Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XX (2020), 217-237

Segre's regularity bound for fat point schemes

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Abstract. Motivated by questions in interpolation theory and on linear systems of rational varieties, one is interested in upper bounds for the Castelnuovo-Mumford regularity of arbitrary subschemes of fat points. An optimal upper bound, named after Segre, was conjectured by Trung and, independently, by Fatabbi and Lorenzini. It is shown that this conjecture is true. Furthermore, a generalized regularity bound is established that improves the Segre bound in some cases. Among the arguments is a new partition result for matroids.

Mathematics Subject Classification (2010): 14C20 (primary); 13D40, 14N20, 13D02, 05B35 (secondary).

1. Introduction

Given s distinct points $P_1, \ldots, P_s$ of projective space and positive integers $m_1, \ldots, m_s$, we consider homogeneous polynomials that vanish at P_i to order m_i for $i = 1, \ldots, s$. Equivalently, these are the polynomials such that P_i is a root of all partial derivatives of order less than m_i for all i. The set of all these polynomials is the homogeneous I_X of the fat point scheme $X = \sum_{i=1}^{s} m_i P_i$. The vector space dimension of the degree d polynomials in I_X is known if d is large. In geometric language, the fat points scheme X imposes independent conditions on forms of degree $d \gg 0$. The least integer d such that this is true for degree d forms is called the *regularity index* of X, denoted $r(X)$. It was conjectured by Trung (as reported in [16]) and, independently, by Fatabbi and Lorenzini in [12] that $r(X) \leq \operatorname{Seg} X$, where $\operatorname{Seg} X$ is

$$\operatorname{Seg} X := \max \left\{ \left\lceil \frac{-1 + \sum_{P_i \in L} m_i}{\dim L} \right\rceil \mid L \subseteq \mathbb{P}^n \text{ a positive-dimensional linear subspace} \right\}.$$

The first author was partially supported by Simons Foundation grant #317096.

Received February 09, 2017; accepted in revised form April 04, 2018.
Published online March 2020.

The number Seg X (see also Remark 4.3) is called the *Segre bound* because B. Segre [14] proved the conjecture in the case where the given points are in a projective plane and no three of them are collinear. Segre's result was extended to $\mathbb{P}^n$ under the assumption that the given points $P_1, \dots, P_s \in \mathbb{P}^n$ are in linearly general position, that is, any subset of $n+1$ of these points spans $\mathbb{P}^n$ (see [7]). Without this assumption, the conjecture has been shown in rather few cases, namely:

- For any fat point subscheme of $\mathbb{P}^2$ in [11,15], independently;
- For any fat point subscheme of $\mathbb{P}^3$ in [12,16], independently;
- If $s \leq n+3$ and the s points span $\mathbb{P}^n$ in [5].

Furthermore, there are partial results for certain fat point subschemes of $\mathbb{P}^4$ (see [3,4]) and for some fat point subschemes of $\mathbb{P}^n$ supported at at most $2n-1$ points (see [6]). In this paper we establish the conjecture in full generality, that is, we show $r(X) \leq \operatorname{Seg} X$ for each fat point subscheme X of some projective space. This bound cannot be improved in general (see Corollary 5.5).

Bounding the regularity index of a fat point scheme X is equivalent to bounding its Castelnuovo-Mumford regularity

$$\operatorname{reg}(X) = \min\left\{m \in \mathbb{Z} \mid H^1(\mathbb{P}^n, \mathcal{I}_X(m-1)) = 0\right\},$$

where $\mathcal{I}_X$ is the ideal sheaf of X, because $r(X) = \operatorname{reg}(X) - 1$ (see, *e.g.*, Lemma 3.1). Thus, by [10, Theorem 4.1] our results have consequences for interpolation problems.

If the points $P_1, \dots, P_s$ are generic, then one expects better bounds for the regularity index. Indeed, for generic points a naive dimension count suggests the precise value of the regularity index. In [2], Alexander and Hirschowitz showed that this naive count is correct in sufficiently large degrees. Moreover, if all points have multiplicity two they completely classified the exceptions in [1]. In all other cases, similarly complete results are not known. In contrast, the Segre bound is true for any fat point scheme. Moreover, we establish a generalization of it (see Theorem 5.6) that improves Segre's bound considerably in some cases. In particular, this is true if many of the points in the support are generic (see Example 5.7).

Let us briefly describe the organization of this paper. In Section 2, a crucial new result on matroid partitions is established. Section 3 discusses refinements of a classical tool, the use of residual subschemes. Both sets of techniques are first combined in order to establish Segre's bound for reduced zero-dimensional schemes. This is carried out in Section 4. The arguments in the case of arbitrary fat point schemes are considerably more involved. This is the subject of Section 5. There, also the optimality and a modification of the Segre bound are discussed.

ACKNOWLEDGEMENTS. Both authors thank Giuliana Fatabbi and Anna Lorenzi for comments that helped improve the exposition. They are also grateful to the referee for further thoughtful comments and suggestions.

2. Matroid partitions

The goal of this section is to establish a result on matroids that will be a key ingredient for our results on the regularity of a fat point scheme. In order to make the paper accessible to a wide audience we recall some basic facts on matroid. For details we refer to [13].

A *matroid* M on a finite *ground set* E is a family of subsets of E, called *independent sets*, that is closed under inclusion, that is, any subset of an independent set is independent, and has the additional property that all maximal independent subsets of any subset $A \subseteq E$ have the same cardinality. This maximum cardinality is called the *rank* of M, denoted $\mathrm{rk}(M)$. More generally, the *rank* of any subset A of E is the maximum cardinality of an independent subset of A. It is denoted by $\mathrm{rk}_M(A)$ or simply $\mathrm{rk}(A)$ if the matroid M is understood. Equivalently, a matroid on E can be described by means of a function $r_M : 2^E \to \mathbb{N}_0$, which has the following three properties:

(i) $0 \leq r_M(A) \leq |E|$ for all A;
(ii) $r_M(A) \leq r_M(B)$ if $A \subseteq B$;
(iii) $r_M(A \cap B) + r_M(A \cup B) \leq r_M(A) + r_M(B)$ for all $A, B \subseteq E$.

Then the subsets I of E with $r_M(I) = |I|$ are the independent subsets of a matroid M and r_M is called the *rank function* of M.

The *closure* or *span* of a subset $A \subseteq E$ is the set

$$\mathrm{Cl}_M(A) = \{e \in E \mid \mathrm{rk}(A + e) = \mathrm{rk}(A)\},$$

where we use the simplified notation $A + e = A \cup \{e\}$. Similarly, we write $C - e$ for $C \setminus \{e\}$.

We will discuss partitions of a ground set into independent sets. The following characterization is due to Edmonds and Fulkerson [9, Theorem 1c].

Theorem 2.1. *Given matroids $M_1, \dots, M_k$ on a ground set E with rank functions $\mathrm{rk}_1, \dots, \mathrm{rk}_k$, there is a partition $E = I_1 \sqcup \cdots \sqcup I_k$ such that each set I_j is independent in M_j if and only if, for each subset $A \subseteq E$, one has $|A| \leq \sum_{j=1}^{k} \mathrm{rk}_j(A)$.*

If all matroids are equal, one obtains the following earlier criterion by Edmonds [8].

Corollary 2.2. *Given a matroid, there is a partition of its ground set E into k independent sets if and only if, for each subset $A \subseteq E$, one has $|A| \leq \mathrm{rk}(A) \cdot k$.*

Strengthening the assumption, one can find a partition with additional properties.

Theorem 2.3. *Let $\tilde{M}$ be a matroid on $\tilde{E} \neq \emptyset$, and let k and p be non-negative integers. Assume there is a subset $E \neq \emptyset$ of $\tilde{E}$ such that*

$$|A| \leq k \cdot \mathrm{rk}_{\tilde{M}} A - p$$

for each non-empty subset $A \subseteq E$, and fix an integer q with $0 \leq q \leq p$. Then, for each q-tuple $(e_1, \dots, e_q) \in \tilde{E}^q$, there are disjoint independent sets $\tilde{I}_1, \dots, \tilde{I}_q$ of E with the following property: If $(a_1, \dots, a_p) \in \tilde{E}^p$ is a p-tuple whose first q entries are $e_1, \dots, e_q$, that is, $a_i = e_i$ if $1 \leq i \leq q$, then there is a partition $E = I_1 \sqcup \cdots \sqcup I_k$ into independent sets such that $a_j \notin \mathrm{Cl}(I_j)$ whenever $1 \leq j \leq p$ and $I_j = \tilde{I}_j$ for $j = 1 \dots, q$.

