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Vitali properties of Banach analytic manifolds

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Abstract. We discuss possible generalizations of Vitali convergence theorem when the source and the target are Banach analytic manifolds. These results are then applied to study the behavior of holomorphic mappings between Banach analytic manifolds. Explicit examples of manifolds having Vitali properties are also provided.

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1. Introduction

The classical Vitali theorem states that a sequence $\{f_k\}$ of holomorphic functions defined on a domain D in $\mathbb{C}$ is uniformly convergent on compact sets if it is locally uniformly bounded and if it converges *pointwise* only on some set having an accumulation point in D. There are two ingredients in the proof. Firstly, by Montel's theorem, the sequence $\{f_k\}$ is relatively compact in the compact open topology and secondly, using the uniqueness property of holomorphic functions, we conclude that two accumulation points of the sequence $\{f_k\}$ must coincide on D. Observe that it is rather straightforward to generalize Vitali's theorem to (scalar valued) holomorphic functions of several variable. For vector-valued holomorphic functions, Montel's theorem is not valid, therefore it does not seem easy to find an analogue of Vitali theorem in this more general setting. Nevertheless, by making use of the notion of weak holomorphicity together with some elementary but quite ingenious arguments, Arendt and Nikolski provide in [1] a correct generalization of Vitali theorem for holomorphic functions defined on domains in $\mathbb{C}$ with values in complex Banach spaces. The aim of this paper is to explore possible versions of Vitali theorems in a general setting where the source and the target spaces are assumed to be Banach analytic manifolds.

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Now, we will shortly review basic notions that are pertaining to our work. By a Banach analytic manifold we mean a *connected* topological space in which each point has a neighborhood homeomorphic to an open set in a Banach space such that the transition maps are holomorphic between open sets of Banach spaces. Thus, Banach analytic manifolds encompass two objects of different character: (finite dimensional) complex manifolds and (infinite dimensional) complex Banach spaces.

Roughly speaking, we say that a Banach analytic manifold X has the Vitali property if for every (connected) Banach analytic manifold A and every sequence $\{f_k\}$ of holomorphic mappings from A into X that converges only pointwise on a "sufficiently large" subset of A then $\{f_k\}$ must converge uniformly on compact sets of A. For clarity of the exposition, we introduce Banach analytic manifolds with weak Vitali property (WVP) and strong Vitali property (SVP) depending on the nature of the set where pointwise convergence of $\{f_k\}$ occurs. Even though, we do not know if the two properties are really different, there are certainly some advantages in studying them. We now briefly outline the content of the paper. The first part concentrates on manifolds having Vitali properties and on applications to the study behavior of holomorphic mappings between Banach analytic manifolds. Our first main results is Theorem 3.1 which says that every Banach analytic manifold having WVP must be (Kobayashi) hyperbolic. This result brings in naturally hyperbolic Banach analytic manifolds into our study. In the opposite direction, we show in Theorem 3.3 that every complete hyperbolic Banach analytic manifold has SVP. The proof relies strongly on a vector valued version of Vitali's theorem which is inspired from the work of Arendt and Nikolski in [1] mentioned above. We also relate our Vitali properties with some sorts of *taut* property of Banach analytic manifolds. Recall that the classical taut property (see [7, page 239]) is defined for (finite dimensional) complex manifolds and it describes the behavior of sequences of holomorphic mappings from the unit disk $\Delta \subset \mathbb{C}$ into the complex manifold under consideration.

Our Vitali properties serve as convenient tools to check tautness of complex manifolds and Banach analytic manifolds. This fact is reflected in Theorem 3.10 which says that every sequence of holomorphic maps from a connected separable Banach analytic manifold A into a Banach analytic manifold X having WVP must contain a subsequence which is either convergent or compactly divergent on an open *dense* subset of A. Under the stronger assumption that the target manifold X has SVP and the source manifold is just the unit disk Δ, we show in Theorem 3.11 that the compactly divergence phenomenon *may* only occur outside a discrete subset of Δ. We should mention that, in the literature, there are some attempts to generalize the classical taut property for Banach analytic manifolds (see [4–6]). It should be, however, noted that our proofs, unlike those in [4–6], are quite constructive, in the sense that we avoid to use Zorn's lemma.

The second part of the work focuses on finding explicit classes of manifolds X having WVP and SVP. The key idea is to impose the existence of certain (nonconstant) negative plurisubharmonic function φ on X, and under certain additional assumptions we get Vitali property of the whole space X if each sublevel set determined by φ has this property. This principle is carried out in Theorem 4.1 (for

WVP of Banach analytic manifolds) and Theorem 4.4 (for SVP of open subsets of Banach spaces). Furthermore, we also give, in the last two results, somewhat complete characterizations for Vitali properties of Hartogs domains (over Banach analytic manifolds) and balanced domains in Banach spaces. The paper ends up by giving a list of open questions that are connected to our work. Finally, we should say that in the recent work [3], the authors also study SVP and WVP in terms of locality of these properties. Moreover, the invariance of SVP and WVP under holomorphic mappings is discussed as well.

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2. Basic notions and notation

We introduce below certain notion that are needed for formulating Vitali properties.

Notation. (a) Let A be a Banach analytic manifold and S be a subset of A. We let

$$S^u := \{z \in A \cap \overline{S} : \forall\, U \text{ connected neighborhood of } z \text{ and every holomorphic function } f : U \longrightarrow \mathbb{C},\ f|_{U\cap S} = 0 \Rightarrow f|_U = 0\}.$$

(b) Given Banach analytic manifolds A and X, by $\mathrm{Hol}(A, X)$ we mean the set of holomorphic mappings from A into X. We then equip $\mathrm{Hol}(A, X)$ with the compact-open topology. According to a result of Palais in [8], a Banach analytic manifold is metrizable if and only if it is paracompact. So, in the case where A and X are both paracompact, the compact-open topology on $\mathrm{Hol}(A, X)$ is equivalent to the topology of locally uniformly convergence.

(c) Let A, X be Banach analytic manifolds and $\{f_k\}$ be a sequence in $\mathrm{Hol}(A, X)$. We denote by $Z_{\{f_k\}}$ the set of points $\lambda \in A$ such that $\{f_k(\lambda)\}$ is convergent to an element in X.

It is easy to check that S^u is a closed subset of $\bar{S}$. Moreover, $S \setminus S^u$ is locally contained in an analytic hypersurface, *i.e.*, for every $a \in S \setminus S^u$, there exists a connected neigborhood U of a and a holomorphic function g on U such that $gt \equiv 0$, $g|_{S\cap U} \equiv 0$.

Now we come the central notions of this paper.

Definition 2.1. Let X be a Banach analytic manifold. We say that:

(a) X has the strong Vitali property (SVP for short) if for every (connected) Banach analytic manifold A and every sequence $\{f_k\}_{k\geq 1} \subset \mathrm{Hol}(A, X)$ such that $Z^u_{\{f_k\}} \neq \emptyset$ we have $\{f_k\}_{k\geq 1}$ is convergent in $\mathrm{Hol}(A, X)$;

(b) X has the weak Vitali property (WVP for short) if for every (connected) Banach analytic manifold A and every sequence $\{f_k\}_{k\geq 1} \subset \mathrm{Hol}(A, X)$ such that $Z_{\{f_k\}} \cap Z^u_{\{f_k\}} \neq \emptyset$ we have $\{f_k\}_{k\geq 1}$ is convergent in $\mathrm{Hol}(A, X)$;

(c) In the particular case where the above properties are true for $A = \Delta$, we say that X has 1-WVP and 1-SVP respectively.

Remarks 2.2. (a) In the special case where $A = \Delta$, $Z^u_{\{f_k\}}$ is exactly the set of accumulation points of $Z_{\{f_k\}}$.

(b) We construct a sequence of polynomials $\{p_k\}$ on $\mathbb{C}$ such that $Z^u_{p_k} \neq \emptyset$ but $Z_{\{p_k\}} \cap Z^u_{\{p_k\}} = \emptyset$. For $k \geq 1$, we let U_k and V_k be disks in $\mathbb{C}$ with disjoint closures such that $0 \in U_k$ and $[1/k, 1] \subset V_k$. By Runge's approximation theorem we can find a polynomial p_k on $\mathbb{C}$ such that

$$|p_k(z)| > 1/2, \quad \forall z \in U_k; \quad \|p\|_{V_k} < 1/k.$$

It is then clear that $0 \in Z^u_{\{p_k\}}$ but $0t \in Z^u_{\{p_k\}}$.

(c) In spite of the above example, we will show in Corollary 3.13 and Theorem 3.14 that in the categories of complex manifolds (respectively bounded domains in Banach spaces), the two notions 1-WVP and 1-SVP (respectively WVP and SVP) are equivalent. Unfortunately, we do not even know if there exists a (unbounded) domain in a (infinite dimensional) Banach space having the 1-WVP, which does not have 1-SVP.

The main technical tool in our paper is the Kobayashi pseudo-distance defined on a Banach analytic manifold X. Analogously as in the case where X is a finite dimensional complex manifold (see [7, page 50]) or a Banach space (see [3, page 81]), the pseudo-distance $\kappa_X(p, q)$ is defined to be the infimum of the length of all holomorphic chains joining $p, q \in X$. More precisely, by a holomorphic chain from p to q we mean a chain of points $p = p_0, p_1, \cdots, p_k = q$ of X, pairs of points $a_1, b_1, \cdots, a_k, b_k$ of Δ and holomorphic maps $f_1, \cdots, f_k \in \mathrm{Hol}(\Delta, X)$ such that

$$f_i(a_i) = p_{i-1}, f_i(b_i) = p_i, \ 1 \leq i \leq k.$$

Denote this chain by α, then the length of α is defined to be

$$l(\alpha) := \rho_\Delta(a_1, b_1) + \cdots \rho_\Delta(a_k, b_k),$$

where ρ_Δ is the Poincare distance on Δ.

Definition 2.3. The Kobayashi pseudo-distance between p and q is defined by

$$\kappa_X(p, q) := \inf_\alpha l(\alpha),$$

where the infimum is taken over all holomorphic chains α connecting p and q.

