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Iterated convolutions and endless Riemann surfaces

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Abstract. We discuss a version of Écalle's definition of resurgence, based on the notion of endless continuability in the Borel plane. We relate this with the notion of Ω-continuability, where Ω is a discrete filtered set or a discrete doubly filtered set, and show how to construct a universal Riemann surface X_Ω whose holomorphic functions are in one-to-one correspondence with Ω-continuable functions. We then discuss the Ω-continuability of convolution products and give estimates for iterated convolutions of the form $\hat{\varphi}_1 * \cdots * \hat{\varphi}_n$. This allows us to handle non-linear operations with resurgent series, *e.g.* substitution into a convergent power series.

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1. Introduction

In this article we deal with the following version of Écalle's definition of resurgence:

Definition 1.1. A convergent power series $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ is said to be *endlessly continuable* if, for every real $L > 0$, there exists a finite subset F_L of $\mathbb{C}$ such that the holomorphic germ at 0 defined by $\hat{\varphi}$ can be analytically continued along every Lipschitz path $\gamma\colon\ [0, 1] \to \mathbb{C}$ of length smaller than L such that $\gamma(0) = 0$ and $\gamma\big((0, 1]\big) \subset \mathbb{C} \setminus F_L$. We denote by $\hat{\mathscr{R}} \subset \mathbb{C}\{\zeta\}$ the space of endlessly continuable functions.

Definition 1.2. A formal series $\tilde{\varphi}(z) = \sum_{j=0}^{\infty} \varphi_j z^{-j} \in \mathbb{C}[[z^{-1}]]$ is said to be *resurgent* if $\hat{\varphi}(\zeta) = \sum_{j=1}^{\infty} \varphi_j \frac{\zeta^{j-1}}{(j-1)!}$ is an endlessly continuable function.

In other words, the space of resurgent series is

$$\tilde{\mathscr{R}} := \mathcal{B}^{-1}(\mathbb{C}\delta \oplus \hat{\mathscr{R}}) \subset \mathbb{C}\big[[z^{-1}]\big],$$

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where $\mathcal{B}\colon \mathbb{C}[[z^{-1}]] \to \mathbb{C}\delta \oplus \mathbb{C}[[\zeta]]$ is the formal Borel transform, defined by $\mathcal{B}\tilde{\varphi} := \varphi_0\delta + \hat{\varphi}(\zeta)$ in the notation of Definition 1.2.

We will also treat the more general case of functions which are "endlessly continuable with respect to bounded direction variation": we will define a space $\hat{\mathscr{R}}^{\mathrm{dv}}$ containing $\hat{\mathscr{R}}$ and, correspondingly, a space $\tilde{\mathscr{R}}^{\mathrm{dv}}$ containing $\tilde{\mathscr{R}}$, but for the sake of simplicity, in this introduction, we stick to the simpler situation of Definitions 1.1 and 1.2.

Note that the radius of convergence of an element of $\tilde{\mathscr{R}}$ may be 0. As for the elements of $\hat{\mathscr{R}}$, we will usually identify a convergent power series and the holomorphic germ that it defines at the origin of $\mathbb{C}$, as well as the holomorphic function which is thus defined near 0. Holomorphic germs with meromorphic or algebraic analytic continuation are examples of endlessly continuable functions, but the functions in $\hat{\mathscr{R}}$ can have a multiple-valued analytic continuation with a rich set of singularities.

The *convolution product* is defined as the Borel image of multiplication and denoted by the symbol $*$: for $\hat{\varphi}, \hat{\psi} \in \mathbb{C}[[\zeta]]$, $\hat{\varphi} * \hat{\psi} := \mathcal{B}(\mathcal{B}^{-1}\hat{\varphi} \cdot \mathcal{B}^{-1}\hat{\psi})$, and δ is the convolution unit (obtained from $(\mathbb{C}[[\zeta]], *)$ by adjunction of unit). As is well known, for convergent power series, convolution admits the integral representation

$$\hat{\varphi} * \hat{\psi}(\zeta) = \int_0^\zeta \hat{\varphi}(\xi)\hat{\psi}(\zeta - \xi)\,\mathrm{d}\xi \tag{1.1}$$

for ζ in the intersection of the discs of convergence of $\hat{\varphi}$ and $\hat{\psi}$.

Our aim is to study the analytic continuation of the convolution product of an arbitrary number of endlessly continuable functions, to check its endless continuability, and also to provide bounds, so as to be able to deal with nonlinear operations on resurgent series. A typical example of nonlinear operation is the substitution of one or several series without constant term $\tilde{\varphi}_1, \dots, \tilde{\varphi}_r$ into a power series $F(w_1, \dots, w_r)$, defined as

$$F(\tilde{\varphi}_1, \dots, \tilde{\varphi}_r) := \sum_{k \in \mathbb{N}^r} c_k\, \tilde{\varphi}_1^{k_1} \cdots \tilde{\varphi}_r^{k_r} \tag{1.2}$$

for $F = \sum_{k\in\mathbb{N}^r} c_k\, w_1^{k_1} \cdots w_r^{k_r}$. One of our main results is:

Theorem 1.3. *Let $r \geq 1$ be an integer. Then, for any convergent power series $F(w_1, \dots, w_r) \in \mathbb{C}\{w_1, \dots, w_r\}$ and for any resurgent series $\tilde{\varphi}_1, \dots, \tilde{\varphi}_r$ without constant term, $F(\tilde{\varphi}_1, \dots, \tilde{\varphi}_r) \in \tilde{\mathscr{R}}$.*

The proof of this result requires suitable bounds for the analytic continuation of the Borel transform of each term in the right-hand side of (1.2). Along the way, we will study the Riemann surfaces generated by endlessly continuable functions. We will also prove similar results for the larger spaces $\hat{\mathscr{R}}^{\mathrm{dv}}$ and $\tilde{\mathscr{R}}^{\mathrm{dv}}$.

Resurgence theory was developed in the early 1980s, with [9] and [11], and has many mathematical applications in the study of holomorphic dynamical systems, analytic differential equations, WKB analysis, etc. (see the references *e.g.* in [20]). More recently, there has been a burst of activity on the use of resurgence in Theoretical Physics, in the context of matrix models, string theory, quantum field theory and also quantum mechanics—see *e.g.* [1–3,5–8,12,16]. In almost all these applications, it is an important fact that the space of resurgent series be stable under nonlinear operations: such stability properties are useful, and at the same time they account for the occurrence of resurgent series in concrete problems.

These stability properties were stated in a very general framework in [11], but without detailed proofs, and the part of [4] which tackles this issue contains obscurities and at least one mistake. It is thus our aim in this article to provide a rigorous treatment of this question, at least in the slightly narrower context of endless continuability. The definitions of resurgence that we use for $\tilde{\mathscr{R}}$ and $\tilde{\mathscr{R}}^{\mathrm{dv}}$ are indeed more restrictive than Écalle's most general definition [11]. In fact, our definition of $\tilde{\mathscr{R}}^{\mathrm{dv}}$ is almost identical to the one used by Pham *et al.* in [4], and our definition of $\tilde{\mathscr{R}}$ is essentially equivalent to the definition used in [18], but the latter preprint has flaws which induced us to develop the results of the present paper. These versions of the definition of resurgence are sufficient for a large class of applications, which virtually contains all the aforementioned ones—see for instance [13] for the details concerning the case of nonlinear systems of differential or difference equations. The advantage of the definitions based on endless continuability is that they allow for a description of the location of the singularities in the Borel plane by means of *discrete filtered sets* or *discrete doubly filtered sets* (defined in Sections 2.1 and 2.5); the notion of discrete (doubly) filtered set, adapted from [4] and [18], is flexible enough to allow for a control of the singularity structure of convolution products.

A more restrictive definition is used in [20] and [17] (see also [9]):

Definition 1.4. Let Σ be a closed discrete subset of $\mathbb{C}$. A convergent power series $\hat{\varphi}$ is said to be Σ-*continuable* if it can be analytically continued along any path which starts in its disc of convergence and stays in $\mathbb{C} \setminus \Sigma$. The space of Σ-continuable functions is denoted by $\hat{\mathscr{R}}_\Sigma$.

This is clearly a particular case of Definition 1.1: any Σ-continuable function is endlessly continuable (take $F_L = \{\omega \in \Sigma \mid |\omega| \leq L\}$). It is proved in [17] that, if Σ' and Σ'' are closed discrete subsets of $\mathbb{C}$, and if also $\Sigma := \{\omega' + \omega'' \mid \omega' \in \Sigma',\ \omega'' \in \Sigma''\}$ is closed and discrete, then $\hat{\varphi} \in \hat{\mathscr{R}}_{\Sigma'},\ \hat{\psi} \in \hat{\mathscr{R}}_{\Sigma''} \Rightarrow \hat{\varphi} * \hat{\psi} \in \hat{\mathscr{R}}_\Sigma$. This is because in formula (1.1), heuristically, singular points tend to add to create new singularities; so, the analytic continuation of $\hat{\varphi} * \hat{\psi}$ along a path which does not stay close to the origin is possible provided the path avoids Σ. In particular, if a closed discrete set Σ is closed under addition, then $\hat{\mathscr{R}}_\Sigma$ is closed under convolution; moreover, in this case, bounds for the analytic continuation of iterated convolutions $\hat{\varphi}_1 * \cdots * \hat{\varphi}_n$ are given in [20], where an analogue of Theorem 1.3 is proved for Σ-continuable functions.

The notion of Σ-continuability is sufficient to cover interesting applications, *e.g.* differential equations of the saddle-node singularity type or difference equations like Abel's equation for one-dimensional tangent-to-identity diffeomorphisms, in which cases one may take for Σ a one-dimensional lattice of $\mathbb{C}$. However, reflecting for a moment on the origin of resurgence in differential equations, one sees that one cannot handle situations beyond a certain level of complexity without replacing Σ-continuability by a more general notion like endless continuability. Let us illustrate this point on two examples.

– The equation $\frac{\mathrm{d}\varphi}{\mathrm{d}z} - \lambda\varphi = b(z)$, where $b(z)$ is given in $z^{-1}\mathbb{C}\{z^{-1}\}$ and $\lambda \in \mathbb{C}^*$, has a unique formal solution in $\mathbb{C}[[z^{-1}]]$, namely $\tilde{\varphi}(z) := -\lambda^{-1}\Big(\mathrm{Id} - \lambda^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{-1} b$, whose Borel transform is $\hat{\varphi}(\zeta) = -(\lambda+\zeta)^{-1}\hat{b}(\zeta)$; here, the Borel transform $\hat{b}(\zeta)$ of $b(z)$ is entire, hence $\hat{\varphi}$ is meromorphic in $\mathbb{C}$, with at worse a pole at $\zeta = -\lambda$ and no singularity elsewhere. Therefore, heuristically, for a nonlinear equation
$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} - \lambda\varphi = b_0(z) + b_1(z)\varphi + b_2(z)\varphi^2 + \cdots$$
with $b(z,w) = \sum b_m(z)w^m \in z^{-1}\mathbb{C}\{z^{-1},w\}$ given, we may expect a formal solution whose Borel transform $\hat{\varphi}$ has singularities at $\zeta = -n\lambda$, $n \in \mathbb{Z}_{>0}$ (because, as an effect of the nonlinearity, the singular points tend to add), *i.e.* $\hat{\varphi}$ will be Σ-continuable with $\Sigma = \{-\lambda, -2\lambda, \dots\}$ (see [19] for a rigorous proof of this), but in the multidimensional case, for a system of r coupled equations with left-hand sides of the form $\frac{\mathrm{d}\varphi_j}{\mathrm{d}z} - \lambda_j\varphi_j$ with $\lambda_1,\dots,\lambda_r \in \mathbb{C}^*$, we may expect that the Borel transforms $\hat{\varphi}_j$ of the components of the formal solution have singularities at the points $\zeta = -(n_1\lambda_1 + \cdots + n_r\lambda_r)$, $n \in \mathbb{Z}^r_{>0}$; this set of possible singular points may fail to be closed and discrete (depending on the arithmetical properties of $(\lambda_1,\dots,\lambda_r)$), hence, in general, we cannot expect these Borel transforms to be Σ-continuable for any Σ. Still, this does not prevent them from being always endlessly continuable, as proved in [13].
– Another illustration of the need to go beyond Σ-continuability stems from parametric resurgence [10]. Suppose that we are given a holomorphic function $b(t)$ globally defined on $\mathbb{C}$, with isolated singularities $\omega \in S \subset \mathbb{C}$, *e.g.* a meromorphic function, and consider the differential equation
$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} - z\lambda\varphi = b(t), \tag{1.3}$$
where $\lambda \in \mathbb{C}^*$ is fixed and z is a large complex parameter with respect to which we consider perturbative expansions. It is easy to see that there is a unique solution which is formal in z and analytic in t, namely $\tilde{\varphi}(z,t) := -\sum_{k=0}^{\infty}\lambda^{-k-1}z^{-k-1}b^{(k)}(t)$, and its Borel transform $\hat{\varphi}(\zeta,t) = -\lambda^{-1}b(t+\lambda^{-1}\zeta)$ is singular at all points of the form $\zeta_{t,\omega} := \lambda(-t+\omega)$, $\omega \in S$. Now, if we add

to the right-hand side of (1.3) a perturbation which is nonlinear in φ, we can expect to get a formal solution whose Borel transform possesses a rich set of singular points generated by the $\zeta_{t,\omega}$'s, which might easily be too rich to allow for Σ-continuability with any Σ; however, we can still hope endless continuability.

These are good motivations to study endless continuable functions. As already alluded to, we will use discrete filtered sets (d.f.s. for short) to work with them. A d.f.s. is a family of sets $\Omega = (\Omega_L)_{L\in\mathbb{R}_{\geq 0}}$, where each Ω_L is a finite set; we will define Ω-continuability when Ω is a d.f.s., thus extending Definition 1.4, and the space of endlessly continuable functions will appear as the totality of Ω-continuable functions for all possible d.f.s. This was already the approach of [4], and it was used in [18] to prove that the convolution product of two endlessly continuable functions is endlessly continuable, hence $\tilde{\mathscr{R}}$ is a subring of $\mathbb{C}[[z^{-1}]]$. However, to reach the conclusions of Theorem 1.3, we will need to give precise estimates on the convolution product of an arbitrary number of endlessly continuable functions, so as to prove the convergence of the series of holomorphic functions $\sum c_k\, \hat{\varphi}_1^{*k_1} * \cdots * \hat{\varphi}_r^{*k_r}$ (Borel transform of the right-hand side of (1.2)) and to check its endless continuability. We will proceed similarly in the case of endless continuability with respect to bounded direction variation, using discrete doubly filtered sets.

Notice that explicit bounds for iterated convolutions can be useful in themselves; in the context of Σ-continuability, such bounds were obtained in [20] and they were used in [14] in a study in WKB analysis, where the authors track the analytic dependence upon parameters in the exponential of the Voros coefficient.

As another contribution to the study of endlessly continuable functions, we will show how to contruct, for each discrete filtered set Ω, a universal Riemann surface X_Ω whose holomorphic functions are in one-to-one correspondence with Ω-continuable functions.

The plan of the paper is as follows:

— Section 2 introduces discrete filtered sets, the corresponding Ω-continuable functions and their Borel pre-images, the Ω-resurgent series, and discusses their relation with Definitions 1.1 and 1.2. The case of discrete doubly filtered sets and the spaces $\hat{\mathscr{R}}^{\mathrm{dv}}$ and $\tilde{\mathscr{R}}^{\mathrm{dv}}$ is in Section 2.5.
— Section 3 discusses the notion of Ω-endless Riemann surface and shows how to construct a universal object X_Ω (Theorem 3.2).
— In Section 4, we state and prove Theorem 4.8 which gives precise estimates for the convolution product of an arbitrary number of endlessly continuable functions. We also show the analogous statement for functions which are endlessly continuable with respect to bounded direction variation.
— Section 5 is devoted to applications of Theorem 4.8: the proof of Theorem 1.3 and even of a more general and more precise version, Theorem 5.2, and an implicit resurgent function theorem, Theorem 5.3.

Some of the results presented here have been announced in [15].

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2. Discrete filtered sets and Ω-continuability

In this section we review the notions concerning discrete filtered sets (usually denoted by the letter Ω), the corresponding Ω-allowed paths and Ω-continuable functions. The relation with endless continuability is established, and sums of discrete filtered sets are defined in order to handle convolution of enlessly continuable functions.

2.1. Discrete filtered sets

We first introduce the notion of discrete filtered sets which will be used to describe singularity structure of endlessly continuable functions (the first part of the definition is adapted from [4] and [18]):

Definition 2.1. We use the notation $\mathbb{R}_{\geq 0} = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$.

1) A *discrete filtered set*, or *d.f.s.* for short, is a family $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ where
   i) Ω_L is a finite subset of $\mathbb{C}$ for each L;
   ii) $\Omega_{L_1} \subseteq \Omega_{L_2}$ for $L_1 \leq L_2$;
   iii) there exists $\delta > 0$ such that $\Omega_\delta = \emptyset$.
2 Let Ω and Ω' be d.f.s. We write $\Omega \subset \Omega'$ if $\Omega_L \subset \Omega'_L$ for every L.
3 We call *upper closure* of a d.f.s. Ω the family of sets $\tilde{\Omega} = (\tilde{\Omega})_{L \in \mathbb{R}_{\geq 0}}$ defined by

$$\tilde{\Omega}_L := \bigcap_{\varepsilon > 0} \Omega_{L+\varepsilon} \quad \text{for } L \in \mathbb{R}_{\geq 0}. \tag{2.1}$$

It is easy to check that $\tilde{\Omega}$ is a d.f.s. and $\Omega \subset \tilde{\Omega}$.

Example 2.2. Given a closed discrete subset Σ of $\mathbb{C}$, the formula

$$\Omega(\Sigma)_L := \{\, \omega \in \Sigma \mid |\omega| \leq L \,\} \quad \text{for } L \in \mathbb{R}_{\geq 0}$$

defines a d.f.s. $\Omega(\Sigma)$ which coincides with its upper closure.

