

The P' -operator, the Q' -curvature, and the CR tractor calculus

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Abstract. We establish an algorithm which computes formulae for the CR GJMS operators, the P' -operator, and the Q' -curvature in terms of CR tractors. When applied to torsion-free pseudo-Einstein contact forms, this algorithm both gives an explicit factorisation of the CR GJMS operators and the P' -operator, and shows that the Q' -curvature is constant, with the constant explicitly given in terms of the Webster scalar curvature. We also use our algorithm to derive local formulae for the P' -operator and Q' -curvature of a five-dimensional pseudo-Einstein manifold. Comparison with Marugame's formulation of the Burns-Epstein invariant as the integral of a pseudohermitian invariant yields new insights into the class of local pseudohermitian invariants for which the total integral is independent of the choice of pseudo-Einstein contact form.

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1. Introduction

An important class of differential operators in CR geometry are the CR GJMS (or Gover-Graham) operators [28]. The CR GJMS operator P_{2k} is a formally self-adjoint differential operator with principal part the k -th power $(-\Delta_b)^k$ of the negative of the sublaplacian; our convention is that $-\Delta_b$ is a positive operator. This operator is defined on any pseudohermitian manifold (M^{2n+1}, H, θ) with $k \leq n+1$ and is CR invariant, $P_{2k} : \mathcal{E}(-\frac{n+1-k}{2}) \rightarrow \mathcal{E}(-\frac{n+1+k}{2})$; i.e. if $\widehat{\theta} = e^\Upsilon \theta$, then

$$e^{\frac{n+1+k}{2}\Upsilon} \widehat{P}_{2k}(f) = P_{2k}(e^{\frac{n+1-k}{2}\Upsilon} f)$$

for all $f \in C^\infty(M)$. Special cases are the CR Laplacian P_2 studied by Jerison and Lee [41] and the CR Paneitz operator P_4 which are, for example, important in the study of the embedding problem in three dimensions [20].

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The critical CR GJMS operators P_{2n+2} are of particular interest. The kernel of P_{2n+2} is nontrivial, containing the space \mathcal{P} of CR pluriharmonic functions [19, 39]; indeed, this characterizes the kernel on the standard CR spheres [6]. As such, Branson’s argument of analytic continuation in the dimension [5] gives rise to the P' -operator. This operator was first identified on the sphere by Branson, Fontana and Morpurgo [6], then on general three-dimensional CR manifolds by Yang and the first-named author [19], and then in general dimensions by Hirachi [39]. As an operator $P' : \mathcal{P} \rightarrow C^\infty(M)$, the P' -operator is not invariant; rather, if $\widehat{\theta} = e^\Upsilon \theta$, then

$$e^{(n+1)\Upsilon} \widehat{P}'(f) = P'(f) + P_{2n+2}(f\Upsilon) \tag{1.1}$$

for all $f \in \mathcal{P}$. In this way, one could think of P' as a Q -curvature operator (cf. [7]). For geometric applications, it is often preferable to regard the P' -operator as a map $P' : \mathcal{P} \rightarrow C^\infty(M)/\mathcal{P}^\perp$. Both $\mathcal{P} \subset \mathcal{E}(0)$ and $\mathcal{P}^\perp \subset \mathcal{E}(-n - 1)$ are CR invariant spaces, while the self-adjointness of P_{2n+2} and the fact $\mathcal{P} \subset \ker P_{2n+2}$ combine with (1.1) to imply that $P' : \mathcal{P} \rightarrow C^\infty(M)/\mathcal{P}^\perp$ is CR invariant. In particular, P' determines a CR invariant pairing $\mathcal{P} \times \mathcal{P} \ni (u, v) \mapsto \int u P'v$.

The natural extension of Branson’s Q -curvature to the CR setting is $Q := P'(1)$ (cf. [25]). While the total Q -curvature is a CR invariant, it is often trivial: The total Q -curvature of a compact three-dimensional CR manifold is always zero [38] and the Q -curvature vanishes identically for any pseudo-Einstein manifold [25]. The latter fact implies that we may again use analytic continuation in the dimension to define the Q' -curvature as a pseudohermitian invariant of pseudo-Einstein manifolds; see [19] in dimension three and [39] in general dimension. Suppose that θ is pseudo-Einstein. Then $\widehat{\theta} = e^\Upsilon \theta$ is pseudo-Einstein if and only if $\Upsilon \in \mathcal{P}$; see [44]. If $\Upsilon \in \mathcal{P}$, we find that

$$e^{(n+1)\Upsilon} \widehat{Q}' = Q' + P'(\Upsilon) + \frac{1}{2} P_{2n+2}(\Upsilon^2), \tag{1.2}$$

where we regard P' and Q' as $C^\infty(M)$ -valued. Regarding instead P' and Q' as $C^\infty(M)/\mathcal{P}^\perp$ -valued, we have the transformation rule

$$e^{(n+1)\Upsilon} \widehat{Q}' = Q' + P'(\Upsilon).$$

It is in this context that the Q' -curvature prescription problem seems solvable; see [18] for progress in the three-dimensional setting.

A key property of the Q' -curvature is that its total integral over a compact pseudo-Einstein manifold is a *secondary invariant*. Following Hirachi [39], by a secondary invariant we mean a pseudohermitian invariant which is not CR invariant, but which is invariant within the distinguished class of pseudo-Einstein contact forms. That the total Q' -curvature is independent of the choice of pseudo-Einstein contact form follows from (1.2) and the self-adjointness of the critical CR GJMS operator and of the P' -operator [19, 39]; that it is not independent of the choice of contact form follows from [19, Proposition 6.1].

As a global secondary invariant, the total Q' -curvature is a biholomorphic invariant of domains in \mathbb{C}^{n+1} (cf. [25]). It is interesting to compare it with the Burns-Epstein invariant [10, 11, 47]. In dimension three, the total Q' -curvature agrees with the Burns-Epstein invariant up to a universal constant [19, 39], whereupon one obtains the Gauss-Bonnet formula

$$\chi(X) = \int_X \left(c_2 - \frac{1}{3}c_1^2 \right) + \frac{1}{16\pi^2} \int_M Q' \theta \wedge d\theta$$

for $X \subset \mathbb{C}^2$ a bounded strictly pseudoconvex domain with boundary $M = \partial X$ and the Chern forms are computed with respect to a complete Kähler-Einstein metric in X . In dimension five, the total Q' -curvature and the Burns-Epstein invariant do not in general agree; see Theorem 1.3 and Proposition 8.9 below for more precise statements.

At present, there are only two ways to study the CR GJMS operators, the P' -operator, and the Q' -curvature. The first is to restrict to dimension three, where local formulae are known and can be used to address questions involving the signs of these objects [19]. The second is to pass to the ambient manifold, where the definitions are relatively simple and may be readily used to prove many formal properties of these objects [39]. However, it is not straightforward to produce local formulae for these operators from the ambient definition, nor is it known how to use the ambient definition to address issues such as the sign of the CR GJMS operators and the P' -operator or the value of the total Q' -curvature.

The goal of this article is to rectify some of these issues by giving a new interpretation of the CR GJMS operators, the P' -operator, and the Q' -curvature. Specifically, we give an interpretation of these objects in terms of the CR tractor calculus, building on the work of Graham and the second-named author on the CR GJMS operators [28]. Our main result is an algorithm, encoded in Theorem 6.7, which produces a tractor formula for these operators in terms of tractor D -operators and the tractor curvature (cf. [29]). As an immediate application, we compute the Q' -curvature and obtain factorisations of the CR GJMS operators and the P' -operator on any Einstein pseudohermitian manifold (cf. [27]); i.e., on any pseudo-Einstein manifold with vanishing torsion.

Theorem 1.1. *Let (M^{2n+1}, H, θ) be an embeddable Einstein pseudohermitian manifold. For any integer $1 \leq k \leq n + 1$, the CR GJMS operator P_{2k} is equal to*

$$P_{2k} = \begin{cases} \prod_{\ell=1}^{\frac{k}{2}} (-\Delta_b + c_\ell i \nabla_0 + d_\ell P) (-\Delta_b - c_\ell i \nabla_0 + d_\ell P) & \text{if } k \text{ is even} \\ Y \prod_{\ell=1}^{\frac{k-1}{2}} (-\Delta_b + c_\ell i \nabla_0 + d_\ell P) (-\Delta_b - c_\ell i \nabla_0 + d_\ell P) & \text{if } k \text{ is odd,} \end{cases} \tag{1.3}$$

where $c_\ell = k - 2\ell + 1$ and $d_\ell = \frac{n^2 - (k - 2\ell + 1)^2}{n}$ and $Y = -\Delta_b + nP$ is the CR Yamabe operator. Moreover, the P' -operator is

$$P'_{2n+2} = n! \left(\frac{2}{n}\right)^{n+1} \prod_{\ell=0}^n (-\Delta_b + 2\ell P) \tag{1.4}$$

and the Q' -curvature is

$$Q'_{2n+2} = (n!)^2 \left(\frac{4P}{n}\right)^{n+1}. \tag{1.5}$$

An alternative proof of Theorem 1.1 has been given by Takeuchi [51] using purely ambient techniques.

Since the standard CR sphere and the Heisenberg group with its standard contact form are both Einstein pseudohermitian manifolds, (1.3) recovers the known formulae for the CR GJMS operators on these manifolds [6, 33]. Moreover, (1.4) recovers the formula for the P' -operator obtained by Branson, Fontana and Morpurgo on the sphere and (1.5) gives a geometric meaning to the constant in their sharp Onofri-type inequality on CR pluriharmonic functions [6].

Einstein pseudohermitian manifolds are equivalent to η -Sasaki-Einstein manifolds (cf. [45, 50]). This observation leads to a wealth of examples to which Theorem 1.1 applies (cf. [3, 4, 50]).

Theorem 1.1 gives factorisations of the CR GJMS operators in terms of Folland-Stein operators and of the P' -operator in terms of the sublaplacian. In particular, the spectrum of the P' -operator is completely understood in terms of the scalar curvature and the spectrum of the sublaplacian of an Einstein pseudohermitian manifold. Likewise, it determines the total Q' -curvature of an Einstein pseudohermitian manifold in terms of its scalar curvature and volume. As a special case of these observations, we have the following corollary.

Corollary 1.2. *Let (M^{2n+1}, H, θ) be a compact embeddable Einstein pseudohermitian manifold with nonnegative CR Yamabe constant. Then $P' \geq 0$, $\ker P' = \mathbb{R}$, and*

$$\int_M Q' \theta \wedge (d\theta)^n \leq \int_{S^{2n+1}} Q'_0 \theta_0 \wedge (d\theta_0)^n, \tag{1.6}$$

where the right-hand side denotes the total Q' -curvature of the standard CR sphere. Moreover, equality holds in (1.6) if and only if (M, H, θ) is CR equivalent to the standard CR sphere.

In three dimensions, the conclusions of Corollary 1.2 are true under weaker hypotheses involving only the CR Paneitz operator and the CR Yamabe constant [19]. It is natural to ask if similar positivity results extend to higher dimensions; Corollary 1.2 suggests that there is scope for such a result. Reasons to be interested in such a result are its characterization of the standard CR sphere and its role in finding metrics of constant Q' -curvature by variational methods (cf. [18]).

In five dimensions, it is straightforward to produce from Theorem 6.7 an explicit tractor formula for the CR GJMS operators, the P' -operator, and the Q' -curvature. In particular, we obtain an explicit local formula for the Q' -curvature in this dimension:

$$\begin{aligned}
 Q' = & 4\Delta_b^2 P + 4\Delta_b |A_{\alpha\beta}|^2 - 16 \operatorname{Im} \nabla^\gamma (A_{\beta\gamma} \nabla^\beta P) + 16i \nabla^\gamma Y_\gamma \\
 & - 16\Delta_b P^2 - 32P |A_{\alpha\beta}|^2 + 32P^3 - 16A^{\alpha\gamma} Q_{\alpha\gamma};
 \end{aligned}
 \tag{1.7}$$

see Section 2 for a description of our notation. We thus obtain a formula for the total Q' -curvature of a compact pseudo-Einstein five-manifold (cf. (8.9) and [40]). On the other hand, Marugame [47] has computed the Burns-Epstein invariant in this setting. By comparing these formulae, we obtain the following Gauss-Bonnet formula for bounded strictly pseudoconvex domains in \mathbb{C}^3 .

Theorem 1.3. *Let $X \subset \mathbb{C}^3$ be a bounded strictly pseudoconvex domain with boundary $M^5 = \partial X$. Let ρ be a defining function for M such that $g = -i\bar{\partial}\partial \log \rho$ is a complete Kähler-Einstein metric in X . Then*

$$\chi(X) = \int_X \left(c_3 - \frac{1}{2}c_1c_2 + \frac{1}{8}c_1^3 \right) + \frac{1}{\pi^3} \int_M (Q' + 16\mathcal{I}') \theta \wedge (d\theta)^2,
 \tag{1.8}$$

where \mathcal{I}' is the pseudohermitian invariant

$$\mathcal{I}' = -\frac{1}{8}\Delta_b \left| S_{\alpha\bar{\beta}\gamma\bar{\sigma}} \right|^2 + |V_{\alpha\bar{\beta}\gamma}|^2 + \frac{1}{2}P |S_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2.
 \tag{1.9}$$

In (1.9), $S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ denotes the Chern tensor – the completely tracefree part of the pseudohermitian curvature – and $V_{\alpha\bar{\beta}\gamma}$ is the CR analogue of the Cotton tensor; see Section 2 for details. The pseudohermitian invariant \mathcal{I}' should be regarded as the analogue in the critical dimension of the nontrivial conformal invariant of weight -6 discovered by Fefferman and Graham (cf. [24, (9.3)]). More precisely, there is a CR invariant \mathcal{I} of weight -3 and of the form $|\nabla_\rho S_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2$ plus terms involving $V_{\alpha\bar{\beta}\delta}$ (see (8.23)) in general dimensions which is a pure divergence in dimension five. Arguing by analytic continuation in the dimension yields, modulo divergences, the pseudohermitian invariant \mathcal{I}' on five-dimensional pseudo-Einstein manifolds; in particular, one expects the total \mathcal{I}' -curvature to be a global secondary invariant. In Proposition 8.9, we give an intrinsic proof of this fact provided $c_2(H^{1,0})$ vanishes in $H^4(M; \mathbb{R})$. If M is the boundary of a Stein manifold, then $c_2(H^{1,0}) = 0$; see Section 8 for details. Note that Marugame has already given an extrinsic proof [47] of this fact, without assuming the vanishing of the second real Chern class. Our study of \mathcal{I}' suggests that the CR analogue of the Deser-Schwimmer conjecture is more subtle than its conformal analogue; see Remark 8.13 for further discussion.

We conclude this introduction by outlining the algorithm for producing tractor formulae for the CR GJMS operators, the P' -operator, and the Q' -curvature contained in Theorem 6.7 and how it is applied to obtain Theorem 1.1. To that end, we first recall the definitions of these objects via the ambient manifold [39].

Suppose that (M, H) is a strictly pseudoconvex CR manifold which is embedded in a complex manifold X ; note that compact strictly pseudoconvex CR manifolds of dimension at least five are automatically embeddable [2, 37, 46]. Let $\rho \in C^\infty(X)$ be a defining function for M which is positive on the pseudoconvex side, and let $\theta = \text{Im } \bar{\partial}\rho|_{TM}$ be the induced contact form. Suppose that, near M in X , there is a $(n + 2)$ -nd root \mathcal{L}_X of the canonical bundle \mathcal{K}_X of X . The *ambient space of M* , which we denote $M_{\mathcal{A}}$, is the total space of $\mathcal{L}_X \setminus \{0\} \rightarrow X$, and the restriction $M_{\mathcal{A}}|_M$ is denoted by \mathcal{F} . Note that \mathcal{F} is a CR manifold of (real) dimension $2n + 3$ with Levi form which is positive definite except in the fibre direction, and the pullback of ρ to $M_{\mathcal{A}}$, also denoted by ρ , is a defining function for \mathcal{F} .

Given $\lambda \in \mathbb{C}^*$, define the *dilation* $\delta_\lambda: M_{\mathcal{A}} \rightarrow M_{\mathcal{A}}$ by fibre-wise scalar multiplication, $\delta_\lambda(\xi) = \lambda\xi$. Given $w \in \mathbb{R}$, denote

$$\tilde{\mathcal{E}}(w) = \left\{ f \in C^\infty(M_{\mathcal{A}}; \mathbb{C}) : \delta_\lambda^* f = |\lambda|^{2w} f \text{ for all } \lambda \in \mathbb{C}^* \right\}.$$

A natural choice of defining function $r \in \tilde{\mathcal{E}}(1)$ for \mathcal{F} is obtained by Fefferman’s construction [23]: it is the unique defining function modulo $O(\rho^{n+3})$ such that

$$\text{Ric}[r] = i\eta r^n \partial r \wedge \bar{\partial} r + O(\rho^{n+1}), \tag{1.10}$$

where $\text{Ric}[r]$ is the Ricci curvature of the *ambient metric* $\tilde{g}[r] = -i\partial\bar{\partial}r$ defined in a neighborhood of \mathcal{F} in $M_{\mathcal{A}}$ and $\eta|_{\mathcal{F}}$ is a CR invariant, the *obstruction function*.

Let $\mathcal{K} = \Lambda^{n+1}(H^{0,1})^\perp$ denote the canonical bundle of M . Note that $\mathcal{K} = \mathcal{K}_X|_M$. Let $\mathcal{L}_M = \mathcal{L}_X|_M$, so that \mathcal{L}_M is a $(n + 2)$ -nd root of \mathcal{K} . Given $w, w' \in \mathbb{C}$ such that $w - w' \in \mathbb{Z}$, we denote

$$\mathcal{E}(w, w') = \mathcal{L}_M^{-w} \otimes \bar{\mathcal{L}}_M^{-w'}.$$

A *CR density of weight $w \in \mathbb{R}$* is a smooth section of the bundle $\mathcal{E}(w) = \mathcal{E}(w, w)$. When clear by context, we also use $\mathcal{E}(w)$ to denote the space of CR densities of weight w . Given a homogeneous function $\tilde{f} \in \tilde{\mathcal{E}}(w)$ on the ambient space, its restriction to \mathcal{F} defines a CR density $f = \tilde{f}|_{\mathcal{F}} \in \mathcal{E}(w)$. We call \tilde{f} an *ambient extension of f* . Such functions are unique up to adding terms of the form ϕr with $\phi \in \tilde{\mathcal{E}}(w - 1)$.

Let $k \in \{0, 1, \dots, n + 1\}$ and set $w = -\frac{n+1-k}{2}$. Given $f \in \mathcal{E}(w)$, define

$$P_{2k} f := (-2\tilde{\Delta})^k \tilde{f}|_{\mathcal{F}}.$$

This definition is independent of the choice of ambient extension \tilde{f} . In particular, $P_{2k}: \mathcal{E}(w) \rightarrow \mathcal{E}(w - k)$ is a conformally covariant operator, the k -th order CR

GJMS operator [28]. Our normalization is such that P_{2k} has leading order term $(-\Delta_b)^k$.

Set $h_\theta = r/\rho$. Define the P -prime operator P'_{2n+2} on \mathcal{P} by

$$P'_{2n+2}f = -(-2\tilde{\Delta})^{n+1} (f \log h_\theta)|_{\mathcal{F}} \in \mathcal{E}(-n-1). \tag{1.11}$$

This operator depends only on f and the choice of contact form θ . Moreover, computing with respect to the contact form $\widehat{\theta} = e^\Upsilon \theta$ yields the transformation formula (1.1).

Suppose now that θ is a pseudo-Einstein contact form; equivalently, suppose that $\log h_\theta|_{\mathcal{F}} \in \mathcal{P}$. Define the Q -prime curvature Q'_{2n+2} by

$$Q'_{2n+2} = \frac{1}{2}(-2\tilde{\Delta})^{n+1} (\log h_\theta)^2|_{\mathcal{F}} \in \mathcal{E}(-n-1). \tag{1.12}$$

This scalar depends only on the choice of contact form θ . Moreover, computing with respect to the pseudo-Einstein contact form $\widehat{\theta} = e^\Upsilon \theta$ yields the transformation formula (1.2).