Remark 2.4. (i) If $\tilde{E} = E$ and $p = 0$, then this result is just Corollary 2.2.

(ii) The assumption of Theorem 2.3 implies $k > p$. Indeed, consider any one-element subset A of E. If its rank is zero, then the assumption gives $1 = |A| \leq -p$, a contradiction to $p \geq 0$. Hence the rank of A must be one, and we obtain $1 \leq k - p$.

The proof of Theorem 2.3 requires some preparation. Recall that matroids can also be characterized by their circuits. A *circuit* of a matroid M on E is a minimal dependent subset $C \subseteq E$, that is, C is dependent, but every proper subset of C is independent.

Fix integers k, p with $k > p \geq 0$ and consider the function $f : 2^E \to \mathbb{Z}$ defined by

$$f(A) = k \cdot \mathrm{rk}(A) - p.$$

Moreover, let

$$\mathcal{C}(f) = \{C \subseteq E \mid \emptyset \neq C \text{ is minimal with } f(C) < |C|\}.$$

By [13, Proposition 12.1.1], there is a matroid on E whose circuits are precisely the elements of $\mathcal{C}(f)$. We denote this matroid by $M_{k,p}$ or $M(f)$. Thus, a non-empty subset $J \subseteq E$ is independent in $M(f)$ if and only if $|A| \leq f(A)$ for every non-empty subset $A \subseteq J$. We also need the following observation.

Lemma 2.5. *If C is a circuit of $M(f)$ and $e \in C$, then $e \in \mathrm{Cl}_M(C)$ and $|C| = k \cdot \mathrm{rk}_M(C) - p + 1$.*

Proof. Since $C - e$ is independent in $M(f)$ we know $f(C - e) \geq |C - e|$ by the above observation. Thus, we obtain

$$|C| > f(C) \geq f(C - e) \geq |C - e| = |C| - 1,$$

which forces $|C| - 1 = f(C - e) = f(C) = k \cdot \mathrm{rk}_M(C) - p$, and thus $\mathrm{rk}_M(C) = \mathrm{rk}_M(C - e)$. □

To simplify notation we will simply write $\mathrm{rk}(A)$ and $\mathrm{Cl}(A)$ if these concepts refer to the original matroid M. We are ready to establish a key result.

Proposition 2.6. *Let M be a matroid on $E \neq \emptyset$, and let k and p be non-negative integers. Assume that*

$$|A| \leq (k + 1) \cdot \mathrm{rk}\, A - (p + 1) \tag{2.1}$$

for each non-empty subset $A \subseteq E$. Then the rank of $M_{k,p}$ satisfies

$$\mathrm{rk}_{M_{k,p}}(E) \geq |E| - \mathrm{rk}(E) + 1.$$

Proof. Notice that by applying Assumption (2.1) to a set with one element, we get $k > p$. Thus, the matroid $M_{k,p} = M(f)$ with $f(A) = k \cdot \mathrm{rk}(A) - p$ is well-defined.

Set $r = \mathrm{rk}\, M = \mathrm{rk}(E)$, and let $I \subseteq E$ be any independent set of $M(f)$. It is enough to show: If $|I| < |E| - r + 1$, then there is some $b \in E - I$ such that $I + b$ is independent in $M(f)$. Indeed, if the latter statement is true we get $\mathrm{rk}_{M(f)}(E) \geq |I + b| = 1 + |I|$. If $|I + b|$ is still less than $|E| - r + 1$, then we repeat the argument until we get an independent set I of $M(f)$ with $|I| \geq |E| - r + 1$, which then implies $\mathrm{rk}_{M(f)}(E) \geq |I| \geq |E| - r + 1$, as desired.

Suppose now the statement in the previous paragraph is not true, that is, its conclusion fails for some I with $|E - I| \geq r$. Then, fixing any subset $B = \{b_1, \ldots, b_r\}$ of r elements in $E - I$, the set $I + b_i$ must be dependent in $M(f)$ for every $b_i \in B$,. Hence, for each i, there is a minimal subset $F_i \subset I$ such that $F_i + b_i$ is dependent in $M(f)$. Thus, $F_i + b_i$ is a circuit of $M(f)$. Using Lemma 2.5, we conclude that, for each i, one has

$$b_i \in \mathrm{Cl}(F_i) \text{ and } |F_i| = k \cdot \mathrm{rk}(F_i) - p.$$

Our next goal is to show the following assertion.

Claim: There are $s \leq r$ subsets $A_1, \ldots, A_s$ of I that satisfy the following conditions:

(i) $|A_i| = k \cdot \mathrm{rk}(A_i) - p$;
(ii) $B \subset \bigcup_{i=1}^{s} \mathrm{Cl}(A_i)$;
(iii) $\mathrm{rk}(A_i \cup A_j) = \mathrm{rk}(A_i) + \mathrm{rk}(A_j)$ if $i \neq j$.

We prove this claim recursively. Initially, put $s = r$ and $A_i = F_i$ for $i = 1, \ldots, r$. Then the set $\{A_1, \ldots, A_s\}$ satisfies conditions (i) and (ii). Thus, the claim follows once we have shown: If a set $\{A_1, \ldots, A_s\}$ satisfies conditions (i) and (ii), but there are elements A_i and A_j with $i \neq j$ and $\mathrm{rk}(A_i \cup A_j) \neq \mathrm{rk}(A_i) + \mathrm{rk}(A_j)$, then, setting $\hat{A}_i = I \cap \mathrm{Cl}(A_i \cup A_j)$, the set $\{A_1, ..., A_s\} - \{A_i, A_j\} + \hat{A}_i$ also satisfies conditions (i) and (ii).

Indeed, repeating this process as many times as necessary will result in a collection of subsets of I that satisfies conditions (i)-(iii) because in each step the number of subsets decreases and condition (iii) is trivially satisfied if $s = 1$.

In order to establish the recursive step, it is enough to show that $|\hat{A}_i| = k \cdot \mathrm{rk}(\hat{A}_i) - p$ because $A_i \cup A_j \subseteq \hat{A}_i$ implies that condition (ii) is satisfied for the modified collection.

To this end notice that $|A_i \cap A_j| \leq f(A_i \cap A_j)$ if $A_i \cap A_j \neq \emptyset$ because $A_i \cap A_j$ is independent in $M(f)$. It follows that $|A_i \cap A_j| \leq k[\mathrm{rk}(A_i) + \mathrm{rk}(A_j) - \mathrm{rk}(A_i \cup$

$A_j)] - p$. Observe that this inequality holds even if $A_i \cap A_j = \emptyset$ because by our hypothesis $\mathrm{rk}(A_i) + \mathrm{rk}(A_j) > \mathrm{rk}(A_i \cup A_j)$, and $k > p$. Hence, we obtain

$$\begin{aligned}|\hat{A}_i| &\geq |A_i \cup A_j| = |A_i| + |A_j| - |A_i \cap A_j| \\ &\geq [k \cdot \mathrm{rk}(A_i) - p] + \big[k \cdot \mathrm{rk}(A_j) - p\big] \\ &\quad - \big[k \cdot (\mathrm{rk}(A_i) + \mathrm{rk}(A_j) - \mathrm{rk}(A_i \cup A_j)) - p\big] \\ &= k \cdot \mathrm{rk}(A_i \cup A_j) - p \\ &= k \cdot \mathrm{rk}(\hat{A}_i) - p.\end{aligned}$$

Since $\hat{A}_i$ is independent in $M(f)$, we also have $|\hat{A}_i| \leq f(\hat{A}_i) = k \cdot \mathrm{rk}(\hat{A}_i) - p$, and the desired equality $|\hat{A}_i| = k \cdot \mathrm{rk}(\hat{A}_i) - p$ follows. Thus, the above claim is shown.