From the definition of d.f.s., we find the following

Lemma 2.3. *For any d.f.s.* Ω*, there exists a real sequence* $(L_n)_{n \geq 0}$ *such that* $0 = L_0 < L_1 < L_2 < \cdots$ *and, for every integer* $n \geq 0$,

$$L_n < L < L_{n+1} \quad \Rightarrow \quad \tilde{\Omega}_{L_n} = \tilde{\Omega}_L = \Omega_L.$$

Proof. First note that (2.1) entails

$$\tilde{\Omega}_L = \bigcap_{\varepsilon>0} \tilde{\Omega}_{L+\varepsilon} \quad \text{for every } L \in \mathbb{R}_{\geq 0} \tag{2.2}$$

(because $\Omega_{L+\varepsilon} \subset \tilde{\Omega}_{L+\varepsilon} \subset \tilde{\Omega}_{L+2\varepsilon}$). Consider the weakly order-preserving integer-valued function $L \in \mathbb{R}_{\geq 0} \mapsto \mathcal{N}(L) := \operatorname{card} \tilde{\Omega}_L$. For each L the sequence $k \mapsto \mathcal{N}(L + \frac{1}{k})$ must be eventually constant, hence there exists $\varepsilon_L > 0$ such that, for all $L' \in (L, L + \varepsilon_L]$, $\mathcal{N}(L') = \mathcal{N}(L + \varepsilon_L)$, whence $\tilde{\Omega}_{L'} = \tilde{\Omega}_{L+\varepsilon_L}$, and in fact, by (2.2), this holds also for $L' = L$. The conclusion follows from the fact that $\mathbb{R}_{\geq 0} = \bigsqcup_{k\in\mathbb{Z}} \mathcal{N}^{-1}(k)$ and each non-empty $\mathcal{N}^{-1}(k)$ is convex, hence an interval, which by the above must be left-closed and right-open, hence of the form $[L, L')$ or $[L, \infty)$. □

Given a d.f.s. Ω, we set

$$\mathcal{S}_\Omega := \big\{(\lambda, \omega) \in \mathbb{R} \times \mathbb{C} \mid \lambda \geq 0 \text{ and } \omega \in \Omega_\lambda\big\} \tag{2.3}$$

and denote by $\overline{\mathcal{S}}_\Omega$ the closure of $\mathcal{S}_\Omega$ in $\mathbb{R} \times \mathbb{C}$. We then call

$$\mathcal{M}_\Omega := \big(\mathbb{R} \times \mathbb{C}\big) \setminus \overline{\mathcal{S}}_\Omega \quad \text{(open subset of } \mathbb{R} \times \mathbb{C}) \tag{2.4}$$

the *allowed open set* associated with Ω.

Lemma 2.4. *One has $\overline{\mathcal{S}}_\Omega = \mathcal{S}_{\tilde{\Omega}}$ and $\mathcal{M}_\Omega = \mathcal{M}_{\tilde{\Omega}}$.*

Proof. Suppose $(\lambda, \omega) \in \mathcal{S}_{\tilde{\Omega}}$. Then $\omega \in \Omega_{\lambda+1/k}$ for each $k \geq 1$, hence $(\lambda + \frac{1}{k}, \omega) \in \mathcal{S}_\Omega$, whence $(\lambda, \omega) \in \overline{\mathcal{S}}_\Omega$.

Suppose $(\lambda, \omega) \in \overline{\mathcal{S}}_\Omega$. Then there exists a sequence $(\lambda_k, \omega_k)_{k\geq 1}$ in $\mathcal{S}_\Omega$ which converges to (λ, ω). If $\varepsilon > 0$, then $\lambda_k \leq \lambda + \varepsilon$ for k large enough, hence $\omega_k \in \Omega_{\lambda+\varepsilon}$, whence $\omega \in \Omega_{\lambda+\varepsilon}$ (because a finite set is closed); therefore $(\lambda, \omega) \in \mathcal{S}_{\tilde{\Omega}}$.

Therefore, $\mathcal{S}_{\tilde{\Omega}} = \overline{\mathcal{S}}_\Omega = \overline{\mathcal{S}}_{\tilde{\Omega}}$ and $\mathcal{M}_{\tilde{\Omega}} = \mathcal{M}_\Omega$. □

2.2. Ω-allowed paths

When dealing with a Lipschitz path $\gamma\colon [a, b] \to \mathbb{C}$, we denote by $L(\gamma)$ its length.

We denote by Π the set of all Lipschitz paths $\gamma\colon [0, t_*] \to \mathbb{C}$ such that $\gamma(0) = 0$, with some real $t_* \geq 0$ depending on γ. Given such a $\gamma \in \Pi$ and $t \in [0, t_*]$, we denote by

$$\gamma_{|t} := \gamma|_{[0,t]} \in \Pi$$

the restriction of γ to the interval $[0, t]$.

Notice that $t \mapsto L(\gamma_{|t})$ is Lipschitz continuous on $[0, t_*]$ since γ' exists a.e. and is essentially bounded by Rademacher's theorem.

Definition 2.5. Given a d.f.s. Ω, we call Ω-*allowed path* any $\gamma \in \Pi$ such that

$$\tilde{\gamma}(t) := \big(L(\gamma_{|t}), \gamma(t)\big) \in \mathcal{M}_\Omega \text{ for all } t.$$

We denote by Π_Ω the set of all Ω-allowed paths.

Notice that, given $t_* \geq 0$,

$$\begin{aligned}&\text{if } t \in [0, t_*] \mapsto \tilde{\gamma}(t) = \big(\lambda(t), \gamma(t)\big) \in \mathcal{M}_\Omega \text{ is a piecewise } C^1 \text{ path} \\ &\text{such that } \tilde{\gamma}(0) = (0,0) \text{ and } \lambda'(t) = |\gamma'(t)| \text{ for a.e. } t, \text{ then } \gamma \in \Pi_\Omega.\end{aligned} \tag{2.5}$$

In view of Lemmas 2.3 and 2.4, we have the following characterization of Ω-allowed paths:

Lemma 2.6. *Let Ω be a d.f.s. Then $\Pi_\Omega = \Pi_{\tilde{\Omega}}$ and, given $\gamma \in \Pi$, the followings are equivalent:*

1) $\gamma \in \Pi_\Omega$;
2) $\gamma(t) \in \mathbb{C} \setminus \tilde{\Omega}_{L(\gamma_{|t})}$ *for every t;*
3) *for every t, there exists n such that $L(\gamma_{|t}) < L_{n+1}$ and $\gamma(t) \in \mathbb{C} \setminus \tilde{\Omega}_{L_n}$*

(using the notation of Lemma 2.3*).*

Proof. Obvious. □

Notation 2.7. For $L, \delta > 0$, we set

$$\mathcal{M}_\Omega^{\delta,L} := \big\{ (\lambda, \zeta) \in \mathbb{R} \times \mathbb{C} \mid \operatorname{dist}\big((\lambda,\zeta), \mathcal{S}_\Omega\big) \geq \delta \text{ and } \lambda \leq L \big\}, \tag{2.6}$$

$$\Pi_\Omega^{\delta,L} := \big\{ \gamma \in \Pi_\Omega \mid \big(L(\gamma_{|t}), \gamma(t)\big) \in \mathcal{M}_\Omega^{\delta,L} \text{ for all } t \big\}, \tag{2.7}$$

where $\operatorname{dist}(\cdot\,,\cdot)$ is the Euclidean distance in $\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$.

Note that

$$\mathcal{M}_\Omega = \bigcup_{\delta, L > 0} \mathcal{M}_\Omega^{\delta,L}, \qquad \Pi_\Omega = \bigcup_{\delta, L > 0} \Pi_\Omega^{\delta,L}.$$

2.3. Ω-continuable functions and Ω-resurgent series

Definition 2.8. Given a d.f.s. Ω, we call Ω-*continuable function* a holomorphic germ $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ which can be analytically continued along any path $\gamma \in \Pi_\Omega$. We denote by $\hat{\mathscr{R}}_\Omega$ the set of all Ω-continuable functions and define

$$\tilde{\mathscr{R}}_\Omega := \mathcal{B}^{-1}\big(\mathbb{C}\delta \oplus \hat{\mathscr{R}}_\Omega\big) \subset \mathbb{C}\big[[z^{-1}]\big]$$

to be the set of Ω-*resurgent series*.

Remark 2.9. Given a closed discrete subset Σ of $\mathbb{C}$, the Σ-continuability in the sense of Definition 1.4 is equivalent to the $\Omega(\Sigma)$-continuability in the sense of Definition 2.8 for the d.f.s. $\Omega(\Sigma)$ of Example 2.2.

Remark 2.10. Observe that $\Omega \subset \Omega'$ implies $\mathcal{S}_\Omega \subset \mathcal{S}_{\Omega'}$, hence $\mathcal{M}_{\Omega'} \subset \mathcal{M}_\Omega$ and $\Pi_{\Omega'} \subset \Pi_\Omega$, therefore

$$\Omega \subset \Omega' \quad \Rightarrow \quad \hat{\mathscr{R}}_\Omega \subset \hat{\mathscr{R}}_{\Omega'}.$$

Remark 2.11. Notice that, for the trivial d.f.s. $\Omega = \emptyset$, $\hat{\mathscr{R}}_\emptyset = \mathscr{O}(\mathbb{C})$, hence $\mathscr{O}(\mathbb{C}) \subset \hat{\mathscr{R}}_\Omega$ for every d.f.s. Ω, *i.e.* entire functions are always Ω-continuable. Consequently, convergent series are always Ω-resurgent: $\mathbb{C}\{z^{-1}\} \subset \tilde{\mathscr{R}}_\Omega$. However, $\hat{\mathscr{R}}_\Omega = \mathscr{O}(\mathbb{C})$ does not imply $\Omega = \emptyset$ (consider for instance the d.f.s. Ω defined by $\Omega_L = \emptyset$ for $0 \le L < 2$ and $\Omega_L = \{1\}$ for $L \ge 2$). In fact, one can show

$$\hat{\mathscr{R}}_\Omega = \mathscr{O}(\mathbb{C}) \quad \Leftrightarrow \quad \forall L > 0,\ \exists L' > L \text{ such that } \Omega_{L'} \subset \{\, \omega \in \mathbb{C} \mid |\omega| < L \,\}.$$

Remark 2.12. In view of Lemma 2.6, we have $\hat{\mathscr{R}}_\Omega = \hat{\mathscr{R}}_{\tilde{\Omega}}$. Therefore, when dealing with Ω-resurgence, we can always suppose that Ω coincides with its upper closure (by replacing Ω with $\tilde{\Omega}$).

We now show the relation between resurgence in the sense of Definition 1.2 and Ω-resurgence in the sense of Definition 2.8.

Theorem 2.13. *A formal series $\tilde{\varphi} \in \mathbb{C}[[z^{-1}]]$ is resurgent if and only if there exists a d.f.s. Ω such that $\tilde{\varphi}$ is Ω-resurgent. In other words,*

$$\hat{\mathscr{R}} = \bigcup_{\Omega\ d.f.s.} \hat{\mathscr{R}}_\Omega, \qquad \tilde{\mathscr{R}} = \bigcup_{\Omega\ d.f.s.} \tilde{\mathscr{R}}_\Omega. \tag{2.8}$$

Before proving Theorem 2.13, we state a technical result.

Lemma 2.14. *Suppose that we are given a germ $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ that can be analytically continued along a path $\gamma\colon [0, t_*] \to \mathbb{C}$ of Π, and that F is a finite subset of $\mathbb{C}$. Then, for each $\varepsilon > 0$, there exists a path $\gamma^*\colon [0, t_*] \to \mathbb{C}$ of Π such that*

- $\gamma^*\big((0, t_*)\big) \subset \mathbb{C} \setminus F$;
- $L(\gamma^*) < L(\gamma) + \varepsilon$;
- $\gamma^*(t_*) = \gamma(t_*)$, *the germ $\hat{\varphi}$ can be analytically continued along γ^* and the analytic continuations along γ and γ^* coincide.*

Proof of Lemma 2.14. Without loss of generality, we can assume that $\gamma\big([0, t_*]\big)$ is not reduced to $\{0\}$ and that $t \mapsto L(\gamma_{|t})$ is strictly increasing.

The analytic continuation assumption allows us to find a finite subdivision $0 = t_0 < \cdots < t_m = t_*$ of $[0, t_*]$ together with open discs $\Delta_0, \ldots, \Delta_m$

For each $k \ge 1$, let us pick $s_k \in (t_{k-1}, t_k)$ such that $\gamma\big([s_k, t_k]\big) \subset \Delta_{k-1} \cap \Delta_k$; increasing the value of s_k if necessary, we can assume $\gamma(s_k) \notin F$. Let us also set $s_0 := 0$ and $s_{m+1} := t_*$, so that

$$0 \le k \le m \ \Rightarrow \ \begin{cases} \gamma\big([s_k, s_{k+1}]\big) \subset \Delta_k \\ \text{the analytic continuation of } \hat{\varphi} \text{ along } \gamma_{|s_k} \text{ is holomorphic in } \Delta_k \\ \gamma(s_k) \notin F \text{ except maybe if } k = 0 \\ \gamma(s_{k+1}) \notin F \text{ except maybe if } k = m. \end{cases}$$

We now define γ^* by specifying its restriction $\gamma^*|_{[s_k,s_{k+1}]}$ for each k so that it has the same endpoints as $\gamma|_{[s_k,s_{k+1}]}$ and,

- if the open line segment $S := \big(\gamma(s_k), \gamma(s_{k+1})\big)$ is contained in $\mathbb{C} \setminus F$, then we let $\gamma^*|_{[s_k,s_{k+1}]}$ start at $\gamma(s_k)$ and end at $\gamma(s_{k+1})$ following S, by setting

$$\gamma^*(t) := \gamma(s_k) + \frac{t - s_k}{s_{k+1} - s_k}\big(\gamma(s_{k+1}) - \gamma(s_k)\big) \quad \text{for } t \in [s_k, s_{k+1}],$$

- if not, then $S \cap F = \{\omega_1, \dots, \omega_\nu\}$ with $\nu \geq 1$ (depending on k); we pick $\rho > 0$ small enough so that

$$\pi\rho < \min\Big\{\frac{1}{2}|\omega_i - \gamma(s_k)|,\ \frac{1}{2}|\omega_i - \gamma(s_{k+1})|,\ \frac{1}{2}|\omega_j - \omega_i|,$$
$$\frac{\varepsilon}{\nu(m+1)} \mid 1 \leq i, j, \leq \nu,\ i \neq j\Big\}$$

  and we let $\gamma^*|_{[s_k,s_{k+1}]}$ follow S except that it circumvents each ω_i by following a half-circle of radius ρ contained in Δ_k.

This way, $\gamma^*|_{[s_k,s_{k+1}]}$ stays in Δ_k; the resulting path $\gamma^*\colon [0, t_*] \to \mathbb{C}$ is thus a path of analytic continuation for $\hat{\varphi}$ and the analytic continuations along γ and γ^* coincide. On the other hand, the length of $\gamma^*|_{[s_k,s_{k+1}]}$ is $< |\gamma(s_k) - \gamma(s_{k+1})| + \frac{\varepsilon}{m+1}$, whereas the length of $\gamma|_{[s_k,s_{k+1}]}$ is $\geq |\gamma(s_k) - \gamma(s_{k+1})|$, hence $L(\gamma^*) < L(\gamma) + \varepsilon$. □

Proof of Theorem 2.13. Suppose first that Ω is a d.f.s. and $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$. Then, for every $L > 0$, $\hat{\varphi}$ meets the requirement of Definition 1.1 with $F_L = \tilde{\Omega}_L$, hence $\hat{\varphi} \in \hat{\mathscr{R}}$. Thus $\hat{\mathscr{R}}_\Omega \subset \hat{\mathscr{R}}$, which yields one inclusion in (2.8).

Suppose now $\hat{\varphi} \in \hat{\mathscr{R}}$. In view of Definition 1.1, the radius of convergence δ of $\hat{\varphi}$ is positive and, for each positive integer n, we can choose a finite set F_n such that

$$\text{the germ } \hat{\varphi} \text{ can be analytically continued along any path } \gamma\colon [0,1] \to \mathbb{C} \text{ of } \Pi \text{ such that } L(\gamma) < (n+1)\delta \text{ and } \gamma\big((0,1]\big) \subset \mathbb{C} \setminus F_n. \tag{2.9}$$

Let $F_0 := \emptyset$. The property (2.9) holds for $n = 0$ too. For every real $L \geq 0$, we set

$$\Omega_L := \bigcup_{k=0}^{n} F_k \qquad \text{with } n := \lfloor L/\delta \rfloor.$$

One can check that $\Omega := (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ is a d.f.s. which coincides with its upper closure. We will show that $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$.

Pick an arbitrary $\gamma\colon [0,1] \to \mathbb{C}$ such that $\gamma \in \Pi_\Omega$. It is sufficient to prove that $\hat\varphi$ can be analytically continued along γ. Our assumption amounts to $\gamma(t) \in \mathbb{C} \setminus \Omega_{L(\gamma_{|t})}$ for each $t \in [0,1]$. Without loss of generality, we can assume that $\gamma\big([0,1]\big)$ is not reduced to $\{0\}$ and that $t \mapsto L(\gamma_{|t})$ is strictly increasing. Let

$$N := \lfloor L(\gamma)/\delta \rfloor.$$

We define a subdivision $0 = t_0 < t_1 < \cdots < t_N \le 1$ by the requirement $L(\gamma_{|t_n}) = n\delta$ and set

$$I_n := [t_n, t_{n+1}) \text{ for } 0 \le n < N, \qquad I_N := [t_N, 1].$$

For each integer n such that $0 \le n \le N$,

$$t \in I_n \quad \Rightarrow \quad n\delta \le L(\gamma_{|t}) < (n+1)\delta, \tag{2.10}$$

thus $\Omega_{L(\gamma_{|t})} = \bigcup_{k=0}^n F_k$, in particular

$$t \in I_n \quad \Rightarrow \quad \gamma(t) \in \mathbb{C} \setminus F_n. \tag{2.11}$$

Let us check by induction on n that $\hat\varphi$ can be analytically continued along $\gamma_{|t}$ for any $t \in I_n$.

If $t \in I_0$, then $\gamma_{|t}$ has length $< \delta$ and the conclusion follows from (2.9).

Suppose now that $1 \le n \le N$ and that the property holds for $n-1$. Let $t \in I_n$. By (2.10)–(2.11), we have $L(\gamma_{|t}) < (n+1)\delta$ and $\gamma\big([t_n, t]\big) \subset \mathbb{C} \setminus F_n$.

- If $\gamma\big((0, t_n)\big) \cap F_n$ is empty, then the conclusion follows from (2.9).
- If not, then, since $\mathbb{C} \setminus F_n$ is open, we can pick $t_* < t_n$ so that $\gamma\big([t_*, t]\big) \subset \mathbb{C} \setminus F_n$, and the induction hypothesis shows that $\hat\varphi$ can be analytically continued along $\gamma_{|t_*}$. We then apply Lemma 2.14 to $\gamma_{|t_*}$ with $F = F_n$ and $\varepsilon = (n+1)\delta - L(\gamma_{|t})$: we get a path $\gamma^*\colon [0, t_*] \to \mathbb{C}$ which defines the same analytic continuation for $\hat\varphi$ as $\gamma_{|t_*}$, avoids F_n and has length $< L(\gamma_{|t_*}) + \varepsilon$. The concatenation of γ^* with $\gamma|_{[t_*, t]}$ is a path γ^{**} of length $< (n+1)\delta$ which avoids F_n, so it is a path of analytic continuation for $\hat\varphi$ because of (2.9), and so is γ itself. □

2.4. Sums of discrete filtered sets

It is easy to see that, if Ω and Ω' are d.f.s., then the formula

$$(\Omega * \Omega')_L := \{\, \omega_1 + \omega_2 \mid \omega_1 \in \Omega_{L_1}, \omega_2 \in \Omega'_{L_2}, L_1 + L_2 = L \,\} \cup \Omega_L \cup \Omega'_L \quad \text{for } L \in \mathbb{R}_{\ge 0} \tag{2.12}$$

defines a d.f.s. $\Omega * \Omega'$. We call it the *sum* of Ω and Ω'.

