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Spatial concavity of solutions to parabolic systems

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Abstract. We investigate spatial log-concavity and spatial power concavity of solutions to parabolic systems with log-concave or power concave initial data in convex domains.

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1. Introduction

In a series of previous papers [18–22], two of the present authors investigated concavity properties of solutions to parabolic equations with respect to space and time variables, introducing also the notion of parabolic concavity. In a recent paper [15], the authors of this paper treated weakly coupled parabolic systems with vanishing initial data and investigated again concavity properties with respect to time and space variables. In this paper we study spatial concavity properties of solutions to parabolic systems with non vanishing initial data.

Concavity properties of solutions to elliptic and parabolic problems are a classical subject of research and have been largely investigated. Here we just refer the reader to the classical monograph by Kawohl [25] and to the papers [1–8,10,12,14–24,26–37], some of which are closely related to this paper and the others include recent developments in this area. However very little is known concerning concavity properties of solutions to elliptic and parabolic systems and the only available results to our knowledge are in [15], which treats power concavity properties with respect to time and space variables for weakly coupled parabolic systems with vanishing initial data. Unfortunately, in order to be able to take account of the time variable, the arguments in [15] are not applicable to the case of non vanishing ini-

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tial data. To our knowledge, this paper is the first one dealing with spatial concavity properties of solutions to parabolic systems with non vanishing initial data.

Let Ω be a bounded convex domain in $\mathbf{R}^N$ $(N \geq 1)$, $D := \Omega \times (0, \infty)$ and $m \in \{1, 2, \dots\}$. We denote by $\mathbf{S}^N$ the space of real $N \times N$ symmetric matrices. Let

$$\mathbf{u} = \big(u^{(1)}, \dots, u^{(m)}\big) \in C^{2,1}(D : \mathbf{R}^m) \cap C(\overline{D} : \mathbf{R}^m)$$

satisfy the parabolic system

$$\begin{cases} \partial_t u^{(k)} + F^{(k)}\big(x, t, \mathbf{u}, \nabla u^{(k)}, \nabla^2 u^{(k)}\big) = 0 & \text{in} \quad D \qquad k = 1, \dots, m \\ u^{(k)}(x,t) > 0 & \text{in} \quad D \qquad k = 1, \dots, m \\ \mathbf{u}(x,t) = 0 & \text{on} \quad \partial\Omega \times [0, \infty) \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & \text{in} \quad \Omega\,, \end{cases} \tag{1.1}$$

where $\mathbf{u}_0 = (u_0^{(1)}, \dots, u_0^{(m)}) \in C(\overline{\Omega} : [0, \infty)^m)$ and

$$u_0^{(j)} > 0 \ \text{ in } \Omega \qquad u_0^{(j)} = 0 \ \text{ on } \ \partial\Omega \ \text{ for } \ j = 1, \dots, m\,.$$

Throughout this paper we assume the following conditions on $\mathbf{F} = (F^{(1)}, \dots, F^{(m)})$:

(A1) $\mathbf{F} = (F^{(1)}, \dots, F^{(m)}) \in C(D \times \mathbf{R}^m \times \mathbf{R}^N \times \mathbf{S}^N : \mathbf{R}^m)$;
(A2) For each $k \in \{1, \dots, m\}$, $F^{(k)}$ is a degenerate elliptic operator, that is $F^{(k)}(x, t, u, \theta, \cdot)$ is non-increasing in $\mathbf{S}^N$ for every fixed $(x, t, u, \theta) \in D \times \mathbf{R}^m \times \mathbf{R}^N$.

Here we refine the technique developed in [15,20] and [22] and investigate spatial concavity properties of the solution $\mathbf{u}$ under conditions (A1) and (A2). Our approach is based on the construction of the spatially concave envelope of the solution and the viscosity comparison principle, and it is different from those of [11,13,24] and [28–35] treating spatial concavity properties of the solutions to parabolic equations.

We state our main theorems in Section 4. Here we state a result on the spatial log-concavity of solutions to parabolic systems which directly descends from them.

Theorem 1.1. *Let Ω be a bounded convex domain in $\mathbf{R}^N$ and $d_1, d_2 > 0$. Let $(u, v) \in C^{2,1}(D : \mathbf{R}^2) \cap C(\overline{D} : \mathbf{R}^2)$ satisfy*

$$\begin{cases} \partial_t u - d_1 \Delta u + f(x, t, u, v, \nabla u) = 0 & \textit{in} \quad D \\ \partial_t v - d_2 \Delta v + g(x, t, u, v, \nabla v) = 0 & \textit{in} \quad D \\ u, v \geq 0 & \textit{in} \quad D \\ u(x,t) = v(x,t) = 0 & \textit{on} \quad \partial\Omega \times [0, \infty) \\ u(x,0) = u_0(x) \qquad v(x,0) = v_0(x) & \textit{in} \quad \Omega\,, \end{cases} \tag{1.2}$$

where f and g are nonnegative continuous functions in $D \times [0, \infty)^2 \times \mathbf{R}^N$. Assume the following conditions:

(i) *The viscosity comparison principle holds for system* (1.2)*;*
(ii) *The functions*

$$\mathfrak{f}_{t,\theta}(x, r, s) := e^{-r} f(x, t, e^r, e^s, e^r\theta)$$
$$\text{and} \quad \mathfrak{g}_{t,\theta}(x, r, s) := e^{-s} g(x, t, e^r, e^s, e^s\theta)$$

are convex in $\Omega \times (0, +\infty)^2$ for every fixed $t > 0$ and $\theta \in \mathbf{R}^N$.

Then $\log u(\cdot, t)$ and $\log v(\cdot, t)$ are concave in Ω for every fixed $t \in [0, \infty)$, provided that $\log u_0$ and $\log v_0$ are concave in Ω.

For the viscosity comparison principle for parabolic systems, see Section 4. As a corollary of Theorem 1.1, we have:

Corollary 1.2. *Let Ω be a bounded convex domain in $\mathbf{R}^N$ and $d_1, d_2 > 0$. Let $(u, v) \in C^{2,1}(D : \mathbf{R}^2) \cap C(\overline{D} : \mathbf{R}^2)$ satisfy*

$$\begin{cases} \partial_t u - d_1 \Delta u + v|\nabla u|^a + c_1 u = 0 & \text{in} \quad D \\ \partial_t v - d_2 \Delta v + u|\nabla v|^b + c_2 v = 0 & \text{in} \quad D \\ u, v \geq 0 & \text{in} \quad D \\ u(x, t) = v(x, t) = 0 & \text{on} \quad \partial\Omega \times [0, \infty) \\ u(x, 0) = u_0(x) \qquad v(x, 0) = v_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where $a \geq 0$, $b \geq 0$, $c_1 > 0$ and $c_2 > 0$. Then $\log u(\cdot, t)$ and $\log v(\cdot, t)$ are concave in Ω for any fixed $t \in [0, \infty)$, provided that $\log u_0$ and $\log v_0$ are concave in Ω.

Next we state a result on the power concavity for porous medium systems.

Theorem 1.3. *Let Ω be a bounded convex domain in $\mathbf{R}^N$ and $d_1, d_2 > 0$. Let $(u, v) \in C^{2,1}(D : \mathbf{R}^2) \cap C(\overline{D} : \mathbf{R}^2)$ satisfy*

$$\begin{cases} \partial_t u - d_1 \Delta (u^\alpha) + f(v) = 0 & \text{in} \quad D \\ \partial_t v - d_2 \Delta (v^\beta) + g(u) = 0 & \text{in} \quad D \\ u, v > 0 & \text{in} \quad D \\ u(x, t) = v(x, t) = 0 & \text{on} \quad \partial\Omega \times [0, \infty) \\ u(x, 0) = u_0(x) \quad v(x, 0) = v_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

where $\alpha, \beta > 1$. *Assume the following:*

(i) *The viscosity comparison principle holds for system* (1.4)*;*
(ii) *The functions*

$$\mathfrak{f}(\xi,\eta) := \xi^{\frac{\alpha-3}{\alpha-1}} f\left(\eta^{\frac{2}{\beta-1}}\right) \quad \textit{and} \quad \mathfrak{g}(\xi,\eta) := \eta^{\frac{\beta-3}{\beta-1}} g\left(\xi^{\frac{2}{\alpha-1}}\right)$$

are convex with respect to $(\xi,\eta) \in (0,\infty)^2$.

Let $p := (\alpha-1)/2$ *and* $q := (\beta-1)/2$. *Then* $u(\cdot,t)^p$ *and* $v(\cdot,t)^q$ *are concave in* Ω *for any* $t > 0$, *provided that* u_0^p *and* v_0^q *are concave in* Ω.

