in C$,
$A := \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$	the ring of regular functions on C outside ∞,
$(R, \gamma \colon A \to R)$	an *A-ring*, that is a ring R with a ring homomorphism $\gamma \colon A \to R$,
$A_R := A \otimes_{\mathbb{F}_q} R$,	
$\sigma := \mathrm{id}_A \otimes \mathrm{Frob}_{q,R}$	the endomorphism of A_R with $a \otimes b \mapsto a \otimes b^q$ for $a \in A$ and $b \in R$,
$\sigma^* M := M \otimes_{R,\,\mathrm{Frob}_{q,R}} R = M \otimes_{A_R,\sigma} A_R$	the Frobenius pullback for an A_R-module M,
$\sigma^* V := V \otimes_{R,\,\mathrm{Frob}_{q,R}} R$	the Frobenius pullback more generally for an R-module V,
$\sigma_V^* v := v \otimes 1 \in \sigma^* V$	for an element $v \in V$,
$\sigma^* f := f \otimes \mathrm{id} \colon \sigma^* M \to \sigma^* N$	for a morphism $f \colon M \to N$ of A_R-modules,
$\mathcal{J} := \ker(\gamma \otimes \mathrm{id}_R \colon A_R \to R) = (a \otimes 1 - 1 \otimes \gamma(a) \colon a \in A) \subset A_R$.	

Note that γ makes R into an $\mathbb{F}_q$-algebra. Further note that $\mathcal{J}$ is a locally free A_R-module of rank 1. Indeed, $\mathcal{J} = I \otimes_{A_A} A_R$ for the ideal $I := (a \otimes 1 - 1 \otimes a \colon a \in A) \subset A_A = A \otimes_{\mathbb{F}_q} A$. The latter is a locally free A_A-module of rank 1 by Nakayama's lemma, because $I \otimes_{A_A} A_A/I = I/I^2 = \Omega^1_{A/\mathbb{F}_q}$ is a locally free module of rank 1 over $A_A/I = A$.

We will sometimes reduce from the ring A to the polynomial ring $\mathbb{F}_q[t]$ by applying the following

Lemma 1.5. *Let $a \in A \smallsetminus \mathbb{F}_q$ and let $\mathbb{F}_q[t]$ be the polynomial ring in the variable t. Then the homomorphism $\mathbb{F}_q[t] \to A$, $t \mapsto a$ makes A into a finite free $\mathbb{F}_q[t]$-module of rank equal to $-[\mathbb{F}_\infty : \mathbb{F}_q]\,\mathrm{ord}_\infty(a)$, where ord_∞ is the normalized valuation of the discrete valuation ring $\mathcal{O}_{C,\infty}$.*

Proof. If $\mathrm{ord}_\infty(a) = 0$ then a would have no pole on the curve C, hence would be constant. Since C is geometrically irreducible this would imply $a \in \mathbb{F}_q$ which was excluded. Therefore a is non-constant and defines a finite surjective morphism of curves $f \colon C \to \mathbb{P}^1_{\mathbb{F}_q}$ with $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{F}_q[t] = \mathbb{P}^1_{\mathbb{F}_q} \smallsetminus \{\infty'\}$, where $\infty' \in \mathbb{P}^1_{\mathbb{F}_q}$ is the pole of t. By [17, Proposition 15.31] its degree can be computed in the fiber $f^{-1}(\infty') = \{\infty\}$ as $\deg f = [\mathbb{F}_\infty : \mathbb{F}_{\infty'}] \cdot e_f(\infty)$ where $\mathbb{F}_{\infty'} = \mathbb{F}_q$ and $e_f(\infty) = \mathrm{ord}_\infty f^*(\frac{1}{t}) = -\mathrm{ord}_\infty(a)$ is the ramification index of f at ∞. Since $\operatorname{Spec} A = f^{-1}(\operatorname{Spec} \mathbb{F}_q[t])$ we conclude that A is a finite (locally) free $\mathbb{F}_q[t]$-module of rank $-[\mathbb{F}_\infty : \mathbb{F}_q]\,\mathrm{ord}_\infty(a)$. □

2. A-Motives

We keep the notation introduced in the introduction and generalize Definition 1.1 to not necessarily effective A-motives.

Definition 2.1. An *A-motive of rank r over an A-ring R* is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_R-module M of rank r and an isomorphism outside the zero locus $\mathrm{V}(\mathcal{J})$ of $\mathcal{J}$ between the induced finite locally free sheaves $\tau_M : \sigma^* M|_{\operatorname{Spec} A_R \smallsetminus \mathrm{V}(\mathcal{J})} \xrightarrow{\sim} M|_{\operatorname{Spec} A_R \smallsetminus \mathrm{V}(\mathcal{J})}$.

A *morphism* $f: (M, \tau_M) \to (N, \tau_N)$ between A-motives is an A_R-homomorphism $f: M \to N$ which satisfies $f \circ \tau_M = \tau_N \circ \sigma^* f$. We write $\operatorname{Hom}_R(\underline{M}, \underline{N})$ for the A-module of morphisms between $\underline{M}$ and $\underline{N}$. The elements of $\operatorname{QHom}_R(\underline{M}, \underline{N}) := \operatorname{Hom}_R(\underline{M}, \underline{N}) \otimes_A Q$ are called *quasi-morphisms*.We also define the endomorphism ring $\operatorname{End}_R(\underline{M}) := \operatorname{Hom}_R(\underline{M}, \underline{M})$ and $\operatorname{QEnd}_R(\underline{M}) := \operatorname{QHom}_R(\underline{M}, \underline{M}) = \operatorname{End}_R(\underline{M}) \otimes_A Q$.

To explain the relation between Definitions 1.1 and 2.1 we begin with a

Lemma 2.2. *Let $f: M \to N$ be a homomorphism between finite locally free A_R-modules M and N of the same rank, and assume that* coker f *is a finitely generated R-module, then f is injective and* coker f *is a finite locally free R-module.*

Proof. To make the proof more transparent, we choose an element $t \in A \smallsetminus \mathbb{F}_q$. Then A is a finite free $\mathbb{F}_q[t]$-module by Lemma 1.5, and M and N are finite locally free modules over $R[t]$. Also t acts as an endomorphism of the finite R-module coker f. By the Cayley-Hamilton Theorem [15, Theorem 4.3] there is a monic polynomial $g \in R[t]$ which annihilates coker f. This implies on the one hand that

$$M/gM \longrightarrow N/gN \longrightarrow \operatorname{coker} f \longrightarrow 0$$

is exact, and therefore coker f is an R-module of finite presentation, because $R[t]/(g)$ is a finite free R-module of rank $\deg_t g$. On the other hand it implies that $M[\frac{1}{g}] \twoheadrightarrow N[\frac{1}{g}]$ is an epimorphism, whence an isomorphism by [17, Corollary 8.12], because M and N are finite locally free over $R[t]$ of the same rank. Since g is a non-zero divisor on $R[t]$ and thus also on M, the localization map $M \to M[\frac{1}{g}]$ is injective, and hence also f is injective.

We obtain the exact sequence $0 \to M \to N \to \operatorname{coker} f \to 0$, which yields for every maximal ideal $\mathfrak{m} \subset R$ with residue field $k = R/\mathfrak{m}$ the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(k, \operatorname{coker} f) \longrightarrow M \otimes_R k \longrightarrow N \otimes_R k \longrightarrow (\operatorname{coker} f) \otimes_R k \longrightarrow 0\,.$$

Again the $k[t]$-modules $M \otimes_R k$ and $N \otimes_R k$ are locally free of the same rank and $(\operatorname{coker} f) \otimes_R k$ is a torsion $k[t]$-module, annihilated by g. Since $k[t]$ is a PID, this implies that $M \otimes_R k \to N \otimes_R k$ is injective and so $\operatorname{Tor}_1^R(k, \operatorname{coker} f) = (0)$. Since coker f is finitely presented, it is locally free of finite rank by Nakayama's Lemma; *e.g.*, [15, Exercise 6.2]. □

For the next proposition note that $\mathcal{J}$ is an invertible sheaf on Spec A_R as we remarked before Lemma 1.5.

Proposition 2.3.

(a) *Let (M,τ_M) be an A-motive. Then there exist integers $e,e'\in\mathbb{Z}$ such that $\mathcal{J}^e\cdot\tau_M(\sigma^*M)\subset M$ and $\mathcal{J}^{e'}\cdot\tau_M^{-1}(M)\subset\sigma^*M$. For any such e,e' the induced A_R-homomorphism $\tau_M\colon\mathcal{J}^e\cdot\sigma^*M\to M$ is injective, and the quotient $M/\tau_M(\mathcal{J}^e\cdot\sigma^*M)$ is a locally free R-module of finite rank, which is annihilated by $\mathcal{J}^{e+e'}$;*
(b) *An A-motive (M,τ_M) is an effective A-motive, if and only if $\tau_M(\sigma^*M)\subset M$;*
(c) *Let (M,τ_M) be an effective A-motive over R. Then $(M,\tau_M|_{\operatorname{Spec}A_R\smallsetminus V(\mathcal{J})})$ is an A-motive. Moreover, $\tau_M\colon\sigma^*M\to M$ is injective and* coker τ_M *is a finite locally free R-module;*
(d) *Let $\underline{M}=(M,\tau_M)$ be an effective A-motive over a field k. Then $\underline{M}$ has dimension* $\dim_k\operatorname{coker}\tau_M$.

Proof. (a) Working locally on affine subsets of $\operatorname{Spec}A_R$ we may assume that $\mathcal{J}$ is generated by a non-zero divisor $h\in\mathcal{J}$. By [14, I, Théorème 1.4.1(d1)] we obtain for every generator m of the A_R-module σ^*M an integer n such that locally $h^n\cdot\tau_M(m)\in M$. Taking e as the maximum of the n when m runs through a finite generating system of σ^*M, yields $\mathcal{J}^e\cdot\tau_M(\sigma^*M)\subset M$. The inclusion $\mathcal{J}^{e'}\cdot\tau_M^{-1}(M)\subset\sigma^*M$ is proved analogously.

Let e and e' be any integers with $\tau_M(\mathcal{J}^e\cdot\sigma^*M)\subset M$ and $\tau_M^{-1}(\mathcal{J}^{e'}\cdot M)\subset\sigma^*M$, whence $\mathcal{J}^{e+e'}\cdot M\subset\tau_M(\mathcal{J}^e\cdot\sigma^*M)$. Then $M/\tau_M(\mathcal{J}^e\cdot\sigma^*M)$ is annihilated by $\mathcal{J}^{e+e'}$, and hence a finite module over $A_R/\mathcal{J}^{e+e'}$ and over R. Therefore the map $\tau_M\colon\mathcal{J}^e\cdot\sigma^*M\to M$ is injective, and the quotient $M/\tau_M(\mathcal{J}^e\cdot\sigma^*M)$ is a finite locally free R-module by Lemma 2.2.

(c) Since $\mathcal{J}^n\cdot\operatorname{coker}\tau_M=(0)$, the map $\tau_M|_{\operatorname{Spec}A_R\smallsetminus V(\mathcal{J})}$ is an epimorphism between locally free sheaves of the same rank, and hence an isomorphism by [17, Corollary 8.12]. Thus $\underline{M}$ is an A-motive and the remaining assertions follow from (a). Also (b) follows directly.

(d) Set $d:=\dim_k\operatorname{coker}\tau_M$. Since every $h\in\mathcal{J}$ which generates $\mathcal{J}$ locally on $\operatorname{Spec}A_k$ is nilpotent on the k-vector space $\operatorname{coker}\tau_M$, it satisfies $h^d=0$ by the Cayley-Hamilton theorem from linear algebra. We conclude that $\mathcal{J}^d\cdot\operatorname{coker}\tau_M=(0)$ and $\underline{M}$ has dimension d. □

Proposition 2.4.

(a) *If S is an R-algebra, then $\underline{M}=(M,\tau_M)\longmapsto\underline{M}\otimes_R S:=(M\otimes_R S,\tau_M\otimes\mathrm{id}_S)$ defines a functor from (effective) A-motives of rank r (and dimension d) over R to (effective) A-motives of rank r (and dimension d) over S;*
(b) *Every A-motive over R and every morphism $f\in\operatorname{Hom}(\underline{M},\underline{N})$ between A-motives over R can be defined over a subring R' of R, which via $\gamma\colon A\to R'\subset R$ is a finitely generated A-algebra, hence noetherian.*

Proof. (a) This is obvious.

(b) Every A-motive $\underline{M} = (M, \tau_M)$ has a presentation given by a short exact sequence $A_R^{\oplus n_1} \xrightarrow{U} A_R^{\oplus n_0} \xrightarrow{\rho} M \longrightarrow 0$. Since M is locally free over A_R, there is a section s of the epimorphism ρ. It corresponds to an endomorphism S of $A_R^{\oplus n_0}$ with $SU = 0$ such that there is a map $W\colon A_R^{\oplus n_0} \to A_R^{\oplus n_1}$ with $S - \mathrm{Id} = UW$. The isomorphism τ_M gives rise to a diagram

$$\begin{array}{ccccccc} (\sigma^* A_R^{\oplus n_1})|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} & \xrightarrow{\sigma^* U} & (\sigma^* A_R^{\oplus n_0})|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} & \xrightarrow{\sigma^* \rho} & \sigma^* M|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} & \longrightarrow & 0 \\ \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle T_0} & & \downarrow{\scriptstyle \tau_M} & & \\ A_R^{\oplus n_1}|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} & \xrightarrow{U} & A_R^{\oplus n_0}|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} & \xrightarrow{\rho} & M|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} & \longrightarrow & 0 \end{array} \tag{2.1}$$

for suitable morphisms T_0 and T_1. Likewise τ_M^{-1} lifts to a similar diagram with vertical morphism T_0' and T_1'. The equations $\tau_M \circ \tau_M^{-1} = \mathrm{id}$ and $\tau_M^{-1} \circ \tau_M = \mathrm{id}$ imply the existence of matrices V and V' in the space of $n_1 \times n_0$-matrices $A_R^{n_1 \times n_0}|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})}$ with $T_0 \circ T_0' - \mathrm{Id} = U \circ V$ and $T_0' \circ T_0 - \mathrm{Id} = \sigma^* U \circ V'$. Let $R' \subset R$ be the A-algebra generated by the finitely many elements of R which occur in the entries of the matrices U, S, W, T_0, T_1, T_0', T_1', V and V'. Define M' as the $A_{R'}$-module which is the cokernel of $U \in A_{R'}^{n_0 \times n_1}$, and define $\tau_{M'}\colon \sigma^* M'|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} \to M'|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})}$ and $\tau_{M'}^{-1}\colon M'|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} \to \sigma^* M'|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})}$ as the $A_{R'}$-homomorphisms given by diagram (2.1) and its analog for τ_M^{-1}. Then M' is via S a direct summand of $A_{R'}^{\oplus n_0}$, hence a finite locally free $A_{R'}$-module, and $\tau_{M'}$ and $\tau_{M'}^{-1}$ are inverse to each other. It follows from diagram (2.1) that $M' \otimes_{R'} R = M$ and $\tau_{M'} \otimes \mathrm{id}_R = \tau_M$.

Finally, the assertion for the morphism $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ follows from a diagram similar to (2.1) for f instead of τ_M. □

We end this section with the following observation in which we denote the residue field of a point $s \in \mathrm{Spec}\, R$ by $\kappa(s)$.

Proposition 2.5. *Let $\underline{M}$ and $\underline{N}$ be A-motives over R and let $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ be a morphism. Then the set X of points $s \in \mathrm{Spec}\, R$ such that $f \otimes \mathrm{id}_{\kappa(s)} = 0$ in $\mathrm{Hom}_{\kappa(s)}\big(\underline{M} \otimes_R \kappa(s), \underline{N} \otimes_R \kappa(s)\big)$ is open and closed, but possibly empty. Let $\mathrm{Spec}\, \widetilde{R} \subset \mathrm{Spec}\, R$ be this set, then $f \otimes \mathrm{id}_{\widetilde{R}} = 0$ in $\mathrm{Hom}_{\widetilde{R}}\big(\underline{M} \otimes_R \widetilde{R}, \underline{N} \otimes_R \widetilde{R}\big)$. In particular if $\mathrm{Spec}\, R$ is connected and $S \neq (0)$ is an R-algebra, then the map $\mathrm{Hom}_R(\underline{M}, \underline{N}) \to \mathrm{Hom}_S(\underline{M} \otimes_R S, \underline{N} \otimes_R S)$, $f \mapsto f \otimes \mathrm{id}_S$ is injective.*

Proof. We fix an element $t \in A \smallsetminus \mathbb{F}_q$. Then A is a finite free $\mathbb{F}_q[t]$-module. By Proposition 2.3 we can find integers e, e' with $\mathcal{J}^e \cdot \tau_N(\sigma^* N) \subset N$ and $\mathcal{J}^{e'} \cdot \tau_M^{-1}(M) \subset \sigma^* M$, such that $d := e + e'$ is a power of q. We obtain morphisms $(t - \gamma(t))^e \tau_N\colon \sigma^* N \to N$ and $(t - \gamma(t))^{e'} \tau_M^{-1}\colon M \to \sigma^* M$. So the equation

$f\circ\tau_M=\tau_N\circ\sigma^*f$ implies $(t^d-\gamma(t)^d)f=(t-\gamma(t))^e\tau_N\circ\sigma^*f\circ(t-\gamma(t))^{e'}\tau_M^{-1}$. We view M and N as modules over $R[t]$ and replace A_R by $R[t]$. Since M and N are finite projective $R[t]$-modules there are split epimorphisms $R[t]^{\oplus n'}\twoheadrightarrow M$ and $R[t]^{\oplus n}\twoheadrightarrow N$. Then $R[t]^{\oplus n'}\twoheadrightarrow M\xrightarrow{f}N\hookrightarrow R[t]^{\oplus n}$ is given by a matrix $F\in R[t]^{n\times n'}$ whose entries are polynomials in t. Let $I\subset R$ be the ideal generated by the coefficients of all these polynomials and set $\widetilde{R}:=R/I$. A prime ideal $\mathfrak{p}\subseteq R$ belongs to the set X if and only if $I\subset\mathfrak{p}$. In particular $X=\mathrm{V}(I):=\operatorname{Spec}\widetilde{R}\subset\operatorname{Spec}R$ is closed.

On the other hand, we consider the map

$$R[t]^{\oplus n}\twoheadrightarrow\sigma^*N\xrightarrow{(t-\gamma(t))^e\tau_N}N\hookrightarrow R[t]^{\oplus n}$$

as a matrix $T\in R[t]^{n\times n}$ and the map

$$R[t]^{\oplus n'}\twoheadrightarrow M\xrightarrow{(t-\gamma(t))^{e'}\tau_M^{-1}}\sigma^*M\hookrightarrow R[t]^{\oplus n'}$$

as a matrix $V\in R[t]^{n'\times n'}$. The formula

$$(t^d-\gamma(t)^d)f=(t-\gamma(t))^e\tau_N\circ\sigma^*f\circ(t-\gamma(t))^{e'}\tau_M^{-1}$$

implies $(t^d-\gamma(t)^d)F=T\,\sigma(F)\,V$, and it follows that the entries of the matrix $(t^d-\gamma(t)^d)F$ are polynomials in t whose coefficients lie in I^q. If $\sum_{i=0}^{\ell}b_it^i$ is an entry of F then $(t^d-\gamma(t)^d)\sum_{i=0}^{\ell}b_it^i=\sum_{i=0}^{\ell+d}(b_{i-d}-\gamma(t)^db_i)t^i$ is the corresponding entry of $(t^d-\gamma(t)^d)F$ and all $b_{i-d}-\gamma(t)^db_i\in I^q$. By descending induction on $i=\ell+d,\dots,0$ we see that all $b_i\in I^q$. It follows that the finitely generated ideal $I\subset R$ satisfies $I=I^q$. By Nakayama's lemma [15, Corollary 4.7] there is an element $b\in 1+I$ such that $b\cdot I=(0)$. Now let $\mathfrak{p}\subset R$ be a prime ideal which lies in X, that is $I\subset\mathfrak{p}$. Then $\mathfrak{p}$ lies in the open subset $\operatorname{Spec}R[\frac{1}{b}]\subset\operatorname{Spec}R$ on which $F=0$ and hence $f=0$. In particular $X\subset\operatorname{Spec}R[\frac{1}{b}]\subset X$. Therefore X is open and closed and $f=0$ on X.

Now let $\operatorname{Spec}R$ be connected and $S\neq(0)$ be an R-algebra. Let $f\in\operatorname{Hom}_R(\underline{M},\underline{N})$ be such that $f\otimes\mathrm{id}_S=0$ in $\operatorname{Hom}_S(\underline{M}\otimes_RS,\underline{N}\otimes_RS)$. Let $s\in\operatorname{Spec}S$ be a point and let $s'\in\operatorname{Spec}R$ be its image. Then $f\otimes\mathrm{id}_{\kappa(s')}=0$ and the set X from above is non-empty. Since it is open and closed and $\operatorname{Spec}R$ is connected, it follows that $X=\operatorname{Spec}R$ and $f=0$. This proves the injectivity. □

Corollary 2.6. *Let $\underline{M}$ and $\underline{N}$ be A-motives over R with* $\operatorname{Spec}R$ *connected. Then* $\operatorname{Hom}_R(\underline{M},\underline{N})$ *is a finite projective A-module of rank less or equal to* $(\operatorname{rk}\underline{M})\cdot(\operatorname{rk}\underline{N})$.

Proof. If $R=k$ is a field and $\underline{M}$ and $\underline{N}$ are effective, the result is due to Anderson [2, Corollary 1.7.2]. For general R we apply Proposition 2.5 with $S=R/\mathfrak{m}$ for $\mathfrak{m}\subset R$ a maximal ideal, and use that over the Dedekind ring A every submodule of a finite projective module is itself finite projective. □

3. Abelian Anderson A-modules

We recall Definition 1.2 of *Abelian Anderson A-modules* from the introduction. Let us give some explanations. All group schemes in this article are assumed to be commutative.

Definition 3.1. Let $\mathcal{O}$ be a commutative unitary ring. An *$\mathcal{O}$-module scheme* over R is a commutative group scheme E over R together with a ring homomorphism $\mathcal{O} \to \operatorname{End}_R(E)$.

For a group scheme E over $\operatorname{Spec} R$ we let $E^n := E \times_R \ldots \times_R E$ be the n-fold fiber product over R. We denote by $e\colon \operatorname{Spec} R \to E$ its zero section and by $\operatorname{Lie} E := \operatorname{Hom}_R(e^*\Omega^1_{E/R}, R)$ the tangent space of E along e. If E is smooth over $\operatorname{Spec} R$, then $\operatorname{Lie} E$ is a locally free R-module of rank equal to the relative dimension of E over R. In particular $\operatorname{Lie} E^n = (\operatorname{Lie} E)^{\oplus n}$. For a homomorphism $f\colon E \to E'$ of group schemes over $\operatorname{Spec} R$ we denote by $\operatorname{Lie} f\colon \operatorname{Lie} E \to \operatorname{Lie} E'$ the induced morphism of R-modules. Also we define the *kernel of f* as the R-group scheme $\ker f := E \underset{f,\, E', e'}{\times} \operatorname{Spec} R$ where $e'\colon \operatorname{Spec} R \to E'$ is the zero section. There is a canonical isomorphism

$$E \underset{f,\, E',\, f}{\times} E \xrightarrow{\sim} E \underset{R}{\times} \ker f \tag{3.1}$$

given by $(P, Q) \mapsto (P, Q - P)$ on T-valued points $P, Q \in E(T)$ for any R-scheme T . If $P \in E(k)$ for a field k and $P' = f(P) \in E'(k)$, pulling back (3.1) under $P\colon \operatorname{Spec} k \to E$ yields an isomorphism of the fiber $\operatorname{Spec} k \underset{P',\, E',\, f}{\times} E$ of f over P' with $\operatorname{Spec} k \times_R \ker f$.