An alternative approach to these definitions can be made through the CR tractor calculus [28]. Specifically, Čap and the second-named author [12,14] have provided a dictionary which effectively equates definitions of CR invariant objects made via the ambient metric with definitions made via the CR tractor calculus. Using this dictionary, we develop in Section 6 an algorithm for generating tractor formulae for the CR GJMS operators, the P' -operators, and the Q' -curvatures in general dimension. This has two benefits. First, it is easy to execute this algorithm in low dimensions, and this allows us to derive (1.7); see Section 8 for further discussion. Second, the algorithm almost immediately yields Theorem 1.1 using the local correspondence between Einstein contact forms and parallel CR standard tractors. More precisely, the algorithm leads to a formula (cf. Theorem 6.7) for the CR GJMS operators in terms of tractor D -operators and the CR Weyl tractor, a tractor version of the curvature tensor of the ambient metric. Since contractions of a parallel CR standard tractor I_A into the curvature necessarily vanish, we obtain a formula for the CR GJMS operators in terms of compositions of $I^A I^{\bar{B}} \mathbb{D}_A \mathbb{D}_{\bar{B}}$ and $I^{\bar{B}} \mathbb{D}_{\bar{B}}$ which, after some reorganization, recovers (1.3) (cf. [27]). The factorisations for P' and Q' then follow from the ‘‘Branson trick,’’ made rigorous using log densities in a manner analogous to the ambient definitions (1.11) and (1.12); see Section 4 and Section 6 for further discussion.

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2. Background

2.1. CR geometry

Recall that an *almost CR structure*, of hypersurface type, on a smooth manifold M of real dimension $2n + 1$ is a rank $2n$ (real) subbundle H of the tangent bundle TM equipped with an almost complex structure $J: H \rightarrow H$. For simplicity, throughout the following we assume M is orientable. We write $q: TM \rightarrow \mathcal{C}$ for the canonical bundle surjection onto the real (quotient) line bundle $\mathcal{C} := TM/H$. For two sections $\xi, \eta \in \Gamma(H)$ the expression $q([\xi, \eta])$ is bilinear over smooth functions, and so there is a skew symmetric bundle map $\mathcal{L}: H \times H \rightarrow \mathcal{C}$ given by $\mathcal{L}(\xi(x), \eta(x)) = q([\xi, \eta](x))$. If this skew form is non-degenerate then the almost CR structure is said to be *non-degenerate*; such non-degeneracy exactly means that H is a contact distribution on M .

We shall write $\mathcal{B}_{\mathbb{C}}$ for the complexification of a real vector bundle \mathcal{B} . Considering now $T_{\mathbb{C}}M$ and $H_{\mathbb{C}} \subset T_{\mathbb{C}}M$, the complex structure on H is equivalent to a splitting of the subbundle $H_{\mathbb{C}}$ into the direct sum of the holomorphic part $H^{1,0}$ and the antiholomorphic part $H^{0,1} = \overline{H^{1,0}}$. The almost CR structure is called *integrable*, or a *CR structure*, if the subbundle $H^{1,0} \subset T_{\mathbb{C}}M$ is involutive; *i.e.* the space of its sections is closed under the Lie bracket. Then, in particular, \mathcal{L} is of type $(1, 1)$, meaning $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in H$. We assume integrability.

Let $q_{\mathbb{C}}$ denote the complex linear extension of q . The *CR Levi form* $\mathcal{L}_{\mathbb{C}}$ of an almost CR structure is the $\mathcal{C}_{\mathbb{C}}$ -valued Hermitian form on $H^{1,0}$ induced by $(\xi, \eta) \mapsto 2iq_{\mathbb{C}}([\xi, \bar{\eta}])$. Note that \mathcal{L} can be naturally identified with the imaginary part of $\mathcal{L}_{\mathbb{C}}$, and so non-degeneracy of the CR structure can be characterised by non-degeneracy of the Levi form.

Choosing a local trivialisation of \mathcal{C} and using the induced trivialisation of $\mathcal{C}_{\mathbb{C}}$, $\mathcal{L}_{\mathbb{C}}$ gives rise to a Hermitian form. If (p, q) is the signature of this form, then one also says that M is non-degenerate of signature (p, q) . If $p \neq q$, then such local trivialisations of \mathcal{C} necessarily fit together to give a global trivialisation. In the case of symmetric signature (p, p) we assume that a global trivialisation of \mathcal{C} exists. A global trivialisation of \mathcal{C} is equivalent to a ray subbundle of the line bundle of contact forms for $H \subset TM$, so it gives a notion of positivity for contact forms.

An important class of CR structures are those which arise from generic real hypersurfaces in complex manifolds, as follows. Let \mathcal{M} be a complex manifold of complex dimension $n + 1$ and let $M \subset \mathcal{M}$ be a smooth real hypersurface. For each point $x \in M$, the tangent space $T_x M$ is a subspace of the complex vector space $T_x \mathcal{M}$ of real codimension one. This implies that the maximal complex subspace H_x of $T_x M$ must be of complex dimension n . These subspaces fit together to define a smooth subbundle $H \subset TM$, equipped with a complex structure. Since the bundle $H^{1,0} \subset T_{\mathbb{C}}M$ can be viewed as the intersection of the involutive subbundles $T_{\mathbb{C}}M$ and $T^{1,0}\mathcal{M}$ of $T_{\mathbb{C}}\mathcal{M}|_M$ we see that we always obtain a CR structure in this way. Generically this structure is non-degenerate, and in this case is referred to as an *embedded CR manifold*.

2.2. CR density bundles

In CR geometry an important role is played by a natural family of line bundles that arise as follows. In the complexified cotangent bundle the annihilator of $H^{0,1}$ has complex dimension $n + 1$, and so its $(n + 1)$ st complex exterior power is a complex line bundle \mathcal{K} ; this is the *canonical bundle*.

It is convenient to assume the existence of certain roots of \mathcal{K} . Specifically we assume that there exists, and we have chosen, a complex line bundle $\mathcal{E}(1, 0) \rightarrow M$ with the property that there is a duality between $\mathcal{E}(1, 0)^{\otimes(n+2)}$ and the canonical bundle \mathcal{K} . Such a bundle may not exist globally, but such a choice is always possible locally. For CR manifolds embedded in \mathbb{C}^{n+1} the canonical bundle is trivial, so such a bundle $\mathcal{E}(1, 0)$ exists globally in this setting. For $w, w' \in \mathbb{R}$ such that $w' - w \in \mathbb{Z}$, the map $\lambda \mapsto |\lambda|^{2w} \bar{\lambda}^{(w'-w)}$ is a well-defined one-dimensional representation of \mathbb{C}^* . Hence we can define a complex line bundle $\mathcal{E}(w, w')$ over M by forming the associated bundle to the frame bundle of $\mathcal{E}(1, 0)$ with respect to this representation. By construction we get $\mathcal{E}(w', w) = \overline{\mathcal{E}(w, w')}$, $\mathcal{E}(-w, -w') = \mathcal{E}(w, w')^*$ and $\mathcal{E}(k, 0) = \mathcal{E}(1, 0)^{\otimes k}$ for $k \in \mathbb{N}$. Finally, by definition $\mathcal{K} \cong \mathcal{E}(0, -n - 2)$. For simplicity, we denote $\mathcal{E}(w) = \mathcal{E}(w, w)$.

2.3. Pseudohermitian structures

For the purposes of explicit calculations on a CR manifold (M, H) it is convenient to use pseudohermitian structures, and we review some basic facts about these. This also serves to fix conventions, which follow [28]. Since M is orientable the annihilator H^\perp of H in T^*M admits a nonvanishing global section. A *pseudohermitian structure* is a choice θ of such a section and, from the non-degeneracy of the CR structure, is a contact form on M . We fix an orientation on H^\perp and restrict consideration to choices of θ which are positive with respect to this orientation. The *Levi form* of θ is the Hermitian form h^θ (or simply h) on $H^{1,0} \subset T_{\mathbb{C}}M$ defined by

$$h(Z, \bar{W}) = -2id\theta(Z, \bar{W}).$$

With the trivialisation of $\mathcal{C}_{\mathbb{C}}$ given by θ , this corresponds to $\mathcal{L}_{\mathbb{C}}$ introduced above.

Given a pseudohermitian structure θ , we define the *Reeb field* T to be the unique vector field on M satisfying

$$\theta(T) = 1 \quad \text{and} \quad i_T d\theta = 0. \tag{2.1}$$

An *admissible coframe* is a set of complex valued forms $\{\theta^\alpha\}$, $\alpha = 1, \dots, n$, which satisfy $\theta^\alpha(T) = 0$, annihilate $H^{0,1}$, and whose restrictions to $H^{1,0}$ are complex linear and form a basis for $(H^{1,0})^*$. We use lower case Greek indices to refer to frames for $T^{1,0}$ or its dual. We shall also interpret these indices abstractly, and use \mathcal{E}^α as an abstract index notation for the bundle $H^{1,0}$ (or its space of smooth sections) and write \mathcal{E}_α for its dual. This notation is extended in an obvious way to the conjugate bundles, and to tensor products of various of these.

There is a natural inclusion of the real line bundle $\mathcal{C} = TM/H$ into the density bundle $\mathcal{E}(1)$ which is defined as follows. For a local nonzero section α of $\mathcal{E}(1, 0)$ recall that one can, by definition, view $\alpha^{-(n+2)}$ as a section of the canonical bundle \mathcal{K} . Then, by [43, Lemma 3.2], there is a unique positive contact form θ with respect to which $\alpha^{-(n+2)}$ is length normalised. From the formula in [43, Lemma 3.2] one sees that, in the other direction, θ determines α up a phase factor, and scaling θ causes the inverse scaling of $\alpha\bar{\alpha}$. Thus the mapping $TM \ni \xi \mapsto \theta(\xi)\alpha\bar{\alpha}$ descends to an inclusion of \mathcal{C} into $\mathcal{E}(1)$ which, by construction, is CR invariant. In particular, the canonical contact form $\boldsymbol{\theta} := \theta \otimes \alpha\bar{\alpha} \in \mathcal{C} \otimes \mathcal{E}(1)$ depends only on the CR structure. Conversely, a *scale* $\alpha\bar{\alpha} \in \mathcal{E}(1)$ determines a contact form θ by this formula.

By integrability and (2.1), we have

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$$

for a smoothly varying Hermitian matrix $h_{\alpha\bar{\beta}}$, which we may interpret as the matrix of the Levi form h determined by θ , in the frame θ^α , or as the Levi form h itself in abstract index notation. Using the inclusion $\mathcal{C} \hookrightarrow \mathcal{E}(1)$ from above, the CR Levi form $\mathcal{L}_{\mathcal{C}}$ can be viewed as a canonical section of $\mathcal{E}_{\alpha\bar{\beta}}(1)$ which we also denote by $h_{\alpha\bar{\beta}}$; this agrees with $h_{\alpha\bar{\beta}}$ if $\mathcal{E}(1)$ is trivialised using θ . By $h^{\alpha\bar{\beta}} \in \mathcal{E}^{\alpha\bar{\beta}}(-1)$ we denote the inverse of $h_{\alpha\bar{\beta}}$ and this will be used to raise and lower indices without further mention.

By ∇ we denote the *Tanaka-Webster connections* (on various bundles) associated to θ . In particular, these satisfy $\nabla\theta = 0, \nabla h = 0, \nabla h = 0, \nabla T = 0$, and $\nabla J = 0$, so the decomposition $T_{\mathbb{C}}M = H^{1,0} \oplus H^{0,1} \oplus \mathbb{C}T$ is invariant under ∇ . On tensors, the Tanaka-Webster connection is determined from the Webster connection forms ω_α^β and the torsion forms $\tau_\gamma = A_{\alpha\gamma}\theta^\alpha$, defined in terms of an admissible coframe by

$$\begin{aligned} d\theta^\alpha &= \theta^\beta \wedge \omega_\beta^\alpha + \theta \wedge \tau^\alpha, \\ dh_{\alpha\bar{\beta}} &= \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha}, \\ A_{\alpha\gamma} &= A_{\gamma\alpha}, \end{aligned}$$

where $\omega_{\bar{\beta}\alpha} := \overline{\omega_{\alpha\bar{\beta}}}$. We call $A_{\alpha\gamma}$ the *torsion* of θ . The *pseudohermitian curvature* $R_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ of θ is obtained from the curvature forms $\Pi_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta$ via the structure equations

$$\begin{aligned} \Pi_\alpha^\beta &= R_{\alpha\mu\bar{\nu}}^\beta \theta^\mu \wedge \theta^{\bar{\nu}} + \nabla^\beta A_{\alpha\mu} \theta^\mu \wedge \theta - \nabla_\alpha A_{\bar{\nu}}^\beta \theta^{\bar{\nu}} \wedge \theta \\ &\quad + ih_{\alpha\bar{\nu}} A_{\bar{\sigma}}^\beta \theta^{\bar{\nu}} \wedge \theta^{\bar{\sigma}} - iA_{\alpha\mu} \theta^\mu \wedge \theta^\beta \end{aligned} \tag{2.2}$$

where $A_{\bar{\alpha}\bar{\beta}} := \overline{A_{\alpha\beta}}$. The *pseudohermitian Ricci tensor* is $R_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}\gamma}^\gamma$ and the *pseudohermitian scalar curvature* is $R = R_\gamma^\gamma$. The *sublaplacian* is

$$\Delta_b = \nabla^\gamma \nabla_\gamma + \nabla_\gamma \nabla^\gamma.$$

A contact form θ on (M^{2n+1}, H) is *pseudo-Einstein* if

$$\begin{cases} R_{\alpha\bar{\beta}} = \frac{1}{n} Rh_{\alpha\bar{\beta}} & \text{if } n > 1 \\ \nabla_{\alpha} R = i \nabla^{\gamma} A_{\alpha\gamma} & \text{if } n = 1. \end{cases}$$

The set of pseudo-Einstein contact forms, when non-empty, forms a distinguished class of contact forms parameterised by \mathcal{P} : If θ is pseudo-Einstein, then $\widehat{\theta} = e^{\Upsilon} \theta$ is pseudo-Einstein if and only if Υ is a CR pluriharmonic function [38,44].

We may decompose a tensor field relative to the splittings of $T_{\mathbb{C}}M$ and its dual. In this way, we may calculate the covariant derivative componentwise. Each of the components may be regarded as a section of a tensor product of \mathcal{E}^{α} or its dual or conjugates thereof. We therefore often restrict consideration to the action of the connection on \mathcal{E}^{α} or \mathcal{E}_{α} . We use indices $\alpha, \bar{\alpha}, 0$ for components with respect to the frame $\{\theta^{\alpha}, \theta^{\bar{\alpha}}, \theta\}$ and its dual, so that the 0-components incorporate weights. If f is a (possibly density-valued) tensor field, we denote components of the (tensorial) iterated covariant derivatives of f in such a frame by preceding ∇ 's; e.g., $\nabla_{\alpha} \nabla_0 \cdots \nabla_{\bar{\beta}}$. As usual, such indices may alternately be interpreted abstractly. For example, if $f_{\beta} \in \mathcal{E}_{\beta}(w, w')$, we consider ∇f as the triple $\nabla_{\alpha} f_{\beta} \in \mathcal{E}_{\alpha\beta}(w, w')$, $\nabla_{\bar{\alpha}} f_{\beta} \in \mathcal{E}_{\bar{\alpha}\beta}(w, w')$, $\nabla_0 f_{\beta} \in \mathcal{E}_{\beta}(w - 1, w' - 1)$.

From the standpoint of CR geometry, it is convenient to consider certain modifications of the curvature $R_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ and its traces. The *CR Schouten tensor* is defined by

$$P_{\alpha\bar{\beta}} = \frac{1}{n+2} (R_{\alpha\bar{\beta}} - Ph_{\alpha\bar{\beta}}),$$

where $P = R/2(n+1)$ is its trace. To describe the tractor connection, it is convenient to introduce the tensors

$$\begin{aligned} T_{\alpha} &= \frac{1}{n+2} (\nabla_{\alpha} P - i \nabla^{\gamma} A_{\alpha\gamma}), \\ S &= -\frac{1}{n} (\nabla^{\alpha} T_{\alpha} + \nabla_{\alpha} T^{\alpha} + P_{\alpha\bar{\beta}} P^{\alpha\bar{\beta}} - A_{\alpha\gamma} A^{\alpha\gamma}) \end{aligned}$$

(cf. [28,43]). The *Chern tensor* is defined by

$$S_{\alpha\bar{\beta}\gamma\bar{\sigma}} = R_{\alpha\bar{\beta}\gamma\bar{\sigma}} - P_{\alpha\bar{\beta}} h_{\gamma\bar{\sigma}} - P_{\alpha\bar{\sigma}} h_{\gamma\bar{\beta}} - P_{\gamma\bar{\beta}} h_{\alpha\bar{\sigma}} - P_{\gamma\bar{\sigma}} h_{\alpha\bar{\beta}}.$$

This tensor is the analogue of the Weyl tensor, in that it is CR invariant, has Weyl-type symmetries, and, when $n \geq 2$, is the obstruction to (M, H) being locally equivalent to the standard CR sphere [22]. Some other important curvature tensors, which together constitute the curvature of the CR tractor connection [28], are

$$\begin{aligned} V_{\alpha\bar{\beta}\gamma} &= \nabla_{\bar{\beta}} A_{\alpha\gamma} + i \nabla_{\gamma} P_{\alpha\bar{\beta}} - iT_{\gamma} h_{\alpha\bar{\beta}} - 2iT_{\alpha} h_{\gamma\bar{\beta}} \\ Q_{\alpha\gamma} &= i \nabla_0 A_{\alpha\gamma} - 2i \nabla_{\gamma} T_{\alpha} + 2P_{\alpha}^{\rho} A_{\rho\gamma} \\ U_{\alpha\bar{\beta}} &= \nabla_{\bar{\beta}} T_{\alpha} + \nabla_{\alpha} T_{\bar{\beta}} + P_{\alpha}^{\rho} P_{\rho\bar{\beta}} - A_{\alpha\rho} A_{\bar{\beta}}^{\rho} + Sh_{\alpha\bar{\beta}} \\ Y_{\alpha} &= \nabla_0 T_{\alpha} - i \nabla_{\alpha} S + 2i P_{\alpha}^{\rho} T_{\rho} - 3A_{\alpha\rho} T^{\rho}. \end{aligned}$$

Note that $V_{\alpha\bar{\beta}\gamma}$ and $U_{\alpha\bar{\beta}}$ are trace-free; this follows from the definitions of T_α and S , respectively. Note also that $V_{\alpha\bar{\beta}\gamma}$ and $Q_{\alpha\gamma}$ are symmetric. Indeed, [44, (2.10)] implies that

$$\nabla_\gamma P_{\alpha\bar{\beta}} - \nabla_\alpha P_{\gamma\bar{\beta}} = T_\alpha h_{\gamma\bar{\beta}} - T_\gamma h_{\alpha\bar{\beta}} \tag{2.3}$$

and [44, (2.9)] implies that

$$\nabla_\alpha T_\gamma - \nabla_\gamma T_\alpha = iP_\alpha{}^\rho A_{\rho\gamma} - iP_\gamma{}^\rho A_{\rho\alpha}; \tag{2.4}$$

these equations imply $V_{\alpha\bar{\beta}\gamma} = V_{\gamma\bar{\beta}\alpha}$ and $Q_{\alpha\gamma} = Q_{\gamma\alpha}$, respectively. When $n = 1$, the *Cartan tensor* $Q_{\alpha\beta}$ is CR invariant and is the obstruction to (M, H) being locally CR equivalent to the standard CR sphere [16, 17]. In order to derive local formulae for the CR GJMS operators, P' -operator, and Q' -curvature, it will be useful to know how these curvature tensors are related via divergence formula. Establishing such relationships needs some important Bianchi and commutator identities proven by Lee [44].