For sufficient conditions for the concavity of the functions $\mathfrak{f} = \mathfrak{f}(\xi,\eta)$ and $\mathfrak{g} = \mathfrak{g}(\xi,\eta)$, see, *e.g.*, [18, Lemma A.1].

The paper is organized as follows. In Section 2 we introduce some notation and recall basic properties of concave functions. In Section 3 we recall some basic viscosity theory for systems and prove a technical lemma. Furthermore, we give a uniqueness result for parabolic systems (see Theorem 3.6) which is enough for the purposes of the next section. In Section 4 we state and prove the main results of this paper, see Theorems 4.1 and 4.3, which are general results on power concavity and log-concavity of solutions to problem (1.1). Theorem 1.1 is a corollary of Theorem 4.3. In Section 5 we apply Theorem 4.1 to the porous medium equation and related systems and prove Theorem 1.3.

2. Preliminaries

Throughout the paper, let N and n be natural numbers and let $\mathbf{S}^N$ denote the space of $N \times N$ real symmetric matrices. If $A, B \in \mathbf{S}^N$, by $A \geq 0$ we mean that A is non-negative definite, while $A \geq B$ means $A - B \geq 0$. For $n \in \{2,3,\dots\}$, we set

$$\Lambda_n := \left\{ \lambda = (\lambda_1,\dots,\lambda_n) \,:\, 0 \leq \lambda_i \leq 1 \;\; (i = 1,\dots,n), \;\; \sum_{i=1}^n \lambda_i = 1 \right\}.$$

For any $r = (r^{(1)},\dots,r^{(n)})$ and $s = (s^{(1)},\dots,s^{(n)}) \in \mathbf{R}^n$, we write

$$r \leq s \quad \text{if } r^{(k)} \leq s^{(k)} \quad \text{for each } k = 1,\dots,n.$$

For any $a = (a_1,\dots,a_n) \in (0,\infty)^n$, $\lambda \in \Lambda_n$ and $p \in [-\infty, +\infty]$, we set

$$\mathbf{M}_p(a;\lambda) := \begin{cases} \left[\lambda_1 a_1^p + \lambda_2 a_2^p + \cdots + \lambda_n a_n^p\right]^{1/p} & \text{if} \quad p \neq -\infty,\ 0,\ +\infty \\ \max\{a_1,\dots,a_n\} & \text{if} \quad p = +\infty \\ a_1^{\lambda_1} \cdots a_n^{\lambda_n} & \text{if} \quad p = 0 \\ \min\{a_1,a_2,\dots,a_n\} & \text{if} \quad p = -\infty, \end{cases}$$

which is the (λ-*weighted*) p-*mean* of a. For $a = (a_1, \dots, a_n) \in [0,\infty)^n$, we define $\mathbf{M}_p(a;\lambda)$ as above if $p \geq 0$ and $\mathbf{M}_p(a;\lambda) = 0$ if $p < 0$ and $\prod_{i=1}^n a_i = 0$. Notice that $\mathbf{M}_p(a;\lambda)$ is a continuous function of the argument a. In the case $n = 2$, for simplicity, we write

$$M_p(a, b; \mu) := \mathbf{M}_p((a, b); (1-\mu, \mu))$$

for $a, b \in [0,\infty)$, $\mu \in [0,1]$ and $p \in [-\infty, \infty]$.

Due to the Jensen inequality, we have

$$\mathbf{M}_p(a;\lambda) \leq \mathbf{M}_q(a;\lambda) \quad \text{if} \quad -\infty \leq p \leq q \leq \infty, \tag{2.1}$$

for any $a \in [0,\infty)^n$ and $\lambda \in \Lambda_n$. Moreover, it easily follows that

$$\lim_{p\to+\infty} \mathbf{M}_p(a;\lambda) = \max\{a_1, \dots, a_n\} \qquad \lim_{p\to-\infty} \mathbf{M}_p(a;\lambda) = \min\{a_1, \dots, a_n\}$$

and $\lim_{p\to 0} \mathbf{M}_p(a;\lambda) = \mathbf{M}_0(a;\lambda)$.

We recall the definition of p-concavity of nonnegative functions in convex sets.

Definition 2.1. Let K be a convex set in $\mathbf{R}^N$, $Q := K \times (0,\infty)$ and $p \in [-\infty,\infty]$. A nonnegative function v is said *spatially p-concave* in Q if, for every fixed $t > 0$,

$$v\big((1-\lambda)x_1 + \lambda x_2, t\big) \geq M_p\big(v(x_1,t), v(x_2,t); \lambda\big)$$

for all $x_1, x_2 \in K$ and $\lambda \in (0,1)$.

Roughly speaking, v is spatially p-concave in Q if

- Case $p = \infty$: for every fixed $t > 0$, $v(\cdot, t)$ is a nonnegative constant function in K;
- Case $p > 0$: for every fixed $t > 0$, $v(\cdot, t)^p$ is concave in K;
- Case $p = 0$: for every fixed $t > 0$, $\log v(\cdot, t)$ is concave in K;
- Case $p < 0$: for every fixed $t > 0$, $v(\cdot, t)^p$ is convex in K;
- Case $p = -\infty$: for every fixed $t > 0$, the level sets $\{x \in K : v(x,t) > d\}$ are convex for every $d \geq 0$.

Then the following hold (see, *e.g.*, [28]):

(a) Let K be a convex set in $\mathbf{R}^N$, $Q := K \times (0,\infty)$ and $-\infty \leq p \leq \infty$. Due to Definition 2.1 and (2.1), if v is spatially p-concave in Q, then v is spatially q-concave in Q for any $-\infty \leq q \leq p$;

(b) Let $\{v_j\}_{j\in\mathbb{N}}$ be nonnegative functions in Q such that, for every $j \in \mathbf{N}$, v_j is spatially p_j-concave in Q for some $p_j \in [-\infty,\infty]$. Let v be the pointwise limit of the sequence v_j in Q and $\lim_{j\to\infty} p_j = p \in [-\infty,\infty]$. If v is continuos with respect to the time variable, then v is spatially p-concave in Q;

(c) Let $p, q \in [0,\infty]$. If v and w are spatially p-concave and q-concave in Q, respectively, then vw is spatially r-concave in Q, where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

3. Viscosity solutions of parabolic systems

In this section we recall the definition of viscosity solutions of elliptic and parabolic systems and some basic related notions and properties. Furthermore, we establish a comparison principle for viscosity solutions of (1.1).

Let Ω be a bounded convex domain in $\mathbf{R}^N$ ($N \geq 1$) and $T > 0$. For any function w in $D_T := \Omega \times (0, T)$, we denote the semi-jets $\mathcal{P}^{2,\pm}w(x,t)$ of w at $(x,t) \in D_T$ by

$$\mathcal{P}^{2,+}w(x,t) := \Big\{(a,\theta,X) \in \mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^n : w(y,s) \leq w(x,t) + a(s-t) + \langle\theta,(y-x)\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|x-y|^2 + |t-s|) \text{ as } D_T \ni (y,s) \to (x,t)\Big\},$$

$$\mathcal{P}^{2,-}w(x,t) := \Big\{(a,\theta,X) \in \mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^n : w(y,s) \geq w(x,t) + a(s-t) + \langle\theta,(y-x)\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o(|x-y|^2 + |t-s|) \text{ as } D_T \ni (y,s) \to (x,t)\Big\}.$$

Furthermore, we define the closures of semi-jets by

$$\overline{\mathcal{P}}^{2,\pm}w(x,t) := \Big\{(a,\theta,X) \in \mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^n : \text{there exists a sequence } \{(x_j,t_j,a_j,\theta_j,X_j)\} \text{ in } D_T \times \mathbf{R}\times\mathbf{R}^N\times\mathbf{S}^n \text{ such that } (a_j,\theta_j,X_j) \in \mathcal{P}^{2,\pm}w(x_j,t_j) \text{ and } (x_j,t_j,a_j,\theta_j,X_j) \to (x,t,a,\theta,X) \text{ as } j \to \infty\Big\}.$$

Then it follows that

$$\overline{\mathcal{P}}^{2,\pm}(\psi + w)(x,t) = (\partial_t\psi(x,t), \nabla\psi(x,t), \nabla^2\psi(x,t)) + \overline{\mathcal{P}}^{2,\pm}w(x,t)$$

for all $\psi \in C^{2,1}(D_T)$.

Definition 3.1. Let $m \in \{1, 2, \dots\}$. Assume (A1) and (A2).

