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Mappings of smallest mean distortion and free-Lagrangians

TADEUSZ IWANIEC AND JANI ONNINEN

Abstract. Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ be bounded domains of the same topological type. We are concerned with mappings $f : \mathbb{X} \to \mathbb{Y}$, predominately orientation preserving homeomorphisms, in the Sobolev space $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^n)$. Thus at almost every $x \in \mathbb{X}$ the linear differential map $Df(x) : \mathbf{T}_x\mathbb{X} \simeq \mathbb{R}^n \longrightarrow \mathbf{T}_y\mathbb{Y} \simeq \mathbb{R}^n$, $y = f(x)$, is represented by the Jacobian matrix $Df(x) \in \mathbb{R}^{n\times n}_+$. Hereafter $\mathbb{R}^{n\times n}_+$ denotes the space of $n \times n$-matrices with positive determinant.

A little reflection on Teichmüller's theory of extremal quasiconformal mappings provokes to study homeomorphisms with smallest $\mathscr{L}^p$-norm of the distortion functions $\mathcal{K}_\ell f \stackrel{\text{def}}{=\!=} \mathcal{K}_\ell[Df(x)]$, $1 \leqslant \ell \leqslant n-1$, $\mathcal{K}_\ell : \mathbb{R}^{n\times n}_+ \to [1,\infty)$. This being so, we seek to compute

$$\mathbf{K}^p_\ell(\mathbb{X}, \mathbb{Y}) \stackrel{\text{def}}{=\!=} \inf_f \int_{\mathbb{X}} \big[\mathcal{K}_\ell \mathbf{M}\big]^p \, \mathrm{d}x \qquad \mathbf{M} = Df(x). \tag{0.1}$$

The infimum is subjected to Sobolev homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with positive Jacobian determinant, $J_f(x) = \det Df(x) > 0$ a.e. Formal change of variables leads to an energy-integral for the inverse mappings $h = f^{-1} : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$. This integral takes the form

$$\mathbf{E}_{\ell,\, p}(\mathbb{Y}, \mathbb{X}) \stackrel{\text{def}}{=\!=} \inf_h \int_{\mathbb{Y}} \big[\mathcal{K}_{n-\ell}\, \mathbf{N}\big]^p \, \det(\mathbf{N})\, \mathrm{d}y \ , \quad \mathbf{N} = Dh(y). \tag{0.2}$$

Equivalence of the minimization problems for f in (0.1) and that for h in (0.2) is a matter of a change of variables for Sobolev homeomorphisms. The concept of *free-Lagrangians* becomes ever more strategic. Broadly speaking, a free Lagrangian is a nonlinear differential n-form $\mathbf{L}(x, f, Df)\mathrm{d}x$, defined for Sobolev mappings $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, whose integral depends only on the homotopy class of the mapping.

Free Lagrangians proved particularly useful in solving the $\mathscr{L}^p$*-Grötzsch problem* for ring domains in $\mathbb{R}^n$. Historically, the Grötzsch problem for $p = \infty$ has been of great interest in Geometric Function Theory (GFT); for example, in the 2-dimensional theory of Teichmüller spaces. In higher dimensions GFT flourished from the pioneering work of ***Fred*** (Frederick William Gehring). Thus our $\mathscr{L}^p-$ approach to GFT commemorates Fred's paper

"Rings and Quasiconformal Mappings in Space"
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Precisely, we ask for homeomorphisms between ring domains having smallest $\mathscr{L}^p$-mean distortion. Call them *$\mathscr{L}^p$-Teichmüller mappings*.We investigate which pairs of ring domains admit $\mathscr{L}^p$-Teichmüller mappings.

It is somewhat surprising that the minimization of the $\mathscr{L}^1$-mean distortion leads to non-surjective mappings. Equivalently, in the variational problem (0.2), we observe the lose of injectivity when passing to the limit of the energy-minimizing sequence of homeomorphisms. In the mathematical models of Nonlinear Elasticity this phenomenon amounts to saying that *interpenetration of matter* may occur when minimizing the energy at (0.2).

More surprisingly, the expected radial symmetry of a minimal mapping turns out to be false already in dimensions $n \geqslant 3$.

In several ways our study here grew out of the conceptual principles of Nonlinear Elasticity and Calculus of Variations. The novelty lies in the proofs, based on rather tricky inequalities; seemingly elementary but in fact challenging.

The art of free Lagrangians is
not to integrate nonlinear differential expressions,
but the correct choice of such expressions.

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Contents

1. Introduction

1.1. Setting

Let $\mathbb{X}$ and $\mathbb{Y}$ be bounded domains in $\mathbb{R}^n$. The following standing assumptions are made on the mappings $f\colon \mathbb{X} \to \mathbb{Y}$ under consideration.

(i) f belongs to the Sobolev spaces $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{R}^n)$ for some exponent $1 \leqslant s \leqslant \infty$, so that the differential matrix $Df(x) \in \mathbb{R}^{n\times n}$ (*deformation gradient*) and its Jacobian determinant $J_f(x) = \det Df(x)$ are well defined almost everywhere;

(ii) The Jacobian determinant $J_f(x) = \det\, Df(x)$ is positive almost everywhere; that is, $Df(x) \in \mathbb{R}^{n\times n}_+$.

The class of homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ which satisfies the conditions (i) and (ii) is denoted by $\mathscr{H}^{1,s}(\mathbb{X}, \mathbb{Y})$.

To every $f \in \mathscr{H}^{1,s}(\mathbb{X}, \mathbb{Y})$ there correspond several distortion functions. As a rule, the map is conformal at the points where its distortion function equals 1. Commonly used distortion functions are:

- The *outer distortion*

$$\mathbf{K}_0 f(x) = \frac{\|Df(x)\|^n}{J_f(x)} \qquad \text{for} \quad f \in \mathscr{H}^{1,1}(\mathbb{X}, \mathbb{Y}) \; ; \tag{1.1}$$

- The *inner distortion*

$$\mathbf{K}_I f(x) = \frac{\|D^\sharp f(x)\|^n}{[J_f(x)]^{n-1}} \qquad \text{for} \quad f \in \mathscr{H}^{1,n-1}(\mathbb{X}, \mathbb{Y}); \tag{1.2}$$

where the entries of the so-called cofactor matrix $D^\sharp f(x) \in \mathbb{R}^{n\times n}$ are the $\pm(n-1) \times (n-1)$-subdeterminants of $Df(x)$. The signs are settled by Cramer's rule, $D^\sharp f(x) = J_f(x)\,[Df(x)]^{-1}$. In (1.1) and (1.2) the notation $\|\cdot\|$ stands for the operator norm of matrices. It should be emphasized that the operator norms will play quite a role in the forthcoming computation. However, to ensure uniqueness of the minimal mappings we deal with the *Frobenius* (Hilbert-Schmidt) norm of matrices, denoted by $|\cdot|$. A commonly used term *K-quasicoformal mapping* refers to a homeomorphism $f \in \mathscr{H}^{1,1}(\mathbb{X}, \mathbb{Y})$ with $K = \|\mathbf{K}_0 f\|_{\mathscr{L}^\infty(\mathbb{X})} < \infty$.

The differential expressions $Df(x)$, $D^\sharp f(x)$ and $J_f(x)$ tell us something about the infinitesimal change of 1-dimensional, $(n-1)$-dimensional and n-dimensional objects. Under the Sobolev regularity assumptions $f \in \mathscr{W}^{1,1}(\mathbb{X},\mathbb{Y})$, $f \in \mathscr{W}^{1,n-1}(\mathbb{X},\mathbb{Y})$ and $f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$, respectively. These nonlinear differential forms are *null-Lagrangians* [7,14,22]. This means that the integrals $\int_{\mathbb{X}} Df(x)\, \mathrm{d}x$, $\int_{\mathbb{X}} D^\sharp f(x)\, \mathrm{d}x$ and $\int_{\mathbb{X}} J_f(x)\, \mathrm{d}x$ depend only on the boundary values of f. However, we shall often be content with the *free-Lagrangian inequality* $\int_{\mathbb{X}} J_f(x)\, \mathrm{d}x \leqslant |\mathbb{Y}|$, whenever $f \in \mathscr{H}^{1,1}(\mathbb{X}, \mathbb{Y})$.

Definition 1.1. To every integer $1 \leqslant \ell \leqslant n-1$ there corresponds so-called $\mathcal{K}_\ell$-distortion function,

$$\mathcal{K}_\ell f(x) = \frac{\binom{n}{\ell}^{\frac{n}{2\ell-2n}} \left|D^\ell_\sharp f(x)\right|^{\frac{n}{n-\ell}}}{[J_f(x)]^{\frac{\ell}{n-\ell}}} \geqslant 1 \qquad \text{for} \quad f \in \mathscr{H}^{1,\ell}(\mathbb{X}, \mathbb{Y}) \tag{1.3}$$

where the notation $D^\ell_\sharp f$ stands for a matrix of size $\binom{n}{\ell} \times \binom{n}{\ell}$ whose entries are $\pm(\ell \times \ell)$-subdeterminants of Df.

Since we work with the Frobenius norm of matrices $\mathbf{M} \in \mathbb{R}^{\binom{n}{\ell}\times\binom{n}{\ell}}$, $|\mathbf{M}| = \sqrt{\operatorname{Tr} \mathbf{M}^*\mathbf{M}}$, it is immaterial which $\pm$ signs of the entries of $D^\ell_\sharp f(x)$ are chosen. We refer to

Chapter 2 for geometric interpretations and full discussion about $\mathcal{K}_\ell f$. Meanwhile it is worth noting that the exponents $\frac{n}{n-\ell}$ and $\frac{\ell}{n-\ell}$, in (1.3), are the least possible for which $\mathcal{K}_\ell$ remains polyconvex function of matrices [6,23].

The present paper is about homeomorphisms $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ between n-dimensional annuli

$$\mathbb{A} \overset{\text{def}}{=\!=} \{x \in \mathbb{R}^n : r < |x| < R\} \quad \text{and} \quad \mathbb{A}^* \overset{\text{def}}{=\!=} \{x \in \mathbb{R}^n : r_* < |x| < R_*\}.$$

Hereafter $0 < r < R$ and $0 < r_* < R_*$ are called the inner and outer radii of $\mathbb{A}$ and $\mathbb{A}^*$, respectively (Figure 1.1).

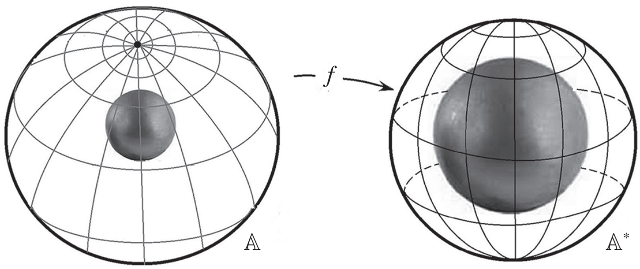

Figure 1.1.

In the pioneering paper [17] Frederick William Gehring opened up a study of ring domains in $\mathbb{R}^3$ as tools for the foundation of the theory of *Quasiconformal Mappings* in higher dimensions. Gehring's approach was new and extremely subtle, incorporating the ideas from PDEs and Geometric Analysis. Nowadays, it has close links with the Calculus of Variations. For example, the concept of extremal length of a family of curves [1,2,12,16,34,55,56] relies on $\mathscr{L}^\infty$-norm of the outer and inner distortions.

In this paper, we seek to minimize the $\mathscr{L}^p$-norm of the $\mathcal{K}_\ell$- distortions, $1 \leqslant \ell \leqslant n-1$,

$$\inf \int_{\mathbb{A}} [\mathcal{K}_\ell f(x)]^p \, \mathrm{d}x \qquad f \in \mathscr{H}^{1,\ell}(\mathbb{A}, \mathbb{A}^*). \tag{1.4}$$

The outcomes depend on the exponent $1 \leqslant p \leqslant \infty$ and the conformal moduli

$$\operatorname{Mod} \mathbb{A} \overset{\text{def}}{=\!=} \log \frac{R}{r} \quad \text{and} \quad \operatorname{Mod} \mathbb{A}^* \overset{\text{def}}{=\!=} \log \frac{R_*}{r_*}. \tag{1.5}$$

Remark 1.2. Observe that there is an orientation preserving conformal diffeomorphism $\Phi \colon \mathbb{A}(r_*, R_*) \xrightarrow{\text{onto}} \mathbb{A}(1/R_*, 1/r_*)$. This transformation does not change the conformal modulus and preserves the distortion functions of f; namely $\mathcal{K}_\ell(\Phi \circ f) \equiv \mathcal{K}_\ell f$ in $\mathbb{A}(r, R)$. But Φ changes the order of the boundary components of the annulus $\mathbb{A}(r_*, R_*)$. Therefore, it involves no loss of generality if we restrict (1.4) to mappings $f \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ so that

$$\begin{cases} |f(x)| = r_* & \text{for } |x| = r \\ |f(x)| = R_* & \text{for } |x| = R. \end{cases} \tag{1.6}$$

In what follows the notation $f \in \mathscr{H}^{1,\ell}(\mathbb{A}, \mathbb{A}^*)$ means, without saying it every time, that f preserves the order of boundary components.

One of our primary questions is whether the mappings of minimal $\mathscr{L}^p$-mean distortion are (modulo rotation) radial mappings.

Let us begin with a relatively effortless case $p = \infty$ (*quasiconformal mappings*).

Theorem 1.3. *Given any pair* $(\mathbb{A}\,,\, \mathbb{A}^*)$ *of annuli, we let* α *denote the ratio* $\operatorname{Mod} \mathbb{A}^* / \operatorname{Mod} \mathbb{A}$. *Then, among all mappings in* $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$, *the power stretching* $f(x) = R_* R^{-\alpha}\, |x|^{\alpha-1} x : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ *has the least supremum norm of* $\mathcal{K}_\ell$*-distortion. However, the uniqueness* (*up to rotation*) *of the distortion-minimal map holds only when* $\alpha = 1$.

It should be emphasized that analogous results for the outer and inner distortions, defined by (1.1) and (1.2), are well known [20,50,52,53]. The novelty of Theorem 1.3 lies in minimizing the $\mathcal{K}_\ell$-distortions and the use of Frobenius norm of matrices. This requires truly new ingredients.

1.2. Minimal radial stretchings

The term *radial stretching* pertains to a homeomorphism $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of the form

$$f(x) = F(|x|)\frac{x}{|x|} \qquad F\colon [r, R] \xrightarrow{\text{onto}} [r_*, R_*] \qquad F(r) = r_* \quad F(R) = R_* \tag{1.7}$$

where F (*normal strain function*) is absolutely continuous on $[r, R]$ and its derivative $\dot{F} = \dot{F}(t)$ is positive at almost every $t \in (r, R)$, thus $f \in \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$. In what follows the notation

$\mathscr{R}(\mathbb{A}, \mathbb{A}^*)$ stands for the class of radial stretchings in $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$.

Definition 1.4. The *minimal radial stretchings* are the ones which have smallest mean distortion within the class $\mathscr{R}(\mathbb{A}, \mathbb{A}^*)$.

They are not always minimal within all homeomorphisms in $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$. It is legitimate to perform inner variation within $\mathscr{R}(\mathbb{A}, \mathbb{A}^*)$. The solutions to the inner variational equation are called *stationary solutions*. Such variational equations are known as *energy-momentum* or *equilibrium* equations, etc. [10,46,51]. As expected, the minimal radial stretchings are $\mathscr{C}^\infty$-diffeomorphisms. Our first statement concerning $p < \infty$ is a summary of Theorems 5.10 and 5.11 in Chapter 5.

Theorem 1.5. *Let* $1 \leqslant \ell \leqslant n-1$ *and* $1 < p < \infty$. *Then to every pair of annuli* $\mathbb{A}$ *and* $\mathbb{A}^*$ *there corresponds unique minimal radial stretching* $\mathfrak{f} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$. *For* $p = 1$, *the minimal radial stretching exists if and only if*

$$\operatorname{Mod} \mathbb{A}^* \leqslant \Xi(\operatorname{Mod} \mathbb{A}). \tag{1.8}$$

Here the function $\Xi : [0, \infty) \xrightarrow{\text{onto}} [0, \infty)$ is determined uniquely by $1 \leqslant \ell \leqslant n-1$, see (5.32). It is continuously increasing from 0 to ∞ and $\Xi(M) > M$, for every $M > 0$.

1.3. Absolute minima (Teichmüller mappings)

We ask the questions:

Question 1.6. Does there exist $\mathfrak{f} \in \mathscr{H}^{1,\ell}(\mathbb{A}, \mathbb{A}^*)$ of smallest $\mathscr{L}^p$-norm of $\mathcal{K}_\ell$-distortion?

We refer to such $\mathfrak{f} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ *as absolute minimizer* or $\mathscr{L}^p_\ell$-*Teichmüller map*.

Question 1.7. Are the radial minimal stretchings absolute minima?

When answering these questions the concept of the *elasticity quotient*

$$\eta_F(x) \stackrel{\text{def}}{=\!=} \frac{|x|\dot{F}(|x|)}{F(|x|)}$$

of a stationary radial solution $\mathfrak{f}(x) = F(|x|)\frac{x}{|x|}$ becomes relevant. The inner variational equation will show that this quotient falls into three categories:

- *Conformal expansion* $[\operatorname{Mod}\mathbb{A}^* > \operatorname{Mod}\mathbb{A}] : \eta_F(x) > 1$ in $\mathbb{A}$;
- *Conformal equivalence* $[\operatorname{Mod}\mathbb{A}^* = \operatorname{Mod}\mathbb{A}] : \eta_F(x) \equiv 1$ in $\mathbb{A}$;
- *Conformal contraction* $[\operatorname{Mod}\mathbb{A}^* < \operatorname{Mod}\mathbb{A}] : \eta_F(x) < 1$ in $\mathbb{A}$.

In either case $1 - \eta_F$ does not change sign within the entire annulus $\mathbb{A}$. For each case different choices of free-Lagrangians will be used, and different phenomena will be observed.

1.4. Conformal equivalence [Mod $\mathbb{A}^*$ = Mod $\mathbb{A}$]

The classical Schottky theorem [48], extended to higher dimensions [30], asserts that two annuli $\mathbb{A}$ and $\mathbb{A}^*$ are conformally equivalent if and only if $\operatorname{Mod}\mathbb{A}^* = \operatorname{Mod}\mathbb{A}$. Furthermore, every conformal map $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ in $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$ takes (upon rotation) the form $\mathfrak{f}(x) = \frac{r_*}{r}x$, so it is a radial stretching. We just conclude that

$$\int_{\mathbb{A}} [\mathcal{K}_\ell g]^p \geqslant \int_{\mathbb{A}} 1 = \int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}]^p \quad \text{for every} \quad g \in \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*). \tag{1.9}$$

For equality we must have $\mathcal{K}_\ell g \equiv 1$. This yields $g = \mathfrak{f}$ (modulo rotation), which is an affirmative answer to Questions 1.6 and 1.7 in this case.

1.5. Conformal expansion [Mod $\mathbb{A}$ < Mod$\mathbb{A}^*$]

In this case our assumption on the annuli is that $\operatorname{Mod}\mathbb{A} < \operatorname{Mod}\mathbb{A}^*$. We shall give complete answer to Question 1.7.

Theorem 1.8. *Suppose that* $\operatorname{Mod}\mathbb{A} < \operatorname{Mod}\mathbb{A}^*$, $1 \leqslant \ell \leqslant n-1$, $1 \leqslant p < \infty$. *Then*

$$\inf_{f \in \mathscr{R}(\mathbb{A},\mathbb{A}^*)} \int_{\mathbb{A}} \mathcal{K}_\ell^p f = \inf_{g \in \mathscr{H}^{1,1}(\mathbb{A},\mathbb{A}^*)} \int_{\mathbb{A}} \mathcal{K}_\ell^p g. \tag{1.10}$$

Furthermore, we have the following existence and uniqueness result.

Theorem 1.9. *Assume that:*

- $\operatorname{Mod}\mathbb{A} < \operatorname{Mod}\mathbb{A}^*$ *when* $p > 1$*;*
- *Or* $\operatorname{Mod}\mathbb{A} < \operatorname{Mod}\mathbb{A}^* \leqslant \Xi(\operatorname{Mod}\mathbb{A})$ *when* $p = 1$.

Then the minimal radial stretching $\mathfrak{f} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ *is an absolute minimizer. Precisely, we have*

$$\int_{\mathbb{A}} [\mathcal{K}_\ell g]^p \geqslant \int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}]^p \quad \text{for every} \quad g \in \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*). \tag{1.11}$$

Equality occurs if and only if $g = \mathfrak{f}$ (*modulo rotation*).

We summarize this case as

Corollary 1.10. *Let* $\operatorname{Mod}\mathbb{A} < \operatorname{Mod}\mathbb{A}^*$. *Then the* $\mathscr{L}_\ell^p$*-Teichmüller map* $\mathfrak{f} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ *does exist if* $p > 1$. *However, if* $p = 1$, *the* $\mathscr{L}_\ell^1$*-Teichmüller map exists if and only if* $\operatorname{Mod}\mathbb{A}^* \leqslant \Xi(\operatorname{Mod}\mathbb{A})$.

1.6. Conformal contraction [$\operatorname{Mod}\mathbb{A}^* < \operatorname{Mod}\mathbb{A}$]

Recall that in this case there always exists a unique minimal radial stretching $\mathfrak{f}$: $\mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ for all $p \geqslant 1$. We shall see that $\mathfrak{f}$ need not be an absolute minimizer.

Let us reveal in advance that the radial symmetry of absolute minimizers is always true in dimension $n = 2$, see also [37]. But it already fails in dimension $n = 3$, in which case answers to Questions 1.6 depend on $\ell \in \{1, 2\}$ and p. The following 3-dimensional cases of subsequent Theorem 1.15 merit separate formulations.

Theorem 1.11. *Let* $n = 3$, $g \in \mathscr{H}^{1,2}(\mathbb{A}, \mathbb{A}^*)$ *and* $p \geqslant 1$, *then*

$$\int_{\mathbb{A}} [\mathcal{K}_2 g]^p \geqslant \int_{\mathbb{A}} [\mathcal{K}_2 \mathfrak{f}]^p. \tag{1.12}$$

Equality occurs if and only if $g = \mathfrak{f}$ (*modulo rotation*).

Concerning $\mathcal{K}_1\mathfrak{f}$, we still have an affirmative answer to Question 1.7 when $p \geqslant 4$; precisely,

Theorem 1.12. *In dimension $n = 3$, for every $p \geqslant 4$ and $g \in \mathscr{H}^{1,2}(\mathbb{A}, \mathbb{A}^*)$ the following inequality holds*

$$\int_{\mathbb{A}} [\mathcal{K}_1 g]^p \geqslant \int_{\mathbb{A}} [\mathcal{K}_1 \mathfrak{f}]^p. \tag{1.13}$$

Equality occurs if and only if $g = \mathfrak{f}$ (modulo rotation).

We now come to the first instance of failure of radial symmetry. This answers Question 1.7 to the negative.

Theorem 1.13. *In dimension $n = 3$, for every exponent $1 \leqslant p < 4$ there are annuli $\mathbb{A}$ and $\mathbb{A}^*$ and a $\mathscr{C}^\infty$-diffeomorphism $g : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ such that*

$$\int_{\mathbb{A}} [\mathcal{K}_1 g]^p < \int_{\mathbb{A}} [\mathcal{K}_1 \mathfrak{f}]^p = \min_{f \in \mathscr{R}(\mathbb{A}, \mathbb{A}^*)} \int_{\mathbb{A}} [\mathcal{K}_1 f]^p. \tag{1.14}$$

Here $\mathfrak{f} \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is the unique minimal radial strecthing.

Actually, negative answers to Question 1.7 (via counterexamples) will be given in all dimensions $n \geqslant 4$ and $1 \leqslant \ell \leqslant n-1$. The following is a direct generalization of Theorem 1.13.

Theorem 1.14. *For every exponent $1 \leqslant p < p_\ell(n) \overset{\text{def}}{=\!=} \frac{n(n+1)^2}{8\ell} - n + 1$, $n \geqslant 4$, there are annuli $\mathbb{A}$ and $\mathbb{A}^*$ and a $\mathscr{C}^\infty$-diffeomorphism $g : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ such that*

$$\int_{\mathbb{A}} [\mathcal{K}_\ell g]^p < \int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}]^p = \min_{f \in \mathscr{R}(\mathbb{A}, \mathbb{A}^*)} \int_{\mathbb{A}} [\mathcal{K}_\ell f]^p. \tag{1.15}$$

Recall that we are dealing with a contracting pair of annuli, so the radial minimal stretching $\mathfrak{f} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ does exist.

In Theorem 1.14 while attempting to reach affirmative answers to Question 1.7, we found ourselves forced to impose a restriction on the size of annuli. It turns out that the target annulus $\mathbb{A}^*$, in addition of being coupled in a conformally contracting pair with $\mathbb{A}$, cannot be conformally too thin relative to $\mathbb{A}$. In fact, we have found explicit lower bound of $\operatorname{Mod} \mathbb{A}^*$; namely,

$$\operatorname{Mod} \mathbb{A} > \operatorname{Mod} \mathbb{A}^* \geqslant \Gamma(\operatorname{Mod} \mathbb{A}). \tag{1.16}$$

Under this condition the minimal radial stretching $\mathfrak{f} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is an absolute minimizer. The function $\Gamma \colon [0, \infty) \to [0, \infty)$ is continuously increasing from 0 to ∞ and $\Gamma(M) < M$, for all $0 < M < \infty$. See (9.7) for an explicit formula of Γ. The sharp lower bound of $\operatorname{Mod} \mathbb{A}^*$ remains unknown. There is also restriction for exponent p. Here is the precise statement.

Theorem 1.15. *Let $n \geqslant 2$, $1 \leqslant \ell \leqslant n-1$ and $p \geqslant p_\circ \stackrel{\text{def}}{=\!=} \frac{(n-1)(n-\ell)}{\ell}$. For any pair of annuli satisfying condition* (1.16) *the minimal radial stretching $\mathfrak{f} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is an absolute minimizer; that is,*

$$\int_{\mathbb{A}} [\mathcal{K}_\ell g]^p \geqslant \int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}]^p \quad \textit{for all} \quad g \in \mathscr{H}^{1,n-1}(\mathbb{A}, \mathbb{A}^*). \tag{1.17}$$

Equality occurs if and only if $g = \mathfrak{f}$ (modulo rotation).

Notice that for dimensions $n \geqslant 4$ there are exponents $p \in [p_\circ, p_\ell(n))$ so the conditions for p in Theorem 1.14 and the conditions for p in Theorem 1.15 partially overlap.

1.7. An open question

Question 1.16. Is it true that for sufficiently large exponents, say $p > P_n$, the infimum

$$\inf \Big\{ \|\mathcal{K}_\ell g\|_{\mathscr{L}^p(\mathbb{A})} : g \in \mathscr{H}^{1,\ell}(\mathbb{A}, \mathbb{A}^*) \Big\} \tag{1.18}$$

is attained at a radial stretching regardless of the conformal moduli of $\mathbb{A}$ and $\mathbb{A}^*$.

As we have seen, this holds in dimensions $n = 2, 3$, where $P_2 = 1$ and $P_3 = 4$.

In light of Theorem 1.5 and Corollary 1.10 we rise the following conjecture.

Conjecture 1.17. For $p > 1$ the $\mathscr{L}^p_\ell$-Teichmüller map does exist for every pair of annuli.

In this paper this conjecture is confirmed when $n = 2$ or $n = 3$ and $\ell = 2$.

1.8. Dirichlet energy revisited

The present paper seeks to do much more than to establish new results concerning mean distortion. It offers new methods (free-Lagrangians) and new viewpoints about modern Calculus of Variation with possible applications to mathematical models of *Nonlinear Elasticity*. In Chapter 11 we use free Lagrangians to solve the 2-dimensional problem of existence of weighted Dirichlet minimal deformations.

1.9. Summarizing comments

1.9.1. *Failure of radial symmetry*

One naturally expected that the deformations between round annuli of smallest $\mathscr{L}^p$-norm of the distortion must always be radially symmetric. For, both the annuli and the distortion functions are rotation invariant. While radial symmetry of the extremal mappings has long been confirmed for the planar annuli [5, 28–30] (also proven here for the case ($n = 3$ and $\ell = 2$)), it is false already when ($n = 3$ and $\ell = 1$), and for all $1 \leqslant \ell \leqslant n-1$ in dimensions $n \geqslant 4$. Moreover, radial symmetry is lost when $p \approx 1$ and $\mathbb{A}^*$ is conformally very thin relative to that of $\mathbb{A}$. Examples are constructed in Chapter 5 (Section 6) via so-called *conformal sliding along spheres*.

1.9.2. *The maximal annulus* $\mathbb{A}^*_{\max}$

Another phenomenon manifests itself in every dimension when $p = 1$ and $\mathbb{A}^*$ is conformally too fat relative to $\mathbb{A}$, precisely when $\operatorname{Mod}\mathbb{A}^* > \operatorname{Mod}\mathbb{A}^*_{\max} = \Xi(\operatorname{Mod}\mathbb{A})$ (beyond the assumption of Theorem 1.9). In this case the limit of a *minimizing sequence* of homeomorphisms $f_j : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ takes $\mathbb{A}$ (homeomorphically) onto an annulus $\mathbb{A}^*_{\max} \subsetneq \mathbb{A}^*$ which is conformally thinner than $\mathbb{A}^*$. This, so-called *squeezing phenomenon*, is caused by lack of uniform convergence of $f_j : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ near the inner boundary of $\mathbb{A}$. Nonetheless, the locally uniform limit $f = \lim f_j : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*_{\max}$ does exist and belongs to $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*_{\max})$. On the other hand, a quick look at the inverse deformations $h_j = f_j^{-1} : \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A}$ reveals that they, together with the first derivatives converge uniformly. But the injectivity of the limit map $h : \mathbb{A}^* \xrightarrow{\text{into}} \mathbb{A}$ is lost. Such phenomena should be scrutinized in the revision of the mathematical models of nonlinear hyperelasticity (NE) [3,6,9,36,49,54]. In fact we should, in connection with the *principle of non-interpenetration of matter* , accept the limits of Sobolev homeomorphisms as legititimate deformations of elastic materials. We observe a *collapse of matter* near the inner boundary of $\mathbb{A}^*$. Precisely, the collapsing phenomenon takes place in the sub-annulus $\Delta^* = \mathbb{A}^* \setminus \mathbb{A}^*_{\max}$, where $h(y) = r\frac{y}{|y|}$. In dimension $n = 2$ the maximal annulus is determined by what we call the *"Nitsche Condition"*

$$\operatorname{Mod}\mathbb{A}^*_{\max} = \cosh^{-1}\big(e^{\operatorname{Mod}\mathbb{A}}\big). \tag{1.19}$$

It is in this way we are led to the n-dimensional analogue of the *Nitsche Conjecture* [43], (once an eminent problem in the theory of minimal surfaces) [44, Section 878], [13, page 138], [4, Conjecture 21.3.2] and [8,35,47]. Nitsche conjecture (now a theorem [24,25]) asks: when does there exist a harmonic homeomorphism between planar annuli? The Dirichlet integral translates into the $\mathscr{L}^1$-mean distortion for the inverse map. Analogously, in higher dimensions the n-harmonic energy translates into the integral of $\mathcal{K}_{n-1}$-distortion. Now, based on Theorem 9.6, with $\ell = n - 1$ the n-dimensional *Nitsche conjecture* reads as follows.

Conjecture 1.18. If $\operatorname{Mod}\mathbb{A}^* > \Xi(\operatorname{Mod}\mathbb{A})$, then there is no n-harmonic homeomorphism $\varphi\colon \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A}$.

Based on Proposition 11.14 one can also formulate a Riemannian metric variant of the Nitsche conjecture. However, in such weighted setting even the question about the non-existence of homeomorphic minimizers is not completely understood today. Theorem 11.17 offers sharp bounds for the non-existence question but only within certain range of annuli. What happens outside this range remains open.

1.10. Some classes of mappings

For the convenience of the reader, we collect basic notation, some of which has been already introduced, some being self-explanatory.

Throughout this text $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ are bounded domains in $\mathbb{R}^n$, $n \geqslant 2$. Their closures are denoted by $\overline{\mathbb{X}}, \overline{\mathbb{Y}}$. Let us begin with:

(I) $\mathscr{L}^p(\mathbb{E})$-the $\mathscr{L}^p$ -space of real valued functions, $1 \leqslant p \leqslant \infty$, defined on a measurable set $\mathbb{E} \subset \mathbb{R}^n$. The norm is denoted by:

$$\|u\|_p = \|u\|_{\mathscr{L}^p(\mathbb{E})};$$

(II) $\mathscr{C}(\mathbb{X})$-real continuous functions in a domain $\mathbb{X}$;
(III) $\mathscr{C}(\mathbb{X}, \mathbb{R}^m)$-continuous mappings $f : \mathbb{X} \to \mathbb{R}^m$;
(IV) $\mathscr{C}(\mathbb{X}, \mathbb{Y})$-continuous mappings $f : \mathbb{X} \to \mathbb{Y}$;
(V) $\mathscr{C}^\infty(\mathbb{X}, \mathbb{R}^m)$-smooth mappings $f : \mathbb{X} \to \mathbb{R}^m$;
(VI) $\mathscr{C}^\infty(\mathbb{X}, \mathbb{Y})$-smooth mappings $f : \mathbb{X} \to \mathbb{Y}$;
(VII) $\mathscr{H}(\mathbb{X}, \mathbb{Y})$-orientation preserving homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$;
(VIII) $\mathcal{D}\mathit{iff}(\mathbb{X}, \mathbb{Y})$-orientation preserving diffeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$;
(IX) $\mathcal{AC}[r, R]$-absolutely continuous functions in the interval $[r, R]$;
(X) $\mathscr{W}^{1,s}(\mathbb{X})$-Sobolev space of real valued functions $(1 \leqslant s \leqslant \infty)$;
(XI) $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{R}^m)$-Sobolev space of mappings $f : \mathbb{X} \to \mathbb{R}^m$;
(XII) $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{R}^n)$-closure of the class of Sobolev mappings from $\mathbb{X} \xrightarrow{\text{into}} \mathbb{Y}$ in norm topology of $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{R}^n)$. Thus, mappings in $\mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$ take $\mathbb{X}$ into the closure of $\mathbb{Y}$;
(XIII) $\mathscr{H}^{1,s}(\mathbb{X}, \mathbb{Y}) = \mathscr{H}(\mathbb{X}, \mathbb{Y}) \cap \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{R}^n)$ $(1 \leqslant s \leqslant \infty)$;
(XIV) $\mathscr{W}_+^{1,s}(\mathbb{X}, \mathbb{Y}) \subset \mathscr{W}^{1,s}(\mathbb{X}, \mathbb{Y})$ -Sobolev mappings $f : \mathbb{X} \xrightarrow{\text{into}} \overline{\mathbb{Y}}$ whose Jacobian determinant is positive almost everywhere;
(XV) $\mathscr{H}_+^{1,s}(\mathbb{X}, \mathbb{Y}) = \mathscr{H}(\mathbb{X}, \mathbb{Y}) \cap \mathscr{W}_+^{1,s}(\mathbb{X}, \mathbb{R}^n)$ $(1 \leqslant s \leqslant \infty)$-Sobolev homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ whose Jacobian determinant is positive almost everywhere;
(XVI) $\overline{\mathscr{H}}_+^{1,s}(\mathbb{X}, \mathbb{Y})$ is the closure of $\mathscr{H}_+^{1,s}(\mathbb{X}, \mathbb{Y})$ in the norm topology of $\mathscr{W}^{1,s}$ $(\mathbb{X}, \mathbb{R}^n)$. Mappings in $\overline{\mathscr{H}}_+^{1,s}(\mathbb{X}, \mathbb{Y})$ have nonnegative Jacobian.

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2. Distortion functions

A distortion function of a nonlinear map $f : \mathbb{X} \to \mathbb{Y}$ measures how much the tangent linear map $\boldsymbol{M} = Df(x) : \mathbf{T}_x\mathbb{X} \to \mathbf{T}_y\mathbb{Y}$, $y = f(x)$ deforms a regular object. In classical presentation of Quasiconformal Mappings (QCM) the regular objects are balls $\mathbf{B} \subset \mathbb{T}_x\mathbb{X}$ deformed into ellipsoids $\mathbf{E} \subset \mathbb{T}_y\mathbb{Y}$ of uniformly bounded eccentricity. Let us call such eccentricity *Spherical distortion* (Figure 2.1).

Thus we should first define the distortion of a linear transformation.

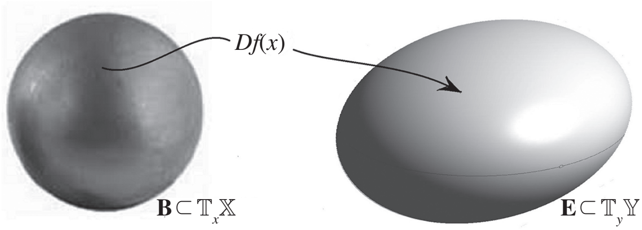

Figure 2.1.

2.1. Outer and inner distortion of a linear map

Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be a matrix of positive determinant. Commonly used distortion functions are defined by the rules:

- The *outer distortion*
$$\mathbf{K}_0\mathbf{M} = \frac{\|\mathbf{M}\|^n}{\det \mathbf{M}}; \tag{2.1}$$
- The *inner distortion*
$$\mathbf{K}_I\mathbf{M} = \frac{\|\mathbf{M}^\sharp\|^n}{(\det \mathbf{M})^{n-1}}. \tag{2.2}$$

Here $\mathbf{M}^\sharp \in \mathbb{R}^{n\times n}$ is the cofactor matrix of $\mathbf{M}$. Its entries are $\pm(n-1)\times(n-1)$-subdeterminants of $\mathbf{M}$. The signs are settled by Cramer's rule, $\mathbf{M}^\sharp = (\det \mathbf{M})\mathbf{M}^{-1}$. The notation $\|\cdot\|$ stands for the operator norm of matrices. It should be emphasized that the operator norms will play quite a role in the forthcoming arguments. However, other distortions will be set up by using the *Frobenius* (Hilbert-Schmidt) norm.

- Note the following symmetry relation:
$$\mathbf{K}_I\mathbf{M} = \mathbf{K}_0\mathbf{M}^{-1}; \tag{2.3}$$
- And the inequalities (sharp)
$$\mathbf{K}_I\mathbf{M} \leqslant (\mathbf{K}_0\mathbf{M})^{n-1} \qquad \mathbf{K}_0\mathbf{M} \leqslant (\mathbf{K}_I\mathbf{M})^{n-1}. \tag{2.4}$$

2.2. Geometric interpretation

Consider the unit ball $\mathbf{B} \subset \mathbf{T}_x\mathbb{X}$ and the ellipsoid $\mathbf{E} = \boldsymbol{M}(\mathbf{B}) \subset \mathbf{T}_y\mathbb{Y}$. Let $\mathbf{B_I}$ be the largest ball inscribed in $\mathbf{E}$ and $\mathbf{B_0}$ the smallest ball containing $\mathbf{E}$. We look at their images under the linear tangent map of the inverse transformation $h = f^{-1}$: $\mathbb{Y} \to \mathbb{R}^n$, $Dh(y) = \boldsymbol{M}^{-1}$; that is, $\boldsymbol{M}^{-1}\mathbf{E} = \mathbf{B}$, $\boldsymbol{M}^{-1}\mathbf{B_0} \overset{\text{def}}{=\!=} \mathbf{E_0}$ and $\boldsymbol{M}^{-1}\mathbf{B_I} \overset{\text{def}}{=\!=} \mathbf{E_I}$. Here are the illustrations in the planar case (Figure 2.2) and the 3-dimensional space (Figure 2.3).

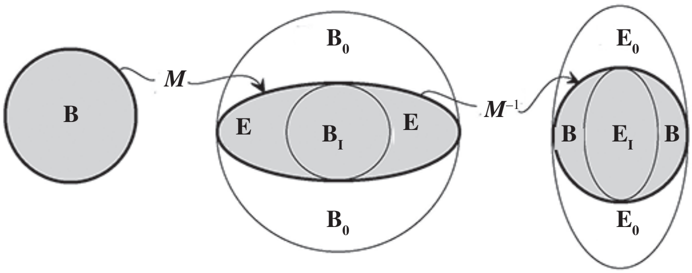

Figure 2.2.

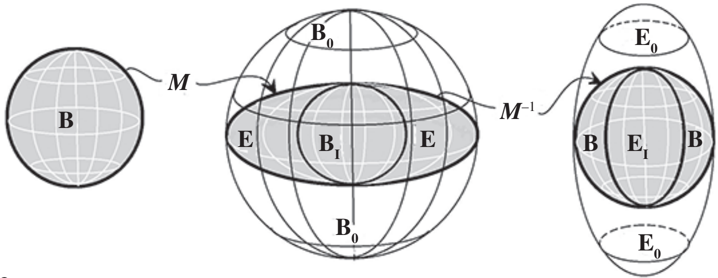

Figure 2.3.

We then see that

$$\mathbf{K}_0 \boldsymbol{M} = \frac{|\mathbf{B_0}|}{|\mathbf{E}|} \qquad \text{and} \qquad \mathbf{K}_I \boldsymbol{M} = \frac{|\mathbf{E}|}{|\mathbf{B_I}|}.$$

We recall a general formula about the change of volume under an affine transformation of measurable sets $\mathbb{A}, \mathbb{B} \subset \mathbb{R}^n$.

$$\frac{|\mathbf{M}^{-1}\mathbb{A}|}{|\mathbf{M}^{-1}\mathbb{B}|} = \frac{|\mathbb{A}|}{|\mathbb{B}|} \quad \text{where} \quad |\cdot| \text{ stands for the Lebesgue measure in } \mathbb{R}^n.$$

Accordingly,

$$\mathbf{K}_0 \boldsymbol{M}^{-1} = \frac{|\mathbf{B}|}{|\mathbf{E}_I|} = \frac{|\boldsymbol{M}^{-1}\mathbf{E}|}{|\boldsymbol{M}^{-1}\mathbf{B}_I|} = \frac{|\mathbf{E}|}{|\mathbf{B}_I|} = \mathbf{K}_I \boldsymbol{M}.$$

Similarly,

$$\mathbf{K}_I \boldsymbol{M}^{-1} = \frac{|\mathbf{E}_0|}{|\mathbf{B}|} = \frac{|\boldsymbol{M}^{-1}\mathbf{B}_0|}{|\boldsymbol{M}^{-1}\mathbf{E}|} = \frac{|\mathbf{B}_0|}{|\mathbf{E}|} = \mathbf{K}_0 \boldsymbol{M}.$$

2.3. Distortions of nonlinear mappings

Consider a homeomorphism $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ and its inverse $h : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$

$$\mathbb{R}^n \supset \mathbb{X} \underset{\longleftarrow\; h \;\longrightarrow}{\overset{\longrightarrow\; f \;\longrightarrow}{}} \mathbb{Y} \subset \mathbb{R}^n \qquad \begin{array}{l} y = f(x) \\ x = h(y). \end{array} \tag{2.5}$$

Definition 2.1 (Regular Points). A point $x \in \mathbb{X}$ is said to be *regular* for $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ if it is differentiable at x and its Jacobian determinant is positive, $J_f(x) = \det[Df(x)] > 0$. Automatically, the point $y = f(x)$ is regular for the inverse map $h : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$.

$$\mathbf{R}^n \simeq T_x\mathbb{X} \quad \underset{\longleftarrow\; Dh(y) \;\text{———}}{\overset{\text{———}\; Df(x) \;\longrightarrow}{}} \quad T_y\mathbb{Y} \simeq \mathbf{R}^n. \tag{2.6}$$

The key role of any distortion function is to measure how far is the map from a conformal one. The map is conformal exactly at the points where its distortion equals 1. In this case we say, by customary abuse of terminology, that there is no distortion at this point. The two foremost distortion functions, defined at the regular points, are:

- The *outer distortion*
$$\mathbf{K}_0 f(x) = \frac{\|Df(x)\|^n}{J_f(x)}; \tag{2.7}$$
- The *inner distortion*
$$\mathbf{K}_I f(x) = \frac{\|D^\sharp f(x)\|^n}{[J_f(x)]^{n-1}}. \tag{2.8}$$

The next step in this vein is to extend these concepts to all Sobolev mappings with nonnegative Jacobian.

Definition 2.2. Let $f \in \mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$. Thus f has well defined differential on the set $\mathbb{E}_f \overset{\text{def}}{=\!=} \{x \in \mathbb{X}$ where $Df(x)$ does exist$\}$ of full measure. The distortions $\mathbf{K}_0 f(x)$ and $\mathbf{K}_I f(x)$ are defined on this set as follows:

- By using formulas (2.7) and (2.8) if $J_f(x) > 0, x \in \mathbb{E}_f$;
- If $J_f(x) = 0$, then

$$\mathbf{K}_0 f(x) \overset{\text{def}}{=\!=} \begin{cases} 1 & \text{if } \; Df(x) = 0 \\ \infty & \text{if } \; Df(x) \;\neq 0 \end{cases} \qquad x \in \mathbb{E}_f$$

$$\mathbf{K}_I f(x) \overset{\text{def}}{=\!=} \begin{cases} 1 & \text{if } \; D^\sharp f(x) = 0 \\ \infty & \text{if } \; D^\sharp f(x) \;\neq 0 \end{cases} \qquad x \in \mathbb{E}_f.$$

Remark 2.3. An indispensable property of the distortion functions is the lower-semicontinuity. This strongly depends on their domain of definition. For this reason we distinguish the following classes of homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$

- The class $\mathscr{H}^{1,1}(\mathbb{X}, \mathbb{Y})$ is the natural domain of definition of $\mathbf{K}_0 f(x)$;
- While $\mathscr{H}^{1,n-1}(\mathbb{X}, \mathbb{Y})$ is the natural domain of definition of $\mathbf{K}_I f(x)$.