By the same proof as in the case of complex manifolds (see [7, Proposition 3.1.7]), we can show that κ_X is decreasing under holomorphic maps *i.e.*, if $f : X \to Y$ is a holomorphic mapping between Banach analytic manifolds X, Y then

$$\kappa_Y(f(p), q(q)) \leq \kappa_X(p, q), \ \forall p, q \in X.$$

Moreover, κ_X is the largest pseudo-distance on X having this property.

Then, as it is customary, we say that X is *hyperbolic* if κ_X is a distance and defines the topology of X. Notice that, in contrast to the case where X is finite dimensional, κ_X may be a distance without defining the topology of X even in the case where X is a domain in a Banach space (see [3, page 93]). Furthermore, X is said to be *complete hyperbolic* if every κ_X- Cauchy sequence in X is convergent. By [2, Proposition 6.9] (see also [6, Proposition 3.6]) we know that every bounded convex domain Ω in a Banach space is complete hyperbolic. Hence, all open subsets of Ω are hyperbolic. In particular, each bounded open subset of a Banach space is hyperbolic.

We recall the notion of normality of holomorphic mappings between Banach analytic manifolds when the target space is of finite dimension. This property will be relevant to our Vitali properties in the category of complex manifolds (see Theorem 3.14 in the next section).

Definition 2.4. Let A be a connected Banach analytic manifolds and X be a complex manifolds. We say that $\mathrm{Hol}(A, X)$ is normal if for every sequence $\{f_k\}_{k\geq 1} \subset \mathrm{Hol}(A, X)$ there is a subsequence $\{f_{k_j}\}$ having one of the following properties:

(a) f_{k_j} is convergent in $\mathrm{Hol}(A, X)$;
(b) f_{k_j} is compactly divergent, *i.e.*, for every compact subsets $K \subset A$ and $L \subset X$, there exists j_0 such that $f_{k_j}(K) \cap L = \emptyset$ for all $j \geq j_0$.

We will see that the notion of normality does not generalize in the expected fashion when X is a general (infinite dimensional) Banach analytic manifold (see the remark following Theorem 3.10).

The final ingredient needed in our work is the concept of plurisubharmonic function on Banach analytic manifold. More precisely, we say that $\varphi : A \to [-\infty, \infty)$ (where A is a Banach analytic manifold) is plurisubharmonic if for every $a \in A$, there exists a neigborhood U of a such that U is isomorphic to a ball B in a Banach space E and that φ, regarded as a function on B, is plurisubharmonic in the classical sense (see [3, page 62]), *i.e.*, φ is upper semicontinuous and the restriction of φ on the intersection of B with each complex line in E is subharmonic. Notice that we allow the function $\varphi \equiv -\infty$ to be plurisubharmonic. We will frequently refer to the following *maximum principle:* Let A be a connected Banach analytic manifold and φ be a plurisubharmonic function on A. Suppose that there exists $x_0 \in A$ such that $\varphi(x_0) = \max_A \varphi$. Then $\varphi|_A \equiv \varphi(x_0)$.

Throughout this paper, for $r > 0$, we will write $\Delta(0, r)$ for the disk in $\mathbb{C}$ with center 0 and radius r.

3. Main results

Our first result states, in spirit, that hyperbolicity of the target manifold is the right substitute for the uniform boundedness assumption given in the classical Vitali theorem.

Theorem 3.1. *Every Banach analytic manifold X having the* 1-WVP *is hyperbolic.*

The proof relies heavily on the following lemma which is a slight modification of a result of Kiernan (see [7, Lemma 5.1.4]).

Lemma 3.2. *Let Y be a Banach analytic manifold and $x \in Y$. Let U, V, W be open subsets of Y such that $x \in V \subset \overline{V} \subset U, \overline{U} \cap \overline{W} = \emptyset$ and U is hyperbolic. Assume that there exists $\delta \in (0,1)$ such that for every $f \in \mathrm{Hol}(\Delta, Y)$ with $f(0) \in V$ we have $f(\Delta(0,\delta)) \subset U$. Then $\kappa_Y(x, W) > 0$.*

Proof. Choose a constant $c(\delta) > 0$ such that

$$\rho_\Delta(0, b) \geq c(\delta)\rho_{\Delta(0,\delta)}(0, b), \ \forall b \in \Delta(0, \delta/2).$$

Fix an arbitrary point $y \in W$ with $\kappa_Y(x, y) < \delta/2$. Consider a holomorphic chain

$$\alpha := \{x = x_0, x_1, \cdots, x_l = y; a_1, b_1, \cdots, a_l, b_l \in \Delta; f_1, \cdots, f_l \in \mathrm{Hol}(\Delta, Y)\}$$

that joins x and y such that

$$l(\alpha) = \rho_\Delta(a_1, b_1) + \cdots + \rho_\Delta(a_l, b_l) < \delta/2.$$

It follows that $\rho_\Delta(a_j, b_j) < \delta/2$ for every $1 \leq j \leq l$. By composing with Möebius transformations of Δ, we may arrange so that $a_1 = \cdots = a_l = 0$, and hence $b_1, \cdots, b_l \in \Delta(0, \delta/2)$. Let k be the integer such that $x_1, \cdots, x_{k-1} \in V$ but $x_k t \in V$. By taking a refinement of α (see [7, page 51]), we may assume further that $x_k \in U$. Since $f_1(0) = x_0, \cdots, f_k(0) = x_{k-1}$ are all in V, by the assumption $f_1(\Delta(0,\delta)), \cdots, f_k(\Delta(0,\delta))$ are all included in U. Hence, the length $l(\alpha)$ of α may be estimated from below as follows:

$$\begin{aligned} l(\alpha) &\geq \sum_{i=1}^{k} \rho_\Delta(0, b_i) \geq c(\delta) \sum_{i=1}^{k} \rho_{\Delta(0,\delta)}(0, b_i) \\ &\geq c(\delta) \sum_{i=1}^{k} \kappa_U(x_{i-1}, x_i) \geq c(\delta)\kappa_U(x, x_k) \geq c(\delta)\kappa_U(x, U \setminus V). \end{aligned}$$

Here, the third inequality follows by applying the distance decreasing property to the map $f_i : \Delta(0,\delta) \to U$. This implies that

$$\kappa_Y(x, y) \geq c'(\delta) := c(\delta)\kappa_U(x, U \setminus V) > 0.$$

The latter estimate follows from the fact that κ_U defines the topology of U. It follows that

$$\kappa_Y(x, W) \geq \min\{\delta/2, c'(\delta)\} > 0.$$

Hence, we are done. □

Proof of Theorem 3.1. First, let $\{x_n\}$ be a sequence in X such that $x_n \to x$ in the initial topology of X. Let U be a neigborhood of x which is isomorphic to some ball in a Banach space. In particular, U is hyperbolic. Then for n large enough, we have

$$\kappa_X(x_n, x) \leq \kappa_U(x_n, x) \to 0, \text{ as } n \to \infty.$$

Here the last statement follows from the hyperbolicity of U. Conversely, we fix $x \in X$ and a sequence $\{x_n\} \in X$ such that $\kappa_X(x_n, x) \to 0$ as $n \to \infty$. We must show that $x_n \to x$ in the original topology of X. Assume this is false, then by passing to a subsequence, we can find an open neigborhood U of x and an open neigborhood W of $\{x_n\}$ such that $\overline{U} \cap \overline{W} = \emptyset$. Furthermore, we can take U to be hyperbolic. Then we have $\kappa_X(x, W) = 0$. We also let $\{V_n\} \subset X$ be a sequence of open neigborhoods of x such that $V_n \downarrow x$. Next, we choose a sequence $\{\delta_n\}_{n \geq 0} \downarrow 0$ such that $\delta_0 = 1/2$, $\delta_1 = 1/3$ and

$$\delta_{n+1} < \min\left\{\frac{1}{n}, r_n := \delta_n \prod_{j=0}^{n-1} \frac{\delta_j - \delta_n}{1 - \delta_j \delta_n}\right\}, \ \forall n \geq 1.$$

It follows that $r_{n+1} < \delta_{n+1} < r_n$. In particular, $r_n \downarrow 0$. Using Lemma 3.2, we obtain a sequence $\{f_n\} \subset \mathrm{Hol}(\Delta, X)$ and points $a_n \in \Delta(0, r_n)$ such that

$$f_n(0) \in V_n, \, f_n(a_n)t \subset U, \quad \forall n \geq 1.$$

We also set for each $n \geq 1$ the following finite Blaschke product

$$\theta_n(\lambda) := \frac{a_n}{r_n} \lambda \prod_{j=0}^{n-1} \frac{\delta_j - \lambda}{1 - \delta_j \lambda}, \quad \forall \lambda \in \Delta.$$

Then $\theta_n \in \mathrm{Hol}(\Delta, \Delta)$. Moreover, we have

$$\theta_n(0) = \theta_n(\delta_j) = 0, \quad \forall 0 \leq j \leq n-1; \quad \theta_n(\delta_n) = a_n.$$

Finally, we define for each $n \geq 1$

$$g_n := f_n \circ \theta_n \in \mathrm{Hol}(\Delta, X).$$

Then by the reasoning used above we have

$$g_n(\delta_j) = g_n(0) = f_n(0), \quad \forall n \geq 1, \forall 0 \leq j \leq n-1.$$

This implies that

$$\lim_{n \to \infty} g_n(0) = \lim_{n \to \infty} g_n(\delta_j) = \lim_{n \to \infty} f_n(0) = x \quad \forall j \geq 0.$$

Since X has the 1-WVP and since $\delta_n \downarrow 0$, we infer that $\{g_n\}_{n \geq 1}$ converges in $\mathrm{Hol}(\Delta, X)$. In particular, there exists a small disk $\Delta(0, r_0)$ such that $g_n(\Delta(0, r_0)) \subset U$ for n large enough. This is impossible, since $g_n(\delta_n) = f_n(a_n)$ stays away from U for every $n \geq 1$. The proof is thereby completed. □

Remarks. (a) There exists a bounded Reinhardt domain X in $\mathbb{C}^2$ that does not have 1-WVP. Indeed, let X is the punctured unit polydisc in $\mathbb{C}^2$, *i.e.*, $X := \Delta^2 \setminus \{(0,0)\}$. Let $\{a_k\} \in \Delta$ be a sequence of distinct points such that $a_k \to a \in \Delta \setminus \{0, 1/2, 1/3, \cdots, 1/j, \cdots\}$. Let $f_k : \Delta \to X, k \geq 1$ be defined by

$$f_k(\lambda) := \left(\frac{a_k - \lambda}{1 - \lambda\bar{a}_k}, \frac{a_{k+1} - \lambda}{1 - \lambda\bar{a}_{k+1}}\right), \quad \lambda \in \Delta.$$

Then $f_k \in \mathrm{Hol}(\Delta, X)$. It is also easy to check that $f_k(\lambda)$ converges uniformly on compact sets of Δ to

$$f(\lambda) := \left(\frac{a - \lambda}{1 - \lambda\bar{a}}, \frac{a - \lambda}{1 - \lambda\bar{a}}\right).$$

In particular $f_k(1/j) \to f(1/j) \in X$, $f_k(0) \to f(0) \in X$ as $k \to \infty$. Notice, however, that $f_k(a) \to f(a) = (0,0)t \in X$. Thus X does not have 1-WVP.