Lemma 2.1. *Let (M^{2n+1}, H, θ) be a pseudohermitian manifold. For any $\tau_\alpha \in \mathcal{E}_\alpha$, it holds that*

$$\nabla_\alpha \nabla_\beta \tau_\gamma = \nabla_\beta \nabla_\alpha \tau_\gamma + iA_{\alpha\gamma} \tau_\beta - iA_{\beta\gamma} \tau_\alpha, \tag{2.5}$$

$$\nabla_{\bar{\alpha}} \nabla_{\bar{\beta}} \tau_\gamma = \nabla_{\bar{\beta}} \nabla_{\bar{\alpha}} \tau_\gamma + ih_{\gamma\bar{\beta}} A_{\bar{\alpha}\bar{\sigma}} \tau^{\bar{\sigma}} - ih_{\gamma\bar{\alpha}} A_{\bar{\beta}\bar{\sigma}} \tau^{\bar{\sigma}}, \tag{2.6}$$

$$\nabla_{\bar{\beta}} \nabla_\gamma \tau_\alpha = \nabla_\gamma \nabla_{\bar{\beta}} \tau_\alpha + ih_{\gamma\bar{\beta}} \nabla_0 \tau_\alpha + R_{\gamma\bar{\beta}\alpha\bar{\sigma}} \tau^{\bar{\sigma}}, \tag{2.7}$$

$$\nabla_{\bar{\beta}} \nabla_0 \tau_\alpha = \nabla_0 \nabla_{\bar{\beta}} \tau_\alpha + A_{\bar{\beta}\bar{\sigma}} \nabla^{\bar{\sigma}} \tau_\alpha + \tau_\gamma \nabla_\alpha A^\gamma_{\bar{\beta}}. \tag{2.8}$$

Since commutators act as derivations, similar formulae hold on general tensor fields. Moreover, we have the following Bianchi identities:

$$\nabla^{\bar{\beta}} P_{\alpha\bar{\beta}} = \nabla_\alpha P + (n - 1)T_\alpha, \tag{2.9}$$

$$\begin{aligned} \nabla^{\bar{\sigma}} S_{\alpha\bar{\beta}\gamma\bar{\sigma}} &= (n + 1)\nabla_\alpha P_{\gamma\bar{\beta}} - \nabla_\gamma P_{\alpha\bar{\beta}} - ni\nabla_{\bar{\beta}} A_{\alpha\gamma} \\ &\quad - (2n + 1)T_\gamma h_{\alpha\bar{\beta}} - (n - 1)T_\alpha h_{\gamma\bar{\beta}}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \nabla_\rho \nabla_{\bar{\beta}} A_{\alpha\gamma} &= \nabla_\gamma \nabla_{\bar{\beta}} A_{\alpha\rho} + ih_{\gamma\bar{\beta}} \nabla_0 A_{\alpha\rho} - ih_{\rho\bar{\beta}} \nabla_0 A_{\alpha\gamma} \\ &\quad + R_{\alpha\bar{\beta}\gamma\bar{\sigma}} A_\rho{}^{\bar{\sigma}} - R_{\alpha\bar{\beta}\rho\bar{\sigma}} A_\gamma{}^{\bar{\sigma}}, \end{aligned} \tag{2.11}$$

$$\nabla_0 P_{\alpha\bar{\beta}} = i\nabla_{\bar{\beta}} T_\alpha - i\nabla_\alpha T_{\bar{\beta}}, \tag{2.12}$$

$$\nabla_\gamma A_{\alpha\beta} = \nabla_\beta A_{\alpha\gamma}. \tag{2.13}$$

Proof. Each of (2.5), (2.6), (2.7), and (2.8) is contained in [44, Lemma 2.3]. Equation (2.9) follows immediately from [44, (2.11)]. Equation (2.10) follows immediately from (2.9) and [44, (2.7)]. Equation (2.11) is exactly [44, (2.9)]. Equation (2.12) follows immediately from [44, (2.12)]. Equation (2.13) is exactly [44, (2.6)]. \square

We now derive the divergence formulae relating the tensors $S_{\alpha\bar{\beta}\gamma\bar{\sigma}}$, $V_{\alpha\bar{\beta}\gamma}$, $Q_{\alpha\gamma}$, $U_{\alpha\bar{\beta}}$, and Y_α :

Lemma 2.2. *Let (M^{2n+1}, H, θ) be a pseudohermitian manifold. Then*

$$\nabla^{\bar{\sigma}} S_{\alpha\bar{\beta}\gamma\bar{\sigma}} = -ni V_{\alpha\bar{\beta}\gamma}, \tag{2.14}$$

$$\nabla^{\bar{\beta}} V_{\alpha\bar{\beta}\gamma} = -(n-1)Q_{\alpha\gamma} + S_{\alpha\bar{\beta}\gamma\bar{\sigma}} A^{\bar{\beta}\bar{\sigma}}, \tag{2.15}$$

$$\nabla^\gamma V_{\alpha\bar{\beta}\gamma} = ni U_{\alpha\bar{\beta}} - i S_{\alpha\bar{\beta}\gamma\bar{\sigma}} P^{\gamma\bar{\sigma}}, \tag{2.16}$$

$$\nabla^\gamma Q_{\alpha\gamma} = -nY_\alpha + 2V_{\alpha\bar{\beta}\gamma} P^{\gamma\bar{\beta}}, \tag{2.17}$$

$$\nabla^{\bar{\beta}} U_{\alpha\bar{\beta}} = -(n-1)iY_\alpha + iV_{\alpha\bar{\beta}\gamma} P^{\gamma\bar{\beta}} + V_{\bar{\beta}\alpha\bar{\sigma}} A^{\bar{\beta}\bar{\sigma}}, \tag{2.18}$$

$$\text{Re } \nabla^\gamma Y_\gamma = \text{Im } A^{\alpha\gamma} Q_{\alpha\gamma}. \tag{2.19}$$

Proof. Writing (2.10) in terms of $V_{\alpha\bar{\beta}\gamma}$ and using the symmetry $V_{\alpha\bar{\beta}\gamma} = V_{\gamma\bar{\beta}\alpha}$ yields (2.14).

Using (2.5) and the conjugate of (2.6), we deduce that

$$\nabla^{\bar{\beta}} \nabla_\gamma P_{\alpha\bar{\beta}} = \nabla_\gamma \nabla^{\bar{\beta}} P_{\alpha\bar{\beta}} + (n-1)i P_{\alpha\bar{\beta}} A^{\bar{\beta}}_\gamma + i P_{\gamma\bar{\beta}} A^{\bar{\beta}}_\alpha - i P A_{\alpha\gamma}.$$

It follows from this, (2.9) and the definitions of $V_{\alpha\bar{\beta}\gamma}$ and T_α that

$$\begin{aligned} \nabla^{\bar{\beta}} V_{\alpha\bar{\beta}\gamma} &= \nabla^{\bar{\beta}} \nabla_{\bar{\beta}} A_{\alpha\gamma} - \nabla_\gamma \nabla_{\bar{\beta}} A^{\bar{\beta}}_\alpha + (2n-1)i \nabla_\gamma T_\alpha - i \nabla_\alpha T_\gamma \\ &\quad - (n-1) P_{\alpha\bar{\beta}} A^{\bar{\beta}}_\gamma - P_{\gamma\bar{\beta}} A^{\bar{\beta}}_\alpha + P A_{\alpha\gamma}. \end{aligned} \tag{2.20}$$

Combining (2.4), (2.11), and (2.20) yields (2.15).

From the symmetry $V_{\alpha\bar{\beta}\gamma} = V_{\gamma\bar{\beta}\alpha}$, we may write

$$\nabla^\gamma V_{\alpha\bar{\beta}\gamma} = \nabla^\gamma \nabla_{\bar{\beta}} A_{\alpha\gamma} + i \nabla^\gamma \nabla_\alpha P_{\gamma\bar{\beta}} - i \nabla_{\bar{\beta}} T_\alpha - 2i \nabla^\gamma T_\gamma h_{\alpha\bar{\beta}}.$$

The commutators (2.6) and (2.7) imply that

$$\begin{aligned} \nabla^\gamma \nabla_{\bar{\beta}} A_{\alpha\gamma} &= \nabla_{\bar{\beta}} \nabla^\gamma A_{\alpha\gamma} - ni A_{\alpha\gamma} A^\gamma_{\bar{\beta}} + i |A_{\gamma\rho}|^2 h_{\alpha\bar{\beta}}, \\ \nabla^\gamma \nabla_\alpha P_{\gamma\bar{\beta}} &= \nabla_\alpha \nabla^\gamma P_{\gamma\bar{\beta}} + i \nabla_0 P_{\alpha\bar{\beta}} - S_{\alpha\bar{\beta}\gamma\bar{\sigma}} P^{\gamma\bar{\sigma}} + n P_{\alpha\gamma} P_{\gamma\bar{\beta}} - |P_{\gamma\bar{\sigma}}|^2 h_{\alpha\bar{\beta}}. \end{aligned}$$

Combining these three displays with (2.12) yields (2.16).

Combining the commutator (2.8) and (2.13) yields

$$\nabla^\gamma \nabla_0 A_{\alpha\gamma} = \nabla_0 \nabla^\gamma A_{\alpha\gamma} + \nabla_\alpha |A_{\gamma\rho}|^2 + A_{\alpha\gamma} \nabla_\rho A^{\gamma\rho}.$$

Writing this in terms of T_α and using the commutator (2.7) on functions yields

$$\begin{aligned} \nabla^\gamma \nabla_0 A_{\alpha\gamma} &= (n+2)i\nabla_0 T_\alpha + \nabla_\alpha \nabla^\gamma T_\gamma - \nabla_\alpha \nabla_\gamma T^\gamma \\ &\quad + \nabla_\alpha |A_{\gamma\rho}|^2 - (n+2)iA_{\alpha\gamma} T^\gamma + 2iA_{\alpha\gamma} \nabla^\gamma P. \end{aligned}$$

In particular, when combined with the definition of $Q_{\alpha\gamma}$, this yields

$$\begin{aligned} \nabla^\gamma Q_{\alpha\gamma} &= -(n+2)\nabla_0 T_\alpha + i\nabla_\alpha \nabla^\gamma T_\gamma - i\nabla_\alpha \nabla_\gamma T^\gamma - 2i\nabla^\gamma \nabla_\gamma T_\alpha \\ &\quad + 2\nabla^\rho (P_\alpha{}^\gamma A_{\gamma\rho}) + i\nabla_\alpha |A_{\gamma\rho}|^2 + (n+2)A_{\alpha\gamma} T^\gamma - 2A_{\alpha\gamma} \nabla^\gamma P. \end{aligned} \quad (2.21)$$

Next, (2.4) and the commutator (2.7) yields

$$\begin{aligned} \nabla^\gamma \nabla_\gamma T_\alpha - \nabla_\alpha \nabla^\gamma T_\gamma &= i\nabla_0 T_\alpha + (n+2)P_\alpha{}^\gamma T_\gamma + PT_\alpha - i\nabla^\rho (P_\alpha{}^\gamma A_{\gamma\rho}) \\ &\quad + i\nabla^\rho (P_\rho{}^\gamma A_{\gamma\alpha}). \end{aligned}$$

Combining this with (2.9) and (2.21) yields (2.17).

Using (2.4) and the commutators (2.6) and (2.7), we deduce that

$$\begin{aligned} \nabla_\beta \nabla_\alpha T^\beta - \nabla_\alpha \nabla_\beta T^\beta &= (n-1)iA_{\alpha\beta} T^\beta, \\ \nabla_\beta \nabla^\beta T_\alpha - \nabla_\alpha \nabla^\beta T_\beta &= -(n-1)i\nabla_0 T_\alpha + iP_\beta{}^\gamma \nabla^\beta A_{\gamma\alpha} - iA_{\gamma\beta} \nabla^\beta P_\alpha{}^\gamma \\ &\quad - P_\alpha{}^\gamma \nabla_\gamma P + (n+2)P_\alpha{}^\gamma T_\gamma + iA_{\alpha\gamma} \nabla^\gamma P \\ &\quad + (n-1)iA_{\alpha\gamma} T^\gamma, \end{aligned}$$

one readily derives (2.18).

From the definitions of T_α , S , and Y_α we compute that

$$\operatorname{Re} \nabla^\gamma Y_\gamma = -P_\alpha{}^\gamma \nabla_0 P_\gamma{}^\alpha + \frac{1}{2} \nabla_0 |A_{\alpha\gamma}|^2 + iP_\alpha{}^\gamma (\nabla^\alpha T_\gamma - \nabla_\gamma T^\alpha) - \operatorname{Re} A_{\alpha\gamma} \nabla^\gamma T^\alpha.$$

Combining this with (2.12) and the definition of $Q_{\alpha\gamma}$ yields (2.19). \square

3. Some tractor calculus

3.1. The CR tractor connection

On a hypersurface type CR structure there is no invariant connection on the tangent bundle, or its contact subbundle. However there is a natural invariant connection on a higher rank natural vector bundle known as the CR cotractor bundle [28]. A

defining feature of this bundle \mathcal{E}_A is that for each choice of pseudohermitian contact form θ this bundle decomposes into a direct sum

$$\mathcal{E}_A \stackrel{\theta}{=} \mathcal{E}(1, 0) \oplus \mathcal{E}_\alpha(1, 0) \oplus \mathcal{E}(0, -1). \tag{3.1}$$

So for a section $v_A \in \Gamma(\mathcal{E}_A)$ we may write $[v_A]_\theta = (\sigma, \tau_\alpha, \rho)$, where $\sigma \in \mathcal{E}(1, 0)$, $\tau_\alpha \in \mathcal{E}_\alpha(1, 0)$, and $\rho \in \mathcal{E}(0, -1)$. When the choice of θ is understood it will be omitted from the notation. A change of contact form to $\hat{\theta} = e^\Upsilon \theta$, where $\Upsilon \in C^\infty(M)$, induces a different identification to the same direct sum bundle, with the components in the $\hat{\theta}$ direct sum related to those in the θ direct sum by the transformation formula

$$[v_A]_{\hat{\theta}} = \begin{pmatrix} \hat{\sigma} \\ \hat{\tau}_\alpha \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \tau_\alpha + \Upsilon_\alpha \sigma \\ \rho - \Upsilon^\beta \tau_\beta - \frac{1}{2}(\Upsilon^\beta \Upsilon_\beta + i\Upsilon_0)\sigma \end{pmatrix}, \tag{3.2}$$

where we have used an obvious notation; *e.g.*, $\Upsilon_\alpha := \nabla_\alpha \Upsilon$. It follows from (3.2) that \mathcal{E}_A has a composition series

$$\mathcal{E}_A = \mathcal{E}(1, 0) \uplus \mathcal{E}_\alpha(1, 0) \uplus \mathcal{E}(0, -1),$$

meaning that in a CR invariant way, $\mathcal{E}(0, -1)$ is a subbundle of \mathcal{E}_A , $\mathcal{E}(1, 0)$ is a quotient bundle of \mathcal{E}_A , and the kernel of the surjection $\mathcal{E}_A \rightarrow \mathcal{E}(1, 0)$ is the reducible subbundle $\mathcal{E}_\alpha(1, 0) \uplus \mathcal{E}(0, -1)$. We write Z_A to denote the canonical bundle injection

$$Z_A : \mathcal{E}(0, -1) \rightarrow \mathcal{E}_A,$$

and also view this as a section $Z_A \in \Gamma(\mathcal{E}_A(0, 1))$. Note that conjugation extends to tractors in the obvious way and, for example, the conjugate tractor bundle has the composition series

$$\mathcal{E}_{\bar{A}} = \mathcal{E}(0, 1) \uplus \mathcal{E}_{\bar{\alpha}}(0, 1) \uplus \mathcal{E}(-1, 0),$$

with the inclusion of $\mathcal{E}(-1, 0)$ denoted

$$Z_{\bar{A}} : \mathcal{E}(-1, 0) \rightarrow \mathcal{E}_{\bar{A}}. \tag{3.3}$$

For calculating one usually works in a pseudohermitian scale exploiting the splitting (3.1) and for $[v_A]_\theta = (\sigma, \tau_\alpha, \rho)$ we may equivalently write

$$v_A = \sigma Y_A + \tau_\alpha W_A^\alpha + \rho Z_A \tag{3.4}$$

where $W_A^\alpha : \mathcal{E}_\alpha(1, 0) \rightarrow \mathcal{E}_A$ and $Y_A : \mathcal{E}(1, 0) \rightarrow \mathcal{E}_A$ are the bundle injections determined by the splitting.

For two sections v_A and v'_B the quantity $\sigma \bar{\rho}' + \rho \bar{\sigma}' + \mathbf{h}^{\alpha\bar{\beta}} \tau_\alpha \bar{\tau}'_\beta$ is independent of the choice of θ ; this is the formula, in a given scale θ , for the CR invariant Hermitian metric $h^{A\bar{B}}$ on \mathcal{E}_A . The *tractor metric* $h_{A\bar{B}}$ is the inverse of this, and it gives the Hermitian metric on the (*standard*) *tractor bundle* \mathcal{E}^A , which by definition is the bundle dual to \mathcal{E}_A . The tractor metric is used to raise and lower tractor indices in the usual way; e.g., $Z^A := h^{A\bar{B}} Z_{\bar{B}}$. Note that from the formula for the metric this gives the surjection

$$Z^A : \mathcal{E}_A \rightarrow \mathcal{E}(1, 0).$$

We refer to Z^A as the (*CR*) *canonical tractor*.

In terms of the tractor splitting (3.1), the CR tractor connection is given by the formulae

$$\begin{aligned} \nabla_\beta v_A &= \begin{pmatrix} \nabla_\beta \sigma - \tau_\beta \\ \nabla_\beta \tau_\alpha + i A_{\alpha\beta} \sigma \\ \nabla_\beta \rho - P_\beta^\alpha \tau_\alpha + T_\beta \sigma \end{pmatrix}, \\ \nabla_{\bar{\beta}} v_A &= \begin{pmatrix} \nabla_{\bar{\beta}} \sigma \\ \nabla_{\bar{\beta}} \tau_\alpha + \mathbf{h}_{\alpha\bar{\beta}} \rho + P_{\alpha\bar{\beta}} \sigma \\ \nabla_{\bar{\beta}} \rho + i A_{\bar{\beta}}^\alpha \tau_\alpha - T_{\bar{\beta}} \sigma \end{pmatrix}, \\ \nabla_0 v_A &= \begin{pmatrix} \nabla_0 \sigma + \frac{i}{n+2} P \sigma - i \rho \\ \nabla_0 \tau_\alpha - i P_\alpha^\beta \tau_\beta + \frac{i}{n+2} P \tau_\alpha + 2i T_\alpha \sigma \\ \nabla_0 \rho + \frac{i}{n+2} P \rho + 2i T^\alpha \tau_\alpha + i S \sigma \end{pmatrix}, \end{aligned} \tag{3.5}$$

where the ∇ 's on the right-hand side refer to the pseudohermitian connection on the appropriate weighted bundles. This connection is canonically determined by the CR structure (and so, in particular, is independent of θ); indeed, it is equivalent to the normal Cartan connection on CR manifold [14]. The tractor connection preserves the tractor metric, $\nabla h_{A\bar{B}} = 0$, and so covariant differentiation commutes with the raising and lowering of tractor indices.

3.2. The tractor *D*-operator

For the construction of differential operators an important tool is the second order *tractor D-operator*

$$\mathbb{D}_A : \mathcal{E}^*(w, w') \rightarrow \mathcal{E}_A \otimes \mathcal{E}^*(w - 1, w'),$$

where $\mathcal{E}^*(w, w')$ indicates any weighted tractor bundle, meaning it is the tensor product of $\mathcal{E}(w, w')$ with any bundle constructed by taking a tensor part of any tensor product of the tractor bundle, its dual, and the conjugates of these. This is

defined by

$$\mathbb{D}_A f = \begin{pmatrix} w(n+w+w')f \\ (n+w+w')\nabla_\alpha f \\ -\left(\nabla^\beta \nabla_\beta f + w i \nabla_0 f + w\left(1 + \frac{w'-w}{n+2}\right) P f\right) \end{pmatrix}$$

in the splitting (3.1) determined by a choice of θ , but is independent of the choice of θ . Here $\nabla_\alpha f$ refers to the tractor connection defined above coupled with the pseudohermitian connection. Conjugation produces the CR invariant operator

$$\mathbb{D}_{\bar{A}} : \mathcal{E}^*(w, w') \rightarrow \mathcal{E}_{\bar{A}} \otimes \mathcal{E}^*(w, w' - 1).$$

By construction both \mathbb{D}_A and $\mathbb{D}_{\bar{A}}$ commute with raising and lowering tractor indices.