Hereafter, the term *natural domain of definition* of a differential operator refers, rather loosely, to the one in which the operator enjoys the designed property.

2.4. Mappings of finite distortion

Quasiconformal mappings, having uniformly bounded distortion, have proven to be fundamental in the study of deformations of Euclidean domains and Riemannian manifolds [1,4,11,23,34,41,45,55,56].

Definition 2.4. A homeomorphism $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$ is *K-quasiconformal*, $1 \leqslant K < \infty$, if

$$\|Df(x)\|^n \leqslant K\, J_f(x) \quad \text{almost everywhere.} \tag{2.9}$$

In particular,

$$\int_{\mathbb{X}} \|Df(x)\|^n \mathrm{d}x \leqslant K|\mathbb{Y}| < \infty \quad \text{hence} \quad f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n).$$

Note that quasiconformal mappings are differentiable almost everywhere and $J_f(x) > 0$ [19, 39]. Consequently, $\mathbf{K}_0 f(x)$ is defined almost everywhere and $\|\mathbf{K}_0 f\|_{\mathscr{L}^\infty(\mathbb{X})} \leqslant K$. The minimal analytic assumptions necessary for a viable theory of more general deformations appear to be as follows [4,21,23,38].

Definition 2.5. A homeomorphism $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ is said to have *finite distortion* if:

- $f \in \mathscr{H}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$. Thus $J_f \in \mathscr{L}^1(\mathbb{X}, \mathbb{R}^n)$ and $\int_{\mathbb{X}} J_f(x)\mathrm{d}x \leqslant |\mathbb{Y}|$;
- There is a measurable function $K = K(x) \geqslant 1$, finite almost everywhere, such that f satisfies the *distortion inequality*

$$\| Df(x)\|^n \leqslant K(x)\, J_f(x) \quad \text{almost everywhere in } \mathbb{X}. \tag{2.10}$$

The smallest such function is none other than the outer distortion of f; that is, $K(x) = \mathbf{K}_0 f(x)$.

Remark 2.6. Inequality (2.10) just amounts to saying that the condition $J_f(x) = 0$ implies $Df(x) = 0 \in \mathbb{R}^{n\times n}$. In what follows some of our equations and estimates rely on the assumption that the mappings in question have finite distortion; such estimates fail otherwise, which will be stressed out in the text.

2.5. $\mathcal{K}_\ell$-distortions

There are many more distortion functions of interest in GFT and nonlinear analysis. We shall make use of Frobenius (Hilbert-Schmidt) norm of matrices. The following figure illustrates the relationship between so-called *Elliptical distortions* (defined via the operator norm) and the *Rectangular distortions* (defined via Hilbert-Schmidt norm). The linear tangent map $Df(x) : \mathbf{T}_x\mathbb{X} \to \mathbf{T}_y\mathbb{Y}$ takes a ball into an ellipsoid and, equivalently, a cube into a parallelotope (Figure 2.4).

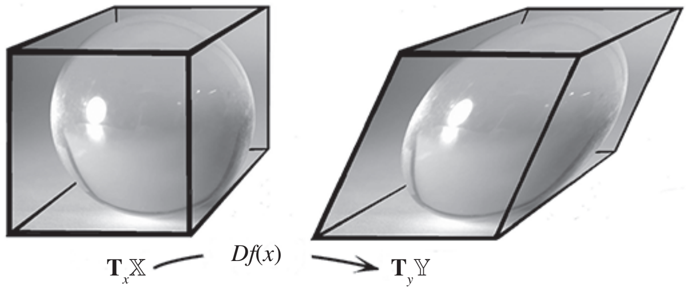

Figure 2.4. Elliptical versus rectangular distortion.

2.5.1. *Geometric description*

Any set of n linearly independent vectors $\mathfrak{v}_1, \dots, \mathfrak{v}_n \in \mathbb{R}^n$ gives rise to an n-dimensional *parallelotope*

$$\mathbb{P} = \left\{ \mathfrak{v} \in \mathbb{R}^n \, ; \; \mathfrak{v} = \alpha_1 \mathfrak{v}_1 + \ldots + \alpha_n \mathfrak{v}_n \quad 0 \leqslant \alpha_1, \ldots, \alpha_n \leqslant 1 \right\}.$$

Given, $1 \leqslant \ell \leqslant n-1$, to every ordered ℓ-tuple I; $1 \leqslant i_1 < \ldots < i_\ell \leqslant n$, there corresponds the *ℓ-dimensional face* of $\mathbb{P}$,

$$\mathbb{P}_I = \left\{ \mathfrak{v} \in \mathbb{P} \, ; \; \mathfrak{v} = \alpha_{i_1} \mathfrak{v}_{i_1} + \ldots + \alpha_{i_\ell} \mathfrak{v}_{i_\ell} \quad 0 \leqslant \alpha_{i_1}, \ldots, \alpha_{i_\ell} \leqslant 1 \right\}.$$

The exterior product $\mathfrak{v}_{i_1} \wedge \cdots \wedge \mathfrak{v}_{i_\ell}$ represents the area/volume of $\mathbb{P}_I$ (length of $\mathfrak{v}_{i_1} \wedge \cdots \wedge \mathfrak{v}_{i_\ell}$). We shall make use of the quadratic mean of areas of all ℓ-dimensional faces:

$$\mathrm{Vol}_\ell \mathbb{P} \overset{\text{def}}{=\!=} \left[\binom{n}{\ell}^{-1} \sum_{1 \leqslant i_1 < \ldots < i_\ell \leqslant n} \left| \mathfrak{v}_{i_1} \wedge \cdots \wedge \mathfrak{v}_{i_\ell} \right|^2 \right]^{\frac{1}{2}}.$$

It will be important later to have various bounds for such quantities. These bounds can be obtained from the general Hadamard type inequality:

$$\sqrt[k]{\mathrm{Vol}_k \mathbb{P}} \leqslant \sqrt[\ell]{\mathrm{Vol}_\ell \mathbb{P}} \quad \text{whenever} \quad 1 \leqslant k < \ell \leqslant n. \tag{2.11}$$

Equality holds if and only if $\mathbb{P}$ is a cube [40]. In particular, we have $[\mathrm{Vol}_n \mathbb{P}]^{\frac{1}{n}} \leqslant [\mathrm{Vol}_\ell \mathbb{P}]^{\frac{1}{\ell}}$ and

$$\mathcal{K}_\ell \mathbb{P} \overset{\text{def}}{=\!=} \frac{[\mathrm{Vol}_\ell \mathbb{P}]^{\frac{n}{n-\ell}}}{[\mathrm{Vol}_n \mathbb{P}]^{\frac{\ell}{n-\ell}}} \geqslant 1 \qquad 1 \leqslant \ell \leqslant n-1.$$

Equality $\mathcal{K}_\ell \mathbb{P} = 1$ occurs if and only if $\mathbb{P}$ is a cube (the most regular shape among all n-parallelopotes). This motivates our calling $\mathcal{K}_\ell \mathbb{P}$ the ℓ-distortion of $\mathbb{P}$.

Next, consider a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}_+$ with positive determinant. It represents a linear transformation, still denoted by $\mathbf{M} : \mathbb{R}^n \to \mathbb{R}^n$. $\mathbf{M}$ takes cubes in $\mathbb{R}^n$

into parallelotopes of ℓ-distortion independent of the cube. We call it $\mathcal{K}_\ell$*-distortion of* $\mathbf{M}$,

$$\mathcal{K}_\ell \mathbf{M} \stackrel{\text{def}}{=\!=} \frac{\left[\binom{n}{\ell}^{-1} \sum_{1 \leqslant i_1 < \ldots < i_\ell \leqslant n} \left| \mathfrak{v}_{i_1} \wedge \cdots \wedge \mathfrak{v}_{i_\ell} \right|^2 \right]^{\frac{n}{2n-2\ell}}}{\left| \mathfrak{v}_1 \wedge \cdots \wedge \mathfrak{v}_n \right|^{\frac{\ell}{n-\ell}}} \geqslant 1 \tag{2.12}$$

where we have chosen $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$ to be the column vectors of the matrix $\mathbf{M}$. These are images of the standard orthonormal vectors $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ in $\mathbb{R}^n$. The following figure illustrates the case $n = 3$. In this case we have two distortion functions. These are analogues of the outer distortion ($\ell = 1$) and inner distortion ($\ell = 2$); simply, by using Frobenius norm of matrices (Figure 2.5).

$$\mathcal{K}_1 \mathbf{M} = \mathcal{K}_1 \mathbf{P} = \left[\frac{[\text{quadratic mean of edges }]^3}{\text{Volume}} \right]^{\frac{1}{2}} = \left[\frac{[a^2 + b^2 + c^2]^3}{27\, |\mathbb{P}\,|^2} \right]^{\frac{1}{4}}$$

$$\mathcal{K}_2 \mathbf{M} = \mathcal{K}_2 \mathbf{P} = \left[\frac{[\text{quadratic mean of faces }]^3}{[\text{Volume}]^2} \right]^{\frac{1}{2}} = \left[\frac{[\mathbf{A}^2 + \mathbf{B}^2 + \mathbf{C}^2]^3}{27 |\mathbb{P}\,|^4} \right]^{\frac{1}{2}} .$$

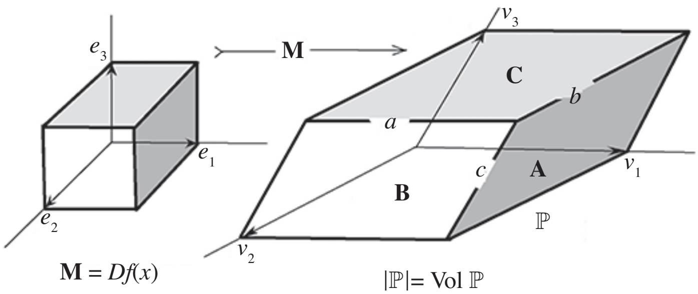

Figure 2.5.

2.5.2. *Definition by Subdeterminants*

Let us write a matrix $\mathbf{M} \in \mathbb{R}^{n \times n}_+$ in terms of its entries.

$$\mathbf{M} = \begin{bmatrix} M_1^1 & M_2^1 & \ldots & M_n^1 \\ M_1^2 & M_2^2 & \ldots & M_n^2 \\ \vdots & \vdots & & \vdots \\ M_1^n & M_2^n & \ldots & M_n^n \end{bmatrix} = \left[M_i^j \right]_{i=1,\ldots,n}^{j=1,\ldots,n} . \tag{2.13}$$

Select and fix an ℓ-tuple of indices, $I;\ 1 \leqslant i_\ell < i_2 < \cdots < i_\ell \leqslant n$, and consider column vectors $\mathfrak{v}_{i_1}, \mathfrak{v}_{i_2} \dots \mathfrak{v}_{i_\ell}$. It gives us an $n \times \ell$-submatrix

$$\mathbf{M}_I = \begin{bmatrix} M^1_{i_1} & M^1_{i_2} & \dots & M^1_{i_\ell} \\ M^2_{i_1} & M^2_{i_2} & \dots & M^2_{i_\ell} \\ \vdots & \vdots & & \vdots \\ M^n_{i_1} & M^n_{i_2} & \dots & M^n_{i_\ell} \end{bmatrix} \in \mathbb{R}^{n\times\ell}. \tag{2.14}$$

The exterior product $\mathfrak{v}_{i_1} \wedge \mathfrak{v}_{i_2} \wedge \dots \wedge \mathfrak{v}_{i_\ell}$ is a vector (also referred to as prime ℓ-vector) in the $\binom{n}{\ell}$-dimensional space $\Lambda^\ell \mathbb{R}^n$ [23]. This space, called *ℓ-exterior power of* $\mathbb{R}^n$, is isometric (via Hodge-star isometry) to the Euclidean space $\mathbb{R}^{\binom{n}{\ell}}$. The coordinates of $\mathfrak{v}_{i_1} \wedge \mathfrak{v}_{i_2} \wedge \dots \wedge \mathfrak{v}_{i_\ell}$ are the $\ell \times \ell$- minors of $\mathbf{M}_I$. To every ℓ-tuple $J;\ 1 \leqslant j_1 < j_2 < \dots < j_\ell \leqslant n$ there corresponds the $\ell \times \ell$- minor:

$$M^J_I = \det \begin{bmatrix} M^{j_1}_{i_1} & M^{j_1}_{i_2} & \dots & M^{j_1}_{i_\ell} \\ M^{j_2}_{i_1} & M^{j_2}_{i_2} & \dots & M^{j_2}_{i_\ell} \\ \vdots & \vdots & & \vdots \\ M^{j_\ell}_{i_1} & M^{j_\ell}_{i_2} & \dots & M^{j_\ell}_{i_\ell} \end{bmatrix}. \tag{2.15}$$

Hence $|\,\mathfrak{v}_{i_1} \wedge \mathfrak{v}_{i_2} \wedge \dots \wedge \mathfrak{v}_{i_\ell}| = \left(\sum_J |M^J_I|^2\right)^{\frac{1}{2}}$. This gives us another (equivalent) definition

Definition 2.7. Given a matrix $\mathbf{M} \in \mathbb{R}^{n\times n}_+$, its *$\mathcal{K}_\ell$-distortion* is defined by

$$\mathcal{K}_\ell \mathbf{M} = \frac{\left[\sqrt{\binom{n}{\ell}^{-1} \sum_{I,\,J} |M^J_I|^2}\,\right]^{\frac{n}{n-\ell}}}{|\det \mathbf{M}\,|^{\frac{\ell}{n-\ell}}}. \tag{2.16}$$

One crucial feature that a distortion function $\mathcal{K}_\ell f$ of a Sobolev mapping $f : \mathbb{X} \to \mathbb{Y}$ might have is the lower-semicontinuity property of its $\mathscr{L}^p$-integrals (quasiconvexity [42]). This heavily depends on the polyconvexity properties of the distortion of the differential matrix $Df(x)$.

2.5.3. *Polyconvexity*

Given a matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, consider the collection of all $\ell \times \ell$-minors with $\ell = 1, 2, \dots, n$. Precisely,

$$\mathbf{M}_{\boxplus} \overset{\text{def}}{=\!=} \left\{ M^J_I\ ;\ \begin{matrix} J;\ 1 \leqslant j_1 < j_2 < \dots < j_\ell \leqslant n \\ I;\ 1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant n \end{matrix} \text{ and } 1 \leqslant \ell \leqslant n \right\}. \tag{2.17}$$

The number of minors in this collection equals $\sum_{\ell=1}^n \binom{n}{\ell}^2 = \binom{2n}{n} - 1$. Choose and fix the order of minors, in which $\det \mathbf{M}$ ($\ell = n$) is the last in the sequence. In this way each $\mathbf{M}_{\boxplus}$ becomes a point of the Euclidean space $\mathbb{R}^{\binom{2n}{n}-1}$. The set $\mathbb{R}_+^{\binom{2n}{n}-1}$ of points having positive last coordinate is convex. Thus, we may speak of convex functions $\mathfrak{F} : \mathbb{R}_+^{\binom{2n}{n}-1} \to \mathbb{R}$.

Definition 2.8 (polyconvexity). A function $\mathcal{K} : \mathbb{R}_+^{n\times n} \to \mathbb{R}$ is said to be *polyconvex* if it can be represented as

$$\mathcal{K}(\mathbf{M}) = \mathfrak{F}(\mathbf{M}_{\boxplus}) \qquad \text{where} \quad \mathfrak{F} : \mathbb{R}_+^{\binom{2n}{n}-1} \to \mathbb{R} \;\text{ is convex.} \tag{2.18}$$

Note that the representation of $\mathcal{K}$ by a convex function $\mathfrak{F}$ in (2.18) need not be unique. It is a quick consequence of [18] and [23, Lemma 8.8.2] that

Proposition 2.9. *For every exponent* $1 \leqslant p < \infty$ *the* p*-powers of the* $\mathcal{K}_\ell$*-distortions,* $\mathcal{K}_\ell^p : \mathbb{R}_+^{n\times n} \to [0,\infty)$, $1 \leqslant \ell \leqslant n-1$ *, are polyconvex.*

Remark 2.10. Polyconvexity of $\mathcal{K}_\ell^p$ fails for every $0 < p < 1$. Actually, this fact motivated our choice of the exponents $\frac{n}{n-\ell}$ and $\frac{\ell}{n-\ell}$ in the formula (2.16) for Definition 2.7; simply because, it gives us the quotient that is homogeneous of degree 0 with respect to Df and its p-power is polyconvex if and only if $\frac{pn}{n-\ell} - \frac{p\ell}{n-\ell} \geqslant 1$, see [18,23].

2.5.4. *The* ℓ*-exterior power of a matrix*

It eventually appears that there is much more algebraic structure in an assembly of the $\ell \times \ell$-minors into a matrix of the form $[\pm M_I^J]$ than one might expect. The ℓ-exterior power of $\mathbf{M} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map $\mathbf{M}_\sharp^\ell : \Lambda^\ell\mathbb{R}^n \to \Lambda^\ell\mathbb{R}^n$ defined on the prime ℓ-vectors by the rule

$$\mathbf{M}_\sharp^\ell\big(\mathfrak{v}_{i_1} \wedge \ldots \wedge \mathfrak{v}_{i_\ell}\big) = \mathbf{M}\mathfrak{v}_{i_1} \wedge \ldots \wedge \mathbf{M}\mathfrak{v}_{i_\ell}$$

and then extended linearly onto the entire space $\Lambda^\ell\mathbb{R}^n$. One should observe that there is no ambiguity in using different representations of the same prime ℓ-vector. The following are basic identities for the exterior powers of matrices.

- If $\mathbf{M} = \mathbf{Id} : \mathbb{R}^n \to \mathbb{R}^n$, then $\mathbf{M}_\sharp^\ell = \mathbf{Id} : \Lambda^\ell\mathbb{R}^n \to \Lambda^\ell\mathbb{R}^n$;
- If $\mathbf{A}, \mathbf{B} : \mathbb{R}^n \to \mathbb{R}^n$, then $\mathbf{A}_\sharp^\ell\mathbf{B}_\sharp^\ell = (\mathbf{AB})_\sharp^\ell : \Lambda^\ell\mathbb{R}^n \to \Lambda^\ell\mathbb{R}^n$;
- Thus for $\mathbf{M} \in \mathbb{R}_+^{n\times n}$ we have $(\mathbf{M}^{-1})_\sharp^\ell = (\mathbf{M}_\sharp^\ell)^{-1}$;
- Hodge star duality operator $* : \Lambda^\ell\mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \Lambda^{n-\ell}\mathbb{R}^n$ is an isometry. For every $\mathbf{M} \in \mathbb{R}^{n\times n}$ a generalization of Cramer's rule reads as:

$$*\mathbf{M}_\sharp^{n-\ell} * \mathbf{M}_\sharp^\ell = \det\mathbf{M} \cdot \mathbf{Id} :\; \Lambda^\ell\mathbb{R}^n \to \Lambda^\ell\mathbb{R}^n;$$

- In particular, we note the following formula for the inverse matrix

$$\left|\mathbf{M}_\sharp^{n-\ell}\right| = (\det\mathbf{M})\left|\big(\mathbf{M}^{-1}\big)_\sharp^\ell\right|;$$

- We reserve the following abbreviation for the matrix of cofactors

$$\mathbf{M}_\sharp^{n-1} = \mathbf{M}^\sharp \stackrel{\text{def}}{=\!=} \mathbf{M}^{-1}\text{det}\mathbf{M};$$

- If $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ are singular values of $\mathbf{M}$, then the products $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_\ell}$, corresponding to every ℓ tuple I ; $1 \leqslant i_1 < i_2 < \ldots < i_\ell \leqslant n$, are the singular values of $\mathbf{M}_\sharp^\ell$.

2.5.5. *Distortion in terms of singular values*

The *Frobenius norm* of a matrix $\mathfrak{M} = [M_I^J] \in \mathbb{R}^{\mathfrak{n}\times\mathfrak{n}}$ is the square root of the sum of squares of its entries.

$$|\mathfrak{M}| \stackrel{\text{def}}{=\!=} \left(\sum_{I,J} |M_I^J|^2\right)^{\frac{1}{2}} = \sqrt{\text{Trace}[\mathfrak{M}^*\mathfrak{M}]}.$$

One may wish to look at the quadratic means of the length of the column vectors, denoted by

$$\lfloor \mathfrak{M} \rfloor \stackrel{\text{def}}{=\!=} \left(\mathfrak{n}^{-1}\sum_{I,\,J} |M_I^J|^2\right)^{\frac{1}{2}} \qquad (= 1 \text{ for orthogonal matrices}).$$

When viewing $\mathfrak{M}$ as a linear transformation, $\mathfrak{M} : \mathbb{R}^\mathfrak{n} \to \mathbb{R}^\mathfrak{n}$, we may also supply it with the operator norm,

$$\|\mathfrak{M}\| \stackrel{\text{def}}{=\!=} \max_{|v|=1} |\mathfrak{M}\, v| \ \geqslant \lfloor \mathfrak{M} \rfloor.$$

Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$ denote the singular values of $\mathbf{M} \in \mathbb{R}^{n\times n}, \mathbf{M} = Df(x)$ (Figure 2.6).

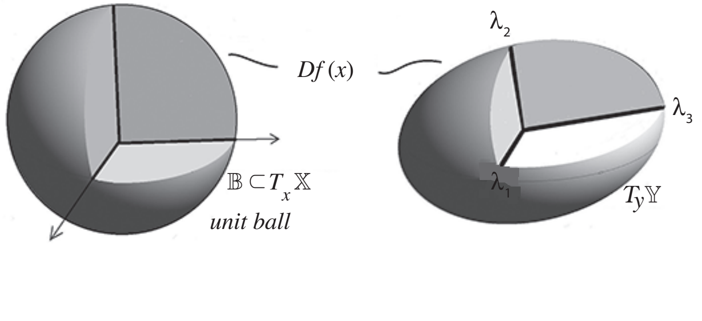

Figure 2.6.

We have

$$
\begin{aligned}
|\mathbf{M}_\sharp^\ell| &= \left(\sum_{1\leqslant i_1<\ldots<i_\ell\leqslant n} \lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_\ell}^2 \right)^{\frac{1}{2}} \\
\lfloor \mathbf{M}_\sharp^\ell \rfloor^2 &= \binom{n}{\ell}^{-1} \sum_{1\leqslant i_1<\ldots<i_\ell\leqslant n} \lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_\ell}^2 \\
\left\| \mathbf{M}_\sharp^\ell \right\| &= \lambda_{n-\ell+1}\,\cdots\,\lambda_n .
\end{aligned}
$$

Hence

$$
\mathcal{K}_\ell\mathbf{M} = \frac{\lfloor \mathbf{M}_\sharp^\ell \rfloor^{\frac{n}{n-\ell}}}{(\det\mathbf{M})^{\frac{\ell}{n-\ell}}} = \frac{\left[\binom{n}{\ell}^{-1} \sum_{1\leqslant i_1<\ldots<i_\ell\leqslant n} \lambda_{i_1}^2\lambda_{i_2}^2\cdots\lambda_{i_\ell}^2 \right]^{\frac{n}{2n-2\ell}}}{\Big(\lambda_1\lambda_2\cdots\lambda_n\Big)^{\frac{\ell}{n-\ell}}} \tag{2.19}
$$

$$
\mathbf{K}_O\mathbf{M} = \frac{\|\mathbf{M}\|^n}{\det\mathbf{M}} = \frac{\lambda_n^n}{\lambda_1\lambda_2\cdots\lambda_n} = \frac{\lambda_n^{n-1}}{\lambda_1\lambda_2\cdots\lambda_{n-1}} \tag{2.20}
$$

$$
\mathbf{K}_I\mathbf{M} = \frac{\|\mathbf{M}^\sharp\|^n}{(\det\mathbf{M})^{n-1}} = \frac{(\lambda_2\lambda_3\cdots\lambda_n)^n}{(\lambda_1\lambda_2\cdots\lambda_n)^{n-1}} = \frac{\lambda_2\lambda_3\cdots\lambda_n}{\lambda_1^{n-1}}. \tag{2.21}
$$

For the inverse matrix, we have the following identities:

$$
\Big[\mathcal{K}_\ell\mathbf{M}^{-1}\Big]^{\frac{1}{\ell}} = \big[\mathcal{K}_{n-\ell}\mathbf{M}\big]^{\frac{1}{n-\ell}} \tag{2.22}
$$

$$
\mathbf{K}_I\mathbf{M}^{-1} = \mathbf{K}_O\mathbf{M}. \tag{2.23}
$$

2.5.6. *Distortions of a nonlinear mapping*

Let $f : \mathbb{X} \to \mathbb{R}^n$ be a mapping of Sobolev class $\mathscr{W}^{1,s}_{\mathrm{loc}}(\mathbb{X}, \mathbb{R}^n)$ with positive Jacobian, $J_f(x) = \det Df(x) > 0$, almost everywhere. Thus, we may apply the formulas for $\mathcal{K}_\ell\mathbf{M}$ to the differential matrix

$$
\mathbf{M} = Df(x) = \left[\frac{\partial f^j}{\partial x_i}\right] = \begin{bmatrix}
\frac{\partial f^1}{\partial x_1} & \frac{\partial f^1}{\partial x_2} & \cdots & \frac{\partial f^1}{\partial x_n} \\
\frac{\partial f^2}{\partial x_1} & \frac{\partial f^2}{\partial x_2} & \cdots & \frac{\partial f^2}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial f^n}{\partial x_1} & \frac{\partial f^n}{\partial x_2} & \cdots & \frac{\partial f^n}{\partial x_n}
\end{bmatrix}. \tag{2.24}
$$

This results in a nonlinear differential expression

$$\mathcal{K}_\ell f \stackrel{\text{def}}{=\!=} \left[\binom{n}{\ell}^{-1} \sum_{\substack{1\leqslant j_1<\cdots<j_\ell\leqslant n\\ 1\leqslant i_1<\cdots<i_\ell\leqslant n}} \left| \frac{\partial(f^{j_1},\dots,f^{j_\ell})}{\partial(x_{i_1},\dots,x_{i_\ell})} \right|^2 \right]^{\frac{n}{2n-2\ell}} \Big/ \left|\det Df\right|^{\frac{\ell}{n-\ell}}.$$

Viewing the ℓ-th order subdeterminants $\frac{\partial(f^{j_1},\dots,f^{j_\ell})}{\partial(x_{i_1},\dots,x_{i_\ell})} \in \mathbb{R}$ and the highest order Jacobian determinant $\det Df = \frac{\partial(f^1,\dots,f^n)}{\partial(x_1,\dots,x_n)} \in \mathbb{R}_+$ as independent variables brings us to a convex function $\mathfrak{F}_\ell : \mathbb{R}^{\binom{n}{\ell}\times\binom{n}{\ell}} \times \mathbb{R}_+ \to [1,\infty)$. The same pertains to the p-powers of $\mathcal{K}_\ell$, with $p \geqslant 1$.

$$[\mathcal{K}_\ell f]^p \stackrel{\text{def}}{=\!=} \left[\binom{n}{\ell}^{-1} \sum_{\substack{1\leqslant j_1<\cdots<j_\ell\leqslant n\\ 1\leqslant i_1<\cdots<i_\ell\leqslant n}} \left| \frac{\partial(f^{j_1},\dots,f^{j_\ell})}{\partial(x_{i_1},\dots,x_{i_\ell})} \right|^2 \right]^{\frac{np}{2n-2\ell}} \Big/ \Big(\det Df\Big)^{\frac{\ell p}{n-\ell}}. \tag{2.25}$$

The Jacobian subdeterminants are rather special nonlinear differential expressions called *null-Lagrangians* [6], see also [7, 14, 22]). They are weakly continuous in various classes of Sobolev mappings, which makes $\int [\mathcal{K}_\ell]^p$ a lower semicontinuous functional. Although we do not explicitly utilize polyconvexity of $[\mathcal{K}_\ell]^p$, this concept is nevertheless behind our construction and discussions of so-called *Free-Lagrangians* in the later chapters.

3. Free-Lagrangians

3.1. Definition

The origin of what is termed free-Lagrangians lies in the study of traction-free problems; that is, energy-minimal deformations $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ (usually homeomorphisms) with no boundary values prescribed up front. Tangential slipping along $\partial\mathbb{X}$ is allowed. This is physically realized by deforming an incompressible material confined in a box. In a simplified way of speaking, free-Lagrangians are defined as follows.

For a pair of domains $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^n$ consider a class of Sobolev homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of finite energy

$$\mathfrak{L}[f] \stackrel{\text{def}}{=\!=} \int_{\mathbb{X}} \mathbf{L}(x, f, Df)\,\mathrm{d}x < \infty \tag{3.1}$$

where $\mathbf{L} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is a given (*stored-energy*) function. The relevant conditions on $\mathbf{L}$ and the class of admissible homeomorphisms, will be specified when required. The present paper is concerned with mappings of positive Jacobian determinant; in symbols, $Df(x) \in \mathbb{R}^{n\times n}_+$. Furthermore, $\mathbf{L}$ will always be $\mathscr{C}^1$-smooth on $\mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n}_+$.

Definition 3.1. The term *free-Lagrangian* refers to a differential n-form $\mathbf{L}(x, f, Df)\,\mathrm{d}x$ whose integral $\mathfrak{L}[f]$ is constant within the same homotopy class of homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, regardless of their boundary values.

Recall, following J. Ball [6] that $\mathbf{L}(x, f, Df)\,\mathrm{d}x$ is a *null-Lagrangian* if $\mathfrak{L}[f]$ depends only on the boundary values of f. Consequently, the variational Lagrange-Euler equation is identically satisfied.

3.1.1. *Invariable free-Lagrangians*

The banal (still useful) example of this is furnished by

$$\mathbf{L}(x, f, Df)\mathrm{d}x = \Phi(x)\mathrm{d}x \quad \text{where} \quad \Phi \in \mathscr{L}^1(\mathbb{X}). \tag{3.2}$$

3.2. Volume free-Lagrangians

The next example in order of complexity (still elementary) is the volume integral defined for orientation preserving homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,n}(\mathbb{X}, \mathbb{Y})$. Let us begin with

$$\mathcal{V}[f] = \int_{\mathbb{X}} J_f(x)\,\mathrm{d}x = |\mathbb{Y}| \qquad J_f(x) \overset{\text{def}}{=\!=} \det Df(x) \geqslant 0. \tag{3.3}$$

One encounters further examples by introducing a weight in the target domain $\mathbb{Y}$. The following *coarea inequality* will come into play.

Proposition 3.2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be domains in $\mathbb{R}^n$ and $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ a homeomorphism in the Sobolev space $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$. Given a nonnegative function $\Phi \in \mathscr{L}^1(\mathbb{Y}) \cap \mathscr{C}(\mathbb{A})$, we have*

$$\mathcal{V}_\Phi[f] \overset{\text{def}}{=\!=} \int_{\mathbb{X}} \Phi\big(f(x)\big)|J_f(x)|\,\mathrm{d}x \leqslant \int_{\mathbb{Y}} \Phi(y)\,\mathrm{d}y. \tag{3.4}$$

Equality occurs, for example, if $f \in \mathscr{W}^{1,n}(\mathbb{X}, \mathbb{R}^n)$. Thus the differential n-form

$$\mathbf{L}(x, f, Df)\,\mathrm{d}x \overset{\text{def}}{=\!=} \Phi\big(f(x)\big)|J_f(x)|\,\mathrm{d}x \tag{3.5}$$

is a free-Lagrangian within homeomorphisms $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ in the Sobolev space $\mathscr{W}^{1,n}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$.

It is tempting at this point to bring up a simple example that illustrates the utility of free-Lagrangians.

Corollary 3.3 (Neohookean Energy). *Consider all homeomorphisms* $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{X}$ *in the Sobolev space* $\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{X}, \mathbb{R}^n)$ *having nonnegative Jacobian. Then the following Neohookean type energy*

$$\mathcal{N}[f] \stackrel{\text{def}}{=\!=} \int_{\mathbb{X}} \frac{dx}{J_f(x)}$$

assumes its minimum value at the identity map $f(x) \equiv x$ *(and of course at any homeomorphism with* $J_f \equiv 1$*).*

Proof. The key is to find a point-wise lower bound of the integrand by means of free-Lagrangians in which equality occurs at $f(x) \equiv x$. Accordingly, we write

$$\frac{1}{J_f(x)} \geqslant -J_f(x) + 2.$$

Integrating over $\mathbb{X}$, with the aid of inequality (3.4), Corollary 3.3 follows. □

3.2.1. *The class* $\mathscr{H}^{1,s}(\mathbb{A}, \mathbb{A}^*)$

Throughout this text we are dealing with homeomorphisms $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ between annuli in the space $\mathscr{W}^{1,s}(\mathbb{A}, \mathbb{A}^*)$.

$$\mathbb{A} = \left\{x \in \mathbb{R}^n;\; r < |x| < R\right\} \quad \text{and} \quad \mathbb{A}^* = \left\{y \in \mathbb{R}^n;\; r_* < |y| < R_*\right\} \tag{3.6}$$

where $0 \leqslant r < R \leqslant \infty$ and $0 \leqslant r_* < R_* \leqslant \infty$. We make two standing assumptions on the mappings. First, that they are orientation preserving. Second, they also preserve the order of the boundary components. Namely,

$$\lim_{|x| \searrow r} |f(x)| = r_* \quad \text{and} \quad \lim_{|x| \nearrow R} |f(x)| = R_*.$$

These limits can easily be shown to exist and the above assumptions involve no loss of generality; compose f with an appropriate isometry if necessary.

Conformal Modulus of an annulus $\mathbb{A} = A(r, R) = \{x \in \mathbb{R}^n;\; r < |x| < R\}$ is defined by

$$\text{Mod}\, \mathbb{A} = \log \frac{R}{r} = \frac{1}{\omega_{n-1}} \int_{\mathbb{A}} \frac{\mathrm{d}x}{|x|^n} \tag{3.7}$$

where, as usual, ω_{n-1} denotes the $(n-1)$-surface area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. As an example, note the folowing free-Lagrangian identity.

$$\frac{1}{\omega_{n-1}} \int_{\mathbb{A}} \frac{J_f(x)\mathrm{d}x}{|f(x)|^n} = \frac{1}{\omega_{n-1}} \int_{\mathbb{A}^*} \frac{\mathrm{d}y}{|y|^n} = \text{Mod}\, \mathbb{A}^* \quad \text{for} \quad f \in \mathscr{H}^{1,n}(\mathbb{A}, \mathbb{A}^*)\,.$$

3.3. Radial free-Lagrangians

Let $f \in \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$. We have well defined (almost everywhere) the *radial gradient* (or normal derivative) $f_{\mathbf{N}}(x) = Df(x)\frac{x}{|x|} \in \mathbb{R}^n$. Moreover,

$$|f|_{\mathbf{N}} = \left\langle \nabla |f|\,,\, \frac{x}{|x|} \right\rangle \leqslant |f_{\mathbf{N}}(x)| \leqslant \|Df(x)\|. \tag{3.8}$$

Here we recall that $\|\cdot\|$ stands for the operator norm of a matrx. Equality holds for radial mappings, say $f(x) = F(|x|)\frac{x}{|x|}$, where the *normal strain* $F : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ is absolutely continuous and has nonnegative derivative

$$\dot{F}(|x|) = |f|_{\mathbf{N}} = |f_{\mathbf{N}}(x)|. \tag{3.9}$$

Lemma 3.4. *Given any function $\Phi \in \mathscr{C}(r_*, R_*) \cap \mathscr{L}^1(r_*, R_*)$. The following differential n-form*

$$\mathbf{L}(x, f, Df)\, dx \stackrel{\text{def}}{=\!=} \Phi(|f|)\, |f|_{=N} \frac{dx}{|x|^{n-1}} \tag{3.10}$$

is a free -Lagrangian within the class $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$. In fact, we have

$$\int_{\mathbb{A}} \Phi(|f|)\, |f|_N \frac{dx}{|x|^{n-1}} = \omega_{n-1} \int_{r_*}^{R_*} \Phi(s)\, ds. \tag{3.11}$$

Proof. Using the polar coordinates of $x = \rho \cdot \omega$, $\rho \in [r, R]$, $\omega \in \mathbb{S}^{n-1}$ we write the left hand side of (3.11) as

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \left(\int_r^R \Phi(|f|) |f|_\rho \,\mathrm{d}\rho \right) \mathrm{d}\omega &= \int_{\mathbb{S}^{n-1}} \int_r^R \frac{\mathrm{d}}{\mathrm{d}\rho} \big(\Psi(|f|)\big) \mathrm{d}\rho \,\mathrm{d}\omega \\
&= \int_{\mathbb{S}^{n-1}} [\Psi(|f(R\omega)|) - \Psi(|f(r\omega)|)]\, \mathrm{d}\omega \\
&= \int_{\mathbb{S}^{n-1}} [\Psi(R_*) - \Psi(r_*)]\, \mathrm{d}\omega \\
&= \omega_{n-1} [\Psi(R_*) - \Psi(r_*)] = \omega_{n-1} \int_{r_*}^{R_*} \Phi(s)\, \mathrm{d}s.
\end{aligned}$$

Here we used the antiderivative of Φ; that is, $\Psi'(s) = \Phi(s)$. □

3.4. Spherical free-Lagrangians

Consider an oriented (smooth) hypersurface $\mathbf{S} \subset \mathbb{R}^n$ of dimension $n-1$ and its $(n-1)$-area form, denoted by $\mathrm{d}S = \mathrm{d}S(x)$. Let $\mathbf{N} = \mathbf{N}(x) = (N_1, \dots, N_n)$ denote the positively oriented unit normal vector field on $\mathbf{S}$. In terms of the (local) coordinates in $\mathbb{R}^n$, we have for $x \in \mathbf{S}$:

$$N_k \,\mathrm{d}S(x) = (-1)^k \,\mathrm{d}x_1 \wedge \dots \widehat{\mathrm{d}x_k} \dots \wedge \mathrm{d}x_n\,, \quad k = 1, \dots, n$$

where, as usual, the hat $\widehat{(\cdot)}$ indicates that the term $(\cdot)$ has to be omitted. In particular, given any vector field $\mathbf{V} = (V_1, \dots, V_n) : \mathbf{S} \to \mathbb{R}^n$, we obtain

$$\langle\, \mathbf{N} \,|\, \mathbf{V} \,\rangle \,\mathrm{d}S(x) = \sum_{k=1}^{n} (-1)^k \, V_k \;\; \mathrm{d}x_1 \wedge \dots \; \widehat{\mathrm{d}x_k} \; \dots \wedge \mathrm{d}x_n.$$

Next, consider a smooth mapping $f = f(x) = (f^1, \dots, f^n)$ defined in a neighborhood of $\mathbf{S}$, its covectors $\mathrm{d}f^1, \dots, \mathrm{d}f^n$ and the matrix of cofactors $[D^\sharp f] = [Df(x)]^\sharp$. Let $\mathbf{F} = (F^1, \dots, F^n) : \mathbf{S} \to \mathbb{R}^n$ be any vector field. We apply the above identity to $\mathbf{V} = [D^\sharp f]\mathbf{F}$, whence it is readily inferred that

$$\Big\langle \mathbf{N} | [D^\sharp f]\mathbf{F} \Big\rangle \,\mathrm{d}S(x) = \sum_{k=1}^{n} (-1)^k F^k \;\; \mathrm{d}f^1 \wedge \dots \widehat{\mathrm{d}f^k} \dots \wedge \mathrm{d}f^n.$$

Suppose that $f : \mathbf{S} \to \mathbb{R}^n \setminus \{0\}$, so we may take $\mathbf{F} = f/|f|^n$ to obtain

$$\begin{aligned} &|f|^{-n} \Big\langle \mathbf{N} \;|\; [D^\sharp f] f \Big\rangle \,\mathrm{d}S(x) \\ &= |f|^{-n} \sum_{k=1}^{n} (-1)^k \, f^k \;\; \mathrm{d}f^1 \wedge \dots \; \widehat{\mathrm{d}f^k} \dots \wedge \mathrm{d}f^n. \end{aligned} \tag{3.12}$$

The right hand side is none other than the pullback under f of the closed differential $(n-1)$-form $\omega = |y|^{-n} \sum_{k=1}^{n} (-1)^k \, y_k \,\mathrm{d}y_1 \wedge \dots \; \widehat{\mathrm{d}y_k} \; \dots \wedge \mathrm{d}y_n$ in $\mathbb{R}^n \setminus \{0\}$. Such pullback is traditionally denoted by $f^\star(\omega)$.

From now on $\mathbf{S}$ will be a topological $(n-1)$-sphere. The integral $\int_{\mathbf{S}} f^\star(\omega)$ is homotopy invariant. Precisely,

$$\begin{aligned} \int_{\mathbf{S}} f^\star(\omega) &= \int_{\mathbf{S}} |f|^{-n} \sum_{k=1}^{n} (-1)^k f^k \;\; \mathrm{d}f^1 \wedge \dots \widehat{\mathrm{d}f^k} \;\; \dots \wedge \mathrm{d}f^n \\ &= \omega_{n-1} \deg f \end{aligned} \tag{3.13}$$

where $\deg f$ stands for the topological degree of the map $f/|f| : \mathbf{S} \to \mathbb{S}^{n-1}$. We actually have the following formula for all differential $(n-1)$-forms ω on $\mathbb{S}^{n-1}$.

$$\int_{\mathbf{S}} f^\star(\omega) = \deg f \int_{\mathbb{S}^{n-1}} \omega. \tag{3.14}$$

For formulas (3.13) and (3.14), and related topics we refer the interested reader to [15] and [30].

On the other hand Hopf's Theorem tells us that two maps $\Phi, \Psi : \mathbf{S} \to \mathbb{S}^{n-1}$ are homotopic if and only if $\deg \Phi = \deg \Psi$. Now let $\mathbf{S} = \mathbb{S}_t^{n-1} = \{x \in \mathbb{R}^n \,;\, |x| = t\}$, and let $f : A(r, R) \xrightarrow{\text{into}} \mathbb{R}^n \setminus \{0\}$ be any smooth mapping of degree $\mathtt{d}$. This means that f restricted to any (or just one) sphere $\mathbb{S}_t^{n-1}$, $r < t < R$, has degree $\mathtt{d}$. Then applying formula (3.12) to $\mathbf{N} = \frac{x}{|x|}$ and , in view of the identity (3.13), we obtain for every $r < t < R$:

$$\int_{|x|=t} f_{\mathbf{S}}(x)\,\mathrm{d}x = \omega_{n-1}\,\mathtt{d} \qquad \text{where} \quad f_{\mathbf{S}} \stackrel{\text{def}}{=\!=} \left\langle \frac{x}{|x|} \,\middle|\, \frac{[D^\sharp f] f}{|f|^n} \right\rangle. \tag{3.15}$$

The advantage of using this formula for general (noninjective) mappings, as oppose to homeomorphisms, lies in the fact that we may approximate f with smooth mappings of a given degree. Suppose f is an orientation preserving homeomorphism of $\mathbb{A} = A(r, R) \xrightarrow{\text{into}} \mathbb{R}^n \setminus \{0\}$ in the Sobolev space $\mathscr{W}^{1,n-1}(\mathbb{A})$. Thus its topological degree equals 1. We may, and do, approximate f with smooth mappings $f^j : \mathbb{A} \xrightarrow{\text{into}} \mathbb{R}^n \setminus \{0\}$, converging c-uniformly and in the norm of $\mathscr{W}^{1,n-1}(\mathbb{A})$ (standard convolution mollification procedure). It is by no means clear (except in dimension $n = 2$, see [26,27]) whether a smooth approximation can be made with homeomorphisms. Nonetheless, for j sufficiently large, we still have $\deg f^j = 1$, because of uniform convergence. Thus

$$\int_{|x|=t} f^j_{\mathbf{S}}(x)\,\mathrm{d}x = \omega_{n-1} \qquad \text{where} \quad f^j_{\mathbf{S}} \stackrel{\text{def}}{=\!=} \left\langle \frac{x}{|x|} \,\middle|\, \frac{[D^\sharp f^j] f^j}{|f^j|^n} \right\rangle. \tag{3.16}$$

Passing to the limit, by Fubini's Theorem for Sobolev functions, we conclude that for almost every $t \in (r, R)$

$$\int_{|x|=t} f_{\mathbf{S}}(x)\,\mathrm{d}x = \omega_{n-1} \qquad \text{where} \quad f_{\mathbf{S}} \stackrel{\text{def}}{=\!=} \left\langle \frac{x}{|x|} \,\middle|\, \frac{[D^\sharp f] f}{|f|^n} \right\rangle. \tag{3.17}$$

We are now ready to introduce (with the aid of polar coordinates) the so-called *spherical free-Lagrangians* .