(b) The above domain is not pseudoconvex. In fact, we will show that every domain having WVP in $\mathbb{C}^n$ must be pseudoconvex. See the remark after Theorem 3.14. On the other hand, there exists a bounded pseudoconvex domain in $\mathbb{C}^2$ which does not have WVP (see the remark following Proposition 4.5).

Our next main result provides, on the positive side, a partial converse to Theorem 3.1.

Theorem 3.3. *Let X be a complete hyperbolic Banach analytic manifold. Then X has* SVP. *In particular, every compact hyperbolic manifold has* SVP.

For the proof of Theorem 3.3 we need some lemmas. The first one is essentially taken from [6].

Lemma 3.4. *Let A, X be Banach analytic manifolds and $\{f_k\} \subset \mathrm{Hol}(A, X)$. Assume that X is hyperbolic. Then the sequence $\{f_k(\lambda)\}$ is a κ_X-Cauchy sequence in X for every $\lambda \in \overline{Z_{\{f_k\}}}$.*

Proof. Choose a sequence $\{\lambda_j\} \subset Z_{\{f_k\}}$ such that $\lim_{j\to\infty} \lambda_j = \lambda$. By the decreasing property of Kobayashi distance, for every $k, j, m \geq 1$ we obtain

$$\begin{aligned}\kappa_X\big(f_k(\lambda), f_m(\lambda)\big) &\leq \kappa_X\big(f_k(\lambda), f_k(\lambda_j)\big) + \kappa_X\big(f_k(\lambda_j), f_m(\lambda_j)\big) + \kappa_X\big(f_m(\lambda_j), f_m(\lambda)\big)\\ &\leq 2\kappa_A(\lambda_j, \lambda) + \kappa_A\big(f_k(\lambda_j), f_m(\lambda_j)\big).\end{aligned}$$

Hence, $\{f_k(\lambda)\}$ is a κ_X-Cauchy sequence in X. □

The next lemma, a variant of Vitali's theorem for holomorphic vector-valued functions, is essentially contained in [1, Theorem 2.1]. We include it here for the sake of completeness.

Lemma 3.5. *Let E, F be Banach spaces and Ω be an open subset of E. Let $\{f_k\}$ be a sequence in* $\mathrm{Hol}(\Omega, F)$ *that satisfies the following conditions:*

(i) $\{f_k\}$ *is locally uniformly bounded on Ω;*
(ii) $Z_{\{f_k\}}$ *is a set of uniqueness for* $\mathrm{Hol}(\Omega, \mathbb{C})$, i.e., *every holomorphic function $g : \Omega \to \mathbb{C}$ that vanishes on $Z_{\{f_k\}}$ must be identically* 0.

Then $\{f_k\}$ converges in $\mathrm{Hol}(\Omega, F)$.

Proof. Let $l^\infty(F)$ be the space of bounded sequences in F equipped with the sup norm and $c(F)$ be the closed subspace of convergent sequences in F. Define the map

$$f : \Omega \to l^\infty(F), \quad f(z) = (f_1(z), \cdots, f_k(z), \cdots).$$

We split the proof into some steps.

Step 1. We show that f is holomorphic on Ω. First, we treat the case where $n = 1$. Fix $z_0 \in \Delta$. Since $\{f_k\}$ is locally uniformly bounded, by Cauchy's inequalities we infer that $\alpha := (f_1'(z_0), \cdots, f_k'(z_0), \cdots) \in l^\infty(F)$. Now we claim that $f'(z_0) = \alpha$. Using Cauchy integral formula we obtain, for h small enough

$$\frac{1}{h}(f_k(z_0 + h) - f_k(z_0)) - f_k'(z_0) = \frac{h}{2\pi i} \int_{|z|=r} \frac{f_k(z)}{(z - z_0)^2(z - z_0 - h)} dz,$$

where $r \in (0, 1)$ is chosen such that $|z_0| < r, |z_0| + |h| < r$. Since f_k is locally uniformly bounded, we have

$$M := \sup_{k \geq 1, |z|=r} \|f_k(z)\| < \infty.$$

It follows that

$$\left\| \frac{1}{h}(f_k(z_0 + h) - f_k(z_0)) - f_k'(z_0) \right\| \leq \frac{|h|M}{(r - |z_0|)^2(r - |z_0| - |h|)}, \ \forall k \geq 1.$$

So

$$\lim_{h \to 0} \sup_{k \geq 1} \left\| \frac{1}{h}(f_k(z_0 + h) - f_k(z_0)) - f_k'(z_0) \right\| = 0.$$

This implies $f'(z_0) = \alpha$ as claimed. Thus f is holomorphic on Ω. For the general case, by the above argument, f is Gâteaux holomorphic on Ω. Since f is locally bounded, we infer that f is indeed holomorphic on Ω, see [2, Corollary II.5.5].

Step 2. Let $\theta : l^\infty(F) \to l^\infty(F)/c(F)$ be the canonical projection map. We will show that the map $f^* := \theta \circ f$ from Ω to the Banach space $\tilde{F} := l^\infty(F)/c(F)$ is identically 0. Indeed, obviously f^* is holomorphic. Moreover, by the assumption we easily get $f^* = 0$ on $Z_{\{f_k\}}$. It follows $\mu \circ f^* = 0$ on $Z_{\{f_k\}}$ for every $\mu \in \tilde{F}'$, the dual space of $\tilde{F}$. Since $\mu \circ f^*$ is a scalar holomorphic function on Ω and since

$Z_{\{f_k\}}$ is a set of uniqueness for $\mathrm{Hol}(\Omega, \mathbb{C})$, we conclude that $\mu \circ f^* \equiv 0$ on Ω. By the Hahn-Banach theorem, $f^* = 0$ on Ω.

Step 3. f_k is convergent in $\mathrm{Hol}(\Omega, F)$. By Step 2, the sequence $\{f_k\}$ is pointwise convergence on Ω. Since $\{f_k\}$ is locally uniformly bounded on Ω, we infer that the sequence $\{f_k\}$ is equicontinuous on every compact subset of Ω. Therefore, $\{f_k\}$ converges to f in $\mathrm{Hol}(\Omega, F)$. □

The next lemma is quite standard, it says roughly that a family of holomorphic mappings into a hyperbolic Banach analytic manifold is equicontinuous.

Lemma 3.6. *Let A, X be Banach analytic manifolds and $\{f_k\}$ be a sequence in Hol(A, X). Assume X is hyperbolic and there exists a sequence $\{\lambda_k\} \to \lambda_0 \in A$ such that $f_k(\lambda_k) \to x_0 \in X$. Then the following assertions hold:*

(i) *For every neigborhood V of $x_0 \in X$, there exists an open neigborhood U of λ_0 in A and $k_0 \geq 1$ such that*

$$f_k(U) \subset V, \forall k \geq k_0;$$

(ii) $f_k(\lambda_0) \to x_0$ *as* $k \to \infty$.

Proof. (i) Assume the conclusion is false. Then we may choose a sequence $\{\beta_j\} \to \lambda_0$ and $k_j \uparrow \infty$ such that

$$f_{k_j}(\beta_j)t \in V \ \forall j \geq 1.$$

By the decreasing property of Kobayashi distance we get

$$\kappa_X(f_{k_j}(\beta_j), f_{k_j}(\lambda_j)) \leq \kappa_A(\beta_j, \lambda_j) \to 0 \text{ as } j \to \infty.$$

Using the triangle inequality, it follows that

$$\kappa_X(f_{k_j}(\beta_j), x_0) \leq \kappa_X(f_{k_j}(\beta_j), f_{k_j}(\lambda_j)) + \kappa_X(f_{k_j}(\lambda_j), x_0) \to 0 \text{ as } j \to \infty.$$

This contradicts the hyperbolicity of X.

(ii) Applying again the triangle inequality we obtain for $k \geq 1$ the following estimates

$$\begin{aligned}\kappa_X(f_k(\lambda_0), x_0) &\leq \kappa_X(f_k(\lambda_k), x_0) + \kappa_X(f_k(\lambda_0), f_k(\lambda_k)) \\ &\leq \kappa_X(f_k(\lambda_k), x_0) + \kappa_A(\lambda_0, \lambda_k).\end{aligned}$$

This implies that $\kappa_X(f_k(\lambda_0), x_0) \to 0$ as $k \to \infty$. The desired conclusion now follows from the hyperbolicity of X. □

Using a standard compactness argument and Lemma 3.6 (ii) we obtain easily the following result that will be needed later on.

Lemma 3.7. *Let A, X be Banach analytic manifolds, and suppose X is hyperbolic. Let $\{f_k\}$ be a sequence in* $\mathrm{Hol}(A, X)$ *which is not compactly divergent. Then there exists $\lambda_0 \in A$ and a subsequence $\{f_{k_j}\}$ such that $f_{k_j}(\lambda_0) \to x_0 \in X$ as $j \to \infty$.*

The following useful fact about propagation of domains on which a sequence of holomorphic maps is compactly divergent will only be used at the end of this section.