On $\mathbb{G}_{a,R} = \operatorname{Spec} R[x]$ the elements $b \in R$, and in particular $\gamma(a) \in R$ for $a \in \mathbb{F}_q$, act via $b^*\colon R[x] \to R[x]$, $x \mapsto bx$. This makes $\mathbb{G}_{a,R}$ into an $\mathbb{F}_q$-module scheme. In addition let $\tau := \operatorname{Frob}_{q,\mathbb{G}_{a,R}}$ be the relative q-Frobenius endomorphism of $\mathbb{G}_{a,R} = \operatorname{Spec} R[x]$ given by $x \mapsto x^q$. It satisfies $\operatorname{Lie} \tau = 0$ and $\tau \circ b = b^q \circ \tau$. We let

$$R\{\tau\} := \left\{ \sum_{i=0}^{n} b_i \tau^i : n \in \mathbb{N}_0, b_i \in R \right\} \quad \text{with} \quad \tau b = b^q \tau \tag{3.2}$$

be the non-commutative polynomial ring in τ over R. For an element $f = \sum_i b_i \tau^i \in R\{\tau\}$ we set $f(x) := \sum_i b_i x^{q^i}$.

Lemma 3.2. *There is an isomorphism of R-modules*

$$R\{\tau\}^{d' \times d} \xrightarrow{\sim} \operatorname{Hom}_{R\text{-groups}, \mathbb{F}_q\text{-lin}}\bigl(\mathbb{G}^d_{a,R}, \mathbb{G}^{d'}_{a,R}\bigr),$$

which sends the matrix $F = (f_{ij})_{i,j}$ to the $\mathbb{F}_q$-equivariant morphism $f\colon \mathbb{G}^d_{a,R} \to \mathbb{G}^{d'}_{a,R}$ of group schemes over R with $f^*(y_i) = \sum_j f_{ij}(x_j)$ where $\mathbb{G}^d_{a,R} = \operatorname{Spec} R[x_1, \ldots, x_d]$ and $\mathbb{G}^{d'}_{a,R} = \operatorname{Spec} R[y_1, \ldots, y_{d'}]$. Under this isomorphism the map $f \mapsto \operatorname{Lie} f$ is given by the map $R\{\tau\}^{d' \times d} \to R^{d' \times d}$, $F = \sum_n F_n \tau^n \mapsto F_0$.

Proof. This is straight forward to prove using Lucas's theorem [23] on congruences of binomial coefficients which states that $\binom{pn+t}{pm+s} \equiv \binom{n}{m}\binom{t}{s} \bmod p$ for all $n, m, t, s \in \mathbb{N}_0$, and implies that $\binom{n}{i} \equiv 0 \bmod p$ for all $0 < i < n$ if and only if $n = p^e$ for an $e \in \mathbb{N}_0$. □

Remark 3.3. The affine group scheme E and its multiplication map $\Delta\colon E \times_R E \to E$ are described by its coordinate ring $B_E := \Gamma(E, \mathcal{O}_E)$ together with the comultiplication $\Delta^*\colon B_E \to B_E \otimes_R B_E$. If we write $\mathbb{G}_{a,R} = \operatorname{Spec} R[\xi]$ the map

$$\begin{aligned} M(\underline{E}) &\xrightarrow{\sim} \left\{ x \in B_E\colon \Delta^* x = x \otimes 1 + 1 \otimes x \text{ and } \varphi_a^* x = \gamma(a) x \text{ for all } a \in \mathbb{F}_q \right\} \\ m &\longmapsto m^*(\xi) \end{aligned}$$

is an isomorphism of A_R-modules. Choosing an element $\lambda \in \mathbb{F}_q$ with $\mathbb{F}_q = \mathbb{F}_p(\lambda)$ we obtain an exact sequence of R-modules

$$\begin{array}{ccccc} 0 \to M(\underline{E}) & \longrightarrow & B_E & \longrightarrow & B_E \otimes_R B_E \;\oplus\; B_E \\ m & \longmapsto & m^*(\xi)\,, \quad x & \mapsto & \big(\Delta^* x - x \otimes 1 - 1 \otimes x\,,\ \varphi_\lambda^* x - \gamma(\lambda) x\big). \end{array} \tag{3.3}$$

This shows that for every flat R-algebra R' we have a canonical isomorphism $M(\underline{E}) \otimes_R R' = M(\underline{E} \times_R \operatorname{Spec} R')$, because $\Gamma(E \times_R R', \mathcal{O}_{E \times R'}) = B_E \otimes_R R'$. In particular, if R' satisfies condition (a) of Definition 1.2 then $M(\underline{E}) \otimes_R R' \cong R'\{\tau\}^{1 \times d}$ by Lemma 3.2.

From this we see that for any R-algebra S the tensor product of the sequence (3.3) with S stays exact and $M(\underline{E}) \otimes_R S = M(\underline{E} \times_{\operatorname{Spec} R} \operatorname{Spec} S)$. Namely, we choose a faithfully flat morphism $R \to R'$ as in Definition 1.2(a) and tensor (3.3) with $S \otimes_R R'$. This tensor product stays exact by Lemma 3.2 because $M(\underline{E}) \otimes_R R' \cong R'\{\tau\}^{1 \times d}$. Since $S \to S \otimes_R R'$ is faithfully flat, already the tensor product of (3.3) with S was exact.

Definition 3.4. If $\underline{E}$ is an Abelian Anderson A-module we consider in addition on $M(\underline{E})$ the map $\tau\colon m \mapsto \mathrm{Frob}_{q,\mathbb{G}_{a,R}} \circ m$. Since $\tau(bm) = b^q \tau(m)$ the map τ is σ-semilinear and induces an A_R-linear map $\tau_M\colon \sigma^* M \to M$. We set $\underline{M}(\underline{E}) := \big(M(\underline{E}), \tau_M\big)$ and call it the *(effective) A-motive associated with $\underline{E}$*.

This definition is justified by the following relative version of Anderson's theorem [2, Theorem 1].

Theorem 3.5. *Let R be a fixed A-ring. If $\underline{E} = (E, \varphi)$ is an Abelian Anderson A-module of rank r and dimension d over R, then $\underline{M}(\underline{E}) = (M, \tau_M)$ is an effective A-motive of rank r and dimension d over R. There is a canonical isomorphism of R-modules*

$$\operatorname{coker} \tau_M \xrightarrow{\sim} \operatorname{Hom}_R(\operatorname{Lie} E, R), \quad m \bmod \tau_M(\sigma^* M) \longmapsto \operatorname{Lie} m\,. \tag{3.4}$$

The contravariant functor $\underline{E} \mapsto \underline{M}(\underline{E})$ between Abelian Anderson A-modules over R and A-motives over R is fully faithful. Its essential image consists of all effective

A-motives $\underline{M} = (M, \tau_M)$ *over* R *of some dimension* d*, for which there exists a faithfully flat ring homomorphism* $R \to R'$ *such that* $M \otimes_R R'$ *is a finite free left* $R'\{\tau\}$*-module under the map* $\tau\colon M \to M,\ m \mapsto \tau_M(\sigma_M^* m)$.

Proof. We first establish the isomorphism (3.4). If $m = \tau_M(\sum_i m_i \otimes b_i) = \sum_i b_i \circ \mathrm{Frob}_{q,\mathbb{G}_{a,R}} \circ m_i$ with $m_i \in M$ and $b_i \in R$, then $\mathrm{Lie}\, m = 0$ because $\mathrm{Lie}\,\mathrm{Frob}_{q,\mathbb{G}_{a,R}} = 0$. So the map (3.4) is well defined. To prove that it is an isomorphism one can apply a faithfully flat base change $R \to R'$, see [14, Section 0_{I}.6.6], such that $E \otimes_R R' \cong \mathbb{G}_{a,R'}^d$ and $\mathrm{Lie}\, E \otimes_R R' \cong (R')^{\oplus d}$. Then $M \otimes_R R' \cong R'\{\tau\}^{1\times d}$ by Remark 3.3, and the inverse map is given by the natural inclusion $(R')^{1\times d} \subset R'\{\tau\}^{1\times d}$, $F_0 \mapsto F_0\tau^0$.

As a consequence, $\mathrm{coker}\, \tau_M$ is a locally free R-module of rank equal to $d = \dim \underline{E}$ and annihilated by $\mathcal{J}^d$ because of condition (b) in Definition 1.2. This implies $\mathrm{coker}\, \tau_M|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} = (0)$, and therefore the morphism $\tau_M\colon \sigma^* M|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})} \to M|_{\mathrm{Spec}\, A_R \smallsetminus \mathrm{V}(\mathcal{J})}$ is surjective. By [17, Corollary 8.12] it is an isomorphism, because M and $\sigma^* M$ are finite locally free over A_R of the same rank. Thus $\underline{M}(\underline{E})$ is an A-motive and even an effective A-motive of dimension d by Proposition 2.3.

Let $\underline{E} = (E, \varphi)$ and $\underline{E}' = (E', \varphi')$ be two Abelian Anderson A-modules over R and let $\underline{M}(\underline{E})$ and $\underline{M}(\underline{E}')$ be the associated effective A-motives. To prove that the map

$$\mathrm{Hom}_R\big(\underline{E}, \underline{E}'\big) \longrightarrow \mathrm{Hom}_R\big(\underline{M}(\underline{E}'), \underline{M}(\underline{E})\big), \quad f \longmapsto (m' \mapsto m' \circ f) \qquad (3.5)$$

is bijective, we again apply a faithfully flat base change $R \to R'$, over which there are isomorphisms $E \otimes_R R' \cong \mathbb{G}_{a,R'}^d$ and $E' \otimes_R R' \cong \mathbb{G}_{a,R'}^{d'}$. Then

$$\mathrm{Hom}_{R'}(\underline{E} \otimes_R R', \underline{E}' \otimes_R R') \cong \big\{F \in R'\{\tau\}^{d'\times d} : \varphi'_a \circ F = F \circ \varphi_a\ \forall a \in A\big\}$$

by Lemma 3.2. Also $\underline{M}(\underline{E}) \otimes_R R' \cong R'\{\tau\}^{1\times d}$ and $\underline{M}(\underline{E}') \otimes_R R' \cong R'\{\tau\}^{1\times d'}$. The condition $h \circ \tau_{M'} = \tau_M \circ \sigma^* h$ on an element $h \in \mathrm{Hom}_{R'}\big(\underline{M}(\underline{E}') \otimes_R R', \underline{M}(\underline{E}) \otimes_R R'\big)$ implies that $h\colon R'\{\tau\}^{1\times d'} \to R'\{\tau\}^{1\times d}$ is a homomorphism of left $R'\{\tau\}$-modules, hence given by multiplication on the right by a matrix $H \in R'\{\tau\}^{d'\times d}$. Then $m' \circ \varphi'_a \circ H = h\big((a \otimes 1) \cdot m'\big) = (a \otimes 1) \cdot h(m') = m' \circ H \circ \varphi_a$ implies $\varphi'_a \circ H = H \circ \varphi_a$ for all $a \in A$. It follows that the map (3.5) is bijective over R'. So every element $h \in \mathrm{Hom}_R\big(\underline{M}(\underline{E}'), \underline{M}(\underline{E})\big)$ gives rise to a morphism $f' \in \mathrm{Hom}_{R'}(\underline{E} \otimes_R R', \underline{E}' \otimes_R R')$ which carries a descent datum because h was defined over R. Since by [7, Section 6.1, Theorem 6(a)] the descent of morphisms relative to the faithfully flat morphism $R \to R'$ is effective, f' descends to the desired $f \in \mathrm{Hom}_R(\underline{E}, \underline{E}')$. This shows that the functor $\underline{E} \mapsto \underline{M}(\underline{E})$ is fully faithful.

Let $\underline{M} = (M, \tau_M)$ be an effective A-motive of dimension d over R for which there exists a faithfully flat ring homomorphism $R \to R'$ such that $M \otimes_R R' \cong R'\{\tau\}^{1\times d}$. Observe that $\mathrm{coker}(\tau_M \otimes \mathrm{id}_{R'}) \cong (R'\{\tau\}/R'\{\tau\}\tau)^{1\times d} = (R')^{1\times d}$. For all $a \in A$ we have $\tau \cdot (a \otimes 1)m = \sigma(a \otimes 1) \cdot \tau(m) = (a \otimes 1)\tau m$. Therefore the map $m \mapsto (a \otimes 1)m$ is a homomorphism of left $R'\{\tau\}$-modules, and hence given by $(a \otimes 1)m = m \cdot \varphi'_a$ for a matrix $\varphi'_a \in R'\{\tau\}^{d \times d}$. Then $\underline{E}' := (E' = \mathbb{G}^d_{a,R'}, \varphi' \colon A \to R'\{\tau\}^{d\times d}, a \mapsto \varphi'_a)$ satisfies $\underline{M}(\underline{E}') = \underline{M} \otimes_R R'$. Again $\big(a\otimes 1 - 1\otimes\gamma(a)\big)^d = 0$ on coker τ_M implies $\big(\operatorname{Lie}\varphi'_a - \gamma(a)\big)^d = 0$ on Lie E'. So $\underline{E}'$ is an Abelian Anderson A-module over R' with $\underline{M}(E') \cong \underline{M} \otimes_R R'$. Consider the ring $R'' := R' \otimes_R R'$ and the two maps $p_1, p_2 \colon R' \to R''$ given by $p_1(b') = b' \otimes 1$ and $p_2(b') = 1 \otimes b'$. The canonical isomorphism $p_1^*(\underline{M} \otimes_R R') = p_2^*(\underline{M} \otimes_R R')$ induces an isomorphism $p_1^*\underline{E}' \cong p_2^*\underline{E}'$ which is a descend datum on $\underline{E}'$ relative to $R \to R'$. Since faithfully flat descend on affine schemes is effective by [7, Section 6.1, Theorem 6(b)] there exists a group scheme E over R with a ring homomorphism $\varphi \colon A \to \operatorname{End}_{R\text{-groups}}(E)$ such that $(E, \varphi) \otimes_R R' \cong \underline{E}'$. By [14, IV$_2$, Proposition 2.7.1 and IV$_4$, Corollaire 17.7.3] the group scheme E is affine and smooth over R and hence (E, φ) is an Abelian Anderson A-module with $\underline{M}(E, \varphi) \cong \underline{M}$. □

The theorem implies the following:

Corollary 3.6. *The assertions of Proposition* 2.5 *and Corollary* 2.6 *also hold for Abelian Anderson A-modules. In particular, for Abelian Anderson A-modules $\underline{E}$ and $\underline{E}'$ over R, the A-module* $\operatorname{Hom}_R(\underline{E}, \underline{E}')$ *is finite projective of rank less or equal to* $(\operatorname{rk} \underline{E}) \cdot (\operatorname{rk} \underline{E}')$. □

An important class of examples are Drinfeld modules. We recall their definition from [12, Section 5] and [27, Section 1].

Definition 3.7. A *Drinfeld A-module of rank $r \in \mathbb{N}_{>0}$ over R* is a pair $\underline{E} = (E, \varphi)$ consisting of a smooth affine group scheme E over Spec R of relative dimension 1 and a ring homomorphism $\varphi \colon A \to \operatorname{End}_{R\text{-groups}}(E)$, $a \mapsto \varphi_a$ satisfying the following conditions:

(a) Zariski-locally on Spec R there is an isomorphism $\alpha \colon E \xrightarrow{\sim} \mathbb{G}_{a,R}$ of $\mathbb{F}_q$-module schemes such that
(b) the coefficients of the τ-polynomial $\Phi_a := \alpha \circ \varphi_a \circ \alpha^{-1} = \sum_{i\ge 0} b_i(a)\tau^i \in \operatorname{End}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\mathbb{G}_{a,R}) = R\{\tau\}$ satisfy $b_0(a) = \gamma(a)$, $b_{r(a)}(a) \in R^\times$ and $b_i(a)$ is nilpotent for all $i > r(a) := -r\,[\mathbb{F}_\infty : \mathbb{F}_q]\operatorname{ord}_\infty(a)$.

If $b_i(a) = 0$ for all $i > r(a)$ we say that $\underline{E}$ is *in standard form.*

It is well known that every Drinfeld A-module over R can be put in standard form; see [12, Section 5] or [24, Section 4.2]. This is a consequence of the following lemma of Drinfeld [12, Propositions 5.1 and 5.2] which we will need again below. For the convenience of the reader we recall the proof.

Lemma 3.8.

(a) *Let $b = \sum_{i=0}^n b_i\tau^i \in R\{\tau\}$ and let r be a positive integer such that $b_r \in R^\times$ and b_i is nilpotent for all $i > r$. Then there is a unique unit $c = \sum_{i\ge 0} c_i\tau^i \in R\{\tau\}^\times$ with $c_0 = 1$ and c_i nilpotent for $i > 0$, such that $c^{-1}bc = \sum_{i=0}^r b'_i\tau^i$ with $b'_r \in R^\times$;*

(b) *Let* Spec R *be connected and let* $b = \sum_{i=0}^{m} b_i\tau^i$ *and* $c = \sum_{i=0}^{n} c_i\tau^i \in R\{\tau\}$ *with* $m, n > 0$ *and* $b_m, c_n \in R^\times$. *Let* $d \in R\{\tau\} \smallsetminus \{0\}$ *satisfy* $db = cd$. *Then* $m = n$ *and* $d = \sum_{i=0}^{r} d_i\tau^i$ *with* $d_r \in R^\times$.

Proof. (a) was also reproved in [22, Lemma 1.1.2] and [24, Proposition 1.4].

(b) We write $d = \sum_{i=0}^{r} d_i\tau^i$ with $d_r \neq 0$. The equation $db = cd$ implies $\sum_j (d_{i-j}b_j^{q^{i-j}} - c_j d_{i-j}^{q^j}) = 0$ for all i, where the sum runs over $j = \max\{0, i-r\}, \ldots, \min\{i, \max\{m, n\}\}$. We now distinguish three cases.

If $m > n$ then $i = m + r$ yields $d_r b_m^{q^r} = 0$, whence $d_r = 0$ which is a contradiction.

If $m < n$ then $i = n+r$ yields $c_n d_r^{q^n} = 0$, whence $d_r \in \mathfrak{p}$ for every prime ideal $\mathfrak{p} \subset R$. For $n+r > i \geq n$ we obtain $c_n d_{i-n}^{q^n} = \sum_{0\leq j<n}(d_{i-j}b_j^{q^{i-j}} - c_j d_{i-j}^{q^j})$ and by descending induction on i it follows that $d_{i-n} \in \mathfrak{p}$ for every prime ideal $\mathfrak{p} \subset R$ for all $i - n = r, \ldots, 0$. So the ideal $I := (d_i : 0 \leq i \leq r) \subset R$ is contained in every prime ideal $\mathfrak{p} \subset R$. Now $i = m + r$ yields $d_r b_m^{q^r} = \sum_{j=m}^{m+r} c_j d_{m+r-j}^{q^j}$, whence $d_r \in I^q$. For $m + r > i \geq m$ we obtain $d_{i-m}b_m^{q^{i-m}} = \sum_{0\leq j<m} d_{i-j}b_j^{q^{i-j}} - \sum_{0\leq j\leq n} c_j d_{i-j}^{q^j}$ and by descending induction on i it follows that $d_{i-m} \in I^q$ for all $i - m = r, \ldots, 0$. Therefore the finitely generated ideal I satisfies $I = I^q$ and by Nakayama's lemma [15, Corollary 4.7] there is an element $f \in 1 + I$ such that $f \cdot I = (0)$. Since $I \subset \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset R$, the element $1 - f$ is a unit in R and $I = 0$. Therefore $d_i = 0$ for all i which is a contradiction.

If $m = n$ then $c_m d_r^{q^m} = d_r b_m^{q^r}$ and we consider the ideal $I = (d_r) \subset R$. Again $I = I^{q^m}$ and by [15, Corollary 4.7] there is an element $f \in 1 + I$ such that $f \cdot d_r = 0$. Now assume that $d_r \in \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset R$. Then $f \notin \mathfrak{p}$, whence $\mathfrak{p} \in \operatorname{Spec} R[\frac{1}{f}] \subset \operatorname{Spec} R$ and $d_r = 0$ on the open neighborhood $\operatorname{Spec} R[\frac{1}{f}]$ of $\mathfrak{p}$. Since the set of prime ideals $\mathfrak{p} \subset R$ with $d_r \in \mathfrak{p}$ is closed in Spec R and the latter is connected, it follows that $d_r = 0$ on all of Spec R. This is a contradiction and so our assumption was false. In particular d_r is not contained in any prime ideal and so $d_r \in R^\times$ as desired. □

Theorem 3.9. *The Abelian Anderson A-modules of dimension* 1 *and rank r over R are precisely the Drinfeld A-modules of rank r over R.*

Proof. Let $\underline{E}$ be a Drinfeld A-module of rank r over R. Choose a Zariski covering as in Definition 3.7(a) such that $\underline{E}$ is in standard form. Since Spec R is quasi-compact this Zariski covering can be refined to a covering by finitely many affines. Their disjoint union is of the form Spec R' and the ring homomorphism $R \to R'$ is faithfully flat. So $\underline{E}$ satisfies conditions (a) and (b) of Definition 1.2. Choose an element $t \in A \smallsetminus \mathbb{F}_q$. Then A is a finite free $\mathbb{F}_q[t]$-module of rank equal to $-[\mathbb{F}_\infty : \mathbb{F}_q]\operatorname{ord}_\infty(t)$ by Lemma 1.5. Writing $\Phi_t = \sum_{i=0}^{r(t)} b_i(t)\tau^i$ with

$r(t) = -r\,[\mathbb{F}_\infty : \mathbb{F}_q]\operatorname{ord}_\infty(t)$ and $b_{r(t)}(t) \in (R')^\times$, we make the following

Claim. As an $R'[t]$-module $$M(\underline{E}) \otimes_R R' = \bigoplus_{\ell=0}^{r(t)-1} R'[t]\cdot\tau^\ell. \tag{3.6}$$

By Remark 3.3 and Lemma 3.2 we have $M(\underline{E}) \otimes_R R' = M(\underline{E} \times_{\operatorname{Spec} R} \operatorname{Spec} R') = R'\{\tau\}$. We prove by induction on n that for every $c = \sum_{i=0}^n c_i\tau^i \in R'\{\tau\} = M(\underline{E})$ there are uniquely determined elements $f_\ell(t) \in R'[t]$ with $c = \sum_{\ell=0}^{r(t)-1} f_\ell(t)\cdot\tau^\ell$. If $n < r(t)$ then we take $f_\ell(t) = c_\ell$. If $n \geq r(t)$, dividing c by Φ_t on the right produces uniquely determined $g = \sum_{i=0}^{n-r(t)} g_i\tau^i$ and $h = \sum_{\ell=0}^{r(t)-1} h_\ell\tau^\ell \in R'\{\tau\}$ with $c = g\Phi_t + h$. Namely, starting with $g_i = 0$ for $i > n - r(t)$ we can and must take $g_i = b_{r(t)}^{-q^i}\big(c_{i+r(t)} - \sum_{j=i+1}^{i+r(t)} g_j\, b_{i+r(t)-j}^{q^j}\big)$ for $i = n - r(t), \ldots, 0$ and $h_\ell = c_\ell - \sum_{j=0}^{\ell} g_j\, b_{\ell-j}^{q^j}$ for $\ell = r(t) - 1, \ldots, 1$. The induction hypothesis implies $g = \sum_{\ell=0}^{r(t)-1} \tilde{f}_\ell(t)\cdot\tau^\ell$. Now $f_\ell(t) := \tilde{f}_\ell(t)\cdot t + h_\ell$ satisfies $c = \sum_{\ell=0}^{r(t)-1} f_\ell(t)\cdot\tau^\ell$. This proves the claim.