Proposition 3.5. *To every function $\Phi \in \mathscr{L}^1(r, R)$, $0 < r < R < \infty$, there corresponds a free-Lagrangian defined for continuous mappings $f : \mathbb{A} \to \mathbb{R}^n \setminus \{0\}$ in the Sobolev space $\mathscr{W}^{1,n-1}(\mathbb{A}, \mathbb{R}^n)$ by the rule*

$$\mathbf{L}(x, f, Df)\,dx \stackrel{\text{def}}{=\!=} \Phi(|x|) f_{\mathbf{S}}(x)\,dx \qquad \textit{where} \quad f_{\mathbf{S}} \stackrel{\text{def}}{=\!=} \left\langle \frac{x}{|x|} \,\middle|\, \frac{[D^\sharp f] f}{|f|^n} \right\rangle. \tag{3.18}$$

The integral mean of $\mathbf{L}(x, f, Df)\,dx$ is a homotopy invariant. In fact we have

$$\int_{\mathbb{A}} \mathbf{L}(x, f, Df)\,dx = \omega_{n-1} deg f \int_r^R \Phi(t)\,dt. \tag{3.19}$$

In particular, for every $g \in \mathscr{H}^{1,n-1}(\mathbb{A}, \mathbb{A}^*)$ *it holds that*

$$\int_{\mathbb{A}} \Phi(|x|)\, g_{\mathbf{S}}(x)\, dx = \omega_{n-1} \int_r^R \Phi(t)\, dt. \tag{3.20}$$

This motivates our calling (3.18) *spherical* (*or tangential*) *free-Lagrangian*. We shall refer to the nonlinear differential expression $f_{\mathbf{S}}(x)$ as *spherical gradient* (or tangential component of $D^\sharp f$). For the later use let us record the following inequality

$$f_{\mathbf{S}}(x) \leqslant \frac{\| D^\sharp f(x) \|}{|f(x)|^n} \qquad \text{where} \quad \| \cdot \| \quad \text{stands for the operator norm.} \tag{3.21}$$

4. Inner variation

4.1. Definition

Let a given energy integral

$$\mathscr{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x \tag{4.1}$$

be subjected to a class of sense-preserving Sobolev homeomorphisms $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ with $J_h(x) = \det Dh(x) > 0$. We assume that $\mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ is continuous and $\mathscr{C}^1$-smooth with respect to the variables $x \in \mathbb{X}$ and $\xi \in \mathbb{R}^{n\times n}_+$. Precisely, the derivatives $\mathbf{E}_x = \mathbf{E}_x(x, y, \xi) \in \mathbb{R}^n$ and $\mathbf{E}_\xi(x, y, \xi) \in \mathbb{R}^{n\times n}$ are also continuous. Here, as usual, the partial gradient $\mathbf{E}_x$ stands for a vector field defined by the rule

$$\langle \mathbf{E}_x \mid v \rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \mathbf{E}(x + \epsilon\, v,\ y,\ \xi) \qquad \text{for all} \quad v \in \mathbb{R}^n.$$

Similarly, $\mathbf{E}_\xi$ stands for the matrix field defined by

$$\langle \mathbf{E}_\xi \mid \zeta \rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \mathbf{E}(x, y, \xi + \epsilon\zeta) \qquad \text{for all} \quad \zeta \in \mathbb{R}^{n\times n}.$$

Minimization of energy leads to variational equations. As a side remark, for traction-free problems, like search for smallest mean distortion, the Euler-Lagrange equations may not be available. Simply because the usual variation $h_\epsilon = h(x) + \epsilon\, \phi(x)$, $\phi \in \mathscr{C}^\infty_\circ(\mathbb{X}, \mathbb{R}^n)$ is not admissible; the mappings h_ϵ must remain injective. We have to rely on somewhat weaker equations derived from inner variation $h_\epsilon = h(\varphi_\epsilon(x))$, where $\{\varphi_\epsilon\}_{\epsilon \sim 0}$ is a one-parameter family of diffeomorphisms $\varphi_\epsilon : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{X}}$, $\varphi_0(x) \equiv x$. This amounts to smooth permutation of points in $\mathbb{X}$ but does not change the set of values of h. It only shuffles them around within the target domain $\mathbb{Y}$. More importantly for us, injectivity is also not lost.

Clearly, the vector field $v \overset{\text{def}}{=\!=} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0}\varphi_\epsilon \,:\, \overline{\mathbb{X}} \to \mathbb{R}^n$ is tangent to $\partial\mathbb{X}$ at every point in $\partial\mathbb{X}$. Hereafter $\partial\mathbb{X}$ is assumed to be $\mathscr{C}^1$-smooth. Now the derivation of the inner-variational equations goes as follows. We write the integral for $\mathscr{E}[h_\epsilon]$, perform change of variables $x = \varphi_\epsilon^{-1}(z)$, and apply $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0}$. It results in the following integral equation

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}\mathscr{E}[h_\epsilon] = \int_{\mathbb{X}} \Big[-\langle \mathbf{E}_x \mid v\rangle + \langle \mathbf{E}_\xi \mid Dh\, Dv\rangle - \mathbf{E}\cdot \operatorname{Tr} Dv \Big] \mathrm{d}x = 0. \tag{4.2}$$

The reader might want to compare this procedure with the proof of Lemma 4.3 below, where we work out a fairly detailed computation for 1-dimensional integrals.

Now recall Green's formula for a $\mathscr{C}^1$-smooth matrix field $\mathbf{M} : \overline{\mathbb{X}} \to \mathbb{R}^{n\times n}$ and a $\mathscr{C}^1$-smooth vector field $v : \overline{\mathbb{X}} \to \mathbb{R}^n$.

$$\int_{\mathbb{X}} \langle \mathbf{M} \mid Dv\rangle \,\mathrm{d}x = -\int_{\mathbb{X}} \langle \operatorname{Div}\mathbf{M} \mid v\rangle \,\mathrm{d}x + \int_{\partial\mathbb{X}} \big\langle \mathbf{M}(x)\, \mathfrak{n}(x) \mid v(x)\big\rangle \,\mathrm{d}\sigma(x) \tag{4.3}$$

where $\mathfrak{n}(x)$ is the outer normal unit vector field and $\mathrm{d}\sigma(x)$ stands for the $(n-1)$-measure on $\partial\mathbb{X}$. The divergence of a matrix field, denoted by $\operatorname{Div}\mathbf{M}$, is a vector field whose coordinates are obtained as divergence of the column vectors of $\mathbf{M}$. To put Green's formula into effect, we write

$$\langle \mathbf{E}_\xi \mid Dh\, Dv\rangle - \mathbf{E}\cdot \operatorname{Tr} Dv = \langle\, D^*h\mathbf{E}_\xi - \mathbf{E}\cdot \mathbf{I} \mid Dv\rangle \overset{\text{def}}{=\!=} \langle \mathbf{M} \mid Dv\rangle.$$

This gives the following integral equation

$$\int_{\mathbb{X}} \Big\langle \mathbf{E}_x + \operatorname{Div}[D^*h\mathbf{E}_\xi - \mathbf{E}\cdot\mathbf{I}] \Big| v\Big\rangle \,\mathrm{d}x = \int_{\partial\mathbb{X}} \langle\, [D^*h\mathbf{E}_\xi - \mathbf{E}\cdot\mathbf{I}]\, \mathfrak{n}(x) \big| v(x)\rangle \,\mathrm{d}\sigma(x).$$

First we test it with $v \in \mathscr{C}_0^\infty(\mathbb{X}, \mathbb{R}^n)$, which amounts to setting $\varphi_\epsilon(x) = x + \epsilon v(x)$. In this way we arrive at the inner-variational equation on $\mathbb{X}$.

$$\mathbf{E}_x + \operatorname{Div}[D^*h\mathbf{E}_\xi - \mathbf{E}\cdot\mathbf{I}] = 0. \tag{4.4}$$

Thus we are left with the boundary integral. Since $\langle \mathfrak{n}(x) \mid v(x)\rangle = 0$, the equation reduces to

$$\int_{\partial\mathbb{X}} \langle [D^*h\mathbf{E}_\xi\,]\, \mathfrak{n}(x) \;\big|\; v(x)\rangle \,\mathrm{d}\sigma(x) = 0. \tag{4.5}$$

At this point it is important to realize that every tangent vector field $v_\circ(x)$ on $\partial\mathbb{X}$ that is supported in a sufficiently small neighborhood of a given point $x_\circ \in \partial\mathbb{X}$ can be accomplished as $v(x)$ for some variational family $\{\varphi_\epsilon\}_{\epsilon\sim 0}$. This means that (4.5) is possible only when $[D^*h\mathbf{E}_\xi\,]\,\mathfrak{n}(x)$ is orthogonal to $\mathbf{T}_x\partial\mathbb{X}$ at every point $x \in \partial\mathbb{X}$. In symbols:

$$D^*h(x)\mathbf{E}_\xi(x, h, Dh) : \mathbf{N}_x\partial\mathbb{X} \to \mathbf{N}_x\partial\mathbb{X}. \tag{4.6}$$

Equivalently, for the transpose matrix, it reads as,

$$\mathbf{E}^*_\xi(x, h, Dh)\, Dh(x) : \mathbf{T}_x \partial\mathbb{X} \to \mathbf{T}_x \partial\mathbb{X}. \tag{4.7}$$

The above computation maybe summarized as follows:

Proposition 4.1. *Let* $\mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{n\times n}_+ \to \mathbb{R}$ *be continuous and* $\mathscr{C}^1$*-smooth with respect to the variables* $x \in \mathbb{X}$ *and* $\xi \in \mathbb{R}^{n\times n}_+$. *Then the traction free energy minimal solution satisfies both the inner-variational equation* (4.4) *in* $\mathbb{X}$ *and the boundary conditions* (4.6)*; the latter being equivalent to* (4.7).

It is generally a highly nontrivial question as to whether the variational equation (4.4) together with the boundary condition (4.6) suffice for h to be a traction-free minimizer. An affirmative answer is given in [31] for the Dirichlet energy. However, this result does not seem to generalize for all traction-free problems. We reserve the term *traction-free stationary solution* to the mappings $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ which satisfy the equations (4.4) and (4.6).

Proposition 4.2. *Let* $\mathbf{L}(x, f, Df)\,\mathrm{d}x$ *be a free Lagrangian defined on a given class of Sobolev homeomorphisms* $f : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$. *Then every* f *in this class is a traction-free stationary solution.*

That is to say, the variational equation (4.4) and the boundary condition (4.6) are freely satisfied. It is desirable to explore Proposition 4.2 and check to what extent it characterizes free Lagrangians.

4.2. Inner variation of 1-dimensional integrals

Consider a general variational integral

$$\mathcal{E}[H] = \int_a^b L(t, H, \dot{H})\,\mathrm{d}t \tag{4.8}$$

subject to all absolutely continuous functions $H : [a, b] \xrightarrow{\text{onto}} [c, d]$ for which $\dot{H}(t) > 0$ almost everywhere. Here we assume that the stored energy function

$$L : [a, b] \times [c, d] \times \mathbb{R}_+ \to \mathbb{R} \qquad \big(E = E(t, H, \xi)\big)$$

is continuous together with its partial derivatives $L_t \overset{\text{def}}{=\!=} \frac{\partial L}{\partial t}$ and $L_\xi \overset{\text{def}}{=\!=} \frac{\partial L}{\partial \xi}$. No regularity of L with respect to H-variable is needed.

Lemma 4.3. *The inner variational equation for* $\mathcal{E}[H]$ *takes the form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\xi L_\xi - L\big) + L_t = 0 \tag{4.9}$$

where the operator $\frac{\mathrm{d}}{\mathrm{d}t}$ *is understood in the sense of distributions.*

For $\mathscr{C}^1$-smooth solutions of (4.9), the function $\dot{H}L_\xi(t, H, \dot{H}) - L(t, H, \dot{H})$ turns out to be $\mathscr{C}^1$-smooth. Such a gain of regularity is typical for variational type PDEs.

Proof. Choose and fix a test function $\varphi \in \mathscr{C}^\infty_\circ(r, R)$ and consider the diffeomorphism $t \to t + \varepsilon\varphi(t)$ of (a, b) onto itself, with sufficiently small $\varepsilon \in \mathbb{R}$. The inner variation of H is defined by $H_\varepsilon(t) = H(t + \varepsilon\varphi(t))$. We have $\dot{H}_\epsilon(t) = [1 + \varepsilon\dot{\varphi}(t)] \cdot \dot{H}(t + \varepsilon\varphi(t))$

$$\mathcal{E}[H_\varepsilon] = \int_a^b L(t, H(t + \varepsilon\varphi), (1 + \varepsilon\dot{\varphi})\dot{H}(t + \varepsilon\varphi))\,\mathrm{d}t.$$

We are going to compute $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}[H_\varepsilon]$ at $\varepsilon = 0$ so the only terms up to order $O(\varepsilon)$ count. But before, let us perform an integration by substitution; namely, $\tau = t + \varepsilon\varphi(t)$. Hence, modulo higher powers of ε, we write $t \approx \tau - \varepsilon\varphi(\tau)$, $\varepsilon\dot{\varphi}(t) \approx \varepsilon\dot{\varphi}(\tau)$, and $\mathrm{d}t \approx [1 - \varepsilon\dot{\varphi}(\tau)]\,\mathrm{d}\tau$. Therefore,

$$\mathcal{E}[H_\varepsilon] \approx \int_a^b L(\tau - \varepsilon\varphi, H(\tau), (1 + \varepsilon\dot{\varphi}(\tau))\dot{H}(\tau))[1 - \varepsilon\dot{\varphi}(\tau)]\,\mathrm{d}\tau.$$

Now the condition $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\mathcal{E}[H_\varepsilon] = 0$ reads as

$$0 = \int_a^b \left[-\varphi L_t + \dot{\varphi}\dot{H}L_\xi - \dot{\varphi}L\right]\,\mathrm{d}\tau$$

which is none other than the integral form of the distributional equation (4.9). $\square$

We now come to a useful special case of Lemma 4.3.

Lemma 4.4. *Consider the variational integral of the form*

$$\mathcal{E}[H] = \int_a^b \mathcal{P}(\eta)\, t^{n-1}\mathrm{d}t \qquad \eta_H \stackrel{\mathrm{def}}{=\!=} \frac{t\dot{H}}{H}.$$

Here we assume that the function $\mathcal{P} \colon \mathbb{R}_+ \to \mathbb{R}$ *is* $\mathscr{C}^1$*-smooth. Then* (4.9) *reduces to the first order ODEs*

$$t^n\dot{\mathcal{P}}(\eta) \equiv \textit{constant}, \qquad \textit{where} \quad \eta(t) = \frac{t\dot{H}}{H}. \tag{4.10}$$

Proof. Setting $L(t, H, \dot{H}) = \mathcal{P}\big(\frac{t\dot{H}}{H}\big)\, t^{n-1}$ in (4.9) the reader might want to verify that

$$\frac{\eta}{t}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^n\, \dot{\mathcal{P}}(\eta)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\big(\xi L_\xi - L\big) + L_t = 0.$$

$\square$

5. Radial stretchings

In this section we discuss in more detail the radial mappings $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ between open annuli $\mathbb{A} = A(r, R)$ and $\mathbb{A}^* = A(r_*, R_*)$

$$f(x) = F(|x|)\frac{x}{|x|} \qquad F : [r, R] \xrightarrow{\text{onto}} [r_*, R_*] \qquad \begin{cases} F(r) = r_* \\ F(R) = R_*. \end{cases} \tag{5.1}$$

Recall the notation

$$\mathscr{R}(\mathbb{A}, \mathbb{A}^*) \stackrel{\text{def}}{=\!=} \Big\{\text{radial stretchings } f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^* \text{ in } \mathscr{W}^{1,1}(\mathbb{A}, \mathbb{A}^*)\Big\}. \tag{5.2}$$

5.1. The elasticity quotient

When speaking of radial stretchings the concept of elasticity quotient arises naturally. It tells us, among other things, how the radial derivative compares with the derivatives in the spherical directions. Choose and fix a point $x \in \mathbb{A}$ that lies on a sphere $\mathbb{S}^{n-1}_\rho = \{x;\, |x| = \rho\}$. The unit normal vector at this point is $\mathbf{N} = \frac{x}{|x|}$. Now consider an arbitrary unit tangent vector $\mathbf{T}$ to $\mathbb{S}^{n-1}_\rho$ at x. This is a unit vector orthogonal to $\frac{x}{|x|}$. We have well defined directional derivatives $f_{\mathbf{N}} \stackrel{\text{def}}{=\!=} Df(x)\mathbf{N} = \dot{F}(|x|)\mathbf{N}$ and $f_{\mathbf{T}} \stackrel{\text{def}}{=\!=} Df(x)\mathbf{T} = \frac{F(|x|)}{|x|}\,\mathbf{T}$. To verify this latter formula, consult with the forthcoming equation (5.6). The *elasticity quotient* does not depend on the choice of the tangent vector $\mathbf{T}$. It is defined by the rule:

$$\eta_F = \eta_F(|x|) \stackrel{\text{def}}{=\!=} \frac{|f_{\mathbf{N}}|}{|f_{\mathbf{T}}|} = \frac{|x|\dot{F}(|x|)}{F(|x|)}. \tag{5.3}$$

Directly from this definition it follows that $\eta_{Id} \equiv 1$ and we have the following composition rule:

$$\eta_{G\circ F}(|x|) = \eta_G(|y|) \cdot \eta_F(|x|) \qquad |y| = F(|x|). \tag{5.4}$$

In particular, if $H = H(|y|)$ is the inverse of $F = F(|x|)$ then

$$\eta_H(|y|) = \frac{1}{\eta_F(|x|)} \qquad \text{where} \quad |y| = F(|x|). \tag{5.5}$$

The elasticity quotient of a radial stretching $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ tells us something about conformal modulus of $\mathbb{A}^*$ in relation to that of $\mathbb{A}$.

$$\operatorname{Mod} \mathbb{A}^* = \int_r^R \frac{\dot{F}(t)}{F(t)}\,\mathrm{d}t = \int_r^R \eta_F(t)\frac{\mathrm{d}t}{t} \begin{cases} < \operatorname{Mod}\mathbb{A} & \text{if } \eta_F < 1 \text{ in } \mathbb{A} \\ = \operatorname{Mod}\mathbb{A} & \text{if } \eta_F \equiv 1 \text{ in } \mathbb{A} \\ > \operatorname{Mod}\mathbb{A} & \text{if } \eta_F > 1 \text{ in } \mathbb{A}. \end{cases}$$

Actually, the ratio $\operatorname{Mod}\mathbb{A}^*/\operatorname{Mod}\mathbb{A}$ is none other than the integral average of η_F with respect to the conformal density in $\mathbb{A}$; namely,

$$\frac{\operatorname{Mod}\mathbb{A}^*}{\operatorname{Mod}\mathbb{A}} = \frac{1}{\mu(\mathbb{A})}\int_{\mathbb{A}} \eta_F(x)\,\mathrm{d}\mu \qquad \mathrm{d}\mu \;=\; \frac{\mathrm{d}x}{|x|^n}.$$

We shall see that radially minimal stretchings (having smallest $\mathscr{L}^p$-mean distortion) always fall into exactly one of the above three cases.

5.1.1. *Some computation*

The differential matrix of a radial map $f(x) = F(|x|)\frac{x}{|x|}$ takes the form

$$Df(x) = \frac{F}{|x|}\left[\mathbf{I} + (\eta_F - 1)\frac{x\otimes x}{|x|^2}\right] \qquad \text{where} \quad \eta_F \overset{\text{def}}{=\!=} \frac{|x|\dot F(|x|)}{F(|x|)} \tag{5.6}$$

where the tensor product of vectors $x = (x_1, \dots, x_n)$ is the matrix

$$x\otimes x = \left[x_i x_j\right]_{i,j=1,\dots,n}$$

see [23, Section 6.5]. Hence its matrix of cofactors is given by:

$$D^\sharp f(x) = \frac{F^{\,n-1}}{|x|^{n-1}}\left[\eta_F\mathbf{I} + (1-\eta_F)\frac{x\otimes x}{|x|^2}\right]. \tag{5.7}$$

To see this, just verify Cramer's rule,

$$[Df]\cdot[D^\sharp f] = \left(\frac{F^n}{|x|^n}\,\eta_F\right)\mathbf{I} = \left(\frac{F^{n-1}}{|x|^{n-1}}\,\dot F\right)\mathbf{I} \qquad \text{where} \quad \frac{F^{n-1}}{|x|^{n-1}}\,\dot F = \det Df.$$

It has already been observed in (3.9) that the normal derivative of a radial stretching is the derivative of its strain function. However, as expected, the spherical gradient does not depend on the strain function at all. Precisely, for all $r < \rho < R$, we have

$$f_{\mathbf{S}} = \left\langle \frac{x}{|x|}\,\middle|\,\frac{[D^\sharp f]\,f}{|f|^{n-1}}\right\rangle = \frac{1}{|x|^{n-1}} \quad \text{so} \quad \int_{\mathbb{S}^{n-1}_\rho} f_{\mathbf{S}}(x)\mathrm{d}\sigma(x) = \omega_{n-1}. \tag{5.8}$$

The singular values of $Df(x)$ (*principal stretchings of* f) are:

$$\underbrace{\frac{F}{|x|},\dots,\frac{F}{|x|}}_{n-1},\,\dot F.$$

The operator norm of Df and its cofactor matrix are:

$$\|Df(x)\| = \max\left\{\frac{F}{|x|},\,\dot F\right\} \quad \text{and} \quad \|D^\sharp f(x)\| = \max\left\{\frac{F^{n-1}}{|x|^{n-1}},\,\frac{F^{n-2}\dot F}{|x|^{n-2}}\right\}.$$

For later use, we record the following formulas:

- If $\eta_F(x) \geqslant 1$ then $\|Df(x)\| = \dot F(|x|) = |f_{\mathbf{N}}| = |f|_{\mathbf{N}}$;
- If $\eta_F(x) \leqslant 1$ then $\|D^\sharp f(x)\| = \frac{F^{n-1}}{|x|^{n-1}} = f_{\mathbf{S}}|f|^{n-1}$.

5.1.2. *Distortions of radial mappings*

The Frobenius norm of the ℓ-th exterior power of Df is computed as:

$$\left|D_\ell^\sharp f(x)\right|^2 = \binom{n}{\ell}\left[\frac{\ell}{n}\dot{F}^2 + \left(1-\frac{\ell}{n}\right)\frac{F^2}{t^2}\right]\left(\frac{F}{t}\right)^{2\ell-2} \qquad t = |x|.$$

Hence

$$\mathcal{K}_\ell f = \frac{\left[\frac{\ell}{n}\,\eta_F^2 + 1 - \frac{\ell}{n}\right]^{\frac{n}{2n-2\ell}}}{\eta_F^{\frac{\ell}{n-\ell}}} \qquad \eta_F = \frac{t\dot{F}(t)}{F(t)} \tag{5.9}$$

and its p-power is:

$$\mathcal{K}_\ell^p f \overset{\text{def}}{=\!=} [\mathcal{K}_\ell f]^p = \frac{\left[\frac{\ell}{n}\,\eta_F^2 + 1 - \frac{\ell}{n}\right]^{\frac{np}{2n-2\ell}}}{\eta_F^{\frac{\ell p}{n-\ell}}}. \tag{5.10}$$

5.2. Minimal radial stretchings

It is natural to first establish the existence of radial mappings of smallest mean distortion and examine their basic properties. Precise conclusions of this section are stated in Theorem 5.10 for $p > 1$ and in Theorem 5.11 for $p = 1$.

5.2.1. *The functions $\mathcal{P} = \mathcal{P}(\eta)$ and $\mathcal{Q} = \mathcal{Q}(\eta)$*

The key to the proof of Theorems 5.10 and 5.11 is the convexity (with respect η_F) of the expression in (5.10).

Lemma 5.1. *The following function*

$$\mathcal{P}(\eta) = \mathcal{P}_{\ell,\,p}(\eta) \overset{\text{def}}{=\!=} \frac{\left[\frac{\ell}{n}\,\eta^2 + 1 - \frac{\ell}{n}\right]^{\frac{np}{2n-2\ell}}}{\eta^{\frac{\ell p}{n-\ell}}} \qquad 0 < \eta < \infty \tag{5.11}$$

is convex (*for every $p \geqslant 1$ and $1 \leqslant \ell \leqslant n-1$*). *See the graphs of* $\mathcal{P}$ *in Figure* 5.1.

We denote by $C = C(\ell, n) = \left(\frac{\ell}{n}\right)^{\frac{n}{2n-2\ell}}$ the slope of the asymptote to the graph of $P_\ell(\eta)$ with $p = 1$.

Proof. Direct computation shows that the first derivative of $\mathcal{P}$ is given by:

$$\dot{\mathcal{P}}(\eta) = \frac{\ell p}{n}\frac{\left[\frac{\ell}{n}\,\eta^2 + 1 - \frac{\ell}{n}\right]^{\frac{np}{2n-2\ell}-1}}{\eta^{\frac{\ell p}{n-\ell}+1}}\left(\eta^2 - 1\right) \overset{\text{def}}{=\!=} \mathcal{Q}_{\ell,p}(\eta) = \mathcal{Q}(\eta). \tag{5.12}$$

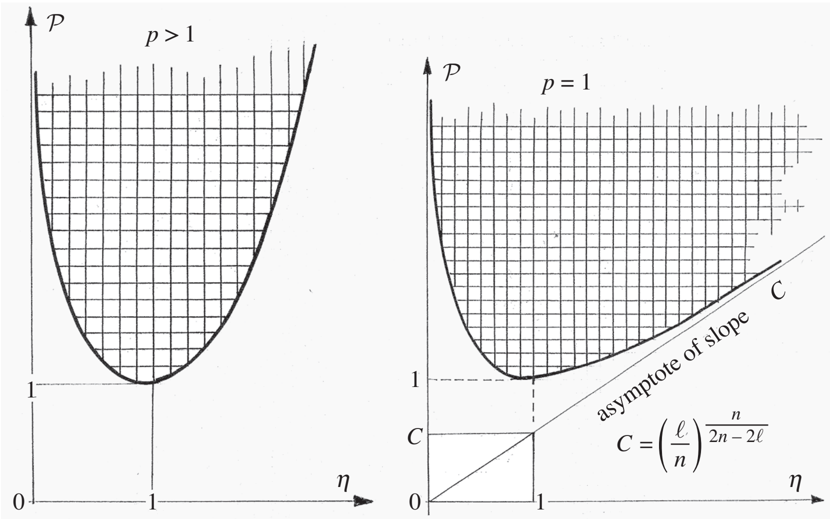

Figure 5.1. The convex functions $\mathcal{P}_{\ell,p}$.

For $p = 1$ the second derivative is positive; indeed, we have

$$\ddot{\mathcal{P}}(\eta) = \frac{\ell}{n}\left[\frac{\ell}{n} + \left(1 - \frac{\ell}{n}\right)\eta^{-2}\right]^{\frac{n}{2n-2\ell}-2}\left(1+\eta^{-2}\right)\eta^{-3}. \tag{5.13}$$

This implies that $\mathcal{P}(\eta)$ is convex for all $p \geqslant 1$. Hence the lemma follows. $\square$

Corollary 5.2. *The function $\mathcal{Q} = \dot{\mathcal{P}}(\eta)$ is strictly increasing.*
See Figure 5.2 *below.*

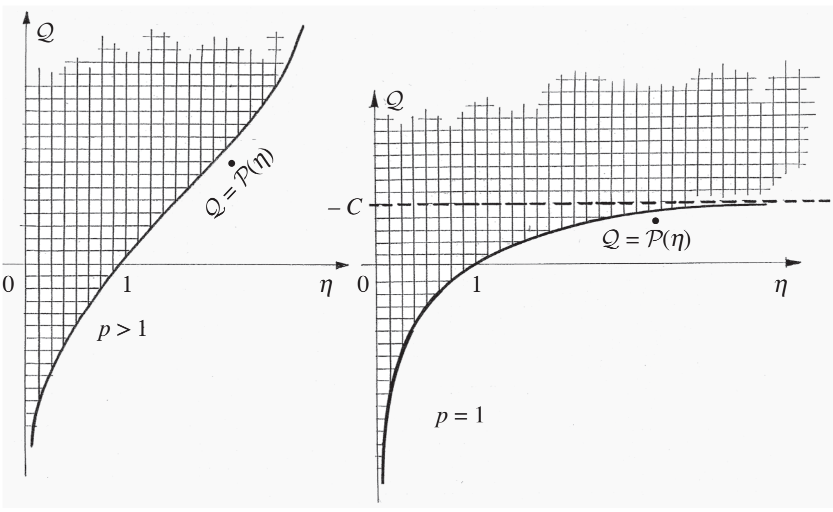

Figure 5.2. The increasing function $\mathcal{Q}$.

Integration in polar coordinates results in a line integral for the $\mathscr{L}^p$-mean distortion

$$\int_{\mathbb{A}} [\mathcal{K}_\ell f(x)]^p \,\mathrm{d}x = \omega_{n-1} \int_r^R \mathcal{P}_{\ell,\,p}\,(\eta_F(t))\, t^{n-1}\,\mathrm{d}t. \tag{5.14}$$

By Lemma 4.4 we obtain:

Corollary 5.3. *Suppose $f = F(|x|)\frac{x}{|x|}$ has smallest $\mathscr{L}^p$-mean distortion among all radial mappings of an annulus $\mathbb{A} \overset{\text{onto}}{\longrightarrow} \mathbb{A}^*$. Then there is a constant $\lambda \in (-\infty, +\infty)$ such that*

$$t^n \mathcal{Q}(\eta_F) \equiv \lambda \qquad (\textit{for all } r < t < R). \tag{5.15}$$

5.2.2. *Radial stationary solutions*

A radial stretching $f = F(|x|)\frac{x}{|x|}$ which satisfies Equation (5.15) will hereafter be referred to as *radial stationary solution* or *stationary solution* for short. We shall denote it by f_λ whenever the parameter λ needs to be indicated. The foremost implication from (5.12) and (5.15) is that $1 - \eta_F$ does not change sign.

Lemma 5.4. *For every stationary solution it holds that,*

- $0 < \eta_F(t) < 1$ *for all* $r < t < R$ $\quad (\lambda < 0)$;
- $0 < \eta_F(t) = 1$ *for all* $r < t < R$ $\quad (\lambda = 0)$;
- $1 < \eta_F(t) < \infty$ *for all* $r < t < R$ $\quad (\lambda > 0)$.

Next notice that

$$\operatorname{Mod}\mathbb{A}^* = \log\frac{R_*}{r_*} = \int_r^R \frac{\dot F(t)}{F(t)}\,\mathrm{d}t = \int_r^R \eta_F(t)\frac{\mathrm{d}t}{t} = \operatorname{Mod}\mathbb{A} + \int_r^R \big[\eta_F(t) - 1\big]\frac{\mathrm{d}t}{t}.$$

In summary:

Proposition 5.5. *There are exactly three possible cases:*

- $\operatorname{Mod}\mathbb{A}^* < \operatorname{Mod}\mathbb{A}$; *$f_\lambda$ is a Conformal Contraction*

$$0 < \eta_F(t) < 1 \qquad \textit{for all } r < t < R \qquad (\lambda < 0);$$

- $\operatorname{Mod}\mathbb{A}^* = \operatorname{Mod}\mathbb{A}$; *$f_\lambda$ is Conformal*

$$\eta_F(t) = 1 \qquad \textit{for all } r < t < R \qquad (\lambda = 0);$$

- $\operatorname{Mod}\mathbb{A}^* > \operatorname{Mod}\mathbb{A}$; *$f_\lambda$ is a Conformal Expansion*

$$1 < \eta_F(t) < \infty \qquad \textit{for all } r < t < R \qquad (\lambda > 0).$$

Remark 5.6. In general, stationary solutions need not enjoy the least mean distortion (just vice versa). We shall nevertheless use equation (5.15) as a tool to foresee the minimal mappings. The idea is to first work out the algebraic equation (5.15) for $\eta = \eta_F$ and then solve the differential equation $\eta(t) = \frac{t\dot F(t)}{F(t)}$ for the strain function $F = F(t)$. The elasticity quotient $\eta(t)$ determines F uniquely up to a multiplicative constant (that is; up to the conformal modulus of the target).

5.2.3. *The function* $\Phi = \Phi(\tau)$

Stationary solutions can be expressed explicitly by integral formulas which involve the inverse function to $\mathcal{Q} = \mathcal{Q}_{\ell,p}$. This function, denoted by $\Phi = \Phi_{\ell,p}$, will play a key role in the sequel.

$$\mathcal{Q}\big(\Phi(\tau)\big) \equiv \tau \qquad \text{or} \qquad \Phi(\mathcal{Q}(\eta)) \equiv \eta \qquad \text{for } 0 < \eta < \infty. \tag{5.16}$$

The range of arguments τ depends on whether $p > 1$ or $p = 1$.

Case $p > 1$. The function $\Phi\colon (-\infty, \infty) \to \mathbb{R}_+$ is strictly increasing from 0 to ∞. See Figure 5.3 below.

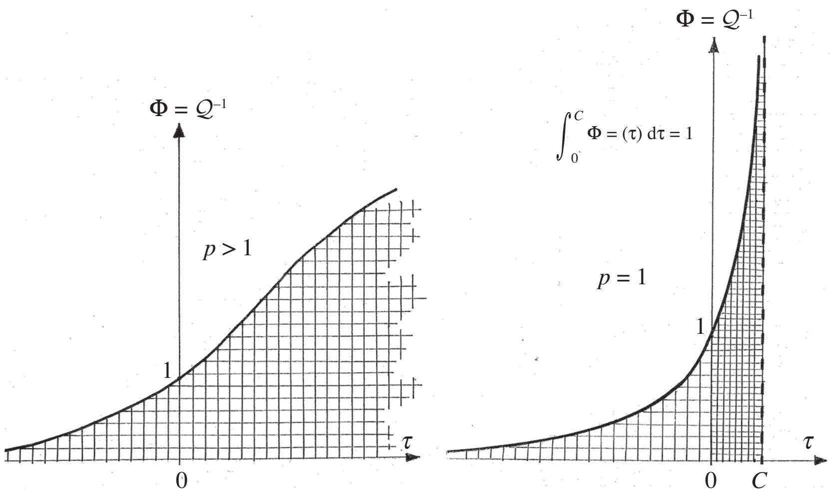

Figure 5.3. The inverse of $\mathcal{Q}$ and constant C.

Case $p = 1$. The function $\Phi\colon (-\infty, C) \to \mathbb{R}_+$ is strictly increasing from $0 = \Phi(-\infty)$ to $\infty = \Phi(C)$. Recall the slope constant $C = \left(\frac{\ell}{n}\right)^{\frac{n}{2n-2\ell}}$, see Figure 5.3. Let us note that

$$\int_0^C \Phi(\tau)\mathrm{d}\tau = 1. \tag{5.17}$$

Proof. To see this we observe that $\lim_{T\to\infty}[T\,\mathcal{Q}(T) - \mathcal{P}(T)] = 0$ and $\mathcal{Q}(1) = 0$. Integrating by substitution, and then by parts, yields

$$\begin{aligned}
&\lim_{L\to C}\int_0^L \Phi(\tau)\mathrm{d}\tau = \lim_{T\to\infty}\int_1^T \Phi(\mathcal{Q}(\eta))\,\dot{\mathcal{Q}}(\eta)\mathrm{d}\eta = \lim_{T\to\infty}\int_1^T \eta\,\dot{\mathcal{Q}}(\eta)\mathrm{d}\eta \\
&= \lim_{T\to\infty}\left[\eta\mathcal{Q}(\eta)\Big|_1^T - \int_1^T \mathcal{Q}(\eta)\mathrm{d}\eta\right] = \lim_{T\to\infty}\left[\eta\mathcal{Q}(\eta)\Big|_1^T - \mathcal{P}(\eta)\Big|_1^T\right] \\
&= \mathcal{P}(1) = 1.
\end{aligned}$$

□

5.2.4. *The family* $\{f_\lambda\}$ *of stationary solutions*

Throughout this text the domain annulus $\mathbb{A} = \{x : r < |x| < R\}$ is fixed whereas the target annuli $\mathbb{A}^* = \mathbb{A}^*_\lambda = \{y : r_*(\lambda) < |y| < R_*\}$ will vary. But their outer radius R_* remains fixed as well.

Proposition 5.7. *The solutions* $f_\lambda(x) = F_\lambda(|x|)\frac{x}{|x|}$, $F_\lambda(R) = R_*$, *to the inner variational equation* (5.15) *are given explicitly by the following formula*

$$F_\lambda(t) = R_* \exp\left[-\int_t^R \Phi\left(\frac{\lambda}{s^n}\right)\frac{\mathrm{d}s}{s}\right] \qquad r \leqslant t \leqslant R. \tag{5.18}$$

Hereafter the parameter λ *runs as follows*

$$-\infty < \lambda < +\infty \qquad \textit{when} \quad p > 1$$

$$-\infty < \lambda \leqslant \lambda_{\max} \overset{\text{def}}{=\!=} C\, r^n = \left(\frac{\ell}{n}\right)^{\frac{n}{2n-2\ell}} r^n \qquad \textit{when} \quad p = 1.$$

Proof. It follows directly from (5.18) that $\frac{t\,\dot{F}_\lambda(t)}{F_\lambda(t)} = \Phi\left(\frac{\lambda}{t^n}\right)$. Thus,

$$\Phi(\lambda\,|x|^{-n}) = \eta_{F_\lambda}(|x|). \tag{5.19}$$

Applying the inverse function $\mathcal{Q}$ yields the desired variational equation.

$$\lambda\,|x|^{-n} \equiv \mathcal{Q}\big(\eta_{F_\lambda}\big) = \frac{\ell p}{n}\,\frac{\left[\frac{\ell}{n}\,\eta^2_{F_\lambda} + 1 - \frac{\ell}{n}\right]^{\frac{np}{2n-2\ell}-1}}{\big(\eta_{F_\lambda}\big)^{\frac{\ell p}{n-\ell}+1}}\,\Big(\eta^2_{F_\lambda} - 1\Big). \tag{5.20}$$

□

5.2.5. *The annuli* $\mathbb{A}^*_\lambda$

The target annulus for f_λ will be denoted by

$$\mathbb{A}^*_\lambda = f_\lambda(\mathbb{A}) = \{y : r_*(\lambda) < |y| < R_*\}. \tag{5.21}$$

Its inner radius is determined uniquely from the equation

$$\int_r^R \Phi\left(\frac{\lambda}{s^n}\right)\frac{\mathrm{d}s}{s} = \log\frac{R_*}{r_*(\lambda)} = \operatorname{Mod}\mathbb{A}^*_\lambda. \tag{5.22}$$

Note the inclusions

$$\mathbb{A}^*_\alpha \subset \mathbb{A}^*_\beta \qquad \text{whenever } \alpha \leqslant \beta.$$

Case $p > 1$. As λ increases from $-\infty$ to $+\infty$ the annuli $\mathbb{A}^*_\lambda$ increase continuously from the degenerate annulus $\mathbb{A}^*_{-\infty} \overset{\text{def}}{=\!=} \{y\,;\ |y| = R_*\}$ to the punctured ball $\mathbb{A}^*_\infty \overset{\text{def}}{=\!=} \{y\,;\ 0 < |y| < R_*\}$.

Case $p = 1$. The largest annulus in the family $\{\mathbb{A}^*_\lambda\}$ is denoted by

$$\mathbb{A}^*_{\max} = \{x;\; r^+_* < |x| < R_*\}\ ,\ \text{it corresponds to } \lambda = C\, r^n. \tag{5.23}$$

Its conformal modulus is determined from the equation

$$\begin{aligned} \operatorname{Mod} \mathbb{A}^*_{\max} &\stackrel{\text{def}}{=\!=} \int_r^R \Phi\left(\frac{C\, r^n}{s^n}\right) \frac{\mathrm{d}s}{s} \\ &= \int_{Cr^n/R^n}^{C} \Phi(\tau)\, \frac{\mathrm{d}\tau}{\tau} \leqslant \frac{R^n}{Cr^n} \int_0^C \Phi(\tau)\, \mathrm{d}\tau = \frac{R^n}{Cr^n} \end{aligned} \tag{5.24}$$

by Equation (5.17).

A close inspection of the formula (5.12) in dimension $n = 2$ with $p = 1$ and $\ell = 1$, reveals that

$$Q(\eta) = \frac{\eta^2 - 1}{2\eta^2}.$$

Solving the equation $Q(\eta) = \tau$ we find the inverse function

$$\eta \stackrel{\text{def}}{=\!=} \Phi(\tau) = \frac{1}{\sqrt{1 - 2\tau}} \quad \text{for} \quad -\infty < \tau < \frac{1}{2} \quad \text{and} \quad \lambda_{\max} = \frac{1}{2} r^2.$$

Hence, the maximal annulus is determined by so-called *Nitsche condition*

$$\frac{R}{r} = \frac{1}{2}\left(\frac{R_*}{r^+_*} = \frac{r^+_*}{R_*}\right).$$

Let us display the notation introduced above; as it will be frequently used throughout this text.

$$f_{\max} = f_{\lambda_{\max}} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*_{\max} \qquad f_{\max} = F_{\max}(|x|) \frac{x}{|x|}. \tag{5.25}$$

We also summarize it as follows:

Corollary 5.8. *Let an integer $1 \leqslant \ell \leqslant n - 1$ and exponent $p > 1$ be given. For every pair of annuli $\mathbb{A}$ and $\mathbb{A}^*$ there exists unique stationary solution $f_\lambda : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$. The parameter $-\infty < \lambda < \infty$ is determined by the equation*

$$\int_r^R \Phi\left(\frac{\lambda}{s^n}\right) \frac{\mathrm{d}s}{s} = \operatorname{Mod} \mathbb{A}^*.$$

The same holds when $p = 1$, provided $\mathbb{A}^*$ is not too fat; namely,

$$\operatorname{Mod} \mathbb{A}^* \leqslant \int_r^R \Phi\left(\frac{\lambda_{\max}}{s^n}\right) \frac{\mathrm{d}s}{s} = \operatorname{Mod} \mathbb{A}^*_{\max} \qquad \lambda_{\max} \stackrel{\text{def}}{=\!=} C\, r^n.$$

For later use, we state the following consequence.

Corollary 5.9. *Let $1 \leqslant \ell \leqslant n-1$ and $p \geqslant 1$ be fixed. Given any closed interval $[\eta_-, \eta_+] \subset (0,1)$ there exist annuli $\mathbb{A}$ and $\mathbb{A}^*$ and the radial stationary solution $f_\lambda \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$, $f_\lambda = F_\lambda(|x|)\frac{x}{|x|}$, such that*

$$\eta_- \leqslant \eta_{F_\lambda}(|x|) \leqslant \eta_+ \qquad \textit{for all } x \in \mathbb{A}. \tag{5.26}$$

Proof. For ℓ and p given, we have the function Φ defined in (5.19) by the rule

$$\eta_{F_\lambda}\big(|x|\big) = \Phi\left(\frac{\lambda}{|x|^n}\right).$$

Choose and fix any negative parameter λ to ensure that $0 < \Phi\big(\frac{\lambda}{|x|^n}\big) < 1$. Inequality (5.26) reads as $\eta_- \leqslant \Phi\big(\frac{\lambda}{|x|^n}\big) \leqslant \eta_+$. Equivalently, applying the inverse function $\mathcal{Q} = \Phi^{-1}$, these inequalities take the form

$$\mathcal{Q}(\eta_-) \leqslant \frac{\lambda}{|x|^n} \leqslant \mathcal{Q}(\eta_+).$$

Since $\mathcal{Q}(\eta) < 0$ for $0 < \eta < 1$, we define the radii of the annulus $\mathbb{A} = \{x \colon r < |x| < R\}$ as follows:

$$r = \left[\frac{\lambda}{\mathcal{Q}(\eta_-)}\right]^{\frac{1}{n}} \qquad \text{and} \qquad R = \left[\frac{\lambda}{\mathcal{Q}(\eta_+)}\right]^{\frac{1}{n}}. \tag{5.27}$$

Note that the conformal modulus of $\mathbb{A}$ does not depend on the choice of the parameter $-\infty < \lambda < 0$; namely, $\operatorname{Mod}\mathbb{A} = \frac{1}{n}\log\frac{\mathcal{Q}(\eta_-)}{\mathcal{Q}(\eta_+)}$. We then take $\mathbb{A}^* = f_\lambda(\mathbb{A})$. $\square$

5.2.6. *Minimal radial mappings*

We now demonstrate that among all radial stretchings the stationary solutions have the least $\mathscr{L}^p$-norm of the $\mathcal{K}_\ell$-distortion. This statement is a converse of Corollary 5.3.

Consider an arbitrary radial stretching $f \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ and an arbitrary stationary solution $f_\lambda \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*_\lambda$. Thus either $\mathbb{A}^*_\lambda = \{y \colon r_*(\lambda) < |y| < R_*\}$ contains $\mathbb{A}^* = \{y \colon r_* < |y| < R_*\}$, when $r_*(\lambda) \leqslant r_*$, or opposite inclusion holds. We begin with the formula:

$$\int_{\mathbb{A}} [\mathcal{K}_\ell f]^p - \int_{\mathbb{A}} [\mathcal{K}_\ell f_\lambda]^p = \omega_{n-1}\int_r^R \big[\mathcal{P}(\eta_F) - \mathcal{P}(\eta_{F_\lambda})\big]\, t^{n-1}\, \mathrm{d}t. \tag{5.28}$$

Since $\mathcal{P}(\eta)$ is convex (for all $0 < \eta < \infty$) we have the tangent line inequality

$$\mathcal{P}(\eta_F) - \mathcal{P}(\eta_{F_\lambda}) \geqslant \mathcal{Q}(\eta_{F_\lambda}) \cdot (\eta_F - \eta_{F_\lambda}) \quad \text{in } \mathbb{A} \qquad \mathcal{Q} = \dot{\mathcal{P}}. \tag{5.29}$$

Equality holds (for all points in $\mathbb{A}$) if and only if $\eta_F \equiv \eta_{F_\lambda}$, which yields $F(t) \equiv$ constant $\cdot\, F_\lambda(t)$. We now appeal to Equation (5.15) to obtain

$$\begin{aligned}\int_{\mathbb{A}} [\mathcal{K}_\ell f]^p - \int_{\mathbb{A}} [\mathcal{K}_\ell f_\lambda]^p &\geqslant \omega_{n-1} \int_r^R t^n \mathcal{Q}(\eta_{F_\lambda}) \left(\frac{\dot F}{F} - \frac{\dot F_\lambda}{F_\lambda} \right) \mathrm{d}t \\ &= \omega_{n-1}\lambda \left[\log \frac{F(R)}{F(r)} - \log \frac{F_\lambda(R)}{F_\lambda(r)} \right] \\ &= \omega_{n-1}\lambda \left[\operatorname{Mod} \mathbb{A}^* - \operatorname{Mod} A^*_\lambda \right].\end{aligned} \tag{5.30}$$

Equality holds if and only if $f = f_\lambda$. Let us first conclude in the case $p > 1$.