The proof of the following lemma is left to the reader.

Lemma 2.15. *The law $*$ on the set of all d.f.s. is commutative and associative. The formula* $\Omega^{*n} := \underbrace{\Omega * \cdots * \Omega}_{n \text{ times}}$ *(for $n \geq 1$) defines an inductive system, which gives rise to a d.f.s.*

$$\Omega^{*\infty} := \varinjlim_{n} \Omega^{*n}.$$

As shown in [4] and [18], the sum of d.f.s. is useful to study the convolution product:

Theorem 2.16 ([18]). *Assume that Ω and Ω' are d.f.s. and $\hat{\varphi} \in \hat{\mathscr{R}}_{\Omega}$, $\hat{\psi} \in \hat{\mathscr{R}}_{\Omega'}$. Then the convolution product $\hat{\varphi} * \hat{\psi}$ is $\Omega * \Omega'$-continuable.*

Remark 2.17. Note that the notion of Σ-continuability in the sense of Definition 1.4 does not give such flexibility, because there are closed discrete sets Σ and Σ' such that $\Omega(\Sigma) * \Omega(\Sigma') \not\subset \Omega(\Sigma'')$ for any closed discrete Σ'' (take *e.g.* $\Sigma = \Sigma' = (\mathbb{Z}_{>0}\sqrt{2}) \cup \mathbb{Z}_{<0}$), and in fact there are Σ-continuable functions $\hat{\varphi}$ such that $\hat{\varphi} * \hat{\varphi}$ is not Σ''-continuable for any Σ''.

In view of Theorem 2.13, a direct consequence of Theorem 2.16 is that the space of endlessly continuable functions $\hat{\mathscr{R}}$ is stable under convolution, and the space of resurgent formal series $\tilde{\mathscr{R}}$ is a subring of the ring of formal series $\mathbb{C}[[z^{-1}]]$.

Given $\tilde{\varphi} \in \tilde{\mathscr{R}}_{\Omega} \cap z^{-1}\mathbb{C}[[z^{-1}]]$, Theorem 2.16 guarantees the Ω^{*k}-resurgence of $\tilde{\varphi}^k$ for every integer k, hence its $\Omega^{*\infty}$-resurgence. This is a first step towards the proof of the resurgence of $F(\tilde{\varphi})$ for $F(w) = \sum c_k w^k \in \mathbb{C}\{w\}$, *i.e.* Theorem 1.3 in the case $r = 1$, however some analysis is needed to prove the convergence of $\sum c_k \tilde{\varphi}^k$ in some appropriate topology. What we need is a precise estimate for the convolution product of an arbitrary number of endlessly continuable functions, and this will be the content of Theorem 4.8. In Section 5, the substitution problem will be discussed in a more general setting, resulting in Theorem 5.2, which is more general and more precise than Theorem 1.3.

2.5. Discrete doubly filtered sets and a more general definition of resurgence

We now define the spaces $\hat{\mathscr{R}}^{\mathrm{dv}}$ and $\tilde{\mathscr{R}}^{\mathrm{dv}}$ which were alluded to in the introduction. We first require the notion of "direction variation" of a $C^{1+\mathrm{Lip}}$ path.

We denote by Π^{dv} the set of all C^1 paths γ belonging to Π, such that γ' is Lipschitz and never vanishes. By Rademacher's theorem, γ'' exists a.e. on the interval of definition $[0, t_*]$ of γ and is essentially bounded. We can thus define the *direction variation* $V(\gamma)$ of $\gamma \in \Pi^{\mathrm{dv}}$ by

$$V(\gamma) := \int_0^{t_*} \left| \mathrm{Im} \frac{\gamma''(t)}{\gamma'(t)} \right| \mathrm{d}t$$

(notice that one can write $\gamma'(t) = |\gamma'(t)|\, \mathrm{e}^{\mathrm{i}\theta(t)}$ with a real-valued Lipschitz function θ, and then $\mathrm{Im} \frac{\gamma''(t)}{\gamma'(t)} = \theta'$, hence $V(\gamma)$ is nothing but the length of the path θ). Note that the function $t \mapsto V(\gamma_{|t})$ is Lipschitz.

Now, we introduce the notion of endlessly continuable functions with respect to bounded direction variation:

Definition 2.18. A convergent power series $\hat{\varphi} \in \mathbb{C}\{\zeta\}$ is said to be *endlessly continuable with respect to bounded direction variation* (and we write $\hat{\varphi} \in \hat{\mathscr{R}}^{\mathrm{dv}}$) if, for every real $L, M > 0$, there exists a finite subset $F_{L,M}$ of $\mathbb{C}$ such that $\hat{\varphi}$ can be analytically continued along every path $\gamma\colon\ [0,1] \to \mathbb{C}$ such that $\gamma \in \Pi^{\mathrm{dv}}$, $L(\gamma) < L$, $V(\gamma) < M$, and $\gamma\big((0,1]\big) \subset \mathbb{C} \setminus F_{L,M}$.

We also set $\tilde{\mathscr{R}}^{\mathrm{dv}} := \mathcal{B}^{-1}(\mathbb{C}\delta \oplus \hat{\mathscr{R}}^{\mathrm{dv}})$. Note that $\hat{\mathscr{R}} \subset \hat{\mathscr{R}}^{\mathrm{dv}} \subset \mathbb{C}\{\zeta\}$ and $\tilde{\mathscr{R}} \subset \tilde{\mathscr{R}}^{\mathrm{dv}} \subset \mathbb{C}[[z^{-1}]]$.

Definition 2.19. A *discrete doubly filtered set*, or *d.d.f.s.* for short, is a family $\Omega = (\Omega_{L,M})_{L,M\in\mathbb{R}_{\geq 0}}$ that satisfies

i) $\Omega_{L,M}$ is a finite subset of $\mathbb{C}$ for each L and M;
ii) $\Omega_{L_1,M_1} \subseteq \Omega_{L_2,M_2}$ when $L_1 \leq L_2$ and $M_1 \leq M_2$;
iii) there exists $\delta > 0$ such that $\Omega_{\delta,M} = \emptyset$ for all $M \geq 0$.

Notice that a d.f.s. Ω can be regarded as a d.d.f.s. Ω^{dv} by setting $\Omega^{\mathrm{dv}}_{L,M} := \Omega_L$ for $L, M \geq 0$.

For a d.d.f.s. Ω, we set $\mathcal{S}_\Omega := \big\{(\mu, \lambda, \omega) \in \mathbb{R}^2 \times \mathbb{C} \mid \mu \geq 0, \lambda \geq 0 \text{ and } \omega \in \Omega_{\lambda,\mu}\big\}$ and $\mathcal{M}_\Omega := \big(\mathbb{R}^2 \times \mathbb{C}\big) \setminus \overline{\mathcal{S}}_\Omega$, where $\overline{\mathcal{S}}_\Omega$ is the closure of $\mathcal{S}_\Omega$ in $\mathbb{R}^2 \times \mathbb{C}$. We call Ω-*allowed path* any $\gamma \in \Pi^{\mathrm{dv}}$ such that

$$\tilde{\gamma}^{\mathrm{dv}}(t) := \big(V(\gamma_{|t}), L(\gamma_{|t}), \gamma(t)\big) \in \mathcal{M}_\Omega \quad \text{for all } t. \tag{2.13}$$

We denote by Π^{dv}_Ω the set of all Ω-allowed paths. Finally, the set of Ω-*continuable functions* (respectively Ω-*resurgent series*) is defined in the same way as in Definition 2.8, and denoted by $\hat{\mathscr{R}}^{\mathrm{dv}}_\Omega$ (respectively $\tilde{\mathscr{R}}^{\mathrm{dv}}_\Omega$).

Arguing as for Theorem 2.13, one obtains

$$\hat{\mathscr{R}}^{\mathrm{dv}} = \bigcup_{\Omega \text{ d.d.f.s.}} \hat{\mathscr{R}}^{\mathrm{dv}}_\Omega, \qquad \tilde{\mathscr{R}}^{\mathrm{dv}} = \bigcup_{\Omega \text{ d.d.f.s.}} \tilde{\mathscr{R}}^{\mathrm{dv}}_\Omega. \tag{2.14}$$

The *sum* $\Omega * \Omega'$ of two d.d.f.s. Ω and Ω' is the d.d.f.s. defined by setting, for $L, M \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} &(\Omega * \Omega')_{L,M} \\ &:= \{\, \omega_1 + \omega_2 \mid \omega_1 \in \Omega_{L_1,M}, \omega_2 \in \Omega'_{L_2,M}, L_1 + L_2 = L \,\} \cup \Omega_{L,M} \cup \Omega'_{L,M}. \end{aligned} \tag{2.15}$$

3. The endless Riemann surface associated with a d.f.s.

We introduce the notion of Ω-endless Riemann surfaces for a d.f.s. Ω as follows:

Definition 3.1. We call Ω-*endless Riemann surface* any triple $(X, \mathfrak{p}, \underline{0})$ such that X is a connected Riemann surface, $\mathfrak{p}\colon X \to \mathbb{C}$ is a local biholomorphism, $\underline{0} \in \mathfrak{p}^{-1}(0)$, and any path $\gamma\colon [0,1] \to \mathbb{C}$ of Π_Ω has a lift $\underline{\gamma}\colon [0,1] \to X$ such that $\underline{\gamma}(0) = \underline{0}$. A *morphism of* Ω-*endless Riemann surfaces* is a local biholomorphism $\mathfrak{q} : (X, \mathfrak{p}, \underline{0}) \to (X', \mathfrak{p}', \underline{0}')$ that makes the following diagram commutative:

$$\begin{tikzcd}
(X, \underline{0}) \arrow[rr, "\mathfrak{q}"] \arrow[dr, "\mathfrak{p}"'] & & (X', \underline{0}') \arrow[dl, "\mathfrak{p}'"] \\
& (\mathbb{C}, 0) &
\end{tikzcd}$$

In this section we prove the existence of an initial object $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ in the category of Ω-endless Riemann surfaces:

Theorem 3.2. *There exists an* Ω*-endless Riemann surface* $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ *such that, for any* Ω*-endless Riemann surface* $(X, \mathfrak{p}, \underline{0})$*, there is a unique morphism*

$$\mathfrak{q}\colon\ (X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega) \to (X, \mathfrak{p}, \underline{0}).$$

The Ω*-endless Riemann surface* $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ *is unique up to isomorphism and* X_Ω *is simply connected.*

3.1. Construction of X_Ω

We first define the "skeleton" of Ω:

Definition 3.3. Let $V_\Omega \subset \bigcup_{n=1}^{\infty} (\mathbb{C} \times \mathbb{Z})^n$ be the set of vertices

$$v := ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n)) \in (\mathbb{C} \times \mathbb{Z})^n$$

that satisfy the following conditions:

1) $(\omega_1, \sigma_1) = (0, 0)$ and $(\omega_j, \sigma_j) \in \mathbb{C} \times (\mathbb{Z} \setminus \{0\})$ for $j \geq 2$;
2) $\omega_j \neq \omega_{j+1}$ for $j = 1, \cdots, n-1$;
3) $\omega_j \in \tilde{\Omega}_{L_j(v)}$ with $L_j(v) := \sum_{i=1}^{j-1} |\omega_{i+1} - \omega_i|$ for $j = 2, \cdots, n$.

Let $E_\Omega \subset V_\Omega \times V_\Omega$ be the set of edges $e = (v', v)$ that satisfy one of the following conditions:

i) $v = ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n))$ and $v' = ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n), (\omega_{n+1}, \pm 1))$;

ii) $v = ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n))$ and $v' = ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n + 1))$ with $\sigma_n \geq 1$;

iii) $v = ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n))$ and $v' = ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n - 1))$ with $\sigma_n \leq -1$.

We denote the directed tree diagram (V_Ω, E_Ω) by Sk_Ω and call it *skeleton* of Ω.

Notation 3.4. For $v \in V_\Omega \cap (\mathbb{C} \times \mathbb{Z})^n$, we set $\omega(v) := \omega_n$ and $L(v) := L_n(v)$.

From the definition of Sk_Ω, we find the following:

Lemma 3.5. *For each $v \in V_\Omega \setminus \{(0,0)\}$, there exists a unique vertex $v_\uparrow \in V_\Omega$ such that $(v, v_\uparrow) \in E_\Omega$.*

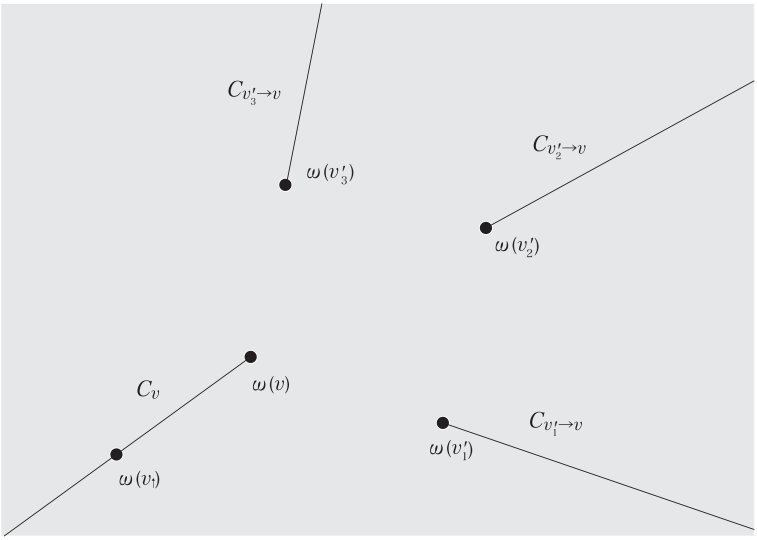

Figure 3.1. The set U_v.

To each $v \in V_\Omega$ we assign a cut plane U_v, defined as the open set

$$U_v := \mathbb{C} \setminus \Big(C_v \cup \bigcup_{v' \underset{\mathrm{i}}{\to} v} C_{v' \to v} \Big),$$

where $\bigcup_{v' \underset{\mathrm{i}}{\to} v}$ is the union over all the vertices $v' \in V_\Omega$ that have an edge $(v', v) \in E_\Omega$ of type i),

$$C_v := \begin{cases} \emptyset & \text{when } v = (0,0) \\ \{\omega_n - s(\omega_n - \omega_{n-1}) \mid s \in \mathbb{R}_{\geq 0}\} & \text{when } v \neq (0,0) \end{cases}$$

$$C_{v' \to v} := \{\omega_{n+1} + s(\omega_{n+1} - \omega_n) \mid s \in \mathbb{R}_{\geq 0}\}.$$

We patch the U_v's along the cuts according to the following rules:

Suppose first that (v', v) is an edge of type i), with $v' = (v, (\omega_{n+1}, \sigma_{n+1})) \in V_\Omega$. To it, we assign a line segment or a half-line $\ell_{v'\to v}$ as follows: If there exists $u = (v, (\omega'_{n+1}, \pm 1)) \in V_\Omega$ such that $\omega'_{n+1} \in C_{v'\to v} \setminus \{\omega_{n+1}\}$, take $u^{(0)} = (v, (\omega^{(0)}_{n+1}, \pm 1)) \in V_\Omega$ so that $|\omega^{(0)}_{n+1} - \omega_{n+1}|$ gives the minimum of $|\omega'_{n+1} - \omega_{n+1}|$ for such vertices and assign an open line segment $\ell_{v'\to v} := \{\omega_{n+1} + s(\omega^{(0)}_{n+1} - \omega_{n+1}) \mid s \in (0,1)\}$ to (v', v). Otherwise, we assign the open half-line $\ell_{v'\to v} := C_{v'\to v} \setminus \{\omega_{n+1}\}$ to (v', v). Since each Ω_L $(L \geq 0)$ is finite, we can take a connected neighborhood $U_{v'\to v}$ of $\ell_{v'\to v}$ so that

$$U_{v'\to v} \setminus \ell_{v'\to v} = U^+_{v'\to v} \cup U^-_{v'\to v} \quad \text{and} \quad U^\pm_{v'\to v} \subset U_v \cap U_{v'}, \tag{3.1}$$

where

$$U^\pm_{v'\to v} := \{\zeta \in U_{v'\to v} \mid \pm\mathrm{Im}(\zeta \cdot \overline{\zeta'}) > 0 \text{ for } \zeta' \in \ell_{v'\to v}\}.$$

Then, if $\sigma_{n+1} = 1$, we glue U_v and $U_{v'}$ along $U^-_{v'\to v}$, whereas if $\sigma_{n+1} = -1$ we glue them along $U^+_{v'\to v}$.

Suppose now that (v', v) is an edge of type ii) or iii). As in the case of i), if there exists $u = (v, (\omega'_{n+1}, \pm 1)) \in V_\Omega$ such that $\omega'_{n+1} \in C_v \setminus \{\omega_n\}$, then we take $u^{(0)} = (v, (\omega^{(0)}_{n+1}, \pm 1)) \in V_\Omega$ so that $|\omega^{(0)}_{n+1} - \omega_n|$ is minimum and assign $\ell_{v'\to v} := \{\omega_n + s(\omega^{(0)}_{n+1} - \omega_n) \mid s \in (0,1)\}$ to (v', v). Otherwise, we assign $\ell_{v'\to v} := C_v \setminus \{\omega_n\}$ to (v', v). Then, we take a connected neighborhood $U_{v'\to v}$ of $\ell_{v'\to v}$ satisfying (3.1), and glue U_v and $U_{v'}$ along $U^-_{v'\to v}$ in case ii), and along $U^+_{v'\to v}$ in case iii).

Patching the U_v's and the $U_{v'\to v}$'s according to the above rules, we obtain a Riemann surface X_Ω, in which we denote by $\underline{0}_\Omega$ the point corresponding to $0 \in U_{(0,0)}$. The map $\mathfrak{p}_\Omega \colon X_\Omega \to \mathbb{C}$ is naturally defined using the affine coordinate of U_v or $U_{v'\to v}$.

Let $\underline{U}_e, \underline{\ell}_e$ $(e \in E_\Omega)$ and $\underline{U}_v$ $(v \in V_\Omega)$ respectively denote the subsets of X_Ω defined by U_e, ℓ_e and U_v. Notice that each $\underline{\zeta} \in X_\Omega$ belongs to one of the $\underline{\ell}_e$'s or $\underline{U}_v$'s $(e \in E_\Omega$ or $v \in V_\Omega)$. Therefore, we have the following decomposition of X_Ω:

$$X_\Omega = \bigsqcup_{v \in V_\Omega} \underline{U}_v \sqcup \bigsqcup_{e \in E_\Omega} \underline{\ell}_e.$$

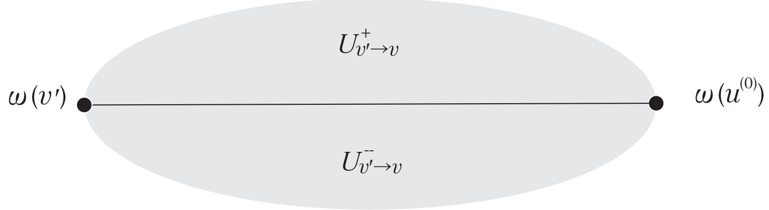

Figure 3.2. The set $U_{v'\to v}$.