Lemma 3.8. *Let A, X be Banach analytic manifolds and let $\{\Omega\}_{\alpha\in I}$ be a family of open subsets of A. Let $\{f_k\}$ be a sequence in* $\mathrm{Hol}(A, X)$ *which is compactly divergent on Ω_α, $\forall\alpha \in I$. Then $\{f_k\}$ is compactly divergent on $\Omega := \cup_{\alpha\in I}\Omega_\alpha$.*

Proof. Suppose that $\{f_k\}$ is not compactly divergent on Ω. Then, there exist compact sets $K \subset \Omega$, $L \subset X$ and a subsequence $\{f_{k_j}\}$ such that $f_{k_j}(K) \cap L \neq \emptyset$ for every j. Using compactness, we can find a sequence $\{\lambda_j\} \to \lambda_0 \in K$ such that $f_{k_j}(\lambda_j) \in L$ for every j and $f_{k_j}(\lambda_j) \to x_0 \in L$. Choose $\alpha_0 \in I$ such that $\lambda_0 \in \Omega_{\alpha_0}$. We may assume that the compact set $K' := \{\lambda_j\} \cup \{\lambda_0\} \subset \Omega_{\alpha_0}$. Hence $f_{k_j}(K') \cap L \neq \emptyset$ for every j. It follows that $\{f_k|_{\Omega_{\alpha_0}}\}$ is *not* compactly divergent. □

Now we proceed to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let A be a connected Banach analytic manifold and $\{f_k\} \subset \mathrm{Hol}(A, X)$ be such that there exists some point $\lambda_0 \in Z^u_{f_k}$. By Lemma 3.4 and the assumption that X is complete hyperbolic, we have $f_k(\lambda_0) \to x_0 \in X$ as $k \to \infty$. Take a neigborhood V of x_0 in X which is isomorphic to some ball in a Banach space. Since X is hyperbolic, by Lemma 3.6(i), we can find an open neigborhood U_0 of $\lambda_0 \in A$ and $k_0 \geq 1$ such that

$$f_k(U_0) \subset V, \ \forall k \geq k_0.$$

Now we apply Lemma 3.5 to deduce that the sequence $\{f_k|_{U_0}\}$ is convergent in $\mathrm{Hol}(U_0, X)$. Put

$$\Omega := \bigcup \{U \subset A : \{f_k|_U\} \text{ is convergent in } \mathrm{Hol}(U, X)\}.$$

Clearly Ω is open and $U_0 \subset \Omega$ by the above proof. It suffices to show that Ω is closed. Assume otherwise, then we can find $\lambda_1 \in \partial\Omega$. Using again Lemma 3.4 we find that $f_k(\lambda_1) \to x_1 \in X$ as $k \to \infty$. Repeating the above argument, we can find a neigborhood U_1 of λ_1 in A such that $\{f_k|_{U_1}\}$ is convergent in $\mathrm{Hol}(U_1, X)$. It follows that $\lambda_1 \in U_1 \subset \Omega$, which is a contradiction. Thus $\Omega = A$. The proof is complete. □

Our next result says that 1-WVP is in fact equivalent to WVP. We do not know if the analogous statement is true for SVP and 1-SVP.

Theorem 3.9. *Let X be a Banach analytic manifold. If X has* 1-WVP *then X has* WVP.

Proof. Let A be a Banach analytic manifold and $\{f_k\}$ be a sequence in $\mathrm{Hol}(A, X)$ such that there exists $\lambda_0 \in Z^u_{\{f_k\}}$ satisfying $f_k(\lambda_0) \to x_0 \in X$. Choose a neigborhood V of $x_0 \in X$ such that V is isomorphic to some ball in a Banach space. By Theorem 3.1, X is hyperbolic, so we may apply Lemma 3.6(i) to find an open neigborhood U_0 of x_0 such that $f_k(U_0) \subset V$ for k large enough. Next, by Lemma 3.5 we see that $\{f_k|_{U_0}\}$ is convergent in $\mathrm{Hol}(U_0, X)$.

Now we set

$$\Omega := \bigcup \{U \subset A : \{f_k|_U\} \text{ is convergent in } \mathrm{Hol}(U, X)\}.$$

Clearly Ω is open and non-empty. It remains to check that Ω is closed. Assume otherwise, then there exists $\lambda_1 \in \partial\Omega$. Choose a small neigborhood $\mathbb{B}$ of λ_1 for which we may assume to be a ball in some Banach space E. Pick $r > 0$ and $\lambda_2 \in \Omega$ such that

$$\lambda_1 \in \mathbb{B}(\lambda_2, r) \subset \mathbb{B} \subset E.$$

Let l be the complex line l joining λ_1 and λ_2. We may identify $\Delta' := l \cap \mathbb{B}(\lambda_2, r)$ with Δ. Consider the restriction $g_k := f_k\big|_{\Delta'}$. By the definition of Ω we see that g_k is pointwise convergent on the nonempty open subset $\Delta' \cap \Omega$ of the disk Δ'. Since X has $1-$WVP we infer that g_k is pointwise convergent on Δ'. In particular $f_k(\lambda_1) = g_k(\lambda_1)$ is convergent in X. Note that $\lambda_1 \in \partial\Omega$, so $\lambda_1 \in Z^u_{\{f_k\}}$. Using the same reasoning as in the beginning of the proof, we see that there exists some small neigborhood U_1 of λ_1 such that $\{f_k|_{U_1}\}$ is convergent in $\mathrm{Hol}(U_1, X)$. Thus $\lambda_1 \in U_1 \subset \Omega$. This contradicts the fact that $\lambda_1 \in \partial\Omega$. Hence $\Omega = A$ and the proof is thereby completed. □

The next result relates weak Vitali property of a Banach analytic manifold with the usual taut property.

Theorem 3.10. *Let X be a Banach analytic manifold. Then the following statements are equivalent:*

(i) *X has* WVP*;*
(ii) *X is hyperbolic and every sequence $\{f_k\} \subset \mathrm{Hol}(A, X)$ where A is a separable Banach analytic manifold, contains a subsequence which is either convergent in* $\mathrm{Hol}(A, X)$ *or compactly divergent on an open dense subset of A.*

For the ease of the exposition, we introduce the following notation: If A, X are Banach analytic manifolds and $\{f_k\}_{k\geq 1}$ is a sequence in $\mathrm{Hol}(A, X)$ then for every subset $N = \{k_1 < k_2 < \cdots\}$ of $\mathbb{N}$ we will write $\{f_k\}_{k\in N}$ for the subsequence $\{f_{k_i}\}_{i\geq 1}$.

Proof. (i) $\Rightarrow$ (ii). Fix a separable Banach analytic manifold A and a sequence $\{f_k\}_{k\geq 1} \subset \mathrm{Hol}(A, X)$. Suppose that $\{f_k\}$ contains no convergent subsequence. Since A is separable, we can choose a topological base $\{U_j\}_{j\geq 1}$ for A. With no loss of generality, we can assume that each U_j is a ball in some Banach space. Fix $j \geq 1$, we claim that there exists an open non empty subset V_j of U_j and a subsequence $\{f_k\}_{k\in N_j}$ which is a compactly divergent subsequence on V_j. Indeed, assume this is false. Then we let $\{x_{j,l}\}$ be a countable dense subset of U_j. For each $l \geq 1$, we choose a small ball $\mathbb{B}_{j,l} := \mathbb{B}(x_{j,l}, r_l) \subset U_j$ with $r_l \to 0$ as $l \to \infty$. Then, since X is hyperbolic, we may apply Lemma 3.7 to find, on each ball $\mathbb{B}_{j,l}$ a point $y_{j,l}$ and a subsequence $\{f_k\}_{k\in N_{j,l}}$ such that $f_k(y_{j,l})$ is convergent as $k \to \infty$ and $(k \in N_{j,l})$. Moreover, we can choose these sequences in such a way that $N_{j,l+1} \subset N_{j,l}, \forall l \geq 1$, *i.e.*, $\{f_k\}_{k\in N_{j,l}}$ is used to construct the further subsequence $\{f_k\}_{k\in N_{j,l+1}}$. Hence, after a diagonal process, we can build a subsequence $\{f_k\}_{k\in N_j}$ which is *pointwise convergent* on the countable set $A_j = \{y_{j,l}\}$ which is also *dense* in U_j. By applying WVP of X to $\{f_k|_{U_j}\}_{k\in N_j}$ we see that the sequence $\{f_k\}_{k\in N_j}$ is convergent in $\mathrm{Hol}(U_j, X)$. Thus, this sequence must converge also in $\mathrm{Hol}(A, X)$ since X has WVP. This is contradictory. Therefore, for each $j \geq 1$, we can find an open subset V_j of U_j and a subsequence $\{f_k\}_{k\in M_j}$ which is compactly divergent on V_j. As before, we may also arrange so that $M_{j+1} \subset M_j$ for every $j \geq 1$. Then, using one more diagonal process, we can construct a subsequence $\{f_k\}_{k\in J}$ which is compactly divergent on *each* V_j. Now we let $\Omega := \cup_{j\geq 1} V_j$. Then obviously Ω is open. Moreover, $\bar{\Omega} = A$, since otherwise we would find $j_0 \geq 1$ such that

$$V_{j_0} \subset U_{j_0} \subset A \setminus \bar{\Omega},$$

which is impossible. Finally, by Lemma 3.8, we conclude that $\{f_k\}_{k\in J}$ is compactly divergent on Ω.