Theorem 5.10. *Fix an integer $1 \leqslant \ell \leqslant n-1$, an exponent $p > 1$, and a pair of annuli $\mathbb{A}$, $\mathbb{A}^*$. Let $f_\lambda = F_\lambda(|x|)\frac{x}{|x|}$ be the (unique) stationary solution which takes $\mathbb{A}$ onto $\mathbb{A}^*$. Then for every radial stretching $f\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ we have*

$$\int_{\mathbb{A}} [\mathcal{K}_\ell f]^p \geqslant \int_{\mathbb{A}} [\mathcal{K}_\ell f_\lambda]^p.$$

Equality holds if and only if $f = f_\lambda$.

This theorem extends to the case $p = 1$, but only if the target annulus $\mathbb{A}^*$ is not, conformally, too large. Precisely, we have,

Theorem 5.11. *Fix an integer $1 \leqslant \ell \leqslant n-1$ and a pair of annuli $\mathbb{A} = \{x;\ r < |x| < R\}$, $\mathbb{A}^* = \{y;\ r_* < |y| < R_*\}$, where*

$$\operatorname{Mod} \mathbb{A}^* \leqslant \operatorname{Mod} \mathbb{A}^*_{\max} \overset{\text{def}}{=\!=} \int_r^R \Phi\left(\frac{C\, r^n}{s^n}\right) \frac{ds}{s} \qquad C = \left(\frac{\ell}{n}\right)^{\frac{n}{2n-2\ell}}. \tag{5.31}$$

Let $f_\lambda = \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ be the (unique) stationary solution. Then for every radial stretching $f\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ we have

$$\int_{\mathbb{A}} \mathcal{K}_\ell f \ \geqslant \int_{\mathbb{A}} \mathcal{K}_\ell f_\lambda.$$

Equality holds if and only if $f = f_\lambda$.

Remark 5.12. Note that the latter integral is a function of the ratio $\frac{R}{r}$, thus a function of Mod $\mathbb{A}$. We write it as

$$\int_r^R \Phi\left(\frac{cr^n}{s^n}\right) \frac{\mathrm{d}s}{s} \overset{\text{def}}{=\!=} \Xi(\operatorname{Mod} \mathbb{A}) \tag{5.32}$$

for some function $\Xi\colon [0,\infty) \xrightarrow{\text{onto}} [0,\infty)$ that is continuously increasing and $\Xi(M) > M$ for $M > 0$.

We emphasize that Condition (5.31) is always satisfied in the case of conformal contraction; that is, when $\operatorname{Mod} \mathbb{A}^* \leqslant \operatorname{Mod} \mathbb{A}$. This is because $\operatorname{Mod} \mathbb{A} < \operatorname{Mod} \mathbb{A}^*_{\max}$.

5.2.7. *Squeezing phenomenon for radial minimal mappings*

It is of particular interest to look beyond the maximal annulus. That is, when $\operatorname{Mod}\mathbb{A}^* > \operatorname{Mod}\mathbb{A}^*_{\max} = \Xi(\operatorname{Mod}\mathbb{A})$. We wish to identify the infimum of $\mathscr{L}^1$-norms of the $\mathcal{K}_\ell$-distortion of the radial stretchings $f\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$.

Theorem 5.13. *The infimum is not attained within radial homeomorphisms. Its exact value is given by the formula*

$$\inf_{\mathscr{R}(\mathbb{A},\mathbb{A}^*)} \int_{\mathbb{A}} \mathcal{K}_\ell f = \int_{\mathbb{A}} \mathcal{K}_\ell f_{\max} + \omega_{n-1}\,\lambda_{\max}\left[\operatorname{Mod}\mathbb{A}^* - \operatorname{Mod}\mathbb{A}^*_{\max}\right]. \tag{5.33}$$

Remark 5.14. This theorem will later be extended to all homeomorphisms in $\mathscr{H}^{1,1}(\mathbb{A},\mathbb{A}^*)$, see Theorem 9.6.

Proof. Suppose, on the contrary, that a radial stretching $f(x) = F(|x|)\frac{x}{|x|}$ attains the infimum of $\mathcal{L}^1$-norm of the $\mathcal{K}_\ell$-distortion. Thus, in view of Corollary 5.15, it satisfies the inner-variational equation,

$$\mathcal{Q}(\eta_F) = \frac{\text{const}}{t^n} = \frac{\lambda}{t^n} \quad \text{for some } \lambda \in \mathbb{R}. \tag{5.34}$$

Furthermore, since $\operatorname{Mod}\mathbb{A}^* > \operatorname{Mod}\mathbb{A}^*_{\max} > \operatorname{Mod}\mathbb{A}^*_0 = \operatorname{Mod}\mathbb{A}$, by Proposition 5.5 we infer that f is a conformal expansion; that is, $\eta_F(t) > 1$ for all $r < t < R$. This yields $\mathcal{Q}(\eta_F) > 0$. Therefore, the constant λ in (5.34) is positive and

$$\lambda \leqslant t^n \sup_{\eta\geqslant 1} \mathcal{Q}(\eta) = t^n C \qquad \text{for every } t \in (r,R).$$

In particular, $\lambda \leqslant Cr^n = \lambda_{\max}$. Consequently, $f = f_\lambda$ for some $0 < \lambda \leqslant \lambda_{\max}$. But then $\mathbb{A}^* = f_\lambda(\mathbb{A}) \subset \mathbb{A}^*_{\max}$, a contradiction.

Concerning Equation (5.33), in view of Inequality (5.30) with $\lambda = \lambda_{\max}$, we have

$$\int_{\mathbb{A}} \mathcal{K}_\ell f \geqslant \int_{\mathbb{A}} \mathcal{K}_\ell f_{\max} + \omega_{n-1}\,\lambda_{\max}\left[\operatorname{Mod}\mathbb{A}^* - \operatorname{Mod}\mathbb{A}^*_{\max}\right].$$

Therefore, we need only construct a sequence of radial stretchings $f_j\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$, $f_j(x) = F_j(|x|)\frac{x}{|x|}$ such that

$$\lim_{j\to\infty} \int_{\mathbb{A}} \mathcal{K}_\ell f_j = \int_{A} \mathcal{K}_\ell f_{\max} + \omega_{n-1}\lambda_{\max}\left[\operatorname{Mod}\mathbb{A}^* - \operatorname{Mod}\mathbb{A}^*_{\max}\right].$$

Choose a sequence of radii $r_j \searrow r$ and redesign $F_{\max}$ linearly near the inner radius.

$$F_j(t) = \begin{cases} F_{\max}(t) & \text{for } r_j \leqslant t < R \\ a_j\,t + b_j & \text{for } r \leqslant t < r_j \end{cases}$$

where

$$a_j = \frac{F_{\max}(r_j) - r_*}{r_j - r} \quad \text{and} \quad b_j = \frac{r_* r_j - r F_{\max}(r_j)}{r_j - r}.$$

Hence we find that

$$\frac{\dot{F}_j(t)}{F_j(t)} = \frac{F_{\max}(r_j) - r_*}{(r_j - r)F_j(t)} \leqslant \frac{R_* - r_*}{(r_j - r)r_*} \quad \text{for all } r < t \leqslant r_j. \tag{5.35}$$

Thus

$$\int_{\mathbb{A}} \mathcal{K}_\ell f_j - \int_{\mathbb{A}} \mathcal{K}_\ell f_{\max} = \omega_{n-1} \int_r^{r_j} \mathcal{P}\big(\eta_{F_j}\big) t^{n-1}\, \mathrm{d}t - \omega_{n-1} \int_r^{r_j} \mathcal{P}\big(\eta_{F_{\max}}\big) t^{n-1} \mathrm{d}t.$$

The latter term goes to zero, because $\int_r^R \mathcal{P}(\eta_{F_{\max}})\, t^{n-1}\, \mathrm{d}t < \infty$. Concerning the first term, we notice that

$$\mathcal{P}(\eta_{F_j})(t) = C\, \eta_{F_j} + \mathcal{O}(1) \qquad \text{as } t \searrow r \qquad C = \left(\frac{\ell}{n}\right)^{\frac{n}{2n-2\ell}}$$

by formula (5.11) applied to $p = 1$. Hence

$$\begin{aligned}\lim_{j\to\infty} \int_r^{r_j} \mathcal{P}(\eta_{F_j})\, t^{n-1}\, \mathrm{d}t &= C \lim_{j\to\infty} \int_r^{r_j} \eta_{F_j}(t)\, t^{n-1} \mathrm{d}t \\ &= C \lim_{j\to\infty} \int_r^{r_j} \frac{t^n \dot{F}_j(t)}{F_j(t)}\, \mathrm{d}t = C \lim_{j\to\infty} r^n \int_r^{r_j} \frac{\dot{F}_j(t)}{F_j(t)}\, \mathrm{d}t.\end{aligned}$$

Here, in the integral term next to the last, we replaced t^n by r^n and passed to the limit (by Dominated Convergence Theorem). This replacement was legitimate because, in view of (5.35), we have a uniform bound.

$$0 \leqslant (t^n - r^n)\frac{\dot{F}_j(t)}{F_j(t)} \leqslant n\, r_j^{n-1}(r_j - r) \cdot \frac{R_* - r_*}{(r_j - r)r_*} \leqslant n R^{n-1}\left(\frac{R_*}{r_*} - 1\right).$$

Therefore,

$$\begin{aligned}\lim_{j\to\infty} \omega_{n-1} \int_r^{r_j} \mathcal{P}_\ell(\eta_{F_j})\, t^{n-1}\, \mathrm{d}t &= \omega_{n-1} C \lim_{j\to\infty} \big[\log F_j(r_j) - \log F_j(r)\big] \\ &= \omega_{n-1} C \big[\log F_{\max}(r) - \log r_*\big] \\ &= \omega_{n-1} C \left[\log \frac{R_*}{r_*} - \log \frac{R_*}{F_{\max}(r)}\right] \\ &= \omega_{n-1} C \big[\operatorname{Mod} \mathbb{A}^* - \operatorname{Mod} A^*_{\max}\big].\end{aligned}$$

□

6. Proof of Theorems 1.13 and 1.14. Minimal radial stretchings need not be absolutely minimal

This phenomenon occurs for $n \geqslant 3$ in the case of conformal contraction; that is when $\operatorname{Mod} \mathbb{A}^* < \operatorname{Mod} \mathbb{A}$. Theorems 1.13 and 1.14 follow from:

Theorem 6.1. *Suppose that $n \geqslant 3$, $1 \leqslant \ell \leqslant n-1$ and*

$$1 \leqslant p < p_\ell(n) \stackrel{\text{def}}{=\!=} \frac{n(n+1)^2}{8\,\ell} - (n-1). \tag{6.1}$$

Then there exists a pair of annuli $\mathbb{A}$, $\mathbb{A}^*$ ($\operatorname{Mod} \mathbb{A}^* < \operatorname{Mod} \mathbb{A}$) and a diffeomorphism $g : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ such that

$$\int_{\mathbb{A}} |\, \mathcal{K}_\ell\, g(x)\, |^p \, dx < \inf_{f \in \mathscr{R}(\mathbb{A}, \mathbb{A}^*)} \int_{\mathbb{A}} |\, \mathcal{K}_\ell\, f(x)\, |^p \, dx. \tag{6.2}$$

Here the infimum over radial stretchings $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is actually attained.

Note one particular case: $n = 3$, $\ell = 1$ and $1 \leqslant p < 4$. All other cases are in dimension $n \geqslant 4$. Also note that the exponents defined by (6.1) are decreasing in ℓ.

$$p_1(n) > \cdots > p_\ell(n) > \cdots > p_{n-1}(n). \tag{6.3}$$

For the proof of Theorem 6.1 we need some preliminary considerations.

6.1. Spherical sliding

This is a generalization of the radial stretching in the following way

$$\mathfrak{f}(x) = F(|x|)E(x/|x|) \qquad \text{where } E \colon \mathbb{S}^{n-1} \xrightarrow{\text{onto}} \mathbb{S}^{n-1}. \tag{6.4}$$

As with the radial stretchings, the normal strain F is absolutely continuous on $[r, R]$ and satisfies:

$$r_* = \min_{r \leqslant t \leqslant R} F(t) \leqslant \max_{r \leqslant t \leqslant R} F(t) = R_*.$$

The sliding map $E \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ (*tangential, or spherical, tension*) is continuous and weakly differentiable in the Sobolev sense; that is, $E \in \mathscr{W}^{1,s}(\mathbb{S}^{n-1}, \mathbb{S}^{n-1})$.

The following lemma will help us to answer the questions of uniqueness.

Lemma 6.2. *If a spherical sliding* (6.4) *shares the same principal strechings $\{\frac{F}{|x|}, \ldots, \frac{F}{|x|}, \dot{F}\}$ with the radial map $F(|x|)\frac{x}{|x|}$, then $E \colon \mathbb{S}^{n-1} \xrightarrow{\text{onto}} \mathbb{S}^{n-1}$ is a rotation.*

Proof. For almost every $\omega \in \mathbb{S}^{n-1}$ there is well defined linear tangent map and its adjoint (determined via the standard inner product),

$$[E'(\omega)] : \mathbf{T}_\omega \mathbb{S}^{n-1} \to \mathbf{T}_{E(\omega)} \mathbb{S}^{n-1} \qquad \text{and} \qquad [E'(\omega)]^* : \mathbf{T}_{E(\omega)} \mathbb{S}^{n-1} \to \mathbf{T}_\omega \mathbb{S}^{n-1}.$$

We aim to show that for almost every $\omega \in \mathbb{S}^{n-1}$ the following nonlinear system of PDEs holds:

$$\big[E'(\omega)\big]^* \, \big[E'(\omega)\big] = \mathbf{I} : \mathbf{T}_\omega \mathbb{S}^{n-1} \to \mathbf{T}_\omega \mathbb{S}^{n-1}. \tag{6.5}$$

This just amounts to saying that E is an isometry of $\mathbb{S}^{n-1}$ onto itself , thus a rotation (quintessence of Riemannian geometry). These equations involve only the first order derivatives. The derivation of (6.5) becomes somewhat simpler if we choose and fix an orthonormal basis at $0 \neq x \in \mathbb{R}^n$ $\{T_1, \dots, T_{n-1} \,,\; N\}$, where $N = \frac{x}{|x|}$ whereas the vectors $\{T_1, \dots, T_{n-1}\}$ form an orthonormal basis in $\mathbf{T}_\omega \mathbb{S}^{n-1}$, $\omega = \frac{x}{|x|}$.

Let $E_i = E_i(\omega) \in \mathbf{T}_\omega \mathbb{S}^{n-1}$ denote the T_i -directional derivative of E , $i = 1, \dots, n-1$. Then the linear map $\big[E'(\omega)\big]^* \big[E'(\omega)\big]$ (*tangential Cauchy-Green tensor*) is represented by the Gram matrix, which we denote by $\mathbf{G} \in \mathbb{R}^{(n-1)\times(n-1)}$,

$$\big[E'(\omega)\big]^* \big[E'(\omega)\big] \overset{\text{def}}{=\!=} \mathbf{G} = \begin{pmatrix} \langle E_1 \mid E_1\rangle & \langle E_1 \mid E_2\rangle & \dots & \langle E_1 \mid E_{n-1}\rangle \\ \langle E_2 \mid E_1\rangle & \langle E_2 \mid E_2\rangle & \dots & \langle E_2 \mid E_{n-1}\rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle E_{n-1} \mid E_1\rangle & \langle E_{n-1} \mid E_2\rangle & \dots & \langle E_{n-1} \mid E_{n-1}\rangle \end{pmatrix}.$$

On the other hand, let $\mathfrak{f}_{T_1}, \dots, \mathfrak{f}_{T_{n-1}}, \mathfrak{f}_N$ denote the derivatives of $\mathfrak{f}(x)$ in the directions $\{T_1, \dots, T_{n-1}, N\}$, respectively. Then the full *Cauchy-Green tensor* $D^*\mathfrak{f}(x)$ $D\mathfrak{f}(x)$ is represented by the $n \times n$ Gram matrix

$$D^*\mathfrak{f}(x) D\mathfrak{f}(x) = \begin{pmatrix} \langle \mathfrak{f}_{T_1} \mid \mathfrak{f}_{T_1}\rangle & \langle \mathfrak{f}_{T_1} \mid \mathfrak{f}_{T_2}\rangle & \dots & \langle \mathfrak{f}_{T_1} \mid \mathfrak{f}_{T_{n-1}}\rangle & \langle \mathfrak{f}_{T_1} \mid \mathfrak{f}_N\rangle \\ \langle \mathfrak{f}_{T_2} \mid \mathfrak{f}_{T_1}\rangle & \langle \mathfrak{f}_{T_2} \mid \mathfrak{f}_{T_2}\rangle & \dots & \langle \mathfrak{f}_{T_2} \mid \mathfrak{f}_{T_{n-1}}\rangle & \langle \mathfrak{f}_{T_2} \mid \mathfrak{f}_N\rangle \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \langle \mathfrak{f}_{T_{n-1}} \mid \mathfrak{f}_{T_1}\rangle & \langle \mathfrak{f}_{T_{n-1}} \mid \mathfrak{f}_{T_2}\rangle & \dots & \langle \mathfrak{f}_{T_{n-1}} \mid \mathfrak{f}_{T_{n-1}}\rangle & \langle \mathfrak{f}_{T_{n-1}} \mid \mathfrak{f}_N\rangle \\ \langle \mathfrak{f}_N \mid \mathfrak{f}_{T_1}\rangle & \langle \mathfrak{f}_N \mid \mathfrak{f}_{T_2}\rangle & \dots & \langle \mathfrak{f}_N \mid \mathfrak{f}_{T_{n-1}}\rangle & \langle \mathfrak{f}_N \mid \mathfrak{f}_N\rangle \end{pmatrix}.$$

It should be noted that $\mathfrak{f}_{T_i}(x) = \frac{F(|x|)}{|x|} E_i(\frac{x}{|x|})$ and $\mathfrak{f}_N(x) = \dot F(|x|) E(\frac{x}{|x|})$. Moreover $\langle E \mid E_i\rangle = 0$, because $\langle E \mid E\rangle \equiv 1$. Hence $\langle \mathfrak{f}_{T_i} \mid \mathfrak{f}_N\rangle = 0$, for $i = 1, \dots, n-1$. Therefore the matrix $D^*\mathfrak{f}(x)\, D\mathfrak{f}(x)$ takes the form

$$D^*\mathfrak{f}(x)\, D\mathfrak{f}(x) = \begin{pmatrix} \lceil & \text{------} & \rceil & 0 \\ | & \frac{F^2}{|x|^2}\mathbf{G} & | & 0 \\ \lfloor & \text{------} & \rfloor & 0 \\ & & & \\ 0 & 0 & 0 & \dot F^2 \end{pmatrix}. \tag{6.6}$$

Here, we have assumed that the eigenvalues of this matrix are equal to $\left\{\frac{F^2}{|x|^2}, \dots, \frac{F^2}{|x|^2}, \dot{F}^2\right\}$. This implies that the eigenvalues of $\mathbf{G}$ are all equal to 1 and, because of symmetry, $\mathbf{G} \equiv \mathbf{I}$. The proof of the lemma is complete. □

6.2. Conformal sliding

Rather than discuss spherical sliding in full generality let us consider a particular case of a continuous family of conformal mappings $E_\tau : \mathbb{S}^{n-1} \xrightarrow{\text{onto}} \mathbb{S}^{n-1}$ defined by the rule

$$E_\tau(\omega) = \Pi^{-1}\left[\tau\, \Pi(\omega)\right] \quad \omega = \frac{x}{|x|} \quad 0 < \tau < \infty. \tag{6.7}$$

Here $\Pi \colon \mathbb{S}^{n-1} \to \widehat{\mathbb{R}}^{n-1}$ stands for the stereographic projection of $\mathbb{S}^{n-1}$ onto one-point compactified $(n-1)$-dimensional hyperplane $\widehat{\mathbb{R}}^{n-1} \subset \widehat{\mathbb{R}}^n$. We call E_τ *conformal sliding* along $\mathbb{S}^1$. There is no sliding at $\tau = 1$, $E_1 = \mathrm{Id} \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, in which case we are dealing with radial stertchings. Let $\mathcal{J}_E$ denote the Jacobian of a conformal sliding $E = E_\tau \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$; that is, pullback of the standard $(n-1)$-area form $\mathrm{d}\omega$ on $\mathbb{S}^{n-1}$ via E.

It should be noted that for every conformal sliding $E = E_\nu$ its *integral average* of the Jacobian equals 1:

$$\fint_{\mathbb{S}^{n-1}} \mathcal{J}_E(\omega)\mathrm{d}\omega = 1 \quad \text{and} \quad J_E \not\equiv 1 \quad \text{for} \quad E \neq E_1 = \mathrm{Id}. \tag{6.8}$$

The singular values of the differential matrix $D\mathfrak{f}(x)$ and its determinant, are computed as follows. First, the singular values $\lambda_1, \dots, \lambda_{n-1}$ of $\frac{F^2}{|x|^2}\mathbf{G}$ are all equal, because $E \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is conformal. The λ_n-singular value equals $\dot{F}(|x|)$, see (6.6). The product $\lambda_1\lambda_2 \dots \lambda_{n-1}$ equals

$$\det\left[\frac{F(|x|)}{|x|^2}\mathbf{G}(\omega)\right] = \left(\frac{F^2(|x|)}{|x|^2}\right)^{n-1} \mathcal{J}_E^2(\omega).$$

Hence,

$$\lambda_1 = \cdots = \lambda_{n-1} = \frac{F(|x|)}{|x|} \sqrt[n-1]{\mathcal{J}_E}(\omega) \quad \text{and} \quad \lambda_n = \dot{F}(|x|)$$

$$\det D\mathfrak{f}(x) = \lambda_1 \cdots \lambda_{n-1}\lambda_n = \dot{F}(|x|)\left(\frac{F(|x|)}{|x|}\right)^{n-1} \mathcal{J}_E(\omega).$$

Now consider a family $\{\mathfrak{f}_\tau\} \subset \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$, where

$$\mathfrak{f}_\tau(x) = F(|x|)\, E_\tau\left(\frac{x}{|x|}\right).$$

It is a matter of direct computation to see that for all $1 \leqslant \ell \leqslant n-1$,

$$\mathcal{K}_\ell \mathfrak{f}_\tau(x) = \frac{\left(\mathcal{J}^{\frac{2}{n-1}} + \frac{\ell}{n-\ell}\eta_F^2\right)^{\frac{n}{2n-2\ell}}}{\left(\frac{n}{n-\ell}\right)^{\frac{n}{2n-2\ell}} \mathcal{J}^{\frac{1}{n-1}} \eta_F^{\frac{\ell}{n-\ell}}} \tag{6.9}$$

where

$$\eta_F(|x|) = \frac{|x|\dot{F}(|x|)}{F(|x|)} \quad \text{and} \quad \mathcal{J} = \mathcal{J}_{E_\tau}\left(\frac{x}{|x|}\right).$$

Thus, for every $p \geqslant 1$

$$\int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}_\tau]^p = \omega_{n-1}\left(\frac{n-\ell}{n}\right)^{\frac{np}{2n-2\ell}} \int_r^R \fint_{|x|=t} \frac{\left(\mathcal{J}^{\frac{2}{n-1}} + \frac{\ell}{n-\ell}\eta_F^2\right)^{\frac{np}{2n-2\ell}}}{\mathcal{J}^{\frac{p}{n-1}} \eta_F^{\frac{\ell p}{n-\ell}}} t^{n-1} \mathrm{d}t. \tag{6.10}$$

Here the symbol $\fint$ stands, as usual, for the integral average. The map $\mathfrak{f}_1 : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is none other than the radial streching $\mathfrak{f}_1 = F(|x|)\frac{x}{|x|}$. From now on, $\mathfrak{f}_1$ will be the one of smallest $\mathscr{L}^p$-norm of $\mathcal{K}_\ell$-distortion. The existence and uniqueness of such stretching is ensured by Theorem 5.31 and Remark 5.12, because $\operatorname{Mod}\mathbb{A}^* < \operatorname{Mod}\mathbb{A}$. Proposition 5.5 tells us that

$$0 \leqslant \eta_F(|x|) < 1 \qquad \text{for every} \quad x \in \mathbb{A}\,. \tag{6.11}$$

Our nearest objective is to find the exponents, say $1 \leqslant p < p_\ell(n)$, for which there are annuli $\mathbb{A}$, $\mathbb{A}^*$ and a conformal sliding map $\mathfrak{f}_\tau : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ with $\tau \neq 1$ (arbitrarily close to the identity) such that

$$\int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}_\tau]^p < \int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}_1]^p \qquad \left(= \inf_{f \in \mathscr{R}^{1,1}(\mathbb{A},\mathbb{A}^*)} \int_{\mathbb{A}} |\,\mathcal{K}_\ell\, f(x)\,|^p \,\mathrm{d}x \right). \tag{6.12}$$

To prove this inequality we need the following considerations.

6.3. Concavity argument

Suppose that for some number $0 \leqslant \eta < 1$ the function

$$G(S) \stackrel{\text{def}}{=\!=} \frac{\left(S^{\frac{2}{n-1}} + \frac{\ell}{n-\ell}\eta^2\right)^{\frac{np}{2n-2\ell}}}{S^{\frac{p}{n-1}}} \qquad \text{defined for } S \approx 1 \tag{6.13}$$

is strictly concave at $S = 1$. It then remains concave for all $S \approx 1$ and, in particular, everywhere in $\mathbb{S}^{n-1}$ for $S \approx \mathcal{J}_{E_\tau}$ (the Jacobian of $E_\tau : \mathbb{S}^{n-1} \xrightarrow{\text{onto}} \mathbb{S}^{n-1}$) whenever

τ is sufficiently close to 1. Concavity will also be preserved upon small alteration of η, say within an interval

$$\eta_- \leqslant \eta \leqslant \eta_+ \qquad \text{where} \quad 0 \leqslant \eta_- < \eta_+ < 1. \tag{6.14}$$

Here we should emphasized that, by Corollary 5.9, there always exist annuli $\mathbb{A}$ and $\mathbb{A}^*$ for which the distortion-minimal radial map $\mathfrak{f}_1 = F(|x|)\frac{x}{|x|}$ satisfies

$$\eta_- \leqslant \eta_F(|x|) \leqslant \eta_+ \qquad \text{for all } x \in \mathbb{A}. \tag{6.15}$$

Now, with the aid of Jensen's inequality, since $\mathcal{J}_{E_\tau} \not\equiv 1$, we would conclude that

$$\begin{aligned}
\int_{\mathbb{A}} [\mathcal{K}_\ell \mathfrak{f}_\tau]^p &= \omega_{n-1} \left(\frac{n-\ell}{n}\right)^{\frac{np}{2n-2\ell}} \int_r^R \fint_{|x|=t} \frac{\left[\left(\mathcal{J}_{E_\tau}\right)^{\frac{2}{n-1}} + \frac{\ell}{n-\ell}\eta_F^2\right]^{\frac{np}{2n-2\ell}}}{\left(\mathcal{J}_{E_\tau}\right)^{\frac{p}{n-1}} \cdot \eta_F^{\frac{\ell p}{n-\ell}}} t^{n-1}\, \mathrm{d}t \\
&< \omega_{n-1} \left(\frac{n-\ell}{n}\right)^{\frac{np}{2n-2\ell}} \int_r^R \frac{\left[\left(\fint_{|x|=t} \mathcal{J}_{E_\tau}\right)^{\frac{2}{n-1}} + \frac{\ell}{n-\ell}\, \eta_F^2\right]^{\frac{np}{2n-2\ell}}}{\left(\fint_{|x|=t} \mathcal{J}_{E_\tau}\right)^{\frac{p}{n-1}} \cdot \eta_F^{\frac{\ell p}{n-\ell}}}\, t^{n-1}\, \mathrm{d}t \\
&= \omega_{n-1} \left(\frac{n-\ell}{n}\right)^{\frac{np}{2n-2\ell}} \int_r^R \frac{\left[1 + \frac{\ell}{n-\ell}\eta_F^2\right]^{\frac{np}{2n-2\ell}}}{\eta_F^{\frac{\ell p}{n-\ell}}} t^{n-1}\, \mathrm{d}t = \int_{\mathbb{A}} [\mathcal{K}_\ell f_1]^p.
\end{aligned}$$

6.4. Further computation

Therefore, we are naturally led to an investigation of the second derivative of the function $G = G(S)$ at $S = 1$. We shall not bother the reader with laborious, rather straightforward, computation. We just state the result of our computation as follows

$$\begin{aligned}
&\frac{(n-1)^2(n-\ell)^2}{p\,\ell}\, S^{\frac{2n+p-2}{n-1}} \left(S^{\frac{2}{n-1}} + \frac{\ell}{n-\ell}\,\eta^2\right)^{2-\frac{np}{2n-2\ell}} \ddot{G}(S) \\
&= [p\ell - (n-l)(n-1)]S^{\frac{4}{n-1}} \\
&\quad + [(n+1)n - 2\ell(n-1) - 2p\ell]S^{\frac{2}{n-1}}\eta^2 \\
&\quad + [p\ell + (n-1)\ell]\eta^4.
\end{aligned} \tag{6.16}$$

The condition $\ddot{G}(1) < 0$ reduces, equivalently, to the following upper bound for the exponent p.

$$\begin{aligned}
1 \leqslant p &< \frac{[n^2 - \ell n - n + \ell] + [2\ell n - 2\ell - n^2 - n]\eta^2 + [-\ell n + \ell]\eta^4}{\ell(1-\eta^2)^2} \\
&= \frac{n}{\ell}\left[\frac{n+1}{1-\eta^2} - \frac{2}{(1-\eta^2)^2}\right] - n + 1.
\end{aligned} \tag{6.17}$$

The expression in the right hand side assumes its maximum at the point $\eta = \sqrt{\frac{n-3}{n+1}}$, and only at this point, see Figure 6.1.

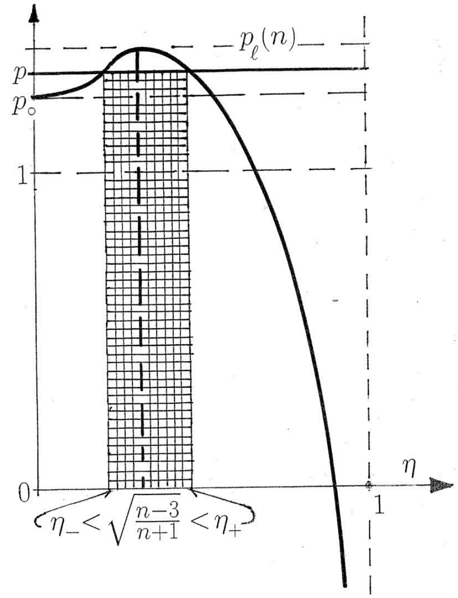

Figure 6.1. The maximum of η.

Let us denote its maximum value by

$$p_\ell(n) = \frac{n(n+1)^2}{8\ell} + 1 - n. \tag{6.18}$$

We just arrived at the critical exponents stated in Theorem 6.1 for which the radial symmetry fails.

7. Absolute minimizers

7.1. Lower bounds of $\mathcal{K}_\ell$-distortions

The Frobenius norm of the differential matrix (used in the definition of $\mathcal{K}_\ell$-distortions) is not suitable for the radial strechings $f(x) = F(|x|)\frac{x}{|x|}$. We need to establish adequate lower bounds of the $\mathcal{K}_\ell$-distortions in terms of the outer and inner distortion which are formulated via the operator norms. There are subtle adjustments necessary to ensure that such lower bounds turn into equalities once tested by the radial stationary solutions. We present two such lower bounds. The first one works in the case of contraction; that is, when $\operatorname{Mod}\mathbb{A}^* \leqslant \operatorname{Mod}\mathbb{A}$ (meaning that $\eta_F(|x|) \leqslant 1$). The second works in the case of conformal expansion; that is, when $\operatorname{Mod}\mathbb{A}^* \geqslant \operatorname{Mod}\mathbb{A}$ (meaning that $\eta_F(|x|) \geqslant 1$).

Proposition 7.1. *Given $1 \leqslant \ell \leqslant n-1$, an exponent $N > 0$ and a parameter $0 < \tau \leqslant 1$ such that:*

$$\left[1 + \frac{\ell}{n-\ell}\tau^2\right]^{N-1} \left(1-\tau^2\right) < 1. \tag{7.1}$$

Then for every matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ it holds that

$$\left(\mathcal{K}_\ell \mathbf{M}\right)^{\frac{2N(n-\ell)}{n}} \geqslant \left(\mathbf{K}_I \mathbf{M}\right)^{\frac{2\ell N}{n(n-1)}} \mathcal{A} + \mathcal{B} \tag{7.2}$$

where

$$\mathcal{A} = \mathcal{A}_\tau(N) = \left(1 - \frac{\ell}{n}\right)^N \left[1 + \frac{\ell}{n-\ell}\tau^2\right]^{N-1} (1-\tau^2) < \left(1 - \frac{\ell}{n}\right)^N \tag{7.3}$$

$$\mathcal{B} = \mathcal{B}_\tau(N) = \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\tau^2\right]^{N-1} \tau^{2-2\ell N/n}. \tag{7.4}$$

Equality occurs in (7.2) *if and only if the singular values of M are a scalar multiple of* $\{1, \dots, 1, \tau\}$.

We apply this proposition to $\mathbf{M}^{-1}$ in place of $\mathbf{M}$, to the integer $n-\ell$ in place of ℓ, and to the parameter $1/\tau$ in place of τ. This time we assume that $\tau \geqslant 1$. In view of (2.22) and (2.23) this results in the following

Proposition 7.2. *Given $1 \leqslant \ell \leqslant n-1$, an exponent $N > 0$ and a parameter $\tau \geqslant 1$ such that:*

$$\left[1 + \frac{n-\ell}{\ell}\tau^{-2}\right]^{N-1} \left(1-\tau^{-2}\right) < 1. \tag{7.5}$$

Then for every matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ it holds that

$$\left(\mathcal{K}_\ell \mathbf{M}\right)^{\frac{2N(n-\ell)}{n}} \geqslant \left(\mathbf{K}_O \mathbf{M}\right)^{\frac{2N(n-\ell)}{n(n-1)}} \mathcal{A} + \mathcal{B} \tag{7.6}$$

where

$$\mathcal{A} = \mathcal{A}_\tau(N) = \frac{\ell}{n}\left[1 - \frac{\ell}{n} + \frac{\ell}{n}\tau^2\right]^{N-1} \left(\tau^2 - 1\right)\tau^{-2N}$$

and

$$\mathcal{B} = \mathcal{B}_\tau(N) = \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\tau^2\right]^{N-1} \tau^{-2\ell N/n}.$$

Equality occurs in (7.6) *if and only if the singular values of* $\mathbf{M}$ *are a scalar multiple of* $\{1, \dots, 1, \tau\}$.

Remark 7.3. Condition (7.5) is satisfied for all $\tau \geqslant 1$ whenever $0 < N \leqslant \frac{n}{n-\ell}$, just because the function $x \rightarrowtail \left[1 + \frac{n-\ell}{\ell}x\right]^{N-1}(1-x)$ is strictly decreasing in $\mathbb{R}_+$.

For the proof of Proposition 7.1 we first reformulate (7.6) in terms of the singular values of $\mathbf{M}$ and then examine the critical points of (7.6). The following key ingredient concerning symmetric functions might be of independent interest.

7.1.1. *Critical points of a symmetric function*

Let $\mathbb{R}^m_+ \subset \mathbb{R}^m$, $m \geqslant 2$, denote the set of points whose coordinates are positive. Consider a symmetric function $\Phi\colon \mathbb{R}^m_+ \to \mathbb{R}$ of the form

$$\Phi(X_1, X_2, \dots, X_m) = W(s_1, s_2, \dots, s_m)$$

where $W\colon \mathbb{R}^m_+ \to \mathbb{R}$ is $\mathscr{C}^1$-smooth with respect to the symmetric monomials

$$\begin{cases} s_1 = X_1 + X_2 + \cdots + X_m \\ s_\ell = \sum\limits_{1 \leqslant i_1 < \cdots < i_\ell \leqslant m} X_{i_1} \cdots X_{i_\ell} \\ s_m = X_1 \cdot X_2 \cdots X_m. \end{cases}$$

We assume that the partial derivatives $W_{s_1}, W_{s_2}, \dots, W_{s_{m-1}}$ are nonnegative in $\mathbb{R}^m_+$ and that $W_{s_1} + W_{s_2} + \cdots + W_{s_{m-1}} > 0$ (similar condition $W_{s_m} \geqslant 0$ is not required).

Lemma 7.4. *Suppose $X^\circ = (X_1^\circ, \dots, X_m^\circ) \in \mathbb{R}^m_+$ is a critical point of Φ. Then*

$$X_1^\circ = X_2^\circ = \cdots = X_m^\circ.$$

Proof. The gradient $\nabla\Phi(X)$ vanishes at $X = X^\circ$. This gives us a nonlinear system of m-equations in which $X = (X_1^\circ, \dots, X_m^\circ)$ is a solution.

$$\frac{\partial \Phi}{\partial X_i} = W_{s_1}\frac{\partial s_1}{\partial X_i} + \cdots + W_{s_\ell}\frac{\partial s_\ell}{\partial X_i} + \cdots + W_{s_m}\frac{\partial s_m}{\partial X_i} = 0 \quad (i = 1, 2 \dots, m). \tag{7.7}$$

Consider the terms

$$T_\ell^i \overset{\text{def}}{=\!=} X_i \frac{\partial s_\ell}{\partial X_i} = \sum_{\substack{1 \leqslant i_1 < \cdots < i_\ell \leqslant m \\ i \in \{i_1, \dots, i_\ell\}}} X_{i_1} \dots X_{i_\ell} \qquad (i, \ell = 1, 2, \dots, m).$$

It should be noted that for $\ell = m$ the terms T_m^i do not depend on i; indeed, we have $T_m^i = X_1 \cdot X_2 \cdots X_m$. Multiply (7.7) by X_i and subtract the analogous equation for an index $j \neq i$,

$$X_i \frac{\partial \Phi}{\partial X_i} - X_j \frac{\partial \Phi}{\partial X_j} = \sum_{\ell=1}^{m-1} \left(T_\ell^i - T_\ell^j\right) W_{s_\ell} = 0. \tag{7.8}$$

A short computation reveals that

$$T_\ell^i - T_\ell^j = (X_i - X_j) \times (\text{a positive factor}).$$

To identify this factor, we write

$$T_\ell^i - T_\ell^j = \sum_{\substack{1 \leqslant i_1 < \cdots < i_\ell \leqslant m \\ i \in \{i_1, \dots, i_\ell\}}} X_{i_1} \dots X_{i_\ell} - \sum_{\substack{1 \leqslant j_1 < \cdots < j_\ell \leqslant m \\ i \in \{j_1, \dots, j_\ell\}}} X_{j_1} \dots X_{j_\ell}.$$

We may subtract from each sum the same term; say,

$$\sum_{\substack{1\leqslant k_1<\cdots<k_\ell\leqslant m\\ i,j\in\{k_1,\dots,k_\ell\}}} X_{k_1}\dots X_{k_\ell}$$

which results in the equation

$$\begin{aligned} T_\ell^i - T_\ell^j &= \sum_{\substack{1\leqslant k_1<\cdots<k_\ell\leqslant m\\ i\in\{k_1,\dots,k_\ell\}\\ j\notin\{k_1,\dots,k_\ell\}}} X_{k_1}\dots X_{k_\ell} - \sum_{\substack{1\leqslant k_1<\cdots<k_\ell\leqslant m\\ i\notin\{k_1,\dots,k_\ell\}\\ j\in\{k_1,\dots,k_\ell\}}} X_{k_1}\dots X_{k_\ell} \\ &= (X_i - X_j)\sum_{\substack{1\leqslant k_1<\cdots<k_{\ell-1}\leqslant m\\ i,j\notin\{k_1,\dots,k_{\ell-1}\}}} X_{k_1}\dots X_{k_{\ell-1}} \stackrel{\text{def}}{=} (X_i - X_j)\, C_{ij}^\ell . \end{aligned}$$

The last sum (denoted by C_{ij}^ℓ) is the above-mentioned positive factor, where we adhere to the convention that $C_{ij}^1 = 1$. Now Equation (7.8) reads as

$$\Big[W_{s_1} C_{ij}^1 + \cdots + W_{s_\ell} C_{ij}^\ell + \dots W_{s_{m-1}} C_{ij}^{m-1}\Big]\cdot (X_i - X_j) = 0.$$

Hence $X_i = X_j$, as claimed. □

7.1.2. *Reduction to singular values*

Returning to Proposition 7.1 we let $0<\lambda_1,\dots,\lambda_n<\infty$ denote the singular values of $\mathbf{M}$. In the left hand side of (7.2) we have the term

$$\big(\mathcal{K}_\ell \mathbf{M}\big)^{\frac{2N(n-\ell)}{n}} = \left[\binom{n}{\ell}^{-1} \sum_{1\leqslant i_1<\cdots<i_\ell\leqslant n} \lambda_{i_1}^2\dots\lambda_{i_\ell}^2\right]^N \Big/ (\lambda_1\cdots\lambda_n)^{\frac{2\ell N}{n}}$$

wheras on the right hand side

$$\big(\mathbf{K}_I \mathbf{M}\big)^{\frac{2\ell N}{n(n-1)}} = \big[\lambda_1\cdots\lambda_{n-1}\big]^{\frac{2\ell N}{n-1}} \Big/ (\lambda_1\cdots\lambda_n)^{\frac{2\ell N}{n}}$$

provided λ_n is the least singular value. Proposition 7.1 will follow once we show that

$$\begin{aligned} &\left[\binom{n}{\ell}^{-1} \sum_{1\leqslant i_1<\cdots<i_\ell\leqslant n} \lambda_{i_1}^2\dots\lambda_{i_\ell}^2\right]^N \\ &\qquad \geqslant (\lambda_1\cdots\lambda_{n-1})^{\frac{2\ell N}{n-1}}\,\mathcal{A} + (\lambda_1\cdots\lambda_n)^{\frac{2\ell N}{n}}\,\mathcal{B} \end{aligned} \tag{7.9}$$

for all positive variables $\lambda_1,\dots,\lambda_n$, regardless of the size of λ_n. We also need to show that equality occurs in (7.9) if and only if

$$\lambda_1 = \cdots = \lambda_{n-1} = \lambda_n/\tau.$$

We first examine the special case where $\lambda_1 = \cdots = \lambda_{n-1} = 1$ while $\lambda_n = X \geqslant 0$ is a free variable. In this case Inequality (7.9) boils down to the following.

Lemma 7.5. *For every $X \geqslant 0$ the following inequality holds*

$$\binom{n}{\ell}^{-N}\left[\binom{n-1}{\ell-1}X^2+\binom{n-1}{\ell}\right]^N=\left(1-\frac{\ell}{n}+\frac{\ell}{n}X^2\right)^N \geqslant \mathcal{A}+\mathcal{B}\,X^{\frac{2\ell N}{n}}. \quad (7.10)$$

Equality occurs if and only if $X=\tau$.