Definition 3.6. We define a function $L : X_\Omega \to \mathbb{R}_{\geq 0}$ by the following formula:

$$L(\underline{\zeta}) := L(v) + |\mathfrak{p}(\underline{\zeta}) - \omega(v)| \quad \text{when} \quad \underline{\zeta} \in \underline{U}_v \sqcup \underline{\ell}_{v \to v_\uparrow}.$$

We call $L(\underline{\zeta})$ *the canonical distance* of $\underline{\zeta}$ from $\underline{0}_\Omega$.

We obtain from the construction of L the following:

Lemma 3.7. *The function $L : X_\Omega \to \mathbb{R}_{\geq 0}$ is continuous and satisfies the following inequality for every Lipschitz path $\underline{\gamma}\colon [0,1] \to X_\Omega$ such that $\underline{\gamma}(0) = \underline{0}_\Omega$:*

$$L(\underline{\gamma}(t)) \leq L(\gamma_{|t}) \quad \textit{for} \quad t \in [0,1],$$

where $\gamma := \mathfrak{p} \circ \underline{\gamma}$.

We now show the fundamental properties of X_Ω.

Lemma 3.8. *The Riemann surface X_Ω constructed above is simply connected.*

Proof. We first note that, since Sk_Ω is connected, X_Ω is path-connected. Let $\underline{\gamma} : [0,1] \to X_\Omega$ be a path such that $\underline{\gamma}(0) = \underline{\gamma}(1)$. Since the image of $\underline{\gamma}$ is a compact set in X_Ω, we can take a finite number of vertices $\{v_j\}_{j=1}^p \subset V_\Omega$ and edges $\{e_j\}_{j=1}^q \subset E_\Omega$ so that $v_1 = (0,0)$ and the image of $\underline{\gamma}$ is covered by $\{\underline{U}_{v_j}\}_{j=1}^p$ and $\{\underline{U}_{e_j}\}_{j=1}^q$. Since each of $\{v_j\}_{j=2}^p$ and $\{e_j\}_{j=1}^q$ has a path to v_1 that contains it, interpolating finitely many of the vertices and the edges if necessary, we may assume that the diagram $\widetilde{Sk}$ defined by $\{v_j\}_{j=1}^p$ and $\{e_j\}_{j=1}^q$ is connected in Sk_Ω. Now, let $\underline{U}$ be the union of $\{\underline{U}_{v_j}\}_{j=1}^p$ and $\{\underline{U}_{e_j}\}_{j=1}^q$. Since all of the open sets are simply connected and $\widetilde{Sk}$ is acyclic, we can inductively confirm using van Kampen's theorem that $\underline{U}$ is simply connected. Therefore, the path $\underline{\gamma}$ is contracted to the point $\underline{0}_\Omega$. This proves the simple connectedness of X_Ω. □

Lemma 3.9. *The Riemann surface X_Ω constructed above is Ω-endless.*

Proof. Take an arbitrary Ω-allowed path γ and $\delta, L > 0$ so that $\gamma \in \Pi_\Omega^{\delta,L}$. Let $V_\Omega^{\delta,L}$ denote the set of vertices $v = ((\omega_1,\sigma_1),\cdots,(\omega_n,\sigma_n)) \in V_\Omega$ that satisfy

$$L^\delta(v) := L_n(v) + \sum_{j=2}^n (|\sigma_j| - 1)\delta \leq L$$

and set $E_\Omega^{\delta,L} := \{(v, v_\uparrow) \in E_\Omega \mid v \in V_\Omega^{\delta,L}\}$. Notice that $V_\Omega^{\delta,L}$ and $E_\Omega^{\delta,L}$ are finite. We set for $\varepsilon > 0$ and $v \in V_\Omega^{\delta,L}$

$$U_v^{\delta,L,\varepsilon} := \left\{\zeta \in U_v \mid \inf_{(v',v)\in E_\Omega} |\zeta - \omega(v')| \geq \delta,\ D_\zeta^\varepsilon \subset U_v\right\} \cap D_{\omega(v)}^{L - L^\delta(v)},$$

where $D_\zeta^r := \{\tilde{\zeta} \in \mathbb{C} \mid |\tilde{\zeta} - \zeta| \leq r\}$ for $\zeta \in \mathbb{C}, r > 0$. We also set for $\varepsilon > 0$ and $(v, v_\uparrow) \in E_\Omega^{\delta,L}$

$$U_{v\to v_\uparrow}^{\delta,L,\varepsilon} := \left\{\zeta \in U_{v\to v_\uparrow} \mid \min_{j=1,2} |\zeta - \tilde{\omega}_j| \geq \delta,\ \inf_{\tilde{\zeta}\in\ell_{v\to v_\uparrow}} |\zeta - \tilde{\zeta}| \leq \varepsilon\right\} \cap D_{\omega(v)}^{L - L^\delta(v_\uparrow)},$$

where $\tilde{\omega}_1 := \omega(v)$ and $\tilde{\omega}_2$ is the other endpoint of $\ell_{v\to v_\uparrow}$ if it exists and $\tilde{\omega}_2 := \omega(v)$ otherwise. Since $E_\Omega^{\delta,L}$ is a finite set, we can take $\varepsilon > 0$ sufficiently small so that $D_\zeta^\varepsilon \subset U_{v\to v_\uparrow}$ for all $\zeta \in U_{v\to v_\uparrow}^{\delta,L,\varepsilon}$ and $(v, v_\uparrow) \in E_\Omega^{\delta,L}$. We fix such a number $\varepsilon > 0$.

Now, let I be the maximal interval such that the restriction of γ to I has a lift $\underline{\gamma}$ on X_Ω. Obviously, $I \neq \emptyset$ and I is open. Assume that $I = [0, a)$ for $a \in (0, 1]$. We take $b \in (0, a)$ so that $L(\gamma_{|a}) - L(\gamma_{|b}) < \varepsilon$. Then, notice that, since $\gamma \in \Pi_\Omega^{\delta,L}$ and $\gamma_{|b}$ has a lift on X_Ω, $\underline{\gamma}(b)$ is in $\underline{U}_v^{\delta,L,\varepsilon}$ for $v \in V_\Omega^{\delta,L}$ or $\underline{U}_e^{\delta,L,\varepsilon}$ for $e \in E_\Omega^{\delta,L}$. Since $D_{\gamma(b)}^\varepsilon \subset U_v$ (respectively, $D_{\gamma(b)}^\varepsilon \subset U_e$) when $\underline{\gamma}(b) \in \underline{U}_v^{\delta,L,\varepsilon}$ (respectively, $\underline{\gamma}(b) \in \underline{U}_e^{\delta,L,\varepsilon}$), we obtain a lift of $\gamma|_{[0,a]}$ by concatenating $\underline{\gamma}_{|b}$ and $\gamma|_{[b,a]}$ in the coordinate. This contradicts the maximality of I, and hence, $I = [0, 1]$. $\square$

3.2. Proof of Theorem 3.2

We first show the following:

Lemma 3.10. *For all $\varepsilon > 0$ and $\underline{\zeta} \in X_\Omega$, there exists an Ω-allowed path γ such that $L(\gamma) < L(\underline{\zeta}) + \varepsilon$ and its lift $\underline{\gamma}$ on X_Ω satisfies $\underline{\gamma}(0) = \underline{0}_\Omega$ and $\underline{\gamma}(1) = \underline{\zeta}$.*

Proof. Let $\underline{\zeta} \in \underline{U}_v$ for $v = ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n))$. We consider a polygonal curve $P_{\underline{\zeta}}^0$ obtained by connecting line segments $[\omega_j, \omega_{j+1}]$ $(j = 1, \cdots, n)$, where we set $\omega_{n+1} := \mathfrak{p}_\Omega(\underline{\zeta})$ for the sake of notational simplicity. Now, collect all the points $\omega_{j,k}$ on (ω_j, ω_{j+1}) such that $(L_{j,k}, \omega) \in \overline{\mathcal{S}}_\Omega$, where $L_{j,k} := L_j(v) + |\omega_{j,k} - \omega_j|$. Since

$$\overline{\mathcal{S}}_\Omega \cap \{\lambda \in \mathbb{R}_{\geq 0} \mid |\lambda| \leq L\} \times \mathbb{C} \text{ is written for each } L > 0 \text{ as the union of a finite number of line segments of the form } \{\lambda \in \mathbb{R}_{\geq 0} \mid \tilde{L} \leq \lambda \leq L\} \times \{\omega\} \ (\tilde{L} > 0, \omega \in \mathbb{C}), \tag{3.2}$$

there are at most finitely many such points. We order ω_j and $\omega_{j,k}$ so that $L_j(v)$ and $L_{j,k}$ increase along the order and denote the sequence by $(\omega'_1, \omega'_2, \cdots, \omega'_{n'})$. We set $L'_j := \sum_{i=1}^{j-1} |\omega'_{i+1} - \omega'_i|$. We extend v to $v' = ((\omega'_1, \sigma'_1), \cdots, (\omega'_{n'}, \sigma'_{n'}))$ by setting $\sigma'_j = 1$ (respectively, $\sigma'_j = -1$) when $(\omega'_j, L'_j) = (\omega_{i,k}, L_{i,k})$ for some i, k and $\sigma_{i+1} \geq 1$ (respectively, $\sigma_{i+1} \leq -1$). Then, in view of (3.2), we can take $\delta > 0$ so that

$$\{(L'_j + |\zeta' - \omega'_j| + \delta, \zeta') \mid \zeta' \in (\omega'_j, \omega'_{j+1})\} \cap \overline{\mathcal{S}}_\Omega = \emptyset,$$

$$\{(L'_j + \delta, \zeta') \mid 0 < |\zeta' - \omega'_j| < \delta\} \cap \overline{\mathcal{S}}_\Omega = \emptyset$$

hold for $j = 1, \cdots, n'$. Let $\omega'_{j,-}$ (respectively, $\omega'_{j,+}$) be the intersection point of $[\omega'_{j-1}, \omega'_j]$ (respectively, $[\omega'_j, \omega'_{j+1}]$) and $C_{\omega'_j}^{\varepsilon'} := \{\zeta' \in \mathbb{C} \mid |\zeta' - \omega'_j| = \varepsilon'\}$ for sufficiently small $\varepsilon' > 0$. We replace the part $[\omega'_{j,-}, \omega'_j] \cup [\omega'_j, \omega'_{j,+}]$ of ℓ with a path that goes anti-clockwise (respectively, clockwise) along $C_{\omega'_j}^{\varepsilon'}$ from $\omega'_{j,-}$ to $\omega'_{j,+}$

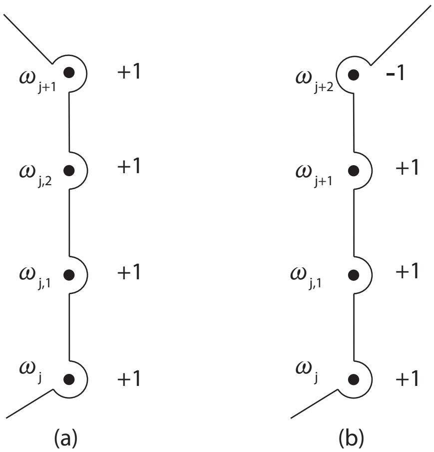

Figure 3.3.

and turns around ω'_j $(|\sigma'_j|-1)$-times when $\sigma'_j \geq 1$ (respectively, when $\sigma'_j \leq -1$). Let $P^{\varepsilon'}_{\underline{\zeta}}$ denote a path obtained from $P^0_{\underline{\zeta}}$ by the modification. Then, $P^{\varepsilon'}_{\underline{\zeta}}$ defines an Ω-allowed path and its lift $\underline{P}^{\varepsilon'}_{\underline{\zeta}}$ on X_Ω satisfies the conditions. Further, by taking ε' sufficiently small so that $2\pi\varepsilon' \sum_{j=2}^{n'} |\sigma'_j| < \varepsilon$, we find $L(P^{\varepsilon'}_{\underline{\zeta}}) < L(\underline{\zeta}) + \varepsilon$, hence one can take $\gamma = P^{\varepsilon'}_{\underline{\zeta}}$. When $\underline{\zeta} \in \underline{\ell}_e$ for an edge $e = (v, v_\uparrow) \in E_\Omega$, we can construct such a path $P^{\varepsilon'}_{\underline{\zeta}} \in \Pi_\Omega$ by totally the same discussion. $\square$

Notice that, since the sequence v' in the proof of Lemma 3.10 is uniquely determined by $\underline{\zeta} \in X_\Omega$, the choice of the path $P^{\varepsilon'}_{\underline{\zeta}}$ depends only on the radius ε' of the circles $C^{\varepsilon'}_{\omega'_j}$. Further, from the construction of the path $P^{\varepsilon'}_{\underline{\zeta}}$, we can extend Lemma 3.10 as follows:

Lemma 3.11. *For all $\varepsilon > 0$ and $\underline{\zeta} \in X_\Omega$, there exist a neighborhood $\underline{U}_{\underline{\zeta}}$ of $\underline{\zeta}$ and, for ε' small enough, a continuous deformation $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'} \in \Pi_\Omega$ $(\underline{\zeta}' \in \underline{U}_{\underline{\zeta}})$ of the path $\gamma = P^{\varepsilon'}_{\underline{\zeta}}$ constructed in the proof of Lemma* 3.10 *such that $L(Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}) < L(\underline{\zeta}') + 2\varepsilon$ for each $\underline{\zeta}' \in \underline{U}_{\underline{\zeta}}$ and the lift $\underline{Q}^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}$ on X_Ω satisfies $\underline{Q}^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}(0) = \underline{0}$ and $\underline{Q}^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}(1) = \underline{\zeta}'$.*

Indeed, the deformation of $P^{\varepsilon'}_{\underline{\zeta}}$ is concretely given as follows:

– When $\underline{\zeta} \in \underline{U}_v$ for $v \in V_\Omega$, taking a neighborhood $\underline{U}_{\underline{\zeta}} \subset \underline{U}_v$ of $\underline{\zeta}$ sufficiently small, we find that the family of the paths $P^{\varepsilon'}_{\underline{\zeta}'}$ $(\underline{\zeta}' \in \underline{U}_{\underline{\zeta}})$ constructed in the proof of Lemma 3.10 gives such a deformation.

– When $\underline{\zeta} \in \underline{\ell}_e$ for $e \in E_\Omega$, we can take a neighborhood $\underline{U}_{\underline{\zeta}} \subset \underline{U}_e$ of $\underline{\zeta}$ so that $[\omega'_{n',+}(\underline{\zeta}'), \mathfrak{p}_\Omega(\underline{\zeta}')] \subset \underline{U}_e$ for all $\underline{\zeta}' \in \underline{U}_{\underline{\zeta}}$, where $\omega'_{n',+}(\underline{\zeta}')$ is the intersection point of $[\omega'_{n'}, \mathfrak{p}_\Omega(\underline{\zeta}')]$ and $C^{\varepsilon'}_{\omega'_{n'}}$. Define a deformation $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}$ ($\underline{\zeta}' \in \underline{U}_e$) of $P^{\varepsilon'}_{\underline{\zeta}}$ by continuously varying the arc of $C^{\varepsilon'}_{\omega'_{n'}}$ from $\omega'_{n',-}$ to $\omega'_{n',+}(\underline{\zeta}')$ and the line segment $[\omega'_{n',+}(\underline{\zeta}'), \mathfrak{p}_\Omega(\underline{\zeta}')]$ and fixing the other part of $P^{\varepsilon'}_{\underline{\zeta}}$. Then, shrinking $\underline{U}_{\underline{\zeta}}$ if necessary, we find that $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}$ satisfies $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'} \in \Pi_\Omega$ and $L(Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}) < L(\underline{\zeta}')+2\varepsilon$ for each $\underline{\zeta}' \in \underline{U}_{\underline{\zeta}}$.

Beware that, when the edge $(v, v_\uparrow)$ is of type i), $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}$ is different from $P^{\varepsilon'}_{\underline{\zeta}'}$ for $\underline{\zeta} \in \underline{\ell}_{v\to v_\uparrow}$ and $\underline{\zeta}' \in \underline{U}_{\underline{\zeta}} \cap \underline{U}_{v_\uparrow}$. On the other hand, $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'} = P^{\varepsilon'}_{\underline{\zeta}'}$ holds for $\underline{\zeta}' \in \underline{U}_{\underline{\zeta}} \cap \underline{U}_v$. When the edge $(v, v_\uparrow)$ is of type ii) or iii), $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'} = P^{\varepsilon'}_{\underline{\zeta}'}$ holds for $\underline{\zeta} \in \underline{\ell}_{v\to v_\uparrow}$ and $\underline{\zeta}' \in \underline{U}_{\underline{\zeta}}$.

Let $(X, \mathfrak{p}, \underline{0})$ be an Ω-endless Riemann surface. For each $\underline{\zeta} \in X_\Omega$, take $\gamma \in \Pi_\Omega$ such that $\underline{\gamma}(1) = \underline{\zeta}$ and let $\underline{\gamma}_X$ be its lift on X. Then, define a map $\mathfrak{q} : X_\Omega \to X$ by $\mathfrak{q}(\underline{\zeta}) = \underline{\gamma}_X(1)$. We now show the well-definedness of $\mathfrak{q}$. For that purpose, it suffices to prove the following:

Proposition 3.12. *Let $\gamma_0, \gamma_1 \in \Pi_\Omega$ such that $\underline{\gamma_0}(1) = \underline{\gamma_1}(1)$. Then, there exists a continuous family $(H_s)_{s\in[0,1]}$ of Ω-allowed paths satisfying the conditions:*

1. *$H_s(0) = 0$ and $H_s(1) = \gamma_0(1)$ for all $s \in [0, 1]$;*
2. *$H_j = \gamma_j$ for $j = 0, 1$.*

The proof of Proposition 3.12 is reduced to the following:

Lemma 3.13. *For each $\gamma \in \Pi_\Omega$ and $\varepsilon' > 0$ sufficiently small, there exists a continuous family $(\tilde{H}_s)_{s\in[0,1]}$ of Ω-allowed paths satisfying the following conditions:*

1. *$L\big(\tilde{H}_s\big) \le L(\gamma_{|s})$ and $\underline{\tilde{H}}_s(1) = \underline{\gamma}(s)$ for all $s \in [0, 1]$;*
2. *$\tilde{H}_s = P^{\varepsilon'}_{\underline{\gamma}(s)}$ for $s = 0, 1$.*

Notice that, since $\underline{\gamma}(0) = \underline{0}_\Omega$, $P^{\varepsilon'}_{\underline{\gamma}(0)}$ is the constant map $P^{\varepsilon'}_{\underline{\gamma}(0)} = 0$.

Reduction of Proposition 3.12 *to Lemma* 3.13. For each $\gamma \in \Pi_\Omega$ and $s \in (0, 1]$, define H_s using $\tilde{H}_s$ constructed in Lemma 3.13 as follows:

$$H_s(t) := \begin{cases} \tilde{H}_s(t/s) & \text{when} \quad t \in [0, s] \\ \gamma(t) & \text{when} \quad t \in [s, 1]. \end{cases}$$

It extends continuously to $s = 0$ and gives a continuous family $(H_s)_{s\in[0,1]}$ of Ω-allowed paths satisfying the assumption in Proposition 3.12 with $\gamma_0 = \gamma$ and $\gamma_1 = P^{\varepsilon'}_{\underline{\gamma}(1)}$.