(ii) $\Rightarrow$ (i). In view of Theorem 3.9, it suffices to prove that X has 1-WVP. For this, let $\{g_k\} \in \mathrm{Hol}(\Delta, X)$ be a sequence such that $Z_{\{g_k\}} \cap Z^u_{\{g_k\}} \neq \emptyset$. We have to show that $\{g_k\}$ is convergent in $\mathrm{Hol}(\Delta, X)$. First, we claim that $\{g_k\}$ has a convergent subsequence in $\mathrm{Hol}(\Delta, X)$. Suppose otherwise, then there exist a dense open subset Ω of A and a subsequence $\{g_{k_j}\}$ which is compactly divergent on Ω. Fix $\lambda_0 \in Z_{\{g_k\}} \cap Z^u_{\{g_k\}}$. Then $g_{k_j}(\lambda_0) \to x_0 \in X$. Since X is hyperbolic, by Lemma 3.6 (i), we can choose a complete hyperbolic neigborhood V of x_0 in X and a neigborhood U of λ_0 in A such that $g_{k_j}(U) \subset V$ for j large enough. By Theorem 3.3, V has SVP. Hence $\{g_{k_j}\}$ is convergent in $\mathrm{Hol}(U, V)$. This yields a contradiction to compactly divergence of $\{g_{k_j}\}$ on the open set $U \cap \Omega$ which is non-empty since Ω is dense in A. The claim now follows. Finally, it remains to check that two (arbitrary) accumulation points g and g' of $\{g_k\}$ must coincide. For this, it suffices to note that $g = g'$ on $Z_{\{g_k\}}$. Hence, the desired conclusion now follows from the assumption that $Z_{\{g_k\}} \cap Z^u_{\{g_k\}} \neq \emptyset$. The proof is complete. □

Remark. The structure of the "exceptional" set $S := A \setminus \Omega$ may depend on the sequence $\{f_k\}$ even in the case in which A and X are nice manifolds. Indeed, let $A := \Delta$ be the unit disk in $\mathbb{C}$ and X be the unit ball of a infinite dimensional Banach space. Let S be a discrete subset of Δ such that $\sum_{a\in S}(1 - |a|) < \infty$. We will construct a sequence $\{f_k\} \in \mathrm{Hol}(A, X)$ which is compactly divergent on $V := A \setminus S$. For this, we pick sequence $\{x_k\}$ in X such that $\{x_k\}$ has no convergent subsequence. We also let f be an infinite Blaschke product associated to S. Then $f \in \mathrm{Hol}(\Delta, \Delta)$ and f vanishes exactly on S. Then $f_k(\lambda) := f(\lambda)x_k$ is the desired sequence.

The statement (ii) in the above theorem can be considerably strengthened in the special case $A = \Delta$ and X has 1-SVP.

Theorem 3.11. *Let X be Banach analytic manifolds. Then the following assertions are equivalent:*

(i) *X has* 1-SVP*;*
(ii) *Every sequence $\{f_k\} \in \mathrm{Hol}(\Delta, X)$ contains a subsequence which is either convergent in $\mathrm{Hol}(\Delta, X)$ or compactly divergent outside a discrete subset of Δ.*

Proof. (i) $\Rightarrow$ (ii). Assume that there exists no subsequence of $\{f_k\}$ which is convergent in $\mathrm{Hol}(\Delta, X)$. The following statement is the key to our proof:

> *For every relatively compact open subset G of Δ there exist a finite (possibly empty) set S of G and $N \subset \mathbb{N}$ such that the subsequence $\{f_k\}_{k\in N}$ is pointwise convergent on S and compactly divergent on $G \setminus S$.*

Indeed, if the entire sequence $\{f_k\}$ is compactly divergent on G then we can take $S := \emptyset$ and $N := \mathbb{N}$. Otherwise, by Theorem 3.1 X is hyperbolic, so we use Lemma 3.7 to find $N_1 \subset \mathbb{N}$, $a_1 \in G$ and a subsequence $\{f_k\}_{k\in N_1}$ such that $f_k(a_1)$ is convergent as $k \to \infty$, for $k \in N_1$. Now, if the above subsequence is compactly divergent on $G \setminus \{a_1\}$ then we can choose $S_1 := \{a_1\}$ and $N := N_1$. Otherwise, we may apply again Lemma 3.7 to get $a_2 \in G \setminus \{a_1\}$ and a further subsequence $\{f_k\}_{k\in N_2}$, where $N_2 \subset N_1$ is such that $f_k(a_2)$ is convergent as $k \to \infty$, for $k \in N_2$. We claim that this process cannot be infinite. Assume it is infinite, then we would get a sequence $\{a_l\} \subset G$ of distinct points, a collection of subsequences $\{f_k\}_{k\in N_l}$, where $N_{l+1} \subset N_l$ is such that $f_k(a_l)$ is convergent as $k \to \infty$, for $k \in N_l$ for every $l \geq 1$. Thus, using a diagonal process, we could obtain a subsequence $\{f_k\}_{k\in M}$ such that $f_k(a_l)$ is convergent for each $l \geq 1$ as $k \to \infty$, $k \in M$. After, passing to a subsequence we may assume that $a_l \to a \in \bar{G} \subset \Delta$. Thus, using 1-SVP of X we could infer that the sequence $\{f_k\}_{k\in M}$ is convergent in $\mathrm{Hol}(\Delta, X)$, a contradiction. Hence, the procedure described above must be finite. Thus, we have proved the desired statement. □

Next, we let $\{r_j\}_{j\geq 1}$ be a sequence of positive numbers with $r_j \uparrow 1$, such that $r_j > 1/j$. Set $\Delta_j := \Delta(0, r_j)$. We will prove by induction on j the following claim: There exist a *finite* (possibly empty) set $S_j \subset \Delta_j$, an open disk Δ'_j such that $\Delta(0, r_j - 1/j) \Subset \Delta'_j \Subset \Delta_j$ and a subsequence $\{f_k\}_{k\in N_j}$ of $\{f_k\}$ such that:

(a) $\{f_k\}_{k\in N_j}$ is compactly divergent on $\Delta_j \setminus S_j$;
(b) $N_{j+1} \subset N_j$;
(c) $S_j \subset S_{j+1}$, $S_{j+1} \setminus S_j \subset \Delta_{j+1} \setminus \bar{\Delta}'_j$.

For $j = 1$, we may apply the fact proved at the beginning of the proof to obtain a finite (possibly empty) set $S_1 \subset \Delta_1$ and $N_1 \subset \mathbb{N}$ such that $\{f_k\}_{k\in N_1}$ is compactly divergent on $\Delta_1 \setminus S_1$. Next, suppose that there exist finite subsets $S_1, \cdots, S_j$ of Δ and subsequences $\{f_k\}_{k\in N_1}, \cdots, \{f_k\}_{k\in N_j}$ that satisfy the conditions (a), (b) and (c). Then we choose a disk $\Delta'_j \Subset \Delta_j$ centered at 0 with radius $> r_j - 1/j$ such that $S_j \subset \Delta'_j$. Then by applying the preceding argument, this time, to $\{f_k\}_{k\in N_j}$ instead of the original one $\{f_k\}$ and G is replaced by the annulus $\Delta_{j+1} \setminus \bar{\Delta}'_j$, we obtain a subsequence $\{f_k\}_{k\in N_{j+1}}$, $N_{j+1} \subset N_j$ and a finite set $S'_j \subset \Delta_{j+1} \setminus \bar{\Delta}'_j$ such that $\{f_k\}_{k\in N_{j+1}}$ is compactly divergent on $\Delta_{j+1} \setminus (\bar{\Delta}'_j \cup S'_j)$. Then we set $S_{j+1} := S_j \cup S'_j$. Since

$$\Delta_{j+1} \setminus S_{j+1} = \big(\Delta_{j+1} \setminus \big(\bar{\Delta}'_j \cup S'_j\big)\big) \cup \big(\Delta_j \setminus S_j\big),$$

by Lemma 3.8, we infer that $\{f_k\}_{k\in N_{j+1}}$ is compactly divergent on $\Delta_{j+1} \setminus S_{j+1}$.

This completes the proof of our claim. Hence, in view of (a) and (b) we may apply a diagonal process to obtain a subsequence $\{f_k\}_{k\in I}$ which is compactly divergent on each domain $\Delta_j \setminus S_j$, $j \geq 1$. Finally, we set $S := \bigcup_{j\geq 1} S_j$. At this point, using (c) we can check that S is a discrete (possibly empty) subset of Δ. Moreover, since $\Delta \setminus S = \cup_{j\geq 1}(\Delta_j \setminus S_j)$, using Lemma 3.8, we deduce that $\{f_k\}_{k\in I}$ is compactly divergent on $\Delta \setminus S$. The desired conclusion now follows.

(ii) $\Rightarrow$ (i). Let $\{f_k\}$ be a sequence in $\mathrm{Hol}(\Delta, X)$ such that $Z^u_{\{f_k\}} \neq \emptyset$. Suppose that $\{f_k\}$ contains no subsequence which is convergent in $\mathrm{Hol}(\Delta, X)$. Then, there exists a subsequence $\{f_{k_j}\}$ which is compactly divergence outside a discrete subset S of Δ. It follows that $Z_{\{f_k\}} \subset Z_{\{f_{k_j}\}} \subset S$. Hence $Z_{\{f_k\}} = \emptyset$, a contradiction. Thus, $\{f_k\}$ contains a convergent subsequence in $\mathrm{Hol}(\Delta, X)$. It remains to show that any two accumulations points g and g' of this sequence must be identical. For this, it suffices to note that $g = g'$ on $Z_{\{f_k\}}$ and that $Z^u_{\{f_k\}} \neq \emptyset$.

Remarks. (a) In [6], a Banach analytic manifold with the property described in (ii) is termed *weakly taut*. Thus, Theorem 3.11 essentially generalizes (with a simpler proof) in [6, Theorem 4.1], since the latter result is proved in the case where X is a *finite* dimensional complex space.

(b) It was proved in [6, Theorem 3.4] that every complete hyperbolic Banach analytic manifold is weakly taut. This statement also follows from our Theorem 3.3 and Theorem 3.11. Notice that our proofs do not use Zorn's lemma as in [6].

Our next two results contain simple observations about inheritance of Vitali properties under inclusion.

Proposition 3.12. *Let X be an open subset of a Banach analytic manifold Y. Assume that X has* WVP *and Y has* SVP. *Then X has* SVP.

Proof. Let A be a Banach analytic manifold and $\{f_k\} \subset \mathrm{Hol}(A, X)$ be such that $Z^u_{\{f_k\}} \neq \emptyset$. Since Y has SVP we deduce that $\{f_k\}$ is convergent to $f \in \mathrm{Hol}(A, Y)$. Put $\Omega := f^{-1}(X)$. Then Ω is open and non-empty since $Z_{\{f_k\}} \subset \Omega$. Notice that $\Omega \subset Z_{\{f_k\}} \cap Z^u_{\{f_k\}}$. Since X has WVP, $\{f_k\}$ is convergent in $\mathrm{Hol}(A, X)$. Thus, X actually has SVP. □

Corollary 3.13. *Let X be an open subset of a complete hyperbolic Banach analytic manifold Y. Then X has* WVP *if and only if X has* SVP. *In particular,* WVP *and* SVP *are equivalent in the classes of bounded open subsets in Banach spaces.*

Proof. Observe that Y has SVP by Theorem 3.3. So the first assertion follows from Proposition 3.12. Finally, since every ball in a Banach space is complete hyperbolic, we get the last statement of the corollary. □

This section ends up with the following result which says roughly that 1-WVP is not much weaker than SVP in the class of (finite dimensional) complex manifolds.