(i) Let $\mathbf{u} = (u^{(1)}, \dots, u^{(m)})$ be a vector of upper semi-continuous functions in D_T. We say that u is a viscosity subsolution of (1.1) if

$$a + F^{(k)}(x, t, \mathbf{u}(x,t), \theta, X) \leq 0$$

for $(x,t) \in D_T$, $k \in \{1, \dots, m\}$ and $(a,\theta,X) \in \overline{\mathcal{P}}^{2,+}u^{(k)}(x,t)$;

(ii) Let $\mathbf{u} = (u^{(1)}, \dots, u^{(m)})$ be a vector of lower semi-continuous functions in D_T. We say that u is a viscosity supersolution of (1.1) if

$$a + F^{(k)}(x, t, \mathbf{u}(x,t), \theta, X) \geq 0$$

for $(x,t) \in D_T$, $k \in \{1, \dots, m\}$ and $(a,\theta,X) \in \overline{\mathcal{P}}^{2,-}u^{(k)}(x,t)$;

(iii) We say that $\mathbf{u}$ is a viscosity solution of (1.1) if $\mathbf{u}$ is both a viscosity subsolution and supersolution of (1.1).

The following trivial lemma and its corollary are crucial to the proof of our main results (see Section 4).

Lemma 3.2. *Let $k \in \{1, \ldots, m\}$ and $(x,t) \in D_T$. Assume that there exists $(\bar{a}, \bar{\theta}, \bar{X}) \in \mathcal{P}^{2,-}u^{(k)}(x,t)$ such that $\bar{a} + F^{(k)}(x,t,\mathbf{u}(x,t), \bar{\theta}, \bar{X}) \le 0$. Then $a = \bar{a}$, $\theta = \bar{\theta}$ and $X \ge \bar{X}$ for every $(a, \theta, X) \in \mathcal{P}^{2,+}u^{(k)}(x,t)$.*

Proof. If $(a, \theta, X) \in \overline{\mathcal{P}}^{2,+}u^{(k)}(x,t)$, then

$$\begin{aligned} &w(x,t) + \bar{a}(s-t) + \langle \bar{\theta}, (y-x)\rangle + \frac{1}{2}\langle \bar{X}(y-x), y-x\rangle + o(|x-y|^2 + |t-s|) \\ &\le w(y,s) \le w(x,t) + a(s-t) + \langle \theta, (y-x)\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle \\ &\quad + o(|x-y|^2 + |t-s|) \end{aligned}$$

for all (y,s) in a neighborhood of (x,t). This implies Lemma 3.2. □

Corollary 3.3. *Assume* (A1) *and* (A2). *If, for every $(x,t) \in D_T$, there exists $\phi = (\phi^{(1)}, \ldots, \phi^{(m)})$ of class C^2 touching $\mathbf{u}$ by above at (x,t)* (i.e. $\phi(x,t) = \mathbf{u}(x,t)$ *while $\phi(y,s) \ge \mathbf{u}(y,s)$ for (y,s) in neighborhood of (x,t)*), *such that*

$$\partial_t \phi^{(k)}(x,t) + F^{(k)}\Big(x, t, \mathbf{u}(x,t), \nabla\phi^{(k)}(x,t), \nabla^2\phi^{(k)}(x,t)\Big) \le 0 \quad \textit{for } k = 1, \ldots, m,$$

then $\mathbf{u}$ is a viscosity subsolution of (1.1).

Proof. Set

$$\bar{a} = \partial_t \phi^{(k)}(x,t) \qquad \bar{\theta} = \nabla\phi^{(k)}(x,t) \qquad \bar{X} = \nabla^2\phi^{(k)}(x,t),$$

and apply the previous lemma for every $(x,t) \in D_T$ and $k = 1, \ldots, m$. Then Corollary 3.3 follows from Definition 3.1 (i), (A1) and (A2). □

Following [23], we introduce the following two conditions on $\mathbf{F} = (F_1, \ldots, F_m)$.

(C1) There exists $\lambda > 0$ such that, if $\mathbf{v} = (v^{(1)}, \ldots, v^{(m)})$, $\mathbf{w} = (w^{(1)}, \ldots, w^{(m)}) \in \mathbf{R}^m$, $\max_{k\in\{1,\ldots,m\}}(v^{(k)} - w^{(k)}) > 0$ and $(x,t,\theta) \in D_T \times \mathbf{R}^N$, then there exists $\ell \in \{1, \ldots, m\}$ such that

$$v^{(\ell)} - w^{(\ell)} = \max_{k\in\{1,\ldots,m\}} \Big(v^{(k)} - w^{(k)}\Big) > 0$$

and

$$F^{(\ell)}(x,t,\mathbf{v},\theta,X) - F^{(\ell)}(x,t,\mathbf{w},\theta,X) \ge \lambda\Big(v^{(\ell)} - w^{(\ell)}\Big)$$

for all $X \in \mathbf{S}^n$;

(C2) There is a nonnegative continuous function ω on $[0, \infty)$ with $\omega(0) = 0$ such that, if $X, Y \in \mathbf{S}^n$, $\sigma > 1$ and

$$-3\sigma \begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le 3\sigma \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then

$$\begin{aligned} &F^{(k)}(y, s, \mathbf{r}, \sigma(x-y), -Y) - F^{(k)}(x, t, \mathbf{r}, \sigma(x-y), X) \\ &\le \omega\big(\sigma(|x-y| + |t-s|)^2 + 1/\sigma\big) \end{aligned}$$

for all $k \in \{1, \dots, m\}$, $t, s \in [0, \infty)$, $x, y \in \Omega$ and $\mathbf{r} \in \mathbf{R}^m$.

Remark 3.4. (C2) implies (A2). See [9, Remark 3.4].

Similarly to [23, Theorem 4.7], we can prove the following comparison principle.

Theorem 3.5. *Let Ω be a bounded domain in $\mathbf{R}^N$, $T > 0$ and $D_T := \Omega \times (0, T)$. Assume* (A1), (C1) *and* (C2). *Let $\mathbf{u} = (u^{(1)}, \dots, u^{(m)})$ and $\mathbf{v} = (v^{(1)}, \dots, v^{(m)})$ be upper semi-continuous and lower semi-continuous on $\overline{\Omega} \times [0, T)$, respectively. If $\mathbf{u}$ is a viscosity subsolution of* (1.1) *and $\mathbf{v}$ is a viscosity supersolution of* (1.1) *such that $\mathbf{u} \le \mathbf{v}$ on $\partial\Omega \times [0, T)$ and $\Omega \times \{0\}$, then $\mathbf{u} \le \mathbf{v}$ in D_T.*

Proof. See the proof of [15, Theorem 3.1]. □

However, to apply our main results, contained in the next Section 4, only the following weak comparison principle is needed.

(WCP) *If $\mathbf{u}$ is a viscosity subsolution of* (1.1) *and $\mathbf{v}$ is a viscosity supersolution of* (1.1) *such that $\mathbf{u} \ge \mathbf{v}$ in $\overline{\Omega} \times [0, T)$, while $\mathbf{u} = \mathbf{v}$ on $\partial\Omega \times [0, T)$ and $\Omega \times \{0\}$, then $\mathbf{u} = \mathbf{v}$ in D_T.*

Sufficient conditions for (WCP) to hold are given in the following theorem.

Theorem 3.6. *Let Ω be a bounded domain in $\mathbf{R}^N$, $T > 0$ and $D_T := \Omega \times (0, T)$. Assume* (A1), (C2) *and the following:*

(C3) *There exists $\lambda > 0$ such that, if $(x, t, \theta) \in D_T \times \mathbf{R}^N$ and $\mathbf{v} = (v^{(1)}, \dots, v^{(m)})$, $\mathbf{w} = (w^{(1)}, \dots, w^{(m)}) \in \mathbf{R}^m$ with $v \ge w$ and $v \ne w$, then there exists $\ell \in \{1, \dots, m\}$ such that*

$$v^{(\ell)} - w^{(\ell)} = \max_{k \in \{1,\dots,m\}} \left(v^{(k)} - w^{(k)}\right) > 0$$

and

$$F^{(\ell)}(x, t, \mathbf{v}, \theta, X) - F^{(\ell)}(x, t, \mathbf{w}, \theta, X) \ge \lambda \left(v^{(\ell)} - w^{(\ell)}\right)$$

for all $X \in \mathbf{S}^n$.