Now, let γ_0 and γ_1 be the Ω-allowed paths satisfying the assumption in Proposition 3.12. Applying the above discussion to each of γ_0 and γ_1, we obtain two families of Ω-allowed paths connecting them to $P^{\varepsilon'}_{\underline{\gamma}_0(1)}$ and, concatenating the deformations at $P^{\varepsilon'}_{\underline{\gamma}_0(1)}$, we obtain a deformation $(H_s)_{s\in[0,1]}$ satisfying the conditions in Proposition 3.12. $\square$

Proof of Lemma 3.13. Take $\delta, L > 0$ so that $\gamma \in \Pi^{\delta,L}_{\Omega}$. We first show the following:

When $\underline{\gamma}(t_0) \in \underline{U}_{v\to(0,0)}$ for $t_0 \in (0, 1]$ and $v = ((0, 0), (\omega_2, \sigma_2))$, the following estimate holds for $t \in [t_0, 1]$: (3.3)

$$L(\underline{\gamma}(t)) + \sqrt{|\omega_2|^2 + \delta^2} - |\omega_2| \le L(\gamma_{|t}).$$

Notice that, since $\gamma \in \Pi^{\delta,L}_{\Omega}$, the length $L(\gamma_{|t_0})$ of $\gamma_{|t_0}$ must be longer than that of the polygonal curve C obtained by concatenating the line segments $[0, \omega_2 + \delta e^{i\theta}]$ and $[\omega_2 + \delta e^{i\theta}, \gamma(t_0)]$, where $\theta = \arg(\omega_2) - \sigma_2\pi/2$. Then, we find that, for an arbitrary $\varepsilon > 0$, taking $\varepsilon' > 0$ sufficiently small, the path $\tilde{\gamma}^{\varepsilon'}$ obtained by concatenating the paths $P^{\varepsilon'}_{\underline{\gamma}(t_0)}$ and $\gamma|_{[t_0,1]}$ satisfies $\tilde{\gamma}^{\varepsilon'} \in \Pi_\Omega$, $\underline{\tilde{\gamma}}^{\varepsilon'}(t) = \underline{\gamma}(t)$ and $L(\tilde{\gamma}^{\varepsilon'}_{|t}) \le L(\tilde{\gamma}^{0}_{|t}) + \varepsilon$ for $t \in [t_0, 1]$. Therefore, we have

$$L(\underline{\gamma}(t)) \le L(\tilde{\gamma}^{0}_{|t}) \quad \text{for} \quad t \in [t_0, 1].$$

Since $L(C) \ge \sqrt{|\omega_2|^2 + \delta^2} + |\gamma(t_0) - \omega_2|$, we find

$$L(\gamma_{|t}) = L(\tilde{\gamma}^{0}_{|t}) + L(\gamma_{|t_0}) - L([0, \gamma(t_0)]) \ge L(\underline{\gamma}(t)) + \sqrt{|\omega_2|^2 + \delta^2} - |\omega_2|$$

holds for $t \in [t_0, 1]$, and hence, we obtain (3.3).

Now, we shall construct $(H_s)_{s\in[0,1]}$. Let $\varepsilon > 0$ be given. We assign the path $P^{\varepsilon'_t}_{\underline{\gamma}(t)}$ ($\varepsilon'_t > 0$) to each $t \in [0, 1]$ and take a neighborhood $\underline{U}_{\underline{\gamma}(t)}$ of $\underline{\gamma}(t)$ and the deformation $Q^{\varepsilon'_t}_{\underline{\gamma}(t),\underline{\zeta}'}$ ($\underline{\zeta}' \in \underline{U}_{\underline{\gamma}(t)}$) of $P^{\varepsilon'_t}_{\underline{\gamma}(t)}$ constructed in Lemma 3.11. Then, we can cover $[0, 1]$ by a finite number of intervals $I_j = [a_j, b_j]$ ($j = 1, 2, \cdots, k$) satisfying the following conditions:

- The interior I°_j of I_j satisfies $I^{\circ}_{j_1} \cap I^{\circ}_{j_2} \ne \emptyset$ when $|j_1 - j_2| \le 1$ and $I_{j_1} \cap I_{j_2} = \emptyset$ otherwise;
- There exists $t_j \in I_j$ such that $t_j < t_{j+1}$ for $j = 1, \cdots, k-1$ and $\underline{\gamma}(I_j) \subset \underline{U}_{\underline{\gamma}(t_j)}$.

Notice that, since $\underline{U}_{\underline{\gamma}(t)}$ is taken for each $t \in [0, 1]$ so that it is contained in one of the charts $\underline{U}_v$ ($v \in V_\Omega$) or $\underline{U}_e$ ($e \in E_\Omega$), one of the followings holds:

- $\underline{\gamma}(t_j) \in \underline{U}_v$ and $\underline{\gamma}(I_j) \subset \underline{U}_v$ ($v \in V_\Omega$);
- $\underline{\gamma}(t_j) \in \underline{\ell}_e$ and $\underline{\gamma}(I_j) \subset \underline{U}_e$ ($e \in E_\Omega$).

We set $\varepsilon' = \min_j\{\varepsilon'_{t_j} \mid \underline{\gamma}(t_j) \notin \underline{U}_{(0,0)}\}$. Then, $P^{\varepsilon'}_{\underline{\gamma}(t_j)}$ and its deformation $Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\zeta}'}$ ($\underline{\zeta}' \in \underline{U}_{\underline{\gamma}(t_j)}$) also satisfy the conditions in Lemma 3.10 and Lemma 3.11. Let $J_E \subset \{1, \cdots, k\}$ denote the set of suffixes satisfying the condition that there exists $e \in E_\Omega$ such that $\underline{\gamma}(t_j) \in \underline{\ell}_e$ and let j_0 be the minimum of J_E. Shrinking the neighborhood $\underline{U}_{\underline{\gamma}(t)}$ for each $t \in [0, 1]$ at the first, we may assume without loss of generality that,

- $|\gamma(t) - \gamma(t_j)| \leq \varepsilon$ for $t \in I_j$ and $j = 1, \cdots, k$;
- if $j, j+1 \in J_E$, there exists an edge $e \in E_\Omega$ such that $\underline{\gamma}(t_j), \underline{\gamma}(t_{j+1}) \in \underline{\ell}_e$.

Recall that, from the construction of $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}$,

$$Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(t)} = Q^{\varepsilon'}_{\underline{\gamma}(t_{j+1}),\underline{\gamma}(t)} \quad \text{for} \quad t \in I_j \cap I_{j+1}$$

except for the cases where there exists an edge $e = (v, v_\uparrow) \in E_\Omega$ of the type i) such that

- $\underline{\gamma}(t_j) \in \underline{U}_e$ and $\underline{\gamma}(t_{j+1}) \in \underline{U}_{v_\uparrow}$;
- $\underline{\gamma}(t_j) \in \underline{U}_{v_\uparrow}$ and $\underline{\gamma}(t_{j+1}) \in \underline{U}_e$.

In the first case, the difference between $Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(t)}$ and $Q^{\varepsilon'}_{\underline{\gamma}(t_{j+1}),\underline{\gamma}(t)}$ is the part from $\omega^t(v_\uparrow)$ to $\gamma(t)$, where $\omega^t(v_\uparrow)$ is the intersection point of $C^{\varepsilon'}_{\omega(v_\uparrow)}$ and $[\omega(v_\uparrow), \gamma(t)]$: Let $\omega_{e,i}$ ($i = 0, \cdots, m+1$) be the points on the line segment $[\omega(v_\uparrow), \omega(v)]$ satisfying the conditions $(L_{e,i}, \omega_{e,i}) \in \overline{\mathcal{S}}_\Omega$ and $L_{e,i} < L_{e,i+1}$, where $L_{e,i} := L(v_\uparrow) + |\omega_{e,i} - \omega(v_\uparrow)|$. Then, the part of $Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(t)}$ from $\omega^t(v_\uparrow)$ to $\gamma(t)$ is given by concatenating the arcs of $C^{\varepsilon'}_{\omega_{e,i}}$ ($i = 0, \cdots, m+1$), the intervals of the line segment $[\omega(v_\uparrow), \omega(v)]$ and $[\omega^t(v), \gamma(t)]$, where $\omega^t(v)$ is the intersection point of $C^{\varepsilon'}_{\omega(v)}$ and $[\omega(v), \gamma(t)]$. (See Figure 3.4 (a).) On the other hand, $Q^{\varepsilon'}_{\underline{\gamma}(t_{j+1}),\underline{\gamma}(t)}$ goes directly from $\omega^t(v_\uparrow)$ to $\gamma(t)$. (See Figure 3.4 (d).)

Now, let $\omega^t_{i,+}$ (respectively $\omega^t_{i,-}$) be the intersection point of $C^{\varepsilon'}_{\omega_{e,i}}$ and $[\omega^t(v_\uparrow), \omega^t(v)]$ that is the closer to $\omega^t(v)$ (respectively $\omega^t(v_\uparrow)$). While t moves on $I_j \cap I_{j+1}$, we first deform the part of $Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(t)}$ from $\omega^t(v_\uparrow)$ to $\omega^t(v)$ to the line segment $[\omega^t(v_\uparrow), \omega^t(v)]$ by shrinking the part of $Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(t)}$ from $\omega^t_{i,-}$ to $\omega^t_{i,+}$ (respectively from $\omega^t_{i,+}$ to $\omega^t_{i+1,-}$) to the line segment $[\omega^t_{i,-}, \omega^t_{i,+}]$ (respectively

$[\omega^t_{i,+}, \omega^t_{i+1,-}])$ for each i. (See Figure 3.4 (b) and (c).) Then, further shrinking the polygonal line given by concatenating $[\omega^t(v_\uparrow), \omega^t(v)]$ and $[\omega^t(v), \gamma(t)]$ to the line segment $[\omega^t(v_\uparrow), \gamma(t)]$, we obtain a continuous family of Ω-allowed paths $\big(\tilde{H}_s\big)_{s\in[t_j,t_{j+1}]}$ satisfying the following conditions:

- $\tilde{H}_s = Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(s)}$ when $s \in [t_j, t_{j+1}] \setminus I_{j+1}$,
- $\tilde{H}_s = Q^{\varepsilon'}_{\underline{\gamma}(t_{j+1}),\underline{\gamma}(s)}$ when $s \in [t_j, t_{j+1}] \setminus I_j$,
- $L\big(\tilde{H}_s\big) \le L\big(Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(s)}\big)$ and $\underline{\tilde{H}}_s(1) = \underline{\gamma}(s)$ when $s \in I_j \cap I_{j+1}$.

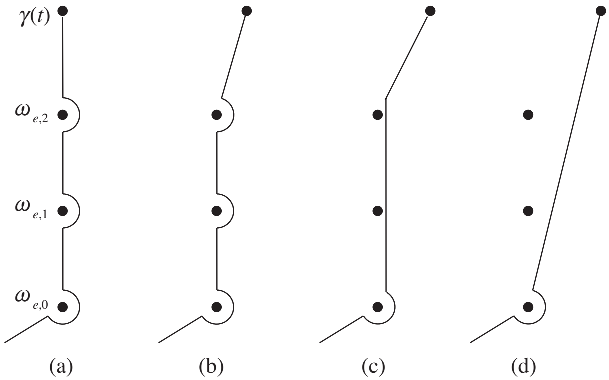

Figure 3.4.

For the second case, we can also construct a continuous family of Ω-allowed paths $\big(\tilde{H}_s\big)_{s\in[t_j,t_{j+1}]}$ satisfying the first and the second conditions above and

- $L\big(\tilde{H}_s\big) \le L\big(Q^{\varepsilon'}_{\underline{\gamma}(t_{j+1}),\underline{\gamma}(s)}\big)$ and $\underline{\tilde{H}}_s(1) = \underline{\gamma}(s)$ when $s \in I_j \cap I_{j+1}$.

Then, we can continuously extend $\tilde{H}_s$ to $[0, 1]$ by interpolating it by $Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(s)}$ so that it satisfies

$$L\big(\tilde{H}_s\big) \le \max_j \big\{L\big(Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(s)}\big) \mid s \in I_j\big\} \text{ and } \underline{\tilde{H}}_s(1) = \underline{\gamma}(s) \text{ for all } s \in [0,1]. \tag{3.4}$$

Since I_{j_0} is taken so that $|\gamma(t) - \gamma(t_{j_0})| \le \varepsilon$ holds on I_{j_0}, applying (3.3) with $t_0 = t_{j_0}$, we have the following estimates:

$$L(\underline{\gamma}(t)) + \sqrt{|\omega_2|^2 + \delta^2} - |\omega_2| - \varepsilon \le L(\gamma_{|t}) \quad \text{for} \quad t \in [a_{j_0}, 1].$$

On the other hand, since $\underline{\gamma}(t) \in \underline{U}_{(0,0)}$ for $t \in [0, a_{j_0}]$, we find $L\big(Q^{\varepsilon'}_{\underline{\gamma}(t_j),\underline{\gamma}(t)}\big) = L(\underline{\gamma}(t))$ holds for $t \in I_j$ and $j < j_0$ from the construction of $Q^{\varepsilon'}_{\underline{\zeta},\underline{\zeta}'}$. Therefore, taking $\varepsilon > 0$ sufficiently small so that

$$3\varepsilon \le \sqrt{|\omega_2|^2 + \delta^2} - |\omega_2|,$$

we obtain the following estimates from Lemma 3.10 and (3.4):

$$L\big(\tilde{H}_s\big) \le L(\gamma_{|s}) \quad \text{for} \quad s \in [0, 1].$$

Finally, from the construction of $\tilde{H}_s$, we find that $\tilde{H}_s$ satisfies $\tilde{H}_s = P^{\varepsilon'}_{\underline{\gamma}(s)}$ for $s = 0, 1$. □

Since $\mathfrak{p}_\Omega = \mathfrak{p} \circ \mathfrak{q}$ and $\mathfrak{p}$ is isomorphic near $\underline{0}$, all the maps $\mathfrak{q} : X_\Omega \to X$ must coincide near $\underline{0}_\Omega$, and hence, uniqueness of $\mathfrak{q}$ follows from the uniqueness of the analytic continuation of $\mathfrak{q}$. Finally, X_Ω is unique up to isomorphism because X_Ω is an initial object in the category of Ω-endless Riemann surfaces.

3.3. Supplement to the properties of X_Ω

Let $\mathscr{O}_X$ denote the sheaf of holomorphic functions on a Riemann surface X and consider the natural morphism $\mathfrak{p}_\Omega^*\colon \mathfrak{p}_\Omega^{-1}\mathscr{O}_{\mathbb{C}} \to \mathscr{O}_{X_\Omega}$ induced by $\mathfrak{p}_\Omega : X_\Omega \to \mathbb{C}$. Since X_Ω is simply connected, we obtain the following:

Proposition 3.14. *Let $\hat{\varphi} \in \mathscr{O}_{\mathbb{C},0}$. Then the followings are equivalent:*

i) *$\hat{\varphi} \in \mathscr{O}_{\mathbb{C},0}$ is Ω-continuable;*
ii) *$\mathfrak{p}_\Omega^*\hat{\varphi} \in \mathscr{O}_{X_\Omega,\underline{0}_\Omega}$ can be analytically continued along any path on X_Ω;*
iii) *$\mathfrak{p}_\Omega^*\hat{\varphi} \in \mathscr{O}_{X_\Omega,\underline{0}_\Omega}$ can be extended to $\Gamma(X_\Omega, \mathscr{O}_{X_\Omega})$.*

Therefore, we find

$$\mathfrak{p}_\Omega^*\colon \hat{\mathscr{R}}_\Omega \xrightarrow{\sim} \Gamma(X_\Omega, \mathscr{O}_{X_\Omega}).$$

Notation 3.15. For $L, \delta > 0$, using $\Pi_\Omega^{\delta,L}$ of (2.7), we define a compact subset $K_\Omega^{\delta,L}$ of X_Ω by

$$K_\Omega^{\delta,L} := \big\{\, \underline{\zeta} \in X_\Omega \mid \exists\gamma \in \Pi_\Omega^{\delta,L} \text{ such that } \underline{\zeta} = \underline{\gamma}(1) \,\big\}. \tag{3.5}$$

Notice that X_Ω is exhausted by $(K_\Omega^{\delta,L})_{\delta,L>0}$. Therefore, the family of seminorms $\|\cdot\|_\Omega^{\delta,L}$ $(\delta, L > 0)$ defined by

$$\|\hat{f}\|_\Omega^{\delta,L} := \sup_{\underline{\zeta}\in K_\Omega^{\delta,L}} |\hat{f}(\underline{\zeta})| \quad \text{for } \hat{f} \in \Gamma(X_\Omega, \mathscr{O}_{X_\Omega})$$

induces a structure of Fréchet space on $\Gamma(X_\Omega, \mathscr{O}_{X_\Omega})$.

Definition 3.16. We introduce a structure of Fréchet space on $\tilde{\mathscr{R}}_\Omega$ by a family of seminorms $\|\cdot\|_\Omega^{\delta,L}$ $(\delta, L > 0)$ defined by

$$\|\tilde{\varphi}\|_\Omega^{\delta,L} := |\varphi_0| + \|\mathfrak{p}_\Omega^*\hat{\varphi}\|_\Omega^{\delta,L} \quad \text{for} \quad \tilde{\varphi} \in \tilde{\mathscr{R}}_\Omega,$$

where $\mathcal{B}(\tilde{\varphi}) = \varphi_0\delta + \hat{\varphi} \in \mathbb{C}\delta \oplus \hat{\mathscr{R}}_\Omega$.

Let Ω' be a d.f.s. such that $\Omega \subset \Omega'$. Since $\Pi_{\Omega'} \subset \Pi_\Omega$, X_Ω is Ω'-endless. Therefore, Theorem 3.2 yields a morphism

$$\mathfrak{q} : (X_{\Omega'}, \mathfrak{p}_{\Omega'}, \underline{0}_{\Omega'}) \to (X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega),$$

which induces a morphism $\mathfrak{q}^* : \mathfrak{q}^{-1}\mathscr{O}_{X_\Omega} \to \mathscr{O}_{X_{\Omega'}}$. Since $\mathfrak{q}(K_{\Omega'}^{\delta,L}) \subset K_\Omega^{\delta,L}$, we have

$$\|\mathfrak{q}^*\hat{f}\|_{\Omega'}^{\delta,L} \le \|\hat{f}\|_\Omega^{\delta,L} \quad \text{for} \;\; \hat{f} \in \Gamma(X_\Omega, \mathscr{O}_{X_\Omega}),$$

and hence,

$$\|\tilde{\varphi}\|_{\Omega'}^{\delta,L} \le \|\tilde{\varphi}\|_\Omega^{\delta,L} \quad \text{for} \;\; \tilde{\varphi} \in \tilde{\mathscr{R}}_\Omega.$$

In view of Theorem 4.8 below, the product map $\tilde{\mathscr{R}}_\Omega \times \tilde{\mathscr{R}}_{\Omega'} \to \tilde{\mathscr{R}}_{\Omega * \Omega'}$ is continuous and hence, when $\Omega * \Omega = \Omega$, $\tilde{\mathscr{R}}_\Omega$ is a Fréchet algebra.