By faithfully flat descent [14, IV$_2$, Proposition 2.5.2] with respect to $R[t] \to R'[t]$ and by the claim, $M(\underline{E})$ is finite, locally free over $R[t]$ and in particular flat over R. We next show that it is finitely presented over A_R. Namely, let $(m_i)_{i\in I}$ be a finite generating system of $M(\underline{E})$ over $R[t]$. Using it as a generating system over A_R we obtain an epimorphism $\rho\colon A_R^I \twoheadrightarrow M(\underline{E})$, where $A_R^I = \bigoplus_{i\in I} A_R$. Since A_R is a finite free $R[t]$-module, also A_R^I is a finite free $R[t]$-module and so the kernel of ρ is a finitely generated $R[t]$-module, whence a finitely generated A_R-module. This shows that $M(\underline{E})$ is a finitely presented A_R-module. From [14, IV$_3$, Théorème 11.3.10] it follows that $M(\underline{E})$ is finite locally free over A_R, because for every point $s \in \operatorname{Spec} R$ the finite $A_{\kappa(s)}$-module $M(\underline{E}) \otimes_R \kappa(s)$ is a free $\kappa(s)[t]$-module and hence a torsion free and flat $A_{\kappa(s)}$-module. Its rank is r as can be computed by comparing the ranks of $A_{R'}$ and $M(\underline{E}) \otimes_R R'$ over $R'[t]$. This proves that $\underline{E}$ is an Abelian Anderson A-module of dimension 1 and rank r over R.

Conversely let $\underline{E} = (E, \varphi)$ be an Abelian Anderson A-module of dimension 1 and rank r over R. Let $R \to R'$ be a faithfully flat ring homomorphism and let $\alpha\colon E \times_R \operatorname{Spec} R' \xrightarrow{\sim} \mathbb{G}_{a,R'}$ be an isomorphism of $\mathbb{F}_q$-module schemes as in Definition 1.2(a). For $a \in A$ write

$$\Phi_a := \sum_{i=0}^{n(a)} b_i(a)\tau^i := \alpha \circ \varphi_a \circ \alpha^{-1} \in \operatorname{End}_{R'\text{-groups},\mathbb{F}_q\text{-lin}}(\mathbb{G}_{a,R'}) = R'\{\tau\},$$

where $n(a) \in \mathbb{N}_0$ and $b_i(a) \in R'$. For $a \in \mathbb{F}_q$ we obtain $\Phi_a = \gamma(a)\cdot\tau^0$. For $t := a \in A \smallsetminus \mathbb{F}_q$ we consider A as a finite free $\mathbb{F}_q[t]$-module of rank $-[\mathbb{F}_\infty : \mathbb{F}_q]\operatorname{ord}_\infty(a)$ by Lemma 1.5. Then $M(\underline{E})$ is a finite locally free $R[t]$-module of rank $r(a) := -r\,[\mathbb{F}_\infty : \mathbb{F}_q]\operatorname{ord}_\infty(a)$ by condition (c) of Definition 1.2. Let $\mathfrak{p} \subset R'$ be a prime ideal, set $k = \operatorname{Frac}(R'/\mathfrak{p})$, and consider the Abelian Anderson A-module $\underline{E} \times_R \operatorname{Spec} k$ over k and the free $k[t]$-module $M(\underline{E}) \otimes_R k = M(\underline{E} \times_R \operatorname{Spec} k)$ of rank

$r(a)$. By an argument similarly to our claim (3.6) we see that $\deg_\tau\big(\Phi_a \otimes_{R'} 1_k\big) = r(a)$, that is $b_{r(a)}(a)\otimes 1_k \in k^\times$ and $b_i(a)\otimes 1_k = 0$ for all $i > r(a)$. This implies that $b_{r(a)}(a) \in (R')^\times$ and $b_i(a)$ is nilpotent for all $i > r(a)$ by [15, Corollary 2.12]. By Lemma 3.8(a) we may change the isomorphism α such that $\Phi_a = \sum_{i=0}^{r(a)} b_i(a)\tau^i$ with $b_{r(a)}(a) \in (R')^\times$ for one $a \in A$, and by Lemma 3.8(b) this then holds for all $a \in A$, because $\Phi_a\Phi_b = \Phi_{ab} = \Phi_b\Phi_a$. By condition (b) of Definition 1.2 we have $b_0(a) = \gamma(a)$. Thus $\underline{E} \times_R \operatorname{Spec} R'$ is a Drinfeld A-module of rank r over R' in standard form.

It remains to show that we can replace the faithfully flat covering $\operatorname{Spec} R' \to \operatorname{Spec} R$ by a Zariski covering. For this purpose consider $R'' := R' \otimes_R R'$ and the two projections $pr_i\colon \operatorname{Spec} R'' \to \operatorname{Spec} R'$ onto the i-th factor for $i = 1, 2$. Then $h := \sum_{i\geq 0} h_i\tau^i := pr_2^*\alpha \circ pr_1^*\alpha^{-1} \in R''\{\tau\}^\times$ satisfies $h_0 \in (R'')^\times$ and h_i is nilpotent for all $i > 0$; see [24, Proposition 1.4]. By Lemma 3.8(b) the equation $pr_2^*\Phi_a \circ h = h \circ pr_1^*\Phi_a$ implies that $h_i = 0$ for all $i > 0$ and $h = h_0 \in (R'')^\times \subset R''\{\tau\}^\times$. The cocycle $\underline{h} := (\operatorname{Spec} R' \to \operatorname{Spec} R,\, h)$ defines an element in the Čech cohomology group $\check{H}^1_{fpqc}(\operatorname{Spec} R, \mathbb{G}_m)$. By Hilbert 90, see [25, Proposition III.4.9] we have $\check{H}^1_{fpqc}(\operatorname{Spec} R, \mathbb{G}_m) = \check{H}^1_{Zar}(\operatorname{Spec} R, \mathbb{G}_m)$. This means that there is a Zariski covering $\operatorname{Spec} \widetilde{R} \to \operatorname{Spec} R$, where $\operatorname{Spec} \widetilde{R} = \coprod_i \operatorname{Spec} \widetilde{R}_i$ is a disjoint union of open affine subschemes $\operatorname{Spec} \widetilde{R}_i \subset \operatorname{Spec} R$, and a unit $\tilde{h} = (\tilde{h}_{ij})_{i,j} \in (\widetilde{R} \otimes_R \widetilde{R})^\times = \prod_{i,j}(\widetilde{R}_i \otimes_R \widetilde{R}_j)^\times$, such that $(\operatorname{Spec} \widetilde{R} \to \operatorname{Spec} R,\, \tilde{h}) = \underline{h}$. Let $\widetilde{E}$ be the smooth affine group and $\mathbb{F}_q$-module scheme over $\operatorname{Spec} R$ with $\beta_i\colon \widetilde{E}|_{\operatorname{Spec} \widetilde{R}_i} \xrightarrow{\sim} \mathbb{G}_{a,\widetilde{R}_i}$ and $\beta_j = \tilde{h}_{ij} \circ \beta_i$ on $\operatorname{Spec} \widetilde{R}_i \otimes_R \widetilde{R}_j$. Then over $\operatorname{Spec} R' \otimes_R \widetilde{R} = \coprod_i \operatorname{Spec} R' \otimes_R \widetilde{R}_i$ we have an isomorphism $\tilde{\alpha} := (\beta_i^{-1} \circ \alpha)_i\colon E \xrightarrow{\sim} \widetilde{E}$. Let $p_i\colon \operatorname{Spec}(R' \otimes_R \widetilde{R}) \otimes_R (R' \otimes_R \widetilde{R}) \to \operatorname{Spec} R' \otimes_R \widetilde{R}$ be the projection onto the i-th factor for $i = 1, 2$. Then $p_2^*\tilde{\alpha} \circ p_1^*\tilde{\alpha}^{-1} = (\tilde{h}_{ij}^{-1} h)_{i,j} = 1$. This shows that $\tilde{\alpha}$ descends to an isomorphism $\tilde{\alpha}\colon E \xrightarrow{\sim} \widetilde{E}$ over $\operatorname{Spec} R$ by [7, Section 6.1, Theorem 6(a)]. On $\operatorname{Spec} \widetilde{R}_i$, now $\beta_i \circ \tilde{\alpha}\colon E \xrightarrow{\sim} \mathbb{G}_{a,\widetilde{R}_i}$ is an isomorphism of $\mathbb{F}_q$-module schemes. Moreover $\widetilde{\Phi}_a := \beta_i\tilde{\alpha} \circ \varphi_a \circ \tilde{\alpha}^{-1}\beta_i^{-1} \in \widetilde{R}_i\{\tau\}$ satisfies $\widetilde{\Phi}_a \otimes 1_{R'} = \Phi_a \otimes 1_{\widetilde{R}_i}$ in $(R' \otimes_R \widetilde{R}_i)\{\tau\} \supset \widetilde{R}_i\{\tau\}$ and by what we proved for Φ_a above, this implies that $\underline{E}$ is a Drinfeld A-module of rank r over R which by $\widetilde{R}$ and $(\beta_i \circ \tilde{\alpha})_i$ is put in standard form. □

4. Review of the finite shtuka equivalence

In preparation for our main results in Sections 5 and 6 we need to recall Drinfeld's functor [13, Section 2] and the equivalence it defines between finite $\mathbb{F}_q$-shtukas and finite locally free strict $\mathbb{F}_q$-module schemes; see also [1], [31, Section 1], [22, Section B.3] and [19, Sections 3-5].

Definition 4.1. A *finite* $\mathbb{F}_q$*-shtuka* over R is a pair $\underline{V} = (V, F_V)$ consisting of a finite locally free R-module V and an R-module homomorphism $F_V\colon \sigma^*V \to V$.

A *morphism* $f\colon (V, F_V) \to (V', F_{V'})$ of finite $\mathbb{F}_q$-shtukas is an R-module homomorphism $f\colon V \to V'$ satisfying $f \circ F_V = F_{V'} \circ \sigma^* f$.

We say that F_V is *nilpotent* if there is an integer n such that the composition $F_V^n := F_V \circ \sigma^* F_V \circ \ldots \circ \sigma^{(n-1)*} F_V = 0$. A finite $\mathbb{F}_q$-shtuka over R is called *étale* if F_V is an isomorphism. If $\underline{V} = (V, F_V)$ is étale, we define for any R-algebra R' the *τ-invariants of $\underline{V}$ over R'* as the $\mathbb{F}_q$-vector space

$$\underline{V}^\tau(R') := \left\{ v \in V \otimes_R R' \colon v = F_V(\sigma_V^* v) \right\}. \tag{4.1}$$

Recall that an R-group scheme $G = \operatorname{Spec} B$ is *finite locally free* if B is a finite locally free R-module. By [14, $\mathrm{I}_{\mathrm{new}}$, Proposition 6.2.10] this is equivalent to G being finite, flat and of finite presentation over $\operatorname{Spec} R$. Every finite locally free R-group scheme $G = \operatorname{Spec} B$ is a relative complete intersection by [29, III.4.15]. This means that locally on $\operatorname{Spec} R$ we can choose a presentation $B = R[X_1, \ldots, X_n]/I$ where the ideal I is generated by a regular sequence; compare [14, IV_4, Proposition 19.3.7]. The zero section $e\colon \operatorname{Spec} R \to G$ defines an augmentation $e_B := e^*\colon B \twoheadrightarrow R$ of the R-algebra B. Set $I_B := \ker e_B$. For the polynomial ring $R[\underline{X}] = R[X_1, \ldots, X_n]$ set $I_{R[\underline{X}]} = (X_1, \ldots, X_n)$ and $e_{R[\underline{X}]}\colon R[\underline{X}] \twoheadrightarrow R$, $X_\nu \mapsto 0$. Faltings [16] and Abrashkin [1] consider the deformation $B^\flat := R[\underline{X}]/(I \cdot I_{R[\underline{X}]})$ and the canonical epimorphism $B^\flat \twoheadrightarrow B$. They remark that there is a unique morphism

$$\Delta^\flat\colon B^\flat \longrightarrow (B \otimes_R B)^\flat := R\big[\underline{X} \otimes 1, 1 \otimes \underline{X}\big] / \big(I \otimes 1 + 1 \otimes I\big)\big(I_{R[\underline{X}]} \otimes 1 + 1 \otimes I_{R[\underline{X}]}\big)$$

lifting the comultiplication $\Delta\colon B \to B \otimes_R B$ and satisfying $(\mathrm{id}_{B^\flat} \otimes e_B^\flat) \circ \Delta^\flat = \mathrm{id}_{B^\flat} = (e_B^\flat \otimes \mathrm{id}_{B^\flat}) \circ \Delta^\flat$, where $e_B^\flat\colon B^\flat \twoheadrightarrow R$ is the augmentation map; see [1, Section 1.2] or [19, remark after Definition 3.5]. It satisfies $\Delta^\flat(x) - x \otimes 1 - 1 \otimes x \in I_{B^\flat} \otimes I_{B^\flat}$ for all $x \in I_{B^\flat}$. Set $\mathcal{G} = (G, G^\flat) := (\operatorname{Spec} B, \operatorname{Spec} B^\flat)$. The *co-Lie complex of $\mathcal{G}$ over* $\operatorname{Spec} R$ (that is, the fiber at the zero section of G of the cotangent complex; see [20, Section VII.3.1]) is the complex of finite locally free R-modules of rank n

$$\ell^\bullet_{\mathcal{G}/\operatorname{Spec} R}\colon \quad 0 \longrightarrow (I/I^2) \otimes_{B, e_B} R \xrightarrow{\;d\;} \Omega^1_{R[\underline{X}]/R} \otimes_{R[\underline{X}], e_{R[\underline{X}]}} R \longrightarrow 0 \tag{4.2}$$

concentrated in degrees -1 and 0 with d being the differential map. Note that $(I/I^2) \otimes_{B, e_B} R = \ker(B^\flat \twoheadrightarrow B)$ and $\Omega^1_{R[\underline{X}]/R} \otimes_{R[\underline{X}], e_{R[\underline{X}]}} R = \ker(e_B^\flat)/\ker(e_B^\flat)^2$ can be computed from $(B, B^\flat)$. Up to homotopy equivalence it only depends on G and not on the presentation $B = R[\underline{X}]/I$. The *co-Lie module of G over R* is defined as $\omega_G := \mathrm{H}^0(\ell^\bullet_{\mathcal{G}/\operatorname{Spec} R}) := \operatorname{coker} d$. We can now recall the definition of strict $\mathbb{F}_q$-module schemes from Faltings [16] and Abrashkin [1]; see also [19, Section 4].

Definition 4.2. Let $(G, [\,.\,])$ be a pair, where $G = \operatorname{Spec} B$ is an affine flat commutative group scheme over R which is a relative complete intersection and where $[\,.\,]\colon \mathbb{F}_q \to \operatorname{End}_{R\text{-groups}}(G)$, $a \mapsto [a]$ is a ring homomorphism. Then $(G, [\,.\,])$ is called a *strict $\mathbb{F}_q$-module scheme* if there exists a presentation $B = R[\underline{X}]/I$ and a lift $[\,.\,]^\flat\colon \mathbb{F}_q \to \mathrm{End}_{R\text{-algebras}}(B^\flat)$, $a \mapsto [a]^\flat$ of the $\mathbb{F}_q$-action on G, such that the induced action on $\ell^{\bullet}_{\mathcal{G}/\operatorname{Spec} R}$ is equal to the scalar multiplication via $\gamma\colon \mathbb{F}_q \to R$, and such that $[1]^\flat = \mathrm{id}_{B^\flat}$ and $[0]^\flat = e_B^\flat$, as well as $[a\tilde{a}]^\flat = [a]^\flat \circ [\tilde{a}]^\flat$ and $[a+\tilde{a}]^\flat = m \circ ([a]^\flat \otimes [\tilde{a}]^\flat) \circ \Delta^\flat$, where $m\colon (B \otimes_R B)^\flat \to B^\flat$ is induced by the multiplication map $B^\flat \otimes_R B^\flat \to B^\flat$ in the ring $B^\flat$ and the homomorphism $[a]^\flat \otimes [\tilde{a}]^\flat\colon B^\flat \otimes_R B^\flat \to B^\flat \otimes_R B^\flat$ induces a homomorphism $(B \otimes_R B)^\flat \to (B \otimes_R B)^\flat$ denoted again by $[a]^\flat \otimes [\tilde{a}]^\flat$. If G is finite locally free, such a lift $a \mapsto [a]^\flat$ then exists for every presentation and is uniquely determined by [19, Lemmas 4.4 and 4.7].

Example 4.3. The group scheme $\mathbb{G}^d_{a,R}$ is a strict $\mathbb{F}_q$-module scheme for any d, because we can choose $B = R[X_1, \dots, X_d]$ and so $I = (0)$ and $B^\flat = B$, and $a \in \mathbb{F}_q$ acts as $[a]^* X_i = a \cdot X_i$. Moreover, every $\mathbb{F}_q$-linear group homomorphism $\mathbb{G}^d_{a,R} \to \mathbb{G}^{d'}_{a,R}$ is strict in the sense of [16, Definition 1], meaning that the homomorphism lifts to a homomorphism between the $B^\flat$ which is equivariant for the $\mathbb{F}_q$-action via $[\,.\,]^\flat$.

Lemma 4.4. *Let G be a finite locally free group scheme over R, let $\mathbb{F}_q \to \mathrm{End}_{R\text{-groups}}(G)$ be a ring homomorphism, and let $R \to R'$ be a faithfully flat ring homomorphism. Then G is a strict $\mathbb{F}_q$-module scheme if and only if $G \times_R R'$ is.*

Proof. Let $pr\colon \operatorname{Spec} R' \to \operatorname{Spec} R$ be the induced morphism and let $pr_i\colon \operatorname{Spec} R' \otimes_R R' \to \operatorname{Spec} R'$ be the projection onto the i-th factor. Let $G = \operatorname{Spec} B$, let $R'[\underline{X}] \twoheadrightarrow B \otimes_R R'$ be a presentation, and let $\mathbb{F}_q \to \mathrm{End}_{R\text{-algebras}}\big((B \otimes_R R')^\flat\big)$, $a \mapsto [a]^\flat$ be a lift of the $\mathbb{F}_q$-action on G as in Definition 4.2, which makes $G \times_R R'$ into a strict $\mathbb{F}_q$-module scheme over R'. Moreover, let $f\colon R[\underline{Y}] \twoheadrightarrow B$ be an arbitrary presentation and let $\widetilde{\mathcal{G}} = \big(\operatorname{Spec} B,\ \operatorname{Spec} R[\underline{Y}]/(\underline{Y}) \cdot \ker(f)\big)$ be the corresponding deformation. By [19, Lemmas 4.4 and 4.7] there exists a unique lift $a \mapsto \widetilde{[a]}^\flat$ on the deformation $\widetilde{\mathcal{G}} \times_R R' = pr^*\widetilde{\mathcal{G}}$. By the uniqueness the two lifts $pr_1^*\widetilde{[a]}^\flat$ and $pr_2^*\widetilde{[a]}^\flat$ on the deformation $pr_1^*\, pr^*\widetilde{\mathcal{G}} = pr_2^*\, pr^*\widetilde{\mathcal{G}}$ coincide. By faithfully flat descent [7, Section 6.1, Theorem 6] this lift descends to a lift on the deformation $\widetilde{\mathcal{G}}$, which makes G into a strict $\mathbb{F}_q$-module scheme over R. □

To explain the equivalence between finite $\mathbb{F}_q$-shtukas and finite locally free strict $\mathbb{F}_q$-module schemes over R we recall Drinfeld's functor.

Definition 4.5. Let $\underline{V} = (V, F_V)$ be a pair consisting of a (not necessarily finite locally free) R-module V and a morphism $F_V\colon \sigma^* V \to V$ of R-modules. Following Drinfeld [13, Section 2] we define

$$\mathrm{Dr}_q(\underline{V}) := \operatorname{Spec}\Big(\bigoplus_{n \ge 0} \mathrm{Sym}^n_R V\Big)\Big/ I$$

where the ideal I is generated by the elements $v^{\otimes q} - F_V(\sigma_V^* v)$ for all $v \in V$. (Here $v^{\otimes q}$ lives in $\mathrm{Sym}^q V$ and $F_V(\sigma_V^* v)$ in $\mathrm{Sym}^1 V$.) Then $\mathrm{Dr}_q(\underline{V})$ is a group scheme over R via the comultiplication $\Delta\colon v \mapsto v \otimes 1 + 1 \otimes v$ and an $\mathbb{F}_q$-module scheme via $[a]\colon v \mapsto av$ for $a \in \mathbb{F}_q$. It has a canonical deformation

$$\mathrm{Dr}_q(\underline{V})^\flat := \mathrm{Spec}\Big(\bigoplus_{n\geq 0} \mathrm{Sym}^n_R V\Big)/(I \cdot I_0),$$

where $I_0 = \bigoplus_{n\geq 1} \mathrm{Sym}^n_R V$ is the ideal generated by the $v \in V$. This deformation is equipped with the comultiplication $\Delta^\flat\colon v \mapsto v \otimes 1 + 1 \otimes v$ and the $\mathbb{F}_q$-action $[a]^\flat\colon v \mapsto av$. We set $\mathcal{D}r_q(\underline{V}) := (\mathrm{Dr}_q(\underline{V}), \mathrm{Dr}_q(\underline{V})^\flat)$. On its co-Lie complex $[a]$ acts by scalar multiplication with a because $(av)^{\otimes q} - F_V(\sigma_V^*(av)) = a^q(v^{\otimes q} - F_V(\sigma_V^* v))$. Therefore $\mathrm{Dr}_q(\underline{V})$ is a finite locally free strict $\mathbb{F}_q$-module scheme if V is a finite locally free R-module. Every morphism $(V, F_V) \to (W, F_W)$, that is, every R-homomorphism $f\colon V \to W$ with $f \circ F_V = F_W \circ \sigma^* f$, induces a morphism $\mathrm{Dr}_q(f)\colon \mathrm{Dr}_q(W, F_W) \to \mathrm{Dr}_q(V, F_V)$. So Dr_q is a contravariant functor. If f is surjective then $\mathrm{Dr}_q(f)$ is a closed immersion.

Conversely, with a (not necessarily finite locally free) $\mathbb{F}_q$-module scheme G over R we associate the pair $\underline{M}_q(G) := \big(M_q(G), F_{M_q(G)}\big)$ consisting of the R-module

$$M_q(G) := \mathrm{Hom}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(G, \mathbb{G}_{a,R})$$

and the R-homomorphism $F_{M_q(G)}\colon \sigma^* M_q(G) \to M_q(G)$ which is induced from $M_q(G) \to M_q(G), m \mapsto \mathrm{Frob}_{q,\mathbb{G}_{a,R}} \circ m$. Every morphism of $\mathbb{F}_q$-module schemes $f\colon G \to G'$ induces an R-homomorphism $\underline{M}_q(G') \to \underline{M}_q(G)$, $m' \mapsto m' \circ f$. Note that by an argument as in Remark 3.3 we have $\underline{M}_q(G) \otimes_R S = \underline{M}_q(G \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, S)$ for every R-algebra S.