Then (WCP) *holds.*

Proof. The proof is again the same of [15, Theorem 3.1], just using (C3) in place of (C2). □

Remark 3.7. We pick the occasion to point out that in [15, Theorem 3.1] was wrongly stated. Indeed condition (A1) in [15] coincides with condition (C3) here, which gives Theorem 3.6, but it is not sufficient for Theorem 3.5 (which instead requires the stronger assumption (C1)). On the other hand, this does not affect the results of [15], since (WCP) is enough for [15, Theorem 4.1].

4. Spatial concavity

Let Ω be a bounded convex smooth domain in $\mathbf{R}^N$, $D := \Omega \times (0,\infty)$ and $m \in \{1,2,\dots\}$. Let $\mathbf{u} = (u^{(1)},\dots,u^{(m)}) \in C^{2,1}(D:\mathbf{R}^m) \cap C(\overline{D}:\mathbf{R}^m)$ satisfy

$$\begin{cases} \partial_t u^{(k)} + F^{(k)}\big(x,t,\mathbf{u},\nabla u^{(k)},\nabla^2 u^{(k)}\big) = 0 & \text{in} \quad D \quad k=1,\dots,m\,, \\ u^{(k)}(x,t) > 0 & \text{in} \quad D \quad k=1,\dots,m\,, \\ \mathbf{u}(x,t) = 0 & \text{on} \quad \partial\Omega \times [0,\infty). \end{cases} \tag{4.1}$$

Let $\lambda \in \Lambda_{N+1}$, $k \in \{1,\dots,m\}$ and $p \in [-\infty,\infty]$. Define

$$\begin{aligned} &U^{(k)}_{p,\lambda}(x,t) \\ &:= \sup\left\{ M_p\Big(u^{(k)}(y_1,t),\dots,u^{(k)}(y_{n+1},t);\lambda\Big) : \{y_i\}_{i=1}^{n+1} \subset \overline{\Omega},\ x = \sum_{i=1}^{n+1} \lambda_i y_i \right\} \end{aligned} \tag{4.2}$$

for $(x,t) \in \overline{D}$. Then we easily see that

$$U^{(k)}_{p,\lambda} \in C(\overline{D}) \quad U^{(k)}_{p,\lambda} \geq u^{(k)}(x,t) > 0 \ \text{ in } \ D \quad U^{(k)}_{p,\lambda} = 0 \ \textit{on} \ \partial\Omega \times [0,\infty)\,. \tag{4.3}$$

We denote by $U^{(k)}_p$ the *spatially p-concave envelope* of $u^{(k)}$ defined by

$$U^{(k)}_p(x,t) := \sup_{\lambda \in \Lambda_{n+1}} U^{(k)}_{p,\lambda}(x,t),$$

which is the smallest spatially p-concave function greater than or equal to $u^{(k)}$. Clearly, $u^{(k)}$ is spatially p-concave in D if and only if $u^{(k)} = U^{(k)}_p$ in D; since $U^{(k)}_p \geq u^{(k)}$ by construction, to have equality we just need to get the opposite inequality $U^{(k)}_p \leq u^{(k)}$, which can be obtained via Comparison Principle if $U^{(k)}$ turns to be a subsolution of the problem at hands. Thus in this section we give a sufficient condition for

$$\mathbf{U}_{p,\lambda} := \Big(U^{(1)}_{p,\lambda},\dots,U^{(m)}_{p,\lambda}\Big)$$

to be a viscosity subsolution of (4.1) in the case of $0 \leq p \leq 1$ and study spatial concavity properties of the solutions of (4.1).

4.1. Case of $0 < p \le 1$

In this subsection we focus on the case of $0 < p \le 1$ and prove the following theorem.

Theorem 4.1. *Let Ω be a bounded convex smooth domain in $\mathbf{R}^N$, $D := \Omega \times (0, \infty)$, $m \in \{1, 2, \dots\}$ and $0 < p \le 1$. Assume* (A1), (A2) *and the following condition:*

(F3) *Let $k \in \{1, \dots, m\}$. For any fixed $\theta \in \mathbf{R}^N$ and $t_* > 0$,*

$$\mathcal{F}^{(k)}_{\theta,t_*}\left(x, v^{(1)}, \dots, v^{(m)}, A\right) := \left(v^{(k)}\right)^{1-\frac{1}{p}} F^{(k)}\left(x, t_*, \left(v^{(1)}\right)^{\frac{1}{p}}, \dots, \left(v^{(m)}\right)^{\frac{1}{p}}, \left(v^{(k)}\right)^{\frac{1}{p}-1}\theta, \left(v^{(k)}\right)^{\frac{1}{p}-3} A\right)$$

is convex with respect to $(x, v^{(1)}, \dots, v^{(m)}, A) \in \Omega \times [0, \infty)^m \times \mathbf{S}^N$.

Let $\mathbf{u} = (u^{(1)}, \dots, u^{(m)}) \in C^{2,1}(D : \mathbf{R}^m) \cap C(\overline{D} : \mathbf{R}^m)$ satisfy (4.1) *and*

$$\lim_{\rho \to 0+} \rho^{-1/p} u^{(k)}(x + \nu(x)\rho, t) = \infty \quad \text{for} \quad (x,t) \in \partial\Omega \times (0,\infty) \quad k = 1, \dots, m, \tag{4.4}$$

where $\nu = \nu(x)$ is the inner unit normal vector to $\partial\Omega$ at x. Then $\mathbf{U}_{p,\lambda}$ is a viscosity subsolution of (4.1).

Proof. Let $(x_*, t_*) \in D$, $\lambda = (\lambda_1, \dots, \lambda_{n+1}) \in \Lambda_{n+1}$ and $k \in \{1, \dots, m\}$. Since $u^{(k)} = 0$ on $\partial\Omega \times (0, \infty)$ and $0 < p \le 1$, by (4.2) and (4.4) we can find $\{x_i^{(k)}\}_{i=1}^{n+1} \subset D$ such that

$$x_* = \sum_{i=1}^{n+1} \lambda_i x_i^{(k)} \qquad U^{(k)}_{p,\lambda}(x_*, t_*) = M_p\left(u^{(k)}\left(x_1^{(k)}, t_*\right), \dots, u^{(k)}\left(x_{n+1}^{(k)}, t_*\right); \lambda\right).$$

Furthermore, the Lagrange multiplier theorem assures that

$$\begin{aligned} \theta &:= u^{(k)}\left(x_1^{(k)}, t_*\right)^{p-1} \nabla u^{(k)}\left(x_1^{(k)}, t_*\right) = \cdots \\ &= u^{(k)}\left(x_{n+1}^{(k)}, t_*\right)^{p-1} \nabla u^{(k)}\left(x_{n+1}^{(k)}, t_*\right). \end{aligned} \tag{4.5}$$

Let $\{a_i^{(k)}\}_{i=1}^{n+1} \subset [0, \infty)$ be such that $\sum_{i=1}^{n+1} \lambda_i a_i^{(k)} = 1$. Set

$$U_*^{(k)} := U^{(k)}_{p,\lambda}(x_*, t_*) \qquad u_i^{(k)} := u^{(k)}\left(x_i^{(k)}, t_*\right) \qquad y_i^{(k)}(x) := x_i^{(k)} + a_i^{(k)}(x - x_*),$$

$$\mathbf{U}_* := \left(U_*^{(1)}, \dots, U_*^{(m)}\right) \quad \mathbf{u}_i := \left(u_i^{(1)}, \dots, u_i^{(m)}\right).$$

It follows that

$$U_*^{(k)} = M_p\left(u_1^{(k)}, \dots, u_{n+1}^{(k)}; \lambda\right) \qquad x = \sum_{i=1}^{n+1} \lambda_i y_i^{(k)}(x). \tag{4.6}$$

For $k = 1, \ldots, m$, we define

$$\varphi^{(k)}(x,t) := M_p(u^{(k)}\left(y_1^{(k)}(x),t),\ldots,u^{(k)}(y_{n+1}^{(k)}(x),t);\lambda\right), \tag{4.7}$$

which is a $C^{2,1}$-function in a neighborhood of $(x_*, t_*) \in D$ and satisfies

$$\varphi^{(k)}(x_*,t_*) = M_p\left(u_1^{(k)},\ldots,u_{n+1}^{(k)};\lambda\right) = U_*^{(k)} = U_{p,\lambda}^{(k)}(x_*,t_*). \tag{4.8}$$

Moreover, it follows from the definition of $U_{p,\lambda}$ and (4.6) that

$$U_{p,\lambda}^{(k)}(x,t) \geq \varphi^{(k)}(x,t)$$

in a neighborhood of (x_*, t_*).