Theorem 3.14. *Let X be a complex manifold. Then the following statements are equivalent:*

(i) *X has* 1-WVP (*and hence X has* WVP *by Theorem* 3.6)*;*
(ii) Hol(A, X) *is normal for every connected, locally separable Banach analytic manifold A;*
(iii) *X has* SVP *for source spaces A having the property described in* (ii).

Proof. (i) $\Rightarrow$ (ii). Assume that X has 1-WVP. Fix a connected, locally separable Banach analytic manifold A. Let $\{f_k\}$ be a sequence in Hol(A, X). Suppose that $\{f_k\}$ is not compactly divergent. Then, by Lemma 3.7, we can find a sequence $\lambda_j \to \lambda_0 \in A$ and a subsequence f_{k_j} such that $f_{k_j}(\lambda_j) \to x_0 \in X$ as $j \to \infty$. Let V be a neigborhood of x_0 which is isomorphic to some ball in an Euclidean space $\mathbb{C}^N$. By Theorem 3.1, X is hyperbolic, so using Lemma 3.6 (i), we can find a neigborhood U of λ_0 and $j_0 \geq 1$ such that

$$f_{k_j}(U) \subset V, \ \forall j \geq j_0.$$

Since A is locally separable, after shrinking U if necessary, we can find a countable dense subset Z_{λ_0} of U. By a diagonal process, we can find a further subsequence $\{f_{k_{j_l}}\}$ which is pointwise convergence on Z_{λ_0}. It follows that $\lambda_0 \in Z_{\{f_{k_{j_l}}\}} \cap Z^u_{\{f_{k_{j_l}}\}}$. Hence, $\{f_{k_{j_l}}\}$ is convergent in Hol(A, X).

(ii) $\Rightarrow$ (iii). Let A be a connected, locally separable Banach analytic manifold. Fix a sequence $\{f_k\}$ in Hol(A, X) such that $Z^u_{\{f_k\}} \neq \emptyset$. In particular, $\{f_k\}$ is pointwise convergence at some point of A. Since Hol(A, X) is normal, we infer that $\{f_k\}$ is relatively compact in Hol(A, X). Notice that any two accumulation points of the sequence $\{f_k\}$ must be identical on A in view of the assumption that $Z^u_{\{f_k\}} \neq \emptyset$. Therefore $\{f_k\}$ is convergent in Hol(A, X) as desired.

(iii) $\Rightarrow$ (i) follows by taking $A = \Delta$. □

Remarks. (a) In view of the implication (i) $\Rightarrow$ (ii), we infer that every complex manifold X having WVP is necessarily taut. In particular, X must be pseudoconvex at least in the case where it is a domain in $\mathbb{C}^n$ (see [7, Theorem 5.2.1]).

(b) The assumption on local separability of A cannot be omitted in the implication (i) $\Rightarrow$ (ii). To see this, we consider the case where A is the unit ball of l^∞ and $X = \Delta$. Then, we consider the sequence of (linear) projections

$$f_k : A \to X, \, f_k(\lambda) := \lambda_k, \ \lambda = (\lambda_1, \cdots, \lambda_k, \cdots), k \geq 1.$$

Since $\{f_k\}$ contains no subsequence which is *pointwise* convergence on A and since $\{f_k\}$ is convergent at the origin, we infer that Hol(A, X) is not normal.

4. Some classes of spaces having WVP and SVP

In this section we will investigate sufficient conditions so that a Banach analytic manifolds has Vitali properties. For this purpose, we introduce the following terminology.

Definition 4.1. An open subset Ω of a Banach analytic manifold X is said to have the quasi strong Vitali property (respectively quasi weak Vitali property) if for every connected Banach analytic manifold A and every sequence $\{f_k\} \subset \mathrm{Hol}(A, \Omega)$ with $Z^u_{\{f_k\}} \neq \emptyset$ (respectively with $Z_{\{f_k\}} \cap Z^u_{\{f_k\}} \neq \emptyset$), the sequence $\{f_k\}$ is convergent in $\mathrm{Hol}(A, X)$.

These properties will be abbreviated as QSVP (respectively QWVP).

Remarks. (a) Obviously, every open subset of a Banach analytic manifold with QSVP (respectively QWVP) also has this property.

(b) By Theorem 3.3, we know that every open subset of a complete hyperbolic open subset of a Banach analytic manifold has QSVP. In particular, since every ball in a Banach space is complete hyperbolic, we conclude that all open *bounded* subsets of a Banach space have QSVP.

(c) Each hyperbolic relatively compact open subset Ω of a complex manifold X has QWVP. For this, we let $\{f_k\}$ be a sequence in $\mathrm{Hol}(\Delta, \Omega)$ such that $Z^u_{\{f_k\}} \neq \emptyset$. Notice that, $\{f_k(z)\}$ is relatively compact in X for every $z \in \Delta$. Furthermore, since Ω is hyperbolic, the family $\{f_k\}$ is equicontinuous. By Arzela-Ascoli's theorem, $\{f_k\}$ is relatively compact in $\mathrm{Hol}(\Delta, X)$. By the assumption that $Z^u_{\{f_k\}} \neq \emptyset$, we deduce that two accumulation points of $\{f_k\}$ must coincide on Δ. This implies that $\{f_k\}$ converges to some $f \in \mathrm{Hol}(\Delta, X)$.

The first result of this section provides a class of Banach analytic manifolds having WVP. This is a reminiscence of the well known fact that every bounded hyperconvex domain in $\mathbb{C}^n$ is taut (see [9, Corollary 5] and [7, Proposition 5.2.2]).

Theorem 4.2. *Let X be a Banach analytic manifold and φ be a negative plurisubharmonic function on X. Then the following assertions are equivalent:*

(i) *X is hyperbolic and for every $c < 0$, the sublevel set*

$$X_c := \{z \in X : \varphi(z) < c\}$$

has QWVP*;*

(ii) *X has* WVP.

Proof. (ii) $\Rightarrow$ (i) follows directly from Theorem 3.1.

(i) $\Rightarrow$ (ii) By Theorem 3.8, it is enough to show that X has 1-WVP. Fix a sequence $\{f_k\} \subset \mathrm{Hol}(\Delta, X)$ such that $Z_{\{f_k\}} \cap Z^u_{\{f_k\}} \neq \emptyset$. We must show that $\{f_k\}$ is convergent in $\mathrm{Hol}(\Delta, X)$. Choose $\lambda_0 \in Z_{\{f_k\}} \cap Z^u_{\{f_k\}}$. By an argument as in the proof of Theorem 3.5, we can find open neigborhoods $U_0 \subset \Delta$ of λ_0 and $V_0 \subset X$

of $\lim_{k\to\infty} f_k(\lambda_0)$ and $k_0 \geq 1$ such that V_0 is isomorphic to a ball in some Banach space and that

$$f_k(U_0) \subset V_0, \forall k \geq k_0.$$

Using Lemma 3.5, we conclude that $\{f_k\}$ is convergent in $\mathrm{Hol}(U_0, X)$. Now we set

$$\Omega := \bigcup \{U \subset \Delta : \{f_k|_U\} \text{ is convergent in } \mathrm{Hol}(U, X)\}.$$

Clearly Ω is open and $U_0 \subset \Omega$ by the above proof. It suffices to show that Ω is closed. Assume this is not true, then we can find $\lambda_1 \in \partial\Omega$. Notice that $\{f_k\}$ converges to $f \in \mathrm{Hol}(\Omega, X)$. Now we set

$$\psi(z) := \sup_{k\geq k_0} (\varphi \circ f_k)(z), \quad \forall z \in \Delta.$$

By the assumption on φ we infer that the upper regularization ψ^* is non-positive and subharmonic on Δ. Furthermore, by the choice of U_0 we have

$$\sup_{U_0} \psi^* \leq \sup_{V_0} \varphi < 0.$$

Thus the maximum principle yields $\psi^* < 0$ entirely on Δ. In particular $\psi^*(\lambda_1) < 0$. Fix $c \in (\psi^*(\lambda_1), 0)$. Choose an open disk U_1 around λ_1 such that $\sup_{U_1} \psi^* < c$. It follows that

$$\sup_{U_1} \varphi \circ f_k < c, \ \forall k \geq k_0.$$

Therefore f_k map U_1 into X_c for every $k \geq k_0$. Notice that

$$\emptyset \neq U_1 \cap \Omega \subset U_1 \cap Z_{\{f_k\}} \cap Z^u_{\{f_k\}}.$$

Since X_c has QWVP, we deduce that the sequence $\{f_k\}$ is convergent in $\mathrm{Hol}(U_1, X)$. Thus $\lambda_1 \in \Omega$. This contradicts our choice that $\lambda_1 \in \partial\Omega$. The proof is therefore complete. □

Corollary 4.3. *Let X be a hyperbolic complex manifold. Assume that there exists a negative exhaustion function φ for X,* i.e., *$X_c := \{z \in X : \varphi(z) < c\}$ is relatively compact in X for every $c < 0$. Then X has* WVP.

Proof. By the remark (c) following Definition 4.1 and the assumption on hyperbolicity of X, we see that all sublevel sets X_c have QWVP. The desired conclusion now follows immediately from Theorem 4.1. □

The theorem below gives a sufficient condition to guarantee Vitali properties for open subsets of Banach analytic manifolds.

Theorem 4.4. *Let Y be an open subset of a Banach analytic manifold X. Assume that X has* WVP *and that there exists a negative plurisubharmonic function φ on Y such that $\lim_{z\to\xi} \varphi(z) = 0$ for every $\xi \in \partial Y$. Then Y has* WVP.