3.4. The endless Riemann surface associated with a d.d.f.s.

In this section we discuss the construction of the endless Riemann surfaces associated with an arbitrary d.d.f.s. Ω. Let us first define the skeleton of Ω:

Definition 3.17. Let $V_\Omega \subset \bigcup_{n=1}^{\infty}(\mathbb{C} \times \mathbb{Z})^n$ be the set of vertices

$$v := ((\omega_1, \sigma_1), \cdots, (\omega_n, \sigma_n)) \in (\mathbb{C} \times \mathbb{Z})^n$$

that satisfy the conditions 1) and 2) in Definition 3.3 and

3′) $\big(M_j(v), L_j(v), \omega_j\big) \in \overline{\mathcal{S}}_\Omega$ for $j = 2, \cdots, n$,

with $L_j(v) := \sum_{i=1}^{j-1} |\omega_{i+1} - \omega_i|$ $(j = 2, \cdots, n)$,

$$M_j(v) := \begin{cases} 0 & (j = 2) \\ \displaystyle\sum_{i=2}^{j-1} \big(A_i(v) + 2\pi(|\sigma_i| - 1)\big) & (j = 3, \cdots, n) \end{cases}$$

and

$$A_i(v) := \begin{cases} |\theta_i| & \text{if } \theta_i\sigma_i \ge 0 \\ 2\pi - |\theta_i| & \text{if } \theta_i\sigma_i < 0, \end{cases}$$

where

$$\theta_i := \arg\frac{\omega_{i+1} - \omega_i}{\omega_i - \omega_{i-1}}$$

is taken so that $\theta_i \in (-\pi, \pi]$. Let $E_\Omega \subset V_\Omega \times V_\Omega$ be the set of edges $e = (v', v)$ that satisfy one of the conditions i) $\sim$ iii) in Definition 3.3. We denote the directed tree diagram (V_Ω, E_Ω) by Sk_Ω and call it *skeleton* of Ω.

Now, assigning a cut plane U_v (respectively an open set U_e) to each $v \in V_\Omega$ (respectively each $e \in E_\Omega$ of type i)) defined by totally the same way with Section 3.1 and patching them as in Section 3.1, we obtain an initial object $(X_\Omega, \mathfrak{p}_\Omega, \underline{0}_\Omega)$ in the category of Ω-endless Riemann surfaces associated with a d.d.f.s. Ω. We denote the lift of $\gamma \in \Pi_\Omega^{\mathrm{dv}}$ on X_Ω by $\underline{\gamma}$.

4. Estimates for the analytic continuation of iterated convolutions

In this section our aim is to prove the following theorem, which is the analytical core of our study of the convolution product of endlessly continuable functions.

Theorem 4.1. *Let $\delta, L > 0$ be real numbers. Then there exist $c, \delta' > 0$ such that, for every d.f.s. Ω such that $\Omega_{4\delta} = \emptyset$, for every integer $n \geq 1$ and for every $\hat{f}_1, \ldots, \hat{f}_n \in \hat{\mathscr{R}}_\Omega$, the function $1 * \hat{f}_1 * \cdots * \hat{f}_n$ (which is known to belong to $\hat{\mathscr{R}}_{\Omega^{*n}}$) satisfies*

$$\begin{aligned} &\left| \mathfrak{p}^*_{\Omega^{*n}}\big(1 * \hat{f}_1 * \cdots * \hat{f}_n\big)(\underline{\zeta}) \right| \\ &\leq \frac{c^n}{n!} \sup_{L_1+\cdots+L_n=L} \| \mathfrak{p}^*_\Omega \hat{f}_1 \|_\Omega^{\delta', L_1} \cdots \| \mathfrak{p}^*_\Omega \hat{f}_n \|_\Omega^{\delta', L_n} \quad \text{for } \underline{\zeta} \in K^{\delta, L}_{\Omega^{*n}} \end{aligned} \tag{4.1}$$

(with notation (3.5)*).*

Using the Cauchy inequality, the identity $\frac{\mathrm{d}}{\mathrm{d}\zeta}(1 * \hat{f}_1 * \cdots * \hat{f}_n) = \hat{f}_1 * \cdots * \hat{f}_n$ and the inverse Borel transform, one easily deduces the following:

Corollary 4.2. *Let $\delta, L > 0$ be real numbers. Then there exist $c, \delta', L' > 0$ such that, for every d.f.s. Ω such that $\Omega_{4\delta} = \emptyset$, for every integer $n \geq 1$ and for every $\tilde{f}_1, \ldots, \tilde{f}_n \in \tilde{\mathscr{R}}_\Omega$ without constant term, the formal series $\tilde{f}_1 \cdots \tilde{f}_n$ (which is known to belong to $\tilde{\mathscr{R}}_{\Omega^{*n}}$) satisfies*

$$\| \tilde{f}_1 \cdots \tilde{f}_n \|^{\delta, L}_{\Omega^{*n}} \leq \frac{c^{n+1}}{n!} \| \tilde{f}_1 \|_\Omega^{\delta', L'} \cdots \| \tilde{f}_n \|_\Omega^{\delta', L'}.$$

In fact, one can cover the case $\hat{f}_1 \in \hat{\mathscr{R}}_{\Omega_1}, \ldots, \hat{f}_n \in \hat{\mathscr{R}}_{\Omega_n}$ with different d.f.s.'s $\Omega_1, \ldots, \Omega_n$ as well—see Theorem 4.8—, but we only give details for the case of one d.f.s. so as to lighten the presentation.

4.1. Notation and preliminaries

We fix an integer $n \geq 1$ and a d.f.s. Ω. In view of Remark 2.12, without loss of generality, we can suppose that Ω coincides with its upper closure:

$$\Omega = \tilde{\Omega}. \tag{4.2}$$

Let $\rho > 0$ be such that $\Omega_{3\rho} = \emptyset$. We set

$$U := \{\, \zeta \in \mathbb{C} \mid |\zeta| < 3\rho \,\}.$$

For each $\zeta \in U$, the path $\gamma_\zeta\colon\ t \in [0,1] \mapsto t\zeta$ is Ω-allowed and hence has a lift $\underline{\gamma}_\zeta$ on X_Ω starting at $\underline{0}_\Omega$. Then $\mathscr{L}(\zeta) := \underline{\gamma}_\zeta(1)$ defines a holomorphic function on U and induces an isomorphism

$$\mathscr{L}\colon\ U \xrightarrow{\sim} \underline{U}, \qquad \text{where} \quad \underline{U} := \mathscr{L}(U) \subset X_\Omega, \tag{4.3}$$

such that $\mathfrak{p}_\Omega \circ \mathscr{L} = \mathrm{Id}$.

Let us denote by Δ_n the n-dimensional simplex

$$\Delta_n := \{\, (s_1, \dots, s_n) \in \mathbb{R}^n_{\geq 0} \mid s_1 + \dots + s_n \leq 1 \,\}$$

with the standard orientation, and by $[\Delta_n] \in \mathscr{E}_n(\mathbb{R}^n)$ the corresponding integration current. For $\zeta \in U$, we define a map $\vec{\mathscr{D}}(\zeta)$ on a neighbourhood of Δ_n in $\mathbb{R}^n$ by

$$\vec{\mathscr{D}}(\zeta)\colon\ \vec{s} = (s_1, \dots, s_n) \mapsto \vec{\mathscr{D}}(\zeta, \vec{s}\,) := \big(\mathscr{L}(s_1\zeta), \dots, \mathscr{L}(s_n\zeta)\big) \in \underline{U}^n \subset X_\Omega^n$$

and denote by $\vec{\mathscr{D}}(\zeta)_\#[\Delta_n] \in \mathscr{E}_n(X_\Omega^n)$ the push-forward of $[\Delta_n]$ by $\vec{\mathscr{D}}(\zeta)$. (See [20] for the notations and notions related to integration currents.)

As in [20], our starting point will be

Lemma 4.3. *Let* $\hat{f}_1, \dots, \hat{f}_n \in \hat{\mathscr{R}}_\Omega$ *and* $\beta := (\mathfrak{p}_\Omega^* \hat{f}_1)\big(\underline{\zeta}_1\big) \cdots (\mathfrak{p}_\Omega^* \hat{f}_n)\big(\underline{\zeta}_n\big)\, \mathrm{d}\underline{\zeta}_1 \wedge \dots \wedge \mathrm{d}\underline{\zeta}_n$, *where we denote by* $\mathrm{d}\underline{\zeta}_1 \wedge \dots \wedge \mathrm{d}\underline{\zeta}_n$ *the pullback by* $\mathfrak{p}_\Omega^{\otimes n}\colon\ X_\Omega^n \to \mathbb{C}^n$ *of the n-form* $\mathrm{d}\zeta_1 \wedge \dots \wedge \mathrm{d}\zeta_n$. *Then*

$$1 * \hat{f}_1 * \dots * \hat{f}_n(\zeta) = \vec{\mathscr{D}}(\zeta)_\#[\Delta_n](\beta) \quad \textit{for } \zeta \in U.$$

Proof. This is just another way of writing the formula

$$1 * \hat{f}_1 * \dots * \hat{f}_n(\zeta) = \zeta^n \int_{\Delta_n} \hat{f}_1(\zeta s_1) \cdots \hat{f}_n(\zeta s_n)\, \mathrm{d}s_1 \cdots \mathrm{d}s_n. \tag{4.4}$$

See [20] for the details. □

Notation 4.4. We set

$$\mathcal{N}(\zeta) := \big\{ \big(\underline{\zeta}_1, \dots, \underline{\zeta}_n\big) \in X_\Omega^n \mid \mathfrak{p}_\Omega\big(\underline{\zeta}_1\big) + \dots + \mathfrak{p}_\Omega\big(\underline{\zeta}_n\big) = \zeta \big\} \ \text{ for } \zeta \in \mathbb{C}, \tag{4.5}$$

$$\mathcal{N}_j := \big\{ \big(\underline{\zeta}_1, \dots, \underline{\zeta}_n\big) \in X_\Omega^n \mid \underline{\zeta}_j = \underline{0}_\Omega \big\} \ \text{ for } 1 \leq j \leq n. \tag{4.6}$$

4.2. γ-adapted deformations of the identity

Let us consider a path $\gamma\colon [0,1] \to \mathbb{C}$ in $\Pi_{\Omega^{*n}}$ for which there exists $a \in (0,1)$ such that

$$\gamma(t) = \tfrac{t}{a}\gamma(a) \text{ for } t \in [0,a], \quad |\gamma(a)| = \rho, \quad \gamma|_{[a,1]} \text{ is } C^1. \tag{4.7}$$

We now introduce the notion of γ-adapted deformation of the identity, which is a slight generalization of the γ-adapted origin-fixing isotopies which appear in [20, Definition 5.1].

Definition 4.5. A *γ-adapted deformation of the identity* is a family $(\Psi_t)_{t\in[a,1]}$ of maps

$$\Psi_t\colon \underline{V} \to X^n_\Omega, \quad \text{for } t \in [a,1],$$

where $\underline{V} := \vec{\mathscr{D}}\big(\gamma(a)\big)(\Delta_n) \subset X^n_\Omega$, such that $\Psi_a = \mathrm{Id}$, the map $\big(t, \vec{\underline{\zeta}}\,\big) \in [a,1] \times \underline{V} \mapsto \Psi_t\big(\vec{\underline{\zeta}}\,\big) \in X^n_\Omega$ is locally Lipschitz, and for any $t \in [a,1]$ and $j = 1, \dots, n$,

$$\Psi_t\big(\underline{V} \cap \mathcal{N}\big(\gamma(a)\big)\big) \subset \mathcal{N}\big(\gamma(t)\big), \qquad \Psi_t\big(\underline{V} \cap \mathcal{N}_j\big) \subset \mathcal{N}_j \tag{4.8}$$

(with the notation (4.5)–(4.6)).

Let $\underline{\gamma}$ denote the lift of γ in X_Ω starting at $\underline{0}_\Omega$. The analytic continuation along $\underline{\gamma}$ of a convolution product can be obtained as follows:

Proposition 4.6 ([20]). *If $(\Psi_t)_{t\in[a,1]}$ is a γ-adapted deformation of the identity, then*

$$\mathfrak{p}^*_{\Omega^{*n}}\big(1 * \hat f_1 * \cdots * \hat f_n\big)\big(\underline{\gamma}(t)\big) = \big(\Psi_t \circ \vec{\mathscr{D}}\big(\gamma(a)\big)\big)_\#[\Delta_n](\beta) \qquad \textit{for } t \in [a,1] \tag{4.9}$$

for any $\hat f_1, \dots, \hat f_n \in \hat{\mathscr{R}}_\Omega$, with β as in Lemma 4.3.

Proof. See the proof of [20, Proposition 5.2]. □

Note that the right-hand side of (4.9) must be interpreted as

$$\int_{\Delta_n} (\mathfrak{p}^*_\Omega \hat f_1)\big(\underline{\zeta}^t_1\big) \cdots (\mathfrak{p}^*_\Omega \hat f_n)\big(\underline{\zeta}^t_n\big) \det\left[\frac{\partial \zeta^t_i}{\partial s_j}\right]_{1\le i,j\le n} \mathrm{d}s_1 \cdots \mathrm{d}s_n \tag{4.10}$$

with the notation

$$\big(\underline{\zeta}^t_1, \dots, \underline{\zeta}^t_n\big) := \Psi_t \circ \vec{\mathscr{D}}\big(\gamma(a)\big), \qquad \zeta^t_i := \mathfrak{p}_\Omega \circ \underline{\zeta}^t_i \ \text{ for } 1 \le i \le n \tag{4.11}$$

(each function ζ^t_i is Lipschitz on Δ_n and Rademacher's theorem ensures that it is differentiable almost everywhere on Δ_n, with bounded partial derivatives).

The following is the key estimate:

Theorem 4.7. *Let $\delta \in (0,\rho)$ and $L > 0$. Let $\gamma \in \Pi^{\delta,L}_{\Omega*n}$ satisfy* (4.7) *and let*

$$\delta'(t) := \rho\, \mathrm{e}^{-2\sqrt{2}\delta^{-1}L(\gamma|_{[a,t]})}, \quad c(t) := \rho\, \mathrm{e}^{3\delta^{-1}L(\gamma|_{[a,t]})} \quad \textit{for } t \in [a,1]. \tag{4.12}$$

Then there exists a γ-adapted deformation of the identity $(\Psi_t)_{t\in[a,1]}$ such that

$$\Psi_t \circ \vec{\mathscr{D}}\big(\gamma(a)\big)(\Delta_n) \subset \bigcup_{L_1+\cdots+L_n=L(\gamma|_t)} K^{\delta'(t),L_1}_{\Omega} \times \cdots \times K^{\delta'(t),L_n}_{\Omega} \quad \textit{for } t\in[a,1]. \tag{4.13}$$

Further, with the notation (4.11)*, the partial derivatives $\partial\zeta_i^t/\partial s_j$ satisfy*

$$\left|\det\left[\frac{\partial \zeta_i^t}{\partial s_j}\right]_{1\le i,j\le n}\right| \le \big(c(t)\big)^n \quad \textit{a.e. on } \Delta_n \tag{4.14}$$

for each $t \in [a,1]$.

Proof that Theorem 4.7 *implies Theorem* 4.1. Let $\delta, L > 0$. We will show that (4.1) holds with

$$\delta' := \min\big\{\delta, \rho\, \mathrm{e}^{-4\sqrt{2}(1+\delta^{-1}L)}\big\}, \quad c := \max\big\{2\rho, \rho\, \mathrm{e}^{6(1+\delta^{-1}L)}\big\}, \quad \text{where } \rho := \tfrac{4}{3}\delta.$$

Let Ω be a d.f.s. such that $\Omega_{4\delta} = \emptyset$. Without loss of generality we may suppose that $\Omega = \tilde{\Omega}$.

In view of formula (4.4), the inequality (4.1) holds for $\underline{\zeta} \in K^{\delta,L}_{\Omega*n} \cap \underline{U}$, where $\underline{U}$ is defined by (4.3), because the Lebesgue measure of Δ_n is $1/n!$.

Let $\underline{\zeta} \in K^{\delta,L}_{\Omega*n} \setminus \underline{U}$. We can write $\underline{\zeta} = \underline{\gamma}(1)$ with $\gamma \in \Pi^{\delta,L}_{\Omega*n}$, assuming without loss of generality that the first two conditions in (4.7) hold. If the third condition in (4.7) does not hold, *i.e.* if $\gamma|_{[a,1]}$ is not C^1, then we use a sequence of paths $\gamma_k \in \Pi^{\delta/2,L+\delta}_{\Omega*n}$ such that $\gamma_k|_{[0,a]} = \gamma|_{[0,a]}$, $\gamma_k(1) = \gamma(1)$, $\gamma_k|_{[a,1]}$ is C^1 and $\sup_{t\in[a,1]}|\gamma(t)-\gamma_k(t)| \to 0$ as $k\to\infty$; for k large enough one has $\underline{\gamma_k}(1) = \underline{\zeta}$, thus one then can replace γ by γ_k. Hence we can assume that (4.7) holds. Let $(\Psi_t)_{[t\in[a,1]]}$ denote the γ-adapted deformation of the identity provided by Theorem 4.7, possibly with (δ, L) replaced by $(\delta/2, L+\delta)$. Proposition 4.6 shows that, for $\hat{f}_1,\ldots,\hat{f}_n \in \hat{\mathscr{R}}_\Omega$, $\mathfrak{p}^*_{\Omega*n}\big(1*\hat{f}_1*\cdots*\hat{f}_n\big)(\underline{\zeta})$ can be written as (4.10) with $t=1$, and (4.13)–(4.14) then show that (4.1) holds because $\delta'(t) \ge \delta'$ and $c(1) \le c$. Therefore, (4.1) holds on $K^{\delta,L}_{\Omega*n} \setminus \underline{U}$ too. □

In fact, in view of the proof of Theorem 4.7 given below, one can give the following generalization of Theorem 4.1:

Theorem 4.8. *Let δ, L be positive real numbers. Then there exist positive constants c and δ' such that, for every integer $n \ge 1$ and for all d.f.s. $\Omega_1,\ldots,\Omega_n$*

with $\Omega_{j,4\delta} = \emptyset$ $(j = 1, \cdots, n)$ *and* $\hat{f}_1 \in \hat{\mathscr{R}}_{\Omega_1}, \ldots, \hat{f}_n \in \hat{\mathscr{R}}_{\Omega_n}$, *the function* $1 * \hat{f}_1 * \cdots * \hat{f}_n$ *belongs to* $\hat{\mathscr{R}}_{\Omega}$, *where* $\Omega := \Omega_1 * \cdots * \Omega_n$, *and*

$$\begin{aligned}&\left|\mathfrak{p}^*_{\Omega}\big(1 * \hat{f}_1 * \cdots * \hat{f}_n\big)(\underline{\zeta})\right| \\ &\le \frac{c^n}{n!} \sup_{L_1+\cdots+L_n=L} \left\| \mathfrak{p}^*_{\Omega_1} \hat{f}_1 \right\|^{\delta',L_1}_{\Omega_1} \cdots \left\| \mathfrak{p}^*_{\Omega_n} \hat{f}_n \right\|^{\delta',L_n}_{\Omega_n} \quad \textit{for } \underline{\zeta} \in K^{\delta,L}_{\Omega}.\end{aligned} \tag{4.15}$$

4.3. Proof of Theorem 4.7

We suppose that we are given $n \ge 1$, $\rho > 0$, a d.f.s. Ω such that $\Omega = \tilde{\Omega}$ and $\Omega_{3\rho} = \emptyset$, and $\gamma \in \Pi^{\delta,L}_{\Omega^{*n}}$ satisfying (4.7) with $\delta \in (0, \rho)$ and $L > 0$.