Proof. The reader may wish to verify the following identity (the case $X=\tau$) directly from the definitions of the coefficients $\mathcal{A}$ in (7.3) and $\mathcal{B}$ in (7.4).

$$\left(1-\frac{\ell}{n}+\frac{\ell}{n}\tau^2\right)^N=\mathcal{A}+\mathcal{B}\tau^{\frac{2\ell N}{n}}. \quad (7.11)$$

Thus we are left to showing that the minimum of the function

$$\mathcal{F}(X)=\left(1-\frac{\ell}{n}+\frac{\ell}{n}X^2\right)^N-\mathcal{A}-\mathcal{B}\,X^{\frac{2\ell N}{n}}$$

is attained at $X=\tau$, and only at this point. First take a quick look at the endpoints

$$\mathcal{F}(0)=\left(1-\frac{\ell}{n}\right)^N-\mathcal{A}_\tau(N)>0 \qquad \text{(by 7.3)}$$

$$\lim_{X\to\infty}\mathcal{F}(X)=\infty \quad \text{whereas } \mathcal{F}(\tau)=0.$$

This shows that $\mathcal{F}$ must assume its minimum value at certain critical point $X\in\mathbb{R}_+$; neither at $X=0$ nor at $X=\infty$. The equation $\mathcal{F}'(X)=0$ for the critical point takes the form

$$\left(1-\frac{\ell}{n}+\frac{\ell}{n}X^2\right)^{N-1}X^{2-\frac{2\ell N}{n}}=\mathcal{B}_\tau(N).$$

It follows (by differentiation) that the left hand side, regarded as a function in X, is monotonically increasing from 0 to ∞. In view of formula (7.4) for $\mathcal{B}_\tau(N)$ we conclude that $X=\tau$. This is the only critical point. Therefore, τ must be a point of minimum of $\mathcal{F}$. □

Proof of Inequality (7.9). Because of homogeneity we may fix λ_n, say $\lambda_n=\tau$ while letting the other parameters $\lambda_1,\dots,\lambda_{n-1}$ vary (not necessarily larger than τ). □

The problem reduces to establishing the inequality

$$\binom{n}{\ell}^{-N}\left[\tau^2\sum_{1\leqslant i_1<\cdots<i_{\ell-1}\leqslant n-1}\lambda_{i_1}^2\dots\lambda_{i_{\ell-1}}^2+\sum_{1\leqslant i_1<\cdots<i_\ell\leqslant n-1}\lambda_{i_1}^2\dots\lambda_{i_\ell}^2\right]^N \quad (7.12)$$
$$\geqslant\tau^{\frac{n}{n-\ell}}(\lambda_1\cdots\lambda_{n-1})^{\frac{2\ell N}{n-1}}\mathcal{A}+\tau^{\frac{2\ell N}{n}}(\lambda_1\cdots\lambda_{n-1})^{\frac{2\ell N}{n}}\ \mathcal{B}$$

for all $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{R}_+$. Hereafter, by convention, the first sum equals 1 if $\ell = 1$. We also need to show that equality holds only for $\lambda_1 = \cdots = \lambda_{n-1} = 1$. For this purpose we investigate a continuous function of positive $(n-1)$ variables, $X = (X_1, \dots, X_{n-1}) \in \mathbb{R}_+^{n-1}$.

$$\begin{aligned}\Psi(X) = &\left[\tau^2 \sum_{1 \leqslant i_1 < \cdots < i_{\ell-1} \leqslant n-1} X_{i_1} \dots X_{i_{\ell-1}} + \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant n-1} X_{i_1} \dots X_{i_\ell}\right] \\ &- \left[\binom{n}{\ell}^N (X_1 \cdots X_{n-1})^{\frac{\ell N}{n-1}} \cdot \mathcal{A} + \binom{n}{\ell}^N \tau^{\frac{2\ell N}{n}} (X_1 \cdots X_{n-1})^{\frac{\ell N}{n}} \cdot \mathcal{B}\right]^{\frac{1}{N}}.\end{aligned} \tag{7.13}$$

The objective is to show that $\Psi(X) \geqslant 0$ for all $X \in \overline{\mathbb{R}_+^{n-1}}$. There is an advantage to reformulate Inequality (7.12) in this way. The point is to separate all product terms $X_{i_1} \cdots X_{i_k}$ that appear with positive coefficients from those with negative coefficients. The latter include (luckily) only the full product $X_1 \cdots X_{n-1}$.

The inequality $\Psi(X_1, \dots, X_{n-1}) \geqslant 0$ is certainly true if one of the variables vanishes; that is, at the boundary of the domain $\mathbb{R}_+^{n-1}$. We wish the inequality $\Psi(X_1, \dots, X_{n-1}) \geqslant 0$ is also true at infinity.

Proposition 7.6.

$$\liminf_{||X|| \to \infty} \Psi(X_1, \dots, X_{n-1}) \geqslant 0. \tag{7.14}$$

This fact, being not obvious, requires the following two lemmas.

Lemma 7.7.

$$\Theta_\tau \stackrel{\text{def}}{=\!=} \binom{n}{\ell}^N A_\tau \sup_{X \in \mathbb{R}_+^{n-1}} \frac{\big(X_1 \dots X_{n-1}\big)^{\frac{\ell N}{n-1}}}{\left[\sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant n-1} X_{i_1} \dots X_{i_\ell}\right]^N} < 1. \tag{7.15}$$

Proof. We make use of Hadamard's inequality

$$(X_1 \dots X_{n-1})^{\frac{\ell}{n-1}} \leqslant \frac{1}{\binom{n-1}{\ell}} \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant n-1} X_{i_1} \dots X_{i_\ell}.$$

Therefore, by Inequality (7.3)

$$\Theta_\tau \leqslant \binom{n}{\ell}^N A_\tau(N) \binom{n-1}{\ell}^{-N} = \left(\frac{n}{n-\ell}\right)^N A_\tau(N) < 1. \qquad \square$$

Lemma 7.8. *Let* $\|X\| \stackrel{\text{def}}{=\!=} \max\{X_1, \dots, X_{n-1}\} \geqslant 1$ *and* $0 < \tau \leqslant 1$. *Then*

$$\begin{aligned} &\frac{(X_1 \dots X_{n-1})^{\frac{\ell}{n}}}{\tau^2 \displaystyle\sum_{1 \leqslant i_1 < \dots < i_{\ell-1} \leqslant n-1} X_{i_1} \dots X_{i_{\ell-1}} + \sum_{1 \leqslant i_1 < \dots < i_\ell \leqslant n-1} X_{i_1} \dots X_{i_\ell}} \\ &\leqslant \frac{1}{\tau^2 \|X\|^{1/n}}. \end{aligned} \tag{7.16}$$

Here, by our convention, the first sum in the denominator equals 1 if $\ell = 1$.

Proof. One needs to verify this inequality only in the worse case of $\tau = 1$. Since the quotient in the left hand side is invariant under permutation of variables, we may assume that $\|X\| = X_1 \geqslant X_2 \geqslant \dots \geqslant X_{n-1} > 0$. We shall take into account only the contribution from the largest of the products under the sums in the denominator; others will be neglected. The problem now reduces to establishing that

$$\frac{(X_1 \dots X_{n-1})^\ell}{(X_1 \dots X_{\ell-1} + X_1 \dots X_\ell)^n} \leqslant \frac{1}{\|X\|}. \tag{7.17}$$

Let $k = k(X) \in \{1, \dots, n-1\}$ denote the largest subscript such that $\|X\| = X_1 \geqslant X_2 \geqslant \dots \geqslant X_k \geqslant 1$. This means, in particular, that $X_{n-1} \leqslant \dots \leqslant X_{k+1} < 1$, whenever $1 \leqslant k < n-1$.

Case 1. $1 \leqslant \ell \leqslant k \leqslant n-1$. We neglect the first product in the denominator of (7.17) and proceed as follows

$$\begin{aligned} \frac{(X_1 \dots X_{n-1})^\ell}{(X_1 \dots X_{\ell-1} + X_1 \dots X_\ell)^n} &\leqslant \frac{(X_1 \dots X_{n-1})^\ell}{(X_1 \dots X_\ell)^n} \\ &= (X_1 \dots X_\ell)^{\ell-n} (X_{\ell+1} \dots X_{n-1})^\ell \\ &\leqslant (X_1 \dots X_\ell)^{\ell-n+n-\ell-1} = \frac{1}{X_1 \dots X_\ell} \leqslant \frac{1}{X_1} = \frac{1}{\|X\|}. \end{aligned}$$

Here the second inequality holds because each term $X_{\ell+1}, \dots X_{n-1}$ does not exceed $(X_1, \dots X_\ell)^{1/\ell}$ and we have $n - \ell - 1$ such terms.

Case 2. $1 \leqslant k \leqslant \ell - 1$, so $2 \leqslant \ell \leqslant n-1$. This time we neglect the second product in the denominator of (7.17) and compute

$$\begin{aligned} \frac{(X_1 \dots X_{n-1})^\ell}{(X_1 \dots X_{\ell-1} + X_1 \dots X_\ell)^n} &\leqslant \frac{(X_1 \dots X_{n-1})^\ell}{(X_1 \dots X_{\ell-1})^n} = \frac{1}{(X_1 \dots X_k)^{n-\ell}} \frac{(X_\ell \dots X_{n-1})^\ell}{(X_{k+1} \dots X_{\ell-1})^{n-\ell}} \\ &\leqslant \frac{1}{\|X\|^{n-\ell}} \frac{X_\ell^{(n-\ell)\ell}}{X_{\ell-1}^{(\ell-k-1)(n-\ell)}} \\ &\leqslant \frac{X_\ell^{(n-\ell)(k+1)}}{\|X\|^{n-\ell}} \leqslant \frac{1}{\|X\|^{n-\ell}} \leqslant \frac{1}{\|X\|}. \end{aligned}$$

Here we adhere to the convention that $X_{k+1}\cdots X_{\ell-1} = 1$ for $k = \ell - 1$. The following inequalities were used:

- $X_1 \cdots X_k \geqslant X_1 = \|X\|$;
- $X_\ell \cdots X_{n-1} \leqslant X_\ell \cdots X_\ell = X_\ell^{n-\ell}$;
- $X_{k+1} \cdots X_{\ell-1} \geqslant X_{\ell-1} \cdots X_{\ell-1} = X_{\ell-1}^{\ell-k-1} \geqslant X_\ell^{\ell-k-1}$;
- $X_\ell^{(n-\ell)(k+1)} \leqslant 1$.

The proof of Lemma 7.8 is complete. □

Proof of Proposition 7.6. To see (7.14) we consider the N-th power of the term in formula (7.13) for $\Psi(X)$; that is,

$$L \stackrel{\text{def}}{=\!=} \left[\tau^2 \sum_{1 \leqslant i_1 < \cdots < i_{\ell-1} \leqslant n-1} X_{i_1} \ldots X_{i_{\ell-1}} + \sum_{1 \leqslant i_1 < \cdots < i_\ell \leqslant n-1} X_{i_1} \ldots X_{i_\ell} \right]^N .$$

We then estimate the negative terms in (7.13) as follows: Inequality (7.15) tells us that for $X_1, \ldots, X_{n-1}$ we have

$$\binom{n}{\ell}^N (X_1 \cdots X_{n-1})^{\frac{\ell N}{n-1}} \cdot \mathcal{A} \leqslant \theta_\tau L \quad \text{where} \quad 0 \leqslant \theta_\tau < 1 .$$

On the other hand, inequality (7.16) yields

$$\binom{n}{\ell}^N \tau^{\frac{2\ell N}{n}} (X_1 \cdots X_{n-1})^{\frac{\ell N}{n}} \cdot \mathcal{B} \leqslant C(\tau) ||X||^{-\frac{N}{n}} L.$$

Summing up we conclude that the negative terms in formula (7.13) do not exceed L; namely,

$$\left[\theta_\tau + C(\tau) ||X||^{-\frac{N}{n}} \right] L \leqslant L$$

for sufficiently large $||X||$ as desired. □

Lemma 7.9. *We have*

$$\inf \left\{ \Psi(X_1, \ldots, X_{n-1}) \colon X \in \mathbb{R}_+^{n-1} \right\} = 0. \tag{7.18}$$

The infimum is attained only when $X_1 = \cdots = X_{n-1} = 1$.

Proof. Since $\Psi(1, \ldots, 1) = 0$, we see that the infimum is non-positive. Suppose that, contrary to our claim, the infimum is negative. Since we have $\Psi(X_1, \ldots, X_{n-1}) \geqslant 0$ when one of the variables vanishes and by Proposition 7.6. it follows that the infimum of Ψ is attained in $\mathbb{R}_+^{n-1}$, say at the point $X^\circ = (X_1^\circ, \ldots, X_{n-1}^\circ)$. This is a critical point of the function Ψ in the domain in which Ψ is $\mathscr{C}^\infty$-smooth.

We now may appeal to Lemma 7.4 which asserts that $X_1^\circ = \cdots = X_{n-1}^\circ \stackrel{\text{def}}{=\!=} t^2 > 0$. The value of Ψ at such point can easily be computed,

$$\begin{aligned}
\Psi(t^2, \dots, t^2) &= \tau^2 \binom{n-1}{\ell-1} t^{2\ell-2} + \binom{n-1}{\ell} t^{2\ell} \\
&\quad - \binom{n}{\ell} \left[t^{2\ell N} \mathcal{A} \,+\, \tau^{\frac{2\ell N}{n}} t^{\frac{2\ell N(n-1)}{n}} \mathcal{B} \right]^{\frac{1}{N}} \\
&= \binom{n-1}{\ell} t^{2\ell} \left[\frac{\ell}{n-\ell} \left(\frac{\tau}{t}\right)^2 + 1 \right] - \binom{n}{\ell} t^{2\ell} \left[\mathcal{A} \,+\, \left(\frac{\eta}{t}\right)^{\frac{2\ell N}{n}} \mathcal{B} \right]^{\frac{1}{N}} \\
&\geqslant \binom{n}{\ell} t^{2\ell} \inf_{X>0} \left\{ \frac{n-\ell}{n} [1 + \frac{\ell}{n-\ell} X^2] - [\mathcal{A} \,+\, X^{\frac{2\ell N}{n}} \mathcal{B}]^{\frac{1}{N}} \right\} \\
&\geqslant 0
\end{aligned}$$

by Lemma 7.5. Lemma 7.5 also tells us that the infimum is attained at $X = \tau$, but nowhere else. Thus we conclude that $\inf_{\mathbb{R}_+^{n-1}} \Psi = 0$ and that the infimum of $\Psi(t^2, \dots, t^2)$ is attained only when $\frac{\tau}{t} = X = \tau$. Hence $t = 1$. □

This completes the proof of Lemma 7.9.

Proof of Proposition 7.1. Lemma 7.9 applied to

$$X_1 = \lambda_1^2, \dots, X_{n-1} = \lambda_{n-1}^2$$

gives inequality (7.9). This, in turn, is none other than Inequality (7.2) expressed in terms of the singular values of matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$. The proof of Proposition 7.1 is complete. □

7.2. Proof of Theorem 1.3 (the case $p = \infty$)

In [30, Chapter 15] new sharp estimates for quasiconformal mappings $g : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ have been established as an application of free-Lagrangians. [30, Theorem 15.1] asserts that

$$\|\mathbf{K}_I g\|_{\mathscr{L}^\infty(\mathbb{A})}^{-1} \leqslant \left(\frac{\operatorname{Mod} \mathbb{A}^*}{\operatorname{Mod} \mathbb{A}}\right)^{n-1} \leqslant \|\mathbf{K}_O g\|_{\mathscr{L}^\infty(\mathbb{A})}. \tag{7.19}$$

New ingredients are needed to treat the $\mathcal{K}_\ell$-distortions, because of Hilbert-Schmidt norms involved. These ingredients are the lower bounds in Proposition 7.1 and Proposition 7.2 , both with the exponent $N = 1$. Let

$$\alpha = \frac{\operatorname{Mod} \mathbb{A}^*}{\operatorname{Mod} \mathbb{A}}.$$

Recall the power stretching

$$f(x) = F\big(|x|\big)\frac{x}{|x|} = R_* R^{-\alpha}|x|^{\alpha-1}x$$

for which $\eta_F \equiv \alpha$. Consider an arbitrary quasiconformal map $g\colon \mathbb{A} \to \mathbb{A}^*$.

Case of conformal contraction $\boldsymbol{\alpha < 1}$. By Proposition 7.1 we have a point-wise inequality

$$\big|\,\mathcal{K}_\ell g(x)\,\big|^{\frac{2n-2\ell}{n}} \geqslant \mathcal{A}\big|\mathbf{K}_I g(x)\big|^{\frac{2\ell}{n(n-1)}} + \mathcal{B}$$

with constant coefficients

$$\mathcal{A} = \left(1 - \frac{\ell}{n}\right)\left(1 - \alpha^2\right) \;\text{ and }\; \mathcal{B} = \alpha^{2-2\ell/n}.$$

Therefore, by the left hand inequality in (7.19) it follows that

$$\Big\|\,\mathcal{K}_\ell g\,\Big\|^{\frac{2n-2\ell}{n}}_{\mathscr{L}^\infty(\mathbb{A})} \geqslant \mathcal{A}\Big\|\mathbf{K}_I g\Big\|^{\frac{2\ell}{n(n-1)}}_{\mathscr{L}^\infty(\mathbb{A})} + \mathcal{B} \geqslant \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\alpha^2\right]\alpha^{-2\ell/n}.$$

Hence we conclude that

$$\begin{aligned}\Big\|\,\mathcal{K}_\ell g\Big\|_{\mathscr{L}^\infty(\mathbb{A})} &\geqslant \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\alpha^2\right]^{\frac{n}{2n-2\ell}} \alpha^{-\ell/(n-\ell)} \\ &= \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\eta_F^2\right]^{\frac{n}{2n-2\ell}} \eta_F^{-\ell/(n-\ell)} = \Big\|\,\mathcal{K}_\ell f\Big\|_{\mathscr{L}^\infty(\mathbb{A})}.\end{aligned}$$

Equality holds for the power stretching $f(x) = F(|x|)\,\frac{x}{|x|} = R_* R^{-\alpha}|x|^{\alpha-1}x$.

Case of conformal expansion $\boldsymbol{\alpha > 1}$. By Proposition 7.2 we have a point-wise inequality

$$\big|\,\mathcal{K}_\ell g(x)\,\big|^{\frac{2n-2\ell}{n}} \geqslant \mathcal{A}\big|\mathbf{K}_O g(x)\big|^{\frac{2n-2\ell}{n(n-1)}} + \mathcal{B}$$

with constant coefficients

$$\mathcal{A} = \frac{\ell}{n}\frac{\alpha^2 - 1}{\alpha^2} \;\text{ and }\; \mathcal{B} = \alpha^{-2\ell/n}.$$

Therefore, by the right hand inequality in (7.19) it follows that

$$\Big\|\,\mathcal{K}_\ell g\Big\|^{\frac{2n-2\ell}{n}}_{\mathscr{L}^\infty(\mathbb{A})} \geqslant \mathcal{A}\;\Big\|\mathbf{K}_O g\Big\|^{\frac{2n-2\ell}{n(n-1)}}_{\mathscr{L}^\infty(\mathbb{A})} + \mathcal{B} \geqslant \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\alpha^2\right]\alpha^{-2\ell/n}.$$

Hence, as before, we conclude with the desired inequality.

$$\begin{aligned}\Big\|\,\mathcal{K}_\ell g\,\Big\|_{\mathscr{L}^\infty(\mathbb{A})} &\geqslant \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\alpha^2\right]^{\frac{n}{2n-2\ell}} \alpha^{-\ell/(n-\ell)} \\ &= \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\eta_F^2\right]^{\frac{n}{2n-2\ell}} \eta_F^{-\ell/(n-\ell)} = \Big\|\,\mathcal{K}_\ell f\Big\|_{\mathscr{L}^\infty(\mathbb{A})}.\end{aligned}$$

Again equality holds for the power stretching

$$f(x) = F(|x|)\,\frac{x}{|x|} = R_* R^{-\alpha} |x|^{\alpha-1} x\,.$$

We refer to [30, Theorem 5.2] for construction of other extremal maps. The case $\alpha = 1$ is obvious.

8. Proof of Theorem 1.9

We are looking for homeomorphisms $g\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ within the class $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$ having smallest $\mathscr{L}^p$-mean distortions, $p \geqslant 1$. Recall the subclass $\mathscr{R}^{1,1}(\mathbb{A}, \mathbb{A}^*) \subset \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$ of radial stretchings.

The minimal radial map $\mathfrak{f}$ in Theorem 1.9 will be none other than f_λ. Let us reformulate Theorem 1.9 as

Theorem 8.1. *Let* $\operatorname{Mod}\mathbb{A}^* \geqslant \operatorname{Mod}\mathbb{A}$ *and* $p > 1$. *Then for every homeomorphism* $g \in \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$, *we have*

$$\int_{\mathbb{A}} \big[\mathcal{K}_\ell g\big]^p \geqslant \int_{\mathbb{A}} \big[\mathcal{K}_\ell f_\lambda\big]^p. \tag{8.1}$$

Equality holds if and only if $g = f_\lambda$ (*modulo rotation*). *In case* $p = 1$, *such conclusions remain valid if and only if*

$$\operatorname{Mod}\mathbb{A}^* \leqslant \operatorname{Mod}\mathbb{A}^*_{\max} \qquad (\textit{see } (5.24) \textit{ for the definition of } \mathbb{A}^*_{\max}). \tag{8.2}$$

Proof. By virtue of Proposition 5.5 our assumption $\operatorname{Mod}\mathbb{A}^* \geqslant \operatorname{Mod}\mathbb{A}$ amounts to saying that

$$\eta_F(x) = \frac{|x|\dot F(|x|)}{F(|x|)} \geqslant 1 \qquad \text{for all } x \in \mathbb{A}\,. \tag{8.3}$$

As in the case of conformal contraction, we consider a weight function

$$\omega = \omega(|x|) = |\mathcal{K}_\ell f_\lambda|^{p-1} = \frac{\left[\frac{\ell}{n}\,\eta_F^2 + 1 - \frac{\ell}{n}\right]^{\frac{np-n}{2n-2\ell}}}{\big[\eta_F\big]^{\frac{\ell p-\ell}{n-\ell}}}. \tag{8.4}$$

It suffices to show the following weighted $\mathscr{L}^1$-inequality.

Lemma 8.2. *For every* $g \in \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$ *we have,*

$$\int_{\mathbb{A}} \mathcal{K}_\ell g(x)\omega(|x|)\,\mathrm{d}x \geqslant \int_{\mathbb{A}} \mathcal{K}_\ell f_\lambda(x)\omega(|x|)\,\mathrm{d}x. \tag{8.5}$$

Equality holds if and only if $g = f_\lambda$ (*modulo rotation*).

Indeed, (8.5) implies (8.1) via Hölder's inequality:

$$\int_{\mathbb{A}} |\mathcal{K}_\ell f_\lambda|^p = \int_{\mathbb{A}} \omega\, \mathcal{K}_\ell f_\lambda \leqslant \int_{\mathbb{A}} \omega\, \mathcal{K}_\ell g \leqslant \left(\int_{\mathbb{A}} |\mathcal{K}_\ell f_\lambda|^p\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{A}} |\mathcal{K}_\ell g|^p\right)^{\frac{1}{p}}.$$

This is the same as (8.1).

Proof of Lemma 8.2. We make use of inequality (7.6) in Proposition 7.2. There we put $\mathbf{M} = Dg(x)$, $N = \frac{n}{2n-2\ell}$ and $\tau = \eta_F(|x|)$. According to Remark 7.3 the condition (7.5) is satisfied for $\eta_F(|x|) \geqslant 1$. This makes it legitimate to write the following lower bound of $\mathcal{K}_\ell g$:

$$\mathcal{K}_\ell g \geqslant \mathcal{A} \|\mathbf{K}_O g\|^{\frac{1}{n-1}} + \mathcal{B} \tag{8.6}$$

where

$$\mathcal{A} = \mathcal{A}(|x|) = \frac{\frac{\ell}{n}\left[\frac{\ell}{n}\eta_F^2 + 1 - \frac{\ell}{n}\right]^{\frac{2\ell-n}{2n-2\ell}} (\eta_F^2 - 1)}{[\eta_F]^{\frac{n}{n-\ell}}} \tag{8.7}$$

and

$$\mathcal{B} = \mathcal{B}(|x|) = \frac{\left[\frac{\ell}{n}\eta_F^2 + 1 - \frac{\ell}{n}\right]^{\frac{2\ell-n}{2n-2\ell}}}{[\eta_F]^{\frac{\ell}{n-\ell}}}. \tag{8.8}$$

Equality holds in (8.6) if and only if the singular values of $Dg(x)$ are a scalar multiple of

$$\{1, \dots, 1, \tau\} \approx \left\{ \frac{F(|x|)}{|x|}, \dots, \frac{F(|x|)}{|x|}, \dot{F}(|x|) \right\}. \tag{8.9}$$

It is at this point of arguments that we appeal to the variational equation (5.20), to obtain the following identity

$$\omega(|x|)\mathcal{A}(|x|) = \frac{\frac{\ell}{n}\left[\frac{\ell}{n}\,\eta_F^2 + 1 - \frac{\ell}{n}\right]^{\frac{np}{2n-2\ell}-1} (\eta_F^2 - 1)}{[\eta_F]^{\frac{\ell p}{n-\ell}+1}} = \mathcal{Q}(\eta_F) = \frac{\lambda}{|x|^n} \tag{8.10}$$

where this time $\lambda \geqslant 0$. Multiplying (8.6) by the weight ω yields

$$\omega(|x|)\ \mathcal{K}_\ell g(x) \geq \frac{\lambda}{|x|^n}\ \|\mathbf{K}_O g(x)\|^{\frac{1}{n-1}} + \omega(|x|)\,\mathcal{B}(|x|). \tag{8.11}$$

In order to arrive at free-Lagrangians we need to estimate the expression

$$\|\mathbf{K}_O g\|^{\frac{1}{n-1}} = \frac{\|Dg\|^{\frac{n}{n-1}}}{J_g^{\frac{1}{n-1}}}$$

linearly in terms of $\|Dg\|$ and the Jacobian determinant J_g. Before proceeding let us introduce a function

$$\mathcal{C} = \mathcal{C}(|g(x)|) = \frac{1}{|g(x)|}. \tag{8.12}$$

We apply Young's inequality in the following way

$$\begin{aligned}\|Dg\| &= \frac{\|Dg\|}{\left[|x|^{n-1}\mathcal{C}^{n-1}J_g\right]^{\frac{1}{n}}} \cdot \left[|x|^{n-1}\mathcal{C}^{n-1}J_g\right]^{\frac{1}{n}} \\ &\leqslant \frac{n-1}{n}\frac{|Dg|^{\frac{n}{n-1}}}{|x|\mathcal{C}J_g^{\frac{1}{n-1}}} + \frac{1}{n}|x|^{n-1}\mathcal{C}^{n-1}J_g.\end{aligned}$$

Hence

$$\|\mathbf{K}_O g\|^{\frac{1}{n-1}} \geqslant \frac{n}{n-1}|x|\mathcal{C}\,\|Dg\| - \frac{1}{n-1}|x|^n\mathcal{C}^n J_g. \tag{8.13}$$

It should be noted that equality holds in (8.13) if and only if $\|Dg(x)\| = \mathcal{C}^{n-1}(x)|x|^{n-1}J_g(x)$. This is the case when $g = f_\lambda = F(|x|)\frac{x}{|x|}$. Indeed we have,

$$\mathcal{C}^{n-1}(x)|x|^{n-1}J_{f_\lambda}(x) = \dot{F}(|x|) = \|Df_\lambda(x)\|. \tag{8.14}$$

Now substituting (8.13) into (8.11) we obtain

$$\omega(|x|)\mathcal{K}_\ell g(x) \geqslant \frac{\lambda n}{n-1}\frac{\mathcal{C}}{|x|^{n-1}}\|Dg(x)\| - \frac{\lambda\mathcal{C}^n}{n-1}J_g(x) + \omega(|x|)\mathcal{B}(|x|). \tag{8.15}$$

The last step toward free-Lagrangians is to use the point-wise inequality

$$\|Dg\| \geqslant |g_N| \geqslant |g|_N \tag{8.16}$$

where $|g|_N$ stands for the radial derivative (the derivative in the direction of $\frac{x}{|x|}$). This latter inequality turns into equality again for $g \equiv f_\lambda = F(|x|)\frac{x}{|x|}$; because $\dot{F}(|x|)$, being the largest singular value of $Df(x)$, equals the operator norm of $Df(x)$. We are now in a position to integrate free-Lagrangians

$$\begin{aligned}\int_{\mathbb{A}} \omega(|x|)\mathcal{K}_\ell g(x)\,\mathrm{d}x &\geqslant \frac{\lambda\, n}{n-1}\int_{\mathbb{A}} \mathcal{C}\big(|g|\big)\,|g|_N \frac{\mathrm{d}x}{|x|^{n-1}} \\ &\quad - \frac{\lambda}{n-1}\int_{\mathbb{A}} \mathcal{C}^n\big(|g|\big)\,J_g(x)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{A}} \omega(|x|)\,\mathcal{B}(|x|)\,\mathrm{d}x \\ &\geqslant \frac{\lambda\, n}{n-1}\omega_{n-1}\int_{r_*}^{R_*} \mathcal{C}(s)\,\mathrm{d}s \qquad \text{by (3.11)} \\ &\quad - \frac{\lambda}{n-1}\int_{\mathbb{A}^*} \mathcal{C}^n(|y|)\,\mathrm{d}y \qquad \text{by (3.4)} \\ &\quad + \int_{\mathbb{A}} \omega(|x|)\,\mathcal{B}(|x|)\,\mathrm{d}x = \int_{\mathbb{A}} \omega(|x|)\mathcal{K}_\ell f_\lambda(x)\,\mathrm{d}x.\end{aligned} \tag{8.17}$$

Indeed, when $g \equiv f_\lambda$ all the inequalities above become equalities.

Uniqueness. Reversing the steps above, we can see that (8.17) holds as equality only when g is a rotation of f_λ. First, the following conditions on g must be satisfied.

(i) Singular values of $Dg(x)$ must by a multiple of $\left\{\frac{F(|x|)}{|x|}, \dots, \frac{F(|x|)}{|x|}, \dot{F}(|x|)\right\}$, in view of (8.9);

(ii) $\|Dg(x)\| = \frac{|x|^{n-1} J_g(x)}{|g(x)|^{n-1}}$, in view of (8.12) and (8.14);

(iii) $\|Dg(x)\| = |g_N(x)| = |g(x)|_N$, in view of (8.16).

We start with an observation that the condition $|g_N| = |g|_N$ alone implies that

$$g(x) = |g(x)|\, E\,(x/|x|) \qquad \text{where} \quad E : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}.$$

Indeed, we always have $|g|\,|g|_N = \langle g \mid g_N \rangle \leqslant |g|\,|g_N|$, so the equality $|g_N| = |g|_N$ is possible only when $|g|\, g_N = |g_N|\, g$. Then we find that

$$\left(\frac{g}{|g|}\right)_N = \frac{|g| g_N \; - \; g|g|_N}{|g|^2} = 0.$$

Hence $\frac{g}{|g|}$ is a function in $\frac{x}{|x|}$, say equal to $E\,(x/|x|)$.

Proceeding further, let $0 < \lambda_1(x) \leqslant \dots \leqslant \lambda_{n-1}(x) \leqslant \lambda_n(x) = \|Dg(x)\|$ be the singular values of $Dg(x)$. The first $(n-1)$ singular values must be equal, say $\lambda_1(x) = \dots = \lambda_{n-1}(x) \overset{\text{def}}{=\!=} \lambda(x)$, and by (ii) $\lambda_n = \left(\frac{|x|}{|g|}\right)^{n-1} \lambda^{n-1} \lambda_n$. Hence

$$0 < \lambda_1(x) = \dots = \lambda_{n-1}(x) = \lambda(x) = \frac{|g(x)|}{|x|}.$$

Again in view of (i) combined with (iii), we see that

$$\frac{|x| \dot{F}(|x|)}{F(|x|)} = \frac{\lambda_n(x)}{\lambda(x)} = \frac{|x|\; |g(x)|_N}{|g(x)|}.$$

Therefore, for $\omega \in \mathbb{S}^{n-1}$ fixed, we arrived at the differential equation

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t}\, |g(t\,\omega)|}{|g(t\,\omega)|} = \frac{\dot{F}(t)}{F(t)} \qquad \text{in the interval} \quad r < t < R. \tag{8.18}$$

At the endpoint $t = R$ both functions $|g(t\,\omega)|$ and $F(t)$ assume the same value R_*. It then follows that $|g(x)| = F(|x|)$ and hence,

$$g(x) = F(|x|) E(x/|x|) \qquad \text{for all} \quad x \in \mathbb{A}.$$

In other words g is a spherical sliding. Finally, we appeal to Lemma 6.2 to deduce that $E : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is a rotation. The proof of Lemma 8.2 is complete. □

This also ends the proof of Theorem 8.1. □

9. Proof of Theorem 1.15

By Theorems 5.10, 5.11 and Remark 5.12 there always exists (unique) stationary solution $f_\lambda \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$, $\lambda < 0$, which among all radial stretchings $f \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ assumes the smallest $\mathscr{L}^p$-mean $\mathcal{K}_\ell$-distortion, $p \geqslant 1$. We have $\eta_F(|x|) < 1$ in $\mathbb{A}$. The question arises whether f_λ is the *absolute minimizer*; that is,

$$\int_{\mathbb{A}} \mathcal{K}_\ell^p f_\lambda = \inf_{f \in \mathscr{H}^{1,p}(\mathbb{A},\mathbb{A}^*)} \int_{\mathbb{A}} \mathcal{K}_\ell^p f. \tag{9.1}$$

This is not always the case. We have already constructed (Theorem 6.1) counterexamples to (9.1) for some pair of annuli if the exponents p are too small; precisely, if

$$1 \leqslant p < p_\ell(n) \stackrel{\text{def}}{=\!=} \frac{n(n+1)^2}{8\ell} - (n-1) \qquad n \geqslant 3.$$

Let us observe that

$$p_\ell(n) \geqslant \frac{(n-1)(n-\ell)}{\ell} \stackrel{\text{def}}{=\!=} p_\circ = p_\circ(\ell, n). \tag{9.2}$$

In what follows the exponent $p_\circ$ will be critical to our arguments. Then the results for $p_\circ$ will easily be extended to hold for all $p \geqslant p_\circ$. However, the condition $p \geqslant p_\circ$ alone is still not sufficient for (9.1) to hold for every pair of annuli in dimensions $n \geqslant 4$. We must impose in addition to $p \geqslant p_\circ$ a lower bound on $\operatorname{Mod} \mathbb{A}^*$; namely,

$$\Gamma(\operatorname{Mod} \mathbb{A}) \leqslant \operatorname{Mod} \mathbb{A}^* \leqslant \operatorname{Mod} \mathbb{A} \qquad (\Gamma \equiv 0 \text{ for } n = 2, 3). \tag{9.3}$$

Here $\Gamma = \Gamma_{\ell,n,p} \colon [0, \infty) \to [0, \infty)$ will be a function continuously increasing from 0 to ∞. It grows to ∞ sublinearly; exactly, $\Gamma(M) < M$ for all $0 < M < \infty$.

9.1. Definition of Γ

The idea behind the construction of Γ is to ensure that under Condition (9.3) the elasticity function $\eta_F(|x|)$ of the stationary solution $f_\lambda \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ for the $\mathcal{L}^p$-mean distortion satisfies

$$\left(1 + \frac{\ell}{n-\ell}\, \eta_F^2\right)^{\frac{n(n-1)}{2\ell} - 1} (1 - \eta_F^2) < 1 \quad \text{everywhere in } \mathbb{A}. \tag{9.4}$$

Be aware that the exponent p is implicitly involved in this inequality through the elasticity function η_F. The reader may verify that Inequality (9.4) holds for all $0 < \eta_F \leqslant 1$ in dimensions $n = 2, 3$. That is why $\Gamma \equiv 0$ in dimensions $n = 2, 3$. For $n \geqslant 4$, we examine the function:

$$\varphi(x) = \varphi_{\ell,n}(x) = \left(1 + \frac{\ell}{n-\ell}\, x^2\right)^{\frac{n(n-1)}{2\ell} - 1} (1 - x^2) \qquad \text{for } 0 \leqslant x \leqslant 1$$

and its derivative

$$\varphi'(x) = \frac{n(n-1)}{n-\ell} x \left(1 + \frac{\ell}{n-\ell} x^2\right)^{\frac{n(n-1)}{2\ell}-2} \left[\frac{n-3}{n-1} - x^2\right].$$

See the graph of φ (Figure 9.1) below

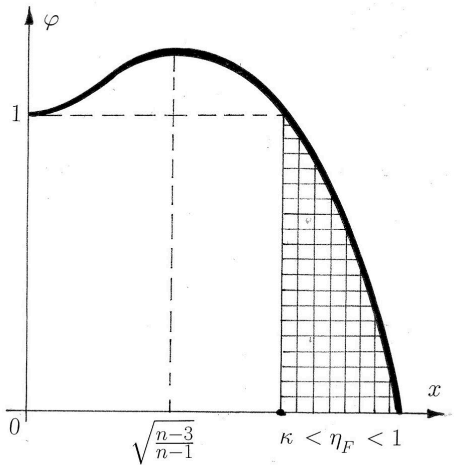

Figure 9.1. Graph of φ.

We have unique solution of the equation

$$\begin{gathered} \varphi(\kappa) = 1 \quad \text{for } 0 < \kappa = \kappa_{n,\ell} < 1 \\ \left(1 + \frac{\ell}{n-\ell}\kappa^2\right)^{\frac{n(n-1)}{2\ell}-1} (1-\kappa^2) = 1. \end{gathered} \tag{9.5}$$

Now condition (9.4) is satisfied whenever

$$\kappa_{n,\ell} < \eta_F(|x|) \leqslant 1 \quad \text{for all } x \in \mathbb{A}. \tag{9.6}$$

Here are two explicit numerical values of κ,

$$\kappa_{4,3} = \sqrt{\frac{2}{3}} \qquad \kappa_{4,2} = \sqrt{\frac{\sqrt{5}-1}{2}}\,. \quad \text{It always holds that } \kappa_{n,\ell} > \sqrt{\frac{n-3}{n-1}}\,.$$

We invoke the function $Q = Q_{\ell,n,p} = \dot{\mathcal{P}}_{\ell,n,p}$ in (5.12) to produce the lower bound at (9.3). Keep in mind that $Q(\kappa) < 0$.

Definition 9.1. The function Γ is defined by the rule

$$\begin{cases} \Gamma \equiv 0 \quad \text{for } n = 2, 3 \text{ and} \\ \Gamma(M) = \Gamma_{\ell,n,p}(M) = \int_0^M \Phi\left[Q(\kappa)e^{-ns}\right] \mathrm{d}s \; < M \text{ for } n \geqslant 4. \end{cases} \tag{9.7}$$

The latter inequality holds because $\Phi(\tau) < 1$ for $\tau < 0$. Obviously Γ is increasing from 0 to ∞ , and we have $\kappa M < \Gamma(M) < M$, where $1 > \kappa > \sqrt{\frac{n-3}{n-1}}$. This is straightforward from $\kappa = \Phi(Q(\kappa)) < \Phi(Q(\kappa)e^{-ns})$.

Lemma 9.2. *Condition* (9.3) *yields* (9.4).

Proof. Let $\lambda < 0$ be the parameter for which $f_\lambda \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is the stationary solution. Integration by substitution shows that

$$\int_r^R \Phi\left(\frac{\lambda}{s^n}\right)\frac{\mathrm{d}s}{s} = \operatorname{Mod}\mathbb{A}^* \geqslant \Gamma(\operatorname{Mod}\mathbb{A}) = \int_0^{\log\frac{R}{r}} \Phi\left(Q(\kappa)e^{-ns}\right)\,\mathrm{d}s$$
$$= \int_r^R \Phi\left(\frac{Q(\kappa)r^n}{s^n}\right)\frac{\mathrm{d}s}{s}.$$

Since Φ is increasing the parameter λ must satisfy $\lambda \geqslant Q(\kappa)r^n$. Finally, we invoke the formula $\eta_F(t) = \Phi\left(\frac{\lambda}{t^n}\right)$ which shows that η_F is increasing because $\lambda < 0$. Therefore,

$$\eta_F(t) > \eta_F(r) = \Phi\left(\frac{\lambda}{r^n}\right) \geqslant \Phi\big(Q(\kappa)\big) = \kappa$$

as desired. $\square$

We now proceed to the proof of Theorem 1.15, which reads as:

Theorem 9.3. *Suppose we are given an exponent*

$$p \geqslant p_\circ \stackrel{\text{def}}{=\!=} \frac{(n-1)(n-\ell)}{\ell} \geqslant 1 \tag{9.8}$$

and a pair of annuli $\mathbb{A}$ *and* $\mathbb{A}^*$ *such that*

$$\Gamma(\operatorname{Mod}\mathbb{A}) \leqslant \operatorname{Mod}\mathbb{A}^* \leqslant \operatorname{Mod}\mathbb{A} \tag{9.9}$$

where $\Gamma = \Gamma_{\ell,n,p}$ *is defined by* (9.7). *Then for every* $f \in \mathscr{H}^{1,p}(\mathbb{A},\mathbb{A}^*)$ *we have*

$$\int_{\mathbb{A}} \mathcal{K}_\ell^p f \;\geqslant \int_{\mathbb{A}} \mathcal{K}_\ell^p f_\lambda = \inf_{g\in\mathscr{H}^{1,p}(\mathbb{A},\mathbb{A}^*)} \int_{\mathbb{A}} \mathcal{K}_\ell^p g. \tag{9.10}$$

Equality occurs iff $f \equiv f_\lambda$, *modulo rotation.*

Remark 9.4. Condition (9.9) always holds in dimensions $n = 2, 3$.

Proof of Theorem 9.3. The case $p = p_\circ$ will be crucial in the proof. We will be reduced to this case via weighted inequality. The weight function is given by

$$\omega = \omega(|x|) = (\mathcal{K}_\ell f_\lambda)^{p-p_\circ} = \frac{\left[\frac{\ell}{n}\eta_F^2 + 1 - \frac{\ell}{n}\right]^{\frac{n(p-p_\circ)}{2(n-\ell)}}}{\eta_F^{\frac{\ell(p-p_\circ)}{n-\ell}}} \tag{9.11}$$

where $\eta_F = \eta_F(t) = \frac{t\dot{F}(t)}{F(t)}$ stands for the elasticity function of the stationary solution f_λ (note it depends on p). Theorem 9.3 is a concequence of the following weighted inequality:

Proposition 9.5. *For every $g \in \mathscr{H}^{1,n-1}(\mathbb{A}, \mathbb{A}^*)$ we have*

$$\int_{\mathbb{A}} \left(\mathcal{K}_\ell^{p_\circ} g\right) \omega \geqslant \int_{\mathbb{A}} \left(\mathcal{K}_\ell^{p_\circ} f_\lambda\right) \omega. \tag{9.12}$$

Equality holds if and only if $g = f_\lambda$ (modulo rotation).

Before proceeding to the proof of this proposition let us show how does it imply Theorem 9.3. We have,

$$\int_{\mathbb{A}} \mathcal{K}_\ell^p f_\lambda = \int_{\mathbb{A}} \omega \mathcal{K}_\ell^{p_\circ} f_\lambda \leqslant \int_{\mathbb{A}} \omega \mathcal{K}_\ell^{p_\circ} g \leqslant \left(\int_{\mathbb{A}} \mathcal{K}_\ell^p f_\lambda\right)^{\frac{p-p_\circ}{p}} \left(\int_{\mathbb{A}} \mathcal{K}_\ell^p g\right)^{\frac{p_\circ}{p}}$$

which is the same as

$$\int_{\mathbb{A}} \mathcal{K}_\ell^p f_\lambda \leqslant \int_{\mathbb{A}} \mathcal{K}_\ell^p g \qquad \text{for every} \quad g \in \mathscr{H}^{1,n-1}(\mathbb{A}, \mathbb{A}^*)$$

as desired.