Two other important operators on weighted tractor bundles are the weight operator \mathbf{w} and its conjugate \mathbf{w}' . The weight operators are defined to be the unique derivations such that

$$\mathbf{w}f = wf \quad \text{and} \quad \mathbf{w}'f = w'f$$

for all $f \in \mathcal{E}^*(w, w')$. One can express the tractor D -operator on arbitrary weighted tractor bundles using the weight operator and tractor connection. A similar formula for the composition $\mathbb{D}_A \mathbb{D}_{\bar{B}}$ is of particular importance in this article and is given in Proposition 3.1 below.

Related to the tractor- D operator are the CR double- D operators. These are defined generally in [14], but as differential operators on weighted tractor fields they may be defined by

$$\mathbb{D}_{AB} := 2Z_{[B} Y_{A]} \mathbf{w} + 2Z_{[B} W_{A]}^\alpha \nabla_\alpha, \tag{3.6}$$

where $[\dots]$ indicates the skew part over the enclosed indices, and

$$\begin{aligned} \mathbb{D}_{A\bar{B}} := & Z_{\bar{B}} Y_A \mathbf{w} - Z_A Y_{\bar{B}} \mathbf{w}' + Z_{\bar{B}} W_A^\alpha \nabla_\alpha - Z_A W_{\bar{B}}^{\bar{\beta}} \nabla_{\bar{\beta}} \\ & - Z_A Z_{\bar{B}} \left(i \nabla_0 + \frac{P}{n+2} (\mathbf{w}' - \mathbf{w}) \right), \end{aligned} \tag{3.7}$$

and their conjugates, cf. [14, Theorem 4.10]. For both we have used (3.4) to simplify the presentation.

As mentioned above, the following formula for the composition $\mathbb{D}_A \mathbb{D}_{\bar{B}}$ is of particular importance in this article.

Proposition 3.1. *Let (M^{2n+1}, H) be a CR manifold. Given any scale $\sigma \bar{\sigma} \in \mathcal{E}(1)$, the operator $\mathbb{D}_A \mathbb{D}_{\bar{B}}^B = h^{B\bar{B}} \mathbb{D}_A \mathbb{D}_{\bar{B}}$ acts on elements of $\mathcal{E}(w)$, $w \in \mathbb{R}$, by*

$$\mathbb{D}_A \mathbb{D}_{\bar{B}}^B = \begin{pmatrix} C_3 & (C_2)^\beta & C_1 \\ (C_5)_\alpha & (C_4)_\alpha^\beta & (C_2)_\alpha \\ C_6 & (C_5)^\beta & \bar{C}_3 \end{pmatrix},$$

where

$$\begin{aligned}
 C_1 &= \mathbf{w}^2(n + 2\mathbf{w})(n + 2\mathbf{w} - 1), \\
 (C_2)^\beta &= \nabla^\beta \mathbf{w}(n + 2\mathbf{w})(n + 2\mathbf{w} - 1), \\
 C_3 &= -\frac{1}{2}(\Delta_b - i\nabla_0(n + 2\mathbf{w}) + 2P\mathbf{w})\mathbf{w}(n + 2\mathbf{w} - 1), \\
 (C_4)_{\alpha\beta} &= \left(\nabla_\alpha \nabla^\beta - \frac{1}{n} h_{\alpha\beta} \nabla_\gamma \nabla^\gamma \right) (n + 2\mathbf{w})(n + 2\mathbf{w} - 1) \\
 &\quad + P_{\alpha\beta} \mathbf{w}(n + 2\mathbf{w})(n + 2\mathbf{w} - 1) + \frac{1}{n} h_{\alpha\beta} (\Delta_b - nP) \mathbf{w}(n + 2\mathbf{w} - 1), \\
 (C_5)_\alpha &= -\frac{1}{n} (\nabla_\alpha \nabla_\gamma \nabla^\gamma + ni A_{\alpha\gamma} \nabla^\gamma) (n + 2\mathbf{w})(n + 2\mathbf{w} - 1) \\
 &\quad + \left(\frac{1}{n} \nabla_\alpha \Delta_b - \nabla_\alpha P - T_\alpha(n + 2\mathbf{w}) \right) \mathbf{w}(n + 2\mathbf{w} - 1), \\
 C_6 &= \frac{1}{n^2} \nabla^\gamma (\nabla_\gamma \nabla_\alpha \nabla^\alpha + ni A_{\alpha\gamma} \nabla^\alpha) (n + 2\mathbf{w})^2 + P^{(\alpha\bar{\beta})_0} \nabla_\alpha \nabla_{\bar{\beta}} (n + 2\mathbf{w}) \\
 &\quad - \frac{1}{n^2} \Delta_b^2 \mathbf{w}(n + \mathbf{w}) + \frac{2}{n} \text{Im} \nabla^\gamma A_{\beta\gamma} \nabla^\beta \mathbf{w}(n + 2\mathbf{w}) - \frac{1}{n} P \Delta_b \mathbf{w}(n - 1 + 2\mathbf{w}) \\
 &\quad + \frac{4}{n} \text{Re} \nabla^\gamma P \nabla_\gamma \mathbf{w}(n + \mathbf{w}) - \frac{2}{n} \text{Re}(\nabla^\gamma P + nT^\gamma) \nabla_\gamma \mathbf{w}(n + 2\mathbf{w}) \\
 &\quad + \frac{1}{n} (\nabla^\gamma (\nabla_\gamma P + nT_\gamma)) \mathbf{w}(n + 2\mathbf{w}) + |P_{(\alpha\bar{\beta})_0}|^2 \mathbf{w}(n + 2\mathbf{w}) \\
 &\quad + \frac{1}{n} \left((n + 2)P^2 - (\Delta_b P) - nS(n + 2\mathbf{w}) \right) \mathbf{w}^2,
 \end{aligned}$$

$P_{(\alpha\bar{\beta})_0} = P_{\alpha\bar{\beta}} - \frac{P}{n} h_{\alpha\bar{\beta}}$ is the trace-free part of the CR Schouten tensor, the term T_α in the definition of $(C_5)_\alpha$ acts as a multiplication operator, the terms $(\nabla^\gamma (\nabla_\gamma P + nT_\gamma))$ and $(\Delta_b P)$ in the definition of C_6 act as multiplication operators, and all other operators in the definitions of C_1, \dots, C_6 act to the right.

Remark 3.2. We raise an index and write $\mathbb{D}_A \mathbb{D}^B$ to make clear the meaning of the rows and columns in our matrix representation for the composition $\mathbb{D}_A \mathbb{D}_{\bar{B}}$.

Proof. Using the definition of the tractor D -operator and [28, Proposition 2.2], we observe that if $f \in \mathcal{E}(w)$, then

$$\mathbb{D}^B f = \left(-\frac{1}{2} \Delta_b f + \frac{n + 2w}{2} i \nabla_0 f - w P f, (n + 2w) \nabla^\beta f, w(n + 2w) f \right). \quad (3.8)$$

Using this and the formula

$$\mathbb{D}_A \mathbb{D}^B f = \begin{pmatrix} w(n+2w-1)D^B f \\ (n+2w-1)\nabla_\alpha D^B f \\ -\left(\nabla^\gamma \nabla_\gamma D^B f + wi\nabla_0 D^B f + \frac{n+1}{n+2}wPD^B f\right) \end{pmatrix}$$

yields the expressions for the operators C_1 , $(C_2)^\beta$, and C_3 .

Next observe that

$$\frac{1}{2}(\Delta_b f - (n+2w)i\nabla_0 f + 2wPf) = \frac{1}{n}((n+2w)\nabla_\beta \nabla^\beta f - w\Delta_b f + wnPf).$$

Combining this with (3.8) and the expression for the tractor connection yields the expressions for the operators $(C_4)_\alpha^\beta$ and $(C_5)_\beta$.

Finally, note that

$$\nabla^\gamma \nabla_\gamma I_{\bar{B}} = \begin{pmatrix} \nabla^\gamma \nabla_\gamma \sigma - \nabla_\gamma \tau^\gamma - P\sigma - n\rho \\ \nabla^\gamma \nabla_\gamma \tau_{\bar{\beta}} + P_{\gamma\bar{\beta}} \nabla^\gamma \sigma + \sigma \left(\nabla_{\bar{\beta}} P + (n-1)T_{\bar{\beta}} \right) + \nabla_{\bar{\beta}} \rho - iA_{\bar{\beta}}{}^\gamma \nabla_\gamma \sigma \\ B \end{pmatrix},$$

where

$$B = \nabla^\gamma \nabla_\gamma \rho - i\nabla^\gamma (A_{\alpha\gamma} \tau^\alpha) - \nabla^\gamma (\sigma T_\gamma) - P_\alpha{}^\gamma (\nabla_\gamma \tau^\alpha + \sigma P_\gamma{}^\alpha + \rho h_\gamma{}^\alpha) + T^\gamma \nabla_\gamma \sigma.$$

Combining this with (3.8) and the definition of the tractor D -operator yields the expression for the operator C_6 . □

4. CR pluriharmonic functions and (pseudo-)Einstein contact forms

4.1. Log densities

In order to study CR pluriharmonic functions via tractors, it is useful to introduce log densities (*cf.* [32]). Let $\tilde{\mathcal{C}} \subset T^*M$ be the \mathbb{R}_+ -bundle of positive elements of H^\perp . Given $w \in \mathbb{R}$, let $\mathcal{E}_\mathbb{R}(w)$ be the bundle associated to $\tilde{\mathcal{C}}$ via the representation $s \mapsto s^{2w}$ of \mathbb{R}_+ . In particular, $\mathcal{E}_\mathbb{R}(w)$ can be identified with a real subbundle of $\mathcal{E}(w, w)$ and $\mathcal{E}_\mathbb{R}(1)$ can be identified with $\tilde{\mathcal{C}}$. Hence $\mathcal{E}_\mathbb{R}(w)$ is trivial as a vector bundle. We likewise let $\mathcal{D}_\mathbb{R}(w)$ be the real line bundle induced by the log representations $s \mapsto w \log s$ of \mathbb{R}_+ . In particular, a section $\lambda \in \mathcal{D}_\mathbb{R}(w)$ is equivalent to a function $\underline{\lambda}: \tilde{\mathcal{C}} \rightarrow \mathbb{R}$ with the equivariance property

$$\underline{\lambda}(t^2\theta) = \underline{\lambda}(\theta) + 2w \log t. \tag{4.1}$$

Note that if τ is a positive section of $\mathcal{E}_\mathbb{R}(w)$ and $\underline{\tau}$ is the corresponding equivariant section of $\tilde{\mathcal{C}}$, then the composition $\log \circ \underline{\tau}$ has the property (4.1), and hence is

equivalent to a section of $\mathcal{D}_{\mathbb{R}}(w)$. We shall denote this section by $\log \tau$. It is clear that a section of $\mathcal{D}_{\mathbb{R}}(1)$ is $\log \tau$ for some positive section $\tau \in \mathcal{E}_{\mathbb{R}}(1)$. We define the operator $\nabla: \mathcal{D}_{\mathbb{R}}(1) \rightarrow T^*M$ by setting

$$\nabla \log \tau = \tau^{-1} \nabla \tau. \tag{4.2}$$

We can extend these definitions by complex linearity and thereby consider $\mathcal{D}(w)$ as a complex bundle; in particular, we obtain the operator $\nabla: \mathcal{D}(w) \rightarrow T_{\mathbb{C}}^*M$.

The requirement that the weight operators \mathbf{w} and \mathbf{w}' satisfy the Leibniz property means that the natural definition of the weight operators on log densities is such that for any $\lambda \in \mathcal{D}(w_0)$, it holds that

$$\mathbf{w}\lambda = w_0 = \mathbf{w}'\lambda. \tag{4.3}$$

Viewing the section λ as a multiplication operator on the space of sections of any density bundle, log density bundle, tractor bundle, or tensor product of these things, we have equivalently that $[\mathbf{w}, \lambda] = w_0 = [\mathbf{w}', \lambda]$.

4.2. CR pluriharmonic functions and tractors

One use of log densities is to provide a rigorous method for carrying out Branson’s method of analytic continuation in the dimension [5,8,32]. For example, the derivation of the formula for the P' -operator in dimension three [19] proceeds by observing that the CR Paneitz operator $P_4: \mathcal{E}(w) \rightarrow \mathcal{E}(w - 2)$ can be written in general dimension as $P_4 = C - wR$ for $w = -\frac{n-1}{2}$ and $C = 4\nabla^\gamma(\nabla_\gamma \nabla_\beta + ni A_{\beta\gamma})\nabla^\beta$; note that C annihilates CR pluriharmonic functions [36]. Thus $-\frac{1}{w}P_4 f = Rf$ makes sense for any $f \in \mathcal{P}$. This expression in the case $n = 1$, corresponding to $w = 0$, yields the operator $P'_4 = R$. As we explain below, by working with log densities and the tractor formula for the CR Paneitz operator, this “division by zero” can be realized through the commutator property (4.3).

To begin, we point out, as an immediate corollary of Proposition 3.1, the tensor formula for the operator $\mathbb{D}_A \mathbb{D}_{\bar{B}}$ on $\mathcal{E}(0)$. From this formula we see that CR pluriharmonic functions are annihilated by $\mathbb{D}_A \mathbb{D}_{\bar{B}}$. A key point is that the tractor formula for the critical CR GJMS operators always factors through this operator; see Theorem 6.7. In particular, $\mathbb{D}_A \mathbb{D}_{\bar{B}}$ acting on $\mathcal{E}(0)$ is the tractor formula for the CR Paneitz operator in dimension three.

Lemma 4.1. *Let (M^{2n+1}, H) be a CR manifold and let $f \in \mathcal{E}(0)$. Given any scale $\sigma \bar{\sigma} \in \mathcal{E}(1)$, it holds that*

$$\mathbb{D}_A \mathbb{D}_{\bar{B}} f = \begin{pmatrix} 0 & 0 & 0 \\ -(n-1)P_\alpha(f) & n(n-1)B_\alpha^\beta(f) & 0 \\ \nabla^\gamma P_\gamma(f) + nP^{\gamma\bar{\sigma}} B_{\gamma\bar{\sigma}}(f) & -(n-1)P^\beta(f) & 0 \end{pmatrix}, \tag{4.4}$$

where

$$\begin{aligned}
 B_{\alpha\bar{\beta}}(f) &= \nabla_\alpha \nabla_{\bar{\beta}} f - \frac{1}{n} \nabla_\gamma \nabla^\gamma f h_{\alpha\bar{\beta}} \\
 P_\alpha(f) &= \nabla_\alpha \nabla_\beta \nabla^\beta f + ni A_{\alpha\bar{\beta}} \nabla^\beta f.
 \end{aligned}$$

In particular, if $f \in \mathcal{P}$, then $\mathbb{D}_A \mathbb{D}_{\bar{B}} f = 0$.

Remark 4.2. When $n > 1$, we readily see that $f \in \mathcal{P}$ if and only if $f \in \ker \mathbb{D}_A \mathbb{D}_{\bar{B}}$.

The fact that CR pluriharmonic functions lie in the kernel of $\mathbb{D}_A \mathbb{D}_{\bar{B}}$ when restricted to $\mathcal{E}(0)$ means that the scale-dependent operator $K_{A\bar{B}}: \mathcal{P} \rightarrow \mathcal{E}_{A\bar{B}}(-1)$ defined by

$$K_{A\bar{B}}(f) = -\mathbb{D}_A \mathbb{D}_{\bar{B}}(f \log \sigma \bar{\sigma}) \tag{4.5}$$

for a given choice of scale $\sigma \bar{\sigma} \in \mathcal{E}(1)$ is well-defined, as follows. We may regard the displayed formula in Proposition 3.1 as giving a formula for $\mathbb{D}_A \mathbb{D}_{\bar{B}}$ acting on log densities (as well as on densities). Using then the fact $\mathcal{P} \subset \ker \mathbb{D}_A \mathbb{D}_{\bar{B}}$, we obtain

$$\mathbb{D}_A \mathbb{D}_{\bar{B}}(f \log \sigma \bar{\sigma}) = [\mathbb{D}_A \mathbb{D}_{\bar{B}}, \log \sigma \bar{\sigma}] f. \tag{4.6}$$

Since $[\mathbf{w}, \log \sigma \bar{\sigma}] = 1$, we see that $[\mathbb{D}_A \mathbb{D}_{\bar{B}}, \log \sigma \bar{\sigma}]$ takes values in linear differential operators mapping $\mathcal{E}(0)$ to $\mathcal{E}_{A\bar{B}}(-1)$. By Lemma 5.4 below, this commutator agrees with the same commutator defined ambiently.

A tractor expression for $K_{A\bar{B}}(f)$ is readily derived using Proposition 3.1; we give here the formula in the scale $\theta = (\sigma \bar{\sigma})^{-1} \theta$.

Lemma 4.3. *Let (M^{2n+1}, H) be a CR manifold and let $\sigma \bar{\sigma} \in \mathcal{E}(1)$ be a scale. Given any $f \in \mathcal{P}$, the function $K_{A\bar{B}}(f)$ is given by*

$$K_A{}^B(f) = \begin{pmatrix} (n-1)\nabla_\gamma \nabla^\gamma f & -n(n-1)\nabla^\beta f & 0 \\ (n-1)C_\alpha(f) & (n-1)C_{\alpha\bar{\beta}}(f) & -n(n-1)\nabla_\alpha f \\ D(f) & (n-1)C^\beta(f) & (n-1)\nabla^\gamma \nabla_\gamma f \end{pmatrix}, \tag{4.7}$$

where

$$\begin{aligned}
 C_{\alpha\bar{\beta}}(f) &= -\frac{1}{n} h_{\alpha\bar{\beta}} \Delta_b f - n f P_{(\alpha\bar{\beta})_0}, \\
 C_\alpha(f) &= -\frac{1}{n} \nabla_\alpha \Delta_b f + P \nabla_\alpha f + f (\nabla_\alpha P + n T_\alpha), \\
 D(f) &= \frac{1}{n} \Delta_b^2 f - 2 \operatorname{Im} \nabla^\beta (A_{\alpha\bar{\beta}} \nabla^\alpha f) - 4 \operatorname{Re} \nabla^\alpha (P \nabla_\alpha f) + \frac{n-1}{n} P \Delta_b f \\
 &\quad + 2 \operatorname{Re} ((\nabla_\alpha P + n T_\alpha) \nabla^\alpha f) - f (n |P_{(\alpha\bar{\beta})_0}|^2 + \nabla^\gamma (\nabla_\gamma P + n T_\gamma)).
 \end{aligned}$$

In particular, if $\sigma \bar{\sigma}$ is a pseudo-Einstein scale, then $1 \in \ker K_{A\bar{B}}$.

Remark 4.4. When $n > 1$, we readily see that $1 \in \ker K_{A\bar{B}}$ if and only if $\sigma\bar{\sigma}$ is a pseudo-Einstein scale.

Proof. By definition, we have that $[\nabla_\alpha, \log \sigma\bar{\sigma}] = 0$ in the scale $\sigma\bar{\sigma}$. It then follows from (4.3) and (4.6) that $K_{A\bar{B}}(f)$ arises as the negative of the coefficient of \mathbf{w} in Proposition 3.1. This yields (4.7). Finally, $\sigma\bar{\sigma}$ is a pseudo-Einstein scale if and only if $P_{(\alpha\bar{\beta})_0} = 0$ and $\nabla_\alpha P + nT_\alpha = 0$, from which the last claim readily follows. \square

In the case $n = 1$, Lemma 4.3 yields $K_{A\bar{B}}(f) = Z_A Z_{\bar{B}} P'_4 f$ for all $f \in \mathcal{P}$, where P'_4 is the operator defined by Yang and the first-named author [19]. This also yields the transformation formula for the P' -operator. The corresponding result in general dimensions is described in Section 6.