We set $\tilde{\gamma}(t) := \big(L(\gamma_{|t}), \gamma(t)\big)$ and define functions

$$\eta : \mathbb{R} \times \mathbb{C} \to \mathbb{R}_{\ge 0}, \qquad D : [a, 1] \times (\mathbb{R} \times \mathbb{C})^n \to \mathbb{R}_{\ge 0}$$

by the formulas

$$\begin{aligned}\eta(v) &:= \operatorname{dist}\big(v, \{(0,0)\} \cup \mathcal{S}_{\Omega}\big), \\ D\big(t, \vec{v}\,\big) &:= \eta(v_1) + \cdots + \eta(v_n) + \left|\tilde{\gamma}(t) - (v_1 + \cdots + v_n)\right|,\end{aligned} \tag{4.16}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3$. The assumptions $\Omega = \tilde{\Omega}$ and $\gamma \in \Pi^{\delta,L}_{\Omega^{*n}}$ yield

Lemma 4.9. *The function D satisfies*

$$D \ge \delta \quad \textit{on } [a, 1] \times (\mathbb{R} \times \mathbb{C})^n. \tag{4.17}$$

Proof. Let $(t, \vec{v}\,) \in [a, 1] \times (\mathbb{R} \times \mathbb{C})^n$. For each $j \in \{1, \ldots, n\}$, pick $u_j \in \{(0,0)\} \cup \overline{\mathcal{S}}_{\Omega}$ so that $\eta(v_j) = |v_j - u_j|$, and let $u := u_1 + \cdots + u_n$. Either all of the u_j's equal $(0,0)$, in which case $u = (0,0)$ too, or $u = (\lambda, \omega)$ is a non-trivial sum of at most n points of the form $u_j = (\lambda_j, \omega_j) \in \overline{\mathcal{S}}_{\Omega}$, in which case we have in fact $\omega_j \in \Omega_{\lambda_j}$ because of Lemma 2.4 and the assuption $\tilde{\Omega} = \Omega$, hence (2.12) then yields $\omega \in \Omega^{*n}_{\lambda}$. We thus find

$$D\big(t, \vec{v}\,\big) = |v_1 - u_1| + \cdots + |v_n - u_n| + \left|\tilde{\gamma}(t) - (v_1 + \cdots + v_n)\right| \ge |\tilde{\gamma}(t) - u| \tag{4.18}$$

with $u \in \{(0,0)\} \cup \mathcal{S}_{\Omega^{*n}}$.

If $u = (0,0)$, then $D\big(t, \vec{v}\,\big) \ge \left|\tilde{\gamma}(t)\right| \ge L(\gamma_{|t}) \ge \rho \ge \delta$ because $t \ge a$.

Otherwise, $u \in \mathcal{S}_{\Omega^{*n}}$ and (4.18) shows that $D\big(t, \vec{v}\,\big) \ge \delta$ because $\gamma \in \Pi^{\delta,L}_{\Omega^{*n}}$. $\square$

Since D never vanishes, we can define a non-autonomous vector field

$$(t, \vec{v}\,) \in [a, 1] \times (\mathbb{R} \times \mathbb{C})^n \mapsto \vec{X}(t, \vec{v}\,) \in T_{\vec{v}}\big((\mathbb{R} \times \mathbb{C})^n\big) \simeq (\mathbb{R} \times \mathbb{C})^n$$

by the formulas

$$\vec{X}(t,\vec{v}\,) = \left| \begin{array}{l} X_1(t,\vec{v}\,) := \dfrac{\eta(v_1)}{D(t,\vec{v}\,)}\tilde{\gamma}'(t) \\ \qquad\qquad \vdots \\ X_n(t,\vec{v}\,) := \dfrac{\eta(v_n)}{D(t,\vec{v}\,)}\tilde{\gamma}'(t). \end{array} \right. \tag{4.19}$$

Note that $\tilde{\gamma}'(t) = \big(|\gamma'(t)|, \gamma'(t)\big)$.

The functions $X_j\colon [a,1]\times(\mathbb{R}\times\mathbb{C})^n \to \mathbb{R}\times\mathbb{C}$ are locally Lipschitz, thus we can apply the Cauchy-Lipschitz theorem on the existence and uniqueness of solutions to differential equations and get a locally Lipschitz flow map

$$(t^*,t,\vec{v}\,) \in [a,1]\times[a,1]\times(\mathbb{R}\times\mathbb{C})^n \mapsto \Phi^{t^*,t}(\vec{v}\,) \in (\mathbb{R}\times\mathbb{C})^n \tag{4.20}$$

(value at time t of the unique maximal solution to $\mathrm{d}\vec{v}/dt = \vec{X}(t,\vec{v}\,)$ whose value at time t^* is $\vec{v}\,$). We construct a γ-adapted deformation of the identity out of the flow map as follows:

Proposition 4.10. *Let* $\vec{\underline{\zeta}} = \big(\mathscr{L}(\zeta_1),\dots,\mathscr{L}(\zeta_n)\big) \in \underline{V}$, i.e. $\zeta_j = s_j\gamma(a)$ *with* $(s_1,\dots,s_n)\in\Delta_n$. *We define* $\vec{v} := \big((|\zeta_1|,\zeta_1),\dots,(|\zeta_n|,\zeta_n)\big) \in (\mathbb{R}\times\mathbb{C})^n$ *and* $\Gamma = (\tilde{\gamma}_1,\dots,\tilde{\gamma}_n)\colon [0,1]\to(\mathbb{R}\times\mathbb{C})^n$ *by*

$$t\in[0,a] \Rightarrow \Gamma(t) := \big(\tfrac{t}{a}(|\zeta_1|,\zeta_1),\dots,\tfrac{t}{a}(|\zeta_n|,\zeta_n)\big), \quad t\in[a,1] \Rightarrow \Gamma(t) := \Phi^{a,t}(\vec{v}\,).$$

Then, for each $j\in\{1,\dots,n\}$, $\tilde{\gamma}_j$ *is a path* $[0,1]\to\mathbb{R}\times\mathbb{C}$ *whose* $\mathbb{C}$*-projection* γ_j *belongs to* Π_Ω, *and the formula*

$$\Psi_t\big(\vec{\underline{\zeta}}\,\big) := \big(\underline{\gamma}_1(t),\dots,\underline{\gamma}_n(t)\big) \in X_\Omega^n \quad \textit{for } t\in[a,1] \tag{4.21}$$

defines a γ*-adapted deformation of the identity.*

Proof. We first prove that $\gamma_1,\dots,\gamma_n\in\Pi_\Omega$. In view of (2.5), we just need to check that, for each $j\in\{1,\dots,n\}$, the path $\tilde{\gamma}_j = (\lambda_j,\gamma_j)$ satisfies

$$t\in[0,1] \quad\Rightarrow\quad \tilde{\gamma}_j(t)\in\mathcal{M}_\Omega \ \text{ and } \ \mathrm{d}\lambda_j/\mathrm{d}t = |\mathrm{d}\gamma_j/\mathrm{d}t|. \tag{4.22}$$

Since $\zeta_j\in U$ and $\gamma_j(t) = \frac{t}{a}\zeta_j$ for $t\in[0,a]$, the property (4.22) holds for $t\in[0,a]$.

For $t\in[a,1]$, the second property in (4.22) follows from the fact that the $\mathbb{R}$-projection of $X_j(t,\vec{v}\,)\in\mathbb{R}\times\mathbb{C}$ coincides with the modulus of its $\mathbb{C}$-projection.

Since $\big(\tilde{\gamma}_1(t),\dots,\tilde{\gamma}_n(t)\big) = \Phi^{a,t}\big(\tilde{\gamma}_1(a),\dots,\tilde{\gamma}_n(a)\big)$ and the first property in (4.22) holds at $t=a$, the first property in (4.22) for $t\in[a,1]$ is a consequence of the inclusion

$$\Phi^{a,t}\big(\mathcal{M}_\Omega^n\big) \subset \mathcal{M}_\Omega^n, \tag{4.23}$$

which can itself be checked as follows: suppose $\vec{v}^* \in (\mathbb{R}\times\mathbb{C})^n \setminus \mathcal{M}^n_\Omega$, then it has at least one component v^*_j in $\overline{\mathcal{S}}_\Omega$ and, in view of the form of the vector field (4.19), the submanifold $\{\, \vec{v} \in (\mathbb{R}\times\mathbb{C})^n \mid v_j = v^*_j \,\}$ is invariant by the maps Φ^{t_1,t_2} (because $\eta(v_j) = 0$ implies that $X_j = 0$ on this submanifold), in particular $\Phi^{t,a}\big((\mathbb{R}\times\mathbb{C})^n \setminus \mathcal{M}^n_\Omega\big) \subset (\mathbb{R}\times\mathbb{C})^n \setminus \mathcal{M}^n_\Omega$, whence (4.23) follows because $\Phi^{a,t}$ and $\Phi^{t,a}$ are mutually inverse bijections.

Therefore the paths $\gamma_1,\dots,\gamma_n$ are Ω-allowed and have lifts in X_Ω starting at $\underline{0}_\Omega$, which allow us to define the maps Ψ_t by (4.21) on $\underline{V}$.

We now prove that $(\Psi_t)_{t\in[a,1]}$ is a γ-adapted deformation of the identity. The map $(t,\vec{v}\,) \mapsto \Psi_t(\vec{v}\,)$ is locally Lipschitz because the flow map (4.20) is locally Lipschitz, and $\Psi_a = \mathrm{Id}$ because $\Phi^{a,a}$ is the identity map of $(\mathbb{R}\times\mathbb{C})^n$; hence, we just need to prove (4.8).

We set

$$\tilde{\mathcal{N}}(w) := \big\{(v_1,\dots,v_n) \in (\mathbb{R}\times\mathbb{C})^n \mid v_1+\dots+v_n = w\big\} \text{ for } w \in \mathbb{R}\times\mathbb{C},$$

$$\tilde{\mathcal{N}}_j := \big\{(v_1,\dots,v_n) \in (\mathbb{R}\times\mathbb{C})^n \mid v_j = (0,0)\big\} \text{ for } 1 \le j \le n.$$

Let $j \in \{1,\dots,n\}$. The second part of (4.8) follows from the inclusion

$$\Phi^{a,t}\big(\tilde{\mathcal{N}}_j\big) \subset \tilde{\mathcal{N}}_j \text{ for } t \in [a,1],$$

which stems from the fact that the jth component of the vector field (4.19) vanishes on $\tilde{\mathcal{N}}_j$ (because $\eta\big((0,0)\big) = 0$).

Since $\zeta_1+\dots+\zeta_n = \gamma(a) \Rightarrow |\zeta_1|+\dots+|\zeta_n| = |\gamma(a)|$ for any $(\zeta_1,\dots,\zeta_n) \in \underline{V}$, the first part of (4.8) follows from the inclusion

$$\Phi^{a,t}\big(\tilde{\mathcal{N}}\big(\tilde{\gamma}(a)\big)\big) \subset \tilde{\mathcal{N}}\big(\tilde{\gamma}(t)\big) \text{ for } t \in [a,1],$$

which can be itself checked as follows: consider first an arbitrary initial condition $\vec{v} \in (\mathbb{R}\times\mathbb{C})^n$ and the corresponding solution $\vec{v}(t) := \Phi^{a,t}(\vec{v}\,)$, and let $v_0(t) := v_1(t)+\dots+v_n(t)$; then (4.19) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\tilde{\gamma}(t) - v_0(t)\big) = \frac{\big|\tilde{\gamma}(t) - v_0(t)\big|}{D(t,\vec{v}(t))}\tilde{\gamma}'(t),$$

hence the Lipschitz function $h(t) := \big|\tilde{\gamma}(t) - v_0(t)\big|$ has an almost everywhere defined derivative which satisfies $|h'(t)| \le \big|\frac{\mathrm{d}}{\mathrm{d}t}\big(\tilde{\gamma}(t) - v_0(t)\big)\big| \le \frac{1}{D(t,\vec{v}(t))}|\tilde{\gamma}'(t)|\,h(t)$, which is $\le \delta^{-1}\sqrt{2}\,|\tilde{\gamma}'(t)|\,h(t)$ by (4.17), whence

$$\big|\tilde{\gamma}(t) - v_0(t)\big| \le \big|\tilde{\gamma}(a) - v_0(a)\big| \exp\big(\delta^{-1}\sqrt{2}\,L(\gamma|_{[a,t]})\big)$$

for all t; now, if $\vec{v} \in \tilde{\mathcal{N}}\big(\tilde{\gamma}(a)\big)$, we find $v_0(a) = \tilde{\gamma}(a)$, whence $v_0(t) = \tilde{\gamma}(t)$ for all t. ☐

We now show that the γ-adapted deformation of the identity that we have constructed in Proposition 4.10 meets the requirements of Theorem 4.7.

In view of (2.6)–(2.7) and (3.5), the inclusion (4.13) follows from:

Lemma 4.11. *Let* $\underline{\tilde{V}} := \{(s_1\tilde{\gamma}(a), \dots, s_n\tilde{\gamma}(a)) \mid (s_1, \dots, s_n) \in \Delta_n\} \in (\mathbb{R} \times \mathbb{C})^n$. *Then*

$$\Phi^{a,t}(\underline{\tilde{V}}) \subset \bigcup_{L_1+\cdots+L_n=L(\gamma_{|t})} \mathcal{M}_\Omega^{L_1,\delta'(t)} \times \cdots \times \mathcal{M}_\Omega^{L_n,\delta'(t)} \tag{4.24}$$

for all $t \in [a, 1]$, *with* $\delta'(t)$ *as in* (4.12).

Proof of Lemma 4.11. Let us consider an initial condition $\vec{v} \in \underline{\tilde{V}}$ and the corresponding solution $\vec{v}(t) := \Phi^{a,t}(\vec{v})$, whose components we write as $v_j(t) = (\lambda_j(t), \zeta_j(t))$ for $j = 1, \dots, n$. We also have $v_j(a) = s_j\tilde{\gamma}(a)$ for some $(s_1, \dots, s_n) \in \Delta_n$, whence $\lambda_1(a) + \cdots + \lambda_n(a) \le |\gamma(a)| = \rho$ and $|v_j(a)| = \rho$ for $j = 1, \dots, n$.

We first notice that

$$\sum_{j=1}^n |\lambda'_j(t)| = \sum_{j=1}^n \frac{\eta(v_j(t))}{D(t, \vec{v}(t))} |\gamma'(t)| \le |\gamma'(t)|,$$

hence $\lambda_1(t) + \cdots + \lambda_n(t) \le \lambda_1(a) + \cdots + \lambda_n(a) + \int_a^t |\gamma'| \le L(\gamma_{|t})$. Therefore, we just need to show that

$$\operatorname{dist}(v_j(t), \mathcal{S}_\Omega) \ge \delta'(t) \quad \text{for } j = 1, \dots, n. \tag{4.25}$$

Let $j \in \{1, \dots, n\}$. Since η is 1-Lipschitz, we can define a Lipschitz function on $[a, 1]$ by the formula $h_j(t) := \eta(v_j(t))$, and its almost everywhere defined derivative satisfies

$$|h'_j(t)| \le |v'_j(t)| = \frac{h_j(t)}{D(t, \vec{v}(t))} |\tilde{\gamma}'(t)| \le g(t)h_j(t), \quad \text{where } g(t) := \delta^{-1}\sqrt{2}\,|\gamma'(t)|.$$

Since $\int_a^t g(\tau)\,\mathrm{d}\tau = \delta^{-1}\sqrt{2}\,L(\gamma|_{[a,t]})$, we deduce that

$$\begin{aligned} \eta(v_j(a))\,\mathrm{e}^{-\delta^{-1}\sqrt{2}\,L(\gamma|_{[a,t]})} &\le \eta(v_j(t)) \\ &\le \eta(v_j(a))\,\mathrm{e}^{\delta^{-1}\sqrt{2}\,L(\gamma|_{[a,t]})} \quad \text{for all } t \in [a, 1]. \end{aligned} \tag{4.26}$$

Let us now fix $t \in [a, 1]$. We conclude by distinguishing two cases.

Suppose first that $\eta(v_j(a)) \ge \rho\,\mathrm{e}^{-\sqrt{2}\,\delta^{-1}L(\gamma|_{[a,t]})}$. Then the first inequality in (4.26) yields $\eta(v_j(t)) \ge \delta'(t)$, and since $\operatorname{dist}(v_j(t), \mathcal{S}_\Omega) \ge \eta(v_j(t))$ we get (4.25).