Now we proceed with the proof of the proposition. Let $A_1, \ldots, A_s$ be a collection of non-empty subsets of I satisfying conditions (i)-(iii) above. Set $B_i = B \cap \mathrm{Cl}(A_i)$. Using $B \subseteq E - I$ and applying the assumption to $A_i \cup B_i$, we get

$$|A_i| + |B_i| = |A_i \cup B_i| \leq (k+1) \cdot \mathrm{rk}(A_i \cup B_i) - (p+1) = (k+1) \cdot \mathrm{rk}(A_i) - (p+1).$$

Hence condition (i) gives $|B_i| \leq \mathrm{rk}(A_i) - 1$. Taking also into account that the sets $A_1, \ldots, A_s$ are necessarily disjoint, we obtain

$$\begin{aligned}|I| &\geq \left|\bigcup_{i=1}^{s} A_i\right| = \sum_{i=1}^{s} |A_i| = \sum_{i=1}^{s} \big[k \cdot \mathrm{rk}(A_i) - p\big] \\ &\geq \sum_{i=1}^{s} \big[k \cdot (|B_i| + 1) - p\big] \\ &\geq k \left(\sum_{i=1}^{s} |B_i|\right) + s(k-p) = k \cdot |B| + s(k-p).\end{aligned}$$

Since $k > p$ and $|B| = r = \mathrm{rk}(E)$, it follows that

$$|I| \geq k \cdot \mathrm{rk}(E) + 1.$$

However, this is impossible because

$$|I| \leq |E - B| = |E| - |B| \leq (k+1) \cdot \mathrm{rk}(E) - (p+1) - |B| = k \cdot \mathrm{rk}(E) - (p+1).$$

Thus, the argument is complete. □

In order to establish a consequence of this result, we need two particular matroid constructions.

Definition 2.7. Let M be a matroid on E.

(i) Suppose M is a submatroid of a matroid $\tilde{M}$ on $\tilde{E}$. For any $e \in \tilde{E} \setminus E$, define a matroid M/e on E by the rank function $\operatorname{rk}_{M/e}(A) = \operatorname{rk}_{\tilde{M}}(A+e)-1$ for subsets $A \subseteq E$. It is called an *elementary quotient* of M. Note that the independent sets of M/e are the independent sets of M whose span does not contain e;

(ii) Let S be any subset of E. Realize the disjoint union $E \sqcup S$ as $(E, 0) \cup (S, 1)$. Denote by M_{+S} the matroid whose independent sets are of the form $(I_1, 0) \cup (I_2, 1)$ with $\operatorname{rk}_M(I_1 \cup I_2) = |I_1| + |I_2|$. The matroid M_{+S} is called the *parallel extension of M by S*.

It is straightforward to check that M_{+S} is indeed a matroid. Its rank is equal to the rank of M. More generally, if $A = (A_1, 0) \cup (A_2, 1)$ is any subset of $E \sqcup S$, then $\operatorname{rk}_{M_{+S}}(A) = \operatorname{rk}_M(A_1 \cup A_2)$.

Corollary 2.8. *Let $\tilde{M}$ be a matroid on $\tilde{E} \neq \emptyset$, and let M be the submatroid induced on a subset $E \neq \emptyset$ of $\tilde{E}$. Assume that, for non-negative integers k and p and each non-empty subset $A \subseteq E$, one has*

$$|A| \leq (k+1) \cdot \operatorname{rk} A - (p+1).$$

Then, for any $e \in \tilde{E}$, there is an independent set $I \subset E$ such that $e \notin \operatorname{Cl}(I)$ and

$$|B| \leq k \cdot \operatorname{rk}(B) - p$$

for each non-empty subset $B \subseteq E - I$.

Proof. Consider the function $f : 2^E \to \mathbb{Z}$ defined by $f(A) = k \cdot \operatorname{rk}(A) - p$, and denote the submatroid of $\tilde{M}$ induced on E by M.

Let $A \neq \emptyset$ be any subset of E. Applying Proposition 2.6 to the submatroid of M induced on A, we get $\operatorname{rk}_{A(f)}(A) \geq |A| - \operatorname{rk}(A) + 1$, and so

$$|A| \leq \operatorname{rk}(A) + \operatorname{rk}_{A(f)}(A) - 1 \leq \operatorname{rk}(A) + \operatorname{rk}_{M(f)}(A) - 1. \tag{2.2}$$

We now consider two cases.

Case 1: Suppose e is not in E. Consider the elementary quotient M/e on E. By definition, for each subset $A \subseteq E$, one has $\operatorname{rk}_{M/e}(A) = \operatorname{rk}_{\tilde{M}}(A+e) - 1$. It follows that $\operatorname{rk}_{M/e}(A) \geq \operatorname{rk}(A) - 1$. Hence, Equation (2.2) gives

$$|A| \leq \operatorname{rk}_{M/e}(A) + \operatorname{rk}_{M(f)}(A).$$

Using Theorem 2.1, we conclude that there is a decomposition $E = I \sqcup J$ such that I is independent in M/e and J is independent in $M(f)$. Be definition of M/e, the span of I does not contain e. Therefore, $E = I \sqcup J$ is a partition with the required properties because, for each subset $B \neq \emptyset$ of J, one has

$$|B| \leq f(B) = k \cdot \operatorname{rk}(B) - p$$

as J is independent in $M(f)$.

Case 2: Suppose e is in E. Then consider first the parallel extension $M_{+\{e\}}$ of M on the set $(E, 0) \cup \{(e, 1)\}$. Second, passing to an elementary quotient of $M_{+\{e\}}$, we get a matroid $M_{+\{e\}}/(e, 1)$ on the ground set $(E, 0)$. To simplify notation, let us denote the latter matroid by M_{+e}/e and identify its ground set with E. Thus, we get for $A \subseteq E$ that

$$\operatorname{rk}_{M_{+e}/e}(A) = \operatorname{rk}_{M_{+\{e\}}}((A, 0) \cup \{(e, 1)\}) - 1 = \operatorname{rk}(A + e) - 1 \geq \operatorname{rk}(A) - 1.$$

Now we conclude as in Case 1, using M_{+e}/e in place of the matroid M/e. □

We are now in a position to establish the announced partition result.

Proof of Theorem 2.3. If $p = 0$, then the assertion is true by Edmond's criterion (Corollary 2.2).

Let $p \geq 1$. First, we construct a suitable partition for a fixed p-tuple $(a_1, \dots, a_p) \in \tilde{E}^p$ step by step. Consider $a_1 \in E$. By Corollary 2.8, there is a partition $E = I_1 \sqcup J_1$ such that I_1 is independent in M, $e_1 \notin \operatorname{Cl}(I_1)$, and $|B| \leq (k - 1) \cdot \operatorname{rk}(B) - (p - 1)$ for each non-empty subset $B \subseteq J_1$. Thus, we are done if $p = 1$. If $p \geq 2$, we apply Corollary 2.8 again, this time to $a_2 \in E$ and the submatroid of M induced on J_1. After p applications of Corollary 2.8, we obtain a partition $E = I_1 \sqcup \ldots \sqcup I_p \sqcup J_p$ such that $I_1, \dots, I_p$ are independent in M, a_j is not in the span of I_j for each j, and $|B| \leq (k - p) \cdot \operatorname{rk}(B)$ for each non-empty subset $B \subseteq J_p$. Applying Corollary 2.2 to the submatroid on J_p, we get a partition $J_p = I_{p+1} \sqcup \ldots \sqcup I_k$ into independent sets of M. This produces a desired partition for a fixed $(a_1, \dots, a_p)$.

Second, we note that in the above construction the first p independent sets are obtained sequentially. Once the sets $I_1, \dots, I_{j-1}$ have been found, the set I_j is determined in the complement of $I_1 \sqcup \ldots \sqcup I_{j-1}$. It depends on the choice of a_j, but not on the elements $a_{j+1}, \dots, a_k$. This shows in particular that the sets $I_1, \dots, I_q$ are independent of the elements $a_{q+1}, \dots, a_k$. Thus, the argument is complete. □

Remark 2.9. (i) Using the notation of the proof of Corollary 2.8, the partition result in Theorem 2.3 can be also stated as follows: There is a partition $E = I_1 \sqcup \cdots \sqcup I_k$ such that $I_{p+1}, \dots, I_k$ are independent in M and, for each $j = 1, \dots, p$, the set I_j is independent in M/a_j if $a_j \notin E$ and independent in M_{+a_j}/a_j if $a_j \in E$, respectively.

(ii) If the ground set E of a matroid can be partitioned into k independent sets, then Edmond's criterion (Corollary 2.2) implies that there is an independent set I such that $|A| \leq (k - 1) \cdot \operatorname{rk} A$ for each subset A of $E \setminus I$. Thus, for a matroid satisfying the assumptions of Theorem 2.3, it is natural to wonder if there is an independent set I of E such that, for each $e \in I$ and each $A \subset (E \setminus I) + e$, one has $|A| \leq (k - 1) \cdot \operatorname{rk}_{\tilde{M}} A - p$. However, this is not always possible, not even for representable matroids, see Example 4.8.