The fact that constants lie in the kernel of $K_{A\bar{B}}$ when $\sigma\bar{\sigma}$ determines a pseudo-Einstein scale means that for such a scale, the tractor $I_{A\bar{B}} \in \mathcal{E}_{A\bar{B}}(-1)$ given by

$$I_{A\bar{B}} = \frac{1}{2} \mathbb{D}_A \mathbb{D}_{\bar{B}} \left((\log \sigma\bar{\sigma})^2 \right) \tag{4.8}$$

is well-defined. Indeed, since $1 \in \ker \mathbb{D}_A \mathbb{D}_{\bar{B}} \cap \ker K_{A\bar{B}}$, we compute that

$$\mathbb{D}_A \mathbb{D}_{\bar{B}} \left((\log \sigma\bar{\sigma})^2 \right) = [[\mathbb{D}_A \mathbb{D}_{\bar{B}}, \log \sigma\bar{\sigma}], \log \sigma\bar{\sigma}]. \tag{4.9}$$

It follows at once from Lemma 5.4 below that $I_{A\bar{B}}$ agrees with the same expression defined ambiently. On a fixed CR manifold this tractor field is determined entirely by θ . A tractor expression for $I_{A\bar{B}}$ is readily derived; we give here the formula in the scale θ .

Lemma 4.5. *Let (M^{2n+1}, H) be a CR manifold and let $\sigma\bar{\sigma} \in \mathcal{E}(1)$ be a pseudo-Einstein scale. Then $I_{A\bar{B}}$ is given by*

$$I_A{}^B = \begin{pmatrix} -(n-1)P & 0 & n(n-1) \\ \frac{2(n-1)}{n} \nabla_\alpha P & \frac{2(n-1)}{n} P h_{\alpha\beta} & 0 \\ -\frac{2}{n} \Delta_b P - |A_{\alpha\beta}|^2 + \frac{n+3}{n} P^2 & \frac{2(n-1)}{n} \nabla_{\bar{\beta}} P & -(n-1)P \end{pmatrix}. \tag{4.10}$$

Proof. Note that (4.3) implies $[\mathbf{w}^2, \log^2 \sigma\bar{\sigma}] = 2$. Since $[\nabla_\alpha, \log \sigma\bar{\sigma}] = 0$ and $\sigma\bar{\sigma}$ is a pseudo-Einstein scale, we thus need only consider the coefficient of \mathbf{w}^2 in Proposition 3.1. \square

In the case $n = 1$, Lemma 4.5 yields $I_{A\bar{B}} = Z_A Z_{\bar{B}} Q'$ and also the transformation formula for the Q' -curvature. The corresponding result in general dimensions is described in Section 6.

Finally, let us comment on our normalisations. Suppose that $\sigma\bar{\sigma}, s\bar{s} \in \mathcal{E}(1)$ are two scales and $s\bar{s} = e^{-\Upsilon} \sigma\bar{\sigma}$; thus the contact forms $\theta = (\sigma\bar{\sigma})^{-1} \boldsymbol{\theta}$ and $\widehat{\theta} = (s\bar{s})^{-1} \boldsymbol{\theta}$ are related by $\widehat{\theta} = e^{\Upsilon} \theta$. Let $K_{A\bar{B}}, \widehat{K}_{A\bar{B}}: \mathcal{P} \rightarrow \mathcal{E}_{A\bar{B}}(-1)$ be the operators defined in terms of $\sigma\bar{\sigma}$ and $s\bar{s}$, respectively, by (4.5). It follows immediately from (4.5) that

$$\widehat{K}_{A\bar{B}}(f) = K_{A\bar{B}}(f) + \mathbb{D}_A \mathbb{D}_{\bar{B}}(\Upsilon f). \tag{4.11}$$

In particular, this normalisation recovers the familiar transformation formula (1.1) for the P' -operator in dimension three.

Suppose additionally that θ and $\widehat{\theta}$ are both pseudo-Einstein. Then $\Upsilon \in \mathcal{P}$ and the tractors $I_{A\bar{B}}$ and $I_{A\bar{B}}$ defined in terms of $\sigma\bar{\sigma}$ and $s\bar{s}$, respectively, by (4.8) are well-defined. Moreover, (4.8) gives

$$\widehat{I}_{A\bar{B}} = I_{A\bar{B}} + K_{A\bar{B}}(\Upsilon) + \frac{1}{2}\mathbb{D}_A\mathbb{D}_{\bar{B}}(\Upsilon^2). \tag{4.12}$$

In particular, this normalisation recovers the familiar transformation formula (1.2) for the Q' -curvature in dimension three.

4.3. (Pseudo-)Einstein manifolds and (partially) parallel tractors

From the tractor perspective, a natural reason to study pseudo-Einstein and Einstein structures on CR manifolds is that they correspond to the existence of holomorphic and parallel standard tractors, respectively (for the latter cf. [14, 45]). Parallel tractors are especially useful; we use them in Section 7 to derive simple local formulae for P , P' and Q' from tractor formulae in Einstein scales.

Proposition 4.6. *Let (M, H) be a CR manifold. Suppose that θ is a pseudo-Einstein contact form. Then locally there exists a $\sigma \in \mathcal{E}(1, 0)$, unique up to multiplication by a constant $\lambda \in \mathbb{C}$ with $|\lambda|^2 = 1$, such that $\theta = (\sigma\bar{\sigma})^{-1}\boldsymbol{\theta}$ and $\mathbb{D}_A\sigma$ is holomorphic; i.e. $\nabla_{\bar{\beta}}\mathbb{D}_A\sigma = 0$. Conversely, if $I_A \in \mathcal{E}_A$ is holomorphic, then $\theta = (\sigma\bar{\sigma})^{-1}\boldsymbol{\theta}$ is pseudo-Einstein wherever $\sigma = Z^A I_A$ is nonzero.*

Proof. Suppose that θ is a pseudo-Einstein contact form. By [38, Lemma 7.2] and [44, Theorem 4.2], locally there exists a closed form $\zeta \in \mathcal{K}$ with respect to which θ is volume-normalized. Let $\sigma \in \mathcal{E}(1, 0)$ be a $-(n + 2)$ -nd root of ζ . Then $\bar{\partial}_b\sigma = 0$. By [28, Proposition 2.4], it holds that

$$-\frac{1}{n+1} \left(\nabla^\gamma \nabla_\gamma \sigma + i \nabla_0 \sigma + \frac{n+1}{n+2} P \sigma \right) = -i \nabla_0 \sigma + \frac{P}{n+2} \sigma.$$

It is now straightforward to check that $\mathbb{D}_A\sigma$ is holomorphic.

Conversely, suppose that I_A is holomorphic and suppose that $\sigma = Z^A I_A$ is holomorphic. Set $\zeta = \sigma^{-(n+2)} \in \mathcal{K}$ and let θ be the unique contact form which is volume-normalized with respect to ζ . Since I_A is holomorphic, ζ is closed, and hence θ is pseudo-Einstein [38, 44]. \square

Proposition 4.7. *Let (M, H) be a CR manifold. Suppose that θ is an Einstein contact form. Then locally there exists a $\sigma \in \mathcal{E}(1, 0)$, unique up to multiplication by a constant $\lambda \in \mathbb{C}$ with $|\lambda|^2 = 1$, such that $\theta = (\sigma\bar{\sigma})^{-1}\boldsymbol{\theta}$ and $\mathbb{D}_A\sigma$ is parallel; i.e. $\nabla_\beta\mathbb{D}_A\sigma = 0$ and $\nabla_{\bar{\beta}}\mathbb{D}_A\sigma = 0$. Conversely, if $I_A \in \mathcal{E}_A$ is parallel, then $\theta = (\sigma\bar{\sigma})^{-1}\boldsymbol{\theta}$ is Einstein wherever $\sigma := Z^A I_A$ is nonzero.*

Proof. Suppose that θ is Einstein. Let σ be as in Proposition 4.6, so that $I_A = \frac{1}{n+1} \mathbb{D}_A \sigma$ is holomorphic. In the scale θ , we have that $|\sigma|^2$ is parallel and hence, since $\bar{\partial}_b \sigma = 0$, it holds that $d_b \sigma = 0$. It is then clear from (3.5) that $\nabla_\beta I_A = 0$.

Conversely, suppose that I_A is parallel. Set $\sigma = Z^A I_A$. By Proposition 4.6, $\theta = (\sigma \bar{\sigma})^{-1} \theta$ is pseudo-Einstein. Evaluating $\nabla_\beta I_A = 0$ in the scale θ yields $A_{\alpha\beta} = 0$, and hence θ is Einstein. \square

5. Tractors and the Fefferman ambient metric

On a CR manifold the tractor calculus provides the basic invariant calculus. It is the CR analogue of the calculus surrounding the Levi-Civita connection in Riemannian geometry. We need to link this to the Fefferman ambient metric for two reasons: First the CR GJMS operators and the related basic objects are *defined* in terms of the ambient metric. Second the ambient metric provides a powerful tool for simplifying tractor calculus computations; this works well because the ambient metric is effectively a non-linear extension of the tractor bundle and connection that captures these in terms of a Kähler metric (of mixed signature) and connection, see in particular Theorem 5.1 below.

Most components of this link between tractors and the ambient metric are available in the literature. To adapt and extend these as required for our current purposes it is useful to first understand the principal bundle structure equivalent to the tractor connection, namely the Cartan connection. This provides a conceptual framework for the tractor connection and its use. In particular, it enables us below to construct and understand the Fefferman space and the ambient connection from this perspective. To understand the groups involved we first recall the model for CR geometry.

5.1. The Cartan connection and the model

Fix a complex vector space \mathbb{V} of complex dimension $n + 2$, equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle$ of signature $(p + 1, q + 1)$, where $p + q = n$. Let $\mathcal{N} \subset \mathbb{V}$ be the cone of nonzero null vectors in \mathbb{V} . Then the image S of \mathcal{N} in the complex projectivisation $\mathcal{P}\mathbb{V} \cong \mathbb{C}P^{n+1}$ has a CR structure, and this provides the usual flat model for hypersurface type CR geometry.

Denote by $G \cong SU(p + 1, q + 1)$ the special unitary group of $(\mathbb{V}, \langle \cdot, \cdot \rangle)$. Note that G acts transitively on S . Thus S may be naturally identified with the homogeneous space G/P , where $P \subset G$ is the isotropy subgroup of a nominated point on S . Note that P stabilises a complex 1-dimensional subspace \mathbb{V}^1 in \mathbb{V} .

Restricting the standard representation of G to the subgroup P , we obtain the associated bundle $\mathcal{T} = G \times_P \mathbb{V}$. Since \mathbb{V} carries a representation of G , the map $G \times V \ni (g, v) \mapsto (gP, g \cdot v) \in (G/P) \times \mathbb{V}$ descends to a canonical trivialisation of $\mathcal{T} = G \times_P \mathbb{V}$. Thus \mathcal{T} has canonical connection and this is the specialisation to S of the standard tractor connection described (as a complex vector bundle) in (3.5) above. In the flat homogeneous setting of $S = G/P$ this tractor connection may be

viewed as arising as an associated connection from the Maurer-Cartan form ω_{MC} on G .

On a general curved (hypersurface type) CR manifold M one can construct a principal bundle \mathcal{G} with fibre P , $P \rightarrow \mathcal{G} \rightarrow M$, and this is canonically equipped with a structure ω , the *Cartan connection* [15,22,52]. Indeed \mathcal{G} can be recovered as an adapted frame bundle for the tractor bundle \mathcal{T} and then the Cartan connection derived from the tractor connection of (3.5), see [12,14]. The Cartan connection should be viewed as a curved analogue of the Maurer-Cartan form, with just weaker equivariance properties. Its characterising properties are as follows. First, ω is a $\mathfrak{g} = \text{Lie}(G)$ valued 1-form field on \mathcal{G} that provides a trivialisation of $T\mathcal{G}$. Second, this trivialisation is P -equivariant and reproduces the generators of fundamental vector fields. Finally, there is a notion of curvature for any such Cartan connection and one requires that this satisfy a normalisation condition defined in terms of Lie algebra cohomology. With these properties satisfied, the pair (\mathcal{G}, ω) is uniquely determined up to isomorphism and is then called the *normal Cartan connection*. The tractor connection (3.5) is normal in this sense, in that the equivalent Cartan connection is normal.

Given the Cartan bundle \mathcal{G} and any representation of P , we may form associated vector bundles. For example, the tangent bundle is $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, where $\mathfrak{p} = \text{Lie}(P)$ and the representation on $\mathfrak{g}/\mathfrak{p}$ is induced by the restriction to P of the adjoint representation. Although in general the Cartan connection does not induce a linear connection on such associated bundles, it does induce a connection on $\mathcal{T}^{\mathbb{W}} := \mathcal{G} \times_P \mathbb{W}$ if \mathbb{W} is the restriction (to P) of a G -representation, which we shall denote ρ . A section $t \in \Gamma(\mathcal{T}^{\mathbb{W}})$ is represented by function $\tilde{t} : \mathcal{G} \rightarrow \mathbb{W}$ satisfying the equivariance property $\tilde{t}(u \cdot q) = \rho(q^{-1})\tilde{t}(u)$, for all $u \in \mathcal{G}$, and $q \in P$. The tractor connection is given by

$$\nabla_{Dp\xi} t(x) = \underline{u}(\xi \cdot \tilde{t}(u) + \rho'(\omega(\xi))(\tilde{t}(u))), \tag{5.1}$$

where $p : \mathcal{G} \rightarrow M$ is the bundle map, $\xi \in T_u\mathcal{G}$ is any tangent vector, $\rho' : \mathfrak{g} \rightarrow \text{End}(\mathbb{W})$ denotes the representation of \mathfrak{g} on \mathbb{W} , and $\underline{u} : \mathbb{W} \rightarrow \mathcal{T}_x^{\mathbb{W}}$, is the isomorphism from \mathbb{W} to the fiber of $\mathcal{T}^{\mathbb{W}}$ over $x \in M$ determined by u . Thus such bundles are called tractor bundles, and note that the standard tractor bundle is induced from \mathbb{V} . So the connection (3.5) induces a connection on any such tractor bundle.

We conclude this subsection by discussing some other groups linked to the geometry of the CR model. Let $\mathcal{N} \subset \mathbb{V}$ be the cone of non-zero null vectors. The CR manifold of the model S is the image of this under complex projectivisation. We are interested also in the real projectivisation.

Let $M_{\mathcal{F}}$ be the space of all real rays in \mathbb{V} which are null for the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, the real part of $\langle \cdot, \cdot \rangle$. The space $M_{\mathcal{F}}$ is a smooth hypersurface in $\mathcal{P}_{\mathbb{R}_+}\mathbb{V} \cong \mathbb{R}P^{2n+3}$, and we have an obvious projection $\mathcal{N} \rightarrow M_{\mathcal{F}}$, which is a principal bundle with fibre group \mathbb{R}_+ .

Any real null ray generates a complex null line containing it. Thus there is a smooth projection $M_{\mathcal{F}} \rightarrow M$ which is a fibre bundle over $M = S$, with fibre the space S^1 of real rays in \mathbb{C} .

Let \tilde{G} be the orthogonal group of $(\mathbb{V}, \langle \cdot, \cdot \rangle_{\mathbb{R}})$, and let $P_{\mathcal{F}} \subset \tilde{G}$ be the stabiliser of a real null ray in \mathbb{V} . In this case we observe that there is a transitive action of \tilde{G} on $M_{\mathcal{F}}$ which leads to an identification $M_{\mathcal{F}} \cong \tilde{G}/P_{\mathcal{F}}$. By construction, \tilde{G} acts by conformal isometries on $M_{\mathcal{F}}$. It is well known that this action identifies $\tilde{G}/Z(\tilde{G})$ with the group of conformal isometries of $M_{\mathcal{F}}$.

Now the subgroup $G \subset \tilde{G}$ acts transitively on \mathcal{N} , so it also acts transitively on $M_{\mathcal{F}}$. Taking a real null ray and the complex null line generated by it as the base points of $M_{\mathcal{F}}$ and M , respectively, we see that $G \cap P_{\mathcal{F}} \subset P$, and $G \cap P_{\mathcal{F}}$ is the stabiliser of a real null line, so we also obtain the identification $M_{\mathcal{F}} \cong G/(G \cap P_{\mathcal{F}})$. This is the model structure for the Fefferman space that we describe below. Again using that G acts transitively on \mathcal{N} we see that \mathcal{N} can be identified with $G/(G \cap \tilde{Q})$, where \tilde{Q} is the subgroup of $P_{\mathcal{F}}$ fixing a nominated point in the ray defining $P_{\mathcal{F}}$.

5.2. The Fefferman space and ambient metric

To a given CR manifold M there are associated two equivalent geometric structures, both due to Fefferman [23]; these are the Fefferman space and the Fefferman ambient metric. The latter associates to M a Kähler manifold $(M_{\mathcal{A}}, J^{M_{\mathcal{A}}}, h^{M_{\mathcal{A}}})$ that is, in a suitable sense, approximately Ricci flat. The construction of this space is recalled in Subsection 5.4. By construction, the Fefferman ambient metric admits an action by a \mathbb{C}^* -parametrised family of homotheties and is equipped with a distinguished real hypersurface embedding $\iota : \mathcal{F} \rightarrow M_{\mathcal{A}}$ that is equivariant with respect to this action. We identify \mathcal{F} with its image in $M_{\mathcal{A}}$ and note that (as will become clear) it is a generalising analogue of the cone \mathcal{N} (of non-zero null vectors in \mathbb{V}) described above in connection with the CR model.

Considering the \mathbb{C}^* -action on \mathcal{F} , the orbit space \mathcal{F}/\mathbb{C}^* is naturally identified with M . That is, $M = \mathcal{F}/\mathbb{C}^*$ and we write $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow M$ for the natural quotient map. To handle the link between tensorial structures along \mathcal{F} , in $M_{\mathcal{A}}$, and the corresponding objects on M , we use that the tractor connection on M can be recovered from the Levi-Civita connection on $M_{\mathcal{A}}$ as follows. For $x \in M$, we denote by \mathcal{F}_x the fiber of \mathcal{F} over x , and we view \mathcal{F}_x as a 2-dimensional submanifold of $M_{\mathcal{A}}$ via $\mathcal{F}_x \subset \mathcal{F} \subset M_{\mathcal{A}}$.

Then $T(M_{\mathcal{A}})|_{\mathcal{F}_x}$ denotes the tangent bundle to $M_{\mathcal{A}}$ restricted to the submanifold \mathcal{F}_x , a (real) rank $2n + 4$ vector bundle over \mathcal{F}_x . Again using the construction of the ambient manifold, it follows that the restriction of the ambient Levi-Civita connection is flat without holonomy; so $T(M_{\mathcal{A}})|_{\mathcal{F}_x}$ may be globally trivialised by parallel sections.

The standard tractor bundle on M may be realised as the complex rank $n + 2$ vector bundle $\mathcal{T} \rightarrow M$ with fibre

$$\mathcal{T}_x = \left\{ U \in \Gamma(T^{1,0}(M_{\mathcal{A}})|_{\mathcal{F}_x}) : \nabla_v^A U = 0 \text{ for all } v \in \Gamma(T\mathcal{F}) \text{ vertical} \right\}, \quad (5.2)$$

where ∇^A is the Levi-Civita connection of the ambient metric and the condition $v \in \Gamma(T\mathcal{F})$ vertical means that v is a generator of the \mathbb{C}^* -action. Thus a section of

\mathcal{T} on M is a vector field in $M_{\mathcal{A}}$, defined along \mathcal{F} , which is constant in the vertical directions. It is easily verified that the Hermitian metric and Levi-Civita connection on $M_{\mathcal{A}}$ induce a Hermitian metric and connection on \mathcal{T} .

Theorem 5.1. *The metric and connection on \mathcal{T} , induced by the metric and connection on $M_{\mathcal{A}}$ agree, up to isomorphism, with the standard (normal) CR tractor metric and connection introduced above.*

This theorem may be proved by analogy with the treatments of corresponding conformal results, as in [13,29]. However we can recover the entire picture, and exploit results in the existing literature, by using the Fefferman space as an intermediate step.

Writing \mathbb{R}_+ to denote a chosen ray in \mathbb{C}^* , we can view \mathbb{C}^* as a direct group product $\mathbb{C}^* = S^1 \times \mathbb{R}_+$. Then the map $\pi_{\mathcal{F}} : \mathcal{F} \rightarrow M$ factors into the composition of $\pi_{M_{\mathcal{F}}} : M_{\mathcal{F}} \rightarrow M$ with $\pi : \mathcal{F} \rightarrow M_{\mathcal{F}}$, where $M_{\mathcal{F}} = \mathcal{F}/\mathbb{R}_+$ is the \mathbb{R}_+ orbit space of \mathcal{F} . Then $\pi_{M_{\mathcal{F}}} : M_{\mathcal{F}} \rightarrow M$ has fibre S^1 and $M_{\mathcal{F}}$ is the Fefferman space; this has a canonical conformal structure induced by the CR structure on M . In fact, $M_{\mathcal{F}}$ is very easy to construct directly via the Cartan and tractor bundle on M . This provides a nice conceptual picture, but also a route to the recovery of the ambient metric, the proof of Theorem 5.1, and other related identifications that we need.