We prove

$$\partial_t\varphi^{(k)}(x_*,t_*) + F^{(k)}(x_*,t_*,\mathbf{U}_*,\nabla\varphi^{(k)}\left(x_*,t_*),\nabla^2\varphi^{(k)}(x_*,t_*)\right) \leq 0 \tag{4.9}$$

for $k = 1, \ldots, m$. Let $\nabla' := (\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial t)$. By (4.5) and (4.7) we have

$$\nabla'\varphi^{(k)}(x,t) = \varphi^{(k)}(x,t)^{1-p}\sum_{i=1}^{n+1}\lambda_i\, u^{(k)}\left(y_i^{(k)}(x),t\right)^{p-1}\nabla' u^{(k)}\left(y_i^{(k)}(x),t\right)$$

and

$$\begin{aligned}\nabla^2\varphi^{(k)}(x,t) = {}& \varphi^{(k)}(x,t)^{1-p}\sum_{i=1}^{n+1}\lambda_i\left(a_i^{(k)}\right)^2 u^{(k)}\left(y_i^{(k)}(x),t\right)^{p-1}\nabla^2 u^{(k)}\left(y_i^{(k)}(x),t\right)\\ &+ (1-p)\varphi^{(k)}(x,t)^{-p}\nabla\varphi^{(k)}(x,t)\\ &\otimes\sum_{i=1}^{n+1}\lambda_i a_i^{(k)}\, u^{(k)}\left(y_i^{(k)}(x),t\right)^{p-1}\nabla u^{(k)}\left(y_i^{(k)}(x),t\right)\\ &-(1-p)\varphi^{(k)}(x,t)^{1-p}\sum_{i=1}^{n+1}\lambda_i\left(a_i^{(k)}\right)^2 u^{(k)}\left(y_i^{(k)}(x),t\right)^{p-2}\nabla u^{(k)}\left(y_i^{(k)}(x),t\right)\\ &\otimes\nabla u^{(k)}(y_i^{(k)}(x),t)\end{aligned}$$

in a neighborhood of (x_*, t_*). Since $y_i^{(k)}(x_*) = x_i^{(k)}$ and $\lambda \in \Lambda_{n+1}$, by (4.5) and (4.8) we obtain

$$
\begin{aligned}
\partial_t\varphi^{(k)}(x_*,t_*)&=\left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i u^{(k)}\!\left(x_i^{(k)},t\right)^{p-1}\partial_t u^{(k)}\left(x_i^{(k)},t\right),\\
\nabla\varphi^{(k)}(x_*,t_*)&=\left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i u^{(k)}\!\left(x_i^{(k)},t\right)^{p-1}\nabla u^{(k)}\left(x_i^{(k)},t\right)=\left(U_*^{(k)}\right)^{1-p}\theta
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
\nabla^2\varphi^{(k)}(x_*,t_*) &= \left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i\left(a_i^{(k)}\right)^2\left(u_i^{(k)}\right)^{p-1}\nabla^2u^{(k)}\left(x_i^{(k)},t_*\right)\\
&\quad+(1-p)\left(U_*^{(k)}\right)^{-1}\nabla\varphi^{(k)}(x_*,t_*)\otimes\nabla\varphi^{(k)}(x_*,t_*)\\
&\quad\times\left(1-\left(U_*^{(k)}\right)^{p}\sum_{i=1}^{n+1}\lambda_i\left(a_i^{(k)}\right)^2\left(u_i^{(k)}\right)^{-p}\right).
\end{aligned}
\tag{4.11}
$$

Taking

$$
a_i^{(k)} = \left(u_i^{(k)}/U_*^{(k)}\right)^p \qquad i=1,\dots,n+1,
$$

we deduce from (4.8) that

$$
\begin{aligned}
&\left(U_*^{(k)}\right)^{p}\sum_{i=1}^{n+1}\lambda_i\left(a_i^{(k)}\right)^2\left(u_i^{(k)}\right)^{-p}\\
&=\left(U_*^{(k)}\right)^{-p}\sum_{i=1}^{n+1}\lambda_i\left(u_i^{(k)}\right)^{p}=\left(U_*^{(k)}\right)^{-p}M_p\left(u_1^{(k)},\dots,u_{n+1}^{(k)};\lambda\right)^p=1.
\end{aligned}
$$

This together with (4.11) implies that

$$
\begin{aligned}
\nabla^2\varphi^{(k)}(x_*,t_*) &= \left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i\left(a_i^{(k)}\right)^2\left(u_i^{(k)}\right)^{p-1}\nabla^2u^{(k)}\left(x_i^{(k)},t_*\right)\\
&=\left(U_*^{(k)}\right)^{1-3p}\sum_{i=1}^{n+1}\lambda_i\left(u_i^{(k)}\right)^{3p-1}\nabla^2u^{(k)}\left(x_i^{(k)},t_*\right)\\
&=\left(U_*^{(k)}\right)^{1-3p}\sum_{i=1}^{n+1}\lambda_i A_i,
\end{aligned}
\tag{4.12}
$$

where

$$
A_i = \left(u_i^{(k)}\right)^{3p-1}\nabla^2u^{(k)}(x_i^{(k)},t_*) \qquad i=1,\dots,n+1.
$$

Then, by (4.10) and (4.12) we obtain

$$\begin{aligned}
&\partial_t\varphi^{(k)}(x_*, t_*) + F^{(k)}\left(x_*, t_*, \mathbf{U}_*, \nabla\varphi^{(k)}(x_*, t_*), \nabla^2\varphi^{(k)}(x_*, t_*)\right)\\
&= \left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i\frac{\partial_t u^{(k)}\left(x_i^{(k)}, t_*\right)}{\left(u_i^{(k)}\right)^{1-p}}\\
&\quad + F^{(k)}\left(x_*, t_*, \mathbf{U}_*, \left(U_*^{(k)}\right)^{1-p}\theta, \left(U_*^{(k)}\right)^{1-3p}\sum_{i=1}^{n+1}\lambda_i A_i\right)\\
&= -\left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i\frac{F^{(k)}\left(x_i^{(k)}, t_*, \mathbf{u}_i, \left(u_i^{(k)}\right)^{1-p}\theta, \left(u_i^{(k)}\right)^{1-3p}A_i\right)}{\left(u_i^{(k)}\right)^{1-p}}\\
&\quad + F^{(k)}\left(x_*, t_*, \mathbf{U}_*, \left(U_*^{(k)}\right)^{1-p}\theta, \left(U_*^{(k)}\right)^{1-3p}\sum\lambda_i A_i\right).
\end{aligned} \tag{4.13}$$

On the other hand, it follows from (F3) that

$$\begin{aligned}
&\left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i\frac{F^{(k)}\left(x_i^{(k)}, t_*, \mathbf{u}_i, \left(u_i^{(k)}\right)^{1-p}\theta, \left(u_i^{(k)}\right)^{1-3p}A_i\right)}{\left(u_i^{(k)}\right)^{1-p}}\\
&= \left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i\left(v_i^{(k)}\right)^{1-\frac{1}{p}}F^{(k)}\left(x_i^{(k)}, t_*, \mathbf{v}_i^{\frac{1}{p}}, \left(v_i^{(k)}\right)^{\frac{1}{p}-1}\theta, \left(v_i^{(k)}\right)^{\frac{1}{p}-3}A_i\right)\\
&= \left(U_*^{(k)}\right)^{1-p}\sum_{i=1}^{n+1}\lambda_i\mathcal{F}_{\theta,t_*}^{(k)}\left(x_i^{(k)}, v_i^{(1)}, \dots, v_i^{(m)}, A_i\right)\\
&\ge \left(U_*^{(k)}\right)^{1-p}\mathcal{F}_{\theta,t_*}^{(k)}\left(\sum_{i=1}^{n+1}\lambda_i x_i^{(k)}, \sum_{i=1}^{n+1}\lambda_i v_i^{(1)}, \dots, \sum_{i=1}^{n+1}\lambda_i v_i^{(m)}, \sum_{i=1}^{n+1}\lambda_i A_i\right),
\end{aligned}$$

where $v_i^{(k)} := \left(u_i^{(k)}\right)^p$ and $v_i^{1/p} := \left(\left(v_i^{(1)}\right)^{1/p}, \dots, \left(v_i^{(m)}\right)^{1/p}\right)$. Since

$$\sum_{i=1}^{n+1}\lambda_i x_i^{(k)} = x_* \qquad \sum_{i=1}^{n+1}\lambda_i v_i^{(j)} = M_p\left(u_1^{(k)}, \dots, u_{n+1}^{(k)}; \lambda\right)^p = \left(U_*^{(j)}\right)^p,$$

where $j = 1, \dots, m$, we deduce that

$$\begin{aligned}
\left(U_*^{(k)}\right)^{1-p} &\ge \left(U_*^{(k)}\right)^{1-p}\mathcal{F}_{\theta,t_*}^{(k)}\left(x_*, \left(U_*^{(1)}\right)^p, \dots, \left(U_*^{(m)}\right)^p, \sum_{i=1}^{n+1}\lambda_i A_i\right)\\
&= F^{(k)}\left(x_*, t_*, \mathbf{U}_*, \left(U_*^{(k)}\right)^{1-p}\theta, \left(U_*^{(k)}\right)^{1-3p}\sum_{i=1}^{n+1}\lambda_i A_i\right).
\end{aligned}$$

This together with (4.13) implies (4.9). Since (x_*, t_*) is arbitrary, by (4.3) and Corollary 3.3 we see that $\mathbf{U}_{\lambda,p}$ is a viscosity subsolution of (4.1). Thus Theorem 4.1 follows. □

Combing Theorem 4.1 with Theorem 3.6, we obtain the following.