There is a natural morphism $\underline{V} \to \underline{M}_q(\mathrm{Dr}_q(\underline{V}))$, $v \mapsto f_v$, where $f_v\colon \mathrm{Dr}_q(\underline{V}) \to \mathbb{G}_{a,R} = \mathrm{Spec}\, R[\xi]$ is given by $f_v^*(\xi) = v$. There is also a natural morphism of group schemes $G \to \mathrm{Dr}_q(\underline{M}_q(G))$ given by $\bigoplus_{n\geq 0} \mathrm{Sym}^n_R M_q(G)/I \to \Gamma(G, \mathcal{O}_G)$, $m \mapsto m^*(\xi)$, which is well defined because $F_{M_q(G)}(\sigma^* m)^*(\xi) = (\mathrm{Frob}_{q,\mathbb{G}_{a,R}} \circ m)^*(\xi) = m^*(\xi^q) = (m^*(\xi))^q$.

Example 4.6. For example if $\underline{E} = (E, \varphi)$ is an Abelian Anderson A-module of dimension d, then $\underline{M}_q(\underline{E}) = (M_q(\underline{E}), F_{M_q(\underline{E})})$ was denoted $\underline{M}(\underline{E}) = (M(\underline{E}), \tau_{M(\underline{E})})$ in Definition 1.2. There is a canonical isomorphism $\underline{E} \xrightarrow{\sim} \mathrm{Dr}_q(\underline{M}_q(\underline{E}))$ which is constructed as follows. We set $\mathbb{G}_{a,R} = \mathrm{Spec}\, R[\xi]$ and consider for each $m \in M_q(\underline{E}) = \mathrm{Hom}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(E, \mathbb{G}_{a,R})$ the element $m^*(\xi) \in \Gamma(E, \mathcal{O}_E)$. We claim that

$$\begin{aligned}&\Big(\bigoplus_{n\geq 0} \mathrm{Sym}^n_R M_q(\underline{E})\Big)\Big/\Big(m^{\otimes q} - F_{M_q(\underline{E})}(\sigma^*_{M_q(\underline{E})} m) : m \in M_q(\underline{E})\Big)\\ &\xrightarrow{\sim} \Gamma(E, \mathcal{O}_E), \qquad m \mapsto m^*(\xi)\end{aligned} \tag{4.3}$$

is an isomorphism of R-algebras. To prove that it is an isomorphism we may apply a faithfully flat base change $R \to R'$ over which we have an $\mathbb{F}_q$-linear isomorphism $\alpha\colon E \otimes_R R' \xrightarrow{\sim} \mathbb{G}^d_{a,R'} = \mathrm{Spec}\, R'[x_1, \ldots, x_d]$. Let $m_i := pr_i \circ \alpha \in$

$M_q(\underline{E}) \otimes_R R'$ where $pr_i : \mathbb{G}^d_{a,R'} \to \mathbb{G}_{a,R'}$ is the projection onto the i-th factor. Then $M_q(\underline{E}) \otimes_R R' = \bigoplus_{i=0}^d R'\{\tau\} \cdot m_i$ by Remark 3.3 and the inverse of (4.3) sends $\alpha^*(x_i)$ to m_i. This is indeed the inverse, because (4.3) sends each of the generators $\tau^j m_i = \mathrm{Frob}_{q^j,\mathbb{G}_{a,R}} \circ m_i$ of the R'-module $M_q(\underline{E}) \otimes_R R'$ to $(\mathrm{Frob}_{q^j,\mathbb{G}_{a,R}} \circ m_i)^*(\xi) = m_i^*(\xi^{q^j}) = \alpha^*(x_i)^{q^j}$, and this inverse sends it back to $m_i^{\otimes q^j} = \mathrm{Frob}_{q^j,\mathbb{G}_{a,R}} \circ m_i = \tau^j m_i$.

The following theorem goes back to Abrashkin [1, Theorem 2]. Statements (b)-(d) were proved in [19, Theorem 5.2].

Theorem 4.7.

(a) *The contravariant functors* Dr_q *and* $\underline{M}_q$ *are mutually quasi-inverse anti-equivalences between the category of finite* $\mathbb{F}_q$*-shtukas over* R *and the category of finite locally free strict* $\mathbb{F}_q$*-module schemes over* R*. Both functors are* $\mathbb{F}_q$*-linear and exact.*

Let $\underline{V} = (V, F_V)$ *be a finite* $\mathbb{F}_q$*-shtuka over* R *and let* $G = \mathrm{Dr}_q(\underline{V})$*. Then*

(b) *The* $\mathbb{F}_q$*-module scheme* $\mathrm{Dr}_q(\underline{V})$ *is étale over* R *if and only if* $\underline{V}$ *is étale;*
(c) *The natural morphisms* $\underline{V} \to \underline{M}_q(\mathrm{Dr}_q(\underline{V}))$, $v \mapsto f_v$ *and* $G \to \mathrm{Dr}_q(\underline{M}_q(G))$ *are isomorphisms;*
(d) *The co-Lie complex* $\ell^{\bullet}_{\mathcal{D}r_q(\underline{V})/S}$ *is canonically isomorphic to the complex of* R*-modules* $0 \to \sigma^* V \xrightarrow{F_V} V \to 0$.

5. Isogenies

Definition 5.1. A morphism $f \in \mathrm{Hom}_R(\underline{E}, \underline{E}')$ between two Abelian Anderson A-modules $\underline{E}$ and $\underline{E}'$ over R is an *isogeny* if $f : E \to E'$ is finite and surjective. If there exists an isogeny between $\underline{E}$ and $\underline{E}'$ then they are called *isogenous*. (Being isogenous is an equivalence relation; see Corollary 5.16 below.)

An isogeny $f : \underline{E} \to \underline{E}'$ is *separable* if f is étale, or equivalently if the group scheme $\ker f$ is étale over R. Indeed, since f is flat by Proposition 5.2(b) it suffices to see that all fibers of f over E' are étale by [7, Section 2.4, Proposition 8]. Now all fibers are isomorphic to $\ker f$ by the remarks after (3.1).

We recall the following well known criterion for being an isogeny. For the convenience of the reader we include a proof.

Proposition 5.2. *Let* $f : E \to E'$ *be a morphism between two affine, smooth* R*-group schemes* E *of relative dimension* d *and* E' *of relative dimension* d'*, such that the fibers of* E' *over all points of* $\operatorname{Spec} R$ *are connected. Then the following are equivalent:*

(a) f *is finite and faithfully flat, that is flat and surjective; see* [14, 0_{I}.6.7.8]*;*
(b) $\ker f$ *is finite and* f *is flat;*

(c) $\ker f$ *is finite and* f *is surjective;*
(d) $\ker f$ *is finite and* $d = d'$*;*
(e) $\ker f$ *is finite and* f *is an epimorphism of sheaves for the fpqc-topology.*

If $R = k$ *is a field, then these conditions are equivalent to*

(f) f *is surjective and* $d = d'$.

Proof. We show that (a) implies all other conditions. This is obvious for (b), (c) and (e). To prove that $d = d'$ let $\mathfrak{m} \subset R$ be a maximal ideal and consider the base change to $k = R/\mathfrak{m}$. Then $f \times \mathrm{id}_k \colon E \times_R k \to E' \times_R k$ is a finite surjective morphism, and hence $d = \dim E \times_R k = \dim E' \times_R k = d'$; see [15, Corollary 9.3].

Conversely, clearly (e)$\Longrightarrow$(c). We now show (f)$\Longrightarrow$(c) and (b)$\Longrightarrow$(c)$\Longrightarrow$(d)$\Longrightarrow$(b)$\Longrightarrow$(a). Generally note that by the remarks after (3.1) all non-empty fibers of f are isomorphic to $\ker f$.

First assume (f) and note that when $R = k$ is a field, the ring $\Gamma(E', \mathcal{O}_{E'})$ is an integral domain by our assumptions on E'. The surjectivity of f implies that $f^* \colon \Gamma(E', \mathcal{O}_{E'}) \hookrightarrow \Gamma(E, \mathcal{O}_E)$ is injective of relative transcendence degree $d - d' = 0$. Since all fibers of f are isomorphic to $\ker f$, [15, Corollary 14.6] implies that $\ker f$ is finite over $\operatorname{Spec} k$ and (c) holds.

We next show for general R that (b) implies (c). Namely, f is of finite presentation by [14, IV_1, Proposition 1.6.2(v)], because E and E' are of finite presentation over R. Therefore (b) implies that f is universally open by [14, IV_2, Théorème 2.4.6]. In particular $(f \times \mathrm{id}_k)(E \times_R k) \subset E' \times_R k$ is open for every point $\operatorname{Spec} k \to \operatorname{Spec} R$ of $\operatorname{Spec} R$. Since $E' \times_R k$ was assumed to be connected, it possesses no proper open subgroup, and hence $f \times \mathrm{id}_k$ is surjective. This establishes (c).

To prove that (c) implies (d) again consider the morphism $f \times \mathrm{id}_k \colon E \times_R k \to E' \times_R k$ over a point $\operatorname{Spec} k \to \operatorname{Spec} R$ of $\operatorname{Spec} R$. Since the map $f \times \mathrm{id}_k$ is surjective, $f^* \otimes \mathrm{id}_k \colon \Gamma(E', \mathcal{O}_{E'}) \otimes_R k \hookrightarrow \Gamma(E, \mathcal{O}_E) \otimes_R k$ is injective, because otherwise its kernel would define a proper closed subscheme of $E' \times_R k$ through which $f \times \mathrm{id}_k$ factors. Since all fibers of f are isomorphic to $\ker f$, and hence finite, [15, Corollary 13.5] shows that $d' = \dim \Gamma(E', \mathcal{O}_{E'}) \otimes_R k = \dim \Gamma(E, \mathcal{O}_E) \otimes_R k = d$.

We prove the implication (d)$\Longrightarrow$(b). Consider the fiber $f \times \mathrm{id}_k \colon E \times_R k \to E' \times_R k$ over a point $\operatorname{Spec} k \to \operatorname{Spec} R$ of $\operatorname{Spec} R$ and the inclusion $\big(\Gamma(E', \mathcal{O}_{E'}) \otimes_R k\big)/\ker(f^* \otimes \mathrm{id}_k) \hookrightarrow \Gamma(E, \mathcal{O}_E) \otimes_R k$. Since all fibers of f are finite, [15, Corollary 13.5] implies $\dim \Gamma(E', \mathcal{O}_{E'}) \otimes_R k = d' = d = \dim \Gamma(E, \mathcal{O}_E) \otimes_R k = \dim\big(\Gamma(E', \mathcal{O}_{E'}) \otimes_R k\big)/\ker(f^* \otimes \mathrm{id}_k)$. It follows that $\ker(f^* \otimes \mathrm{id}_k) = (0)$ and $f^* \otimes \mathrm{id}_k \colon \Gamma(E', \mathcal{O}_{E'}) \otimes_R k \hookrightarrow \Gamma(E, \mathcal{O}_E) \otimes_R k$ is injective. Let $\mathfrak{m} \subset \Gamma(E, \mathcal{O}_E) \otimes_R k$ be a maximal ideal. Then $(f^* \otimes \mathrm{id}_k)^{-1}(\mathfrak{m}) \subset \Gamma(E', \mathcal{O}_{E'}) \otimes_R k$ is a maximal ideal by [15, Theorem 4.19]. Since the fiber of f over $\mathfrak{m}$ is finite, [15, Theorem 18.16(b)] implies that $f \otimes \mathrm{id}_k$ is flat at $\mathfrak{m}$. Since E and E' are smooth over R it follows from [14, IV_3, Théorème 11.3.10] that f is flat.

Finally we show that (b) and (c) together imply (a). By (b) and (c) the morphism $f : E \to E'$ is faithfully flat. Whether f is finite can by [14, IV$_2$, Proposition 2.7.1] be tested after the faithfully flat base change $E \to E'$. By (3.1) the finiteness of the projection $E \times_{E'} E \to E$ onto the first factor follows from the finiteness of $\ker f$ over $\operatorname{Spec} R$. This proves (a). □

Corollary 5.3. *Let $f \in \operatorname{Hom}_R(\underline{E}, \underline{E}')$ be an isogeny between Abelian Anderson A-modules over R. Then*

(a) *The kernel* $\ker f$ *of f is a finite locally free group scheme and a strict $\mathbb{F}_q$-module scheme over R;*
(b) *f induces an isomorphism between E' and the quotient $E/\ker f$.*

Proof. (a) Since f is flat of finite presentation by [14, IV$_1$, Proposition 1.6.2(v)], $\ker f$ is flat of finite presentation over R. Since it is also finite, it is finite locally free. Over a faithfully flat R-algebra R' both E and E' become isomorphic to powers of $\mathbb{G}_{a,R'}$ and hence are strict $\mathbb{F}_q$-module schemes by Example 4.3. Therefore $(\ker f) \otimes_R R'$ is a strict $\mathbb{F}_q$-module scheme over R' by [16, Proposition 2] and $\ker f$ is a strict $\mathbb{F}_q$-module scheme over R by Lemma 4.4.

(b) This follows from [29, Théorème V.4.1]. □

Note that two isogenous Abelian Anderson A-modules have the same dimension by Proposition 5.2. We will see in Corollary 5.10 below that they also have the same rank. For Drinfeld modules there is a further characterization of isogenies as follows.

Proposition 5.4.

(a) *If $\underline{E}$ and $\underline{E}'$ are Drinfeld A-modules over R with* $\operatorname{Spec} R$ *connected and $f \in$* $\operatorname{Hom}_k(\underline{E}, \underline{E}')$*, then f is an isogeny if and only if $f \neq 0$;*
(b) *If this is the case then f is separable if and only if* $\operatorname{Lie} f \in R^\times$.

Proof. (a) Let $f : \underline{E} \to \underline{E}'$ be an isogeny, then $f \neq 0$ because the zero morphism is not surjective. Conversely let $f \neq 0$. By Proposition 5.2(d) we must show that $\ker f$ is finite. This question is local on $\operatorname{Spec} R$, so we may assume that $E = E' = \mathbb{G}_{a,R}$ and that $\underline{E} = (E, \varphi)$ and $\underline{E}' = (E', \psi)$ are in standard form. Let $t \in A \smallsetminus \mathbb{F}_q$, and hence $\deg_\tau \varphi_t > 0$ and $\deg_\tau \psi_t > 0$. By Lemma 3.8(b) applied to $f \circ \varphi_t = \psi_t \circ f$ we have $f = \sum_{i=0}^n f_i \tau^i \in R\{\tau\}$ with $f_n \in R^\times$. It follows that $\ker f = \operatorname{Spec} R[x]/(\sum_{i=0}^n f_i x^{q^i})$ which is finite over R.

(b) By the Jacobi criterion [7, Section 2.2, Proposition 7],

$$\ker f = \operatorname{Spec} R[x]/\left(\sum_{i=0}^n f_i x^{q^i}\right)$$

is étale if and only if $\operatorname{Lie} f = f_0 = \frac{\partial f(x)}{\partial x} \in R^\times$. □

Next we turn to A-motives.

Definition 5.5. A morphism $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ between A-motives over R is an *isogeny* if f is injective and $\mathrm{coker}\, f$ is finite and locally free as R-module. If there exists an isogeny between $\underline{M}$ and $\underline{N}$ then they are called *isogenous*. (Being isogenous is an equivalence relation; see Corollary 5.16 below.) A quasi-morphism $f \in \mathrm{QHom}_R(\underline{M}, \underline{N})$ which is of the form $g \otimes c$ for an isogeny $g \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ and a $c \in Q$ is called a *quasi-isogeny*.

If f is an isogeny and $\underline{M}$ and $\underline{N}$ are effective, then the snake lemma yields the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \ker(\tau_{\mathrm{coker}\, f}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma^* M & \xrightarrow{\sigma^* f} & \sigma^* N & \longrightarrow & \sigma^*(\mathrm{coker}\, f) & \longrightarrow & 0 \\
 & & \downarrow \tau_M & & \downarrow \tau_N & & \downarrow \tau_{\mathrm{coker}\, f} & & \\
0 & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & \mathrm{coker}\, f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to \ker(\tau_{\mathrm{coker}\, f}) & \to & \mathrm{coker}\, \tau_M & \to & \mathrm{coker}\, \tau_N & \to & \mathrm{coker}(\tau_{\mathrm{coker}\, f}) & \to & 0\,.
\end{array} \tag{5.1}$$

Namely, by local freeness of $\mathrm{coker}\, f$ the upper row is again exact and identifies $\sigma^*(\mathrm{coker}\, f)$ with $\mathrm{coker}(\sigma^* f)$.

An isogeny $f : \underline{M} \to \underline{N}$ between effective A-motives is *separable* if $\tau_{\mathrm{coker}\, f} : \sigma^*(\mathrm{coker}\, f) \to \mathrm{coker}\, f$ is an isomorphism.

Remark 5.6. If $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ is an isogeny and S is an R-algebra, then the base change $f \otimes \mathrm{id}_S \in \mathrm{Hom}_S(\underline{M} \otimes_R S, \underline{N} \otimes_R S)$ of f to S is again an isogeny. This follows from the exact sequence $0 \longrightarrow \underline{M} \xrightarrow{f} \underline{N} \longrightarrow \mathrm{coker}\, f \longrightarrow 0$ because $\mathrm{coker}\, f$ is a flat R-module.

Example 5.7. For $0 \neq a \in A$ the morphism $a : \underline{M} \to \underline{M}$ is an isogeny with $\mathrm{coker}\, a = M/aM$. Let $\underline{M}$ be effective. Then a is separable if and only if $\ker(\tau_{\mathrm{coker}\, a}) = \mathrm{coker}(\tau_{\mathrm{coker}\, a}) = (0)$. That is, if and only if multiplication with a is an automorphism of $\mathrm{coker}\, \tau_M$. Since $a - \gamma(a)$ is nilpotent on $\mathrm{coker}\, \tau_M$ this is the case if and only if $\gamma(a) \in R^\times$. For the corresponding result about Abelian Anderson A-modules see Corollary 5.11.

Proposition 5.8. *Let $\underline{M}$ and $\underline{N}$ be A-motives over R. If $\underline{M}$ and $\underline{N}$ are isogenous then* $\mathrm{rk}\, \underline{M} = \mathrm{rk}\, \underline{N}$, *and if, moreover, $\underline{M}$ and $\underline{N}$ are effective, then* $\mathrm{rk}_R\, \mathrm{coker}\, \tau_M = \mathrm{rk}_R\, \mathrm{coker}\, \tau_N$. *Conversely assume* $\mathrm{rk}\, \underline{M} = \mathrm{rk}\, \underline{N}$ *and let $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ be a morphism such that* $\mathrm{coker}\, f$ *is a finitely generated R-module. Then f is an isogeny.*

Proof. Let $f : \underline{M} \to \underline{N}$ be an isogeny. Since M, respectively $\mathrm{coker}\, \tau_M$, are finite locally free over A_R, respectively over R, we can compute their ranks by choosing a maximal ideal $\mathfrak{m} \subset R$ and applying the base change from R to $k = R/\mathfrak{m}$. Then $f \otimes \mathrm{id}_k$ is still an isogeny by Remark 5.6. Since $\mathrm{coker}(f \otimes \mathrm{id}_k)$ is a torsion A_k-module it follows that

$$\mathrm{rk}\,\underline{M} \;=\; \mathrm{rk}_{A_R} M \;=\; \mathrm{rk}_{A_k}(M \otimes_R k) \;=\; \mathrm{rk}_{A_k}(N \otimes_R k) \;=\; \mathrm{rk}_{A_R} N \;=\; \mathrm{rk}\,\underline{N}\,.$$

If $\underline{M}$ and $\underline{N}$ are effective, we consider diagram (5.1) for the isogeny $f \otimes \mathrm{id}_k$. Since $\mathrm{coker}(f \otimes \mathrm{id}_k)$ and $\sigma^* \mathrm{coker}(f \otimes \mathrm{id}_k)$ are finite dimensional k-vector spaces of the same dimension, the right vertical column and the bottom row of diagram (5.1) imply that

$$\mathrm{rk}_R\, \mathrm{coker}\,\tau_M = \dim_k \mathrm{coker}(\tau_M \otimes \mathrm{id}_k) = \dim_k \mathrm{coker}(\tau_N \otimes \mathrm{id}_k) = \mathrm{rk}_R\, \mathrm{coker}\,\tau_N\,.$$

The converse follows from Lemma 2.2. □

After these preparations we are now able to formulate and prove our main theorem.

Theorem 5.9. *Let $f \in \mathrm{Hom}_R(\underline{E}, \underline{E}')$ be a morphism between Abelian Anderson A-modules over R, and let $\underline{M}(f) \in \mathrm{Hom}_R(\underline{M}', \underline{M})$ be the associated morphism between the associated effective A-motives $\underline{M} = \underline{M}(\underline{E})$ and $\underline{M}' = \underline{M}(\underline{E}')$ over R. Then*

(a) *f is an isogeny if and only if $\underline{M}(f)$ is an isogeny;*
(b) *f is a separable isogeny if and only if $\underline{M}(f)$ is a separable isogeny;*
(c) *If f is an isogeny there are canonical A-equivariant isomorphisms of finite $\mathbb{F}_q$-shtukas*

$$\big(\mathrm{coker}\,\underline{M}(f),\, \tau_{\mathrm{coker}\,\underline{M}(f)}\big) \;\xrightarrow{\;\sim\;}\; \underline{M}_q(\ker f)$$

and of finite locally free R-group schemes

$$\mathrm{Dr}_q\big(\mathrm{coker}\,\underline{M}(f)\big) \;\xrightarrow{\;\sim\;}\; \ker f\,.$$

Proof. In the beginning we do neither assume that f nor that $\underline{M}(f)$ is an isogeny. We denote by ι the inclusion $\ker f \hookrightarrow E$. Consider the A_R-homomorphism $\underline{M}(\underline{E}) \to \underline{M}_q(\ker f)$, $m \mapsto m \circ \iota$, which is compatible with the Frobenius maps $\tau_{M(\underline{E})}$ and $F_{M_q(\ker f)}$. Since $m = \underline{M}(f)(m') = m' \circ f$ implies $m' \circ f \circ \iota = 0$, it factors over

$$\mathrm{coker}\,\underline{M}(f) \;\longrightarrow\; \underline{M}_q(\ker f), \quad m \bmod \mathrm{im}\,\underline{M}(f) \mapsto m \circ \iota\,. \tag{5.2}$$

On the other hand we claim that there are A-equivariant morphisms

$$\mathrm{Dr}_q\big(\underline{M}_q(\ker f)\big) \longrightarrow \mathrm{Dr}_q(\mathrm{coker}\,\underline{M}(f)) \hookrightarrow \ker f \hookrightarrow E\,, \tag{5.3}$$

where the last two morphisms are closed immersions. The first morphism is obtained from (5.2). Moreover, the epimorphism $\underline{M}(\underline{E}) \twoheadrightarrow \mathrm{coker}\,\underline{M}(f)$ induces by Example 4.6 an A-equivariant closed immersion $\alpha : \mathrm{Dr}_q(\mathrm{coker}\,\underline{M}(f)) \hookrightarrow \mathrm{Dr}_q(\underline{M}(\underline{E})) = \underline{E}$.