3. Inductive techniques

We now begin considering zero-dimensional subschemes of projective space. In this section we collect some facts that are used in subsequent parts of this note.

Let K be an arbitrary field, and let X be any projective subscheme of some projective space $\mathbb{P}^n = \mathbb{P}^n_K$. For short, we often write $H^1(\mathcal{I}_X(j))$ instead of $H^1(\mathbb{P}^n, \mathcal{I}_X(j))$ for the first cohomology of its ideal sheaf $\mathcal{I}_X$. We use $R = K[x_0, \dots, x_n]$ to denote the coordinate ring of $\mathbb{P}^n$.

Lemma 3.1. *Let $X \subset \mathbb{P}^n$ be a zero-dimensional subscheme.*

(a) *Then $r(X) = \min\{j \in \mathbb{Z} \mid H^1(\mathcal{I}_X(j)) = 0\}$;*
(b) *For any zero-dimensional subscheme Z of X, one has that $r(Z) \le r(X)$.*

Proof. These results are known to specialists. We include a proof for the convenience of the reader. Part (a) is a consequence of

$$h_X(j) - \deg X = -\dim_K H^1(\mathcal{I}_X(j)).$$

This relation also shows that $h_X(j) \le \deg X$ for all integers j and that equality is true if and only if $j \ge r(X)$. Hence, the exact sequence $0 \to I_Z/I_X \to R/I_X \to R/I_Z \to 0$ gives that $h_X(j) = \deg X$ implies $h_Z(j) = \deg Z$. Now (b) follows. □

A special case of Lemma 3.1(b) has been shown in [17, Proposition 3.2]. We also need the following fact about the Castelnuovo-Mumford regularity, which can be found, *e.g.*, in [10, Corollary 4.4].

Lemma 3.2. *If $A \neq 0$ is an artininan graded K-algebra, then one has*

$$\operatorname{reg}(A) = \max\{j \mid [A]_j \neq 0\}.$$

The following observation is an extension of [7, Lemma 1].

Lemma 3.3. *Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme, and let $P \in \mathbb{P}^n$ be a point that is not in the support of Z. Then one has, for every integer $m \ge 1$,*

$$r(Z + mP) = \max\{m-1,\ r(Z),\ 1 + \operatorname{reg}(R/(I_Z + I_P^m))\}.$$

Proof. The argument is essentially given in [7]. We recall it for the reader's convenience.

Consider the Mayer-Vietoris sequence

$$0 \to R/I_{Z+mP} \to R/I_Z \oplus R/I_{mP} \to R/(I_Z + I_P^m) \to 0.$$

Since $\deg(Z + mP) = \deg Z + \deg(mP)$ and $r(mP) = m - 1$, it shows that $h_{Z+mP}(j) = \deg(Z+mP)$ if and only if $h_Z(j) = \deg Z$, $h_{mP}(j) = \deg mP$, and $[R/(I_Z+I_P^m)]_j = 0$. Since $R/(I_Z+I_P^m)$ is artinian we conclude by Lemma 3.2. □

The following result follows from a standard residual sequence (see [12, Theorem 3.2] for a special case).

Lemma 3.4 (Inductive Technique 1). *Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme, and let $F \subset \mathbb{P}^n$ be a hypersurface defined by a form $f \in R$. Denote by $\emptyset \neq W \subset \mathbb{P}^n$ the residual of Z with respect to F (defined by $I_Z : f$). If $Z \cap F \neq \emptyset$, then one has*

$$r(Z) \leq \max\{r(W) + \deg F,\ r(Z \cap F)\}.$$

Proof. Let $d = \deg F$. Multiplication by f induces the following exact sequence of ideal sheaves

$$0 \to \mathcal{I}_W(-d) \to \mathcal{I}_Z \to \mathcal{I}_{Z \cap F} \to 0.$$

Its long exact cohomology sequence gives, for all integers j,

$$H^1(\mathcal{I}_W(j-d)) \to H^1(\mathcal{I}_Z(j)) \to H^1(\mathcal{I}_{Z\cap F}(j)).$$

Now the claim follows because $r(Z) = \min\{j \in \mathbb{Z} \mid H^1(\mathcal{I}_Z(j)) = 0\}$ (see Lemma 3.1). □

If a hypersurface F is defined by a form f, then we also write $\mathrm{Res}_f(Z)$ for $\mathrm{Res}_F(Z)$.

For induction on the multiplicity of a point in the support of a fat point scheme, the statement below will be useful.

Lemma 3.5 (Inductive Technique 2). *Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme, and let $P \in \mathbb{P}^n$ be a point that is not in the support of Z. Fix integers m and k with $1 \leq k \leq m-1$. Set $t = \binom{n-1+k}{k}$. Assume there are homogeneous polynomials $g_1, \dots, g_t \in R$ and $f_1, \dots, f_t \in R$ such that $I_P^k = (g_1, \dots, g_t)$, $f_i(P) \neq 0$, and*

$$r\left(\mathrm{Res}_{g_i f_i}(Z + mP)\right) \leq b - k - \deg f_i$$

for all $i \in \{1, 2, \dots, t\}$ and some integer $b \geq m-1$. If $r(Z + (m-1)P) \leq b$, then $r(Z + mP) \leq b$.

Proof. Note that it is enough to show $[R/(I_Z + I_P^m)]_b = 0$. Indeed, $[R/(I_Z + I_P^m)]_b = 0$ implies $1 + \mathrm{reg}(R/(I_Z + I_P^m)) \leq b$ by Lemma 3.2. Furthermore, the assumption $r(Z + (m-1)P) \leq b$ gives $r(Z) \leq b$ by Lemma 3.1(b). Since we also assume $m - 1 \leq b$, Lemma 3.3 shows $r(Z + mP) \leq b$.

In order to prove $[R/(I_Z + I_P^m)]_b = 0$ observe that

$$\dim_K \left[R/\left(I_Z + I_P^m\right)\right]_b = \sum_{j=0}^{m-1} \dim_K \left[\left(I_Z + I_P^j\right)/\left(I_Z + I_P^{j+1}\right)\right]_b.$$

By assumption and Lemma 3.1, we know $r(Z + jP) \leq b$ if $0 \leq j < m$. Hence Lemma 3.3 gives $[I_Z + I_P^j]_b = [R]_b$. It follows that

$$\dim_K \left[R/\left(I_Z + I_P^m\right)\right]_b = \dim_K \left[\left(I_Z + I_P^{m-1}\right)/\left(I_Z + I_P^m\right)\right]_b.$$

Thus, we are done once we have shown

$$\left[I_Z + I_P^{m-1}\right]_b = \left[I_Z + I_P^m\right]_b. \tag{3.1}$$

Let $\ell \in R$ be any linear form that does not vanish at P. Then $(x_0, \dots, x_n) = (\ell, I_P)$. Since I_P^{m-1} is generated by polynomials of degree $m-1$, it follows that Equality (3.1) is true if and only if

$$\ell^{b-m+1} \cdot \left[I_P^{m-1}\right]_{m-1} \subset I_Z + I_P^m. \tag{3.2}$$

Observe that, for each $i \in [t] = \{1, 2, \dots, t\}$, the scheme $W_i := \operatorname{Res}_{g_i f_i}(Z + mP))$ is defined by $I_{Z+mP} : (g_i f_i)$ and has multiplicity $m-k$ at P because $f_i(P) \neq 0$ and g_i vanishes precisely to order k at P by assumption. Denote by J_i the homogeneous ideal of $W_i - (m-k)P$. Thus, $I_{W_i} = J_i \cap I_P^{m-k}$. Hence, Lemma 3.3 gives

$$r(W_i) = \max\left\{m-k-1,\ r(W_i - (m-k)P),\ 1 + \operatorname{reg}\left(R/\left(J_i + I_P^{m-k}\right)\right)\right\}.$$

Since $r(W_i) \le b - k - d_i$ by assumption, where $d_i = \deg f_i$, we get as above, for each $i \in [t]$,

$$0 = \dim_K \left[R/\left(J_i + I_P^{m-k}\right)\right]_{b-k-d_i} = \sum_{j=0}^{m-k-1} \dim_K \left[\left(J_i + I_P^j\right)\Big/\left(J_i + I_P^{j+1}\right)\right]_{b-k-d_i}.$$

In particular, this yields $[J_i + I_P^{m-k-1}]_{b-k-d_i} = [J_i + I_P^{m-k}]_{b-k-d_i}$. We conclude

$$\ell^{b-d_i-m+1} \cdot \left[I_P^{m-k-1}\right]_{m-k-1} \subset J_i + I_P^{m-k} \tag{3.3}$$

because $b - d_i - m + 1 \ge 0$. The latter estimate follows from $(m-k)P \subset W_i$, which implies $0 \le m-k-1 = r((m-k)P) \le r(W_i) \le b-k-d_i$ (see Lemma 3.1).