Corollary 4.2. *Assume the same conditions as in Theorem* 4.1. *Furthermore, assume* (C2) *and* (C3). *Let* $0 < p \le 1$ *and* $\mathbf{u} = (u^{(1)}, \dots, u^{(m)})$ *satisfy* (4.1) *and* (4.4). *If the initial datum* $u_0^{(k)}$ *is* p*-concave in* $\overline{\Omega}$ *for* $k = 1, \dots, m$*, then* $u^{(k)}$ *is spatially* p*-concave in* D *for* $k \in \{1, \dots, m\}$.

Proof. Let $k \in \{1, \dots, m\}$. Due to the p-concavity of $u_0^{(k)}$, we have

$$U_{p,\lambda}^{(k)}(x, 0) = U_p^{(k)}(x, 0) = u_0^{(k)}(x) \qquad x \in \overline{\Omega},$$

for every $\lambda \in \Lambda_{n+1}$. Then, by Theorem 4.1 we see that $\mathbf{U}_{p,\lambda}$ is a viscosity subsolution of (1.1) for every $\lambda \in \Lambda_{n+1}$. Applying Theorem 3.6, by (4.3) we see that $\mathbf{U}_{p,\lambda} \le \mathbf{u}$ in D, which implies that $\mathbf{U}_p \le \mathbf{u}$ in D. On the other hand, it follows from the definition of $\mathbf{U}_p$ that $\mathbf{U}_p \ge \mathbf{u}$ in D. Therefore we deduce that $\mathbf{U}_p = \mathbf{u}$ in D. Then $u^{(k)}$ is spatially p-concave in D for every $k \in \{1, \dots, m\}$. □

4.2. Case of $p = 0$

In the next theorem we give a sufficient condition for $U_{0,\lambda} = (U_{0,\lambda}^{(1)}, \dots, U_{0,\lambda}^{(m)})$ to be a viscosity subsolution of (4.1).

Theorem 4.3. *Let* Ω *be a bounded convex smooth domain in* $\mathbf{R}^N$, $D := \Omega \times (0, \infty)$ *and* $m \in \{1, 2, \dots\}$. *Assume* (A1), (A2) *and the following condition:*

(F4) *Let* $k \in \{1, \dots, m\}$. *For any fixed* $\theta \in \mathbf{R}^N$ *and* $t_* > 0$,

$$\mathcal{F}_{\theta,t_*}^{(k)}\Big(x, v^{(1)}, \dots, v^{(m)}, A\Big) := e^{-v^{(k)}} F^{(k)}\left(x, t_*, e^{v^{(1)}}, \dots, e^{v^{(m)}}, e^{v^{(k)}}\theta, e^{v^{(k)}} A\right)$$

is convex with respect to $(x, v^{(1)}, \dots, v^{(m)}, A) \in D \times \mathbf{R}^m \times \mathbf{S}^N$.

Then $\mathbf{U}_{0,\lambda} := (U_{0,\lambda}^{(1)}, \dots, U_{0,\lambda}^{(m)})$ *is a viscosity subsolution of* (4.1).

Proof. Let $(x_*, t_*) \in D$ and $\lambda = (\lambda_1, \dots, \lambda_{n+1}) \in \Lambda_{n+1}$. Thanks to the boundary conditions and to the regularity of u and of the geometric mean, we can find $\{x_i^{(k)}\}_{i=1}^{n+1} \subset D$ such that

$$x_* = \sum_{i=1}^{n+1} \lambda_i x_i^{(k)},$$

$$U_{0,\lambda}^{(k)}(x_*, t_*) = M_0\left(u^{(k)}\left(x_1^{(k)}, t_*\right), \dots, u^{(k)}\left(x_{n+1}^{(k)}, t_*\right); \lambda\right) = \prod_{i=1}^{n+1} u^{(k)}\left(x_i^{(k)}, t_*\right)^{\lambda_i}.$$

Notice that the Lagrange multiplier theorem assures that

$$\theta := \frac{\nabla u^{(k)}\big(x_1^{(k)}, t_*\big)}{u^{(k)}\big(x_1^{(k)}, t_*\big)} = \cdots = \frac{\nabla u^{(k)}\big(x_{n+1}^{(k)}, t_*\big)}{u^{(k)}\big(x_{n+1}^{(k)}, t_*\big)}.$$

Set

$$U_*^{(k)} := U_{0,\lambda}^{(k)}(x_*, t_*) \qquad u_i^{(k)} := u^{(k)}\left(x_i^{(k)}, t_*\right) \quad y_i^{(k)}(x) := x_i^{(k)} + (x - x_*),$$
$$\mathbf{U}_* := \left(U_*^{(1)}, \dots, U_*^{(m)}\right) \ \mathbf{u}_i := \left(u_i^{(1)}, \dots, u_i^{(m)}\right) \ A_i := \frac{\nabla^2 u^{(k)}\big(x_i^{(k)}, t_*\big)}{u_i^{(k)}}.$$

It follows that

$$x = \sum_{i=1}^{n+1} \lambda_i y_i^{(k)}(x). \tag{4.14}$$

For $k = 1, \dots, m$, we define

$$\varphi^{(k)}(x,t) := M_0\Big(u^{(k)}\Big(y_1^{(k)}(x),t\Big), \dots, u^{(k)}\Big(y_{n+1}^{(k)}(x),t\Big);\lambda\Big) = \prod_{i=1}^{n+1} u^{(k)}\Big(y_i^{(k)}(x), t\Big)^{\lambda_i},$$

which is a $C^{2,1}$-function in a neighborhood of $(x_*, t_*) \in D$ and satisfies

$$\varphi^{(k)}(x_*, t_*) = \prod_{i=1}^{n+1} \left[u_i^{(k)}\right]^{\lambda_i} = U_*^{(k)} = U_{0,\lambda}^{(k)}(x_*, t_*).$$

Moreover, it follows from the definition of U_λ and (4.14) that

$$U_{0,\lambda}^{(k)}(x, t) \geq \varphi^{(k)}(x, t)$$

in a neighborhood of (x_*, t_*).

We apply the same argument in the proof of Theorem 4.1 with $p = 0$, and prove

$$\partial_t \varphi^{(k)}(x_*, t_*) + F^{(k)}(x_*, t_*, \mathbf{U}_*, \nabla\varphi^{(k)}\left(x_*, t_*), \nabla^2\varphi^{(k)}(x_*, t_*)\right) \leq 0 \tag{4.15}$$

for $k = 1, \dots, m$. Similarly to (4.10) and (4.12), we have

$$\partial_t \varphi^{(k)}(x_*, t_*) = \varphi^{(k)}(x_*, t_*) \sum_{i=1}^{n+1} \lambda_i \frac{\partial_t u^{(k)}\big(x_i^{(k)}, t_*\big)}{u^{(k)}\big(x_i^{(k)}, t_*\big)} = U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i \frac{\partial_t u^{(k)}\big(x_i^{(k)}, t_*\big)}{u_i^{(k)}},$$
$$\nabla\varphi^{(k)}(x_*, t_*) = U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i \frac{\nabla u^{(k)}\big(x_i^{(k)}, t_*\big)}{u^{(k)}\big(x_i^{(k)}, t_*\big)} = \varphi^{(k)}(x_*, t_*)\,\theta,$$
$$\nabla^2\varphi^{(k)}(x_*, t_*) = U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i \frac{\nabla^2 u^{(k)}\big(x_i^{(k)}, t_*\big)}{u_i^{(k)}} = U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i A_i.$$