Proof. According to Theorem 3.5, it suffices to show that Y has 1-WVP. Fix a sequence $\{f_k\}_{k\geq 1} \subset \mathrm{Hol}(\Delta, Y)$ such that $Z_{\{f_k\}} \cap Z^u_{\{f_k\}} \neq \emptyset$. Then $\{f_k\}_{k\geq 1} \subset \mathrm{Hol}(\Delta, X)$. Thus $\{f_k\}$ is convergent to $f \in \mathrm{Hol}(\Delta, X)$, since X has WVP. The proof would be finished if we could show that $f(\Delta) \subset Y$. To prove this, we set

$$u(z) := \limsup_{k\to\infty}(\varphi \circ f_k)(z), \quad z \in \Delta.$$

Notice that $\Omega := f^{-1}(Y) = Z_{\{f_k\}}$ is an open non-empty subset of Δ. Moreover, the following statements are true:

(a) $u(z) \leq (\varphi \circ f)(z) < 0\ \forall z \in \Delta'$. This follows from upper semicontinuity of φ and the hypothesis that $\varphi < 0$ on Y;
(b) u satisfies the sub-mean value inequality, *i.e.*,

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u\big(z_0 + re^{i\theta}\big)d\theta, \quad \forall z_0 \in \Delta, \quad \forall r > 0 \text{ small enough.}$$

This is an easy consequence of Fatou's lemma.

The problem is to show $\Omega = \Delta$. Assume otherwise, then there exists $x_1 \in \Delta \cap \partial\Omega$. Then $f_k(x_1) \to f(x_1) \in \partial Y$ as $k \to \infty$. This implies that

$$u(x_1) = \limsup_{k\to\infty}(\varphi \circ f_k)(x_1) = 0.$$

Here the last equality follows from the assumption that $\lim_{\xi\to f(x_1)} \varphi(\xi) = 0$. Now we choose $r > 0$ so small such that the closed disk $\bar{\Delta}(x_1, r)$ is included in Δ. Thus

$$\partial\Delta(x_1, r) \cap \Omega \neq \emptyset.$$

It follows that there exist $\theta_0 \in (0, 2\pi)$ and $\delta > 0$ such that

$$x_1 + re^{i\theta} \in \Omega\ \forall\theta \in (\theta_0 - \delta, \theta_0 + \delta).$$

By (b) we obtain

$$0 = u(x_1) \leq \frac{1}{2\pi}\int_0^{2\pi} u\big(x_1 + re^{i\theta}\big)d\theta \leq \frac{1}{2\pi}\int_{\theta_0-\delta}^{\theta_0+\delta} u\big(x_1 + re^{i\theta}\big)d\theta < 0.$$

The last inequality follows from (a). This is contradictory. Thus $\Omega = \Delta$. Hence $f(\Delta) \subset Y$. □

Now we consider conditions that imply SVP of open subsets of a Banach space. This result will be used to characterize SVP of balanced domain in Banach spaces.

Theorem 4.5. *Let Ω be an open subset of a Banach space X. Assume that there exists a negative plurisubharmonic function φ on Ω such that*

$$\lim_{z\to\xi} \varphi(z) = 0, \ \forall\xi \in \partial\Omega.$$

Then the following assertions are equivalent:

(i) Ω *is hyperbolic and for every* $c < 0$ *the open set* $\Omega_c := \{z \in \Omega : \varphi(z) < c\}$ *has* QSVP*;*
(ii) Ω *has* SVP.

Proof. (ii) $\Rightarrow$ (i) follows from Theorem 3.1.

(i) $\Rightarrow$ (ii). Fix a connected Banach analytic manifold A. Let $\{f_k\}$ be a sequence in $\mathrm{Hol}(A, X)$ such that $Z^u_{\{f_k\}} \neq \emptyset$. Fix $\lambda_0 \in Z_{\{f_k\}}$. Then we have $f_k(\lambda_0) \to x_0 \in X$. By upper-semicontinuity of φ, we can choose a neigborhood V of $x_0 \in X$ such that $\sup_V \varphi < 0$. Using hyperbolicity of X, by Lemma 3.6(i), we can find a neigborhood U of λ_0 in A and $k_0 \geq 1$ such that

$$f_k(U) \subset V \ \forall k \geq k_0.$$

Set

$$\psi(z) := \sup_{k \geq k_0} (\varphi \circ f_k)(z), \ \forall z \in A.$$

Since φ is negative and plurisubharmonic on A, we infer that the upper regularization ψ^* is also plurisubharmonic on A and $\psi^* \leq 0$ there. Moreover, by the choice of U and V we have

$$\psi^*(\lambda_0) \leq \sup_V \varphi < 0.$$

Hence the maximum principle yields $\psi^* < 0$ entirely on A. Next, we choose an arbitrary point $\lambda_1 \in Z^u_{\{f_k\}}$. We claim that f_k is uniformly bounded on a small open neigborhood of λ_1. To see this, choose $c \in \mathbb{R}$ such that $\psi^*(\lambda_1) < c < 0$. Then, there exists a open neigborhood W of λ_1 such that $\sup_W \psi^* < c$. It follows that $f_k(W) \subset \Omega_c$ for every $k \geq k_0$. Since Ω_c has QSVP, we can find an open subset Y of X that contains Ω_c such that Y has SVP. Hence, the sequence $\{f_k\big|_W\}$ is convergent in $\mathrm{Hol}(W, Y)$. Thus f_k must be locally uniformly bounded near λ_1 for $k \geq k_0$. The claim is proved. It means that we can find an open neigborhood $\mathbb{B}$ of λ_1 on which f_k is uniformly bounded for $k \geq k_0$. Now, we use Lemma 3.4 to conclude that $\{f_k\}$ is convergent to f in $\mathrm{Hol}(\mathbb{B}, X)$. We claim that $f(\mathbb{B}) \subset \Omega$. Assume this is not true, then there exists $x_0 \in \mathbb{B}$ such that $f(x_0) \in \partial\Omega$. Since ψ tends to 0 at $f(x_0)$ we must have $\psi(x_0) = 0$. This is a contradiction. Thus $f(\mathbb{B}) \subset \Omega$ as desired. Now we let

$$A' := \bigcup \{U \subset A : \{f_k|_U\} \text{ is convergent in } \mathrm{Hol}(U, \Omega)\}.$$

Clearly A' is open, and by the above reasoning A' is also non-empty. It remains to check that A' is closed. Assume the contrary holds, then there exists $\lambda_2 \in \partial A'$. Then $\lambda_2 \in Z^u_{\{f_k\}}$. Repeating the preceding argument, we see that there exists a small ball $\mathbb{B}'$ around λ_2 such that f_k is uniformly convergent in $\mathrm{Hol}(\mathbb{B}', \Omega)$. Hence $\lambda_2 \in A'$. This is impossible. The proof is therefore complete. ☐

We now discuss Vitali properties for special classes of Banach analytic manifolds in the rest of this section. The first objects to consider are Hartogs domains over Banach analytic manifolds.

Recall that, given a Banach analytic manifold X and an upper semicontinuous function $\varphi : X \to [-\infty, \infty)$, the Hartogs domain $\Omega_\varphi(X)$ is defined as

$$\Omega_\varphi(X) := \left\{(z, w) \in X \times \mathbb{C} : |w| < e^{-\varphi(z)}\right\}.$$

The next result relates Vitali properties of a Hartogs domain and those of its base and radii of fibers.

Proposition 4.6. *The Hartogs domain $\Omega_\varphi(X)$ has* SVP (*respectively* WVP) *if and only if φ is continuous plurisubharmonic on X and X has* SVP (*respectively* WVP).

Remark. Thus, if $X = \Delta$ and φ is bounded subharmonic but not continuous on Δ then $\Omega_\varphi(\Delta)$ is a bounded pseudoconvex domain in $\mathbb{C}^2$ without having WVP.

Proof. We only give the proof for the SVP case, the other case is similar and somewhat simpler.

($\Rightarrow$). Suppose that $\Omega_\varphi(X)$ has SVP. First we check that X has SVP. For this, let A be a connected Banach analytic manifold and $\{f_k\} \in \mathrm{Hol}(A, X)$ be such that $Z^u_{\{f_k\}} \neq \emptyset$. Set $f'_k := (f_k, 0)$. It is then clear that $\{f'_k\} \in \mathrm{Hol}(A, \Omega_\varphi(X))$. Moreover, $Z^u_{\{f'_k\}} \neq \emptyset$. It follows, using SVP of $\Omega_\varphi(X)$ that $\{f'_k\}$ converges in $\mathrm{Hol}(A, \Omega_\varphi(X))$. Thus, so does $\{f_k\}$. Hence X has SVP.

Now we prove continuity of φ. Assume that φ is discontinuous at $x^* \in X$. Then, since φ is upper semicontinuous, we can find a sequence $x_k \to x^*$ and $s \in \mathbb{R}$ such that

$$\varphi(x_k) \leq s < \varphi(x^*), \ \forall k.$$

Next, we set $r := e^{-s}$ and define a sequence $\{g_k\} \subset \mathrm{Hol}(\Delta, \Omega_\varphi(X))$ by

$$g_k(\lambda) := (x_k, r\lambda), \ \forall \lambda \in \Delta.$$

If $|\lambda| < \delta := r^{-1} e^{-\varphi(x^*)}$ then $|\lambda| r < e^{-\varphi(x^*)}$. Hence,

$$g_k(\lambda) \to g(\lambda) := (x_0, r\lambda) \in \Omega_\varphi(X),$$

for $\lambda \in \delta\Delta$. In particular, $0 \in Z^u_{\{f_k\}}$. Since $\Omega_\varphi(X)$ has SVP, the sequence $\{g_k\}$ must converge to $\tilde{g} \in \mathrm{Hol}(\Delta, \Omega_\varphi(X))$. By the above reasoning, f coincides with $\tilde{f}$ on $\delta\Delta$. By uniqueness property of holomorphic maps from Δ to $X \times \mathbb{C}$, we infer that $\tilde{g} = (x_0, r\lambda)$ for all $\lambda \in \Delta$. Hence $(x_0, r\lambda) \in \Omega_\varphi(X)$ for all $\lambda \in \Delta$. This yields a contradiction to the choice of r. Thus φ is continuous on X. It remains to prove that φ is plurisubharmonic on X. To see this, it is enough to show that φ is plurisubharmonic on every open set U which is isomorphic to an open subset of a Banach space. Fix such an open set U and let $\theta : \Delta \to U$ be an arbitrary holomorphic map. It suffices to show that the continuous function $u := \varphi \circ \theta :$