Suppose now that $\eta(v_j(a)) < \rho\,\mathrm{e}^{-\sqrt{2}\,\delta^{-1}L(\gamma|_{[a,t]})}$. Then the second inequality in (4.26) yields $\eta(v_j(t')) < \rho$ for all $t' \in [a, t]$. This implies that $v_j([a, t]) \subset B :$ $= \{ v \in \mathbb{R} \times \mathbb{C} \mid |v| < 3\rho/2 \}$; indeed, if not, since $v_j(a) \in B$, there would exist $t' \in (a, t]$ such that $v_j(t') \in \partial B$, but using $\Omega_{3\rho} = \emptyset$ it is easy to check that

$$v \in \overline{B} \;\Rightarrow\; \operatorname{dist}(v, \mathcal{S}_\Omega) \ge 3\rho/2,$$

hence we would have $\operatorname{dist}(v_j(t'), \mathcal{S}_\Omega) \geq 3\rho/2 > \eta(v_j(t'))$, whence $\eta(v_j(t')) = \operatorname{dist}\big(v_j(t'), (0,0)\big) = 3\rho/2$, which is a contradiction. Therefore $v_j(t) \in B$, whence $\operatorname{dist}(v_j(t), \mathcal{S}_\Omega) \geq 3\rho/2 > \delta'(t)$ and we are done. □

Only the inequality (4.14) remains to be proved. We first show the following:

Lemma 4.12. *For any $t \in [a, 1]$ and $\vec{u}, \vec{v} \in (\mathbb{R} \times \mathbb{C})^n$, the vector field* (4.19) *satisfies*

$$\sum_{j=1}^{n} |X_j(t, \vec{u}\,) - X_j(t, \vec{v}\,)| \leq 3\frac{|\tilde{\gamma}'(t)|}{D(t, \vec{u}\,)} \sum_{j=1}^{n} |u_j - v_j|. \tag{4.27}$$

Proof of Lemma 4.12. We rewrite $X_j(t, \vec{u}\,) - X_j(t, \vec{v}\,)$ as follows:

$$X_j(t, \vec{u}\,) - X_j(t, \vec{v}\,) = \Big(\eta(u_j) - \eta(v_j) + \big(D(t, \vec{v}\,) - D(t, \vec{u}\,)\big)\frac{\eta(v_j)}{D(t, \vec{v}\,)}\Big)\frac{\tilde{\gamma}'(t)}{D(t, \vec{u}\,)}.$$

Since $|\eta(u_j) - \eta(v_j)| \leq |u_j - v_j|$ holds for $j = 1, \dots, n$, we have

$$\begin{aligned}\big|D(t, \vec{u}\,) - D(t, \vec{v}\,)\big| &\leq \sum_{j=1}^{n} \big|\eta(u_j) - \eta(v_j)\big| + \bigg|\Big|\tilde{\gamma}(t) - \sum_{j=1}^{n} u_j\Big| - \Big|\tilde{\gamma}(t) - \sum_{j=1}^{n} v_j\Big|\bigg| \\ &\leq 2\sum_{j=1}^{n} |u_j - v_j|.\end{aligned}$$

Then, summing up $|X_j(t, \vec{u}\,) - X_j(t, \vec{v}\,)|$ in j, we obtain (4.27) from the inequality $\sum_{j=1}^{n} \eta(v_j) \leq D(t, \vec{v}\,)$. □

We conclude by deriving the inequality (4.14) from Lemma 4.12. We use the notation (4.11) to define $\zeta_1^t, \dots, \zeta_n^t \colon \Delta_n \to \mathbb{C}$, and we now define $v_j^t \colon \Delta_n \to \mathbb{R} \times \mathbb{C}$ for $t \in [a, 1]$ by the formulas $v_j^a(\vec{s}\,) := s_j \tilde{\gamma}(a)$ and

$$\vec{v}^{\,t} := (v_1^t, \dots, v_n^t) := \Phi^{a,t} \circ (v_1^a, \dots, v_n^a).$$

Let

$$V(t) := \sum_{j=1}^{n} |\zeta_j^t(\vec{s}\,) - \zeta_j^t(\vec{s}\,')| \quad \text{for } \vec{s}, \vec{s}\,' \in \Delta_n.$$

We obtain from (4.17) and (4.27) the following estimate:

$$\begin{aligned}V(t) &\leq V(a) + \frac{1}{\sqrt{2}} \sum_{j=1}^{n} \int_a^t \big|X_j\big(\tau, \vec{v}^{\,\tau}(\vec{s}\,)\big) - X_j\big(\tau, \vec{v}^{\,\tau}(\vec{s}\,')\big)\big|\, \mathrm{d}\tau \\ &\leq V(a) + \frac{3}{\delta}\int_a^t |\gamma'(\tau)| V(\tau) d\tau.\end{aligned}$$

Therefore, Gronwall's lemma yields $V(t) \le V(a)\,\mathrm{e}^{3\delta^{-1}L(\gamma|_{[a,t]})}$, and hence, since $V(a) = \rho\sum_{j=1}^{n}|s_j - s'_j|$, we have

$$V(t) \le \rho\,\mathrm{e}^{3\delta^{-1}L(\gamma|_{[a,t]})}\sum_{j=1}^{n}|s_j - s'_j|. \tag{4.28}$$

Then, (4.28) entails via Rademacher's theorem that the following estimate holds a.e. on Δ_n:

$$\sum_{i=1}^{n}\left|\frac{\partial\zeta_i^t}{\partial s_j}\right| \le \rho\,\mathrm{e}^{3\delta^{-1}L(\gamma|_{[a,t]})}.$$

Finally, (4.14) follows from the inequality

$$\left|\det\left[\frac{\partial\zeta_i^t}{\partial s_j}\right]_{1\le i,j\le n}\right| \le \prod_{j=1}^{n}\left(\sum_{i=1}^{n}\left|\frac{\partial\zeta_i^t}{\partial s_j}\right|\right).$$

Remark 4.13. Theorem 4.8 is verified by replacing the vector field (4.19) by

$$\vec{X}(t,\vec{v}\,) = \left|\begin{array}{l} X_1 := \dfrac{\eta_1(v_1)}{D(t,\vec{v}\,)}\tilde{\gamma}'(t) \\ \qquad\vdots \\ X_n := \dfrac{\eta_n(v_n)}{D(t,\vec{v}\,)}\tilde{\gamma}'(t), \end{array}\right.$$

where $\eta_j(v) := \operatorname{dist}\big(v, \{(0,0)\}\cup\overline{\mathcal{S}}_{\Omega_j}\big)$, $D\big(t,\vec{v}\,\big) := \eta_1(v_1)+\cdots+\eta_n(v_n)+|\tilde{\gamma}(t) - (v_1+\cdots+v_n)|$.

4.4. The case of endless continuability with respect to bounded direction variation

In this subsection, we extend the estimates of Theorem 4.1 to the case of a d.d.f.s..

Notation 4.14. Given $\delta, M, L > 0$, we denote by $\Pi_{\Omega}^{\delta,M,L}$ the set of all paths $\gamma \in \Pi_{\Omega}^{\mathrm{dv}}$ such that $V(\gamma) \le M$, $L(\gamma) \le L$ and $\inf_{t\in[0,t_*]}\operatorname{dist}_1\big(\tilde{\gamma}^{\,\mathrm{dv}}(t), \overline{\mathcal{S}}_{\Omega}\big) \ge \delta$, where $\tilde{\gamma}^{\,\mathrm{dv}}$ is as in (2.13) and dist_1 is the distance associated with the norm $\|\cdot\|_1$ defined on $\mathbb{R}^2\times\mathbb{C}$ by $\|(\mu,\lambda,\zeta)\|_1 := |\mu| + \sqrt{|\lambda|^2+|\zeta|^2}$.

Let us fix an arbitrary d.d.f.s. Ω. We fix $\rho > 0$ such that $\Omega_{3\rho,M} = \emptyset$ for every $M \ge 0$. We consider a path $\gamma : [0,1] \to \mathbb{C}$ in $\Pi_{\Omega^{*n}}^{\delta,M,L}$, with arbitrary $\delta \in (0,\rho)$ and $L > 0$, satisfying the following condition:

There exists $a \in (0,1)$ such that $\gamma(t) = \frac{t}{a}\gamma(a)$ for $t \in [0,a]$ and $|\gamma(a)| = \rho$. (4.29)

Then, for $t \in [0,1]$ and $v \in \mathbb{R}\times\mathbb{C}$, we set

$$\eta(t,v) := \mathrm{dist}_1\Big(\big(V(\gamma_{|t}),v\big), \big(\mathbb{R}\times\{(0,0)\}\big)\cup\overline{\mathcal{S}}_\Omega\Big)$$

and, for $\vec{v} = (v_1,\cdots,v_n) \in (\mathbb{R}\times\mathbb{C})^n$,

$$D\big(t,\vec{v}\,\big) := \eta(t,v_1)+\cdots+\eta(t,v_n)+|\tilde{\gamma}(t)-(v_1+\cdots+v_n)|.$$

Choosing $(\mu_j,u_j) \in \big(\mathbb{R}\times\{(0,0)\}\big)\cup\overline{\mathcal{S}}_\Omega$ so that $\eta(t,v_j) = \|(V(\gamma_{|t}),v_j)-(\mu_j,u_j)\|_1$ for each j and using $(\mu_{j_0}, u_1+\cdots+u_n) \in \big(\mathbb{R}\times\{(0,0)\}\big)\cup\overline{\mathcal{S}}_{\Omega^{*n}}$ with $\max\limits_{j=1,\cdots,n}\mu_j = \mu_{j_0}$, we see that

$$\begin{aligned} D(t,\vec{v}\,) &= \sum_{j=1}^{n}(|V(\gamma_{|t})-\mu_j|+|v_j-u_j|)+|\tilde{\gamma}(t)-(v_1+\cdots+v_n)| \\ &\ge \big|V(\gamma_{|t})-\mu_{j_0}\big| + |\tilde{\gamma}(t)-(u_1+\cdots+u_n)| \ge \min\{\delta,\rho\} \end{aligned}$$

for $(t,\vec{v}\,) \in [a,1]\times(\mathbb{R}\times\mathbb{C})^n$.

We can thus define a map $(t^*,t,\vec{v}\,) \in [a,1]\times[a,1]\times(\mathbb{R}\times\mathbb{C})^n \mapsto \Phi^{t^*,t}(\vec{v}\,) \in (\mathbb{R}\times\mathbb{C})^n$ as the flow map of

$$\vec{X}(t,\vec{v}\,) = \left| \begin{aligned} &X_1(t,\vec{v}\,) := \frac{\eta(t,v_1)}{D(t,\vec{v}\,)}\tilde{\gamma}'(t) \\ &\qquad\vdots \\ &X_n(t,\vec{v}\,) := \frac{\eta(t,v_n)}{D(t,\vec{v}\,)}\tilde{\gamma}'(t). \end{aligned} \right. \tag{4.30}$$

Let $\vec{v}^{\,t} = (\vec{v}^{\,t}_1,\cdots,\vec{v}^{\,t}_n)$ be the flow of (4.30) with the initial condition $\vec{v}^{\,a}_j := (|\gamma(a)|s_j, \gamma(a)s_j)$ with $\vec{s} \in \Delta_n$. Since $\tilde{\gamma}'(t)$, $\eta(t,v_j)$ and $D(t,\vec{v}\,)$ are Lipschitz continuous on $[a,1]\times(\mathbb{R}\times\mathbb{C})^n$, we find by Rademacher's theorem that $d\zeta^t_j/dt$ is differentiable a.e. on $[a,1]$ and satisfies

$$\frac{d^2\zeta^t_j/dt^2}{d\zeta^t_j/dt} = \frac{1}{\eta(v^t_j)}\frac{d\eta(v^t_j)}{dt} - \frac{1}{D(t,\vec{v}^{\,t})}\frac{dD(t,\vec{v}^{\,t})}{dt} + \frac{\gamma''(t)}{\gamma'(t)}$$

when $s_j \neq 0$. Since $\eta(v^t_j)$ and $D(t,\vec{v}^{\,t})$ are real valued functions, we have

$$\mathrm{Im}\frac{d^2\zeta^t_j/dt^2}{d\zeta^t_j/dt} = \mathrm{Im}\frac{\gamma''(t)}{\gamma'(t)}.$$

Therefore, the following holds for every $t \in [a,1]$:

$$V\big(\zeta^{\cdot}_j|_{[0,t]}\big) = V(\gamma_{|t}) \quad \text{when} \quad s_j \neq 0.$$

Arguing as for Theorem 4.1, we obtain:

Theorem 4.15. *Let $\delta, L, M > 0$ be real numbers. Then there exist $c, \delta' > 0$ such that, for every d.d.f.s. Ω such that $\Omega_{4\delta,M} = \emptyset$ $(M \geq 0)$, for every integer $n \geq 1$ and for every $\hat{f}_1, \ldots, \hat{f}_n \in \hat{\mathscr{R}}^{\mathrm{dv}}_{\Omega}$, the function $1 * \hat{f}_1 * \cdots * \hat{f}_n$ belongs to $\hat{\mathscr{R}}^{\mathrm{dv}}_{\Omega^{*n}}$ and satisfies*

$$\begin{aligned} &\| \mathfrak{p}^*_{\Omega^{*n}}\big(1 * \hat{f}_1 * \cdots * \hat{f}_n\big) \|^{\delta,M,L}_{\Omega^n} \\ &\leq \frac{c^n}{n!} \sup_{L_1+\cdots+L_n=L} \| \mathfrak{p}^*_{\Omega} \hat{f}_1 \|^{\delta',M,L_1}_{\Omega} \cdots \| \mathfrak{p}^*_{\Omega} \hat{f}_n \|^{\delta',M,L_n}_{\Omega}, \end{aligned} \tag{4.31}$$

where the seminorm $\| \cdot \|^{\delta,M,L}_{\Omega^n}$ on $\hat{\mathscr{R}}^{\mathrm{dv}}_{\Omega}$ is defined by the supremum on the set $\big\{\underline{\gamma}(1) \mid \gamma \in \Pi^{\delta,M,L}_{\Omega}\big\}$.

5. Applications

In this section we display some applications of our results of Section 4. We first introduce convergent power series with coefficients in $\tilde{\mathscr{R}}_{\Omega}$:

Definition 5.1. Given Ω a d.f.s. and $r \geq 1$, we define $\tilde{\mathscr{R}}_{\Omega}\{w_1, \cdots, w_r\}$ as the space of all

$$\tilde{F}(z, w_1, \cdots, w_r) = \sum_{k \in \mathbb{Z}^r_{\geq 0}} \tilde{F}_k(z) w_1^{k_1} \cdots w_r^{k_r} \in \tilde{\mathscr{R}}_{\Omega}[[w_1, \cdots, w_r]]$$

such that, for every $\delta, L > 0$, there exists a positive constant C satisfying

$$\| \tilde{F}_k \|^{\delta,L}_{\Omega} \leq C^{|k|+1} \quad \text{for every } k = (k_1, \cdots, k_r) \in \mathbb{Z}^r_{\geq 0},$$

where $|k| := k_1 + \cdots + k_r$ (with the notation of Definition 3.16 for $\| \cdot \|^{\delta,L}_{\Omega}$).

We can now deal with the substitution of resurgent formal series in a context more general than in Theorem 1.3.

Theorem 5.2. *Let $r \geq 1$ be an integer and let $\Omega_0, \ldots, \Omega_r$ be d.f.s. Then for any $\tilde{F}(w_1, \ldots, w_r) \in \tilde{\mathscr{R}}_{\Omega_0}\{w_1, \cdots, w_r\}$ and for any $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r \in \mathbb{C}[[z^{-1}]]$ without constant term, one has*

$$\tilde{\varphi}_1 \in \tilde{\mathscr{R}}_{\Omega_1}, \ldots, \tilde{\varphi}_r \in \tilde{\mathscr{R}}_{\Omega_r} \quad \Rightarrow \quad \tilde{F}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) \in \tilde{\mathscr{R}}_{\Omega_0 * \Omega^{*\infty}},$$

where $\Omega := \Omega_1 * \cdots * \Omega_r$.

Proof. Since the family $\{\Omega_0 * \Omega^{*k} := \Omega_0 * \Omega_1^{*k_1} * \cdots * \Omega_r^{*k_r} \mid k = (k_1, \cdots, k_r) \in \mathbb{Z}^r_{\geq 0}\}$ of d.f.s. satisfies the conditions in Theorem 4.8 for sufficiently small $\delta > 0$, for every $L > 0$, there exist $\delta', L', C > 0$ such that

$$\| \tilde{F}_k \tilde{\varphi}^{k_1} \cdots \tilde{\varphi}^{k_r} \|^{\delta,L}_{\Omega_0 * \Omega^{*\infty}} \leq \frac{C^{|k|+2}}{(|k|+1)!} \| \tilde{F}_k \|^{\delta',L'}_{\Omega_0} \big(\| \tilde{\varphi}_1 \|^{\delta',L'}_{\Omega_0} \big)^{k_1} \cdots \big(\| \tilde{\varphi}_r \|^{\delta',L'}_{\Omega_0} \big)^{k_r}.$$

Therefore, since $\tilde{F}(w_1, \ldots, w_r) \in \tilde{\mathscr{R}}_{\Omega_0}\{w_1, \cdots, w_r\}$, we find that $\tilde{F}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)$ converges in $\tilde{\mathscr{R}}_{\Omega_0 * \Omega^{*\infty}}$ and defines an $\Omega_0 * \Omega^{*\infty}$-resurgent formal series. $\square$

Notice that, in view of Theorem 2.13, Theorem 1.3 is a direct consequence of Theorem 5.2.

Next, we show the following implicit function theorem for resurgent formal series:

Theorem 5.3. *Let* $\tilde{F}(z,w) \in \tilde{\mathscr{R}}_\Omega\{w\}$ *and assume that* $F(x,w) := \tilde{F}(x^{-1},w)$ *satisfies* $F(0,0)=0$ *and* $\partial_w F(0,0) \neq 0$*. Then, the unique solution* $\tilde{\varphi} \in z\mathbb{C}[[z]]$ *of*

$$\tilde{F}(z,\tilde{\varphi}(z)) = 0 \tag{5.1}$$

satisfies $\tilde{\varphi} \in \tilde{\mathscr{R}}_{\Omega*\infty}$.

Proof. We rewrite $\tilde{F}(z,w)$ into the form

$$\tilde{F}(z,w) = \tilde{F}_0(z) + \partial_w F(0,0)w + \sum_{k=1}^{\infty} \tilde{F}_k(z)w^k.$$

Considering (5.1) as the equation for $\tilde{\psi} = z^{-1}(\tilde{\varphi}(z) - \varphi_1 z)$, we can assume that $\tilde{F}_k$ has no constant term for $k = 0,1,\ldots$ Further, we can assume without loss of generality that $\partial_w F(0,0) = -1$. Then, the unique solution $\tilde{\varphi} \in z\mathbb{C}[[z]]$ of (5.1) can be written as $\tilde{\varphi} = \tilde{H}(z,\tilde{F}_0)$, where

$$\tilde{H}(z,w) = \sum_{m\geq 1} \tilde{H}_m(z)w^m, \quad \tilde{H}_m := \sum_{k\geq 1} \frac{(m+k-1)!}{m!k!} \sum_{\substack{n_1+\cdots+n_k=m+k-1\\ n_1,\cdots,n_k\geq 1}} \tilde{F}_{n_1}\cdots\tilde{F}_{n_k}$$

(see proof of Theorem 4 in [20] for the detail). Since $\tilde{F}(z,w) \in \tilde{\mathscr{R}}_\Omega\{w\}$, we obtain from Corollary 4.2 the following estimates: For every $\delta, L > 0$, there exist $\delta', L', C > 0$ such that $\| F_k \|_\Omega^{\delta',L'} \leq C^{k+1}$ and

$$\begin{aligned}
\|\tilde{H}_m\|_{\Omega*\infty}^{\delta,L} &\leq \sum_{k\geq 1} \frac{(m+k-1)!}{m!k!} \sum_{\substack{n_1+\cdots+n_k=m+k-1\\ n_1,\cdots,n_k\geq 1}} \frac{C^{k+1}}{k!} \|\tilde{F}_{n_1}\|_\Omega^{\delta',L'} \cdots \|\tilde{F}_{n_k}\|_\Omega^{\delta',L'} \\
&\leq \sum_{k\geq 1} 2^{m+k} \sum_{\substack{n_1+\cdots+n_k=m+k-1\\ n_1,\cdots,n_k\geq 1}} \frac{C^{m+3k}}{k!} \leq \sum_{k\geq 1} 2^{2m+3k-2} \frac{C^{m+3k}}{k!} \\
&\leq e^{8C^3}(4C)^m.
\end{aligned}$$

This yields $\tilde{H}(z,w) \in \tilde{\mathscr{R}}_{\Omega*\infty}\{w\}$, whence, $\tilde{H}(z,\tilde{F}_0(z)) \in \tilde{\mathscr{R}}_{\Omega*\infty}$. □

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