Then we deduce that

$$
\begin{aligned}
&\partial_t \varphi^{(k)}(x_*, t_*) + F^{(k)}\left(x_*, t_*, \mathbf{U}_*, \nabla\varphi^{(k)}(x_*, t_*), \nabla^2\varphi^{(k)}(x_*, t_*)\right) \\
&= U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i \frac{\partial_t u^{(k)}\big(x_i^{(k)}, t_*\big)}{u_i^{(k)}} + F^{(k)}\left(x_*, t_*, \mathbf{U}_*, U_*^{(k)}\theta, U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i A_i\right) \\
&= -U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i \frac{F\big(x_i^{(k)}, t_*, u_i, u_i^{(k)}\theta, u_i^{(k)} A_i\big)}{u_i^{(k)}} \\
&\quad + F^{(k)}\left(x_*, t_*, \mathbf{U}_*, U_*^{(k)}\theta, U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i A_i\right).
\end{aligned}
\tag{4.16}
$$

On the other hand, it follows from (F4) that

$$
\begin{aligned}
&\sum_{i=1}^{n+1} \lambda_i \frac{F^{(k)}\big(x_i^{(k)}, t_*, \mathbf{u}_i, u_i^{(k)}\theta, u_i^{(k)} A_i\big)}{u_i^{(k)}} = \sum_{i=1}^{n+1} \lambda_i \frac{F^{(k)}\big(x_i^{(k)}, t_*, e^{\mathbf{v}_i}, e^{v_i^{(k)}}\theta, e^{v_i^{(k)}} A_i\big)}{e^{v_i^{(k)}}} \\
&= \sum_{i=1}^{n+1} \lambda_i \mathcal{F}_{\theta, t_*}^{(k)}\left(x_i^{(k)}, v_i^{(1)}, \ldots, v_i^{(m)}, A_i\right) \\
&\geq \mathcal{F}_{\theta, t_*}^{(k)}\left(\sum_{i=1}^{n+1} \lambda_i x_i^{(k)}, \sum_{i=1}^{n+1} \lambda_i v_i^{(1)}, \ldots, \sum_{i=1}^{n+1} \lambda_i v_i^{(m)}, \sum_{i=1}^{n+1} \lambda_i A_i\right),
\end{aligned}
$$

where $v_i^{(k)} := \log u_i^{(k)}$ and $e^{\mathbf{v}_i} := (e^{v_i^{(1)}}, \ldots, e^{v_i^{(m)}})$. Since

$$
\sum_{i=1}^{n+1} \lambda_i x_i^{(k)} = x_* \qquad \sum_{i=1}^{n+1} \lambda_i v_i^{(j)} = \log \prod_{i=1}^{n+1} \left(u_i^{(j)}\right)^{\lambda_i} = \log U_*^{(j)},
$$

we deduce that

$$
\begin{aligned}
&\sum_{i=1}^{n+1} \lambda_i \frac{F^{(k)}\big(x_i^{(k)}, t_*, \mathbf{u}_i, u_i^{(k)}\theta, u_i^{(k)} A_i\big)}{u_i^{(k)}} \\
&\geq \mathcal{F}_{\theta, t_*}^{(k)}\left(x_*, \log U_*^{(1)}, \ldots, \log U_*^{(m)}, \sum_{i=1}^{n+1} \lambda_i A_i\right) \\
&= \frac{1}{U_*^{(k)}} F^{(k)}\left(x_*, t_*, U_*^{(k)}\theta, U_*^{(k)} \sum_{i=1}^{n+1} \lambda_i A_i\right).
\end{aligned}
$$

This together with (4.16) implies (4.15). Since (x_*, t_*) is arbitrary, by (4.3) and Corollary 3.3 we see that $\mathbf{U}_\lambda$ is a viscosity subsolution of (4.1). Thus Theorem 4.3 follows. □

By Theorem 4.3 we apply a similar argument as in the proof of Corollary 4.2 to obtain the following result.

Corollary 4.4. *Assume the same conditions as in Theorem* 4.3. *Furthermore, assume* (C2) *and* (C3). *Let* $\mathbf{u} = (u^{(1)}, \dots, u^{(m)})$ *satisfy* (4.1) *with initial value* $\mathbf{u_0} = (u_0^{(1)}, \dots, u_0^{(m)})$. *If the initial datum* $u_0^{(k)}$ *is log-concave in* $\overline{\Omega}$ *for* $k = 1, \dots, m$, *then* $u^{(k)}$ *is spatially log-concave in* D *for* $k \in \{1, \dots, m\}$.

Theorem 1.1 easily follows from Corollary 4.4. Corollary 1.2 follows from Theorems 1.1 and 3.6. Furthermore, we have the following well known result (see [7,13] and [31]).

Corollary 4.5. *Let* Ω *be a bounded convex domain in* $\mathbf{R}^N$. *Let* $u \in C^2(D) \cap C(\overline{D})$ *satisfy*

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in} \quad D \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times [0,\infty) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.17}$$

where u_0 *is a nonnegative continuous function on* $\overline{\Omega}$. *Then* u *is spatially log-concave in* D *provided that* u_0 *is log-concave in* Ω.

Proof. Let u be a solution of (4.17) and $\lambda > 0$. Then the function $U := e^{-\lambda t} u$ satisfies

$$\partial_t U - \Delta U + \lambda U = 0 \quad \text{in} \quad \Omega \times (0,\infty).$$

Applying Corollary 4.4 to the case where $m = 1$ and $F(x, t, U, \nabla U, \nabla^2 U) = -\Delta U + \lambda U$, we obtain the spatial log-concavity of U in $\Omega \times (0,\infty)$. Thus Corollary 4.5 follows. □

Similarly, we obtain Corollary 1.2.

5. Applications to porous medium equations

We apply our results in the previous section and study concavity properties of porous medium equations and related systems. Concavity properties of solutions to the porous medium equation have been studied in several papers, see, *e.g.*, [11, 17,30,34,35] and references therein (see also a survey book [38] for porous medium equations).

5.1. Porous medium equation

Let Ω be a bounded smooth convex domain in $\mathbf{R}^N$, $D := \Omega \times (0, \infty)$ and $\alpha > 1$. Consider the Cauchy-Dirichlet problem for the porous medium equation

$$\begin{cases} \partial_t u - \Delta(u^\alpha) = 0 & \text{in} \quad D \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.1}$$

where $u_0 \in X := \{w \in C(\overline{\Omega}) : w > 0 \text{ in } \Omega, w = 0 \text{ on } \partial\Omega\}$. Problem (5.1) has a unique classical solution in D (see, *e.g.*, [38, Theorem 5.5 and Proposition 7.21]). In this subsection, as an application of Theorem 4.1, we prove the following theorem, already given in [35].

Theorem 5.1. *Let Ω be a bounded smooth convex domain in $\mathbf{R}^N$ and $\alpha > 1$. Let u be a classical solution of (5.1) with $u_0 \in X$. Then u is spatially $(\alpha - 1)/2$ concave in D provided that u_0 is $(\alpha - 1)/2$ concave in Ω.*

Notice that our approach is completely different from that of [35] and enables us to obtain concavity properties of solutions to general parabolic problems including parabolic systems (see also Subsection 5.2).

For the proof of Theorem 5.1, we prepare the following lemma.

Lemma 5.2. *Let η be a solution of*

$$-\Delta\eta = \eta^{1/\alpha} \quad in \quad \Omega \qquad \eta > 0 \quad in \quad \Omega \qquad \eta = 0 \quad in \quad \partial\Omega.$$

Let $0 < \beta \le 1$ be such that $2\beta \le \alpha(\alpha - 1)$. For any concave function $\psi \in C(\overline{\Omega})$, such that $\psi > 0$ in Ω and $\psi = 0$ on $\partial\Omega$, and for every $\epsilon > 0$, set

$$u_0^\epsilon(x) := \left[u_0(x)^{\frac{\alpha-1}{2}} + \epsilon\psi(x)^\beta\right]^{\frac{2}{\alpha-1}}.$$

Then u_0^ϵ is $(\alpha - 1)/2$ concave in Ω and

$$u_0^\epsilon(x) \ge \delta\eta(x)^\alpha \quad in \quad \Omega \tag{5.2}$$

for some $\delta > 0$.

Proof. Since $u_0^{(\alpha-1)/2}$ and ψ are concave in Ω and $0 < \beta \le 1$, we see that u_0^ϵ is $(\alpha - 1)/2$ concave in Ω. So it suffices to prove (5.2).