$\Delta \to \mathbb{R}$ is subharmonic. Assume the contrary holds, then we can find a closed disk $\Delta' \subset \Delta$ and a holomorphic polynomial p in $\mathbb{C}$ such that

$$u \le \Re p \text{ on } \partial\Delta' \text{ whereas } \varepsilon := \sup_{x\in\Delta'} (u(x) - \Re p(x)) > 0.$$

For $k \ge 1$, we define

$$h_k(\lambda) := \left(\theta(\lambda), e^{-p(\lambda)-\varepsilon-\frac{1}{j}}\right) \ \forall\lambda \in \Delta'.$$

By the choice of ε we can check that $h_k \in \mathrm{Hol}(\Delta', \Omega_\varphi(X))$. Furthermore, we also note

$$h_k(\lambda) \to h(\lambda) := \left(\theta(\lambda), e^{-p(\lambda)-\varepsilon}\right), \ \forall\lambda \in \Delta',$$

with $h \in \mathrm{Hol}(\Delta', X \times \mathbb{C})$. Now we choose $\alpha \in (0, 1)$ such that $\Re p(\lambda) + \varepsilon > u(\lambda)$ if $\alpha \in V_\alpha := \Delta \setminus \Delta(0, \alpha)$. It follows that $h(V_\alpha) \subset \Omega_\varphi(X)$. Since $\Omega_\varphi(X)$ has SVP we deduce that h_k converges to $\tilde{h} \in \mathrm{Hol}(\Delta', \Omega_\varphi(X))$. Since $h = \tilde{h}$ on V_α, using again the uniqueness property of holomorphic maps from Δ' to $X \times \mathbb{C}$ we obtain $h = \tilde{h}$ on Δ'. This implies that $\Re p(\lambda)+\varepsilon > u(\lambda)$ for *all* $\lambda \in \Delta'$. This contradiction to the choice of ε proves plurisubharmonicity of φ on X.

($\Leftarrow$). Assume that φ is continuous plurisubharmonic on X and X has SVP. Fix a connected Banach analytic manifold A and a sequence $\{f_k\} \in \mathrm{Hol}(A, \Omega_\varphi(X))$ satisfying $Z^u_{\{f_k\}} \neq \emptyset$. We write $f_k = (g_k, h_k)$, where $g_k \in \mathrm{Hol}(A, X)$ and $h_k \in \mathrm{Hol}(A, \mathbb{C})$. Then we have

$$Z^u_{\{g_k\}} \neq \emptyset, \quad Z^u_{\{h_k\}} \neq \emptyset.$$

Since X has SVP, we deduce that $\{g_k\}$ converges to $g \in \mathrm{Hol}(A, X)$. Notice also that

$$|h_k(\lambda)| < e^{-\varphi(g_k(\lambda))}, \quad \forall\lambda \in A, \quad \forall j \ge 1.$$

It follows that the sequence $\{h_k\}$ is uniformly bounded on compact sets of A. By Lemma 3.5, we infer that $\{h_k\}$ is convergent to $h \in \mathrm{Hol}(A, \mathbb{C})$, which implies that

$$|h(\lambda)| \le e^{-\varphi(g(\lambda)}, \quad \forall\lambda \in A.$$

The above inequality can be rewritten as follows:

$$f(\lambda) := \log|h(\lambda)| + \varphi(g(\lambda)) \le 0, \quad \forall\lambda \in A.$$

Since g, h are holomorphic functions on A and since φ is plurisubharmonic on X we infer that f is plurisubharmonic on A. Moreover, $f < 0$ on the non-empty set $Z_{\{f_k\}}$. It follows, using the maximum principle, that $f(\lambda) < 0$ for every $\lambda \in A$. Therefore

$$|h(\lambda)| < e^{-\varphi(g(\lambda)}, \quad \forall\lambda \in A.$$

Thus $\{f_k\}$ converges to $(g, h) \in \mathrm{Hol}(A, \Omega_\varphi(X))$. Hence $\Omega_\varphi(X)$ has SVP as desired. □

The next result deals with Vitali properties of balanced domains in Banach spaces.

Definition 4.7. A domain Ω in a Banach space E is said to be balanced if $x \in \Omega$ then λx in Ω for every $\lambda \in \Delta$.

For a balanced domain Ω, since $0 \in \Omega$, we may define the gauge (or Minkowski) functional of Ω as follows

$$h_\Omega(x) := \inf\{\lambda > 0 : x \in \lambda X\}, \quad x \in E.$$

It is clear that h_Ω is *homogeneous*, *i.e.*, $h(\lambda x) = |\lambda| h(x)$ and, since Ω is a domain, h_Ω is upper semicontinuous and

$$\Omega = \{x \in E : h_\Omega(x) < 1\}.$$

We are now able to formulate the final result of this section.

Proposition 4.8. *Let Ω be a balanced domain in a Banach space E. Then the following statements are equivalent:*

(i) Ω *has* WVP*;*
(ii) Ω *is bounded and h_Ω is continuous on E and plurisubharmonic on Ω;*
(iii) Ω *has* SVP.

Proof. (i) $\Rightarrow$ (ii). If Ω has WVP then by Theorem 3.1, Ω is hyperbolic. Thus using the same argument as in the beginning of the proof of [6, Theorem 6.1] we conclude that Ω is bounded. Next, we show that $\log h_\Omega$ is plurisubharmonic on Ω. For this, since h_Ω is upper semicontinuous on Ω, it suffices to show that for every choice $a, b \in E$ the function

$$\lambda \mapsto u(\lambda) := \log h_\Omega(a + \lambda b)$$

is subharmonic on the open set $\Omega_{a,b} \subset \mathbb{C}$ where it is defined. If this is not the case then we can find a closed disk $\Delta' \subset \Omega_{a,b}$ and a holomorphic polynomial p in $\mathbb{C}$ such that

$$u \leq \Re p \text{ on } \partial\Delta' \text{ whereas } \varepsilon := \sup_{x \in \Delta'} (u(x) - \Re p(x)) > 0.$$

For $j \geq 1$, we define

$$f_k(\lambda) := \frac{a + \lambda b}{e^{p(\lambda)+\varepsilon_0+1/j}} \; \forall \lambda \in \Delta'.$$

By the choice of ε we can check that $f_k \in \mathrm{Hol}(\Delta', \Omega)$. Furthermore, we also note

$$f_k(\lambda) \to f(\lambda) := \frac{a + \lambda b}{e^{p(\lambda)+\varepsilon_0+1/j}} \; \forall \lambda \in \Delta'.$$

By the choice of p, we see that there exists an open neigborhood V of $\partial\Delta'$ such that $f(V) \subset \Omega$. Since Ω has WVP we deduce that f_k converges to $\tilde{f} \in \mathrm{Hol}(\Delta', \Omega)$. Since $f = \tilde{f}$ on V, using again the uniqueness property of holomorphic maps from Δ' to E we obtain $f = \tilde{f}$ on Δ'. This implies that $\Re p(\lambda) + \varepsilon > u(\lambda)$ for *all* $\lambda \in \Delta'$. This contradiction to the choice of ε proves plurisubharmonicity of $\log h_\Omega$ on $\Omega_{a,b}$. Thus $\log h_\Omega$ and hence h_Ω is plurisubharmonic on Ω. It remains to check the continuity of h_Ω on E. Suppose h_Ω is discontinuous at $x^* \in E$. Then, since h_Ω is upper semicontinuous at x^*, we can find a sequence $\{x_k\} \to x^*$ and $s > 0$ such that

$$h_\Omega(x_k) < s < h_\Omega(x^*), \quad \forall k \geq 1.$$

For $k \geq 1$, we define

$$g_k(\lambda) := \frac{\lambda}{s} x_k, \quad \forall \lambda \in \Delta.$$

By the choice of s, we have $g_k \in \mathrm{Hol}(\Delta, X)$. Moreover,

$$g_k(\lambda) \to g(\lambda) := \frac{\lambda}{s} x^*, \quad \text{as } k \to \infty, \quad \forall \lambda \in \Delta.$$

Since X contains a neigborhood of 0, there exists $\delta > 0$ such that $g(\Delta(0, \delta)) \subset X$. It follows that $\Delta(0, \delta) \subset Z_{\{g_k\}}$. Since X has WVP, we infer that $g(\Delta)$ must be included in X. Hence, $h_\Omega(x^*) \leq s$. This contradiction proves the continuity (on X) of h_Ω.

(ii) $\Rightarrow$ (i). Suppose that Ω is bounded and h_Ω is continuous plurisubharmonic on Ω. Let $\varphi := h_\Omega - 1$. Then φ is negative plurisubharmonic on Ω. Moreover, fix $\xi \in \partial\Omega$, since h_Ω is continuous at ξ we infer that $\lim_{z\to\xi} \varphi(z) = 0$. Notice also that, being a bounded domain in a Banach space, Ω has QSVP. Therefore, we may apply Theorem 4.4 to obtain that Ω has WVP.

(ii) $\Rightarrow$ (iii). By the above implication Ω has WVP. In view of Corollary 3.13, Ω has SVP.

(iii) $\Rightarrow$ (i) is trivial.

The proof is thereby completed. □

5. Open questions

Before concluding this paper, we wish to point out a few questions that are left open by our methods.

1. Is there a Banach analytic manifold with WVP but without SVP? We conjecture that such a Banach analytic manifold exists.

2. Is there any analogue of Theorem 3.11 in the case where X has SVP, *i.e.*, the sequence $\{f_k\}$ is completely divergent outside a set which is locally contained in *an analytic hypersurface*?

3. Using Proposition 4.5, we see that the Hartogs domain $\Omega_\varphi(\Delta)$ is *unbounded* and has SVP if φ is continuous, subharmonic and satisfies $\inf_\Delta \varphi = -\infty$. Is there any substantial class of unbounded domains (in Banach spaces) having WVP and SVP? More precisely, can we describe WVP and SVP of an unbounded domain in terms of the existence of peak plurisubharmonic functions at finite and *infinite* boundary points? In this direction, we may refer the reader to the recent work [3] where the authors construct explicitly unbounded domains in Banach spaces having SVP.

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