5.3. The Fefferman space

Here we start again on the CR manifold M and build a Fefferman space and then later the ambient metric directly. Our treatment is brief since for the first part of our construction further details may be found in [14], while for the second part mainly similar ideas are used. Related earlier constructions of the Fefferman space exist in [9,42,43]. In our treatment we use the groups defined in Section 5.1 above.

On a CR manifold M , recall that we write $\mathcal{E}(-1, 0)$ for the dual of $\mathcal{E}(1, 0)$ (the chosen $(n + 2)$ nd root of the anticanonical bundle). We define the *Fefferman space* $M_{\mathcal{F}}$ of M to mean the space of real rays in $\mathcal{E}(-1, 0)$ constructed as follows. Let \mathcal{F} be (the total space of) the bundle obtained by removing the zero section in $\mathcal{E}(-1, 0)$. There is a free right action of \mathbb{C}^* on \mathcal{F} which is transitive on each fibre. Restricting this action to the subgroup \mathbb{R}_+ , we define $M_{\mathcal{F}}$ to be the quotient \mathcal{F}/\mathbb{R}_+ . Hence $\pi_{M_{\mathcal{F}}} : M_{\mathcal{F}} \rightarrow M$ is a principal fibre bundle with structure group $\mathbb{C}^*/\mathbb{R}_+ \cong S^1$.

Via the bundle inclusion $\mathcal{E}(-1, 0) \rightarrow \mathcal{T}$ we see that we may identify the total space $\mathcal{E}(-1, 0)$ with $\mathcal{G} \times_P \mathbb{V}^1$. By construction, we can therefore view $M_{\mathcal{F}}$ as the associated fibre bundle $\mathcal{G} \times_P \mathcal{P}_{\mathbb{R}_+}(\mathbb{V}^1)$ with fibre the space of real rays in \mathbb{V}^1 . Since G acts transitively on the cone of nonzero null vectors, P acts transitively on the space of real rays in \mathbb{V}^1 ; the stabiliser of one of these rays is $G \cap P_{\mathcal{F}}$ and the stabiliser of a point in that ray $G \cap \tilde{Q}$, whence $\mathcal{P}_{\mathbb{R}_+} \mathbb{V}^1 \cong P/(G \cap P_{\mathcal{F}})$ and $\mathbb{V}^1 \cong P/(G \cap \tilde{Q})$. Now $M_{\mathcal{F}} = \mathcal{G} \times_P (P/(G \cap P_{\mathcal{F}}))$ and $\mathcal{F} = \mathcal{G} \times_P (P/(G \cap \tilde{Q}))$ are naturally identified with the orbit spaces $\mathcal{G}/(G \cap P_{\mathcal{F}})$ and $\mathcal{G}/(G \cap \tilde{Q})$, respectively. Hence we can view \mathcal{G} as a principal bundle over $M_{\mathcal{F}}$ with structure group $G \cap P_{\mathcal{F}}$ and, alternatively, as a principal bundle over \mathcal{F} with structure group $G \cap \tilde{Q}$.

Now for any closed Lie subgroup $H \subset P$ we have the following observations. As for the cases just described we have a manifold \mathcal{G}/H . The bundle \mathcal{G} is a principal bundle over this with fibre P/H , and the normal CR Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ also provides a Cartan connection on $\mathcal{G} \rightarrow \mathcal{G}/H$. This is clear: The property that ω gives a trivialisation of \mathcal{G} is not dependent on the base; the P -equivariance of this trivialisation restricts to H -equivariance; and the fundamental vector fields for $\mathcal{G} \rightarrow \mathcal{G}/H$ have generators in $\text{Lie}(H) \subset \mathfrak{p}$ so ω provides the map to these generators simply by restriction. Similarly any representation \mathbb{W} of P then determines an associated bundle $\mathcal{G} \times_H \mathbb{W}$ over \mathcal{G}/H that corresponds to the bundle $\mathcal{G} \times_P \mathbb{W}$ over $M = \mathcal{G}/P$. In particular this applies to the case that \mathbb{W} is a G -representation, so corresponding to each tractor bundle \mathcal{W} on M there is a corresponding tractor bundle \mathcal{W}_H on \mathcal{G}/H and the Cartan connection induces a tractor connection on \mathcal{W}_H . Furthermore since sections of the tractor bundle \mathcal{W} on M correspond to functions $\mathcal{G} \rightarrow \mathbb{W}$ that are P -equivariant, it follows at once from the explicit formula (5.1) that these are the same as sections on \mathcal{W}_H that are parallel in the vertical directions of $\mathcal{G}/H \rightarrow M$.

Thinking of the cases that H is $G \cap P_{\mathcal{F}}$ or $G \cap \tilde{Q} \subset G \cap P_{\mathcal{F}}$, we may apply these results in particular to the standard representation \mathbb{V} of G to obtain the associated bundle $\mathcal{T}_H := \mathcal{G} \times_H \mathbb{V} \rightarrow \mathcal{G}/H$. The Hermitian inner product on \mathbb{V} is G -invariant, so it gives rise to a Hermitian bundle metric on \mathcal{T}_H of signature $(p+1, q+1)$. Taking the real part of this defines a real bundle metric h_H of signature $(2p+2, 2q+2)$ on \mathcal{T}_H . The real ray $\mathbb{V}_{\mathbb{R}_+}^1 \subset \mathbb{V}$, stabilised by $G \cap P_{\mathcal{F}}$ (or in the case of $H = G \cap \tilde{Q}$, the point in $\mathbb{V}_{\mathbb{R}_+}^1$) gives rise to an oriented real line subbundle $\mathcal{T}_H^1 \subset \mathcal{T}_H$ (respectively, this line subbundle with also a nowhere zero distinguished section), and each of these lines is null with respect to h_H . Thus, defining \mathcal{T}_H^0 to be the real orthogonal complement of \mathcal{T}_H^1 , we obtain a filtration $\mathcal{T}_H = \mathcal{T}_H^{-1} \supset \mathcal{T}_H^0 \supset \mathcal{T}_H^1$ by smooth subbundles. The real volume form on \mathbb{V} induces a trivialisation of the highest real exterior power $\Lambda^{2n+4} \mathcal{T}_H$.

Specialising to the case $H = G \cap P_{\mathcal{F}}$ we come to the following. We write $\mathcal{T}_{M_{\mathcal{F}}}$, $h_{M_{\mathcal{F}}}$ rather than \mathcal{T}_H , h_H , etc.

Theorem 5.2 ([14]). *Let $(M, H, \mathcal{E}(1, 0))$ be a CR geometry of signature (p, q) . The corresponding Fefferman space $M_{\mathcal{F}}$ canonically carries a conformal structure $c_{M_{\mathcal{F}}}$ of signature $(2p+1, 2q+1)$.*

The Cartan connection ω on \mathcal{G} induces a tractor connection $\nabla^{\mathcal{T}_{M_{\mathcal{F}}}}$ on the bundle $\mathcal{T}_{M_{\mathcal{F}}} \rightarrow M_{\mathcal{F}}$, and $(\mathcal{T}_{M_{\mathcal{F}}}, \mathcal{T}_{M_{\mathcal{F}}}^1, h_{M_{\mathcal{F}}}, \nabla_{M_{\mathcal{F}}}^{\mathcal{T}})$ is a standard tractor bundle for the natural conformal structure on $M_{\mathcal{F}}$. The tractor connection $\nabla_{M_{\mathcal{F}}}^{\mathcal{T}}$ is normal.

Proof. It is straightforward to identify the quotient bundle $\mathcal{T}_{M_{\mathcal{F}}}^0/\mathcal{T}_{M_{\mathcal{F}}}^1$ with a weighted twisting of $TM_{\mathcal{F}}$. Using this, the conformal metric is then seen to be induced by the tractor metric, as in the usual conformal case. For more details (with a slightly different approach) see [14, Theorem 2.1]. The identification of $\mathcal{T}_{M_{\mathcal{F}}}$ and $\nabla_{M_{\mathcal{F}}}^{\mathcal{T}}$ with the usual conformal tractor bundle and connection follows from the characterisation of the latter (see [12]), or is proved in detail and directly in [14, Theorem 2.3]. \square

Remark 5.3. In fact the conformal structure on $M_{\mathcal{F}}$ carries a canonical spin structure, but we do not need that here.

Finally, we note that conformal tractor bundle $(\mathcal{T}_{M_{\mathcal{F}}}, \nabla_{M_{\mathcal{F}}}^{\mathcal{T}})$ inherits a complex structure $\mathbb{J}_{M_{\mathcal{F}}}$ corresponding to multiplication by i on the defining representation \mathbb{V} of G . This is parallel for the tractor connection because G is complex linear and the Cartan connection is \mathfrak{g} -valued. We may complexify \mathcal{T} and identify $\mathcal{T}_{M_{\mathcal{F}}}$ with the part $\mathcal{T}_{M_{\mathcal{F}}}^{1,0}$ in $\mathbb{C} \otimes \mathcal{T}_{M_{\mathcal{F}}}$.

5.4. The ambient metric

We now specialise to the case $H = G \cap \tilde{Q}$. Then $\mathcal{G}/H = \mathcal{F}$ and so over this we have the bundle $\mathcal{T}_{\mathcal{F}} = \mathcal{T}_H$ and this is equipped with the metric $h_{\mathcal{F}} = h_H$ and a connection $\nabla^{\mathcal{F}}$ preserving $h_{\mathcal{F}}$. From the above we see that the sections of $\mathcal{T}_{\mathcal{F}} \rightarrow \mathcal{F}$ that are parallel along the submanifolds generated by the \mathbb{C}^* -action may be identified with sections of $\mathcal{T} \rightarrow M$ and conversely. Similarly one sees that sections of $\mathcal{T}_{\mathcal{F}} \rightarrow \mathcal{F}$ that are parallel in the vertical directions of $\mathcal{F} \rightarrow M_{\mathcal{F}}$ (which coincide with the directions of the \mathbb{R}_+ -action) are the same as sections of $\mathcal{T}_{\mathcal{F}} \rightarrow M_{\mathcal{F}}$.

Now it is easily verified that, at each point p of \mathcal{F} , $\mathcal{T}_{\mathcal{F}}^0$ can be naturally identified with $T_p\mathcal{F} \cong T_p^{1,0}\mathcal{F} \subset T_{\mathbb{C}}\mathcal{F}$, and $\mathcal{T}_{\mathcal{F}}^1$ with the vertical subspace (with respect to $\pi : \mathcal{F} \rightarrow M_{\mathcal{F}}$) therein. We write \mathfrak{g} for the restriction of $h_{\mathcal{F}}$ to $T\mathcal{F} \cong \mathcal{T}_{\mathcal{F}}^0$; i.e., \mathfrak{g} on \mathcal{F} is defined for $X, Y \in T_p\mathcal{F}$ by $\mathfrak{g}(X, Y) = h_x^{\mathcal{F}}(X, Y)$.

Using these observations we have that the quotient space $\mathcal{T}_{\mathcal{F}}^0/\mathcal{T}_{\mathcal{F}}^1$, at p , is naturally identified with $T_{\pi(p)}M_{\mathcal{F}}$. So each point p of \mathcal{F} determines a metric $g_{\pi(p)}$ (from the conformal class of the Fefferman space) by lifting tangent vectors in $T_{\pi(p)}M_{\mathcal{F}}$ to vectors in $T_p\mathcal{F}$ and evaluating using the pairing \mathfrak{g} . The result is independent of the choice of lift because the subbundle $\mathcal{T}_{\mathcal{F}}^1$ is orthogonal to $\mathcal{T}_{\mathcal{F}}^0$ (as mentioned above). For $x \in M_{\mathcal{F}}$, distinct points of the fibre $\pi^{-1}(x)$ determine distinct metrics on $T_xM_{\mathcal{F}}$ intertwining the \mathbb{R}_+ action. Thus \mathcal{F} may be identified with the natural ray bundle

$$\{(x, g_x) : x \in M_{\mathcal{F}}, g \in \mathfrak{c}_{M_{\mathcal{F}}}\} \subset S^2T^*M_{\mathcal{F}}$$

of conformally related metrics over $M_{\mathcal{F}}$; so metrics in the conformal class $\mathfrak{c}_{M_{\mathcal{F}}}$ are sections of the metric bundle \mathcal{F} . Let $\delta_s : \mathcal{F} \rightarrow \mathcal{F}$ denote the dilations defined by $\delta_s(x, g_x) = (x, s^2g_x)$, $s > 0$, and let $T = \frac{d}{ds}\delta_s|_{s=1}$. So T is the infinitesimal generator of the dilations.

Now the ambient space is constructed as follows. Regard \mathcal{F} as a hypersurface in $M_A = \mathcal{F} \times \mathbb{R}$ via $\iota(z) = (z, 0)$, $z \in \mathcal{F}$. The variable in the \mathbb{R} factor is denoted ρ . In the language of [24], a straight pre-ambient metric for $(M_{\mathcal{F}}, \mathfrak{c}_{M_{\mathcal{F}}})$ is a smooth metric g^{M_A} of signature $(p + 1, q + 1)$ on a dilation-invariant neighborhood $\tilde{\mathcal{F}}$ of \mathcal{F} satisfying

- (1) $\delta_s^*g^{M_A} = s^2g^{M_A}$ for $s > 0$;
- (2) $\iota^*g^{M_A} = \mathfrak{g}$;

- (3) $\nabla^A T = Id$, where Id denotes the identity endomorphism and ∇^A the Levi-Civita connection of g^{M_A} .

Now if S_{IJ} is a symmetric 2-tensor field on an open neighborhood of \mathcal{F} in $\mathcal{F} \times \mathbb{R}$ and $m \geq 0$, we write $S_{IJ} = O_{IJ}^+(\rho^m)$ if $S_{IJ} = O(\rho^m)$ and for each point $p \in \mathcal{F}$, the symmetric 2-tensor $(\iota^*(\rho^{-m}S))(p)$ is of the form π^*t for some symmetric 2-tensor t at $x = \pi(p) \in M_{\mathcal{F}}$ satisfying $\text{tr}_{g_x} t = 0$. Since the dimension $2n + 2$ of $M_{\mathcal{F}}$ is even, an ambient metric for $(M_{\mathcal{F}}, \mathbf{c}_{M_{\mathcal{F}}})$ is a straight pre-ambient metric g^{M_A} such that $\text{Ric}(g^{M_A}) = O_{IJ}^+(\rho^n)$. From [24], there exists an ambient metric for $(M_{\mathcal{F}}, \mathbf{c}_{M_{\mathcal{F}}})$ and it is unique up to addition of a term which is $O_{IJ}^+(\rho^{n+1})$ and up to pullback by a diffeomorphism defined on a dilation-invariant neighborhood of \mathcal{F} which commutes with dilations and which restricts to the identity on \mathcal{F} . Since $M_{\mathcal{F}}$ is a Fefferman space, an invariant natural density obstructs the existence of smooth solutions to $\text{Ric}(g^{M_A}) = O(\rho^{n+1})$.

Next we note that the restriction of the ambient Levi-Civita connection ∇^A to $TM_A|_{\mathcal{F}}$ agrees (up to isomorphism) with the normal tractor connection on $\mathcal{T}_{\mathcal{F}}$. This follows by combining the results in [13] (and see also [29]) with Theorem 5.2. The former show that the usual conformal tractor connection is induced by the Fefferman-Graham ambient connection. By the uniqueness of the normal conformal tractor connection, it then follows from Theorem 5.2 that the conformal tractor connection on $M_{\mathcal{F}}$ determines the usual CR tractor connection on M . It is easily verified that both steps are compatible with our claim above Theorem 5.1 for the way in which the tractor bundle arises from $TM_A|_{\mathcal{F}}$ and that the CR tractor connection is then induced in the obvious way from the ambient connection.

Putting the above together we see that Theorem 5.1 is proved.

Finally, we claim that (M_A, g_A) can be taken to be Kähler. Indeed, since M is embedded, Fefferman’s original construction (cf. [39, 40]) produces an ambient metric which is Kähler. By the uniqueness of ambient metrics explained above, this latter metric can be taken to be (M_A, g_A) .

5.5. Tractors via the ambient metric

Theorem 5.1 and its proof show that any section V of the standard tractor bundle on M may be identified with the restriction to \mathcal{F} of a vector field \tilde{V} on M_A that is parallel in the vertical directions of $M_A \supset \mathcal{F} \rightarrow M$. These directions are generated by the \mathbb{C}^* -action on \mathcal{F} mentioned above. Taking tensor powers we may assume that any tractor field on M arises from a tensor field on M_A that is parallel along the \mathbb{C}^* -orbits of \mathcal{F} .

Now in view of our treatment of Theorem 5.1 above, creating the Rosetta stone relating tractor operators and similar objects to their ambient equivalents can be broken into a two step process. First we have the bijection between such objects on M and their equivalents on the Fefferman space $M_{\mathcal{F}}$, then second we have the bijection between these tractor tools on the Fefferman space and their equivalents on the ambient manifold. But the first part of this is treated in [14], see especially Sections 3.2 to 3.7. The second is treated in [13, 29], and [30]. Only some minor

additional input is required to specialise the latter to the case that conformal tractor bundle has a parallel complex structure. In fact conformal manifolds with a parallel complex structure are treated in Sections 4.4 to 4.7 of [14] (cf. [45]).

Putting these results together we can simply read off the correspondence between the canonical CR tractor fields and operators and their (Fefferman CR) ambient metric counterparts. For example, the canonical tractor Z^A giving the inclusion (3.3) corresponds to the $(1, 0)$ -part E of the generator X of the standard \mathbb{R}_+ -action on the ambient manifold. This vector field, which we also denote by Z^A , provides an *Euler operator* E on the ambient manifold, in that if a function $f : M_A \rightarrow \mathbb{C}$ is homogeneous of degree w , with respect to the \mathbb{C}^* -action on \mathcal{F} , then $E \cdot f = Z^A \nabla_A^A f = wf$ along \mathcal{F} . CR densities in $\Gamma(\mathcal{E}(w, w'))$ correspond to functions f on \mathcal{F} that satisfy $Z^A \nabla_A^A f = wf$ and $\bar{E} \cdot f = Z^{\bar{A}} \nabla_{\bar{A}}^{\bar{A}} f = w'f$. We shall always extend such functions to M_A with the same homogeneity. Putting this together with our earlier convention, weighted tractor fields on M are associated with homogeneous tensor fields on M_A that satisfy the same transport equations.

The calculus of the conformal ambient metric is by now well-known from [24,35]; see also [13,29]. From this we can read off useful identities for the case considered here. Since X is a homothetic gradient field such that $\nabla^A X$ is the identity endomorphism field, we have

$$\nabla_A^A Z^B = \delta_A^B \quad \text{and} \quad \nabla_{\bar{A}}^A Z^B = 0. \tag{5.3}$$

Thus

$$[E, Z^A] = Z^A, \quad \text{and} \quad [E, Z^{\bar{A}}] = 0, \tag{5.4}$$

and

$$Z^A R_{A\bar{B}}{}^C{}_D = 0 = Z^{\bar{B}} R_{A\bar{B}}{}^C{}_D.$$

It also follows that

$$[E, \nabla_A^A] = -\nabla_A^A, \quad [E, \nabla_{\bar{B}}^A] = 0, \quad \text{and so} \quad [E, \Delta^A] = -\Delta^A. \tag{5.5}$$

It also follows from the definition of Z^A that $r = Z^A Z_A$ is a defining function for $\mathcal{F} \subset M_A$, in that \mathcal{F} is the zero locus of r , and from (5.3) that $\nabla_A^A r = Z_A$; in particular, $\nabla_A^A r$ is non-vanishing along \mathcal{F} .