We compose it with $f\colon E \to E'$ and show that the composition factors through the zero section $e'\colon \operatorname{Spec} R \to E'$. This will imply that α factors through $\ker f$. We can study this composition after a faithfully flat base change $R \to R'$ over which we have an $\mathbb{F}_q$-linear isomorphism $\beta\colon E' \otimes_R R' \cong \mathbb{G}_{a,R'}^{d'} = \operatorname{Spec} R'[y_1, \dots, y_{d'}]$. Let $m'_i := pr_i \circ \beta \in M(\underline{E}') \otimes_R R'$ where $pr_i\colon \mathbb{G}_{a,R'}^{d'} \to \mathbb{G}_{a,R'} = \operatorname{Spec} R'[\xi]$ is the projection onto the i-th factor. Then $pr_i^*(\xi) = y_i$ and $\alpha^* f^* \beta^*(y_i) = \alpha^* f^* m_i'^*(\xi) = \alpha^* \circ \underline{M}(f)(m'_i)^*(\xi) = 0$ because $\underline{M}(f)(m'_i) = 0$ in $\operatorname{coker} \underline{M}(f)$.

(a) Now assume that f is an isogeny. Then $\ker f$ is a finite locally free group scheme over R, and a strict $\mathbb{F}_q$-module scheme by Corollary 5.3(a). So $\underline{M}_q(\ker f)$ is a finite locally free R-module by Theorem 4.7 and the morphism $\operatorname{Dr}_q(\underline{M}_q(\ker f)) \to \ker f$ in (5.3) is an isomorphism. This shows that $\operatorname{Dr}_q(\operatorname{coker} \underline{M}(f)) \xrightarrow{\sim} \ker f$. We next show that the map (5.2) is an isomorphism. Its cokernel is a finite R-module because $\underline{M}_q(\ker f)$ is. We apply again a faithfully flat base change $R \to R'$ such that $E \otimes_R R' \cong \mathbb{G}_{a,R'}^d$ and $E' \otimes_R R' \cong \mathbb{G}_{a,R'}^{d'}$. Then f is given by a matrix $F \in R'\{\tau\}^{d' \times d}$ by Lemma 3.2. By faithfully flat descent and by Nakayama's lemma [15, Corollaries 2.9 and 4.8] the map (5.2) will be surjective if for all maximal ideals $\mathfrak{m}' \subset R'$ its tensor product with $k := R'/\mathfrak{m}'$ is surjective. By Remark 3.3 and its analog for $\underline{M}_q(\ker f)$ the tensor product of (5.2) with k equals $\operatorname{coker} \underline{M}(f \times \mathrm{id}_k) \to \underline{M}_q(\ker(f \times \mathrm{id}_k))$, where $f \times \mathrm{id}_k\colon \underline{E} \times_R k \to \underline{E}' \times_R k$ is given by the matrix $\overline{F} := F \otimes 1_k$. In particular $\ker(f \times \mathrm{id}_k) = \operatorname{Spec} k[x_1, \dots, x_d]/(f^*(y_\ell)\colon 1 \le \ell \le d)$. Since $\ker f$ is finite, $k[x_1, \dots, x_d]/(f^*(y_\ell)\colon 1 \le \ell \le d)$ is a finite dimensional k-vector space. For fixed i this implies that $\{x_i, x_i^q, x_i^{q^2}, \dots\}$ is linearly dependent and there is a positive integer N and $b_{i,n} \in k$ such that $x_i^{q^{N+1}} = \sum_{n=0}^N b_{i,n} \cdot x_i^{q^n}$ in $k[x_1, \dots, x_d]/(f^*(y_\ell)\colon 1 \le \ell \le d)$. We introduce the new variables $z_{i,n} := x_i^{q^n}$ for $1 \le i \le d$ and $0 \le n \le N$. Then $f^*(y_\ell)$ is a k-linear relation between the $z_{i,n}$. Furthermore

$$k[x_1, \dots, x_d]/(f^*(y_\ell)\colon 1 \le \ell \le d) \cong k[z_{i,n}\colon 1 \le i \le d, \le n \le N]/I \quad \text{with}$$

$$I = \left(f^*(y_i),\ z_{i,N}^q - \sum_{n=0}^N b_{i,n} \cdot z_{i,n},\ z_{i,n}^q - z_{i,n+1}\colon 1 \le i \le d, 0 \le n < N \right).$$

Let $\tilde{z}_1, \dots, \tilde{z}_r$ be a k-basis of $(\bigoplus_{i=1}^d \bigoplus_{n=0}^N k \cdot z_{i,n})/(f^*(y_\ell)\colon 1 \le \ell \le d)$. Then there are elements $c_{ij} \in k$ for $1 \le i, j \le r$ such that

$$k[x_1, \dots, x_d]/(f^*(y_\ell)\colon 1 \le \ell \le d) \cong k[\tilde{z}_1, \dots, \tilde{z}_r] \Big/ \left(\tilde{z}_i^q - \sum_{j=1}^r c_{ij} \tilde{z}_j : 1 \le i \le r \right) =: B\,.$$

Moreover, the group law on $\ker f$ is given by the comultiplication $\Delta^*\colon B \to B \otimes_k B$, $\Delta^*(\widetilde{z}_i) = \widetilde{z}_i \otimes 1 + 1 \otimes \widetilde{z}_i$ and the $\mathbb{F}_q$-action is given by $\varphi_\lambda\colon B \to B$, $\varphi_\lambda^*(\widetilde{z}_i) = \gamma(\lambda) \cdot \widetilde{z}_i$.

We are now ready to compute $\underline{M}_q(\ker(f\times \mathrm{id}_k))$ from (3.3). If $\mathbb{G}_{a,k} = \operatorname{Spec} k[\xi]$ then every element $\widetilde{m} \in \underline{M}_q(\ker(f \times \mathrm{id}_k))$ satisfies $\widetilde{m}^*(\xi) = \sum_{\ell_i \in \{0\ldots q-1\}} d_{\ell_1,\ldots,\ell_r} \cdot \tilde{z}_1^{\ell_1} \cdot \ldots \cdot \tilde{z}_r^{\ell_r}$ with $d_{\ell_1,\ldots,\ell_r} \in k$. Since the $\tilde{z}_1^{\ell_1} \cdot \ldots \cdot \tilde{z}_r^{\ell_r}$ form a k-basis of B, the conditions $\Delta^*\widetilde{m}^*(\xi) = \widetilde{m}^*(\xi)\otimes 1 + 1\otimes \widetilde{m}^*(\xi)$ in $B\otimes_k B$ and $\varphi_\lambda^*\widetilde{m}^*(\xi) = m^*(\gamma(\lambda)\cdot \xi) = \gamma(\lambda)\cdot\widetilde{m}^*(\xi)$ in B for $\lambda \in \mathbb{F}_q$ imply as in Lemma 3.2 that $\widetilde{m}^*(\xi) = d_{1,0\ldots 0}\cdot\tilde{z}_1 + \ldots + d_{0\ldots 0,1}\cdot\tilde{z}_r$. Since $\tilde{z}_i$ is a k-linear combination of the $z_{j,n} = x_j^{q^n}$ the morphism $m\colon E\times_R k \to \mathbb{G}_{a,k}$ with $m^*(\xi) = d_{1,0\ldots 0}\cdot\tilde{z}_1 + \ldots + d_{0\ldots 0,1}\cdot\tilde{z}_r$ belongs to $\underline{M}(E\times_R k)$ and maps to $\widetilde{m}$ under the map $\operatorname{coker}\underline{M}(f\times \mathrm{id}_k) \to \underline{M}_q\big(\ker(f\times\mathrm{id}_k)\big)$. This proves that (5.2) is surjective.

In order to show that (5.2) is injective let $m \in M(\underline{E})$ be an element with $m\circ\iota = 0$. By [29, Théorème V.4.1] the morphism $m\colon E \to \mathbb{G}_{a,R}$ factors through $E/\ker f \xrightarrow{\sim} E'$ (use Corollary 5.3(b)) in the form $m = m'\circ f$ for an $m' \in M(\underline{E}')$. This shows that $m \bmod \operatorname{im}\underline{M}(f) = 0$ in $\operatorname{coker}\underline{M}(f)$. All together we have proved that $\operatorname{coker}\underline{M}(f) \xrightarrow{\sim} \underline{M}_q(\ker f)$ is a finite locally free R-module. Moreover, $\underline{M}(f)$ is injective, because if $m' \in M(\underline{E}')$ satisfies $m'\circ f = \underline{M}(f)(m') = 0$ the surjectivity of f implies $m' = 0$. More precisely, f is an epimorphism of sheaves for the *fpqc*-topology by Proposition 5.2(e). Now the injectivity of $\underline{M}(f)$ follows from the left exactness of the functor $\operatorname{Hom}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\bullet, \mathbb{G}_{a,R})$. This proves that $\underline{M}(f)$ is an isogeny, and it also proves (c).

Conversely assume that $\underline{M}(f)$ is an isogeny. Then $d := \dim\underline{E} = \dim\underline{E}'$ by Theorem 3.5 and Proposition 5.8. We prove that $\ker f$ is finite. For this purpose we apply a faithfully flat base change $R \to R'$ such that $E\otimes_R R' \cong \mathbb{G}_{a,R'}^d = \operatorname{Spec} R'[x_1,\ldots,x_d]$ and $E'\otimes_R R' \cong \mathbb{G}_{a,R'}^d = \operatorname{Spec} R'[y_1,\ldots,y_d]$. Also when we write $\mathbb{G}_{a,R'} = \operatorname{Spec} R'[\xi]$ then $\underline{M}(\underline{E}\times_R R') \cong \bigoplus_{i=1}^d R'\{\tau\}\cdot m_i$ and $\underline{M}(\underline{E}'\times_R R') \cong \bigoplus_{i=1}^d R'\{\tau\}\cdot m_i'$ where $m_i^*(\xi) = x_i$ and $m_i'^*(\xi) = y_i$. Consider the epimorphism of R'-modules

$$\bigoplus_{i=1}^d \bigoplus_{0\le n} R'\cdot\tau^n m_i \;\cong\; \underline{M}(\underline{E}\times_R R') \overset{\delta}{\twoheadrightarrow} \operatorname{coker}\underline{M}(f\otimes\mathrm{id}_{R'}).$$

Since $\operatorname{coker}\underline{M}(f\otimes\mathrm{id}_{R'})$ is finite locally free over R', and hence projective, this epimorphism has a section s whose image lies in $\bigoplus_{i=1}^d\bigoplus_{n=0}^N R'\cdot\tau^n m_i$ for some N. It follows that $\tau^{N+1}m_i - s(\delta(\tau^{N+1}m_i))$ maps to zero in $\operatorname{coker}\underline{M}(f\otimes\mathrm{id}_{R'})$. That is, there are elements $b_{i,j,n} \in R'$ and $\widetilde{m}_i' \in \underline{M}(\underline{E}'\times_R R')$ with $\tau^{N+1}m_i - \sum_{j=1}^d\sum_{n=0}^N b_{i,j,n}\cdot\tau^n m_j = \underline{M}(f)(\widetilde{m}_i')$. Applying this equation to ξ yields

$$x_i^{q^{N+1}} - \sum_{j=1}^d\sum_{n=0}^N b_{i,j,n}\cdot x_j^{q^n} = f^*\widetilde{m}_i'^*(\xi) \in f^*R'[y_1,\ldots,y_d] \cong f^*\Gamma(E',\mathcal{O}_{E'})\otimes_R R'.$$

Thus $f\times\mathrm{id}_{R'}\colon E\times_R R' \to E'\times_R R'$ is finite. By faithfully flat descent [14, IV$_2$, Proposition 2.7.1] also f is finite. By Proposition 5.2(d) this proves that f is an isogeny and establishes (a).

Finally (b) follows from (c) and Theorem 4.7(b). □

Corollary 5.10. *If $\underline{E}$ and $\underline{E}'$ are isogenous Abelian Anderson A-modules over R, then* $\operatorname{rk}\underline{E} = \operatorname{rk}\underline{E}'$.

Proof. This follows directly from Theorems 3.5, 5.9 and Proposition 5.8. □

Corollary 5.11. *Let $\underline{E}$ be an Abelian Anderson A-module over R and let $a \in A$. Then $\varphi_a\colon \underline{E} \to \underline{E}$ is an isogeny. It is separable if and only if $\gamma(a) \in R^\times$.*

Proof. The assertion follows from Theorem 5.9 and Example 5.7. The criterion for separability can also be proved without reference to A-motives; see our proof of Theorem 6.4(b) below. □

We next come to our second main result.

Theorem 5.12. *Let $\underline{M}$ and $\underline{N}$ be two A-motives over R and let $f \in \operatorname{Hom}_R(\underline{M}, \underline{N})$ be a morphism. Then the following are equivalent:*

(a) *f is an isogeny;*
(b) *There is an element $0 \neq a \in A$ such that f induces an isomorphism of $A_R[\frac{1}{a}]$-modules $M[\frac{1}{a}] \xrightarrow{\sim} N[\frac{1}{a}]$.*

In particular, a quasi-morphism $f \in \operatorname{QHom}_R(\underline{M}, \underline{N})$ is a quasi-isogeny if and only if it induces an isomorphism $f\colon M[\frac{1}{a}] \xrightarrow{\sim} N[\frac{1}{a}]$ for an element $a \in A \smallsetminus \{0\}$.

Proof. (b)$\Longrightarrow$(a) Clearly $\operatorname{rk}\underline{M} = \operatorname{rk}\underline{N}$. Since $\operatorname{coker} f$ is a finitely generated A_R-module, $(\operatorname{coker} f) \otimes_A A[\frac{1}{a}] = (0)$ implies that $a^n \cdot \operatorname{coker} f = (0)$ for some positive integer n. Therefore, $\operatorname{coker} f$ is a finitely generated module over $A_R/(a^n) = A/(a^n) \otimes_{\mathbb{F}_q} R$, whence over R. So (a) follows from Proposition 5.8.

(a)$\Longrightarrow$(b) If R is a field this was proved in [6, Corollary 5.4] and also follows from [26, Proposition 3.4.5] and [30, Proposition 3.1.2]. We generalize the proof to the relative situation.

1. If f is an isogeny, then $\operatorname{coker} f$ is a finite locally free R-module, which we may assume to be free after passing to an open affine covering of $\operatorname{Spec} R$. Let $t \in A \smallsetminus \mathbb{F}_q$ and consider the finite flat homomorphism $\widetilde{A} := \mathbb{F}_q[t] \hookrightarrow A$ from Lemma 1.5, under which we view $\underline{M}$ and $\underline{N}$ as $\widetilde{A}$-motives by restriction of scalars. That is, we view M and N as locally free $R[t]$-modules of rank $\tilde{r} = \operatorname{rk}\underline{M} \cdot \operatorname{rk}_{\widetilde{A}} A$ and τ_M and τ_N as $R[t][\frac{1}{t-\gamma(t)}]$-isomorphisms. By multiplying both τ_M and τ_N with $(t-\gamma(t))^e$ for $e \gg 0$ we may assume that $\underline{M}$ and $\underline{N}$ are effective $\widetilde{A}$-motives. Then the equation $f \circ \tau_M = \tau_N \circ \sigma^* f$ is multiplied by $(t-\gamma(t))^e$, and so the map f continues to be an isogeny $f\colon \underline{M} \to \underline{N}$ between the (now effective) $\widetilde{A}$-motives $\underline{M}$ and $\underline{N}$. Let $\mathfrak{a} = \operatorname{ann}_{R[t]}(\operatorname{coker} f) = \ker\bigl(R[t] \to \operatorname{End}_R(\operatorname{coker} f)\bigr)$ be the annihilator of $\operatorname{coker} f$. By the Cayley-Hamilton theorem [15, Theorem 4.3] (applied with $I = R$), the monic characteristic polynomial χ_t of the endomorphism t of $\operatorname{coker} f$ lies in $\mathfrak{a}$.

This shows that $R[t]/\mathfrak{a}$ is a quotient of the finite R-module $R[t]/(\chi_t)$. In particular the closed subscheme $V := \operatorname{Spec} R[t]/\mathfrak{a}$ of $\mathbb{A}^1_R = \operatorname{Spec} R[t]$ is finite over $\operatorname{Spec} R$. On its open complement $f\colon M \to N$ is an isomorphism.

We now consider the exterior powers $\wedge^{\tilde r} M$ and $\wedge^{\tilde r} N$ of the $R[t]$-modules M and N and set $\mathcal{L} := (\wedge^{\tilde r} M)^\vee \otimes \wedge^{\tilde r} N$. These are invertible $R[t]$-modules. The isogeny f induces a global section $\wedge^{\tilde r} f$ of the invertible sheaf $\mathcal{L}$ on $\mathbb{A}^1_R$ which provides an isomorphism $\mathcal{O}_{\mathbb{A}^1_R} \xrightarrow{\sim} \mathcal{L}$, $1 \mapsto \wedge^{\tilde r} f$ on $\mathbb{A}^1_R \smallsetminus V$. Likewise we obtain global sections $\wedge^{\tilde r} \sigma^* f$, respectively $\wedge^{\tilde r} \tau_M$, respectively $\wedge^{\tilde r} \tau_N$ of the invertible sheaves $\sigma^*\mathcal{L}$, respectively $(\wedge^{\tilde r}\sigma^* M)^\vee \otimes \wedge^{\tilde r} M$, respectively $(\wedge^{\tilde r}\sigma^* N)^\vee \otimes \wedge^{\tilde r} N$ by the effectivity assumption on $\underline{M}$ and $\underline{N}$. Diagram (5.1) implies that there is an equality of global sections

$$\wedge^{\tilde r} f \otimes \wedge^{\tilde r} \tau_M \;=\; \wedge^{\tilde r} \tau_N \otimes \wedge^{\tilde r} \sigma^* f \tag{5.4}$$

of $(\wedge^{\tilde r}\sigma^* M)^\vee \otimes \wedge^{\tilde r} N \;=\; \mathcal{L} \otimes (\wedge^{\tilde r}\sigma^* M)^\vee \otimes \wedge^{\tilde r} M\big) \;=\; \big((\wedge^{\tilde r}\sigma^* N)^\vee \otimes \wedge^{\tilde r} N\big) \otimes \sigma^*\mathcal{L}$.

Since V is proper over $\operatorname{Spec} R$ and the projective line $\mathbb{P}^1_R$ is separated, the map $V \hookrightarrow \mathbb{A}^1_R \hookrightarrow \mathbb{P}^1_R$ is a closed immersion which does not meet $\{\infty\} \times_{\mathbb{F}_q} \operatorname{Spec} R$, where $\{\infty\} = \mathbb{P}^1_{\mathbb{F}_q} \smallsetminus \mathbb{A}^1_{\mathbb{F}_q}$. Thus we may glue $\mathcal{L}$ with the trivial sheaf $\mathcal{O}_{\mathbb{P}^1_R \smallsetminus V}$ on $\mathbb{P}^1_R \smallsetminus V$ along the isomorphism $\mathcal{O}_{\mathbb{P}^1_R} \xrightarrow{\sim} \mathcal{L}$, $1 \mapsto \wedge^{\tilde r} f$ over $\mathbb{A}^1_R \smallsetminus V$. In this way we obtain an invertible sheaf $\overline{\mathcal{L}}$ on the projective line $\mathbb{P}^1_R$. By replacing $\overline{\mathcal{L}}$ with $\overline{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^1_R}(m \cdot \infty)$ for a suitable integer m we may achieve that $\overline{\mathcal{L}}$ has degree zero (see [7, Section 9.1, Proposition 2]) and induces an R-valued point of the relative Picard functor $\operatorname{Pic}^0_{\mathbb{P}^1/\mathbb{F}_q}$; *cf.* [7, Section 8.1]. Since $\operatorname{Pic}^0_{\mathbb{P}^1/\mathbb{F}_q}$ is trivial, [7, Section 8.1, Proposition 4] shows that $\overline{\mathcal{L}} \cong \mathcal{K} \otimes_R \mathcal{O}_{\mathbb{P}^1_R}$ for an invertible sheaf $\mathcal{K}$ on $\operatorname{Spec} R$. Replacing $\operatorname{Spec} R$ by an open affine covering which trivializes $\mathcal{K}$ we may assume that there is an isomorphism $\alpha\colon \mathcal{L} \xrightarrow{\sim} R[t]$ of $R[t]$-modules. Let $h := \alpha(\wedge^{\tilde r} f) \in R[t]$.

2. Let $d := \operatorname{rk}_R \operatorname{coker} \tau_M$. We claim that locally on $\operatorname{Spec} R$ there is a positive integer n_0 and for every integer $n \geq n_0$ an isomorphism of $R[t]$-modules

$$\big((\wedge^{\tilde r}\sigma^* M)^\vee \otimes_{R[t]} \wedge^{\tilde r} M\big)^{\otimes q^n} \xrightarrow{\sim} R[t] \quad\text{with}\quad \big(\wedge^{\tilde r} \tau_M\big)^{\otimes q^n} \longmapsto \big(t-\gamma(t)\big)^{q^n d} \tag{5.5}$$

and similarly for $\underline{N}$. To prove the claim we apply Proposition 2.3(c) to the A-motive $\wedge^{\tilde r}\underline{M}$ and derive that $\wedge^{\tilde r}\tau_M\colon\ \wedge^{\tilde r}\sigma^* M \to \wedge^{\tilde r} M$ is injective $\operatorname{coker} \wedge^{\tilde r}\tau_M$ is a finite locally free R-module, annihilated by a power of $t-\gamma(t)$. Consider the exact sequence

$$\begin{aligned} 0 \longrightarrow \wedge^{\tilde r}\sigma^* M \otimes_{R[t]} (\wedge^{\tilde r} M)^\vee &\xrightarrow{\wedge^{\tilde r}\tau_M \otimes \mathrm{id}_{(\wedge^{\tilde r} M)^\vee}} R[t] \\ &\longrightarrow \operatorname{coker} \wedge^{\tilde r}\tau_M \otimes_{R[t]} (\wedge^{\tilde r} M)^\vee \longrightarrow 0\,. \end{aligned} \tag{5.6}$$

Choose an open affine covering of $\operatorname{Spec} R[t]$ which trivializes the locally free $R[t]$-module $\wedge^{\tilde r} M$. Pulling back this covering under the section $\operatorname{Spec} R \xrightarrow{\sim} \operatorname{Spec} R[t]/$

$(t-\gamma(t)) \hookrightarrow \operatorname{Spec} R[t]$ gives an open affine covering of $\operatorname{Spec} R$ on which we may find an isomorphism $\operatorname{coker} \wedge^{\tilde r} \tau_M \otimes_{R[t]} (\wedge^{\tilde r} M)^\vee \xrightarrow{\sim} \operatorname{coker} \wedge^{\tilde r} \tau_M$. We replace $\operatorname{Spec} R$ by this open affine covering and even shrink it further in such a way that $\operatorname{coker} \wedge^{\tilde r} \tau_M$ becomes a free R-module. By [15, Proposition 4.1(b)] the sequence (5.6) is then isomorphic to the sequence

$$0 \longrightarrow R[t] \xrightarrow{\;g\;} R[t] \longrightarrow \operatorname{coker} \wedge^{\tilde r} \tau_M \longrightarrow 0\,, \tag{5.7}$$

where $g \in R[t]$ is a monic polynomial of degree equal to $\operatorname{rk}_R(\operatorname{coker} \wedge^{\tilde r} \tau_M)$. We now tensor sequence (5.7) over R with $k := \operatorname{Frac}(R/\mathfrak{p})$ where $\mathfrak{p} \subset R$ is a prime ideal. It remains exact because $\operatorname{coker} \wedge^{\tilde r} \tau_M$ is free. Since $k[t]$ is a principal ideal domain the elementary divisor theorem applied to

$$0 \longrightarrow \sigma^* M \otimes_R k \xrightarrow{\tau_M \otimes \mathrm{id}_k} M \otimes_R k \longrightarrow \operatorname{coker} \tau_M \otimes_R k \longrightarrow 0$$

allows to write $\tau_M \otimes \mathrm{id}_k$ as a diagonal matrix. This shows that $\operatorname{coker} \wedge^{\tilde r} \tau_M \otimes_R k$ is a k-vector space of dimension equal to $\operatorname{rk}_R(\operatorname{coker} \tau_M) =: d$. Since $t-\gamma(t)$ is nilpotent on this vector space, the Cayley-Hamilton theorem from linear algebra implies $g \bmod \mathfrak{p} = (t-\gamma(t))^d$. In particular the coefficients of the difference $g' := g - (t-\gamma(t))^d$ lie in every prime ideal of R, and hence are nilpotent by [15, Corollary 2.12]. Therefore there is a positive integer n_0 with $(g')^{q^{n_0}} = 0$, whence $g^{q^n} = (t-\gamma(t))^{q^n d}$ for every $n \geq n_0$. The q^n-th tensor power of the isomorphism between (the left entries in) the sequences (5.6) and (5.7) provides the isomorphism in (5.5). This proves the claim.