Proof of Proposition 9.5. Let $f_\lambda = F(|x|)\frac{x}{|x|}$ be the stationary solution for the $\mathscr{L}^p$-mean distortion. We make use of lower bound (7.2) in Proposition 7.1 with $\mathbf{M} = Dg(x)$, $\tau = \eta_F(|x|)$ and $N = \frac{n(n-1)}{2\ell}$. Accordingly,

$$\mathcal{K}_\ell^{p_\circ} g \geqslant \mathcal{A}\ \|\mathbf{K}_I g\| + \mathcal{B} \tag{9.13}$$

where

$$\mathcal{A} = \mathcal{A}(|x|) = \left(1 - \frac{\ell}{n}\right)\left[1 - \frac{\ell}{n} + \frac{\ell}{n}\eta_F^2\right]^{N-1}\left(1 - \eta_F^2\right) \tag{9.14}$$

and

$$\mathcal{B} = \mathcal{B}(|x|) = \left[1 - \frac{\ell}{n} + \frac{\ell}{n}\eta_F^2\right]^{N-1} \eta_F^{3-n} \qquad N = \frac{n(n-1)}{2\ell}. \tag{9.15}$$

The above lower bound is legitimate because of condition (9.4). Equality occurs in (9.13) if and only if the singular values of Dg are scalar multiple of $\{1, \dots, 1, \eta_F\} \approx \left\{\frac{F(|x|)}{|x|}, \dots, \frac{F(|x|)}{|x|}, \dot{F}(|x|)\right\}$. Here $\dot{F}(|x|)$ is the smallest singular value of Df and therefore

$$\|D^\sharp f_\lambda\| = \left(\frac{F(|x|)}{|x|}\right)^{n-1}, \tag{9.16}$$

whereas

$$J_{f_\lambda}(x) = \left(\frac{F(|x|)}{|x|}\right)^{n-1} F(|x|). \tag{9.17}$$

Next we estimate from below the inner distortion of g by using Young's inequality

$$\mathbf{K}_I g = \frac{\|D^\sharp g\|^n}{J_g^{n-1}} \geqslant n\,\mathcal{C}^{n-1}\|D^\sharp g\| - (n-1)\,\mathcal{C}^n J_g. \tag{9.18}$$

This holds for all $\mathcal{C} \geqslant 0$; equality occurs if and only if $\mathcal{C} = \frac{\|D^\sharp g\|}{J_g}$. We shall take

$$\mathcal{C} = \frac{F(|x|)}{|g(x)|\dot{F}(|x|)}.$$

Thus (9.13) takes the form

$$(\mathcal{K}_\ell g)^{p_\circ} \geqslant n\,\mathcal{A}\frac{(F)^{n-1}}{|g|^{n-1}(\dot{F})^{n-1}}\|D^\sharp g\| - (n-1)\,\mathcal{A}\frac{(F)^n}{|g|^n(\dot{F})^n} J_g + \mathcal{B}. \tag{9.19}$$

Equality holds if $g = f_\lambda$. Indeed, in this case we have $\mathcal{C} = 1/\dot{F} = \frac{\|D^\sharp f_\lambda\|}{J_\lambda}$. For a general mapping $g \in \mathscr{H}^{1,n-1}(\mathbb{A}, \mathbb{A}^*)$ it must be required that

$$\frac{\|D^\sharp g\|}{J_g} = \frac{F(|x|)}{|g(x)|\ \dot{F}(|x|)}. \tag{9.20}$$

We are going to multiply (9.19) by the weight function $\omega = \omega(|x|)$ defined at (9.11). Before proceeding let us simplify the product of ω and the $\mathcal{A}$ coefficient. It is at this point of arguments (and only at this point) that we appeal to the variational equation (5.20). Formulas (9.11) and (9.14) yield.

$$\begin{aligned}\omega(|x|)\mathcal{A}(|x|) &= \frac{\left(1-\frac{\ell}{n}\right)\left[1-\frac{\ell}{n}+\frac{\ell}{n}\eta_F^2\right]^{\frac{np}{2n-2\ell}-1}(1-\eta_F^2)}{\eta_F^{1+\frac{\ell p}{n-\ell}}}\cdot\eta_F^n \\ &= \frac{\ell-n}{\ell p}\,Q(\eta_F)\cdot\eta_F^n = \frac{\ell-n}{\ell p}\cdot\lambda\cdot\left(\frac{\dot{F}(|x|)}{F(|x|)}\right)^n.\end{aligned}$$

We indicate that $\frac{\ell-n}{\ell p}\cdot\lambda > 0$ to arrive at the following lower bound:

$$\omega(\mathcal{K}_\ell g)^{p_\circ} \geqslant \frac{n(\ell-n)\lambda}{\ell p}\frac{\dot{F}(|x|)}{F(|x|)}\frac{\|D^\sharp g\|}{|g|^{n-1}} - \frac{(n-1)(\ell-n)\lambda}{\ell p\, n}\frac{J_g}{|g|^n} + \omega(|x|)\mathcal{B}(|x|). \tag{9.21}$$

The last step toward the establishment of free-Lagrangians is to use (3.21); namely,

$$\|D^\sharp g\| \geqslant g_{\mathbf{S}}\,|g|^{n-1}. \tag{9.22}$$

Here again equality holds for the stationary solution $g = f_\lambda$. We substitute (9.22) into (9.21) and integrate free Lagrangians. Using (3.20) and (3.4) we conclude with the desired sharp lower bound.

$$
\begin{aligned}
\int_{\mathbb{A}} \omega(|x|) \left[\mathcal{K}_\ell g(x)\right]^{p_\circ} \, \mathrm{d}x \geq \; & \frac{n(\ell-n)\omega_{n-1}\,\lambda}{\ell\, p} \int_r^R \frac{\dot{F}(t)}{F(t)} \, \mathrm{d}t \\
& - \frac{(n-1)(\ell-n)\lambda}{\ell p\, n} \int_{\mathbb{A}^*} \frac{\mathrm{d}y}{|y|^n} \\
& + \int_{\mathbb{A}} \omega(|x|) \mathcal{B}(|x|) \, \mathrm{d}x \\
= & \int_{\mathbb{A}} \omega(|x|) \left[\mathcal{K}_\ell f_\lambda(x)\right]^{p_\circ} \, \mathrm{d}x .
\end{aligned}
$$

The latter is justified because all the inequalities above turn into equalities when $g \equiv f_\lambda$.

Uniqueness. The proof proceeds along the same lines as the proof of uniqueness in Lemma 8.2, for Theorem 8.1 . It maybe worth pointing out some nuances though. First, to reach equalities the following conditions on g must be imposed.

(i) Singular values of $Dg(x)$ must be a multiple of $\left\{\frac{F(|x|)}{|x|}, \dots, \frac{F(|x|)}{|x|}, \; \dot{F}(|x|)\right\}$;

(ii) $\frac{|g| \cdot \|D^\sharp g\|}{J_g} = \frac{F(|x|)}{\dot{F}(|x|)}$, in view of (9.20);

(iii) $\|D^\sharp g\| = g_{\mathbf{S}} \, |g|^{n-1}$, in view of (9.22).

We start with an observation that the equality $\|D^\sharp g\| = g_{\mathbf{S}} \, |g|^{n-1}$ alone implies

$$
g(x) = |g(x)| E\,(x/|x|) \qquad \text{where} \quad E : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}.
$$

Indeed, we have

$$
g_{\mathbf{S}} \, |g|^{n-1} \stackrel{\text{def}}{=\!=} \left\langle \frac{x}{|x|} \middle| \left[D^\sharp g\right] \frac{g}{|g|} \right\rangle \leqslant \left| \left[D^\sharp g\right] \frac{g}{|g|} \right| \leqslant \|D^\sharp g\|.
$$

For equality to occur it is necessary that $[D^\sharp g] \frac{g}{|g|} = \|D^\sharp g\| \frac{x}{|x|}$. We apply the linear differential map Dg to both sides of this equation, use the identities $[Dg]\,[D^\sharp g] = J_g \mathbf{I}$ and $[Dg] \frac{x}{|x|} = g_N$, to infer that

$$
g_N(x) = \alpha\, g(x) \qquad \text{where} \quad \alpha = \alpha(x) \; = \frac{J_g(x)}{|g(x)| \cdot \|D^\sharp g(x)\|}.
$$

Hence, in particular, $|g|_N = \left\langle \frac{g}{|g|} \; \middle| g_N \right\rangle = \alpha\, |g| \; = |g_N|$. This yields

$$
\left(\frac{g}{|g|}\right)_N = \frac{|g| g_N - g |g|_N}{|g|^2} = 0.
$$

Thus $\frac{g}{|g|} \overset{\text{def}}{=\!=} E(x/|x|)$ is a function in $\frac{x}{|x|}$, as expected. Also note, by (ii), that $\alpha = \frac{\dot{F}(|x|)}{F(|x|)}$. On the other hand $\alpha = \frac{|g|_N}{|g|}$. We just arrived at the same differential equation (8.18) from the proof of uniqueness in Lemma 8.2. It then follows, by the same reasoning, that $|g(x)| = F(|x|)$ and that $g(x) = F(|x|)E(x/|x|)$ is a spherical sliding. Again by Lemma 6.2 we conclude that $E : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is a rotation, completing the proof of Proposition 9.5. □

This also completes the proof of Theorem 9.3. □

9.2. Proof of Theorem 1.8

The case $p > 1$ is immediately from Theorem 1.9, proven in Section 8. Also the case $p = 1$ is covered by Theorem 1.9 if $\operatorname{Mod} \mathbb{A}^* \leqslant \operatorname{Mod} \mathbb{A}^*_{\max}$. It remains to consider that case $p = 1$ and $\operatorname{Mod} \mathbb{A}^* > \operatorname{Mod} \mathbb{A}^*_{\max}$. In this case we observe:

9.3. Squeezing phenomenon for absolute minimizers

This occurs only in the expansion case $\operatorname{Mod}\mathbb{A} < \operatorname{Mod}\mathbb{A}^*$ with $p = 1$. Let us take a quick look at critical case $\mathbb{A}^* = \mathbb{A}^*_{\max}$ and $f(x) = f_{\max}(x) \overset{\text{def}}{=\!=} F(|x|)\frac{x}{|x|}$. The elasticity quotient blows up to infinity as we approach the inner boundary of $\mathbb{A}$. Precisely,

$$\eta_F(x) = \Phi\left(\frac{Cr^n}{|x|^n}\right) \longrightarrow \infty \qquad \text{as} \quad |x| \searrow r \quad \text{where} \quad C = \left(\frac{\ell}{n}\right)^{\frac{n}{2n-2\ell}}.$$

In particular,

$$\mathcal{K}_\ell f_{\max}(x) = C\,\eta_F(x) + \mathcal{O}(1) \longrightarrow \infty \qquad \text{as} \quad |x| \searrow r.$$

It is counterintuitive that the derivative $\dot{F}(|x|)$ of the strain function of a radially minimal map blows up when $|x| \searrow r$.

In the proof of Theorem 5.13 we have constructed a minimizing sequence $f_j : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^* \supsetneq \mathbb{A}^*_{\max}$ of radial stretchings $f_j(x) = F_j(|x|)\frac{x}{|x|}$ which converge to $f_{\max} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*_{\max}$.

The normal strain functions $F_j : [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$ converge to $F : [r, R] \xrightarrow{\text{onto}} [r_*^+, R_*]$ uniformly on closed subintervals of $(r, R]$. Recall that r_*^+ stands for the inner radius of the maximal annulus $\mathbb{A}^*_{\max}$. Approaching the left endpoint with $r_j \searrow r$ we obtain $F_j(r_j) = F(r_j) \to r_*^+ > r_*$. A quick look at the left part of Figure 9.2 helps to visualize this incident.

Let us introduce a notation for the missing annulus

$$\Delta^* \overset{\text{def}}{=\!=} \mathbb{A}^* \setminus \mathbb{A}^*_{\max} = \{y; r_* < |y| < r_*^+\}. \tag{9.23}$$

Precise statement, which generalizes Theorem 5.13 , is the following:

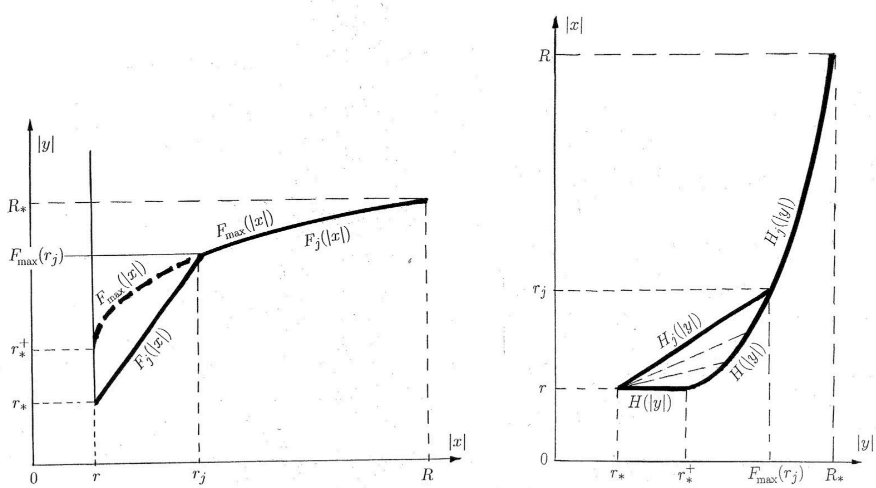

Figure 9.2. The convex functions $\mathcal{P}_{\ell,p}$.

Theorem 9.6. *Let* $\operatorname{Mod} \mathbb{A}^*_{\max} < \operatorname{Mod} \mathbb{A}^*$. *The class* $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$ *does not contain a mapping with smallest* $\mathscr{L}^1$*-norm of* $\mathcal{K}_\ell$*-distortion; the infimum is given by,*

$$\inf_{g \in \mathscr{H}^{1,1}(\mathbb{A},\mathbb{A}^*)} \int_{\mathbb{A}} \mathcal{K}_\ell g = \int_{\mathbb{A}} \mathcal{K}_\ell f_{\max} + \omega_{n-1} \lambda_{\max} \operatorname{Mod} \Delta^*. \tag{9.24}$$

Proof. Given any $g : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of class $\mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$, we shall measure the integral mean of $\mathcal{K}_\ell g$ against that of $\mathcal{K}_\ell f_{\max}$ for $f_{\max} = F(|x|)\frac{x}{|x|} : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*_{\max} \subsetneq \mathbb{A}^*$. For this purpose we again appeal to the point-wise inequality (8.15). Here we take $\omega = 1$, $\lambda = \lambda_{\max}$ and $C = 1/|g(x)|$. The coefficient $\mathcal{B}(|x|)$ is given in (9.15). We then replace, as we may, the operator norm $\|Dg\|$ by the normal derivative of $|g|$.

$$\mathcal{K}_\ell g(x) \geqslant \frac{n\,\lambda_{\max}}{n-1} \frac{|g(x)|_N}{|x|^{n-1}|g(x)|} - \frac{\lambda_{\max}}{n-1} \frac{J_g(x)}{|g(x)|^n} + \mathcal{B}(|x|). \tag{9.25}$$

The following necessary conditions for having equality in (9.25) (almost everywhere in $\mathbb{A}$) will be useful.

(i) Singular values of $Dg(x)$ must by a multiple of $\left\{\frac{F(|x|)}{|x|}, \dots, \frac{F(|x|)}{|x|}, \dot{F}(|x|)\right\}$, with $\dot{F}(|x|) > \frac{F(|x|)}{|x|}$, see (8.9);

(ii) $\|Dg(x)\| = \frac{|x|^{n-1} J_g(x)}{|g(x)|^{n-1}}$, see (8.14);

(iii) $\|Dg(x)\| = |g(x)|_N$.

On the other hand Inequality (9.25) applied to $g = f_{\max}$ holds as equality,

$$\mathcal{K}_\ell f_{\max}(x) = \frac{n\lambda_{\max}}{n-1} \frac{\dot{F}(|x|)}{|x|^{n-1}F(|x|)} - \frac{\lambda_{\max}}{n-1} \frac{J_{f_{\max}}(x)}{|f_{\max}(x)|^n} + \mathcal{B}(|x|). \tag{9.26}$$

We now integrate the difference $\mathcal{K}_\ell g - \mathcal{K}_\ell f_{\max}$ by using free-Lagranians in (3.11) and (3.4) to conclude that

$$\begin{aligned}
&\int_{\mathbb{A}} \mathcal{K}_\ell g(x)\,\mathrm{d}x - \int_{\mathbb{A}} \mathcal{K}_\ell f_{\max}(x)\,\mathrm{d}x \\
&\geqslant + \left[\frac{n\,\lambda_{\max}}{n-1}\,\omega_{n-1} \int_{r_*}^{R_*} \frac{\mathrm{d}s}{s} - \frac{\lambda_{\max}}{n-1} \int_{\mathbb{A}^*} \frac{\mathrm{d}y}{|y|^n} \right] \qquad \big(\equiv \lambda_{\max}\omega_{n-1}\mathrm{Mod}\,\mathbb{A}^*\big) \\
&\quad - \left[\frac{n\,\lambda_{\max}}{n-1}\,\omega_{n-1} \int_{r_*^+}^{R_*} \frac{\mathrm{d}s}{s} - \frac{\lambda_{\max}}{n-1} \int_{\mathbb{A}^*_{\max}} \frac{\mathrm{d}y}{|y|^n} \right] \quad \big(\equiv -\lambda_{\max}\omega_{n-1}\mathrm{Mod}\,\mathbb{A}^*_{\max}\big) \\
&= \lambda_{\max}\,\omega_{n-1}\mathrm{Mod}\ \Delta^*\,,
\end{aligned}$$

as claimed.

For the nonexistence statement in Theorem 9.6 suppose that, on the contrary, the infimum in (9.24) is attained at some $g \in \mathscr{H}^{1,1}(\mathbb{A}, \mathbb{A}^*)$. This forces the conditions itemized above to hold almost everywhere. Let $0 < \lambda_1(x) \leqslant \ldots \leqslant \lambda_{n-1}(x) \leqslant \lambda_n(x) = \|Dg(x)\|$ denote the singular values of $Dg(x)$. In view of item (i) the first $(n-1)$ singular values are the same, say $\lambda_1(x) = \ldots = \lambda_{n-1}(x) \overset{\mathrm{def}}{=\!=} \lambda(x)$. Then, by item (ii) $\lambda_n = \left(\frac{|x|}{|g|}\right)^{n-1} \lambda^{n-1}\lambda_n$. Hence

$$0 < \lambda_1(x) = \ldots = \lambda_{n-1}(x) = \lambda(x) = \frac{|g(x)|}{|x|}.$$

Now, combining (i) and (iii) yields

$$\frac{|x|\dot{F}(|x|)}{F(|x|)} = \frac{\lambda_n(x)}{\lambda(x)} = \frac{|x|\ |g(x)|_N}{|g(x)|}.$$

We look at this identity as an ordinary differential equation for $|g(x)|$ together with the ”initial” boundary condition $|g(x)| = R_* = F(|x|)$ at the outer circle $|x| = R$, like that in (8.18). This yields $|g(x)| = F(|x|)$ for all $x \in \mathbb{A}$. In particular, $|g(x)| = F(|x|) = r_*$ at the inner circle $|x| = r$, which is a clear contradiction of the inclusion $\mathbb{A}^* \supsetneq \mathbb{A}^*_{\max}$. The proof of Theorem 9.6 is complete. □

10. Energy integral for inverse mappings

Another explanation of the squeezing phenomenon and its significance for understanding traction-free problems can be enhanced by looking at the energy-integrals for the inverse mappings.

$$h \overset{\mathrm{def}}{=\!=} f^{-1} : \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A} \qquad h = h(y) \quad y = f(x).$$

In the proof of Theorem 5.13, the minimizing sequence of homeomorphisms $f_j : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ suggests investigating the inverse mappings $h_j = f_j^{-1} : \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A}$.

These are radial stretchings $h_j(y) = H_j(|y|)\frac{y}{|y|}$, $H_j = F_j^{-1} : [r_*, R_*] \xrightarrow{\text{onto}} [r, R]$. The advantage lies in the fact that this time the mappings h_j and their deformation gradients Dh_j converge uniformly on $[r_*, R_*]$. However, the limit map $h_\circ$ is no longer injective, we have

$$h_\circ(y) = \begin{cases} r\, y/|y| \quad \text{in } \mathbb{A}^* \setminus \mathbb{A}^*_{\max} = \Delta^* & \left(\begin{matrix}\text{squeezing (but not folding) type}\\ \text{of an interpenetration of matter}\end{matrix}\right) \\ \\ f^{-1}_{\max}(y) \quad \big(\text{elastic, reversible, deformation in } \mathbb{A}^*_{\max}\big). \end{cases} \tag{10.1}$$

See Figure 9.2.

We have, from Formula (2.22), and the change of variables for diffeomorphisms $f : \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$

$$\int_{\mathbb{A}} \big[\mathcal{K}_\ell f(x)\big]^p \, \mathrm{d}x = \binom{n}{\ell}^{-\frac{np}{2n-2\ell}} \int_{\mathbb{A}^*} \frac{\big|\, D_\sharp^{n-\ell} h(y)\,\big|^{\frac{np}{n-\ell}}}{J_h^{p-1}(y)} \, \mathrm{d}y \overset{\text{def}}{=\!=} \mathscr{E}_p[h] \tag{10.2}$$

where h stands for the inverse of f.

Now the energy integral $\mathscr{E}_p[h]$ can reasonably be extended to noninjective mappings. We should allow for *squeezing of matter*, but not for *folding*. Thus we adopt the weak limits $h : \mathbb{A}^* \xrightarrow{\text{onto}} \mathbb{A}$ of Sobolev homeomorphisms [33]. In dimension $n = 2$ these are none other than monotone Sobolev mappings [32]. Next, we demonstrate this precisely when $n = 2$.

10.1. Planar case

10.1.1. $p = 1$

First, we recall the following existence result for traction free minimizers.

Definition 10.1. Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{C}$ be Lipschitz domains.

- Let $\mathscr{M}^{1,2}\,(\overline{\mathbb{Y}}, \overline{\mathbb{X}})$ denote the class of orientation preserving monotone mappings $h : \overline{\mathbb{Y}} \xrightarrow{\text{onto}} \overline{\mathbb{X}}$ of finite Dirichlet energy;
- Let $\mathscr{H}^{1,2}\,(\mathbb{Y}, \mathbb{X})$ denote the class of orientation preserving homeomorphisms $h : \mathbb{Y} \xrightarrow{\text{onto}} \mathbb{X}$ of finite Dirichlet energy.

Now there always exists $h_* \in \mathscr{M}^{1,2}\,(\overline{\mathbb{Y}}, \overline{\mathbb{X}})$ with smallest Dirichlet energy, see [32]. Furthermore, the energy of h_* equals exactly the infimum of the energy among all $\mathscr{W}^{1,2}$-homeomorphisms:

$$\begin{aligned} \int_{\mathbb{Y}} |Dh_*(y)|^2 \, \mathrm{d}y &= \min_{h \in \mathscr{M}^{1,2}(\overline{\mathbb{Y}}, \overline{\mathbb{X}})} \int_{\mathbb{Y}} |Dh(y)|^2 \mathrm{d}y \\ &= \inf_{h \in \mathscr{H}^{1,2}(\mathbb{Y}, \mathbb{X})} \int_{\mathbb{Y}} |Dh(y)|^2 \, \mathrm{d}y = \inf_{h \in \mathcal{D}\mathrm{iff}(\mathbb{Y}, \mathbb{X})} \int_{\mathbb{Y}} |Dh(y)|^2 \mathrm{d}y \end{aligned} \tag{10.3}$$

where $\mathcal{D}\mathrm{iff}(\mathbb{Y}, \mathbb{X})$ stands for the class of $\mathscr{C}^1$-diffeomorphisms from $\mathbb{Y}$ onto $\mathbb{X}$.

Returning to the limit map $h_\circ$ in (10.1).

Theorem 10.2 ([5,30]). *Let* $\mathbb{A}$ *and* $\mathbb{A}^*$ *be planar annuli and* $\operatorname{Mod}\mathbb{A} < \operatorname{Mod}\mathbb{A}^*$. *Then,*

$$\inf_{h\in\mathscr{H}^{1,2}(\mathbb{A}^*,\,\mathbb{A})}\int_{\mathbb{A}^*}|Dh(y)|^2\,dy = \int_{\mathbb{A}^*}|Dh_\circ(y)|^2\,dy. \tag{10.4}$$

Proof. By (10.3) we have

$$\min_{h\in\mathscr{M}^{1,2}(\overline{\mathbb{A}^*},\overline{\mathbb{A}})}\int_{\mathbb{A}^*}|Dh(y)|^2\,\mathrm{d}y = \inf_{h\in\mathscr{H}^{1,2}(\mathbb{A}^*,\mathbb{A})}\int_{\mathbb{A}^*}|Dh(y)|^2\,\mathrm{d}y\,,$$

and

$$\inf_{h\in\mathscr{H}^{1,2}(\mathbb{A}^*,\,\mathbb{A})}\int_{\mathbb{A}^*}|Dh(y)|^2\,\mathrm{d}y = \inf_{h\in\mathcal{D}\mathrm{iff}(\mathbb{A}^*,\,\mathbb{A})}\int_{\mathbb{A}^*}|Dh(y)|^2\,\mathrm{d}y.$$

By (10.2), with $\ell = 1$ and $n = 2$

$$\inf_{h\in\mathcal{D}\mathrm{iff}(\mathbb{A}^*,\,\mathbb{A})}\int_{\mathbb{A}^*}|Dh(y)|^2\,\mathrm{d}y = 2\inf_{f\in\mathcal{D}\mathrm{iff}(\mathbb{A},\,\mathbb{A}^*)}\int_{\mathbb{A}}\mathcal{K}_1 f(x)\,\mathrm{d}x.$$

Combining these with Theorem 9.6 we have

$$\min_{h\in\mathscr{M}^{1,2}(\overline{\mathbb{A}^*},\overline{\mathbb{A}})}\int_{\mathbb{A}^*}|Dh(y)|^2\,\mathrm{d}y = 2\int_{\mathbb{A}}\mathcal{K}_1 f_{\max} + 4\pi\,\lambda_{\max}\operatorname{Mod}\,\Delta^*.$$

On the other hand,

$$\int_{\mathbb{A}^*}|Dh_\circ(y)|^2\,\mathrm{d}y = \int_{\mathbb{A}^*_{\max}}|Dh_\circ|^2 + \int_{\Delta^*}|Dh_\circ|^2\,.$$

Here

$$|Dh_\circ(y)|^2 = \frac{r^2}{|y|^2}\quad\text{ in } \Delta^*$$

and

$$\int_{\mathbb{A}^*_{\max}}|Dh_\circ|^2 = 2\int_{\mathbb{A}}\mathcal{K}_1 f_{\max}\,,\qquad \text{by (10.2).}$$

Hence (10.4) follows. □

10.1.2. $p > 1$

Fix $1 < p < \infty$. Let $\mathbb{A}$ and $\mathbb{A}^*$ be planar annuli and $\mathfrak{f}:\mathbb{A}\xrightarrow{\text{onto}}\mathbb{A}^*$ be the extremal $\mathscr{L}^p$-Teichmüller map; that is, absolute minimizer for the inverse problem, that is,

$$\int_{\mathbb{A}}[\mathcal{K}_1 g]^p \geqslant \int_{\mathbb{A}}[\mathcal{K}_1\mathfrak{f}]^p \qquad \text{for every}\quad g\in\mathscr{H}^{1,1}(\mathbb{A},\mathbb{A}^*). \tag{10.5}$$

Recall that $\mathfrak{f}\colon\mathbb{A}\xrightarrow{\text{onto}}\mathbb{A}^*$ is a $\mathscr{C}^\infty$-diffeomorphism. We denote the inverse of $\mathfrak{f}$ by $\mathfrak{h}\colon\mathbb{A}^*\xrightarrow{\text{onto}}\mathbb{A}$. Note that $\mathfrak{h}\in\mathscr{H}^{1,2}(\mathbb{A}^*,\mathbb{A})$. We aim to prove the following result.

Theorem 10.3. *For every $h \in \mathscr{H}^{1,2}(\mathbb{A}^*, \mathbb{A})$ it holds that*

$$\mathscr{E}_p[h] = \int_{\mathbb{A}^*} \frac{|Dh(y)|^{2p}}{J_h^{p-1}(y)} \, \mathrm{d}y \geqslant \int_{\mathbb{A}^*} \frac{|D\mathfrak{h}(y)|^{2p}}{J_{\mathfrak{h}}^{p-1}(y)} \, \mathrm{d}y = \mathscr{E}_p[\mathfrak{h}] . \tag{10.6}$$

Proof. Inequality (10.6) is a straightforward consequence of Theorems 1.9 and 1.15 when $h \in \mathscr{H}^{1,2}(\mathbb{A}, \mathbb{A}^*) \cap \mathcal{D}\mathrm{iff}(\mathbb{A}, \mathbb{A}^*)$. Indeed under this assumption we may use the change of variables at (10.2) with $f = h^{-1} \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ to obtain

$$\mathscr{E}_p[h] = 2^p \int_{\mathbb{A}} |\mathcal{K}_1 f(x)|^p \, \mathrm{d}x \geqslant \int_{\mathbb{A}} |\mathcal{K}_1 \mathfrak{f}(x)|^p \, \mathrm{d}x = \mathscr{E}_p[\mathfrak{h}] .$$

For a general homeomorphism $h \in \mathscr{H}^{1,2}(\mathbb{A}^*, \mathbb{A})$ we consider the weighted Dirichlet integral

$$\int_{\mathbb{A}^*} |Dh(y)|^2 \left[\mathcal{K}_1 \mathfrak{f}\left(h(y)\right)\right]^{p-1} \, \mathrm{d}y$$

where we note $\mathcal{K}_1 \mathfrak{f} \in \mathscr{C}(\overline{\mathbb{A}})$. According to [26, 27] the class $\mathscr{H}^{1,2}(\mathbb{A}^*, \mathbb{A}) \cap \mathcal{D}\mathrm{iff}(\mathbb{A}^*, \mathbb{A})$ is dense in $\mathscr{W}^{1,2}(\mathbb{A}^*, \mathbb{A}) \cap \mathscr{C}(\overline{\mathbb{A}^*})$. We have

$$\begin{aligned} &\inf_{h \in \mathscr{H}^{1,2}(\mathbb{A}^*, \mathbb{A})} \int_{\mathbb{A}^*} |Dh(y)|^2 \left[\mathcal{K}_1 \mathfrak{f}\left(h(y)\right)\right]^{p-1} \, \mathrm{d}y \\ &= \inf_{h \in \mathcal{D}\mathrm{iff}(\mathbb{A}^*, \mathbb{A})} \int_{\mathbb{A}^*} |Dh(y)|^2 \left[\mathcal{K}_1 \mathfrak{f}\left(h(y)\right)\right]^{p-1} \, \mathrm{d}y. \end{aligned}$$

It is legitimate to change variables to obtain

$$\begin{aligned} &\inf_{h \in \mathcal{D}\mathrm{iff}(\mathbb{A}^*, \mathbb{A})} \int_{\mathbb{A}^*} |Dh(y)|^2 \left[\mathcal{K}_1 \mathfrak{f}\big(h(y)\big)\right]^{p-1} \, \mathrm{d}y \\ &= 2 \inf_{f \in \mathcal{D}\mathrm{iff}(\mathbb{A}, \mathbb{A}^*)} \int_{\mathbb{A}^*} \mathcal{K}_1 f(x) \left[\mathcal{K}_1 \mathfrak{f}\big(x\big)\right]^{p-1} \, \mathrm{d}x \\ &\geqslant 2 \int_{\mathbb{A}^*} \mathcal{K}_1 \mathfrak{f}(x) \left[\mathcal{K}_1 \mathfrak{f}\big(x\big)\right]^{p-1} \, \mathrm{d}x \\ &= 2 \int_{\mathbb{A}} \left[\mathcal{K}_1 \mathfrak{f}\big(x\big)\right]^{p} \, \mathrm{d}x. \end{aligned}$$

This latter inequality follows from Lemma 8.2 when $\operatorname{Mod} \mathbb{A} \leqslant \operatorname{Mod} \mathbb{A}^*$ and from Proposition 9.5 ($p_\circ = 1$) when $\operatorname{Mod} \mathbb{A}^* < \operatorname{Mod} \mathbb{A}$. To obtain

$$\int_{\mathbb{A}^*} |Dh(y)|^2 \left[\mathcal{K}_1 \mathfrak{f}\big(h(y)\big)\right]^{p-1} \, \mathrm{d}y \geqslant 2 \int_{\mathbb{A}} \mathcal{K}_1^p \mathfrak{f}(x) \, \mathrm{d}x \qquad \text{for all } h \in \mathscr{H}^{1,2}(\mathbb{A}^*, \mathbb{A}).$$

Now, by Hölder's inequality,

$$\int_{\mathbb{A}^*} |Dh(y)|^2 \left[\mathcal{K}_1 \mathfrak{f}\big(h(y)\big)\right]^{p-1} \mathrm{d}y \leqslant \left(\int_{\mathbb{A}^*} \frac{|Dh(x)|^{2p}}{J_h^{p-1}(x)} \mathrm{d}x\right)^{\frac{1}{p}}$$

$$\times \left(\int_{\mathbb{A}^*} \mathcal{K}_1^p \mathfrak{f}\left(h(y)\right) J_h(y)\, \mathrm{d}y\right)^{\frac{p-1}{p}}$$

$$= \left(\int_{\mathbb{A}^*} \frac{|Dh(x)|^{2p}}{J_h^{p-1}(x)} \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{A}^*} \mathcal{K}_1^p \mathfrak{f}\,(x)\, \mathrm{d}x\right)^{\frac{p-1}{p}}.$$

Hence

$$\int_{\mathbb{A}^*} \frac{|Dh(x)|^{2p}}{J_h^{p-1}(x)} \mathrm{d}x \geqslant 2^p \int_{\mathbb{A}^*} \mathcal{K}_1^p \mathfrak{f}\,(x)\, \mathrm{d}x$$

$$= \int_{\mathbb{A}^*} \frac{|D\mathfrak{h}(y)|^{2p}}{J_{\mathfrak{h}}^{p-1}(y)} \mathrm{d}y.$$

The last identity follows by changing variables formula (10.2). □

11. Weighted Dirichlet energy

The usefulness of free Lagrangians, their power and beauty can further be illustrated in the solution of existence of Dirichlet energy-minimal mappings.

11.1. Dirichlet energy for noneuclidean metrics

Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{Y} \subset \mathbb{R}^n$ be domains furnished with Riemannian metric tensors

$$\mathfrak{g}_{\mathbb{X}} = \sum_{i\,j=1}^{n} X_{i\,j}(x) \mathrm{d}x^i \otimes \mathrm{d}x^j \quad \text{and} \quad \mathfrak{g}_{\mathbb{Y}} = \sum_{\alpha\,\beta=1}^{n} Y_{\alpha\beta}(y) \mathrm{d}y^i \otimes \mathrm{d}y^j.$$

This means that $\mathbf{X} = [X_{ij}]_{i\,j=1,\dots,n}$ and $\mathbf{Y} = [Y_{\alpha\beta}]_{\alpha\beta=1,\dots,n}$ are functions valued in the space $\mathbb{R}^{n\times n}_{\mathrm{sym}+}$ of symmetric positive definite $n \times n$-matrices[1]. We consider continuous mappings $h = (h^1, \dots, h^n) \colon \mathbb{X} \to \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X}, \mathbb{Y})$. The energy density (*stored energy function*) is defined

$$\mathsf{E}_h = \frac{1}{2} \mathbf{Tr}_{\mathbb{X}}[D^* h\, \mathbf{Y}\, Dh]\,.$$

[1] In general $\mathbf{X}$ and $\mathbf{Y}$ may only be measurable functions.

That is,

$$\mathsf{E}_h(x) = \frac{1}{2}\,\text{Trace}\,\big\{\mathbf{X}(x)[D^*h(x)\,\mathbf{Y}\big(h(x)\big)\,Dh(x)]\big\}$$

where $Dh \in \mathbb{R}^{n\times n}$ stands for the differential matrix (*deformation gradient*), $Dh = \left[\frac{\partial h^\alpha}{\partial x_i}\right]_{i=1\dots n}^{\alpha=1\dots n}$ and D^*h is its transpose. The stored energy function can be expressed in terms of the entries of those matrices as follows

$$\mathsf{E}_h = \frac{1}{2} \sum_{1\leqslant i\; j\; \alpha\, \beta \leqslant n} X_{i\;j}\, h^j_\alpha\, Y_{\alpha\,\beta}\, h^\beta_i .$$

The Dirichlet energy of h is the integral of E_h over $\mathbb{X}$ with respect to the volume element

$$\mathrm{d}_{\mathbb{X}} = \mathrm{d}_{\mathbb{X}}(x) = \sqrt{\det \mathbf{X}(x)}\,\mathrm{d}x.$$

We denote the energy by

$$\begin{aligned}\mathscr{D}_{\mathbb{X}}[h] &= \frac{1}{2}\int_{\mathbb{X}} \mathsf{E}_h(x)\mathrm{d}_{\mathbb{X}}(x)\\ &= \frac{1}{2}\int_{\mathbb{X}} \text{Trace}\,\big\{\mathbf{X}(x)[D^*h(x)\,\mathbf{Y}\,\big(h(x)\big)\,Dh(x)\big\}\cdot\sqrt{\det\mathbf{X}(x)}\,\mathrm{d}x.\end{aligned}$$

Let us make the following definition.

Definition 11.1. A map $h \in \mathscr{C}(\mathbb{X},\mathbb{Y}) \cap \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X},\mathbb{Y})$ is said to be harmonic with respect to the metrics $\mathfrak{g}_{\mathbb{X}}$ and $\mathfrak{g}_{\mathbb{Y}}$ if it satisfies the Euler-Lagrange equation for $\mathscr{D}_{\mathbb{X}}$.

11.1.1. *Isotropic structures*

Suppose $\mathbb{X}$ and $\mathbb{Y}$ are made of isotropic materials; that is,

$$\mathbf{X} = \lambda^2_{\mathbb{X}}(x)\mathbf{I} \qquad \text{and} \qquad \mathbf{Y} = \lambda^2_{\mathbb{Y}}(y)\mathbf{I}$$

where $\lambda_{\mathbb{X}}\colon \mathbb{X} \to \mathbb{R}_+$ and $\lambda_{\mathbb{Y}}\colon \mathbb{Y} \to \mathbb{R}_+$. The Dirichlet energy is given by

$$\mathscr{D}_{\mathbb{X}}[h] = \frac{1}{2}\int_{\mathbb{X}} |Dh(x)|^2\sigma(x,h)\,\mathrm{d}x \qquad \text{where} \quad \sigma(x,h) = \lambda^{2+n}_{\mathbb{X}}(x)\lambda^2_{\mathbb{Y}}(h).$$

We shall confine ourselves to a specific situation in which $\mathbb{X}$ and $\mathbb{Y}$ are planar annuli, $\lambda_{\mathbb{X}}(x) = \lambda(|x|)$ and $\lambda_{\mathbb{Y}} \equiv 1$.

11.2. λ-Harmonics

Let $\mathbb{A} = A(r,R)$ and $\mathbb{A}^* = A(r_*,R_*)$ be circular annuli in $\mathbb{R}^2$. We consider the weighted Dirichlet energy

$$\mathcal{E}[h] = \frac{1}{2\pi}\int_{\mathbb{A}} |Dh(z)|^2\lambda(|z|)\,\mathrm{d}z \quad 0 < m \leqslant \lambda(|z|) \leqslant M < \infty$$

subject to the class $\mathscr{H}^2(\mathbb{A}, \mathbb{A}^*) \subset \mathscr{W}^{1,2}(\mathbb{A}, \mathbb{R}^2)$ of orientation preserving homeomorphisms $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ which preserve the order of the boundary circles. It is unrealistic to expect that the minimizer will always exist in this class. In fact, passing to the weak $\mathscr{W}^{1,2}$-limit of the energy-minimizing sequence of homeomorphisms we usually loose injectivity. Let $\overline{\mathscr{H}}^2(\mathbb{A}, \mathbb{A}^*)$ denote the closure of $\mathscr{H}^2(\mathbb{A}, \mathbb{A}^*)$ in the norm topology of $\mathscr{W}^{1,2}(\mathbb{A}, \mathbb{R}^2)$. It should be noted that $\overline{\mathscr{H}}^2(\mathbb{A}, \mathbb{A}^*)$ coincides with the class of $\mathscr{W}^{1,2}$-weak limits of homeomorphisms (actually diffeomorphisms) in $\mathscr{H}^2(\mathbb{A}, \mathbb{A}^*)$, see [33] and [26]. The radial stretchings $h(z) = H(|z|)\frac{z}{|z|}$ with $H \in \mathscr{W}^{1,2}(r, R)$ and $\dot{H}(t) \geqslant 0$ for almost every $t \in (r, R)$ are examples of mappings in $\overline{\mathscr{H}}^2(\mathbb{A}, \mathbb{A}^*)$. Since $\mathscr{W}^{1,2}(r, R) \subset \mathscr{C}[r, R]$ the radial mappings $h(z) = H(|z|)\frac{z}{|z|}$ of class $\overline{\mathscr{H}}^2(\mathbb{A}, \mathbb{A}^*)$ are continuous, $H(r) = r_*$ and $H(R) = R_*$.

It is straightforward that:

Proposition 11.2. *Within the class $\overline{\mathscr{H}}^2(\mathbb{A}, \mathbb{A}^*)$ there always exists a mapping $\mathbf{h}\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ of smallest energy.*

The question we are concerned with is whether or not $\mathbf{h}$ is a homeomorphism. The answer is known in case when $\lambda \equiv$ *const.*, see Theorem 10.2. But in general this is a highly nontrivial question.

In this section we are concerned with the variational integral

$$\mathcal{E}_r^R[\mathcal{H}] = \int_r^R \left[t\dot{\mathcal{H}}^2(t) + t^{-1}\mathcal{H}^2(t)\right]\lambda(t)\,\mathrm{d}t \tag{11.1}$$

where $\mathcal{H} = \mathcal{H}(t)$ belongs to $\mathscr{W}^{1,2}(r, R) \subset \mathscr{C}[r, R]$ and the weight $\lambda = \lambda(t)$ is a measurable function such that

$$0 < m \leqslant \lambda(t) \leqslant M < \infty \qquad \text{for almost every } 0 < r \leqslant t \leqslant R < \infty\,. \tag{11.2}$$

This integral represents the energy

$$\mathcal{E}[h] = \frac{1}{2\pi}\int_{r\leqslant|z|\leqslant R} |Dh(z)|^2\lambda\big(|z|\big)\,\mathrm{d}z \tag{11.3}$$

of a radial mapping $h(z) = \mathcal{H}\big(|z|\big)\frac{z}{|z|}$. The Lagrange-Euler equation for (11.1) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t\lambda(t)\dot{\mathcal{H}}(t)\right] = \frac{\lambda(t)}{t}\mathcal{H}(t) \qquad \text{almost everywhere in } (a, b)\,. \tag{11.4}$$

Here the differential operator $\frac{\mathrm{d}}{\mathrm{d}t}\colon \mathscr{L}^2(a, b) \to \mathscr{D}(a, b)$ is understood in the sense of Schwartz distribution; that is,

$$\int_r^R t\lambda(t)\dot{\mathcal{H}}(t)\dot{\Theta}(t)\,\mathrm{d}t = -\int_r^R \frac{\lambda(t)}{t}\mathcal{H}(t)\Theta(t)\,\mathrm{d}t \tag{11.5}$$

for every test function $\Theta \in \mathscr{W}_{\circ}^{1,2}(r,R)$. Minimizing the energy functional (11.1), subject to the given boundary data

$$\begin{cases} \mathcal{H}(r) = r_* \in \mathbb{R} \\ \mathcal{H}(R) = R_* \in \mathbb{R}\,, \end{cases}$$

results in the unique solution of (11.4). Every solution of (11.4) will be called a λ-harmonic function. It is readily seen from (11.4) that every λ-harmonic function $\mathcal{H} = \mathcal{H}(t)$ is Lipschitz continuous on $[r,R]$. Furthermore, in case of zero boundary data, $\mathcal{H}(r) = \mathcal{H}(R) = 0$, we have $\mathcal{H} \equiv 0$. However, if $\mathcal{H}(r) = \mathcal{H}(R) = c \neq 0$ then the constant function is not a solution. To every given λ-harmonic function $\mathcal{H} = \mathcal{H}(t)$ there is associated a function

$$\mathcal{A}(t) \overset{\text{def}}{=\!=} t\lambda(t)\dot{\mathcal{H}}(t) \qquad r \leqslant t \leqslant R.$$

Although the product $\mathcal{A}(t) = t\lambda(t)\dot{\mathcal{H}}(t)$ is Lipschitz continuous the function $\dot{\mathcal{H}}(t) \in \mathscr{L}^\infty(r,R)$ alone fails to be continuous at the points of discontinuity of λ, except for the points $t_\circ \in [r,R]$ where $\mathcal{A}(t_\circ) = 0$. We have the following precise Lipschitz estimate by means of the supremum norm of $\|\mathcal{H}\|_\infty = \|\mathcal{H}\|_{\mathscr{L}^\infty(r,R)}$.

Lemma 11.3. *Define the Lipschitz norm by the rule*

$$||\mathcal{A}||_{\text{Lip}} \overset{\text{def}}{=\!=} \sup_{r \leqslant t \leqslant R} |\mathcal{A}(t)| + \sup_{r \leqslant t_1 \neq t_2 \leqslant R} \frac{|\mathcal{A}(t_1) - \mathcal{A}(t_2)|}{|t_1 - t_2|}. \tag{11.6}$$

Each term in the above formula can be estimated as follows.

$$\sup_{r \leqslant t \leqslant R} |\mathcal{A}(t)| \leqslant M \left(\frac{R}{r} + \frac{r}{R-r} \right) \|\mathcal{H}\|_\infty \tag{11.7}$$

and

$$\sup_{r \leqslant t_1 \neq t_2 \leqslant R} \frac{|\mathcal{A}(t_1) - \mathcal{A}(t_2)|}{|t_1 - t_2|} \leqslant \frac{M}{r} \|\mathcal{H}\|_\infty. \tag{11.8}$$

Hence,

$$\|\dot{\mathcal{H}}\|_\infty \leqslant \frac{M}{m} \left(\frac{R}{r^2} + \frac{1}{R-r} \right) \|\mathcal{H}\|_\infty. \tag{11.9}$$

Proof. For $r \leqslant t_1 < t_2 \leqslant R$ we write:

$$\mathcal{A}(t_1) - \mathcal{A}(t_2) = \int_{t_1}^{t_2} \mathcal{A}'(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \frac{\lambda(t)\mathcal{H}(t)\,\mathrm{d}t}{t}.$$

Hence

$$\frac{|\mathcal{A}(t_1) - \mathcal{A}(t_2)|}{|t_1 - t_2|} \leqslant \frac{M}{r} \|\mathcal{H}\|_\infty. \tag{11.10}$$

To deal with the supremum norm of $\mathcal{A}$ we note that for $r \leqslant \tau \leqslant R$:

$$\begin{aligned}\left|\mathcal{A}(\tau) - \fint_r^R \mathcal{A}(t)\,\mathrm{d}t\right| &= \left|\fint_r^R [\mathcal{A}(\tau) - \mathcal{A}(t)]\,\mathrm{d}t\right| \\ &\leqslant \frac{M}{r}\|\mathcal{H}\|_\infty \cdot \fint_r^R |\tau - t|\,\mathrm{d}t \\ &= \frac{1}{2}\frac{M}{r}(R-r)\|\mathcal{H}\|_\infty.\end{aligned} \tag{11.11}$$

Hereafter the notation $\fint_r^R$ stands for the integral average $\frac{1}{R-r}\int_r^R$. We divide the above inequality by $\tau\lambda(\tau)$ and compute average over the interval $[r, R]$. For this we denote by $A \overset{\text{def}}{=\!=} \fint_r^R \frac{\mathrm{d}\tau}{\tau\lambda(\tau)}$,

$$\left|\fint_r^R \frac{\mathcal{A}(\tau)\,\mathrm{d}\lambda}{\tau\lambda(\tau)} - A\fint_r^R \mathcal{A}(t)\,\mathrm{d}t\right| \leqslant \frac{A}{2}\frac{M}{r}(R-r)\|\mathcal{H}\|_\infty.$$

Since

$$\left|\fint_r^R \frac{\mathcal{A}(\tau)\,\mathrm{d}\lambda}{\tau\lambda(\tau)}\right| = \frac{|\mathcal{H}(R) - \mathcal{H}(r)|}{R-r} \leqslant \frac{2}{R-r}\|\mathcal{H}\|_\infty$$

we see that

$$\left|\fint_r^R \mathcal{A}(t)\,\mathrm{d}t\right| \leqslant \frac{2}{(R-r)A}\|\mathcal{H}\|_\infty + \frac{M}{2r}(R-r)\|\mathcal{H}\|_\infty.$$

We now return to (11.11) to estimate $\mathcal{A}(\tau)$,

$$\begin{aligned}|\mathcal{A}(\tau)| &\leqslant \frac{1}{2}\frac{M}{r}(R-r)\|\mathcal{H}\|_\infty + \frac{2}{(R-r)A}\|\mathcal{H}\|_\infty + \frac{M}{2r}(R-r)\|\mathcal{H}\|_\infty \\ &= M\left(\frac{R}{r} + \frac{r}{R-r}\right)\|\mathcal{H}\|_\infty.\end{aligned}$$

The latter holds because $A = \fint_r^R \frac{\mathrm{d}\tau}{\tau\lambda(\tau)} \geqslant \frac{1}{RM}$.