It follows from [3, Proposition 1] that $\eta \in C^{2+1/\alpha}(\overline{\Omega})$. Then

$$\eta(x) \le C_1 \text{dist}\,(x, \partial\Omega) \quad \text{in} \quad \Omega \tag{5.3}$$

for some constant $C_1 > 0$. On the other hand, since ψ is concave, we see that

$$u_0^\epsilon(x) \ge \epsilon^{\frac{2}{\alpha-1}}\psi(x)^{\frac{2\beta}{\alpha-1}} \ge C_2\epsilon^{\frac{2}{\alpha-1}}\text{dist}\,(x, \partial\Omega)^{\frac{2\beta}{\alpha-1}} \quad \text{in} \quad \Omega \tag{5.4}$$

for some constant C_2. Since $2\beta \le \alpha(\alpha - 1)$, by (5.3) and (5.4) we have (5.2). Thus Lemma 5.2 follows. □

Proof of Theorem 5.1. For any $\epsilon > 0$, there exists a unique classical solution u_ϵ of (5.1) with the initial data u_0^ϵ (see, *e.g.*, [38, Theorem 5.5 and Proposition 7.21]). By Lemma 5.2 we can find $\tau > 0$ such that

$$\tau^{-\frac{1}{\alpha-1}}\eta(x)^\alpha \le u_0^\epsilon(x) \quad \text{in} \quad \Omega. \tag{5.5}$$

Set

$$z(x,t) := [(\alpha-1)t+\tau]^{-\frac{1}{\alpha-1}}\eta(x)^{1/\alpha},$$

which satisfies

$$z_t - \Delta(z^\alpha) = 0 \quad \text{in} \quad D \qquad z = 0 \quad \text{on} \quad \partial\Omega\times(0,\infty).$$

By (5.5) we apply the comparison principle to obtain

$$u_\epsilon(x,t) \ge z(x,t) \quad \text{in} \quad D. \tag{5.6}$$

On the other hand, it follows from the Hopf lemma that

$$\liminf_{\rho\to 0+} \frac{\eta(x+\rho\nu(x))}{\rho} > 0$$

for any $x \in \partial\Omega$. This together with (5.6) and the definition of z implies that

$$\liminf_{\rho\to +0} \rho^{-\frac{1}{\alpha}} u_\epsilon(x+\rho\nu(x),t) > 0 \tag{5.7}$$

for all $(x,t) \in \partial\Omega\times(0,\infty)$.

Let $v_\epsilon := \alpha u_\epsilon^{\alpha-1}$. Then we have

$$\begin{cases} \partial_t v_\epsilon - v_\epsilon\Delta v_\epsilon - \dfrac{1}{\alpha-1}|\nabla v_\epsilon|^2 = 0 & \text{in} \quad D \\ v_\epsilon = 0 & \text{on} \quad \partial\Omega\times(0,\infty) \\ v(x,0) = \alpha\big[u_0^\epsilon(x)\big]^{\alpha-1} & \text{in} \quad \Omega. \end{cases} \tag{5.8}$$

Set

$$F(x,t,w,\theta,A) := -w\,\mathrm{tr}(A) - \frac{1}{\alpha-1}|\theta|^2$$

for $(x,t,w,\theta,A) \in D\times(0,\infty)\times\mathbf{R}^n\times\mathbf{S}^n$.

We apply Corollary 4.2 with $p = 1/2$ to v_ϵ. Then the function

$$\mathcal{F}_{\theta,t}(x,w,A) := w^{-1}F\big(x,t,w^2,w\theta,w^{-1}A\big) = -\mathrm{tr}(A) - \frac{1}{\alpha-1}w|\theta|^2$$

is convex with respect to $(x,w,A) \in \Omega\times[0,\infty)\times\mathbf{S}^N$ for any fixed $\theta\in\mathbf{R}^n$ and $t > 0$. This means that F satisfies condition (F3) with $p = 1/2$. Furthermore, we deduce from (5.7) that

$$\lim_{\rho\to+0} \rho^{-2}v(x+\rho\nu(x),t) = \infty$$

for all $(x, t) \in \partial\Omega \times (0, \infty)$. Therefore, by Corollary 4.2 we see that v_ϵ is spatially $1/2$ concave in D, which means that u_ϵ is spatially $(\alpha - 1)/2$ concave in D.

On the other hand, if $0 < \epsilon_1 < \epsilon_2$, the comparison principle implies that

$$0 < u(x,t) \le u_{\epsilon_1}(x,t) \le u_{\epsilon_2}(x,t) \quad \text{in} \quad D.$$

Then, by [38, Proposition 3.6] we see that

$$\lim_{\epsilon \to 0} u_\epsilon(x,t) = u(x,t) \quad \text{in} \quad D.$$

Therefore we deduce from the spatially $(\alpha - 1)/2$ concavity of u_ϵ in D that u is spatially $(\alpha - 1)/2$ concave in D. Thus Theorem 5.1 follows. □

5.2. Porous medium systems

We discuss spatial concavity properties of the solution of the following nonlinear porous medium system

$$\begin{cases} \partial_t u - d_1 \Delta(u^\alpha) + f(x,t,u,v,\nabla u) = 0 & \text{in} \quad D \\ \partial_t v - d_2 \Delta(v^\beta) + g(x,t,u,v,\nabla v) = 0 & \text{in} \quad D \\ u > 0, \quad v > 0 & \text{in} \quad D \\ u = v = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in} \quad \Omega, \end{cases} \tag{5.9}$$

where $\alpha, \beta > 1, d_1, d_2 > 0$ and $u_0, v_0 \in X$. Assume the following conditions:

(F3′) For any fixed $\theta \in \mathbf{R}^N$ and $t > 0$, the functions

$$\mathfrak{f}_{t,\theta}(x,u) := u^{\frac{\alpha-3}{\alpha-2}} f\left(x, t, u^{\frac{2}{\alpha-1}}, v^{\frac{2}{\beta-1}}, u^{\frac{3-\alpha}{\alpha-1}}\theta\right),$$
$$\mathfrak{g}_{t,\theta}(x,v) := v^{\frac{\beta-3}{\beta-2}} g\left(x, t, v^{\frac{2}{\beta-1}}, u^{\frac{2}{\alpha-1}}, v^{\frac{3-\beta}{\beta-1}}\theta\right)$$

are convex with respect to $(x,u,v) \in \Omega \times (0,\infty)^2$.

Then, setting $U = \alpha u^{\alpha-1}$ and $V = \beta v^{\beta-1}$, we have

$$\begin{cases} \partial_t U - U\Delta U + \tilde{f}(x,t,U,V,\nabla U) - \dfrac{1}{\alpha-1}|\nabla U|^2 = 0 & \text{in} \quad D \\ \partial_t V - V\Delta V + \tilde{g}(x,t,U,V,\nabla V) - \dfrac{1}{\beta-1}|\nabla V|^2 = 0 & \text{in} \quad D, \end{cases} \tag{5.10}$$

where

$$
\begin{aligned}
&\tilde{f}(x,t,U,V,\nabla U)\\
&:= \alpha(\alpha-1)\left(\frac{U}{\alpha}\right)^{\frac{\alpha-2}{\alpha-1}} f\left(x,t,\left(\frac{U}{\alpha}\right)^{\frac{1}{\alpha-1}},\left(\frac{V}{\beta}\right)^{\frac{1}{\beta-1}},\frac{1}{\alpha(\alpha-1)}\left(\frac{U}{\alpha}\right)^{\frac{2-\alpha}{\alpha-1}}\nabla U\right),\\
&\tilde{g}(x,t,U,V,\nabla U)\\
&:= \beta(\beta-1)\left(\frac{V}{\beta}\right)^{\frac{\beta-2}{\beta-1}} g\left(x,t,\left(\frac{U}{\alpha}\right)^{\frac{1}{\alpha-1}},\left(\frac{V}{\beta}\right)^{\frac{1}{\beta-1}},\frac{1}{\beta(\beta-1)}\left(\frac{V}{\beta}\right)^{\frac{2-\beta}{\beta-1}}\nabla V\right).
\end{aligned}
$$

By a similar argument as in the proof of Theorem 5.1 with the aid of (F3′), we can apply Theorem 4.1 with $p = 1/2$ to problem (5.10). Indeed, if the viscosity comparison principle and regularity theorems hold for problem (5.9), then U and V are spatially $1/2$ concave in D, which means that u and v are spatially $(\alpha-1)/2$ concave and $(\beta-1)/2$ concave in D, respectively. (We leave the details to the reader.) Theorem 1.3 is a direct consequence of the consideration above.

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