3. Since $d = \operatorname{rk}_R \operatorname{coker} \tau_M = \operatorname{rk}_R \operatorname{coker} \tau_N$ by Proposition 5.8, equations (5.4) and (5.5) imply that for $n \gg 0$ there is an isomorphism $\beta\colon \sigma^* \mathcal{L}^{\otimes q^n} \xrightarrow{\sim} \mathcal{L}^{\otimes q^n}$ of $R[t]$-modules sending $(t-\gamma(t))^{q^n} (\sigma^* \wedge^{\tilde r} f)^{\otimes q^n}$ to $(t-\gamma(t))^{q^n} (\wedge^{\tilde r} f)^{\otimes q^n}$ and hence $(\sigma^* \wedge^{\tilde r} f)^{\otimes q^n}$ to $(\wedge^{\tilde r} f)^{\otimes q^n}$ because $t-\gamma(t)$ is a non-zero divisor. In particular the isomorphism

$$\alpha^{\otimes q^n} \circ \beta \circ \left(\sigma^* \alpha^{\otimes q^n}\right)^{-1}\colon R[t] \xrightarrow{\sim} \sigma^* \mathcal{L}^{\otimes q^n} \xrightarrow{\sim} \mathcal{L}^{\otimes q^n} \xrightarrow{\sim} R[t]\,,$$

which is given by multiplication with a unit $u \in R[t]^\times$, sends $\sigma(h^{q^n}) = \sigma^* \alpha^{\otimes q^n} (\wedge^{\tilde r} \sigma^* f)^{\otimes q^n}$ to $h^{q^n} = \alpha^{\otimes q^n} (\wedge^{\tilde r} f)^{\otimes q^n}$. We thus obtain the equation $h^{q^n} = u \cdot \sigma(h^{q^n})$ in $R[t]$.

By Lemma 5.13 below, $u = \sum_{i \geq 0} u_i t^i$ with $u_0 \in R^\times$ and $u_i \in R$ nilpotent for all $i \geq 1$. Let $R' = R[v_0]/(v_0^{q-1} u_0 - 1)$ be the finite étale R-algebra obtained by adjoining a $(q-1)$-th root v_0 of u_0^{-1}. Then there is a unit $v = \sum_{i \geq 1} v_i t^i \in R'[t]^\times$ with $v = u \cdot \sigma(v)$. Indeed the latter amounts to the equations

$$v_i = \sum_{j=0}^{i} u_j v_{i-j}^q \qquad \text{and} \qquad \tfrac{v_i}{v_0} = \left(\tfrac{v_i}{v_0}\right)^q + \sum_{j \geq 1} \tfrac{u_j}{u_0} \left(\tfrac{v_{i-j}}{v_0}\right)^q$$

which have the solutions $\frac{v_i}{v_0} = \sum_{n\ge 0}\big(\sum_{j\ge 1}\frac{u_j}{u_0}(\frac{v_{i-j}}{v_0})^q\big)^{q^n}$ because the u_j are nilpotent. Therefore the element $v^{-1}h^{q^n} \in R'[t]$ satisfies $\sigma(v^{-1}h^{q^n}) = v^{-1}h^{q^n}$. Working on each connected component of Spec R' separately, Lemma 5.14 below shows that $a := v^{-1}h^{q^n} \in \mathbb{F}_q[t] \subset A$.

In the ring $R'[t][\frac{1}{a}]$ the element h becomes a unit. Therefore the homomorphism $\alpha^{-1}\circ h\colon R'[t][\frac{1}{a}] \to \mathcal{L}[\frac{1}{a}]$, $1 \mapsto \wedge^{\tilde r} f$ is an isomorphism. This implies that $\wedge^{\tilde r} f\colon \wedge^{\tilde r} M[\frac{1}{a}] \to \wedge^{\tilde r} N[\frac{1}{a}]$ is an isomorphism, and hence also $f\colon M[\frac{1}{a}] \to N[\frac{1}{a}]$ by Cramer's rule (*e.g.*, [8, III.8.6, Formulas (21) and (22)]). Thus we have established (b) étale locally on Spec R. Replacing a by the product of all the finitely many elements a obtained locally, establishes (b) globally on Spec R.

4. To prove the statement about quasi-morphisms $f \in \mathrm{QHom}_R(\underline{M}, \underline{N})$ assume first, that f induces an isomorphism $f\colon M[\frac{1}{a}] \xrightarrow{\sim} N[\frac{1}{a}]$ for some $a \in A \smallsetminus \{0\}$. Then $g := a^n \cdot f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ for $n \gg 0$, because M is finitely generated. In particular g is an isogeny and $f = g \otimes a^{-n}$ is a quasi-isogeny.

Conversely, if f is a quasi-isogeny, that is $f = g \otimes c$ for an isogeny $g \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ and a $c \in Q$, there is an element $a \in A \smallsetminus \{0\}$ such that $g\colon M[\frac{1}{a}] \xrightarrow{\sim} N[\frac{1}{a}]$. If d is the denominator of c it follows that $f\colon M[\frac{1}{ad}] \xrightarrow{\sim} N[\frac{1}{ad}]$. □

To finish the proof of Theorem 5.12 we must demonstrate the following two lemmas.

Lemma 5.13. *An element $u = \sum_{i\ge 0} u_i t^i \in R[t]$ is a unit in $R[t]$ if and only if $u_0 \in R^\times$ and u_i is nilpotent for all $i \ge 1$.*

Proof. If the u_i satisfy the assertion then there is a positive integer n such that $u_i^{q^n} = 0$ for all $i \ge 1$. Therefore $u^{q^n} = u_0^{q^n}$ is a unit in $R[t]$ and so the same holds for u.

Conversely if u is a unit then u_0 must be a unit in R. By [15, Corollary 2.12] the kernel of the map $R \to \prod_{\mathfrak{p}\subset R} R/\mathfrak{p}$ where $\mathfrak{p}$ runs over all prime ideals of R, equals the nil-radical of R. Under this map u is sent to a unit in each factor $R/\mathfrak{p}[t]$. Since $R/\mathfrak{p}$ is an integral domain, the u_i for $i \ge 1$ must be sent to zero in each factor $R/\mathfrak{p}$. This shows that u_i is nilpotent for $i \ge 1$. □

Lemma 5.14. *Assume that R contains no idempotents besides* 0 *and* 1, *that is* Spec R *is connected. Then $R^\sigma := \{x \in R\colon x^q = x\} = \mathbb{F}_q$.*

Proof. Let $\mathfrak{m} \subset R$ be a maximal ideal and let $\bar x \in R/\mathfrak{m}$ be the image of x. Then $\bar x^q = \bar x$ implies that $\bar x$ is equal to an element $\alpha \in \mathbb{F}_q \subset R/\mathfrak{m}$. Now $e := (x-\alpha)^{q-1}$ satisfies $e^2 = (x-\alpha)^{q-2}(x^q - \alpha^q) = (x-\alpha)^{q-1} = e$, that is e is an idempotent. Since $e \in \mathfrak{m}$ we cannot have $e = 1$ and must have $e = 0$. Therefore $x - \alpha = (x-\alpha)^q = (x-\alpha)\cdot e = 0$ in R, that is $x = \alpha \in \mathbb{F}_q$. □

Corollary 5.15. *If $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ is an isogeny between A-motives over R then there is an element $0 \ne a \in A$ and an isogeny $g \in \mathrm{Hom}_R(\underline{N}, \underline{M})$ with*

$f \circ g = a \cdot \mathrm{id}_{\underline{N}}$ and $g \circ f = a \cdot \mathrm{id}_{\underline{M}}$. The same is true for Abelian Anderson A-modules.

Proof. Let $a \in A$ be the element from Theorem 5.12(b). As in the proof of (b)$\Longrightarrow$(a) of this theorem there is a positive integer n such $a^n \cdot \operatorname{coker} f = (0)$. Therefore there is a map $g\colon N \to M$ with $g \circ f = a^n \cdot \mathrm{id}_M$ and $f \circ g = a^n \cdot \mathrm{id}_N$. This implies that g is injective, because a^n is a non-zero divisor on N. From

$$f \circ g \circ \tau_N \;=\; a^n \cdot \tau_N \;=\; \tau_N \circ \sigma^* a^n \cdot \mathrm{id}_N \;=\; \tau_N \circ \sigma^* f \circ \sigma^* g \;=\; f \circ \tau_M \circ g$$

and the injectivity of f we conclude that $g \circ \tau_N = \tau_M \circ \sigma^* g$ and that $g \in \mathrm{Hom}_R(\underline{N}, \underline{M})$. By construction g induces an isomorphism $N[\frac{1}{a}] \xrightarrow{\sim} M[\frac{1}{a}]$ after inverting a. So g is an isogeny by Theorem 5.12. The statement about Abelian Anderson A-modules follows from Theorems 3.5 and 5.9. □

Corollary 5.16. *The relation of being isogenous is an equivalence relation for A-motives and for Abelian Anderson A-modules over R.*

Proof. This follows from Theorem 5.12 and Corollary 5.15. □

Corollary 5.17. *Let $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ be an isogeny between effective A-motives $\underline{M}$ and $\underline{N}$ over R and suppose that $\gamma(A \smallsetminus \{0\}) \subset R^\times$. Then f is separable. The same is true for isogenies between Abelian Anderson A-modules over R.*

Proof. Consider diagram (5.1) and set $K := \operatorname{coker}(\tau_{\operatorname{coker} f})$. As in the proof of Theorem 5.12 there is an element $0 \neq a \in A$ and a positive integer n with $a^n \cdot \operatorname{coker} f = (0)$, and hence $a^n \cdot K = (0)$. Let e be an integer with $q^e \geq \mathrm{rk}_R \operatorname{coker} \tau_N$ and $q^e \geq n$. Then $(a \otimes 1 - 1 \otimes \gamma(a))^{q^e} \cdot \operatorname{coker} \tau_N = (0)$. Therefore

$$0 = (a \otimes 1 - 1 \otimes \gamma(a))^{q^e} \cdot K = \left(a^{q^e} \otimes 1 - 1 \otimes \gamma(a)^{q^e}\right) \cdot K = -\gamma(a)^{q^e} \cdot K\,.$$

Since $\gamma(a) \in R^\times$ we have $K = (0)$, and since $\operatorname{coker} f$ and $\sigma^*(\operatorname{coker} f)$ are finite locally free R modules of the same rank, [17, Corollary 8.12] shows that $\tau_{\operatorname{coker} f}$ is an isomorphism, that is f is separable. The statement about Abelian Anderson A-modules follows from Theorem5.9(b). □

Corollary 5.18. *If $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$ and $g \in \mathrm{Hom}_R(\underline{N}, \underline{M})$ are isogenies between A-motives over R with $f \circ g = a \cdot \mathrm{id}_{\underline{N}}$ and $g \circ f = a \cdot \mathrm{id}_{\underline{M}}$ for an $a \in A$, then there is an isomorphism of Q-algebras $\mathrm{QEnd}_R(\underline{M}) \xrightarrow{\sim} \mathrm{QEnd}_R(\underline{N})$ given by $h \otimes b \mapsto f \circ h \circ g \otimes \frac{b}{a}$ for $h \in \mathrm{End}_R(\underline{M})$.* □

Example 5.19. Let R be an A-ring *of finite characteristic* $\mathfrak{p}$, that is $\gamma\colon A \to R$ factors through $\mathbb{F}_\mathfrak{p} := A/\mathfrak{p}$ for a maximal ideal $\mathfrak{p} \subset A$. Let $\ell \in \mathbb{N}_{>0}$ be divisible by $[\mathbb{F}_\mathfrak{p} : \mathbb{F}_q]$. Then $\sigma^{\ell *}(\mathcal{J}) = (a \otimes 1 - 1 \otimes \gamma(a)^{q^\ell} : a \in A) = \mathcal{J} \subset A_R$, because the elements $\gamma(a) \in \mathbb{F}_\mathfrak{p}$ satisfy $\gamma(a)^{q^\ell} = \gamma(a)$. Let $\underline{M} = (M, \tau_M)$ be an A-motive over R. Then $\sigma^{\ell *}\underline{M} = (\sigma^{\ell *}M, \sigma^{\ell *}\tau_M)$ is also an A-motive over R, because $\sigma^{\ell *}\tau_M$ is an isomorphism outside $\mathrm{V}(\sigma^{\ell *}\mathcal{J}) = \mathrm{V}(\mathcal{J})$. If $\underline{M}$ is effective, then the A_R-homomorphism

$$\mathrm{Fr}_{q^\ell, \underline{M}} := \tau_M^\ell := \tau_M \circ \sigma^* \tau_M \circ \ldots \circ \sigma^{(\ell-1)*}\tau_M \colon \sigma^{\ell *}\underline{M} \longrightarrow \underline{M} \tag{5.8}$$

satisfies $\tau_M \circ \sigma^* \mathrm{Fr}_{q^\ell, \underline{M}} = \mathrm{Fr}_{q^\ell, \underline{M}} \circ \sigma^{\ell *}\tau_M$. Moreover, it is injective and its cokernel is a successive extension of the σ^{i*} coker τ_M for $i = 0, \ldots, \ell - 1$, whence a finitely presented R-module. Therefore $\mathrm{Fr}_{q^\ell, \underline{M}} \in \mathrm{Hom}_R\big(\sigma^{\ell *}\underline{M}, \underline{M}\big)$ is an isogeny, called the *q^ℓ-Frobenius isogeny* of $\underline{M}$. It is always inseparable, because the ℓ-th power of τ_M, which equals $\mathrm{Fr}_{q^\ell, \underline{M}}$ annihilates the cokernel of $\mathrm{Fr}_{q^\ell, \underline{M}}$.

If $\underline{M}$ is not effective, let $n \in \mathbb{N}_{>0}$ be such that $\mathfrak{p}^n = (a)$ is principal. Then $(a \otimes 1) \subset \mathcal{J}$ and $(a \otimes 1) \subset \sigma^{i*}\mathcal{J}$ for all i. This shows that

$$\mathrm{Fr}_{q^\ell, \underline{M}} := \tau_M^\ell := \tau_M \circ \sigma^* \tau_M \circ \ldots \circ \sigma^{(\ell-1)*}\tau_M \colon \sigma^{\ell *}\underline{M}\left[\frac{1}{a}\right] \xrightarrow{\sim} \underline{M}\left[\frac{1}{a}\right] \tag{5.9}$$

is a quasi-isogeny in $\mathrm{QHom}_R\big(\sigma^{\ell *}\underline{M}, \underline{M}\big)$ by Theorem 5.12, called the *q^ℓ-Frobenius quasi-isogeny* of $\underline{M}$.

Finally if $R = k$ is a field contained in $\mathbb{F}_{q^\ell}$ then $\sigma^{\ell *}\underline{M} = \underline{M}$ and $\mathrm{Fr}_{q^\ell, \underline{M}} \in \mathrm{QEnd}_k\big(\underline{M}\big)$, respectively $\mathrm{Fr}_{q^\ell, \underline{M}} \in \mathrm{End}_k\big(\underline{M}\big)$ if $\underline{M}$ is effective. In this case, $A[\pi]$ lies in the center of $\mathrm{End}_k(\underline{M})$ and $Q[\pi]$ lies in the center of $\mathrm{QEnd}_k(\underline{M})$, because every $f \in \mathrm{End}_k(\underline{M})$ satisfies $f \circ \tau_M = \tau_M \circ \sigma^* f$ and $\sigma^{\ell *} f = f$. If $k = \mathbb{F}_{q^\ell}$, the center equals $A[\pi]$, respectively $Q[\pi]$, and the isogeny classes of A-motives are largely controlled by their Frobenius endomorphism; see [5, Theorems 8.1 and 9.1].

6. Torsion points

Definition 6.1. Let $(0) \neq \mathfrak{a} = (a_1, \ldots, a_n) \subset A$ be an ideal and let $\underline{E} = (E, \varphi)$ be an Abelian Anderson A-module over R. Then

$$\underline{E}[\mathfrak{a}] := \ker\big(\varphi_{a_1,\ldots,a_n} := (\varphi_{a_1}, \ldots, \varphi_{a_n}) \colon E \longrightarrow E^n\big)$$

is called the *$\mathfrak{a}$-torsion submodule of $\underline{E}$.*

This definition is independent of the generators $(a_1, \ldots, a_n)$ of $\mathfrak{a}$ by the following

Lemma 6.2.

(a) *If* $(a_1,\ldots,a_n) \subset (b_1,\ldots,b_m) \subset A$ *are ideals then* $\ker(\varphi_{b_1,\ldots,b_m}) \hookrightarrow \ker(\varphi_{a_1,\ldots,a_n})$ *is a closed immersion;*
(b) *If* $(a_1, \ldots, a_n) = (b_1, \ldots, b_m)$ *then* $\ker(\varphi_{b_1,\ldots,b_m}) = \ker(\varphi_{a_1,\ldots,a_n})$*;*
(c) *For any R-algebra S we have* $\underline{E}[\mathfrak{a}](S) = \{\, P \in E(S) \colon \varphi_a(P) = 0$ *for all* $a \in \mathfrak{a}\,\}$*;*
(d) $\underline{E}[\mathfrak{a}]$ *is an $A/\mathfrak{a}$-module via* $A/\mathfrak{a} \to \mathrm{End}_R(\underline{E}[\mathfrak{a}])$, $\bar{b} \mapsto \varphi_b$*;*
(e) $\underline{E}[\mathfrak{a}]$ *is a finite R-group scheme of finite presentation.*

Proof. (a) By assumption there are elements $c_{ij} \in A$ with $a_i = \sum_j c_{ij} b_j$. Therefore $\varphi_{a_i} = \sum_j \varphi_{c_{ij}} \varphi_{b_j}$ and the composition of $\varphi_{b_1,\dots,b_m} \colon E \to E^m$ followed by $(\varphi_{c_{ij}})_{i,j} \colon E^m \to E^n$ equals $\varphi_{a_1,\dots,a_n} \colon E \to E^n$. This proves (a) and clearly (a) implies (b).

To prove (c) let $P \colon \operatorname{Spec} S \to \underline{E}$ be an S-valued point in $\underline{E}(S)$ with $0 = \varphi_a(P) := \varphi_a \circ P$ for all $a \in \mathfrak{a}$. If $\mathfrak{a} = (a_1, \dots, a_n)$ then in particular $\varphi_{a_i} \circ P = 0$ for $i = 1, \dots, n$. Therefore P factors through $\ker \varphi_{a_1,\dots,a_n} = \underline{E}[\mathfrak{a}]$.

Conversely let $P \colon \operatorname{Spec} S \to \underline{E}[\mathfrak{a}]$ be an S-valued point in $\underline{E}[\mathfrak{a}](S)$ and let $a \in \mathfrak{a}$. By (b) we may write $\mathfrak{a} = (a_1, \dots, a_n)$ with $a_1 = a$ to have $\underline{E}[\mathfrak{a}] = \ker \varphi_{a_1,\dots,a_n}$. Therefore $\varphi_a(P) := \varphi_a \circ P = 0$. This proves (c).

(d) The relation $ab = ba$ in A implies $\varphi_a \circ \varphi_b = \varphi_b \circ \varphi_a$. Using that the closed subscheme $\underline{E}[\mathfrak{a}]$ is uniquely determined by (c) it follows that the ring homomorphism $A \to \operatorname{End}_R(\underline{E}[\mathfrak{a}])$, $b \mapsto \varphi_b|_{\underline{E}[\mathfrak{a}]}$ is well defined. If $b \in \mathfrak{a}$ then clearly $\varphi_b|_{\underline{E}[\mathfrak{a}]} = 0$ and so this ring homomorphism factors through $A/\mathfrak{a}$.

(e) If $\mathfrak{a} = (a_1, \dots, a_n)$ then $\underline{E}[\mathfrak{a}] = \ker \varphi_{a_1,\dots,a_n}$ is of finite presentation, because $\varphi_{a_1,\dots,a_n}$ is a morphism of finite presentation between the schemes E and E^n of finite presentation over R by [14, IV$_1$, Proposition 1.6.2]. The finiteness of $\underline{E}[\mathfrak{a}]$ follows for $\mathfrak{a} = (a)$ from Corollaries 5.11 and 5.3, and for general $\mathfrak{a}$ from (a) by considering some $(a) \subset \mathfrak{a}$. □

The following lemma is a version of the Chinese remainder theorem in our context.

Lemma 6.3. *Let* $(0) \neq \mathfrak{a}, \mathfrak{b} \subset A$ *be two ideals with* $\mathfrak{a} + \mathfrak{b} = A$.

(a) *For an Abelian Anderson A-module $\underline{E}$ there is a canonical isomorphism* $\underline{E}[\mathfrak{a}] \times_R \underline{E}[\mathfrak{b}] \xrightarrow{\sim} \underline{E}[\mathfrak{a}\mathfrak{b}]$;

(b) *For an effective A-motive $\underline{M}$ there is a canonical isomorphism* $\underline{M}/\mathfrak{a}\mathfrak{b}\underline{M} \xrightarrow{\sim} \underline{M}/\mathfrak{a}\underline{M} \oplus \underline{M}/\mathfrak{b}\underline{M}$ *of finite $\mathbb{F}_q$-shtukas.*

Proof. By the Chinese remainder theorem there is an isomorphism $A/\mathfrak{a}\mathfrak{b} \xrightarrow{\sim} A/\mathfrak{a} \times A/\mathfrak{b}$ whose inverse is given by $(x_\mathfrak{a}, x_\mathfrak{b}) \mapsto bx_\mathfrak{a} + ax_\mathfrak{b}$ for certain elements $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ which satisfy $a \equiv 1 \bmod \mathfrak{b}$ and $b \equiv 1 \bmod \mathfrak{a}$, and hence $a + b \equiv 1 \bmod \mathfrak{a}\mathfrak{b}$.

(b) follows directly from this, because $\underline{M}/\mathfrak{a}\underline{M} = \underline{M} \otimes_A A/\mathfrak{a}$.