The estimate (11.9) is straightforward. Indeed, we have

$$|\dot{\mathcal{H}}(\tau)| = \left|\frac{\mathcal{A}(\tau)}{\tau\lambda(\tau)}\right| \leqslant \frac{\|\mathcal{A}\|_\infty}{rm} \leqslant \frac{M}{m}\left(\frac{R}{r} + \frac{1}{R-r}\right)\|\mathcal{H}\|_\infty.$$

It also yields

$$\|\dot{\mathcal{H}}\|_\infty = \|\frac{\mathcal{A}(t)}{t\lambda(t)}\|_\infty \leqslant \frac{1}{rm}\|\mathcal{A}\|_\infty. \qquad \square$$

We have seen that if a λ-harmonic function $\mathcal{H} = \mathcal{H}(t)$ vanishes at two distinct points, say $\mathcal{H}(t_1) = \mathcal{H}(t_2) = 0$, then $\mathcal{H}(t) = 0$ for all $t_1 \leqslant t \leqslant t_2$. In fact, a stronger result holds.

Lemma 11.4. *If a λ-harmonic function $\mathcal{H} = \mathcal{H}(t)$ vanishes at two points then $\mathcal{H}(t) = 0$ for all $r \leqslant t \leqslant R$.*

Proof. Let $[a, b] \subset [r, R]$ denote the largest subinterval on which $\mathcal{H} = 0$. Suppose that, contrary to our claim, $[a, b] \subsetneq [r, R]$, say $r \leqslant a < b < R$. The case $r < a < b \leqslant R$ is similar. Restrict λ to an interval $[a, b+\varepsilon] \subset [r, R]$, where $\varepsilon > 0$ is sufficiently small. We have: $\mathcal{H}(t) = 0$ for $a \leqslant t \leqslant b$ while for $b \leqslant t \leqslant b + \varepsilon$ we appeal to inequality (11.9) to infer that

$$
\begin{aligned}
|\mathcal{H}(t)| = \left| \int_b^{b+\varepsilon} \dot{\mathcal{H}}(\tau)\, \mathrm{d}\tau \right| &\leqslant \varepsilon \|\dot{\mathcal{H}}\|_{\mathscr{L}^\infty(a,b+\varepsilon)} \\
&\leqslant \varepsilon \frac{M}{m} \left(\frac{b+\varepsilon}{a^2} + \frac{1}{b+\varepsilon-a} \right) \|\mathcal{H}\|_{\mathscr{L}^\infty(a,b+\varepsilon)} \\
&\leqslant \varepsilon \frac{M}{m} \left(\frac{R}{r^2} + \frac{1}{b-a} \right) \|\mathcal{H}\|_{\mathscr{L}^\infty(a,b+\varepsilon)}.
\end{aligned}
$$

Hence

$$
0 < \|\mathcal{H}\|_{\mathscr{L}^\infty(a,b+\varepsilon)} \leqslant \varepsilon \frac{M}{m} \left(\frac{R}{r^2} + \frac{1}{b-a} \right) \|\mathcal{H}\|_{\mathscr{L}^\infty(a,b+\varepsilon)}
$$

which is impossible for small ε. □

Corollary 11.5 (Strong Unique Continuation). *Two graphs of two different λ-harmonics cannot intersect at two points.*

Lemma 11.6 (Minimum Energy Property). *Let $\mathcal{H}$ be λ-harmonic. Then for every subinterval $[a, b] \subset [r, R]$ we have*

$$
\mathcal{E}_a^b[\mathcal{H}] \leqslant \mathcal{E}_a^b[H] \tag{11.12}
$$

whenever $H \in \mathscr{W}^{1,2}(r, R)$ coincides with $\mathcal{H}$ at the endpoints a, b. Equality occurs in (11.12) *if and only if $H = \mathcal{H}$ on $[a, b]$.*

Proof.

$$
\begin{aligned}
\mathcal{E}_a^b[H] - \mathcal{E}_a^b[\mathcal{H}] &= \int_a^b t(\dot{H}^2 - \dot{\mathcal{H}}^2)\lambda\, \mathrm{d}t + \int_a^b \frac{H^2 - \mathcal{H}^2}{t} \lambda\, \mathrm{d}t \\
&\geqslant 2 \int_a^b t(\dot{H} - \dot{\mathcal{H}})\dot{\mathcal{H}}\lambda\, \mathrm{d}t + 2 \int_a^b \frac{H - \mathcal{H}}{t} \mathcal{H}\lambda\, \mathrm{d}t \\
&= 2 \int_a^b t\lambda \dot{\mathcal{H}} \dot{\Theta}\, \mathrm{d}t + 2 \int_a^b \frac{\lambda \mathcal{H}}{t} \Theta\, \mathrm{d}t = 0.
\end{aligned}
$$

The latter is immediate from equation (11.5) when tested with $\Theta = H - \mathcal{H} \in \mathscr{W}_\circ^{1,2}(a, b)$. □

Lemma 11.7. *If a λ-harmonic function $\mathcal{H} = \mathcal{H}(t)$, $r \leqslant t \leqslant R$, is nonnegative at the endpoints, it is nonnegative everywhere. Actually, $\mathcal{H}$ is strictly positive in (r, R) except for the case $\mathcal{H} \equiv 0$.*

Proof. Suppose that, contrary to our claim, $\mathcal{H}(t_\circ) < 0$ for some $t_\circ \in (r, R)$. Such a point $t_\circ$ is contained in an interval $[r', R'] \subset [r, R]$ such that $\mathcal{H}(r') = \mathcal{H}(R') = 0$. This yields $\mathcal{H}(t) \equiv 0$ on $[r', R']$ and, in particular, $\mathcal{H}(t_\circ) = 0$ which is a contradiction.

Assume, again to get a contradiction, that $\mathcal{H}(t_\circ) = 0$ at some point $t_\circ(r, R)$ while $\mathcal{H} \not\equiv 0$. At least one of the boundary values $\mathcal{H}(r)$ or $\mathcal{H}(R)$ is positive, say $\mathcal{H}(r) > 0$. Consider a λ-harmonic curve $\mathcal{H}_\circ = \mathcal{H}_\circ(t)$ determined by the boundary values $\mathcal{H}_\circ(r) = 0$ and $\mathcal{H}_\circ(R) = \mathcal{H}(R) \geqslant 0$. We have just shown that $\mathcal{H}_\circ(t) \geqslant 0$ in $[r, R]$. The λ-harmonic function $\mathcal{H} - \mathcal{H}_\circ$ is nonnegative at the end-points, so $\mathcal{H}(t) - \mathcal{H}_\circ(t) \geqslant 0$ everywhere in $[r, R]$. We see that $0 \leqslant \mathcal{H}_\circ(t) \leqslant \mathcal{H}(t)$. Since $\mathcal{H}(t_\circ) = 0$ it follows that $\mathcal{H} - \mathcal{H}_\circ$ vanishes at two different points, namely at $t_\circ$ and R. Therefore, by Corollary 11.5, $\mathcal{H} - \mathcal{H}_\circ \equiv 0$ in $[r, R]$. This is a clear contradiction of the fact that $\mathcal{H}(r) - \mathcal{H}_\circ(r) > 0$. □

Lemma 11.8 (Maximum Principle). *Let $\mathcal{H} = \mathcal{H}(t) \not\equiv 0$, $r \leqslant t \leqslant R$, be λ-harmonic and nonnegative at the end-points. Then*

$$0 < \mathcal{H}(t) < \max\{\mathcal{H}(r), \mathcal{H}(R)\} \qquad \textit{for all } t \in (r, R).$$

Proof. Suppose that, on the contrary, $\mathcal{H}$ assumes its maximum at a point $t_\circ \in (r, R)$; that is,

$$\mathcal{H}(t_\circ) \geqslant \mathcal{H}(t) \qquad \text{for all } t \in [r, R].$$

There is a point $t_\circ^+ \in (t_\circ, R)$ for which $\mathcal{H}(t_\circ^+) < \mathcal{H}(t_\circ)$, otherwise we would $\mathcal{H}(t) \equiv \text{const.} = \mathcal{H}(t_\circ) > 0$ in $[t_\circ, R]$, which is impossible due to the equation (11.4). By the same reasoning, there is a point $t_\circ^- \in (r, t_\circ)$ for which $\mathcal{H}(t_\circ^-) < \mathcal{H}(t_\circ)$. Choose and fix any number s between $\max\{\mathcal{H}(t_\circ^-), \mathcal{H}(t_\circ^+)\}$ and $\mathcal{H}(t_\circ)$. It is readily inferred, by Intermediate Value Theorem, that

$$\mathcal{H}(t_-) = s = \mathcal{H}(t_+) \tag{11.13}$$

for some points $r < t_- < t_\circ < t_+ < R$. We take the t_- and t_+ to be the closest to $t_\circ$. This ensures that

$$s < \mathcal{H}(t) \leqslant \mathcal{H}(t_\circ) \qquad \text{for } t_- < t < t_+.$$

Now the usual truncation trick comes into play. Let

$$\mathbf{H}(t) = \begin{cases} s & \text{for } t_- \leqslant t \leqslant t_+ \\ \mathcal{H}(t) & \text{otherwise}. \end{cases}$$

Clearly $\mathbf{H}(r) = \mathcal{H}(r)$ and $\mathbf{H}(R) = \mathcal{H}(R)$. Within the interval (t_-, t_+) the energy integrand of $\mathbf{H}$ is pointwise smaller than that of $\mathcal{H}$, because

$$\mathbf{H}(t) = s < \mathcal{H}(t) \quad \text{and} \quad |\dot{\mathbf{H}}(t)| = 0 \leqslant |\dot{\mathcal{H}}(t)|.$$

This is in contradiction with Lemma 11.6. □

11.3. The minimum point

Lemma 11.9. *Every λ-harmonic function $\mathcal{H} = \mathcal{H}(t)$ which is positive at the end-points assumes its minimum value at exactly one point $r_\circ \in [r, R]$. Moreover, $\mathcal{H}$ is strictly decreasing in $[r, r_\circ]$ and strictly increasing in $[r_\circ, R]$. This includes the cases $r_\circ = r$ and $r_\circ = R$.*

Proof. First observe that $\mathcal{H}$ cannot assume its minimum value at two different points; say at $a \neq b$. Otherwise we would have

$$\mathcal{H}(t) \geqslant \max\{\mathcal{H}(a), \mathcal{H}(b)\} \qquad \text{for } a \leqslant t \leqslant b$$

in contradiction with Lemma 11.8. To identify the minimum point we look at the function $\mathcal{A}(t) = t\lambda(t)\dot{\mathcal{H}}(t)$. Equation 11.4 reads as

$$\mathcal{A}'(t) = \frac{\lambda(t)\mathcal{H}(t)}{t} \geqslant \frac{m}{R} \min_{r\leqslant\tau\leqslant R} \mathcal{H}(\tau) > 0 \qquad \text{a.e. in } [r, R].$$

In particular, $\mathcal{A}$ is Lipschitz continuous and strictly increasing. In particular $\mathcal{A}(r) < \mathcal{A}(R)$. Therefore, we have three cases to consider:

Case 1: $0 \leqslant \mathcal{A}(r)$. Thus $0 < \mathcal{A}(t)$ for every $r < t \leqslant R$, which tells us that $\dot{\mathcal{H}}(t) > 0$ almost everywhere in (r, R). In this case the minimum point is $r_\circ = r$, and $\mathcal{H}$ is strictly increasing in $[r_\circ, R]$.

Case 2. $\mathcal{A}(r) < 0 \leqslant \mathcal{A}(R)$. This means there exists exactly one point $r_\circ \in (r, R)$ at which $\mathcal{A}(r_\circ) = 0$. We have $\mathcal{A}(t) < 0$ for $r \leqslant t < r_\circ$ and $\mathcal{A}(t) > 0$ for $r_\circ < t \leqslant R$. Since $mr \leqslant t\lambda(t) \leqslant MR$, these latter inequalities also apply to $\dot{\mathcal{H}}(t)$ almost everywhere. It shows that $\mathcal{H}$ is strictly increasing in $[r_\circ, R]$. Thus $r_\circ$ is the minimum point of $\mathcal{H}$.

Case 3. $\mathcal{A}(R) < 0$. Thus $\mathcal{A}(t) < 0$ for every $r \leqslant t < R$, so $\dot{\mathcal{H}}(t) < 0$ almost everywhere in (r, R). This means that the minimum point $r_\circ = R$, and $\mathcal{H}$ is strictly decreasing in $[r, R]$. □

Remark 11.10. The property of nonnegative λ-harmonics, stated in Lemma 11.9, can be regarded as the surrogate for convexity. However, Figure 11.1 demonstrates that positive λ-harmonics need not be convex in the usual sense.

11.4. A flow of λ-harmonics

We shall now consider a family of λ-harmonics $\{\mathcal{H}_s\}_{s\in\mathbb{R}}$ defined in $[r, R]$ by the rule

$$\begin{cases} \mathcal{H}_s(r) = s \\ \mathcal{H}_s(R) = R_* \end{cases} \tag{11.14}$$

where $R_* > 0$ is fixed. Thus $\mathcal{H}_s(t) > 0$ for all $r < t \leqslant R$ whenever $s \geqslant 0$.

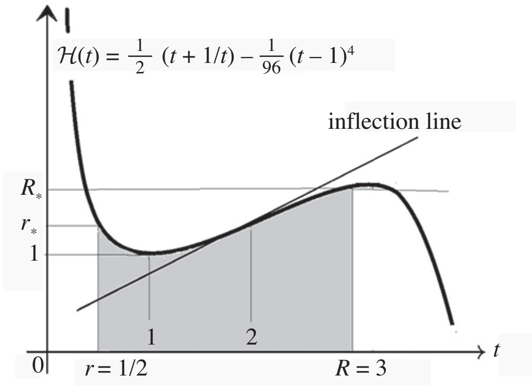

Figure 11.1. A λ-harmonic curve may have an inflection point.

Lemma 11.11. *Whenever $s_1 < s_2$ the following estimates hold*

$$0 < \mathcal{H}_{s_2}(t) - \mathcal{H}_{s_1}(t) < s_2 - s_1 \qquad \textit{for } r \leqslant t < R.$$

Proof. We look at the λ-harmonic function $\mathcal{H} \stackrel{\text{def}}{=\!=} \mathcal{H}_{s_2} - \mathcal{H}_{s_1} \not\equiv 0$, which is nonnegative at the endpoints. Thus $\mathcal{H}(t) > 0$ for $r \leqslant t < R$. By the maximum principle in Lemma 11.8 we conclude with the estimate

$$0 < \mathcal{H}(t) < \max\{\mathcal{H}(r), \mathcal{H}(R)\} = s_2 - s_1 \quad \text{for } r < t < R$$

as desired. □

Lemma 11.12. *For s sufficiently large it holds that*

$$\mathcal{H}_s(t) \geqslant \mathcal{H}_s(R) = R_* \quad \textit{for all } r \leqslant t \leqslant R.$$

Precisely, this is true whenever

$$s > e^T R_* \quad T = \frac{M}{m} \log^2 \frac{R}{r}.$$

Recall that $0 < m \leqslant \lambda(t) \leqslant M < \infty$.

Proof. Let $a \in [r, R]$ denote the minimum point of $\mathcal{H}_s$ in $[r, R]$. Obviously $a > r$. Our goal is to show that $a = R$. Suppose, to the contrary, that $r < a < R$, so $\mathcal{A}(a) = 0$, by Lemma 11.9. To simplify the writing we denote $\mathcal{H}(t) = \mathcal{H}_s(t)$. For every $r \leqslant t \leqslant a$ we can write

$$-t\lambda(t)\dot{\mathcal{H}}(t) = \int_t^a \mathcal{A}'(\tau)\,\mathrm{d}\tau = \int_t^a \frac{\lambda(\tau)\mathcal{H}(\tau)}{\tau}\,\mathrm{d}\tau.$$

Since $\mathcal{H}$ is decreasing in $[t, a]$, by Lemma 11.9 we see that

$$-t\lambda(t)\dot{\mathcal{H}}(t) \leqslant \mathcal{H}(t)\int_t^a \frac{\lambda(\tau)\mathrm{d}\tau}{\tau} \leqslant \mathcal{H}(t)\cdot M\int_r^R \frac{\mathrm{d}\tau}{\tau} = \mathcal{H}(t)M\log\frac{R}{r}.$$

Hence

$$-\frac{\dot{\mathcal{H}}(t)}{\mathcal{H}(t)} \leqslant \frac{M}{m}\frac{1}{t}\log\frac{R}{r}.$$

We integrate from r to a with respect to the variable t to obtain

$$-\log\frac{\mathcal{H}(a)}{\mathcal{H}(r)} \leqslant \frac{M}{m}\log\frac{a}{r}\log\frac{R}{r} \leqslant \frac{M}{m}\log^2\frac{R}{r}.$$

Hence

$$\log\mathcal{H}(r) \leqslant \frac{M}{m}\log^2\frac{R}{r} + \log\mathcal{H}(a).$$

We arrive at a desired contradiction,

$$s = \mathcal{H}(r) \leqslant \mathcal{H}(a)e^{\frac{M}{m}\log^2\frac{R}{r}} \leqslant e^T R_*. \qquad \square$$

Lemma 11.13. *Given* $0 \leqslant r_* < R_*$, *there exists exactly one* λ*-harmonic function* $\mathcal{H} = \mathcal{H}(t) = \mathcal{H}_s(t)$, $r_* \leqslant s < \infty$, *such that*

$$\begin{cases} \mathcal{H}_s(r) = s \\ \mathcal{H}_s(R) = R_* \\ \min_{r\leqslant t\leqslant R}\mathcal{H}_s(r) = r_*. \end{cases} \tag{11.15}$$

Proof. It is impossible that two different nonnegative λ-harmonics cannot have the same minimum value. Otherwise, they graphs must intersect at two different points, one of which is $t = R$, contradicting Lemma 11.7. For the existence we consider the function

$$\mu(s) \overset{\text{def}}{=\!=} \min_{r\leqslant t\leqslant R}\mathcal{H}_s(t) \qquad 0 \leqslant s < \infty.$$

Given $0 \leqslant s_1 < s_2$, we have

$$0 \leqslant \mathcal{H}_{s_2}(t) - \mathcal{H}_{s_1}(t) \leqslant s_2 - s_1 \quad \text{for all } t \in [r, R].$$

Hence

$$0 \leqslant \mu(s_2) - \mu(s_1) \leqslant s_2 - s_1$$

which shows μ is continuous and nonnegative. When $s = r_*$, we have

$$\mu(r_*) = \min_{r\leqslant t\leqslant R}\mathcal{H}_{r_*}(t) \leqslant \mathcal{H}_{r_*}(r) = r_*.$$

On the other hand, when $s \geqslant e^T R_*$, see Lemma 11.12, we have

$$\mu(s) = \min_{r\leqslant t\leqslant R}\mathcal{H}_s(t) = \mathcal{H}_s(R) = R_* > r_*.$$

Therefore, by intermediate value theorem, there exists $s_* \in [r_*, e^T R_*]$ for which $\mu(s_*) = r_*$. The λ-harmonic function $\mathcal{H}_{s_*}(t)$ satisfies (11.15) (Figure 11.2). $\square$

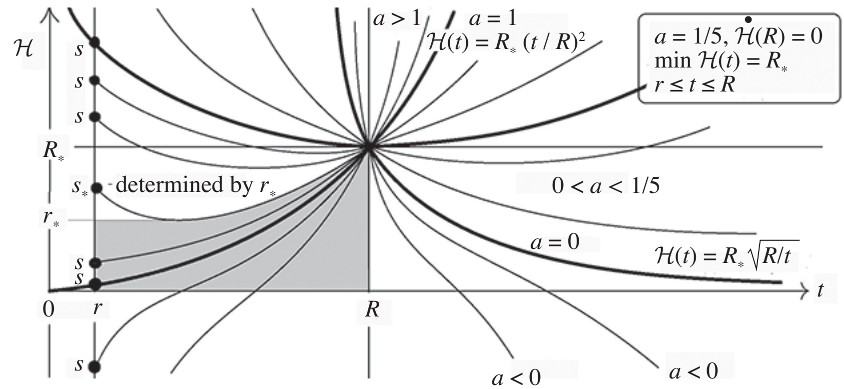

Figure 11.2. λ-harmonic curves normalized by the condition $\mathcal{H}(R) = R_*$.

11.5. Minimizing the energy within monotone functions

In general, λ-harmonic functions fail to be injective. A way to overcome this difficulty is to restrict the energy functional to $\mathscr{W}^{1,2}$-limits of homeomorphisms $H\colon [r, R] \xrightarrow{\text{onto}} [r_*, R_*]$. However, minimizing the energy within this class one looses Lagrange-Euler equation. This is because the outer variation is unjustified. The limits are still nondecreasing, thus have nonnegative derivatives. Let us introduce the following notation

$$\mathscr{W}^{1,2}_{\nearrow}(r, R) = \left\{ H \in \mathscr{W}^{1,2}(r, R)\colon \dot{H}(t) \geqslant 0 \right\}.$$

Since $\mathscr{W}^{1,2}_{\nearrow}(r, R)$ is weakly closed in $\mathscr{W}^{1,2}(r, R)$ the direct method in the calculus of variations still applies. Consequently, we always have a map $H = H(t)$ of smallest energy within the class $\mathscr{W}^{1,2}_{\nearrow}(r, R)$ with any given boundary values. The following Proposition describes such mappings in detail.

Proposition 11.14. *For every* $0 < r_* < R_*$ *there exist unique energy-minimal map* $\mathbf{H} \in \mathscr{W}^{1,2}_{\nearrow}(r, R)$, *such that* $\mathbf{H}(r) = r_* < R_* = \mathbf{H}(R)$. *Precisely, there is unique* $r_\circ \in [r, R)$ *such that*

$$\mathbf{H}(t) = \begin{cases} \mathcal{H}_s(t) & \text{for } r_\circ \leqslant t \leqslant R \\ r_* & \text{for } r \leqslant t \leqslant r_\circ \end{cases} \tag{11.16}$$

where $\mathcal{H}_s$ *is a* λ*-harmonic function given by Lemma* 11.13.

Proof. In (11.16) $r_\circ$ is the minimum point of the λ-harmonic function $\mathcal{H}_s(t)$. We aim to show that

$$\mathcal{E}_r^R[\mathbf{H}] \leqslant \mathcal{E}_r^R[H]$$

for every $H \in \mathscr{W}^{1,2}_{\nearrow}(r, R)$ with the boundary values $H(r) = r_*$ and $H(R) = R_*$. Let us begin with the algebraic identity

$$\left(t\dot{H}^2 + \frac{H^2}{t}\right)\lambda = 2\eta H\dot{H} + \left(\frac{\lambda}{t} - \frac{\eta^2}{t\lambda}\right)H^2 + \left(t\sqrt{\lambda}\dot{H} - \eta\frac{H}{\sqrt{t\lambda}}\right)^2 \tag{11.17}$$

which holds for every measurable function $\eta = \eta(t)$. We take for η the so-called weighted elasticity function of $\mathcal{H}_s$,

$$\eta(t) \stackrel{\text{def}}{=\!=} \lambda(t)\frac{t\dot{\mathcal{H}}_s(t)}{\mathcal{H}_s(t)} \stackrel{\text{def}}{=\!=} \frac{\mathcal{A}(t)}{\mathcal{H}_s(t)}.$$

We have already discussed in some detail the properties of $\mathcal{A}(t) = t\lambda(t)\dot{\mathcal{H}}(t)$. Accordingly, $\mathcal{A}$ is Lipschitz continuous and it vanishes at the minimum point $r_\circ$, provided $r_\circ \in (r, R)$. Thus η is also Lipschitz continuous. The following differential equation for η is straightforward from (11.4)

$$\dot{\eta}(t) = \frac{\lambda(t)}{t} - \frac{\eta^2(t)}{t\lambda(t)} \qquad \text{for } r \leqslant t \leqslant R. \tag{11.18}$$

Indeed,

$$(H\eta)' = \mathcal{A}' = \frac{\lambda H}{t} \qquad \dot{H}\eta + H\dot{\eta} = \frac{\lambda H}{t}.$$

Hence

$$\dot{\eta} = \frac{\lambda}{t} - \eta\frac{\dot{H}}{\lambda H} = \frac{\lambda}{t} - \frac{\eta^2}{t\lambda}.$$

Note that if $r_\circ \in (r, R)$, the solution of (11.18) is uniquely determined by the initial condition $\eta(r_\circ) = 0$. We now estimate the integrand for H

$$\left(t\dot{H}^2 + \frac{H^2}{t}\right)\lambda \geqslant 2\eta H\dot{H} + \dot{\eta}H^2 = \frac{\mathrm{d}}{\mathrm{d}t}\big(\eta H^2\big). \tag{11.19}$$

Equality occurs at a givenpoint t if and only if

$$\frac{t\lambda(t)\dot{H}}{H} = \eta(t).$$

We integrate the above estimate from $\tau \leqslant t \leqslant R$ to obtain

$$\mathcal{E}_\tau^R[H] \geqslant \int_\tau^R \frac{\mathrm{d}}{\mathrm{d}t}\left(\eta H^2\right)\mathrm{d}t = \eta(R)H^2(R) - \eta(\tau)H^2(\tau) \qquad r \leqslant \tau \leqslant R.$$

Equality occurs if and only if $\frac{\dot{H}(t)}{H(t)} = \frac{\dot{\mathcal{H}}_s(t)}{\mathcal{H}_s(t)}$ for every $t \in [\tau, R]$. This, in view of the boundary condition $H(R) = \mathcal{H}_s(R) = R_*$, yields $H(t) = \mathcal{H}_s(t)$ for all $t \in [\tau, R]$, as desired. First consider the case

Case 1. $r_\circ = r$; that is for all $r \leqslant t \leqslant R$. $\mathbf{H}(t) = \mathcal{H}_s(t)$. We take $\tau = r_\circ = r$ to conclude with the desired estimate

$$\mathcal{E}_r^R[H] \geqslant \eta(R)R_*^2 - \eta(r)r_*^2 = \mathcal{E}_r^R[\mathcal{H}_s] = \mathcal{E}_r^R[\mathbf{H}].$$

The right hand side is independent of H and, clearly, equals $\mathcal{E}_r^R[\mathcal{H}_s]$. Equality occurs iff $H = \mathbf{H}$.

Case 2. $r < r_\circ < R$; that is, $\mathbf{H}(t) \equiv r_*$ for $r \leqslant t \leqslant r_\circ$ and $\mathbf{H}(t) = \mathcal{H}_s(t)$ for $r_\circ \leqslant t \leqslant R$. We take $\tau = r_\circ$ to conclude with the estimate

$$\mathcal{E}_{r_\circ}^R[H] \geqslant \eta(R)R_*^2 - \eta(r_\circ)H^2(r_\circ) = \eta(R)R_*^2$$

because $\eta(r_\circ) = 0$. The remaining energy integral is estimated by trivial means

$$\mathcal{E}_r^{r_\circ}[H] = \int_r^{r_\circ} \left(t\dot{H}^2 + \frac{H^2}{t} \right) \lambda \, \mathrm{d}t \geqslant \int_r^{r_\circ} \frac{H^2}{t} \lambda \, \mathrm{d}t \geqslant r_*^2 \int_r^{r_\circ} \frac{\lambda(t) \, \mathrm{d}t}{t}.$$

The right hand side is independent of H and, again, equality occurs iff $H = \mathbf{H}$. Summing up we conclude with the desired lower bound of the energy.

$$\mathcal{E}_r^R[H] \geqslant \eta(R)R_*^2 + r_*^2 \int_r^{r_\circ} \frac{\lambda(t) \, \mathrm{d}t}{t} = \mathcal{E}_r^R[\mathbf{H}]. \qquad \square$$

Remark 11.15. It is worth noting that the expression $2\eta H \dot{H} + \left(\frac{\eta}{t} - \frac{\eta^2}{t\lambda} \right) H^2$ in (11.17) was a free Lagrangian. This was due to a correct choice of η that solves the nonlinear ODE (11.18). In higher dimensions, guessing the correct choice of free Lagrangians is far from obvious. In Section 11.7 we demonstrate this technique in case of a weighted Dirichlet integral for mappings between planar annuli.

11.6. Critical intervals

The so-called *critical domain interval* $[c, R] \subset [r, R]$ is defined by taking the smallest number $c \in [r, R)$ such that

$$\int_c^R \frac{\lambda(t) \, \mathrm{d}t}{t} \leqslant \lambda(R). \tag{11.20}$$

In other words:

- $c = r$ if $\int_r^R \frac{\lambda(t)\,\mathrm{d}t}{t} \leqslant \lambda(R)$;
- Otherwise, c is determined from the equation

$$\int_c^R \frac{\lambda(t) \, \mathrm{d}t}{t} = \lambda(R). \tag{11.21}$$

Recall that to such $c \in (r, R)$ there corresponds unique λ-harmonic function $\mathcal{H} \stackrel{\text{def}}{=\!=} \mathcal{H}^c(t)$ such that

$$\mathcal{H}^c(R) = R_* \quad \text{and} \quad \mathcal{H}^c(c) = \min_{r \leqslant t \leqslant R} \mathcal{H}^c(t). \tag{11.22}$$

For $c = r$, however, there can be many λ-harmonics satisfying (11.22); namely, the ones which assume the minimum value at $t = r$. Such is $\mathcal{H}_0(t)$ defined by $\mathcal{H}_0(0) = 0$. Also recall that for $s_1 < s_2$ we have $\mathcal{H}_{s_1}(t) < \mathcal{H}_{s_2}(t)$ everywhere in $[r, R)$. When $c = r$, we choose for $\mathcal{H}^c(t)$ the largest λ-harmonic function which satisfies the conditions,

$$\mathcal{H}^c(R) = R_* \quad \text{and} \quad \mathcal{H}^c(r) = \min_{r \leqslant t \leqslant R} \mathcal{H}^c(t).$$

In either case, note that the associated function

$$\mathcal{A}^c(t) \stackrel{\text{def}}{=\!=} t\lambda(t)\dot{\mathcal{H}}^c(t) \qquad r \leqslant t \leqslant R$$

is increasing and it vanishes at $t = c$. Thus $\mathcal{A}^c$ is positive for $c < t \leqslant R$. In particular, $\mathcal{H}^c$ is strictly increasing in $[c, R]$. Consider

$$\mathbf{H}^c(t) \stackrel{\text{def}}{=\!=} \begin{cases} \mathcal{H}^c(t) & \text{for } c \leqslant t \leqslant R \\ \mathcal{H}^c(c) & \text{for } r \leqslant t \leqslant r_\circ. \end{cases}$$

According to Proposition 11.14, $\mathbf{H}^c(t)$ is an energy-minimal function of class $\mathscr{W}^{1,2}_{\nearrow}(r, R)$ which takes $[r, R]$ onto $[c_*, R_*]$, where $c_* \stackrel{\text{def}}{=\!=} \mathcal{H}^c(c)$. We refer to $\mathbf{H}^c$ as the *critical energy-minimal function* and to $[c_*, R_*]$ as *critical target interval*. This name is justified by the fact that:

- To every subinterval $[\rho_*, R_*] \subset [c_*, R_*]$ there corresponds unique energy-minimal solution $\mathbf{H} = \mathbf{H}^{\rho_*} \in \mathscr{W}^{1,2}_{\nearrow}(r, R)$ which takes $[r, R]$ onto $[\rho_*, R_*]$. It lies between $\mathbf{H}^c(t)$ and R_*.

Precisely,

- $$\mathbf{H}^c(t) \leqslant \mathbf{H}(t) \leqslant R_* \quad \text{for } r \leqslant t \leqslant R; \tag{11.23}$$

- $\mathbf{H}(t)$ assumes its minimum value at some point $\rho \in [c, R]$ so that

$$\mathcal{A}(t) \stackrel{\text{def}}{=\!=} t\lambda(t)\dot{\mathbf{H}}(t) \qquad \text{vanishes at } t = \rho.$$

Moreover, $\mathbf{H}\colon [\rho, R] \xrightarrow{\text{onto}} [\rho_*, R_*]$ is strictly increasing, thus defines the inverse function $F = F(s)$ for $\rho_* \leqslant s \leqslant R_*$,

$$F\colon [\rho_*, R_*] \xrightarrow{\text{onto}} [\rho, R].$$

Before we proceed to the energy-minimal deformations of annuli let us state and prove a useful technical lemma.

Lemma 11.16. *We have*

$$0 \leqslant \frac{t\dot{\mathbf{H}}(t)}{\mathbf{H}(t)} < 1 \qquad \textit{for almost every } t \in [r, R] \tag{11.24}$$

$$\frac{F(s)\lambda\big(F(s)\big)}{s\dot{F}(s)} < m \qquad \textit{for almost every } s \in (\rho_*, R_*]. \tag{11.25}$$

Proof. For $t \in [\rho, R]$ we have $\mathbf{H}(t) = \mathcal{H}(t)$, where $\mathcal{H}$ is λ-harmonic, so we may exploit the Lagrange-Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}(t) \overset{\text{def}}{=\!=} \frac{\mathrm{d}}{\mathrm{d}t}\Big[t\lambda(t)\dot{\mathcal{H}}(t)\Big] = \frac{\lambda(t)\mathcal{H}(t)}{t}. \qquad \square$$

Since $\mathcal{A}(\rho) = 0$ and $\mathcal{H}$ is strictly increasing in $[\rho, R]$ we see that

$$\begin{aligned}\mathcal{A}(t) &= \int_\rho^t \dot{\mathcal{A}}(\tau)\,\mathrm{d}\tau = \int_\rho^t \frac{\lambda(\tau)\mathcal{H}(\tau)}{\tau}\,\mathrm{d}\tau \\ &< \mathcal{H}(t)\int_\rho^t \frac{\lambda(\tau)\,\mathrm{d}\tau}{\tau} \leqslant \mathcal{H}(t)\int_c^R \frac{\lambda(\tau)\,\mathrm{d}\tau}{\tau} \leqslant \mathcal{H}(t)m \quad \text{by (11.20).}\end{aligned}$$

For $t = \rho$ we have $t\lambda(t)\dot{\mathcal{H}}(t) = 0$. This shows that for $\rho \leqslant t < R$,

$$\frac{t\lambda(t)\dot{\mathcal{H}}(t)}{\mathcal{H}(t)} < m \leqslant \lambda(\tau) \tag{11.26}$$

where τ can be any number in $[r, R]$, because λ is nonincreasing. Letting $\tau = t$ we obtain (11.24). To see (11.25) we express s by the formula $s = \mathcal{H}(t)$ for some $t \in [\rho, R]$. Applying (11.26) to $t = F(s)$ yields

$$\frac{F(s)\lambda\big(F(s)\big)}{s\dot{F}(s)} < m.$$

11.7. Minimal deformations between annuli

We consider an arbitrary domain annulus $\mathbb{A} = A(r, R)$ equipped with a nondecreasing weight $\lambda = \lambda(|z|)$. The target annulus $\mathbb{A}^* = A(r_*, R_*)$, however, will be restricted to the inclusion $[r_*, R_*] \subset [c_*, R_*]$, where $[c_*, R_*]$ stands for the critical target interval defined in Section 11.6. The annulus $\mathbb{A}_c^* = A(c_*, R_*)$ will be called the critical target annulus.

Theorem 11.17. *Consider an annulus* $\mathbb{A} = A(r, R)$ *and a nonincreasing weight function* $\lambda = \lambda\big(|z|\big)$,

$$0 < m \leqslant \lambda(t) \leqslant M < \infty \quad \textit{for } r \leqslant t \leqslant R.$$

Then, for every subannulus $\mathbb{A}^* = A(r_*, R_*) \subset \mathbb{A}^*_c$ *the energy-minimal map* $\mathbf{h}\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ *of class* $\overline{\mathscr{H}}^2(\mathbb{A}, \mathbb{A}^*)$ *is unique (up to rotation) and it takes the form*

$$\mathbf{h} = \mathbf{H}\big(|z|\big)\frac{z}{|z|}, \qquad \mathbf{H} \in \mathscr{W}^{1,2}_{\nearrow}(r, R).$$

Precisely,

$$\mathbf{h}(z) = \begin{cases} r_* \frac{z}{|z|} & \text{for } r \leqslant |z| \leqslant \rho \\ \mathcal{H}\big(|z|\big)\frac{z}{|z|} & \text{for } \rho \leqslant |z| \leqslant R. \end{cases}$$

Here $\mathcal{H}\colon [\rho, R] \xrightarrow{\text{onto}} [r_*, R_*]$ *is a* λ*-harmonic homeomorphism (Figure* 11.3*).*

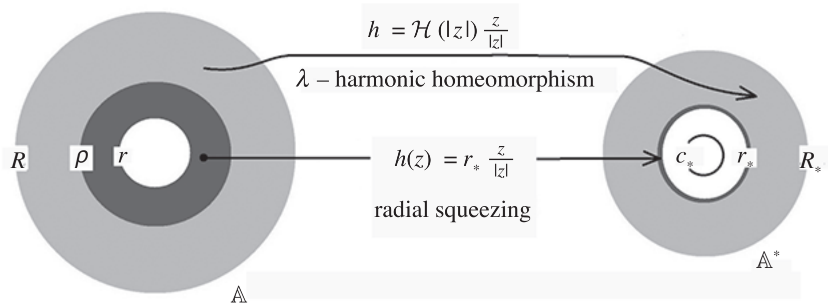

Figure 11.3. Squeezing phenomenon when $r_* > c_*$.

Proof. Let $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ be an arbitrary mapping of class $\mathscr{W}^{1,2}_{\nearrow}(r, R)$. We aim to prove the following inequality

$$\mathcal{E}_{\mathbb{A}}[h] = \int_{\mathbb{A}} |Dh(z)|^2 \lambda\big(|z|\big)\,\mathrm{d}z \geqslant \mathcal{E}[\mathbf{h}] \tag{11.27}$$

and to show that equality occurs if and only if, upon a rotation, $h = \mathbf{h}$. We shall make use of polar coordinates

$$z = te^{i\theta} \qquad r \leqslant t \leqslant R \qquad 0 \leqslant \theta \leqslant 2\pi.$$

The normal (radial) and tangential (angular) derivatives are defined by the rules

$$h_N = \frac{\partial h}{\partial t} \qquad \text{and} \qquad h_T = \frac{1}{t}\frac{\partial h}{\partial \theta}.$$

Thus the weighted energy integrand reads as

$$|Dh(z)|^2\lambda\big(|z|\big) = \big(|h_N|^2 + |h_T|^2\big)\lambda\big(|z|\big). \tag{11.28}$$

11.7.1. *Free-Lagrangian estimate*

The key to the proof of Theorem 11.17 is the following estimate by means of free Lagrangians:

Proposition 11.18. *For every $z \in A(r, R)$, we have*

$$
\begin{aligned}
|Dh(z)|^2\lambda\big(|z|\big) \geqslant\ & \frac{2F(s)\lambda\big(F(s)\big)}{s\dot{F}(s)} J_h(z) && (s = |h(z)|) \\
& - \frac{\lambda(t)}{t^2}\Big[\mathbf{H}^2(t) - t^2\dot{\mathbf{H}}^2(t)\Big] && (t = |z|) \\
& + \frac{2\lambda(t)}{t^2}\Big[\mathbf{H}^2(t) - t^2\dot{\mathbf{H}}^2(t)\Big]\,\mathrm{Im}\,\frac{h_\theta}{h}.
\end{aligned}
\tag{11.29}
$$

As before the notation $F = F(s)$ for $r_* \leqslant s \leqslant R_*$ stands for the inverse of $\mathcal{H} = \mathcal{H}(t)$. This function is a λ-harmonic homeomorphism of $[\rho, R]$ onto $[r_*, R_*]$. Thus $F\colon [r_*, R_*] \xrightarrow{\text{onto}} [\rho, R]$.

Proof. Let us begin with the algebraic inequality

$$
\begin{aligned}
|Dh(z)|^2\lambda\big(|z|\big) =& \Big(|h_N|^2 + |h_T|^2\Big)\,\lambda\big(|z|\big) \\
=& \lambda(t)\big(|h_N| - b|h_T|\big)^2 + \lambda(t)\big(1-b^2\big)\big(B - |h_T|\big)^2 \\
& + 2\lambda(t)b|h_N||h_T| - \lambda(t)\big(1-b^2\big)B^2 + 2\lambda(t)\big(1-b^2\big)B|h_T| \\
\geqslant& 2\lambda(t)\, b\, |h_N|\, |h_T| \quad \Big(\stackrel{\text{def}}{=} \mathcal{I}\Big) \\
& - \lambda(t)(1-b^2)B^2 \quad \Big(\stackrel{\text{def}}{=} \mathcal{II}\Big) \\
& + 2\lambda(t)(1-b^2)B|h_T| \quad \Big(\stackrel{\text{def}}{=} \mathcal{III}\Big).
\end{aligned}
\tag{11.30}
$$

This holds with arbitrary parameters $0 \leqslant b \leqslant 1$ and $B \geqslant 0$. Equality occurs if and only if $B = |h_T|$ and $b|h_T| = |h_N|$. We shall consider both b and B as functions of two variables $t \in [r, R]$ and $s \in [r_*, R_*]$,

$$
\begin{aligned}
b = b(t, s) &= \frac{F(s)\lambda\big(F(s)\big)}{s\dot{F}(s)\lambda(t)} < 1 \\
B = B(t, s) &= \frac{1}{t}\sqrt{\frac{\mathbf{H}^2(t) - t^2\dot{\mathbf{H}}^2(t)}{1 - b^2(t, s)}} > 0.
\end{aligned}
$$

These formulas are legitimate because of Lemma 11.16. Indeed, in view of (11.25) we have $0 \leqslant b(t, s) < 1$ since $\lambda(t) \geqslant m$. Also, by (11.24) we see that $0 \leqslant \frac{t\,\dot{\mathbf{H}}(t)}{\mathbf{H}(t)} < 1$. This means that $B(t, s)$ is well defined and $B(t, s) > 0$ for almost every

$t \in [r, R]$ and $s \in [r_*, R_*]$. Now, concerning, the term $\mathcal{I}$ in (11.30) we observe that $|h_N|\,|h_T| \geqslant J_h(z)$, so

$$\mathcal{I} \geqslant \frac{2\lambda\big(F(s)\big)F(s)}{s\dot{F}(s)} J_h(z) \qquad (s = |h(z)|)$$

as desired. Next

$$\mathcal{II} = -\lambda(t)(1-b^2)B^2 = \frac{-\lambda(t)}{t^2}\Big[\mathbf{H}^2(t) - t^2\dot{\mathbf{H}}^2(t)\Big] \qquad t = |z|.$$

The only term that requires care is $\mathcal{III}$. We write

$$\begin{aligned}\mathcal{III} &= \frac{2\lambda(t)}{t}\sqrt{\big(1-b^2\big)\left[\mathbf{H}^2(t) - t^2\dot{\mathbf{H}}^2(t)\right]} \cdot |h_T| \\ &= \frac{2}{t}\sqrt{\mathbf{H}^2(t) - t^2\dot{\mathbf{H}}^2(t)}\,\sqrt{\Phi(t,s)}\,\frac{|h_T|}{|h|}\end{aligned}$$

where $\Phi(t,s) = \big[s\lambda(t)\big]^2 - \Big[\frac{\lambda(F(s))\,F(s)}{\dot{F}(s)}\Big]^2$.