Note that, for each $i \in [t]$, one has $J_i = I_Z : (g_i f_i)$. Using $g_i \in I_P^k$ this gives

$$g_i f_i \cdot \left(J_i + I_P^{m-k}\right) \subset I_Z + I_P^m.$$

Combined with Inclusion (3.3), we get

$$g_i f_i \ell^{b-d_i-m+1} \cdot \left[I_P^{m-k-1}\right]_{m-k-1} \subset I_Z + I_P^m.$$

Since $f(P_i) \neq 0$, possibly after rescaling, we may write $f_i = h_i + \ell^{d_i}$ for some $h_i \in I_P$. Substituting, we obtain,

$$g_i\left(h_i + \ell^{d_i}\right)\ell^{b-d_i-m+1} \cdot \left[I_P^{m-k-1}\right]_{m-k-1} \subset I_Z + I_P^m.$$

Now $g_i h_i \in I_P^{k+1}$ yields

$$\ell^{b-m+1} g_i \in I_Z + I_P^m \quad \text{for each } i \in [t].$$

Since $\{g_1, \dots, g_t\}$ is a K-basis of $[I_P^k]_k$, this establishes the desired Containment (3.2). ∎

4. Reduced zero-dimensional subschemes

We now establish the Segre bound for an arbitrary finite set of points. To this end we use suitable vector matroids.

Recall that a vector matroid or representable matroid M over a field K is given by an $m \times n$ matrix A with entries in K. Its ground set E is formed by the column vectors of A, and the rank of a subset of E is the dimension of the subspace of K^n they generate. Here we adapt this idea in order to use it in a projective space instead of an affine space.

Definition 4.1.

(i) For a point P of $\mathbb{P}^n$ and an integer $m \geq 1$, denote by $[P]^m$ an $(n+1) \times m$ matrix whose m columns are all equal to a vector $v \in K^{n+1}$, where v is any representative of the point P;

(ii) Let $X = \sum_{i=1}^{s} m_i P_i \subset \mathbb{P}^n$ be a fat point scheme. We write $A_X := \oplus_{i=0}^{s} [P_i]^{m_i}$ for the concatenation of the matrices $[P_i]^{m_i}$. Define the *matroid of* X on the column set E_X of A_X, denoted M_X, as the vector matroid to the matrix A_X. Thus $|V_X| = \sum_{i=1}^{s} m_i$.

Remark 4.2. (i) Since we are only interested in the span of a subset of columns, the above definition does not depend on the choice of coordinate vectors for the points. Abusing notation slightly, we will identify a non-zero vector of K^{n+1} with a point in $\mathbb{P}^n$.

(ii) For consistency of notation, rk will always refer to rank in the matroid sense, that is, to a dimension of a subspace of K^{n+1}, and dim will always refer to dimension in $\mathbb{P}^n$. Hence, if S is a subset of the column set E_X, then $\operatorname{rk}(S) = 1 + \dim_{\mathbb{P}^n} \operatorname{Span}(S)$. Furthermore, we will use Cl to refer to the closure operator in a matroid and Span to refer to the span of the points in $\mathbb{P}^n$.

Recall that the Segre bound of $X = \sum_{i=1}^{s} m_i P_i$ is

$$\operatorname{Seg}(X) = \max\left\{ \left\lceil \frac{w_L(X) - 1}{\dim L} \right\rceil \mid L \subseteq \mathbb{P}^n \text{ a positive-dimensional linear subspace} \right\},$$

where $w_L(X) = \sum_{P_i \in L} m_i$ is the *weight* of $X|_L$.

Remark 4.3. In the literature the Segre bound has also been defined as

$$\operatorname{Seg}(X) = \max\left\{ \left\lfloor \frac{w_L(X) + \dim L - 2}{\dim L} \right\rfloor \mid L \subseteq \mathbb{P}^n \text{ a positive-dimensional linear subspace} \right\}.$$

Obviously, this is equivalent to our definition above.

Lemma 4.4. *If* $X = \sum_{i=1}^{s} m_i P_i$ *is a fat point scheme whose support consists of at least two distinct points, then* $m_i \leq \operatorname{Seg}(X)$ *for all* i *and* $\operatorname{Seg}(X) \geq m_i + m_j - 1$ *whenever* $i \neq j$.

Proof. Let L be a line passing through two distinct points P_i and P_j in the support of X. Then $w_L(X) \geq m_i + m_j$, which implies $\operatorname{Seg}(X) \geq m_i + m_j - 1$. □

Remark 4.5. If $X = m_1 P_1$ is supported at a single point, then $r(X) = \operatorname{Seg} X = m_1 - 1$.

The following is the main result of this section.

Theorem 4.6. *Let $Z \subset \mathbb{P}^n$ be a fat point scheme satisfying $r(Z) \leq \operatorname{Seg}(Z)$. Then, for every point $P \in \mathbb{P}^n$ that is not in the support of Z, one has $r(Z + P) \leq \operatorname{Seg}(Z + P)$.*

Proof. We want to use inductive technique 1. To this end, consider the matrix

$$A = A_Z \oplus [P]^B = \oplus_{i=1}^s [P_i]^{m_i} \oplus [P]^B,$$

where $B = \operatorname{Seg}(Z + P)$ and $Z = \sum_{i=1}^s m_i P_i$. Let M be the vector matroid on the column set V of A. Set $X = Z + P$.

Consider any subset S of V. If $P \notin \operatorname{Span}(S)$, then the definition of weight gives

$$|\operatorname{Cl}(S)| = w_{\operatorname{Span}(S)}(Z) = w_{\operatorname{Span}(S)}(X).$$

If $P \in \operatorname{Span}(S)$, then $w_{\operatorname{Span}(S)}(X) = 1 + w_{\operatorname{Span}(S)}(Z)$, and thus

$$|\operatorname{Cl}(S)| = w_{\operatorname{Span}(S)}(X) + B - 1.$$

In either case we have

$$|S| \leq w_{\operatorname{Span}(S)}(X) + B - 1.$$

Using $\operatorname{rk}(S) = 1 + \dim_{\mathbb{P}^n} S$, the definition of $B = \operatorname{Seg}(X)$ yields, for any subset $S \subset V$ with $\operatorname{rk}(S) \geq 2$,

$$\frac{|S| - B}{\operatorname{rk}(S) - 1} \leq \frac{w_{\operatorname{Span}(S)}(X) - 1}{\dim(\operatorname{Span}(S))} \leq \operatorname{Seg}(X) = B.$$

It follows that

$$|S| \leq \operatorname{rk}(S) \cdot B.$$

This estimate is also true if $\operatorname{rk}(S) \leq 1$ as $B \geq m_i$ for all i (see Lemma 4.4). Therefore Corollary 2.2 gives that there is a partition of the column set V into B linearly independent subsets $I_1, \ldots, I_B$. Note that $P \in I_j$ for each $j \in \{1, 2, \ldots, B\}$ as B columns of the matrix A correspond to the point P. Thus, for each such j, there is a hyperplane H_j such that

$$\operatorname{Span}(I_j \setminus \{P\}) \subset H_j \quad \text{and} \quad P \notin H_j.$$

It follows that the hypersurface $F = H_1 + \cdots + H_B$ does not contain P. However, F does contain Z because any form defining F vanishes at each point P_j to order at least m_j as m_j columns of A correspond to P_j. Hence we get $\operatorname{Res}_F(X) = P$ and $X \cap F = Z$. Now Lemma 3.4 gives $r(X) \leq \max\{B, r(Z)\} = B$, as desired. □

Corollary 4.7. *If X is any reduced zero-dimension subscheme of $\mathbb{P}^n$, then $r(X) \leq$ Seg(X).*

Proof. This is true if X consists of one point (see Remark 4.5). Thus, we conclude by induction on the cardinality of X using the above theorem. □

We conclude this section with an example as promised in Remark 2.9(ii).