(a) By Lemma 6.2(a) the addition Δ on $\underline{E}[\mathfrak{a}\mathfrak{b}]$ defines a canonical morphism $\underline{E}[\mathfrak{a}] \times_R \underline{E}[\mathfrak{b}] \hookrightarrow \underline{E}[\mathfrak{a}\mathfrak{b}] \times_R \underline{E}[\mathfrak{a}\mathfrak{b}] \xrightarrow{\Delta} \underline{E}[\mathfrak{a}\mathfrak{b}]$. Its inverse is described as follows. The elements $a, b \in A$ from above satisfy $a\mathfrak{b} \subset \mathfrak{a}\mathfrak{b}$ and $b\mathfrak{a} \subset \mathfrak{a}\mathfrak{b}$. By Lemma 6.2(c) the endomorphism φ_a of $\underline{E}[\mathfrak{a}\mathfrak{b}]$ factors through $\underline{E}[\mathfrak{b}]$ and φ_b factors through $\underline{E}[\mathfrak{a}]$. So the inverse is the morphism $(\varphi_b, \varphi_a) \colon \underline{E}[\mathfrak{a}\mathfrak{b}] \to \underline{E}[\mathfrak{a}] \times_R \underline{E}[\mathfrak{b}]$. Indeed, for $x \in \underline{E}[\mathfrak{a}\mathfrak{b}]$, we compute $\varphi_b(x) + \varphi_a(x) = \varphi_{a+b}(x) = \varphi_1(x) = x$, because $a + b \equiv 1 \bmod \mathfrak{a}\mathfrak{b}$. On the other hand, for $x \in \underline{E}[\mathfrak{a}]$ and $y \in \underline{E}[\mathfrak{b}]$, we compute $\varphi_b(x + y) = \varphi_b(x) = x$ and $\varphi_a(x + y) = \varphi_a(y) = y$, because $b \equiv 1 \bmod \mathfrak{a}$ and $a \equiv 1 \bmod \mathfrak{b}$. □

Theorem 6.4. *Let $\underline{E}$ be an Abelian Anderson A-module over R and let* $(0) \neq \mathfrak{a} \subset$ *A be an ideal.*

(a) *Then $\underline{E}[\mathfrak{a}]$ is a finite locally free group scheme over* Spec *R and a strict $\mathbb{F}_q$-module scheme;*
(b) *$\underline{E}[\mathfrak{a}]$ is étale over R if and only if $R \cdot \gamma(\mathfrak{a}) = R$, that is if and only if $\mathfrak{a} + \mathcal{J} = A_R$;*
(c) *If $\underline{M} = \underline{M}(\underline{E})$ is the associated effective A-motive then there are canonical A-equivariant isomorphisms*

$$\underline{M}/\mathfrak{a}\underline{M} \xrightarrow{\sim} \underline{M}_q(\underline{E}[\mathfrak{a}]) \qquad \textit{of finite } \mathbb{F}_q\textit{-shtukas and}$$

$$\mathrm{Dr}_q(\underline{M}/\mathfrak{a}\underline{M}) \xrightarrow{\sim} \underline{E}[\mathfrak{a}] \qquad \textit{of finite locally free } R\textit{-group schemes.}$$

Proof. Since A is a Dedekind domain, $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdot \ldots \cdot \mathfrak{p}_r^{e_r}$ for prime ideals $\mathfrak{p}_i \in A$ and positive integers e_i. By Lemma 6.3 and the exactness of the functors Dr_q and $\underline{M}_q$, see Theorem 4.7(a), it suffices to treat the case $\mathfrak{a} = \mathfrak{p}^e$. Let $A_\mathfrak{p}$ be the localization of A at $\mathfrak{p}$. Since $A/\mathfrak{p}^e = A_\mathfrak{p}/\mathfrak{p}^e A_\mathfrak{p}$ there is an element $z \in A$ which is congruent modulo $\mathfrak{a}$ to a uniformizer of $A_\mathfrak{p}$. Moreover, since $\underline{E}[\mathfrak{p}^e]$ is an $A_\mathfrak{p}/\mathfrak{p}^e A_\mathfrak{p}$-module, every φ_s with $s \in A \smallsetminus \mathfrak{p}$ is an automorphism of $\underline{E}[\mathfrak{p}^e]$. Let $0 \leq n \leq e$. We denote the inclusion $\underline{E}[\mathfrak{p}^n] \hookrightarrow \underline{E}[\mathfrak{p}^e]$ of Lemma 6.2(a) by $i_{n,e}$. By Lemma 6.2(c) the endomorphism φ_z^{e-n} of $\underline{E}[\mathfrak{p}^e]$ has kernel $\underline{E}[\mathfrak{p}^{e-n}]$ and factors through the closed subscheme $\underline{E}[\mathfrak{p}^n]$ via a morphism $j_{e,n} \colon \underline{E}[\mathfrak{p}^e] \to \underline{E}[\mathfrak{p}^n]$ with $\varphi_z^{e-n} = i_{n,e} \circ j_{e,n}$. We claim that $j_{e,n}$ is an epimorphism in the category of sheaves on the big *fpqc*-site over Spec R, and we therefore have an exact sequence

$$0 \longrightarrow \underline{E}[\mathfrak{p}^{e-n}] \xrightarrow{\ i_{e-n,e}\ } \underline{E}[\mathfrak{p}^e] \xrightarrow{\ j_{e,n}\ } \underline{E}[\mathfrak{p}^n] \longrightarrow 0\,. \tag{6.1}$$

To prove the claim let S be an R-algebra and let $P\colon \operatorname{Spec} S \to \underline{E}[\mathfrak{p}^n]$ be an S-valued point in $\underline{E}[\mathfrak{p}^n](S)$. Since $\varphi_{z^{e-n}} \colon \underline{E} \to \underline{E}$ is an isogeny by Corollary 5.11, hence an epimorphism of *fpqc*-sheaves by Proposition 5.2(e), there exists a faithfully flat S-algebra S' and a point $P' \in E(S')$ with $\varphi_{z^{e-n}}(P') = P$. We have to show that $P' \in \underline{E}[\mathfrak{p}^e](S')$. For this purpose let $a \in \mathfrak{p}^e$. Then $\frac{a}{1} = \frac{c}{s}(\frac{z}{1})^e$ in $A_\mathfrak{p}$ for $c \in A,\ s \in A \smallsetminus \mathfrak{p}$. We compute

$$\varphi_a(P') = \varphi_s^{-1} \circ \varphi_c \circ \varphi_{z^n} \circ \varphi_{z^{e-n}}(P') = \varphi_s^{-1} \circ \varphi_c \circ \varphi_{z^n}(P) = 0\,,$$

because $z^n \in \mathfrak{p}^n$. This proves our claim and establishes the exactness of (6.1).

We now use that A is a Dedekind domain with finite ideal class group. This means that for the prime ideal $\mathfrak{p} \subset A$ there are (arbitrarily large) integers e such that $\mathfrak{p}^e = (a)$ is principal. Then $\underline{E}[\mathfrak{p}^e] = \ker \varphi_a$ is a finite locally free R-group scheme by Corollaries 5.11 and 5.3. If $0 \leq n \leq e$ then we show that $\underline{E}[\mathfrak{p}^n]$ is flat over R. Namely, using the epimorphism $j_{e,n} \colon \underline{E}[\mathfrak{p}^e] \to \underline{E}[\mathfrak{p}^n]$ from (6.1) and the flatness of $\underline{E}[\mathfrak{p}^e]$ over R, the flatness of $\underline{E}[\mathfrak{p}^n]$ will follow from [14, IV_3, Théorème 11.3.10] once we show that $j_{e,n}$ is flat in each fiber over a point of Spec R. This follows from [11, Section III.3, Corollaire 7.4] and so $\underline{E}[\mathfrak{p}^n]$ is flat over R for all n. By Lemma 6.2(e) this proves that $\underline{E}[\mathfrak{p}^n]$ is a finite locally free group scheme over $\operatorname{Spec} R$. Moreover, it is a strict $\mathbb{F}_q$-module scheme by [16, Proposition 2], because for $\mathfrak{p}^n = (a_1, \dots, a_n)$ the morphism $\varphi_{a_1,\dots,a_n}$ is strict $\mathbb{F}_q$-linear by Example 4.3. So (a) is established.

If $\mathfrak{a} = \mathfrak{p}^e = (a)$ we know from Theorem 5.9(c) applied to the isogeny φ_a and $\operatorname{coker} \underline{M}(\varphi_a) = \underline{M}/a\underline{M}$ that (c) holds. If $0 \le n \le e$ we use the exact sequence (6.1) and the fact that the functors Dr_q and $\underline{M}_q$ are exact by Theorem 4.7. Namely, multiplication with z^{e-n} on $\underline{M}/a\underline{M}$ has cokernel $\underline{M}/\mathfrak{p}^{e-n}\underline{M}$ and image isomorphic to $\underline{M}/\mathfrak{p}^n\underline{M}$. We obtain an exact sequence of finite $\mathbb{F}_q$-shtukas

$$0 \longrightarrow \underline{M}/\mathfrak{p}^n\underline{M} \xrightarrow{\beta_{n,e}} \underline{M}/a\underline{M} \xrightarrow{\alpha_{e,e-n}} \underline{M}/\mathfrak{p}^{e-n}\underline{M} \longrightarrow 0 \tag{6.2}$$

with $\beta_{n,e} \circ \alpha_{e,n} = z^{e-n}$ on $\underline{M}/a\underline{M}$. Applying Dr_q to (6.2), using the exactness of Dr_q, and that $\mathrm{Dr}_q(\underline{M}/a\underline{M}) = \underline{E}[\mathfrak{p}^e]$ and $\mathrm{Dr}_q(z^{e-n}) = \varphi_z^{e-n}$, proves $\mathrm{Dr}_q(\underline{M}/\mathfrak{p}^n\underline{M}) = \underline{E}[\mathfrak{p}^n]$. Conversely applying $\underline{M}_q$ to (6.1), using the exactness of $\underline{M}_q$, and that $\underline{M}/a\underline{M} = \underline{M}(\underline{E}[\mathfrak{p}^e])$ and $z^{e-n} = \underline{M}_q(\varphi_z^{e-n})$, proves $\underline{M}/\mathfrak{p}^n\underline{M} = \underline{M}_q(\underline{E}[\mathfrak{p}^n])$. This establishes (c) in general.

(b) Suppose that $R \cdot \gamma(\mathfrak{a}) = R$, that is there are elements $a_1, \dots, a_n \in \mathfrak{a}$ and $b_1, \dots, b_n \in R$ with $\sum_{i=1}^n b_i \gamma(a_i) = 1$. Then the open subschemes $\operatorname{Spec} R[\frac{1}{\gamma(a_i)}] \subset \operatorname{Spec} R$ cover $\operatorname{Spec} R$ and it suffices to check that $\underline{E}[\mathfrak{a}]$ is étale over $\operatorname{Spec} R[\frac{1}{\gamma(a_i)}]$ for each i. But there $\underline{E}[\mathfrak{a}]$ is a closed subscheme of $\underline{E}[a_i]$ which is étale by Corollary 5.11. This shows that $\underline{E}[\mathfrak{a}]$ is unramified over R. Since it is flat by (a), it is étale as desired.

Conversely assume that $R \cdot \gamma(\mathfrak{a}) \subset \mathfrak{m}$ for a maximal ideal $\mathfrak{m} \subset R$ and set $k = R/\mathfrak{m}$. Over a field extension k' of k we have $E \times_R k = \mathbb{G}_{a,k'}^d = \operatorname{Spec} k'[x_1, \dots, x_d]$. We will show that $\underline{E}[\mathfrak{a}] \times_R k'$ is not étale over k' by applying the Jacobi criterion [7, Section 2.2, Proposition 7]. Let $\mathfrak{a} = (a_1, \dots, a_n)$. Then $\underline{E}[\mathfrak{a}] = \operatorname{Spec} k'[x_1, \dots, x_d]/\big(\varphi_{a_1}^*(x_1, \dots, x_d) \colon j = 1, \dots, n\big)$. The Jacobi matrix is

$$\frac{\partial \varphi_{a_j}^*}{\partial x_i} = \begin{pmatrix} \operatorname{Lie} \varphi_{a_1} \\ \vdots \\ \operatorname{Lie} \varphi_{a_n} \end{pmatrix} \in (k')^{nd \times d}.$$

Since $\gamma(a_i) = 0$ in k' each $\operatorname{Lie} \varphi_{a_i}$ is a nilpotent $d \times d$ matrix. Since $\varphi_{a_i} \circ \varphi_{a_j} = \varphi_{a_i a_j} = \varphi_{a_j} \circ \varphi_{a_i}$ we have $\operatorname{Lie} \varphi_{a_i}(\ker \operatorname{Lie} \varphi_{a_j}) \subset \ker \operatorname{Lie} \varphi_{a_j}$. Therefore all $\ker \operatorname{Lie} \varphi_{a_i}$ have a non-trivial intersection. This shows that the rank of the Jacobi matrix is less than d and $\underline{E}[\mathfrak{a}] \times_R k'$ is not étale over k'. □

Proposition 6.5. *Let $\underline{M} = (M, \tau_M)$ be an A-motive over R of rank r and let $(0) \neq \mathfrak{a} \subset A$ be an ideal with $R \cdot \gamma(\mathfrak{a}) = R$, that is $\mathfrak{a} + \mathcal{J} = A_R$. Let $\bar{s} = \operatorname{Spec} \Omega$ be a geometric base point of $\operatorname{Spec} R$. Then $\underline{M}/\mathfrak{a}\underline{M}$ is an étale finite $\mathbb{F}_q$-shtuka whose τ-invariants $(\underline{M}/\mathfrak{a}\underline{M})^\tau(\Omega)$, see (4.1), form a free $A/\mathfrak{a}$-module of rank r which carries a continuous action of the étale fundamental group $\pi_1^{\text{ét}}(\operatorname{Spec} R, \bar{s})$.*

Proof. This result and its proof are due to Anderson [2, Lemma 1.8.2] for R a field. Let $G := \mathrm{Res}_{A/\mathfrak{a}|\mathbb{F}_q}\, \mathrm{GL}_{r,A/\mathfrak{a}}$ be the Weil restriction with $G(R') = \mathrm{GL}_r(A/\mathfrak{a}\otimes_{\mathbb{F}_q} R')$ for all $\mathbb{F}_q$-algebras R'. Then G is a smooth connected affine group scheme over $\mathbb{F}_q$ by [10, Proposition A.5.9]. Thus by Lang's theorem [21, Corollary on page 557] the Lang map $L\colon G \to G$, $g \mapsto g\cdot\sigma^*g^{-1}$ is finite étale and surjective (although not a group homomorphism if $r > 1$ and $\mathfrak{a} \neq A$).

Since $\mathfrak{a}+\mathcal{J}=A_R$ the isomorphism $\tau_M\colon \sigma^*M|_{\mathrm{Spec}\,A_R\smallsetminus \mathrm{V}(\mathcal{J})} \xrightarrow{\sim} M|_{\mathrm{Spec}\,A_R\smallsetminus \mathrm{V}(\mathcal{J})}$ of $\underline{M}$ induces an isomorphism $\tau_{M/\mathfrak{a}M}\colon \sigma^*M/\mathfrak{a}M \xrightarrow{\sim} M/\mathfrak{a}M$ and makes $\underline{M}/\mathfrak{a}\underline{M}$ into a finite $\mathbb{F}_q$-shtuka, which is étale. After passing to a covering of Spec R by open affine subschemes, we may assume that there is an isomorphism $\alpha\colon (A/\mathfrak{a})^r \otimes_{\mathbb{F}_q} R \xrightarrow{\sim} M/\mathfrak{a}M$ and then $\alpha^{-1}\circ\tau_{M/\mathfrak{a}M}\circ\sigma^*\alpha$ is an element $b \in G(R)$ and corresponds to a morphism $b\colon \mathrm{Spec}\,R \to G$. The fiber product $\mathrm{Spec}\,R \underset{b,G,L}{\times} G$ is finite étale over Spec R and of the form Spec R'. The projection onto the second factor G corresponds to an element $c \in G(R')$ with $c\cdot\sigma^*c^{-1} = b$, that is $c = b\cdot\sigma^*c$. This implies $\alpha\circ c = \tau_{M/\mathfrak{a}M}\circ\sigma^*(\alpha\circ c)$, and thus $\alpha\circ c$ is an isomorphism $(A/\mathfrak{a})^r \xrightarrow{\sim} (\underline{M}/\mathfrak{a}\underline{M})^\tau(R') := \{m \otimes M/\mathfrak{a}M \otimes_R R'\colon m = \tau_M(\sigma_M^* m)\}$. The proposition follows from this. ☐

Theorem 6.6. *Let $\underline{E}$ be an Abelian Anderson A-module over R of rank r and let $\underline{M} = \underline{M}(\underline{E})$ be its associated effective A-motive. Let* $(0) \neq \mathfrak{a} \subset A$ *be an ideal with $R\cdot\gamma(\mathfrak{a}) = R$, that is $\mathfrak{a}+\mathcal{J} = A_R$. Then for every R-algebra R' such that* Spec R' *is connected, there is an isomorphism of $A/\mathfrak{a}$-modules*

$$\underline{E}[\mathfrak{a}](R') \xrightarrow{\sim} \mathrm{Hom}_{A/\mathfrak{a}}\big((\underline{M}/\mathfrak{a}\underline{M})^\tau(R')\,,\ \mathrm{Hom}_{\mathbb{F}_q}(A/\mathfrak{a}, \mathbb{F}_q)\big)\,,$$
$$P \longmapsto \big[\,\overline{m} \longmapsto [\bar{a} \mapsto m\circ\varphi_a(P)]\big]\,.$$

In particular, if $\bar{s}$ = Spec Ω *is a geometric base point of* Spec R*, then $\underline{E}[\mathfrak{a}](\Omega)$ is a free $A/\mathfrak{a}$-module of rank r which carries a continuous action of the étale fundamental group $\pi_1^{\text{ét}}(\mathrm{Spec}\,R, \bar{s})$.*

Proof. This result and its proof are due to Anderson [2, Proposition 1.8.3] for R a field. For general R the proof was carried out in [4, Lemma 2.4 and Theorem 8.6]. The last statement follows from Proposition 6.5. ☐

7. Divisible local Anderson modules

In this section we consider the situation where $\mathfrak{p} \subset A$ is a maximal ideal and the elements of $\gamma(\mathfrak{p}) \subset R$ are nilpotent. Let $\hat{q}$ be the cardinality of the residue field $\mathbb{F}_\mathfrak{p} = A/\mathfrak{p}$ and $f = [\mathbb{F}_\mathfrak{p} : \mathbb{F}_q]$, that is $\hat{q} = q^f$. We fix a uniformizing parameter $z \in \mathrm{Frac}(A)$ at $\mathfrak{p}$. It defines an isomorphism $\mathbb{F}_\mathfrak{p}[\![z]\!] \xrightarrow{\sim} \widehat{A}_\mathfrak{p} := \varprojlim A/\mathfrak{p}^n$. We consider the $\mathfrak{p}$-adic completion $\widehat{A}_{\mathfrak{p},R} := \varprojlim A_R/\mathfrak{p}^n = (\mathbb{F}_\mathfrak{p}\otimes_{\mathbb{F}_q} R)[\![z]\!]$. By continuity the map γ extends to a ring homomorphism $\gamma\colon \widehat{A}_\mathfrak{p} \to R$. We consider the ideals

$$\mathfrak{a}_i = \big(a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_\mathfrak{p}\big) \subset \widehat{A}_{\mathfrak{p},R} \qquad \text{for} \quad i \in \mathbb{Z}/f\mathbb{Z}\,. \tag{7.1}$$

By the Chinese remainder theorem $\widehat{A}_{\mathfrak{p},R}$ decomposes

$$\widehat{A}_{\mathfrak{p},R} = \big(\mathbb{F}_\mathfrak{p} \otimes_{\mathbb{F}_q} R\big)[\![z]\!] \;=\; \prod_{i\in\mathbb{Z}/f\mathbb{Z}} \widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_i\,,$$

and $\widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_i$ is the subset of $\widehat{A}_{\mathfrak{p},R}$ on which $a \otimes 1$ acts as $1 \otimes \gamma(a)^{q^i}$ for all $a \in \mathbb{F}_\mathfrak{p}$. Each factor is canonically isomorphic to $R[\![z]\!]$. The factors are cyclically permuted by σ because $\sigma(\mathfrak{a}_i) = \mathfrak{a}_{i+1}$. In particular $\hat{\sigma} := \sigma^f$ stabilizes each factor and acts on it via $\hat{\sigma}(z) = z$ and $\hat{\sigma}(b) = b^{\hat{q}}$ for $b \in R$. The ideal $\mathcal{J} := (a\otimes 1 - 1\otimes\gamma(a)\colon a \in A) \subset A_R$ decomposes as follows $\mathcal{J}\cdot\widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0 = (z-\gamma(z))$ and $\mathcal{J}\cdot\widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_i = (1)$ for $i \neq 0$. In particular, $\widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0$ equals the $\mathcal{J}$-adic completion of A_R, as $\gamma(z)$ is nilpotent in R; compare also [3, Lemma 5.3]. We also set $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$.

Definition 7.1. A *local $\hat{\sigma}$-shtuka* (or *local shtuka*) *of rank r* over R is a pair $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ consisting of a locally free $R[\![z]\!]$-module $\hat{M}$ of rank r, and an isomorphism $\tau_{\hat{M}}\colon \hat{\sigma}^*\hat{M}[\frac{1}{z-\gamma(z)}] \xrightarrow{\;\sim\;} \hat{M}[\frac{1}{z-\gamma(z)}]$. If $\tau_{\hat{M}}(\hat{\sigma}^*\hat{M}) \subset \hat{M}$ then $\underline{\hat{M}}$ is called *effective*, and if $\tau_{\hat{M}}(\hat{\sigma}^*\hat{M}) = \hat{M}$ then $\underline{\hat{M}}$ is called *étale*.

A *morphism* of local shtukas $f\colon (\hat{M}, \tau_{\hat{M}}) \to (\hat{M}', \tau_{\hat{M}'})$ over R is a morphism of $R[\![z]\!]$-modules $f\colon \hat{M} \to \hat{M}'$ which satisfies $\tau_{\hat{M}'} \circ \hat{\sigma}^* f = f \circ \tau_{\hat{M}}$.

Example 7.2. Let $\underline{M} = (M, \tau_M)$ be an A-motive over R. We consider the $\mathfrak{p}$-adic completion $\underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R} := (M \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}\,,\ \tau_M \otimes 1) = \varprojlim \underline{M}/\mathfrak{p}^n\underline{M}$. We recall the definition of $\mathfrak{a}_0$ from (7.1) and define the *local $\hat{\sigma}$-shtuka at $\mathfrak{p}$ associated with* $\underline{M}$ as $\underline{\hat{M}}_\mathfrak{p}(\underline{M}) := \big(M \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0\,,\ (\tau_M \otimes 1)^f\big)$, where $\tau_M^f := \tau_M \circ \sigma^*\tau_M \circ \ldots \circ \sigma^{(f-1)*}\tau_M$. It equals the $\mathcal{J}$-adic completion of $\underline{M}$ and therefore is effective if and only if $\underline{M}$ is effective, because of Proposition 2.3. Of course if $\mathbb{F}_\mathfrak{p} = \mathbb{F}_q$, and hence $\hat{q} = q$ and $\hat{\sigma} = \sigma$, we have $\widehat{A}_{\mathfrak{p},R} = R[\![z]\!]$ and $\underline{\hat{M}}_\mathfrak{p}(\underline{M}) = \underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}$.