Lemma 11.19. *For $t \in [r, R]$ and $s \in [r_*, R_*]$ it holds*

$$\Phi(t,s) \geqslant \Phi(t, \mathbf{H}(t)) = \lambda^2(t)\Big[\mathbf{H}^2(t) - t^2\dot{\mathbf{H}}^2(t)\Big].$$

Proof. Firts observe that the Lagrange-Euler equation (11.4) translates into an equation for $F = F(s)$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{F(s)\lambda\big(F(s)\big)}{\dot{F}(s)}\right] = \lambda\big(F(s)\big)\frac{s\dot{F}(s)}{F(s)} \qquad s \in (r_*, R_*].$$

This shows, in particular, that the function $s \mapsto \Phi(t,s)$ is Lipschitz continuous. Therefore to find its minimum point it suffices to show that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(t,s)\begin{cases}\leqslant 0 & \text{for almost every } s \in [r_*, \mathbf{H}(t)] \\ \geqslant 0 & \text{for almost every } s \in [\mathbf{H}(t), R_*].\end{cases}$$

Note that the derivative of the weight function does not enter the computation below. Indeed, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\Phi(t,s) &= 2\lambda^2(t)s - 2\frac{F(s)\lambda\big(F(s)\big)}{\dot{F}(s)} \cdot \lambda\big(F(s)\big)\frac{s\dot{F}(s)}{F(s)} \\ &= 2s\Big[\lambda^2(t) - \lambda^2\big(F(s)\big)\Big].\end{aligned}$$

The lemma follows because λ is nonincreasing and F is increasing in $[r_*, R_*]$. $\square$

Proposition 11.18 also follows because

$$\frac{|h_T|}{|h|} \geqslant \operatorname{Im} \frac{h_T}{h}.$$

□

The energy of h can now be estimated from bellow by integrating null-Lagrangians in (11.29). The computation below can easily be seen when $h\colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ is a homeomorphism. For $h \in \overline{\mathscr{H}}^2(\mathbb{A}, \mathbb{A}^*)$ (the $\mathscr{W}^{1,2}$-weak limit of homeomorphisms) the computation is still valid. We have

$$\mathcal{E}[h] \geqslant \mathcal{E}_1[h] + \mathcal{E}_2[h] + \mathcal{E}_3[h]$$

where

$$\begin{aligned}
\mathcal{E}_1[h] &\stackrel{\text{def}}{=\!=} \int_{\mathbb{A}} \frac{2F\big(|h(z)|\big)\,\lambda\big(F(|h(z)|)\big)}{|h(z)|\dot F\big(|h(z)|\big)} J_h(z)\,\mathrm{d}z \\
&= \int_{h(\mathbb{A})=\mathbb{A}^*} \frac{2F\big(|w|\big)\,\lambda\big(F(|w|)\big)}{|w|\dot F(|w|)} \mathrm{d}w \\
&= \int_{A(\rho,R)} \frac{2F\big(|\mathbf{h}(z)|\big)\,\lambda\big(F(|\mathbf{h}(z)|)\big)}{|\mathbf{h}(z)|\dot F\big(|\mathbf{h}(z)|\big)} \cdot J_{\mathbf{h}}(z)\,\mathrm{d}z \\
&= \int_{A(\rho,R)} \frac{2t\lambda(t)\dot{\mathcal{H}}(t)}{\mathcal{H}(t)} \cdot \frac{\mathcal{H}(t)\dot{\mathcal{H}}(t)}{t} \mathrm{d}z = 4\pi \int_\rho^R t\,\dot{\mathcal{H}}^2(t)\,\lambda(t)\,\mathrm{d}t \\
&= 4\pi \int_r^R t\,\dot{\mathbf{H}}^2(t)\,\lambda(t)\,\mathrm{d}t = \mathcal{E}_1[\mathbf{h}] \\
\mathcal{E}_2[h] &\stackrel{\text{def}}{=\!=} 2\pi \int_r^R \left[\frac{\mathbf{H}^2(t)}{t} - t\,\dot{\mathbf{H}}^2(t)\right] \lambda(t)\,\mathrm{d}t = \mathcal{E}_2[\mathbf{h}] \\
\mathcal{E}_3[h] &\stackrel{\text{def}}{=\!=} 2\int_r^R \frac{\lambda(t)}{t^2} \Big[\mathbf{H}^2(t) - t^2\,\dot{\mathbf{H}}^2(t)\Big] \left(\int_0^{2\pi} \operatorname{Im} \frac{h_\theta}{h} \mathrm{d}\theta\right) t\,\mathrm{d}t \\
&= 4\pi \int_r^R \left[\frac{1}{t}\mathbf{H}^2(t) - t\,\dot{\mathbf{H}}^2(t)\right] \lambda(t)\,\mathrm{d}t = \mathcal{E}_3[\mathbf{h}]
\end{aligned}$$

we can now add those free Lagrangian identities to conclude with the desired energy inequality

$$\mathcal{E}[h] \geqslant \mathcal{E}_1[h] + \mathcal{E}_2[h] + \mathcal{E}_3[h] = 2\pi \int_r^R \left[t\,\dot{\mathbf{H}}^2(t) + \frac{\mathbf{H}^2(t)}{t}\right] \lambda(t)\,\mathrm{d}t = \mathcal{E}[\mathbf{h}].$$

11.7.2. *Uniqueness*

The uniqueness statement in Theorem 11.17 follows through backward analysis of the inequalities above. Indeed, for the equality to occur we must ensure that for almost every $z \in \mathbb{A}$ the following equations are satisfied.

(i) $B\big(|z|, |h(z)|\big) = |h_T(z)|$;
(ii) $b\big(|z|, |h(z)|\big) \cdot |h_T(z)| = |h_N(z)|$;
(iii) $|h_N(z)|\,|h_T(z)| = J_h(z) = \operatorname{Im}\Big[\overline{h_N(n)} h_T(z)\Big]$;
(iv) $\left|\frac{h_T(z)}{h(z)}\right| = \operatorname{Im}\frac{h_T(z)}{h(z)}$.

Using polar coordinates $z = te^{i\theta}, r \leqslant t \leqslant R, 0 \leqslant \theta \leqslant 2\pi$, write

$$h(z) = H(t,\theta)e^{i\Phi(t,\theta)}$$

where $r_* \leqslant H(t,\theta) \leqslant R_*$ and $\Phi(t,\theta)$ is 2π-periodic with respect to θ. Condition (iv) yields $\operatorname{Re}\frac{h_\theta}{h} \equiv 0$. Hence

$$\left(H^2\right)_\theta = \left(|h|^2\right)_\theta = h_\theta \overline{h} + h\overline{h_\theta} = |h|^2 \left(\frac{h_\theta}{h} + \frac{\overline{h_\theta}}{\overline{h}}\right) \equiv 0.$$

Thus $H = H(t)$ so

$$h = H(t)e^{i\Phi(t,\theta)}.$$

Furthermore, we see from (iv), that $\operatorname{Im}\frac{h_\theta}{h} \geqslant 0$. This shows that $\Phi_\theta(t,\theta) \geqslant 0$. Then we look at (i) to infer that

$$B(t, H(t)) = \frac{1}{t} H(t)|\Phi_\theta|$$

so

$$\Phi_\theta = \frac{tB(t, H(t))}{H(t)} \stackrel{\text{def}}{=\!=} \lambda(t).$$

Thus $\Phi(t,\theta) = \lambda(t)\theta + c(t)$. Hence

$$h(z) = H(t)e^{i\alpha(t)\theta + ic(t)}.$$

But h is a limit of orientation preserving homeomorphisms, so the increment of $\arg h$ over every circle $|z| = t$ equals 2π. This yields $\alpha(t) \equiv 1$. Thus

$$h(z) = H(t)e^{i\theta + ic(t)}.$$

Finally, we look at the condition (iii) which tells us that $\operatorname{Re}(\overline{h_N} h_\theta) = 0$. This reads as $\operatorname{Re}[\dot{H}(t) - iH\dot{c}(t)][iH(t)] = 0$, meaning that $\dot{c}(t) \equiv 0$, so c is a constant. We obtain,

$$h = e^{ic}H(t)e^{i\theta}.$$

This is none other than a rotation of a radial map. Since $\mathcal{E}[h] = \mathcal{E}[\mathbf{h}]$ we conclude from the uniqueness statement in Proposition 11.14 that $h = e^{ic}\mathbf{h}$. $\square$

11.7.3. *A laminate annulus*

Suppose a cylindrical shell

$$\mathbb{S} = A(r, R) \times [0, \ell] = \{(z, s) \in \mathbb{C} \times \mathbb{R} \colon r \leqslant |z| \leqslant R\,,\ 0 \leqslant s \leqslant \ell\}$$

is manufactured in multiple cylindrical layers of different materials to achieve improved strength or desired properties, see Figure 11.4.

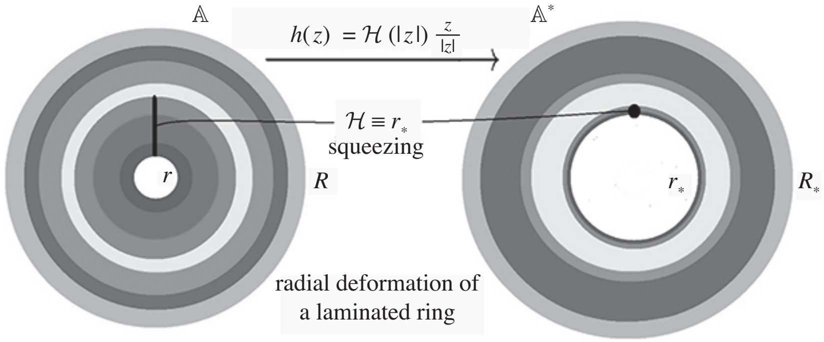

Figure 11.4. Multiple cylindrical layers.

Let

$$\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \cup \cdots \cup \mathbb{S}_n\,, \quad \mathbb{S}_i = \mathbb{A}_i \times [0, \ell]$$

where

$$\mathbb{A}_i = A(r_i, r_{i+1}) \qquad r < r_1 < R_2 < \cdots < r_{n-1} < r_n = R.$$

Our mathematical model of hyperelastic deformations of $\mathbb{S}$ will be furnished by the weighted Dirichlet energy (11.3) in which $\lambda = \lambda\big(|z|\big)$ assumes constant values on each layer $\mathbb{A}_i$, say $\lambda(t) \equiv m_i > 0$, for $r_{i-1} < t < r_i$, $i = 1, 2, \ldots, n$ $(r_\circ = r)$. These constants characterize the resilience of the layer to stretching or squeezing. We extend λ for convenience of the writing, to the entire positive real line $(0, \infty)$ by setting $\lambda(t) = m_1$ whenever $0 < t \leqslant r$ and $\lambda(t) = m_n$ whenever $R \leqslant t < \infty$. On each layer, the weight λ does not appear in the Lagrange-Euler equation (11.4).

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t\dot{\mathcal{H}}(t)\big) = \frac{\mathcal{H}(t)}{t}. \tag{11.31}$$

The general solution of (11.31), called harmonic curve, takes the form

$$\mathcal{H}(t) = at + \frac{b}{t} \qquad a, b \in \mathbb{R}.$$

All λ-harmonics are now obtained by gluing these solutions

$$\mathcal{H}(t) = \begin{cases} \mathcal{H}_1(t) \overset{\text{def}}{=\!=} a_1 t + \frac{b_1}{t} & \text{for } 0 < t < r \\ \mathcal{H}_i(t) \overset{\text{def}}{=\!=} a_i t + \frac{b_i}{t} & \text{for } r_{i-1} < t < r_i \quad i = 1, 2, \dots, n \\ & \vdots \\ \mathcal{H}_n(t) \overset{\text{def}}{=\!=} a_n t + \frac{b_n}{t} & \text{for } t > R. \end{cases}$$

To make $\mathcal{H}$ continuous we must impose the conditions

$$\mathcal{H}_i(r_i) = a_i r_i + \frac{b_i}{r_i} = \mathcal{H}_{i+1}(r_i) = a_{i+1} r_i + \frac{b_{i+1}}{r_i} \quad \text{for } i = 1, 2, \dots, n-1. \tag{11.32}$$

To satisfy the Lagrange-Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[t\lambda(t)\dot{\mathcal{H}}(t)\Big] = \frac{\lambda(t)\mathcal{H}(t)}{t} \qquad \text{for all } 0 < t < \infty$$

we need this equation to be true near each point $t = r_i$, $i = 1, 2, \dots, n-1$ (this equation is automatically satisfied near $t = r_\circ$ and $t = r_n$). This amounts to saying that the function $\mathcal{A}(t) \overset{\text{def}}{=\!=} t\lambda(t)\dot{\mathcal{H}}(t)$ is continuous on $(0, \infty)$. Equivalently,

$$\lim_{t \nearrow r_i} \mathcal{A}(t) = \lim_{t \searrow r_{i+1}} \mathcal{A}(t) \qquad \text{for } i = 1, 2, \dots, n-1.$$

Equivalently,

$$m_i \dot{\mathcal{H}}_i(r_i) = r_i m_i \left(a_i - \frac{b_i}{r_i^2}\right) = m_{i+1} \dot{\mathcal{H}}_{i+1}(r_i) = r_i\, m_{i+1} \left(a_{i+1} - \frac{b_{i+1}}{r_i^2}\right). \tag{11.33}$$

Remark 11.20. It is worth noting that choosing any $\mathcal{H}_k(t) = a_k t + \frac{b_k}{t}$, for some $1 \leqslant k \leqslant n$, the other pieces of $\mathcal{H}$ are uniquely determined. Indeed, we solve the linear systems (11.32) and (11.33) for a_{k+1} and b_{k+1} in terms of a_k and b_k,

$$\begin{cases} 2\,a_{k+1} & = \left(1 + \frac{m_k}{m_{k+1}}\right) a_k + \left(1 - \frac{m_k}{m_{k+1}}\right) \frac{b_k}{r_k^2} \\ 2\,b_{k+1} & = \left(1 - \frac{m_k}{m_{k+1}}\right) a_k\, r_k^2 + \left(1 + \frac{m_k}{m_{k+1}}\right) b_k. \end{cases}$$

Similarly, we can express a_{k-1} and b_{k-1} by means of a_k and b_k

$$\begin{cases} 2\,a_{k-1} & = \left(1 + \frac{m_k}{m_{k-1}}\right) a_k + \left(1 - \frac{m_k}{m_{k-1}}\right) \frac{b_k}{r_{k-1}^2} \\ 2\,b_{k-1} & = \left(1 - \frac{m_k}{m_{k-1}}\right) a_k\, r_{k-1}^2 + \left(1 + \frac{m_k}{m_{k-1}}\right) b_k. \end{cases}$$

We see, by induction, that the λ-harmonics $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{k-1}$ as well as $\mathcal{H}_{k+1}$, $\mathcal{H}_{k+2}, \dots, \mathcal{H}_n$ are uniquely determined by $\mathcal{H}_k$. This observation reflects our more general fact; the unique continuation result in Corollary 11.5.

From now on we assume that every harmonic curve $\mathcal{H}_i = \mathcal{H}_i(t)$, $i = 1, 2, \dots, n$, is positive at the endpoints; thus positive in the entire interval (r_{i_1}, r_i). It is either convex, linear or concave, depending on the sign of the coefficient b_i. To detect the collapsing phenomenon $\mathcal{H}$ must assume its minimum value inside the interval $[r, R]$. This cannot occur at the endpoints, $r_1, r_2, \dots, r_n$, because $\dot{\mathcal{H}}$ does not change sign when passing through the endpoints. Let $t_\circ \in (r_{k-1}, r_k)$ be a minimum point of $\mathcal{H}$ within $[r, R]$ for some $k = 1, 2, \dots, n$. Thus $\dot{\mathcal{H}}_k(t_\circ) = 0$, so

$$\mathcal{H}_k(t) = \frac{r_*}{2}\left(\frac{t}{t_\circ} + \frac{t_\circ}{t}\right) \qquad \text{where } r_* \stackrel{\text{def}}{=\!=} \mathcal{H}_k(t_\circ). \tag{11.34}$$

Consequently, $\mathcal{H}_k$ is convex and we have:

$$(\text{for } r_{k-1} \leqslant t < t_\circ) \qquad \dot{\mathcal{H}}_k(t) < 0 < \dot{\mathcal{H}}_k(t) \qquad (\text{for } t_\circ < t <\leqslant r_k).$$

For $i = k, k+1, \dots, n-1$ the positive sign of $\dot{\mathcal{H}}(t)$ is preserved in the passage from $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$. Similarly, for $i = 1, 2, \dots, k-1$, the negative sign of $\dot{\mathcal{H}}(t)$ is preserved in the passage from $\mathcal{H}_i$ to $\mathcal{H}_{i-1}$. In summary, $\mathcal{H}$ is strictly decreasing in $(0, t_\circ)$ and strictly increasing in $(t_\circ, \infty)$.

However, the sign of the second derivative $\ddot{\mathcal{H}}$ can generally change without any particular pattern. To see this, consider a continuous curve $\mathcal{H}$ defined in $(0, \infty) = (0, r_1] \cup [r_1, r_2] \cup \cdots \cup [r_{n-1}, r_n] \cup [r_n, \infty)$ as composed of harmonic curves $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ (convex, linear or concave) in such a way that $\dot{\mathcal{H}}_i$ and $\dot{\mathcal{H}}_{i+1}$ have the same sign at their common end-point r_i. Then one can always adjust positive numbers $m_1, m_2, \dots, m_n$ so that $\mathcal{H}$ becomes λ-harmonic with $\lambda(t) = m_i$ for $r_{i-1} \leqslant t \leqslant r_i$, see Figure 11.5.

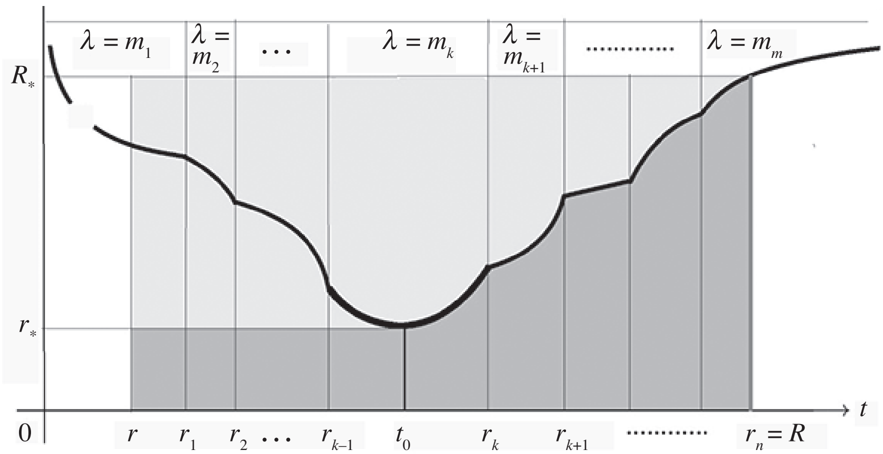

Figure 11.5. It is generally possible that squeezing of matter, which always begins in the inner layer $\mathbb{A}_1$, will continue until $\mathbb{A}_k$.

11.7.4. *Two layers*

To work out fairly detailed picture of squeezing phenomenon under general deformations of class $\mathscr{W}_+^{1,2}(\mathbb{A}, \mathbb{A}^*)$ (weak $\mathscr{W}^{1,2}$-limits of homeomorphism), we now confine ourselves to discussing two layers. Thus we consider a cylindrical shell $\mathbb{A} \times [0, \ell] = A(r, R) \times [0, \ell]$, $0 < r < 1 < R < \infty$, which is manufactured with two different materials. In our Dirichlet-energy model of hyperelastic deformations, this will be reflected by the assumption that the weight $\lambda = \lambda(|z|)$ is a constant, equal to M, in the inner annulus $A(r, 1)$ and equal to another constant m in the outer annulus $A(1, R)$. We assume that $M > m$, so λ is nonincreasing. Let us interprete this assumption in two examples.

Example 11.21. First consider a sliding bearing shell that is made of two cylindrical layers. Naturally, the inner layer is made of hard material to resist the damage of sliding. Consequently, any necessary small adjustment of the dimensions of the bearing shell, by stretching or squeezing, takes more energy (per volume) in the inner layer than that in the outer layer. Thus, in our weighted Dirichlet-energy model we assume that $M > m$.

Example 11.22. Another illustration concerns a low cost finger ring that is made of two materials. The outer thin layer is made of gold for better appearance. Gold is relatively easy to change shape. In particular, squeezing or stretching the ring takes less energy (per volume) in the golden layer than in the inner layer. Thus, in our model we take $m < M$.

Without losing any generality, we normalize λ-harmonics $\mathcal{H}\colon \mathbb{R}_+ \to \mathbb{R}$ at the interface point $t = 1$ by setting $\mathcal{H}(1) = 1$. It results in one-parameter family of λ-harmonic in $(0, \infty)$

$$\mathcal{H}(t) = \begin{cases} \mathcal{H}_M(t) \stackrel{\text{def}}{=\!=} \frac{1}{2}\left[\left(1 + \frac{k}{M}\right)t + \left(1 - \frac{k}{M}\right)t^{-1}\right] & \text{for } 0 < t \leqslant 1 \\ \mathcal{H}_m(t) \stackrel{\text{def}}{=\!=} \frac{1}{2}\left[\left(1 + \frac{k}{m}\right)t + \left(1 - \frac{k}{m}\right)t^{-1}\right] & \text{for } 1 \leqslant t < \infty \end{cases}$$

where the parameter k runs over all real numbers.

The function $\mathcal{A} \stackrel{\text{def}}{=\!=} t\lambda(t)\dot{\mathcal{H}}(t)$ is locally Lipschitz continuous in $\mathbb{R}_+$, because

$$\lim_{t \nearrow 1} \mathcal{A}(t) = M\,\dot{\mathcal{H}}_M(1) = k \qquad \text{and} \qquad \lim_{t \searrow 1} \mathcal{A}(t) = m\,\dot{\mathcal{H}}_m(1) = k.$$

Thus we have the Lagrange-Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t\lambda(t)\dot{\mathcal{H}}(t)\right] = \frac{\lambda(t)\mathcal{H}(t)}{t} \qquad \text{for all } 0 < t < \infty.$$

The case $k = 0$ is rather special. It gives us a curve $\mathcal{H}(t) = \frac{1}{2}\left(t + \frac{1}{t}\right)$ which is λ-harmonic regardless of the values M and m. This is because $M\dot{\mathcal{H}}_M(1) = 0 = m\dot{\mathcal{H}}_m(1)$.

From now on we assume that $\dot{\mathcal{H}}_M(1) \neq \dot{\mathcal{H}}_m(1)$. The condition

$$M\,\dot{\mathcal{H}}_M(1) = m\,\dot{\mathcal{H}}_m(1)$$

tells us that $\dot{\mathcal{H}}_M(1)$ and $\dot{\mathcal{H}}_m(1)$ are non zero and have the same sign. In particular, the minimum value $\mathcal{H}_{\min} \overset{\text{def}}{=\!=} \mathcal{H}(t_\circ)$ within the interval $[r, R]$ is attained at a point $t_\circ \neq 1$.

Case 1. $1 < t_\circ < R$. Thus, the minimal radial map takes the form

$$\mathbf{H}(t) = \begin{cases} r_* & \text{for } r \leqslant t \leqslant r_\circ \\ \mathcal{H}_m(t) & \text{for } t_\circ \leqslant t \leqslant R \end{cases}$$

where $\mathcal{H}_m(t) = \frac{1}{2}\left[(1+\frac{k}{m})t + (1-\frac{k}{m})t^{-1}\right]$. As indicated by Figure 11.6 the range of the parameter k is

$$-m < k < 0.$$

For such k the curve $\mathcal{H}_M$ is uniquely determined. It is strictly decreasing and convex. The slopes of $\mathcal{H}_M$ and $\mathcal{H}_m$ at the intercase point $p = 1$ are negative. Precisely

$$\dot{\mathcal{H}}_M(1) = \frac{m}{M}\dot{\mathcal{H}}_m(1) < 0.$$

In what follow $\mathcal{H}_M$ will be irrelevant to the question of the energy-minimal map.

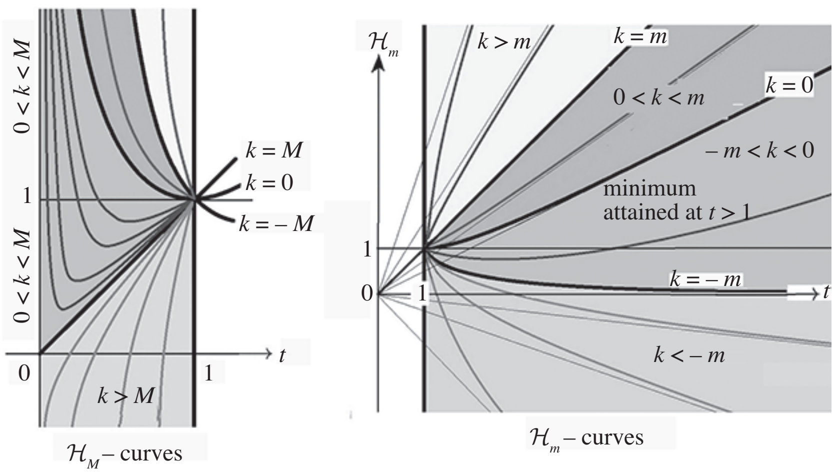

Figure 11.6. λ-harmonics are obtained by gluing an $\mathcal{H}_M$-curve and the corresponding $\mathcal{H}_m$-curve of the same parameter k.

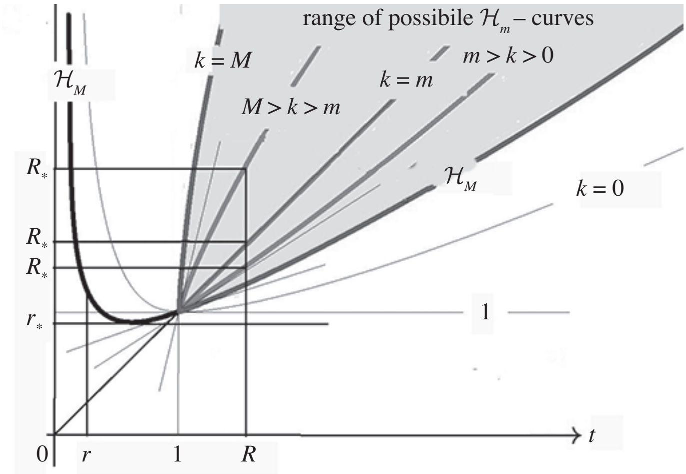

Figure 11.7. Gluing $\mathcal{H}_M$-curves with $\mathcal{H}_m$-curves at $t = 1$.

We find that $\dot{\mathcal{H}}_m(t_\circ) = 0$ at $t_\circ = \sqrt{\frac{m-k}{m+k}}$, so the minimum value of $\mathcal{H}_m$ equals

$$r_* \stackrel{\text{def}}{=\!=} \mathcal{H}_m(t_\circ) = \sqrt{1 - \frac{k^2}{m^2}}.$$

Moreover $R_* \stackrel{\text{def}}{=\!=} \mathcal{H}_m(R) = \frac{1}{2}\left[(1+\frac{k}{m})R + (1-\frac{k}{m})R^{-1}\right]$. Hence

$$\frac{R_*}{r_*} = \frac{1}{2}\left[\sqrt{\frac{m+k}{m-k}}R + \frac{m+km-k^{-1}}{R}\right] = \frac{1}{2}\left[\frac{R}{t_\circ} + \frac{t_\circ}{R}\right].$$

We just recovered the familiar equation for the critical Nitsche map; that is, the one which takes $[t_\circ, R]$ homeomorphically onto $[r_*, R_*]$ and $\dot{\mathcal{H}}_m(t_\circ) = 0$. The map $\mathbf{h} \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^*$ given by

$$\mathbf{h} = \begin{cases} r_* \frac{z}{|z|} & \text{for } r \leqslant |z| \leqslant t_\circ \\ \mathcal{H}_m(|z|)\frac{z}{|z|} & \text{for } t_\circ \leqslant |z| \leqslant R \end{cases}$$

is energy-minimal within the class of radial deformations $\mathscr{W}_+^{1,2}(\mathbb{A}, \mathbb{A}^*)$. Theorem 11.17 tells us that this map is also energy-minimal within the general deformations of class $\mathscr{W}_+^{1,2}(\mathbb{A}, \mathbb{A}^*)$, if

$$\int_{t_\circ}^{R} \frac{\lambda(t)\,\mathrm{d}t}{t} \leqslant \lambda(R) = m. \tag{11.35}$$

This reads as $\frac{R}{t_\circ} < e$. We arrive at the necessary condition

$$\frac{R_*}{r_*} \leqslant \frac{1}{2}\big(e + e^{-1}\big) = \cosh 1.$$

In fact this condition suffices to have squeezing phenomenon. Precisely, we have the following:

Corollary 11.23. *Let $\mathbb{A} = A(r, 1) \cup A(1, R)$, where the outer layer $A(1, R)$ has conformal modulus* $\operatorname{Mod} A(1, R) \overset{\text{def}}{=\!=} \log R \geqslant 1$. *Recall the weight*

$$\lambda(z) = \begin{cases} M & \textit{for } r \leqslant |z| < 1 \\ m & \textit{for } 1 < |z| \leqslant R \end{cases} \qquad M > m > 0.$$

Let $\mathbf{h} \colon \mathbb{A} \xrightarrow{\text{onto}} \mathbb{A}^* = A(r_*, R_*)$ be a general energy-minimal deformation of the class $\mathscr{W}^{1,2}_+(\mathbb{A}, \mathbb{A}^*)$. We assume that $\mathbb{A}^*$ is conformally thin:

$$\operatorname{Mod} \mathbb{A}^* \overset{\text{def}}{=\!=} \log \frac{R_*}{r_*} < \log \frac{1}{2}\big(e + e^{-1}\big) = \log(\cosh 1).$$

Then $\mathbf{h}$ squeezes the inner layer $A(r, 1)$ and a part of the outer layer $A(1, R)$ onto the inner boundary circle of $\mathbb{A}^*$. Precisely, for some $1 < t_\circ < R$, we have (up to rotation)

$$\mathbf{h}(z) = \begin{cases} r_* \frac{z}{|z|} & \textit{for } r \leqslant |z| \leqslant t_\circ \\ \mathcal{H}(|z|)\frac{z}{|z|} & \textit{for } t_\circ \leqslant |z| < R \end{cases}$$

where $\mathcal{H}(|z|)\frac{z}{|z|}$ is a harmonic homeomorphism of $A(t_\circ, R)$ onto $\mathbb{A}^*$, $\mathcal{H}(t_\circ) = r_*$, $\mathcal{H}(R) = R_*$, $\dot{\mathcal{H}}(t_\circ) = 0$.

Proof. Denote by $\mu = \frac{R_*}{r_*}$ so

$$1 < \mu < \frac{1}{2}\left(e + e^{-1}\right).$$

There is a unique $t_\circ \in (\frac{R}{e}, R) \subset (1, R)$ such that

$$\mu = \frac{1}{2}\left(\frac{R}{t_\circ} + \frac{t_\circ}{R}\right)$$

invoke $\mathcal{H}_m(t)$ with the parameter $k \in (-m, 0)$ defined by

$$k = -m\frac{t_\circ^2 - 1}{t_\circ^2 + 1}\,, \text{ equivalently } t_\circ = \sqrt{\frac{m-k}{m+k}}.$$

Denote

$$r'_* \stackrel{\text{def}}{=\!=} \sqrt{1-\frac{k^2}{m^2}} \qquad \text{and} \qquad R'_* = \mu r'_*.$$

Now, we look at the harmonic curve

$$h_m(t) = \frac{1}{2}\left[\left(1+\frac{k}{m}\right)t + \left(1-\frac{k}{m}\right)t^{-1}\right]$$

and see that:

$$\begin{aligned} h_m(1) &= 1\dot{h}_m(t_\circ) = 0 \quad h_m(t_\circ) = r'_* \\ h_m(R) &= \left[\left(1+\frac{k}{m}\right)R + \left(1-\frac{k}{m}\right)R^{-1}\right] \\ &= r'_*\frac{1}{2}\left[\frac{R}{t_\circ}+\frac{t_\circ}{R}\right] = \mu r'_* = R'_*. \end{aligned}$$

For our purpose the extension of h_m to the curve h_M will play no role because $h_M(t) > r'_*$. We obtain a radial energy-minimal map $\mathbf{h}' \colon \mathbb{A} \xrightarrow{\text{onto}} A(r'_*, R'_*)$

$$\mathbf{h}'(z) = \begin{cases} r'_* \frac{z}{|z|} & \text{for } r \leqslant |z| \leqslant t_\circ \\ \mathcal{H}_m(|z|)\frac{z}{|z|} & \text{for } t_\circ \leqslant |z| < R. \end{cases}$$

Actually this map is also energy-minimal deformation of $\mathbb{A}$ onto $A(r'_*, R'_*)$ within the general class $\mathscr{W}^{1,2}_+(\mathbb{A})$. To see this we appeal to Theorem 11.17. Accordingly, this is indeed the case beacuse

$$\int_{t_\circ}^{R} \frac{\lambda(t)}{t} \leqslant \lambda(R) = m.$$

Note that $\lambda(t) \equiv m$ in $(t_\circ, R) \subset (1, R)$, so this above inequality reduces to $\log\frac{R}{t_\circ} \leqslant 1$ which holds because $t \in (\frac{R}{e}, R)$. Finally we rescale $\mathbf{h}'$ to arrive at the desired energy-minimal deformation

$$\mathbf{h} = \frac{r_*}{r'_*}\mathbf{h}' \qquad \mathbf{h}(t_\circ) = r_*$$

and

$$\mathbf{h}(R) = \frac{r_*}{r'_*}\mathbf{h}'(R) = \frac{r_*}{r'_*}\mathcal{H}_m(R) = \frac{r_*}{r'_*}R'_* = r_*\mu = R_*.$$

Case 2. $r < t_\circ < 1$. This is the case when the radial energy minimal

$$\mathbf{h}(z) = \begin{cases} \mathcal{H}_M(t_\circ)\frac{z}{|z|} & \text{for } r \leqslant |z| \leqslant t_\circ \\ \mathcal{H}_M(|z|)\frac{z}{|z|} & \text{for } t_\circ \leqslant |z| \leqslant 1 \\ \mathcal{H}_m(|z|)\frac{z}{|z|} & \text{for } 1 \leqslant |z| \leqslant R \end{cases} \tag{11.36}$$

squeezes only part of the inner layer of $\mathbb{A} = A(r, 1) \cup A(1, R)$. For a given minimum point $t_\circ \in (r, 1)$ we choose the parameter $k \in (0, M)$ so that the function

$$\mathcal{H}_M(t) = \frac{1}{2}\left[\left(1 + \frac{k}{M}\right)t + \left(1 - \frac{k}{M}\right)t^{-1}\right]$$

has vanishing derivative at $t = t_\circ$. This occurs when

$$t_\circ = \sqrt{\frac{M-k}{M+k}} \qquad \text{so } k = \frac{1-t_\circ^2}{1+t_\circ^2}\,M.$$

There are two conditions on $t_\circ$ to ensure that the radial map $\mathbf{h}$ in (11.36) defines general energy-minimal deformation within the class $\mathscr{W}^{1,2}(\mathbb{A}, \mathbb{A}^*)$. First is that $t_\circ > r$, meaning that

$$k > \frac{1-r^2}{1+r^2}\,M.$$

Second condition is to ensure hypotheses of Theorem 11.17; that is,

$$\int_{t_\circ}^{R} \frac{\lambda(t)\,\mathrm{d}t}{t} \leqslant \lambda(R) = m.$$

This reads as

$$M \log \frac{1}{t_\circ} + m \log R \leqslant m$$

so $t_\circ \geqslant \left(\frac{e}{R}\right)^{\frac{m}{M}}$. Note that this yields $R \geqslant e$ (as in Case 1). Further computation of the size of the target annulus $\mathbb{A}^* = A(r_*, R_*)$ goes similar lines as in Case 1, but explicit bounds are more involved. However, without making explicit bounds we see by Theorem 11.17, that the squeezing of the part of the inner layer always occurs if the target annulus is sufficiently thin.

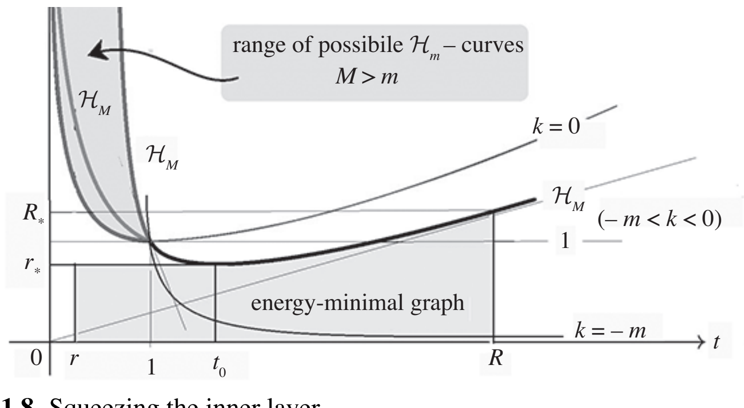

Figure 11.8. Squeezing the inner layer.

□

References

[1] L. V. AHLFORS, "Conformal Invariants: Topics in Geometric Function Theory", McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[2] L. V. AHLFORS and A. BEURLING, *Conformal invariants and function-theoretic null-sets*, Acta Math. **83** (1950), 101–129.

[3] S. S. ANTMAN, "Nonlinear Problems of Elasticity. Applied Mathematical Sciences", Vol. 107, Springer-Verlag, New York, 1995.

[4] K. ASTALA, T. IWANIEC and G. MARTIN, "Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane", Princeton University Press, 2009.

[5] K. ASTALA, T. IWANIEC and G. MARTIN, *Deformations of annuli with smallest mean distortion*, Arch. Ration. Mech. Anal. **195** (2010), 899–921.

[6] J. M. BALL, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal. **63** (1976/77), 337–403.

[7] J. M. BALL, J. C. CURRIE and P. J. OLVER, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal. **41** (1981), 135–174.

[8] D. BSHOUTY and W. HENGARTNER, *Univalent harmonic mappings in the plane*, Workshop on Complex Analysis (Lublin, 1994), Ann. Univ. Mariae Curie-Skłodowska Sect. A **48** (1994), 12–42.

[9] P. G. CIARLET, "Mathematical elasticity. Three-dimensional Elasticity", Vol. I, Studies in Mathematics and its Applications, 20, North-Holland Publishing Co., Amsterdam, 1988.

[10] R. COURANT, "Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces", with an appendix by M. Schiffer, Springer-Verlag, New York-Heidelberg, 1950.

[11] S. K. DONALDSON and D. P. SULLIVAN, *Quasiconformal 4-manifolds*, Acta Math. **163** (1989), 181–252.

[12] R. J. DUFFIN, *The extremal length of a network*, J. Math. Anal. Appl. **5** (1962) 200–215.

[13] P. DUREN, "Harmonic Mappings in the Plane", Cambridge University Press, Cambridge, 2004.

[14] D. G. B. EDELEN, *The null set of the Euler-Lagrange operator*, Arch. Ration. Mech. Anal. **11** (1962), 117–121.

[15] I. FONSECA and W. GANGBO, "Degree Theory in Analysis and Applications", Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[16] B. FUGLEDE, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.

[17] F. W. GEHRING, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.

[18] F. W. GEHRING and T. IWANIEC, *The limit of mappings with finite distortion*, Ann. Acad. Sci. Fenn. Math. **24** (1999), 253–264.

[19] F. W. GEHRING and O. LEHTO, *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Math. **272** (1959), 1–9.

[20] H. GRÖTZSCH, *Über möglichst konforme Abbildungen von schlichten Bereichen*, Ber. Math.-phys. Kl. Sachs. Akad. Wiss. Leipzig **84** (1932), 114–120.

[21] S. HENCL and P. KOSKELA, "Lectures on Mappings of Finite Distortion", Lecture Notes in Math. 2096, Springer, 2014.

[22] T. IWANIEC, *On the concept of the weak Jacobian and Hessian*, Rep. Univ. Jyväskylä Dep. Math. Stat. Univ. Jyväskylä, Jyväskylä, **83** (2001), 181–205.

[23] T. IWANIEC and G. MARTIN, "Geometric Function Theory and Non-linear Analysis", Oxford Mathematical Monographs, Oxford University Press, 2001.

[24] T. IWANIEC, L. V. KOVALEV and J. ONNINEN, *Harmonic mappings of an annulus, Nitsche conjecture and its generalizations*, Amer. J. Math. **132** (2010), 1397–1428.

[25] T. IWANIEC, L. V. KOVALEV and J. ONNINEN, *The Nitsche conjecture*, J. Amer. Math. Soc. **24** (2011), 345–373.

[26] T. IWANIEC, L. V. KOVALEV and J. ONNINEN, *Diffeomorphic approximation of Sobolev homeomorphisms*, Arch. Ration. Mech. Anal. **201** (2011), 1047–1067.

[27] T. IWANIEC, L. V. KOVALEV and J. ONNINEN, *Hopf differentials and smoothing Sobolev homeomorphisms*, Int. Math. Res. Not. IMRN **2012** (2012), 3256–3277.

[28] T. IWANIEC and J. ONNINEN, *Hyperelastic deformations of smallest total energy*, Arch. Ration. Mech. Anal. **194** (2009), 927–986.

[29] T. IWANIEC and J. ONNINEN, *Neohookean deformations of annuli, existence, uniqueness and radial symmetry*, Math. Ann. **348** (2010), 35–55.

[30] T. IWANIEC and J. ONNINEN, "n-Harmonic Mappings between Annuli", Mem. Amer. Math. Soc., Vol. 218, 2012.

[31] T. IWANIEC and J. ONNINEN, *Mappings of least Dirichlet energy and their Hopf differentials*, Arch. Ration. Mech. Anal. **209** (2013), 401–453.

[32] T. IWANIEC and J. ONNINEN, *Monotone Sobolev mappings of planar domains and surfaces*, Arch. Ration. Mech. Anal. (2016), 159–181.

[33] T. IWANIEC and J. ONNINEN, *Limits of Sobolev homeomorphisms*, J. Eur. Math. Soc. (JEMS) **19** (2017), 473–505.

[34] O. LEHTO and K. I. VIRTANEN, "Quasiconformal Mappings in the Plane", Springer-Verlag, New York-Heidelberg, 1973.

[35] A. LYZZAIK, *Univalent harmonic mappings of annuli*, Publ. Inst. Math. (Beograd) (N.S.) **75** (89) (2004), 173–183.

[36] J. E. MARSDEN and T. J. R. HUGHES, "Mathematical Foundations of Elasticity", Dover Publications, Inc., New York, 1994.

[37] G. J. MARTIN and M. MCKUBRE-JORDENS, *Deformations with smallest weighted Lp average distortion and Nitsche-type phenomena*, J. Lond. Math. Soc. (2) **85** (2012), 282–300.

[38] O. MARTIO, V. RYAZANOV, U. SREBRO and E. YAKUBOV, "Moduli in Modern Mapping Theory", Springer, 2009.

[39] D. MENCHOFF, *Sur les differentielles totales des fonctions univalentes*, Math. Ann. **105** (1931), 75–85.

[40] D. S. MITRLOVIC and P. M. VASIC, "Analytic Inequalities", Springer-Verlag, Berlin, New York, 1970.

[41] G. D. MOSTOW, "Strong Rigidity of Locally Symmetric Spaces", Annals of Mathematics Studies Vol. 78, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973.

[42] C. B. MORREY, *Quasi-convexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

[43] J. C. C. NITSCHE, *On the modulus of doubly connected regions under harmonic mappings*, Amer. Math. Monthly **69** (1962), 781–782.

[44] J. C. C. NITSCHE, "Vorlesungen über Minimalflächen", Springer-Verlag, Berlin-New York, 1975.

[45] YU. G. RESHETNYAK, "Space Mappings with Bounded Distortion", Translations of Mathematical Monographs, Vol. 73, American Mathematical Society, Providence, RI, 1989.

[46] E. SANDIER and S. SERFATY, *Limiting vorticities for the Ginzburg-Landau equations*, Duke Math. J. **117** (2003), 403–446.

[47] G. SCHOBER, *Planar harmonic mappings*, In: "Computational Methods and Function Theory", Lecture Notes in Math., Vol. 1435, Springer, Berlin, 1990, 171–176.

[48] F. H. SCHOTTKY, *Über konforme Abbildung von mehrfach zusammenhängenden Fläche*, J. für Math. **83** (1877).

[49] M. ŠILHAVÝ, "The Mechanics and Thermodynamics of Continuous Media", Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[50] K. STREBEL, *Extremal quasiconformal mappings*, Results Math. **10** (1986), 168–210.

[51] A. TAHERI, *Quasiconvexity and uniqueness of stationary points in the multi-dimensional calculus of variations*, Proc. Amer. Math. Soc. **131** (2003), 3101–3107.

[52] O. TEICHMÜLLER, *Extremale quasikonforme abbildungen und quadratische differentiale*, Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **22** (1940).
[53] O. TEICHMÜLLER, *Bestimmung der extremalen quasikonformen abbildungen bei geschlossenen orientierten Riemannschen flächen*, Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. (1943), 1–42.
[54] C. TRUESDELL and W. NOLL, "The Non-linear Field Theories of Mechanics", Edited and with a preface by Stuart S. Antman, Springer-Verlag, Berlin, 2004.
[55] M. VUORINEN, "Conformal Geometry and Quasiregular Mappings", Lecture Notes in Math., 1319, Springer-Verlag, Berlin, 1988.
[56] J. VÄISÄLÄ, "Lectures on n-dimensional Quasiconformal Mappings", Lecture Notes in Math., Vol. 229, Springer-Verlag, Berlin-New York, 1971.

Department of Mathematics
Syracuse University, Syracuse
NY 13244, USA
tiwaniec@syr.edu

Department of Mathematics
Syracuse University, Syracuse
NY 13244, USA
and
Department of Mathematics and Statistics
P.O. Box 35, FI- 40014
University of Jyväskylä, Finland
jkonnine@syr.edu