Example 4.8. Consider any integers $k > p > 0$, and let K be an infinite field. Let $L_1, \dots, L_t \subset K^{t-1}$ be t generic one-dimensional subspaces, where $t \geq \frac{k}{p} + 1$. On each of the lines choose generically $k - p$ points. Let M be the vector matroid on the set E of all these vectors. Then, one has for each non-empty subset $A \subset E$ that $|A| \leq k \cdot \operatorname{rk} A - p$. Indeed, if $A = E$ this follows because $|E| = t(k-p) \leq k \cdot \operatorname{rk} E - p = k \cdot (t-1) - p$ by the assumption on t. If the rank of A is at most $t-2$, then it contains at most $\operatorname{rk} A$ of the lines $L_1, \dots, L_t$, which implies $|A| \leq \operatorname{rk} A \cdot (k-p) \leq k \cdot \operatorname{rk} A - p$, as desired.

Assume now there is an independent $I \subset E$ with at most $t-2$ elements such that for each non-empty subset $B \subset E \setminus I$ one has $|B| \leq (k-1) \cdot \operatorname{rk} B - p$. Thus, $|B| \leq k - 1 - p$ if B has rank one. Consider now $B = E \setminus I$. By assumption on I, we have $|B| \geq t(k-p) - (t-2) = t(k-p-1) + 2$. However, we also obtain $|B| = \sum_{i=1}^{t} |B \cap L_i| \leq t(k-p-1)$. This contradiction shows that M is a matroid as desired in Remark 2.9(ii).

5. Arbitrary fat point schemes

The goal of this section is to establish the conjecture by Trung, Fatabbi, and Lorenzini. We also discuss the sharpness of the Segre bound and establish an alternate regularity estimate.

We need one more preparatory result on the matroid introduced in Definition 4.1.

Lemma 5.1. *Consider the vector matroid M to a fat point scheme $Z = \sum_{j=1}^{s} m_j P_j$ on the column set E_Z. Then, for every subset $S \subset E_Z$ with $\operatorname{rk} S \geq 2$, one has*

$$|S| \leq \operatorname{Seg}(Z) \cdot \{\operatorname{rk}(S) - 1\} + 1.$$

Proof. Recall that $\operatorname{rk}(S) = \dim(\operatorname{Span}(S)) + 1$ for any subset $S \subset E_Z$. Moreover, one has $|S| \leq |\operatorname{Cl}_M(S)| = w_L(Z)$, where $L = \operatorname{Span}(S)$. Hence, if $\operatorname{rk} S \geq 2$ we obtain

$$\frac{|S| - 1}{\operatorname{rk}(S) - 1} \leq \frac{w_L(Z) - 1}{\dim L} \leq \operatorname{Seg}(Z).$$

Now the claim follows. □

The following result allows us to use induction on the cardinality of the support of a fat point scheme.

Proposition 5.2. *Let $Z \subset \mathbb{P}^n$ be a fat point scheme satisfying $r(Z) \leq \operatorname{Seg}(Z)$. Then, for every point $P \in \mathbb{P}^n$ that is not in the support of Z and every integer $m \geq 1$, one has $r(Z + mP) \leq \operatorname{Seg}(Z + mP)$.*

Proof. We want to apply Inductive Technique 2 to $X = Z + mP$, where $Z = \sum_{j=1}^{s} m_j P_j$. This requires some preparation. Consider the vector matroid associated to the matrix

$$A_Z = \oplus_{i=1}^{s} [P_i]^{m_i}$$

with column set E_Z. Define another matroid M on E_Z by setting the rank of any subset $S \subseteq E_Z$ as $\operatorname{rk}_M(S) = \operatorname{rk}(S + P) - 1 = \dim \operatorname{Span}(S + P)$. Thus, we get

$$\operatorname{rk}_M(S) \geq \dim \operatorname{Span}(S) = \operatorname{rk}(S) - 1.$$

In particular, a subset I of E_Z is independent in M if and only if $I + P$ is a linearly independent subset of $\mathbb{P}^n$. Notice that the matroid M is determined by Z and P only and independent of the multiplicity of P in X. We now argue that, for every subset $S \neq \emptyset$ of E_Z, one has

$$|S| \leq \operatorname{Seg}(X) \cdot \operatorname{rk}_M(S) - (m - 1). \tag{5.1}$$

Indeed, given any subset $S \neq \emptyset$ of E_Z, extend S by m copies of P to a subset S' of E_X. Then one has $\operatorname{rk} S' \geq 2$, and thus by applying Lemma 5.1 to S' we obtain

$$|S| + m = |S'| \leq \operatorname{Seg}(X) \cdot \{\operatorname{rk}(S') - 1\} + 1 = \operatorname{Seg}(X) \cdot \operatorname{rk}_M(S) + 1,$$

which completes the argument for Estimate (5.1).

We are now going to show the following key statement.

Claim: Given Z and P as above, suppose that there are integers σ and $m \geq 1$ such that, for every non-empty subset $S \subseteq E_Z$, one has

$$|S| \leq \sigma \cdot \operatorname{rk}_M(S) - (m - 1). \tag{5.2}$$

Then there are $t = \binom{n+m-2}{n-1}$ generators $g_1, \ldots, g_t$ of I_P^{m-1} and degree $\sigma - m + 1$ forms $f_1, \ldots, f_t$ with $f_j(P) \neq 0$ such that

$$g_j f_j \in I_{Z+(m-1)P} \quad \text{for } j = 1, \ldots, t. \tag{5.3}$$

To establish this claim, we use induction on $m \geq 1$. Let $m = 1$. Then Assumption (5.2) is also true for $S = \emptyset$. Hence Corollary 2.2 gives a partition $E_Z = I_1 \sqcup \ldots \sqcup I_\sigma$ into independent sets of M. Thus, P is not in any $\operatorname{Span}(I_j)$, and so there are σ linear forms ℓ_j such that $\ell_j(P) \neq 0$ and $I_j \subset H_j$, where H_j is the hyperplane defined by ℓ_j. It follows that $f = \ell_1 \cdots \ell_\sigma$ is in I_Z and $f(P) \neq 0$, as desired.

Let $m \geq 2$. Choose a point $Q_1 \in \mathbb{P}^n \setminus \{P\}$. Pass from the vector matroid to the matrix $A_Z \oplus [Q_1]$ to a matroid $\widetilde{M}$ on $E_Z \cup \{Q_1\}$ as for M above. That is,

$\operatorname{rk}_{\widetilde{M}}(S) = \operatorname{rk}(S+P) - 1 = \dim \operatorname{Span}(S+P)$ for any subset $S \subseteq E_Z \cup \{Q_1\}$. Due to Assumption (5.2) we can apply Corollary 2.8 to obtain a partition

$$E_Z = I_1 \sqcup J_1,$$

where I_1 is independent in M, $Q_1 \notin \operatorname{Span}(I_1 + P)$, and

$$|B| \le (\sigma - 1) \cdot \operatorname{rk}_M(B) - (m-2) \tag{5.4}$$

for each subset $B \neq \emptyset$ of J_1. Let W_1 be the fat point scheme determined by J_1, that is, $W_1 = \sum_{j=1}^{s} n_j P_j$, where n_j is the number of column vectors in J_1 corresponding to the point P_j. Estimate (5.4) shows that the induction hypothesis applies to W_1. Hence, there are $u = \binom{n+m-3}{n-1}$ generators $h_1^{(1)}, \dots, h_u^{(1)}$ of I_P^{m-2} and degree $\sigma - m + 1$ forms $q_1^{(1)}, \dots, q_u^{(1)}$ with $q_j^{(1)}(P) \neq 0$ such that $h_j^{(1)} q_j^{(1)} \in I_{W_1 + (m-2)P}$ for each j.

Since Q_1 is not in the span of the linearly independent set $I_1 + P$, there is a linear form ℓ_1 such that $\ell_1(Q_1) \neq 0$ and $I_1 + P \subset H_1$, where H_1 is the hyperplane defined by ℓ_1. Taking into account that $E_Z = I_1 \sqcup J_1$, it follows that $\ell_1 h_j^{(1)} q_j^{(1)} \in I_{Z+(m-1)P}$ for each j.