Putting all this together, it follows that the CR double-D operators (3.6) and (3.7) on weighted tractor fields correspond to the ambient operators

$$D_{AB} = Z_B \nabla_A^A - Z_A \nabla_B^A \quad \text{and} \quad D_{A\bar{B}} = Z_{\bar{B}} \nabla_A^A - Z_A \nabla_{\bar{B}}^A,$$

respectively. From this it follows that the tractor-D operator \mathbb{D}_A corresponds to

$$D_A = (n + E + \bar{E} + 1) \nabla_A^A - Z_A \Delta^A$$

on M_A . Note that these ambient operators are defined on all of M_A , but strictly it is only along the hypersurface \mathcal{F} (and restricted to homogeneous tensor fields)

that they correspond precisely to the given tractor operators. Along \mathcal{F} they each act tangentially, meaning that they do not depend on how the given tensor field is smoothly extended off \mathcal{F} . This follows because r is a defining function for \mathcal{F} and

$$[D_{AB}, r] = [D_{A\bar{B}}, r] = 0,$$

while $[D_A, r] = r \cdot \text{Op}$ for some differential operator Op .

We need the following technical result.

Lemma 5.4. *Let $\tilde{\tau}$ be a function on M_A which is homogenous of bidegree (w, w') on M_A and let τ denote the corresponding density on M . Then*

$$[D_A D_{\bar{B}}, \log \tilde{\tau}]|_{\mathcal{F}}$$

is a homogeneous linear differential operator along \mathcal{F} and so, applied to functions homogeneous of degree zero, determines a linear differential operator from sections of $\mathcal{E}(0)$ (over M) to sections of the tractor bundle $\mathcal{T}_{A\bar{B}}(-1)$. This tractor operator agrees with

$$[\mathbb{D}_A \mathbb{D}_{\bar{B}}, \log \tau] : \mathcal{E}(0) \rightarrow \mathcal{T}_{A\bar{B}}(-1)$$

calculated in any scale.

Similarly,

$$[[D_A D_{\bar{B}}, \log \tilde{\tau}], \log \tilde{\tau}](1)|_{\mathcal{F}}$$

(where 1 denotes the unit-valued constant function) is a homogeneous tensor along \mathcal{F} , so descends to a tractor field in $\mathcal{T}_{A\bar{B}}(-1)$, and this agrees with

$$[[\mathbb{D}_A \mathbb{D}_{\bar{B}}, \log \tau], \log \tau](1)$$

as calculated in any scale.

Proof. The first statement is immediate from the definition of the ambient operators D_A and $D_{\bar{B}}$. Note also that, from the tangentiality of these operators, $[D_A D_{\bar{B}}, \log \tilde{\tau}]|_{\mathcal{F}}$ is independent of how $\tilde{\tau}$ extends off \mathcal{F} . Then recall that, by definition, $\log \tau$ means the log-density that corresponds to $\log \tilde{\tau}$.

Next by direct calculation one verifies that

$$[D_{AB}, \log \tilde{\tau}]|_{\mathcal{F}} \quad \text{and} \quad [D_{A\bar{B}}, \log \tilde{\tau}]|_{\mathcal{F}}$$

are also homogeneous and correspond to the tractor commutators

$$[\mathbb{D}_{AB}, \log \tau] \quad \text{and} \quad [\mathbb{D}_{A\bar{B}}, \log \tau]|_{\mathcal{F}}.$$

But it is easily verified that there are algebraic formulae for \mathbb{D}_A and $\mathbb{D}_{\bar{B}}$ in terms of compositions of \mathbb{D}_{AB} and $\mathbb{D}_{A\bar{B}}$ and these are equivalent to corresponding formulae for D_A and $D_{\bar{B}}$ in terms of compositions of D_{AB} and $D_{A\bar{B}}$ on M_A ; cf. [13] for the analogous conformal case. (Using the Leibniz property of \mathbb{D}_{AB} and $\mathbb{D}_{A\bar{B}}$, one sees

that the formulae for \mathbb{D}_A and $\mathbb{D}_{\bar{B}}$ in terms of \mathbb{D}_{AB} and $\mathbb{D}_{A\bar{B}}$ give the usual matrix formulae for \mathbb{D}_A and $\mathbb{D}_{\bar{B}}$ when applied to log densities.) Thus $[D_{\bar{B}}, \log \tilde{\tau}]|_{\mathcal{F}}$ and $[D_A, \log \tilde{\tau}]|_{\mathcal{F}}$ are equivalent to well-defined fields and these are $[\mathbb{D}_{\bar{B}}, \log \tau]$ and $[D_A, \log \tau]$, as calculated in any scale.

The final statement follows similarly. □

In the subsequent sections, we normally omit the superscript M_A in the notation for ambient objects, relying instead on context. Also, we often identify tractor operators with the corresponding tangential ambient space operators without comment.

6. Tractor formulae for CR GJMS operators

It is straightforward to compute from the definitions of r , Z_A and E that

$$[\Delta, r] = n + E + \bar{E} + 2. \tag{6.1}$$

It follows that, as an operator on ambient tensors of weight $-\frac{n+3-k}{2}$,

$$[\Delta^k, r] = r \cdot \text{Op}. \tag{6.2}$$

Thus if $f \in \mathcal{E}\left(-\frac{n+1-k}{2}\right)$, then

$$P_{2k}f = (-2\Delta)^k \tilde{f}|_{\mathcal{F}} \tag{6.3}$$

is independent of the choice of homogeneous extension \tilde{f} of f to M_A . Let $k \leq n + 1$. Since the ambient metric is uniquely determined up to $O^+(\rho^{n+1})$, the operator (6.3) is well-defined, and hence defines the CR GJMS operator of order $2k$. Our goal in this section is to develop an algorithm which converts (6.3) into a tractor formula.

To derive tractor formulae for the CR GJMS operators, we first need to derive some useful identities on the ambient manifold M_A . In the following, Δ is the Kähler Laplacian $\Delta = g^{A\bar{B}}\nabla_A\nabla_{\bar{B}}$ and $R_{A\bar{B}}{}^{\bar{C}}{}_{\bar{D}}$ is the $(1, 1)$ -part of the ambient curvature. We write $R_{A\bar{B}}\sharp$ to denote the usual action of this 2-form-valued endomorphism field on ambient tensors. Since g is Kähler, we have the operator equations on ambient tensors

$$[\nabla_A, \nabla_{\bar{B}}] = 0, \tag{6.4}$$

$$[\nabla_A, \nabla_{\bar{B}}] = R_{A\bar{B}}\sharp; \tag{6.5}$$

e.g., on a vector field T^A we have $[\nabla_A, \nabla_{\bar{B}}]T^C = -R_{A\bar{B}}{}^C{}_D T^D$. With these conventions we obtain the following:

Lemma 6.1. *As operators on arbitrary ambient tensors,*

$$[\Delta, Z_{\bar{B}}] = \nabla_{\bar{B}}, \quad [\Delta, Z_A] = \nabla_A, \tag{6.6}$$

$$[\Delta, \nabla_A] = -R_A^C \sharp \nabla_C + O(r^{n-1}), \quad [\Delta, \nabla_{\bar{B}}] = R_{C\bar{B}} \sharp \nabla^C + O(r^{n-1}), \tag{6.7}$$

$$[\Delta, Z_A Z_{\bar{B}}] = Z_A \nabla_{\bar{B}} + Z_{\bar{B}} \nabla_A + g_{A\bar{B}}, \tag{6.8}$$

$$[\Delta, Z_{\bar{B}} \nabla_A] = \nabla_{\bar{B}} \nabla_A - Z_{\bar{B}} R_A^C \sharp \nabla_C + O(r^{n-1}), \tag{6.9}$$

$$[\Delta, \nabla_A \nabla_{\bar{B}}] = -R_A^C \sharp \nabla_C \nabla_{\bar{B}} + \nabla_A R_{C\bar{B}} \sharp \nabla^C + O(r^{n-2}). \tag{6.10}$$

Moreover, as operators on ambient functions,

$$[\Delta, \nabla_A] = O(r^n), \quad [\Delta, \nabla_{\bar{B}}] = O(r^n), \tag{6.11}$$

$$[\Delta, \nabla_A \nabla_{\bar{B}}] = -R_A^C \sharp \nabla_C \nabla_{\bar{B}} + O(r^{n-1}). \tag{6.12}$$

Remark 6.2. That these formulas hold as operators means that we are writing compositions; e.g., $\nabla_A R_{C\bar{B}} \sharp \nabla^C$ acts on $T^{\bar{B}}$ by

$$\nabla_A R_{C\bar{B}} \sharp \nabla^C T^{\bar{B}} = \nabla_A (R_{C\bar{B}} \sharp \nabla^C T^{\bar{B}}).$$

Proof. Using (5.3) we compute that

$$[\Delta, Z_A] = [\nabla_C, Z_A] \nabla^C + \nabla_C [\nabla^C, Z_A] = \nabla_A,$$

and the other result follows by conjugation. Using (6.4) and (6.5), we compute that

$$[\Delta, \nabla_A] = [\nabla_C, \nabla_A] \nabla^C + \nabla_C [\nabla^C, \nabla_A] = -\nabla_C R_A^C \sharp = -R_A^C \sharp \nabla_C + O(r^{n-1}),$$

where the last equality uses $\text{Ric} = O(r^n)$. Moreover, since $R_A^C \sharp$ annihilates functions, we see that $[\Delta, \nabla_A]$ also annihilates functions. Similarly,

$$[\Delta, \nabla_{\bar{B}}] = R_{C\bar{B}} \sharp \nabla^C,$$

and hence on functions,

$$[\Delta, \nabla_{\bar{B}}] = -R_{C\bar{B}} \nabla^C = O(r^n).$$

Using (5.3) and (6.6) we compute that

$$[\Delta, Z_A Z_{\bar{B}}] = \nabla_A Z_{\bar{B}} + Z_A \nabla_{\bar{B}} = Z_{\bar{B}} \nabla_A + Z_A \nabla_{\bar{B}} + g_{A\bar{B}}.$$

Using (6.6) and (6.7) immediately yields (6.9). Finally, using both parts of (6.7) (respectively of (6.11) on functions) gives (6.10) (respectively (6.12) on functions). \square

The main step in deriving a tractor formula for the CR GJMS operators is the following formula for the difference between $\Delta^k D_A D_{\bar{B}}$ and $Z_A Z_{\bar{B}} \Delta^{k+2}$ as operators on ambient tensors. As in Lemma 6.1, we only identify this difference along the hypersurface \mathcal{F} and identify the order of the error. To simplify the exposition, we do not keep track of the order of the error throughout the proof, but only record the final result; one counts the order of the error terms by using Lemma 6.1, counting additional derivatives, and using the identity (6.1).

Proposition 6.3. *Modulo the addition of terms $O(r^{n-k-1})$ the following holds:*

$$\begin{aligned} \Delta^k D_A D_{\bar{B}} &= Z_A Z_{\bar{B}} \Delta^{k+2} \\ &\quad - (n + E + \bar{E} + k + 1) (Z_{\bar{B}} \nabla_A \Delta^{k+1} + Z_A \nabla_{\bar{B}} \Delta^{k+1} + g_{A\bar{B}} \Delta^{k+1}) \\ &\quad + (n + E + \bar{E} + k + 1) (n + E + \bar{E} + k + 2) \nabla_A \nabla_{\bar{B}} \Delta^k \\ &\quad - Z_A R_{C\bar{B}} \sharp \left(\Delta \delta_E^C - R_E^C \sharp \right)^k D^E \\ &\quad + \sum_{j=0}^{k-1} (j + 1) \nabla_A \Delta^{k-1-j} R_{C\bar{B}} \sharp \left(\Delta \delta_E^C - R_E^C \sharp \right)^j D^E \tag{6.13} \\ &\quad - \sum_{j=0}^{k-1} \left(\Delta \delta_A^C + R_A^C \sharp \right)^j R_C^E \sharp \Delta^{k-1-j} D_E D_{\bar{B}} \\ &\quad + \sum_{j=0}^{k-1} \left(\Delta \delta_A^C + R_A^C \sharp \right)^j D_C R_{E\bar{B}} \sharp \left(\Delta \delta_F^E - R_F^E \sharp \right)^{k-1-j} D^F. \end{aligned}$$

Here the left- and right-hand side are operators acting on arbitrary ambient tensors, and we interpret $(\Delta \delta_C^B - R_C^B \sharp)^k$ at $k = 0$ to mean the identity endomorphism field, while for $k \geq 2$ there is an obvious abuse of the abstract index notation.

Proof. The proof is by induction. To begin, we compute that

$$\begin{aligned} D_A D_{\bar{B}} &= ((n + E + \bar{E} + 1) \nabla_A - Z_A \Delta) ((n + E + \bar{E} + 1) \nabla_{\bar{B}} - Z_{\bar{B}} \Delta) \\ &= Z_{\bar{B}} Z_A \Delta^2 - (n + E + \bar{E} + 1) (Z_{\bar{B}} \nabla_A \Delta + Z_A \nabla_{\bar{B}} \Delta + g_{A\bar{B}} \Delta) \\ &\quad + (n + E + \bar{E} + 1) (n + E + \bar{E} + 2) \nabla_A \nabla_{\bar{B}} \\ &\quad - (n + E + \bar{E} + 2) Z_A R_{C\bar{B}} \sharp \nabla^C, \end{aligned}$$

where the second equality uses Lemma 6.1; for the purposes of determining the order of vanishing of the error, note that the only commutator involving errors which was evaluated in the above derivation is $[\Delta, \nabla_{\bar{B}}]$. Using the definition of (the ambient) D^A and the fact $R_{C\bar{B}} \sharp Z^C = 0$, we see that

$$(n + E + \bar{E} + 2) Z_A R_{C\bar{B}} \sharp \nabla^C = Z_{\bar{B}} R_{C\bar{B}} \sharp D^C.$$

Together, these two displays yield the case $k = 0$.

Suppose now that (6.13) holds. We want to compute $\Delta^{k+1} D_A D_{\bar{B}}$. To that end, we make a number of observations. Note that, by Lemma 6.1, the formulae derived below all hold up to terms of order $O(r^{n-2})$, and up to terms of order $O(r^{n-1})$ on functions. Thus the lowest order error comes from commuting the Laplacian through the error term in $\Delta^k D_A D_{\bar{B}}$, accounting for the loss of a power of r in the order of the error.

First, as an immediate consequence of (6.8), for any k it holds that

$$\Delta Z_A Z_{\bar{B}} \Delta^{k+2} = Z_A Z_{\bar{B}} \Delta^{k+3} + (Z_{\bar{B}} \nabla_A + Z_A \nabla_{\bar{B}} + g_{A\bar{B}}) \Delta^{k+2}. \quad (6.14)$$

Second, as an immediate consequence of (6.5) and (6.9), for any k it holds that

$$\begin{aligned} & \Delta(n + E + \bar{E} + k + 1) (Z_{\bar{B}} \nabla_A + Z_A \nabla_{\bar{B}} + g_{A\bar{B}}) \Delta^{k+1} \\ &= (n + E + \bar{E} + k + 3) \left((Z_{\bar{B}} \nabla_A + Z_A \nabla_{\bar{B}} + g_{A\bar{B}}) \Delta^{k+2} \right. \\ & \quad \left. + (2 \nabla_A \nabla_{\bar{B}} - R_{A\bar{B}} \sharp + Z_A \nabla^C R_{C\bar{B}} \sharp \right. \\ & \quad \left. - Z_{\bar{B}} \nabla_C R_A{}^C \sharp) \Delta^{k+1} \right). \end{aligned} \quad (6.15)$$

Third, it holds for any k that

$$(n + E + \bar{E} + k + 1) \nabla^B \Delta^k = Z^B \Delta^{k+1} + \left(\Delta \delta_C{}^B - R_C{}^B \sharp \right)^k D^C. \quad (6.16)$$

Indeed, (6.16) trivially holds when $k = 0$. Using first (6.7) and then proceeding by induction, one finds that

$$\begin{aligned} & (n + E + \bar{E} + k + 1) \nabla^B \Delta^k \\ &= \left(\Delta \delta_C{}^B - R_C{}^B \sharp \right) (n + E + \bar{E} + k - 1) \nabla^C \Delta^{k-1} \\ &= \left(\Delta \delta_C{}^B - R_C{}^B \sharp \right) \left(Z^C \Delta^k - \nabla^C \Delta^{k-1} + \left(\Delta \delta_E{}^C - R_E{}^C \sharp \right)^{k-1} D^E \right) \\ &= \Delta Z^B \Delta^k - \Delta \nabla^B \Delta^{k-1} + R_C{}^B \sharp \nabla^C \Delta^{k-1} + \left(\Delta \delta_C{}^B - R_C{}^B \sharp \right)^k D^C \\ &= Z^B \Delta^{k+1} + \left(\Delta \delta_C{}^B - R_C{}^B \sharp \right)^k D^C. \end{aligned}$$

Fourth, as an immediate consequence of (5.5), (6.10) and (6.16), for any k it holds that

$$\begin{aligned} & \Delta(n + E + \bar{E} + k + 1)(n + E + \bar{E} + k + 2) \nabla_A \nabla_{\bar{B}} \Delta^k \\ &= (n + E + \bar{E} + k + 3)(n + E + \bar{E} + k + 4) \nabla_A \nabla_{\bar{B}} \Delta^{k+1} \\ & \quad + (n + E + \bar{E} + k + 3) \nabla_A R_{C\bar{B}} \sharp \left(\Delta \delta_E{}^C - R_E{}^C \sharp \right)^k D^E \\ & \quad - R_A{}^C \sharp (n + E + \bar{E} + k + 1)(n + E + \bar{E} + k + 2) \nabla_C \nabla_{\bar{B}} \Delta^k. \end{aligned} \quad (6.17)$$

Note that the last summand is easily rewritten in terms of $\Delta^k D_C D_{\bar{B}}$ using the inductive hypothesis (6.13).

Fifth, as an immediate consequence of (6.6) and the definition of the tractor D operator, it holds that

$$\Delta Z_A R_{C\bar{B}} \# = -D_A R_{C\bar{B}} \# + (n + E + \bar{E} + 2) \nabla_A R_{C\bar{B}} \#. \tag{6.18}$$

Sixth, as an immediate consequence of (5.5), for any k it holds that

$$\begin{aligned} & R_A^C \# (n + E + \bar{E} + k + 1) (Z_C \nabla_{\bar{B}} + Z_{\bar{B}} \nabla_C + g_{C\bar{B}}) \Delta^{k+1} \\ &= (n + E + \bar{E} + k + 3) (Z_{\bar{B}} R_A^C \# \nabla_C + R_{A\bar{B}} \#) \Delta^{k+1}. \end{aligned} \tag{6.19}$$

Seventh, as an immediate consequence of (6.16), for any k it holds that

$$\begin{aligned} & (n + E + \bar{E} + k + 3) Z_A \nabla^C R_{C\bar{B}} \# \Delta^{k+1} \\ &= Z_A R_{C\bar{B}} \# (\Delta \delta_E^C - R_E^C \#)^{k+1} D^E. \end{aligned} \tag{6.20}$$

Eighth, as an immediate consequence of (6.7), it holds that

$$\Delta \nabla_A R_{C\bar{B}} \# = \nabla_A \Delta R_{C\bar{B}} \# - R_A^E \# \nabla_E R_{C\bar{B}} \#. \tag{6.21}$$

It is then straightforward to use the above eight observations and the inductive hypothesis (6.13) to show that $\Delta^{k+1} D_A D_{\bar{B}}$ is given as in (6.13). \square

Lemma 6.1 and the fact that $R_{AC} \#$ and $R_{A\bar{B}} \#$ annihilate functions together simplify (6.13) and yield an improved order of vanishing when acting on functions.

Corollary 6.4. *Modulo the addition of terms $O(r^{n-k})$, the following holds on functions:*

$$\begin{aligned} \Delta^k D_A D_{\bar{B}} &= Z_A Z_{\bar{B}} \Delta^{k+2} \\ &\quad - (n + E + \bar{E} + k + 1) (Z_{\bar{B}} \nabla_A \Delta^{k+1} + Z_A \nabla_{\bar{B}} \Delta^{k+1} + g_{A\bar{B}} \Delta^{k+1}) \\ &\quad + (n + E + \bar{E} + k + 1) (n + E + \bar{E} + k + 2) \nabla_A \nabla_{\bar{B}} \Delta^k \\ &\quad - \sum_{j=0}^{k-1} (\Delta \delta_A^C + R_A^C \#)^j R_C^E \# \Delta^{k-1-j} D_E D_{\bar{B}}. \end{aligned}$$

In particular, when acting on $\mathcal{E} \left(-\frac{n-1-k}{2}, -\frac{n-1-k}{2} \right)$,

$$\Delta^k D_A D_{\bar{B}} = Z_A Z_{\bar{B}} \Delta^{k+2} - \sum_{j=0}^{k-1} (\Delta \delta_A^C + R_A^C \#)^j R_C^E \# \Delta^{k-1-j} D_E D_{\bar{B}}, \tag{6.22}$$

modulo the addition of terms $O(r^{n-k})$.