Also for $f > 1$ the local shtuka $\underline{\hat{M}}_\mathfrak{p}(\underline{M})$ allows to recover $\underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}$ via the isomorphism

$$\bigoplus_{i=0}^{f-1}(\tau_M \otimes 1)^i \bmod \mathfrak{a}_i : \left(\bigoplus_{i=0}^{f-1}\sigma^{i*}(M \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0),\ (\tau_M \otimes 1)^f \oplus \bigoplus_{i\neq 0}\mathrm{id}\right) \xrightarrow{\;\sim\;} \underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}\,,$$

because for $i \neq 0$ the equality $\mathcal{J}\cdot\widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_i = (1)$ implies that $\tau_M \otimes 1$ is an isomorphism modulo $\mathfrak{a}_i$; see [18, Example 2.2] or [6, Propositions 8.8 and 8.5] for more details.

Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ be an effective local shtuka over R. Set $\underline{\hat{M}}_n := (\hat{M}_n, \tau_{\hat{M}_n}) := (\hat{M}/z^n\hat{M}, \tau_{\hat{M}} \bmod z^n)$ and $G_n := \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}_n)$. Then G_n is a finite locally free strict $\mathbb{F}_\mathfrak{p}$-module scheme over R and $\underline{\hat{M}}_n = \underline{M}_{\hat{q}}(G_n)$ by Theorem 4.7. Moreover, G_n inherits from $\underline{\hat{M}}_n$ an action of $\mathbb{F}_\mathfrak{p}[z]/(z^n)$. The canonical epimorphisms $\underline{\hat{M}}_{n+1} \twoheadrightarrow \underline{\hat{M}}_n$ induce closed immersions $i_n \colon G_n \hookrightarrow G_{n+1}$. The inductive limit $\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}) := \varinjlim G_n$ in the category of sheaves on the big *fppf*-site of Spec R is a sheaf of $\mathbb{F}_\mathfrak{p}[\![z]\!]$-modules that is a $\mathfrak{p}$-divisible local Anderson module in the sense of the following

Definition 7.3. A $\mathfrak{p}$*-divisible local Anderson module over* R is a sheaf of $\mathbb{F}_\mathfrak{p}[\![z]\!]$-modules G on the big *fppf*-site of Spec R such that

(a) G is $\mathfrak{p}$*-torsion*, that is $G = \varinjlim G[z^n]$, where $G[z^n] := \ker(z^n \colon G \to G)$;
(b) G is $\mathfrak{p}$*-divisible*, that is $z \colon G \to G$ is an epimorphism;
(c) For every n the $\mathbb{F}_\mathfrak{p}$-module $G[z^n]$ is representable by a finite locally free strict $\mathbb{F}_\mathfrak{p}$-module scheme over R (Definition 4.2);
(d) There exists an integer $d \in \mathbb{Z}_{\geq 0}$, such that $(z - \gamma(z))^d = 0$ on ω_G where $\omega_G := \varprojlim \omega_{G[z^n]}$ and $\omega_{G[z^n]} = e^*\Omega^1_{G[z^n]/\operatorname{Spec} R}$ is the pullback under the zero section $e \colon \operatorname{Spec} R \to G[z^n]$.

Note that the terminology $\mathfrak{p}$-torsion and $\mathfrak{p}$-divisible in relation to z reflects that $\mathbb{F}_\mathfrak{p}[\![z]\!] = \widehat{A}_\mathfrak{p}$ and $\mathfrak{p} \cdot \widehat{A}_\mathfrak{p} = z \cdot \mathbb{F}_\mathfrak{p}[\![z]\!]$.

A *morphism of* $\mathfrak{p}$*-divisible local Anderson modules over* R is a morphism of *fppf*-sheaves of $\mathbb{F}_\mathfrak{p}[\![z]\!]$-modules. The category of divisible local Anderson modules is $\mathbb{F}_\mathfrak{p}[\![z]\!]$-linear. It is shown in [19, Lemma 8.2] that ω_G is a finite locally free R-module and we define the *dimension of* G as $\operatorname{rk} \omega_G$. A $\mathfrak{p}$-divisible local Anderson module is called *étale* if $\omega_G = 0$. Since ω_G surjects onto each $\omega_{G[z^n]}$, this is the case if and only if all $G[z^n]$ are étale, see [19, Lemma 3.7].

Conversely with a $\mathfrak{p}$-divisible local Anderson module G over R one associates the local shtuka $\underline{M}_{\hat{q}}(G) := \varprojlim \underline{M}_{\hat{q}}(G[z^n])$. Multiplication with z on G gives $M_{\hat{q}}(G)$ the structure of an $R[\![z]\!]$-module. In [19, Theorem 8.3] we proved the following:

Theorem 7.4.

(a) *The two contravariant functors* $\mathrm{Dr}_{\hat{q}}$ *and* $\underline{M}_{\hat{q}}$ *are mutually quasi-inverse anti-equivalences between the category of effective local shtukas over R and the category of* $\mathfrak{p}$*-divisible local Anderson modules over R;*
(b) *Both functors are* $\mathbb{F}_\mathfrak{p}[\![z]\!]$*-linear and map short exact sequences to short exact sequences. They preserve étale objects.*

Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ *be an effective local shtuka over S and let* $G = \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})$ *be its associated* $\mathfrak{p}$*-divisible local Anderson module. Then*

(c) *G is a formal Lie group if and only if* $\tau_{\hat{M}}$ *is topologically nilpotent, that is* $\operatorname{im}(\tau^n_{\hat{M}}) \subset z\hat{M}$ *for an integer n;*
(d) *The* $R[\![z]\!]$*-modules* $\omega_{\mathrm{Dr}_{\hat{q}}(\underline{\hat{M}})}$ *and* $\operatorname{coker} \tau_{\hat{M}}$ *are canonically isomorphic.*

We now want to show that for an Abelian Anderson A-module $\underline{E}$ over R the local shtuka $\hat{\underline{M}}_{\mathfrak{p}}\big(\underline{M}(\underline{E})\big)$ corresponds to the $\mathfrak{p}$-power torsion of $\underline{E}$ as in the following

Definition 7.5. Let $\underline{E}$ be an Abelian Anderson A-module over R and assume that the elements of $\gamma(\mathfrak{p}) \subset R$ are nilpotent. We define $\underline{E}[\mathfrak{p}^\infty] := \varinjlim \underline{E}[\mathfrak{p}^n]$ and call it the $\mathfrak{p}$-*divisible local Anderson module* associated with $\underline{E}$.

This definition is justified by the following

Theorem 7.6. *Let $\underline{E} = (E, \varphi)$ be an Abelian Anderson A-module over R and assume that the elements of $\gamma(\mathfrak{p}) \subset R$ are nilpotent. Then*

(a) *All $\underline{E}[\mathfrak{p}^n]$ are finite locally free strict $\mathbb{F}_{\mathfrak{p}}$-module schemes;*
(b) *$\underline{E}[\mathfrak{p}^\infty]$ is a $\mathfrak{p}$-divisible local Anderson module over R;*
(c) *If $\underline{M} = \underline{M}(\underline{E})$ is the associated effective A-motive of $\underline{E}$ and $\hat{\underline{M}} := \hat{\underline{M}}_{\mathfrak{p}}(\underline{M}) = \underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0$ is the local $\hat{\sigma}$-shtuka at $\mathfrak{p}$ associated with $\underline{M}$ from Example* 7.2, *then there are canonical isomorphisms*

$$\underline{M}_{\hat{q}}(\underline{E}[\mathfrak{p}^\infty]) \cong \hat{\underline{M}}_{\mathfrak{p}}(\underline{M}) \quad \text{and} \quad \underline{E}[\mathfrak{p}^\infty] \cong \mathrm{Dr}_{\hat{q}}\big(\hat{\underline{M}}_{\mathfrak{p}}(\underline{M})\big),$$
$$\underline{M}_{q}(\underline{E}[\mathfrak{p}^\infty]) \cong \underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R} \quad \text{and} \quad \underline{E}[\mathfrak{p}^\infty] \cong \mathrm{Dr}_{q}\big(\underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}\big),$$
$$\underline{M}_{\hat{q}}(\underline{E}[\mathfrak{p}^n]) \cong \hat{\underline{M}}/\mathfrak{p}^n\hat{\underline{M}} \quad \text{and} \quad \underline{E}[\mathfrak{p}^n] \cong \mathrm{Dr}_{\hat{q}}(\hat{\underline{M}}/\mathfrak{p}^n\hat{\underline{M}}).$$

Proof. (a) By Lemma 4.4 we may test strictness after applying a faithfully flat base change to R and assume that $E = \mathbb{G}^d_{a,R} = \operatorname{Spec} R[x_1, \ldots, x_d] = \operatorname{Spec} R[\underline{X}]$ and $\underline{M}(\underline{E}) = R\{\tau\}^{1\times d}$. We set $B := \Gamma(\underline{E}[\mathfrak{p}^n], \mathcal{O}_{\underline{E}[\mathfrak{p}^n]})$ and $I = \ker(R[\underline{X}] \twoheadrightarrow B)$ and $I_0 = (x_1, \ldots, x_d)$, and consider the deformation $B^\flat = R[\underline{X}]/I \cdot I_0$. The endomorphisms φ_a of E for $a \in A$ satisfy $\varphi_a^*(I) \subset I$ and $\varphi_a^*(I_0) \subset I_0$. This defines a lift $A \to \mathrm{End}_{R\text{-algebras}}(B^\flat)$, $a \mapsto [a]^\flat := \varphi_a^*$ compatible with addition and multiplication as in Definition 4.2.

Let $N \geq \dim \underline{E}$ be a positive integer which is a power of $\hat{q}$ such that $\gamma(a)^N = 0$ for every $a \in \mathfrak{p}^n$. Choose $\lambda \in \mathbb{F}_{\mathfrak{p}}$ with $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_q(\lambda)$ and let g be the minimal polynomial of λ over $\mathbb{F}_q$. Choose an element $t \in A$ with $t \bmod \mathfrak{p}^n = \lambda$ in $A/\mathfrak{p}^n = \mathbb{F}_{\mathfrak{p}}[\![z]\!]/(z^n)$. Then $g(t) \in \mathfrak{p}^n$, and hence $\gamma(g(t))^N = 0$. On $\operatorname{Lie} E$ the equation $g(t^N) = g(t)^N$ implies $\operatorname{Lie} \varphi_{g(t^N)} = \operatorname{Lie} \varphi^N_{g(t)} - \gamma(g(t))^N = \big(\operatorname{Lie} \varphi_{g(t)} - \gamma(g(t))\big)^N = 0$. So $\varphi_{g(t^N)} \in \mathrm{End}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\mathbb{G}^d_{a,R}) = R\{\tau\}^{d\times d}$ as a polynomial in τ has no constant term. This means that $\varphi^*_{g(t^N)}(x_i) \in I_0^q$. Moreover, since $g(t) \in \mathfrak{p}^n$ we have $\varphi_{g(t)} = 0$ on $\underline{E}[\mathfrak{p}^n]$ and hence $\varphi^*_{g(t)}(x_i) \in I$. Therefore $\varphi^*_{g(t^{\hat{q}N})}(I_0) = \varphi^*_{g(t)} \circ \varphi^*_{g(t^{\hat{q}N-N-1})} \circ \varphi^*_{g(t^N)}(I_0) \subset \varphi^*_{g(t)}(I_0^q) \subset \varphi^*_{g(t)}(I_0)^2 \subset I \cdot I_0$. In other words $[g(t^{\hat{q}N})]^\flat = [0]^\flat$ on $B^\flat$. This shows that the map $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_q[t^{\hat{q}N}]/(g(t^{\hat{q}N})) \to \mathrm{End}_{R\text{-algebras}}(B^\flat)$ lifts the action of $\mathbb{F}_{\mathfrak{p}} \subset \mathbb{F}_{\mathfrak{p}}[\![z]\!]/(z^n)$ on $\underline{E}[\mathfrak{p}^n]$ and is compatible with addition and multiplication.

We compute the induced action on the co-Lie complex $\ell^\bullet_{\mathcal{G}/\operatorname{Spec} R}$ of $\mathcal{G} = (\operatorname{Spec} B, \operatorname{Spec} B^\flat)$. In degree 0 we have $\ell^0_{\mathcal{G}/\operatorname{Spec} R} = \Omega^1_{R[\underline{X}]/R} \otimes_{R[\underline{X}],\, e_{R[\underline{X}]}} R =$

$\bigoplus_{i=1}^d R\cdot x_i = I_0/I_0^2$. From $t-\lambda\in\mathfrak{p}^n$ we obtain $\gamma(t^{\hat q N})-\gamma(\lambda)=\gamma(t-\lambda)^{\hat q N}=0$ in R. On $\operatorname{Lie} E$ this implies $\operatorname{Lie}\varphi_{t^{\hat q N}}-\gamma(\lambda)=(\operatorname{Lie}\varphi_t-\gamma(t))^{\hat q N}=0$ and therefore $\varphi_{t^{\hat q N}}-\gamma(\lambda)\in \operatorname{End}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\mathbb{G}_{a,R}^d)=R\{\tau\}^{d\times d}$ as a polynomial in τ has no constant term. This implies that $\big(\varphi^*_{t^{\hat q N}}-\gamma(\lambda)\big)(I_0)\subset I_0^q\subset I_0^2$. We conclude that $t^{\hat q N}$ acts as the scalar $\gamma(\lambda)$ on I_0/I_0^2.

To compute the action of $t^{\hat q N}$ on $\ell^{-1}_{\mathcal{G}/\operatorname{Spec} R}$ we use that by Theorem 4.7(d), $\ell^{\bullet}_{\mathcal{G}/\operatorname{Spec} R}$ is homotopically equivalent to the complex $0\to\sigma^*M/\mathfrak{p}^n\sigma^*M\xrightarrow{\tau_M} M/\mathfrak{p}^nM\to 0$ where $\underline{M}_q(\underline{E}[\mathfrak{p}^n])=\underline{M}/\mathfrak{p}^n\underline{M}$ and $\underline{M}=\underline{M}(\underline{E})=(M,\tau_M)$; see Theorem 6.4(c). Since $t^{\hat q N}-\gamma(\lambda)=(t\otimes 1-1\otimes\gamma(t))^{\hat q N}=0$ on $\operatorname{coker}\tau_M$ there is an A_R-homomorphism $h\colon M\to\sigma^*M$ with $h\,\tau_M=\big(t^{\hat q N}-\gamma(\lambda)\big)\cdot\mathrm{id}_{\sigma^*M}$ and $\tau_M\,h=\big(t^{\hat q N}-\gamma(\lambda)\big)\cdot\mathrm{id}_M$. This means that $t^{\hat q N}$ is homotopic to the scalar multiplication with $\gamma(\lambda)$ on $0\to\sigma^*M/\mathfrak{p}^n\sigma^*M\xrightarrow{\tau_M} M/\mathfrak{p}^nM\to 0$, and therefore also on $\ell^{\bullet}_{\mathcal{G}/\operatorname{Spec} R}$. Let $h'\colon I_0/I_0^2\to\ell^{-1}_{\mathcal{G}/\operatorname{Spec} R}=:\ell^{-1}$ be this homotopy, that is $(t^{\hat q N}-\gamma(\lambda))|_{\ell^{-1}}=h'd$ and $(t^{\hat q N}-\gamma(\lambda))|_{I_0/I_0^2}=dh'$. But we must show that $t^{\hat q N}$ and $\gamma(\lambda)$ are not only homotopic on $\ell^{\bullet}_{\mathcal{G}/\operatorname{Spec} R}$, but equal.

Since $0=g(t^{\hat q N})=\prod_{i\in\mathbb{Z}/f\mathbb{Z}}(t^{\hat q N}-\gamma(\lambda)^{q^i})$ on $\ell^{\bullet}_{\mathcal{G}/\operatorname{Spec} R}$, we can decompose $\ell^{-1}=\prod_{i\in\mathbb{Z}/f\mathbb{Z}}(\ell^{-1})_i$ where $(\ell^{-1})_i:=\ker(t^{\hat q N}-\gamma(\lambda)^{q^i}\colon\ell^{-1}\to\ell^{-1})$. Since the differential d of $\ell^{\bullet}_{\mathcal{G}/\operatorname{Spec} R}$ is an R-homomorphism and equivariant for the action of $t^{\hat q N}$, it maps $(\ell^{-1})_i$ into $\ker(t^{\hat q N}-\gamma(\lambda)^{q^i}\colon I_0/I_0^2\to I_0/I_0^2)$ which is trivial for $i\neq 0$. This shows that $0=h'd=t^{\hat q N}-\gamma(\lambda)=\gamma(\lambda^{q^i}-\lambda)$ on $(\ell^{-1})_i$, whence $(\ell^{-1})_i=(0)$ for $i\neq 0$, because $\gamma(\lambda^{q^i}-\lambda)\in R^\times$. We conclude that $\ell^{-1}=(\ell^{-1})_0$ and $t^{\hat q N}$ acts as the scalar $\gamma(\lambda)$ on ℓ^{-1}. This proves that $\underline{E}[\mathfrak{p}^n]$ is a finite locally free strict $\mathbb{F}_{\mathfrak{p}}$-module scheme over R.

(b) By construction $\ker(z^n\colon\underline{E}[\mathfrak{p}^\infty]\to\underline{E}[\mathfrak{p}^\infty])=\underline{E}[\mathfrak{p}^n]$ and $\underline{E}[\mathfrak{p}^\infty]$ is $\mathfrak{p}$-torsion. Using the epimorphism $j_{n+1,n}\colon\underline{E}[\mathfrak{p}^{n+1}]\twoheadrightarrow\underline{E}[\mathfrak{p}^n]$ from (6.1) with $i_{n,n+1}\circ j_{n+1,n}=\varphi_z$ we see that $\underline{E}[\mathfrak{p}^\infty]$ is $\mathfrak{p}$-divisible. In (a) we saw that $\underline{E}[\mathfrak{p}^n]$ is representable by a finite locally free strict $\mathbb{F}_{\mathfrak{p}}$-module scheme over R. It remains to verify condition (d) of Definition 7.3. Since $\underline{E}[\mathfrak{p}^n]\hookrightarrow\underline{E}$ is a closed immersion, $\omega_{\underline{E}[\mathfrak{p}^n]}$ is a quotient of $\omega_{\underline{E}}=\operatorname{Hom}_R(\operatorname{Lie}\underline{E},R)$. Since $A/\mathfrak{p}^n=\mathbb{F}_{\mathfrak{p}}[\![z]\!]/(z^n)$, there is an element $a\in A$ with $z-a\in\mathfrak{p}^n$, whence $\varphi_a=\varphi_z$ on $\underline{E}[\mathfrak{p}^n]$. Therefore $(\operatorname{Lie}\varphi_a-\gamma(a))^d=0$ on $\operatorname{Lie}\underline{E}$ implies $(\varphi_z-\gamma(z))^N=(\varphi_a-\gamma(a))^N+\gamma(a-z)^N=0$ on $\omega_{\underline{E}[\mathfrak{p}^n]}$. It follows that $(\varphi_z-\gamma(z))^N=0$ on $\omega_{\underline{E}[\mathfrak{p}^\infty]}:=\varprojlim\omega_{\underline{E}[\mathfrak{p}^n]}$, and that $\underline{E}[\mathfrak{p}^\infty]$ is a $\mathfrak{p}$-divisible local Anderson module over R.

(c) We have $\underline{M}_q(\underline{E}[\mathfrak{p}^n])=\operatorname{Hom}_{R\text{-groups},\mathbb{F}_q\text{-lin}}(\underline{E}[\mathfrak{p}^n],\mathbb{G}_{a,R})=\underline{M}/\mathfrak{p}^n\underline{M}$ and $\underline{E}[\mathfrak{p}^n]=\mathrm{Dr}_q(\underline{M}/\mathfrak{p}^n\underline{M})$ by Theorem 6.4(c). This implies

$$\underline{M}_q\big(\underline{E}[\mathfrak{p}^\infty]\big)=\varprojlim\underline{M}_q\big(\underline{E}[\mathfrak{p}^n]\big)=\varprojlim\underline{M}/\mathfrak{p}^n\underline{M}=\underline{M}\otimes_{A_R}\widehat{A}_{\mathfrak{p},R}$$

and $\underline{E}[\mathfrak{p}^\infty]=\varinjlim\mathrm{Dr}_q(\underline{M}/\mathfrak{p}^n\underline{M})=\mathrm{Dr}_q(\varprojlim\underline{M}/\mathfrak{p}^n\underline{M})=\mathrm{Dr}_q(\underline{M}\otimes_{A_R}\widehat{A}_{\mathfrak{p},R})$.

On $\underline{E}[\mathfrak{p}^n]$ every $\lambda \in \mathbb{F}_\mathfrak{p}$ acts as φ_λ and on $\mathbb{G}_{a,R}$ as $\gamma(\lambda)$. Therefore

$$\begin{aligned} \underline{M}_{\hat{q}}\big(\underline{E}[\mathfrak{p}^n]\big) &= \mathrm{Hom}_{R\text{-groups},\mathbb{F}_\mathfrak{p}\text{-lin}}\big(\underline{E}[\mathfrak{p}^n], \mathbb{G}_{a,R}\big) \\ &= \underline{M}_q\big(\underline{E}[\mathfrak{p}^n]\big)/\mathfrak{a}_0\underline{M}_q\big(\underline{E}[\mathfrak{p}^n]\big) \\ &= \underline{M}/\mathfrak{p}^n\underline{M} \otimes_{\widehat{A}_{\mathfrak{p},R}} \widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0 \\ &= \underline{\hat{M}}/\mathfrak{p}^n\underline{\hat{M}}\,, \end{aligned}$$

where the second equality is due to the fact that $\widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0$ is the summand of $\widehat{A}_{\mathfrak{p},R}$ on which $\lambda \otimes 1$ acts as $1 \otimes \gamma(\lambda)$ for all $\lambda \in \mathbb{F}_\mathfrak{p}$. This implies

$$\underline{M}_{\hat{q}}\big(\underline{E}[\mathfrak{p}^\infty]\big) = \varprojlim \underline{M}/\mathfrak{p}^n\underline{M} \otimes_{\widehat{A}_{\mathfrak{p},R}} \widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0 = \underline{M} \otimes_{A_R} \widehat{A}_{\mathfrak{p},R}/\mathfrak{a}_0 = \underline{\hat{M}}_\mathfrak{p}(\underline{M}) = \underline{\hat{M}}\,.$$

On the other hand, since $\underline{E}[\mathfrak{p}^n]$ is a finite locally free strict $\mathbb{F}_\mathfrak{p}$-module by (a), $\underline{E}[\mathfrak{p}^n] = \mathrm{Dr}_{\hat{q}}\big(\underline{M}_{\hat{q}}(\underline{E}[\mathfrak{p}^n])\big) = \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}/\mathfrak{p}^n\underline{\hat{M}})$ by Theorem 4.7(e), and so $\underline{E}[\mathfrak{p}^\infty] = \varinjlim \mathrm{Dr}_{\hat{q}}(\underline{\hat{M}}/\mathfrak{p}^n\underline{\hat{M}}) = \mathrm{Dr}_{\hat{q}}\big(\underline{\hat{M}}_\mathfrak{p}(\underline{M})\big)$. □

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