Notice that the above construction of the forms $h_1^{(1)}, \dots, h_u^{(1)}, q_1^{(1)}, \dots, q_u^{(1)}$, and ℓ_1, depending on the choice of Q_1, works for any point in $\mathbb{P}^n \setminus \{P\}$. Repeating it $(n-1)$ more times by choosing alltogether points $Q_1, \dots, Q_n \in \mathbb{P}^n \setminus \{P\}$, we obtain linear forms $\ell_1, \dots, \ell_n \in I_P$ as well as n generating sets $\{h_1^{(i)}, \dots, h_u^{(i)}\}$ of I_P^{m-2}, and degree $\sigma - m + 1$ forms $q_j^{(i)}$ with $q_j^{(i)}(P) \neq 0$ such that

$$\ell_i h_j^{(i)} q_j^{(i)} \in I_{Z+(m-1)P} \quad \text{for all } i = 1, \dots, n,\ j = 1, \dots, u. \tag{5.5}$$

The forms $h_1^{(i)}, \dots, h_u^{(i)}, q_1^{(i)}, \dots, q_u^{(i)}$, and ℓ_i depend on the choice of the point Q_i, $i = 1 \dots, n$.

We now claim that by choosing the points $Q_2, \dots, Q_n$ suitably we can additionally achieve that the linear forms $\ell_1, \dots, \ell_n$ are linearly independent. We show this recursively. Let $2 \le i \le n$ and assume that points $Q_1, \dots, Q_{i-1}$ have been found such that the linear forms $\ell_1, \dots, \ell_{i-1}$ are linearly independent. Let H_j be the hyperplane defined by ℓ_j. Since $\dim(\bigcap_{j=1}^{i-1} H_j) \ge 1$, there is a point Q_i in $(\bigcap_{j=1}^{i-1} H_j) \setminus \{P\}$. By construction of H_i, the point Q_i is not contained in H_i. Thus, we get

$$\dim \bigcap_{j=1}^{i} H_j = \dim \bigcap_{j=1}^{i-1} H_j - 1 = n - (i-1) - 1 = n - i.$$

In particular, we have shown that $\dim(\bigcap_{j=1}^{n} H_j) = 0$. Since each of the hyperplanes H_j contains the point P, we conclude that the ideal of this point is $I_P = (\ell_1, \dots, \ell_n)$. Now it follows that $\{\ell_i h_j^{(i)} \mid 1 \le i \le n,\ 1 \le j \le u\}$ is a generating set of $I_P \cdot I_P^{m-2} = I_P^{m-1}$. It is not minimal. However, it contains a minimal generating set $\{f_1, \dots, f_t)$ of I_P^{m-1}, where each f_k is of the form $\ell_i h_j^{(i)}$. Setting $g_k = q_j^{(i)}$, Containment (5.5) implies the claim.

After these preparations we are ready to show $r(Z + mP) \leq \mathrm{Seg}(Z + mP)$. We use induction on $m \geq 1$. If $m = 1$, then we are done by Theorem 4.6.

Let $m \geq 2$. Estimate (5.1) shows that we can apply the above claim with $\sigma = \mathrm{Seg}(X)$ and m being the multiplicity of P in $X = Z + mP$. Adopt the notation of this claim. Since each form g_j vanishes precisely to order $m - 1$ at P, it follows that $I_{Z+mP} : f_j g_j = I_P$, and thus

$$r\left(\mathrm{Res}_{g_j f_j}(Z + mP)\right) = r(P) = 0$$

for each j. Since $Z + (m - 1)P$ is a subscheme of $Z + mP$, the definition of the Segre bound implies $\mathrm{Seg}(Z + (m - 1)P) \leq \mathrm{Seg}(Z + mP) = \mathrm{Seg}(X)$. By the induction hypothesis on m, we know $r(Z + (m - 1)P) \leq \mathrm{Seg}(Z + (m - 1)P)$, and so we get $r(Z + (m - 1)P) \leq \mathrm{Seg}(X)$. Thus, applying Lemma 3.5 we conclude that $r(Z + mP) \leq \mathrm{Seg}(X)$, as desired. □

The regularity bound announced in the introduction follows now easily.

Theorem 5.3. *If X is any fat point subscheme of $\mathbb{P}^n$, then $r(X) \leq \mathrm{Seg}(X)$.*

Proof. This is true if X consists of one point (see Remark 4.5). Thus, we conclude by induction on the cardinality of $\mathrm{Supp}\, X$ using the above proposition. □

We conclude by discussing a modification of the above Segre bound. To this end consider the d-th Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^N$, where $d \in \mathbb{N}$ and $N = \binom{n+d}{d} - 1$. We use it to compare the regularity indices of fat point schemes in $\mathbb{P}^n$ and $\mathbb{P}^N$, respectively.

Proposition 5.4. *Let $X = \sum_{i=1}^s m_i P_i$ be a fat point subscheme of $\mathbb{P}^n$. Define a fat point subscheme $\hat{X}$ of $\mathbb{P}^N$ by $\hat{X} = \sum_{i=1}^s m_i v_d(P_i)$. Then one has $\left\lceil \frac{r(X)}{d} \right\rceil \leq r(\hat{X})$.*

Moreover, if both $n = 1$ and $d(m_j + m_k) \leq 2d - 2 + \sum_{i=1}^s m_i$ for all integers j, k with $1 \leq j < k \leq s$, then this is an equality and $r(\hat{X}) = \left\lceil \frac{-1+\sum_{i=1}^d m_i}{d} \right\rceil$.

Proof. Let $S = \oplus_{j \in \mathbb{N}_0} [R]_{jd}$ be the d-th Veronese subring of $R = K[x_0, ..., x_n]$. It is a polynomial ring in variables y_a, where y_a corresponds to the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ of degree d. Consider the ring homomorphism $\varphi : S \to R$ that maps y_a onto x^a. Observe that, for each point $P \in \mathbb{P}^n$, one has $\varphi(I_{v_d(P)}) \subset I_P$. If follows that $\varphi(I_{\hat{X}}) \subset I_X$, and so $I_{\hat{X}} \subset \varphi^{-1}(I_X)$. Furthermore, the ideal $\varphi^{-1}(I_X)$ of S is saturated. Indeed, if $f \in S$ is a homogeneous polynomial that multiplies a power, say, the k-th power of the ideal generated by all the variables in S into $\varphi^{-1}(I_X)$, then $\varphi(f) \cdot (x_0, ..., x_n)^{kd} \subset I_X$. Since I_X is saturated, this implies $f \in \varphi^{-1}(I_X)$, as desired.

Thus, the ideal $\varphi^{-1}(I_X)$ is the homogenous ideal of a zero-dimensional subscheme $W \subset \mathbb{P}^N$, and one has

$$H^1(\mathbb{P}^n, \mathcal{I}_X(j)) \cong H^1\big(\mathbb{P}^N, \mathcal{I}_W(jd)\big).$$

Hence, Lemma 3.1(a) implies $r(W) = \left\lceil \frac{r(X)}{d} \right\rceil$. Since W is a subscheme of $\hat{X}$, Lemma 3.1(b) gives $r(W) \leq r(\hat{X})$, and now the first assertion follows.

In order to show the second claim, assume $n = 1$. Thus $N = d$, and $\operatorname{Supp} \hat{X}$ lies on a rational normal curve of $\mathbb{P}^d$. It follows that the support of $\hat{X}$ is in linearly general position, that is, any subset of $j + 1 \leq d + 1$ points span a j-dimensional linear subspace of $\mathbb{P}^d$. Therefore, a straightforward computation shows that the Segre bound of $\hat{X}$ is determined by the one-dimensional subspaces and $\mathbb{P}^d$, that is,

$$\operatorname{Seg} \hat{X} = \max \left\{ m_j + m_k - 1, \left\lceil \frac{-1 + \sum_{i=1}^{s} m_i}{d} \right\rceil \mid 1 \leq j < k \leq s \right\}.$$

Combining the assumption and Theorem 5.3, we obtain

$$r(\hat{X}) \leq \operatorname{Seg} \hat{X} = \left\lceil \frac{-1 + \sum_{i=1}^{s} m_i}{d} \right\rceil.$$

Since X is a subscheme of $\mathbb{P}^1$, its homogeneous ideal is a principal ideal of degree $\sum_{i=1}^{s} m_i$. Thus, $r(X) = -1 + \sum_{i=1}^{s} m_i$. Now the first assertion gives the desired equality. □

As a first consequence, we describe instances where the Segre bound in Theorem 5.3 is sharp. The result extends [7, Proposition 7].

Corollary 5.5. *Let $X \subset \mathbb{P}^n$ be a fat point subscheme, and let $L \subset \mathbb{P}^n$ be a positive-dimensional linear subspace such that* $\operatorname{Seg} X = \left\lceil \frac{w_L(X)-1}{\dim L} \right\rceil$. *If the points of* $\operatorname{Supp} X$ *that are in L lie on a rational normal curve of L, then $r(X) = \operatorname{Seg} X$.*

Proof. Consider the fat point subscheme $Y = \sum_{P_i \in L} m_i P_i$ of X such that $w_L(X) = w_L(Y)$. If $\dim L = 1$, then $w_L(Y) - 1 = r(Y) \leq r(X) \leq w_L(X) - 1$, and thus the claim follows.

Assume $\dim L \geq 2$. Considering lines through any two points in the support of X, the assumption on L gives $m_j + m_k - 1 \leq \left\lceil \frac{w_L(Y)-1}{\dim L} \right\rceil$ for all $j < k$. Hence, applying Proposition 5.4 with $\hat{X} = Y$, we conclude $r(Y) = \left\lceil \frac{-1+\sum_{P_i \in L} m_i}{\dim L} \right\rceil = \left\lceil \frac{w_L(Y)-1}{\dim L} \right\rceil = \operatorname{Seg} X$. Since $r(Y) \leq r(X)$, the desired equality follows by Theorem 5.3. □

The second consequence of Proposition 5.4 is a generalized regularity bound. Notice that the following result specializes to Theorem 5.3 if $d = 1$.

Theorem 5.6. *Given any scheme of fat points $X = \sum_{i=1}^{s} m_i P_i \subseteq \mathbb{P}^n$ and any integer $d \geq 1$, the regularity index of X is subject to the bound*

$$r(X) \leq \max \left\{ d \cdot \left\lceil \frac{-1 + \sum\limits_{P_i \in Y} m_i}{\dim_K [R/I_Y]_d - 1} \right\rceil \;\middle|\; Y \subseteq \operatorname{Supp} X \text{ and } |Y| \geq 2 \right\}.$$

Proof. Consider the d-th Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$. As above, let R and S be the coordinate rings of $\mathbb{P}^n$ and $\mathbb{P}^N$, respectively. Notice that the Segre bound of $\hat{X} = \sum_{i=1}^{s} m_i \nu_d(P_i)$ is

$$\operatorname{Seg} \hat{X} = \max \left\{ \left\lceil \frac{-1 + \sum\limits_{\nu_d(P_i) \in L} m_i}{\dim L} \right\rceil \;\middle|\; L \subseteq \mathbb{P}^N \text{ linear}, \dim L \geq 1 \right\}.$$

Consider a linear subspace $L \subset \mathbb{P}^N$ for which the right-hand side above is maximal. Set $Y = \{P_i \in \operatorname{Supp} X \mid \nu_d(P_i) \in L\}$. The assumption on L gives that $\hat{Y} = \nu_d(Y)$ is not contained in a proper subspace of L, that is, $\dim_K [S/I_{\hat{Y}}]_1 - 1 = \dim L$. Since $\dim_K [S/I_{\hat{Y}}]_1 = \dim_K [R/I_Y]_d$, Theorem 5.3 gives

$$r(\hat{X}) \leq \operatorname{Seg} \hat{X} = \left\lceil \frac{-1 + \sum\limits_{P_i \in Y} m_i}{\dim_K [R/I_Y]_d - 1} \right\rceil.$$

Using $\frac{r(X)}{d} \leq r(\hat{X})$ due to Proposition 5.4, the claim follows. □

If one has information on subsets of the points supporting a fat point scheme, then the above result can be used to obtain a better regularity bound than the Segre bound of Theorem 5.3. We illustrate this by a simple example.

Example 5.7. Let $X = \sum_{i=1}^{s} m P_i \subset \mathbb{P}^n$ be a fat point scheme, where all points have the same multiplicity m. Suppose that the support of X consists of five arbitrary points and $\binom{d+n}{n}$ generic points for some $d \geq 5$. Thus, $s = 5 + \binom{d+n}{n}$. Let $L \subset \mathbb{P}^n$ be a linear subspace of dimension k with $1 \leq k < n$. Then $|L \cap \operatorname{Supp} X| \leq k + 4$. It follows that for sufficiently large d (or n)

$$\begin{aligned} \operatorname{Seg} X &= \max \left\{ \left\lceil \frac{(k+4)m - 1}{k} \right\rceil, \left\lceil \frac{[\binom{d+n}{n} + 5]m - 1}{n} \right\rceil \middle| 1 \leq k < n \right\} \\ &= \left\lceil \frac{\binom{d+n}{n} m + 5m - 1}{n} \right\rceil. \end{aligned}$$

Consider now any subset $Y \subset \operatorname{Supp} X$ of $t \geq 2$ points. Since $d \geq 5$, one gets

$$\dim_K [R/I_Y]_d = \begin{cases} t & \text{if } t \leq \binom{n+d}{n} \\ \binom{n+d}{n} & \text{otherwise.} \end{cases}$$

Hence, Theorem 5.6 and a straightforward computation give

$$\begin{aligned} r(X) &\leq d \cdot \max \left\{ \left\lceil \frac{tm-1}{t-1} \right\rceil, \left\lceil \frac{\left[\binom{d+n}{n} + 5\right] m - 1}{\binom{d+n}{n} - 1} \right\rceil \mid 2 \leq t \leq \binom{d+n}{n} \right\} \\ &= d \cdot \max \left\{ 2m-1, \left\lceil \frac{\left[\binom{d+n}{n} + 5\right] m - 1}{\binom{d+n}{n} - 1} \right\rceil \right\}. \end{aligned}$$

For sufficiently large d (or n), this implies $r(X) \leq d(2m-1)$. In comparison, $\operatorname{Seg} X$ is essentially a polynomial function in d of degree n.

References

[1] J. ALEXANDER and A. HIRSCHOWITZ, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), 201–222.

[2] J. ALEXANDER and A. HIRSCHOWITZ, *An asymptotic vanishing theorem for generic unions of multiple points*, Invent. Math. **140** (2000), 303–325.

[3] E. BALLICO, *On the Segre upper bound of the regularity for fat points in* $\mathbb{P}^4$, I, Int. J. Pure Appl. Math. **102** (2015), 281–300.

[4] E. BALLICO, *On the Segre upper bound of the regularity for fat points in* $\mathbb{P}^4$, II, Int. J. Pure Appl. Math. **102** (2015), 301–347.

[5] E. BALLICO, O. DUMITRESCU and E. POSTINGHEL, *On Segre's bound for fat points in* $\mathbb{P}^n$, J. Pure Appl. Algebra **220** (2016), 2307–2323.

[6] G. CALUSSI, G. FATABBI and A. LORENZINI, *The regularity index of up to* $2n-1$ *equimultiple fat points of* $\mathbb{P}^n$, J. Pure Appl. Algebra (2016).

[7] M. V. CATALISANO, N. V. TRUNG and G. VALLA, *A sharp bound for the regularity index of fat points in general position*, Proc. Amer. Math. Soc. **118** (1993), 717–724.

[8] J. EDMONDS, *Minimum partition of a matroid into independent subsets*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 67–72.

[9] J. EDMONDS and D. R. FULKERSON, *Transversals and matroid partition*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 147–153.

[10] D. EISENBUD, "The Geometry of Syzygies", Graduate Texts in Mathematics, Vol. 229, Springer, New York, 2005.

[11] G. FATABBI, *Regularity index of fat points in the projective plane*, J. Algebra **170** (1994), 916–928.

[12] G. FATABBI and A. LORENZINI, *On a sharp bound for the regularity index of any set of fat points*, J. Pure Appl. Algebra **161** (2001), 91–111.

[13] J. OXLEY, "Matroid Theory", Second edition, Oxford Graduate Texts in Mathematics, Vol. 21, Oxford University Press, Oxford, 2011.

[14] B. SEGRE, *Alcune questioni su insiemi finiti di punti in Geometria Algebrica*, In: "Atti del Convegno Internazionale di Geometria Algebrica" (Torino, 1961), Rattero, Turin, 1962, 15–33.

[15] P. V. THIÊN, *On Segre bound for the regularity index of fat points in* $\mathbb{P}^2$, Acta Math. Vietnam. **24** (1999), 75–81.
[16] P. V. THIÊN, *Segre bound for the regularity index of fat points in* $\mathbb{P}^3$, J. Pure Appl. Algebra **151** (2000), 197–214.
[17] P. V. THIÊN, *Lower bound for the regularity index of fat points*, Int. J. Pure Appl. Math. **109** (2016), 745–755.

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