Proof. Let $\tilde{f} \in \tilde{\mathcal{E}}(w, w')$. Then $R_E^C \# D^E \tilde{f} = R_E^C F^E D^F \tilde{f} = O(r^n)$. Using this observation in Proposition 6.3 and the improved order of vanishing in Lemma 6.1 yields the final result. \square

The next step in deriving a tractor formula for the CR GJMS operators is to derive a tractor formula for the summation on the right-hand side of (6.22). To that end, we first observe that the operator acting on $D_A D_{\bar{B}}$ in the right-most summand acts tangentially on $\mathcal{E}^*\left(-\frac{n+1-k}{2}\right)$, where we recall from Subsection 3.2 that $\mathcal{E}^*(w, w')$ denotes a weighted tractor bundle.

Proposition 6.5. *As an operator on $\mathcal{E}^*\left(-\frac{n+1-k}{2}\right)$,*

$$\sum_{j=0}^{k-1} \left(\Delta \delta_A^C + R_A^C \# \right)^j R_C^E \# \Delta^{k-1-j} \tag{6.23}$$

acts tangentially.

Proof. We must show that the commutator of (6.23) with r is of the form $r \cdot \text{Op}$, for some operator Op , when applied to an element of $\mathcal{E}^*\left(-\frac{n+3-k}{2}\right)$. This follows directly from the formula (6.1). \square

The key point of Proposition 6.5 is that it shows that (6.23) is a tangential differential operator of order $2(k - 1)$. Since we assume that $k \leq n + 1$, the operators (6.23) are all subcritical, in the sense that they depend on strictly fewer derivatives of the metric than the number guaranteed to be uniquely determined in the construction of the ambient metric. In particular, we can adapt the arguments from [27, 29, 30] to easily produce tractor formulae for the operators (6.23). (Here and below we identify tractor operators with the corresponding tangential ambient operators.)

Proposition 6.6. *Suppose that $k \leq n - 1$. There is a tractor operator*

$$\Phi_{A\bar{E}} : \mathcal{E}^*\left(-\frac{n+1-k}{2}\right) \rightarrow \mathcal{E}_{A\bar{E}} \otimes \mathcal{E}^*\left(-\frac{n+1+k}{2}\right)$$

such that

$$\Phi_A^E = \sum_{j=0}^{k-1} \left(\Delta \delta_A^C + R_A^C \# \right)^j R_C^E \# \Delta^{k-1-j}.$$

Moreover, for any parallel tractor I^A it holds that $I^A \Phi_A^E = 0$.

Proof. Let $T \in \mathcal{E}^*\left(-\frac{n+1-k}{2}\right)$. Since $k \leq n - 1$, we may extend T harmonically so that $\Delta^j T = 0$ along \mathcal{F} for $j \in \{0, \dots, k - 1\}$. In particular,

$$\sum_{j=0}^{k-1} \left(\Delta \delta_A^C + R_A^C \# \right)^j R_C^E \# \Delta^{k-1-j} T = \left(\Delta \delta_A^C + R_A^C \# \right)^{k-1} R_C^E \# T. \tag{6.24}$$

Moreover, since $k \leq n - 1$, the ambient tensors $\Delta^j R_{A\bar{B}C\bar{E}}$ are well-defined for all $j \in \{0, \dots, k - 1\}$. Using the commutator identity (6.7), we may thus distribute the Laplacians in (6.24) to obtain an equivalent expression which is polynomial in $\nabla^j T$ and $\nabla^r \Delta^s R_{A\bar{B}C\bar{E}}$ for $j, r, s \leq k - 1$. Using the identity $\Delta R_{A\bar{B}E\bar{F}} = -R_{C\bar{B}} \# R_{A\bar{E}\bar{F}}$, we may then rewrite this as a polynomial in $D^j T$ and $D^r R_{A\bar{B}C\bar{E}}$, which is manifestly a tractor formula. Finally, from direct inspection of (6.24), we see that in every term, after expanding and eliminating the Kronecker deltas, the free indices are always on curvature terms $R_{A\bar{B}C\bar{E}}$. Thus $I^A \Phi_{A\bar{E}}$ involves contracting I into the curvature tensor. Since I^A is parallel, such contractions must vanish. \square

By repeating this argument and arguing inductively from Proposition 6.3, we obtain the following general tractor formula for the CR GJMS operators.

Theorem 6.7. *Let $k \leq n - 1$. There is a tractor operator*

$$\Psi_{C_{k+2} \dots C_2 \bar{C}_1}{}^{E_2 \bar{E}_1} : \mathcal{E}_{E_2 \bar{E}_1} \left(\frac{k+1-n}{2} \right) \rightarrow \mathcal{E}_{C_{k+2} \dots C_2 \bar{C}_1} \left(-\frac{k+1+n}{2} \right)$$

such that

$$\begin{aligned} & (-1)^k Z_{C_{k+2}} \dots Z_{C_4} Z_{\bar{C}_3} Z_{C_2} Z_{\bar{C}_1} \Delta^{k+2} \\ &= D_{C_{k+2}} \dots D_{C_4} D_{\bar{C}_3} D_{C_2} D_{\bar{C}_1} + \Psi_{C_{k+2} \dots C_4 \bar{C}_3 C_2 \bar{C}_1}{}^{E_2 \bar{E}_1} D_{E_2} D_{\bar{E}_1} \end{aligned}$$

and the full contraction $I^{C_{k+2}} \dots I^{C_2} I^{\bar{C}_1} \Psi_{C_{k+2} \dots C_2 \bar{C}_1}{}^{E_2 \bar{E}_1}$ vanishes for any parallel tractor I_A .

Proof. By Corollary 6.4 and Proposition 6.6, it holds that

$$\begin{aligned} Z_{C_{k+2}} \dots Z_{C_4} Z_{\bar{C}_3} Z_{C_2} Z_{\bar{C}_1} \Delta^{k+2} &= Z_{C_{k+2}} \dots Z_{C_4} Z_{\bar{C}_3} \Delta^k D_{C_2} D_{\bar{C}_1} \\ &\quad + Z_{C_{k+2}} \dots Z_{C_4} Z_{\bar{C}_3} \Phi_{C_2}{}^E D_E D_{\bar{C}_1}. \end{aligned}$$

Moreover, the second summand on the right-hand side is already a tractor operator satisfying the annihilation by contraction with I_A condition, so we need only consider the first summand. If $k = 0$, we are done. If $k = 1$, we may immediately write $-Z_{\bar{C}_3} \Delta D_{C_2} D_{\bar{C}_1} = D_{\bar{C}_3} D_{C_2} D_{\bar{C}_1}$, so we are again done. Suppose now that $k \geq 2$. By applying Proposition 6.3, we find that

$$\begin{aligned} Z_{C_4} Z_{\bar{C}_3} \Delta^k D_{C_2} D_{\bar{C}_1} &= \Delta^{k-2} D_{C_4} D_{\bar{C}_3} D_{C_2} D_{\bar{C}_1} \\ &\quad + \sum_{j=0}^{k-3} \left(\Delta \delta_{C_4}{}^E + R_{C_4}{}^E \# \right)^j R_E{}^F \# \Delta^{k-1-j} D_F D_{\bar{C}_3} D_{C_2} D_{\bar{C}_1} \\ &\quad - \Psi_{C_4 \bar{C}_3}^{(1)} D_{C_2} D_{\bar{C}_1}, \end{aligned}$$

where $\Psi_{C_4 \bar{C}_3}^{(1)}$ is the action of the operators which factor through a single D^E in (6.13) (i.e. the fourth, fifth and seventh summands). From Proposition 6.5 combined with

Proposition 6.3 we see that $\Psi_{C_4\bar{C}_3}^{(1)}$ is tangential. Here we used that in Proposition 6.3 the Laplacian powers appearing on the left-hand side and in the first term on the right-hand side are each tangential. Hence, by arguing as in Proposition 6.6, $\Psi_{C_4\bar{C}_3}^{(1)}$ admits a tractor formula which is annihilated upon complete contraction with a parallel tractor and its conjugate. Arguing inductively in this manner yields the final conclusion. \square

Note that the proofs of Proposition 6.6 and Theorem 6.7 provide an algorithm for producing tractor formulae for the CR GJMS operators. It is easy to carry out this algorithm at low orders to obtain tractor formulae for P_4 and P_6 as well as the P' -operators and Q' -curvatures of the corresponding order. The formulae for the fourth-order invariants are self-evident; see also [19]. The formulae for the sixth-order invariants are discussed in Section 8.

7. The product formulae

We are now prepared to prove Theorem 1.1. For convenience, we separate the proof into two parts. First, we prove the factorisation of the CR GJMS operators.

Proposition 7.1. *Let (M^{2n+1}, H, θ) be an embeddable Einstein pseudohermitian manifold. For any integer $1 \leq k \leq n + 1$, the CR GJMS operator P_{2k} is equal to*

$$P_{2k} = \begin{cases} \prod_{\ell=1}^{\frac{k}{2}} (-\Delta_b + c_\ell i \nabla_0 + d_\ell P) (-\Delta_b - c_\ell i \nabla_0 + d_\ell P) & \text{if } k \text{ is even} \\ Y \prod_{\ell=1}^{\frac{k-1}{2}} (-\Delta_b + c_\ell i \nabla_0 + d_\ell P) (-\Delta_b - c_\ell i \nabla_0 + d_\ell P) & \text{if } k \text{ is odd,} \end{cases} \tag{7.1}$$

where $c_\ell = k - 2\ell + 1$ and $d_\ell = \frac{n^2 - (k - 2\ell + 1)^2}{n}$ and $Y = -\Delta_b + nP$ is the CR Yamabe operator.

Remark 7.2. To pass from (7.1) to the factorisation given in [6, Proposition 1.1], reindex the product in terms of $\ell' = \frac{k}{2} - \ell$ when k is even, or $\ell' = \frac{k+1}{2} - \ell$ when k is odd. Note that our formula (7.1) in the case when k is odd also corrects a minor typo in [6, Proposition 1.1], where the index of the product incorrectly starts at zero.

Proof. Fix a point $p \in M$. By Proposition 4.7, there is a neighborhood U of p in which there exists a $\sigma \in \mathcal{E}(1, 0)$ such that $\theta = (\sigma\bar{\sigma})^{-1}\theta$ and $\mathbb{D}_A\sigma$ is parallel. Set $I_A = \frac{1}{n+1}\mathbb{D}_A\sigma$ and observe that in the scale θ it holds that $I^{\bar{A}} = (-P\sigma/n, 0, \sigma)$. It follows from Theorem 6.7 that

$$(\sigma\bar{\sigma})^{k/2} P_{2k} = 2^k I^{C_1} I^{\bar{C}_2} \dots I^{C_{k/2-1}} I^{\bar{C}_{k/2}} \mathbb{D}_{C_1} \mathbb{D}_{\bar{C}_2} \dots \mathbb{D}_{C_{k/2-1}} \mathbb{D}_{\bar{C}_{k/2}}$$

if k is even and

$$\sigma(\sigma\bar{\sigma})^{(k-1)/2} P_{2k} = 2^k I^{\bar{C}_1} \dots I^{C_{(k-3)/2}} I^{\bar{C}_{(k-1)/2}} \mathbb{D}_{\bar{C}_1} \dots \mathbb{D}_{C_{(k-3)/2}} \mathbb{D}_{\bar{C}_{(k-1)/2}}$$

if k is odd. Since $I_A \in \mathcal{E}_A$ is parallel, this implies that

$$P_{2k} = \begin{cases} (\sigma\bar{\sigma})^{-k/2} \left(4I^A I^{\bar{B}} \mathbb{D}_A \mathbb{D}_{\bar{B}}\right)^{k/2} & \text{if } k \text{ is even} \\ 2(\sigma\bar{\sigma})^{-(k-1)/2} \sigma^{-1} I^{\bar{C}} \mathbb{D}_{\bar{C}} \left(4I^A I^{\bar{B}} \mathbb{D}_A \mathbb{D}_{\bar{B}}\right)^{(k-1)/2} & \text{if } k \text{ is odd.} \end{cases} \tag{7.2}$$

Denote by D_1 and D_2 the operators

$$D_1 := 2\sigma^{-1} I^{\bar{C}} \mathbb{D}_{\bar{C}} : \mathcal{E}(w) \rightarrow \mathcal{E}(w - 1),$$

$$D_2 := 4I^A I^{\bar{B}} \mathbb{D}_A \mathbb{D}_{\bar{B}} : \mathcal{E}(w) \rightarrow \mathcal{E}(w - 1).$$

Using the definition of the tractor D -operator and Proposition 3.1 and recalling that $\nabla\sigma\bar{\sigma} = 0$ in the scale θ , we readily compute that, in the scale θ ,

$$D_1 = Yf + ai\nabla_0 - \frac{a^2}{n} P, \tag{7.3}$$

$$D_2 = \sigma\bar{\sigma} \left(-\Delta_b + ai\nabla_0 - \frac{4w(n+w)}{n} P\right) \left(-\Delta_b - ai\nabla_0 - \frac{4w(n+w)}{n} P\right) \tag{7.4}$$

for $a = n + 2w$. Inserting (7.3) and (7.4) into (7.2) and recalling that P_{2k} acts on $\mathcal{E}\left(-\frac{n-k+1}{2}\right)$ yields the desired factorisation in U . Since the factorisation is independent of the choice of σ , the factorisation holds in all of M . \square

Second, we prove the factorisation of the P' -operator and compute the Q' -curvature.

Proposition 7.3. *Let (M^{2n+1}, H, θ) be an embeddable Einstein pseudohermitian manifold. Then the P' -operator is given by*

$$P'_{2n+2} = n! \left(\frac{2}{n}\right)^{n+1} \prod_{\ell=0}^n (-\Delta_b + 2\ell P), \tag{7.5}$$

and the Q' -curvature is given by

$$Q'_{2n+2} = (n!)^2 \left(\frac{4P}{n}\right)^{n+1}. \tag{7.6}$$

Proof. Since θ is Einstein, $[\Delta_b, T] = 0$ and the operator C defined in Subsection 4.2 satisfies $C = \Delta_b^2 + n^2 T^2$. A straightforward computation reveals that, for c_ℓ and d_ℓ as in Proposition 7.1,

$$\begin{aligned} & (-\Delta_b + c_\ell i \nabla_0 + d_\ell P) (-\Delta_b - c_\ell i \nabla_0 + d_\ell P) \\ &= \frac{c_\ell^2}{n^2} C + \frac{d_\ell}{n} (-\Delta_b + (n - c_\ell) P) (-\Delta_b + (n + c_\ell) P). \end{aligned}$$

Inserting this into (7.1) and recalling that $\mathcal{P} \subset \ker C$ yields

$$P_{2k}|_{\mathcal{P}} = \begin{cases} \prod_{\ell=1}^{\frac{k}{2}} \frac{d_\ell}{n} (-\Delta_b + (n - c_\ell) P) (-\Delta_b + (n + c_\ell) P) & \text{if } k \text{ is even} \\ Y \prod_{\ell=1}^{\frac{k-1}{2}} \frac{d_\ell}{n} (-\Delta_b + (n - c_\ell) P) (-\Delta_b + (n + c_\ell) P) & \text{if } k \text{ is odd.} \end{cases}$$

Equivalently, we have that

$$P_{2k}|_{\mathcal{P}} = \prod_{\ell=1}^k \frac{n - k - 1 + 2\ell}{n} (-\Delta_b + (n - k - 1 + 2\ell) P). \tag{7.7}$$

Formally we have that

$$P'_{2n+2} = \left(\frac{2}{n - k + 1} P_{2k}|_{\mathcal{P}} \right)_{k=n+1} \tag{7.8}$$

$$Q'_{2n+2} = \left(\frac{4}{(n - k + 1)^2} P_{2k}(1) \right)_{k=n+1}. \tag{7.9}$$

Inserting (7.7) into (7.8) yields (7.5). Inserting (7.7) into (7.9) yields (7.6). As discussed in Subsection 4.2, this argument is made rigorous via log densities. \square

8. Q' -curvature in dimension five

For five-dimensional CR manifolds, Corollary 6.4 implies the ambient formula

$$-Z_{\bar{C}} Z_A Z_{\bar{B}} \Delta^3 = D_{\bar{C}} D_A D_{\bar{B}} - Z_{\bar{C}} R_{A\bar{B}E}{}^F D_F D^E \tag{8.1}$$

acting on $\mathcal{E}(0)$. Obtaining from this a tractor formula for the sixth-order CR GJMS operator and both the P' -operator and the Q' -curvature in dimension five requires establishing a tractor formula for the restriction of the ambient curvature $R_{A\bar{B}C\bar{E}}$ to \mathcal{F} . This tractor is a multiple of the CR Weyl tractor $S_{A\bar{B}C\bar{E}} \in \mathcal{E}_{A\bar{B}C\bar{E}}(-1)$, a tractor defined in all dimensions which has Weyl-type symmetries and whose projecting part is the CR Weyl curvature $S_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ when $n \geq 2$.

The following proposition defines the CR Weyl tractor in terms of the splitting (3.1) and verifies that it satisfies the correct transformation formula.

Proposition 8.1. *Let (M^{2n+1}, H, θ) be an embeddable pseudohermitian manifold. With respect to θ , the CR Weyl tractor is*

$$S_{A\bar{B}C\bar{E}} = (n - 1)W_A^\alpha W_{\bar{B}}^{\bar{\beta}} \Omega_{\alpha\bar{\beta}C\bar{E}} + (n - 1)W_A^\alpha Z_{\bar{B}} \Phi_{\alpha C\bar{E}} \\ + (n - 1)Z_A W_{\bar{B}}^{\bar{\beta}} \Phi_{\bar{\beta}C\bar{E}} + Z_A Z_{\bar{B}} \Psi_{C\bar{E}},$$

where W_A^α is defined in (3.4), $\Omega_{\alpha\bar{\beta}C\bar{E}}$ is the tractor curvature

$$\Omega_{\alpha\bar{\beta}C\bar{E}} = W_C^\gamma W_{\bar{E}}^{\bar{\sigma}} S_{\alpha\bar{\beta}\gamma\bar{\sigma}} + iW_C^\gamma Z_{\bar{E}} V_{\alpha\bar{\beta}\gamma} - iZ_C W_{\bar{E}}^{\bar{\sigma}} V_{\bar{\sigma}\alpha\bar{\beta}} + Z_C Z_{\bar{E}} U_{\alpha\bar{\beta}}, \quad (8.2)$$

$\Phi_{\alpha C\bar{E}}$ and $\Psi_{C\bar{E}}$ are given by

$$\Phi_{\alpha C\bar{E}} = iW_C^\gamma W_{\bar{E}}^{\bar{\sigma}} V_{\gamma\bar{\sigma}\alpha} + iW_C^\gamma Z_{\bar{E}} Q_{\alpha\gamma} + Z_C W_{\bar{E}}^{\bar{\sigma}} U_{\alpha\bar{\sigma}} + iZ_C Z_{\bar{E}} Y_\alpha, \\ \Psi_{C\bar{E}} = (n - 1)W_C^\gamma W_{\bar{E}}^{\bar{\sigma}} U_{\gamma\bar{\sigma}} + (n - 1)iW_C^\gamma Z_{\bar{E}} Y_\gamma \\ - (n - 1)iZ_C W_{\bar{E}}^{\bar{\sigma}} Y_{\bar{\sigma}} + Z_C Z_{\bar{E}} \mathcal{O},$$

$\Phi_{\bar{\beta}C\bar{E}}$ is given by conjugation, and \mathcal{O} is given by

$$\mathcal{O} = -i\nabla^\gamma Y_\gamma + 2P^{\alpha\bar{\beta}} U_{\alpha\bar{\beta}} + A^{\alpha\gamma} Q_{\alpha\gamma}.$$

Remark 8.2. By the symmetries of the CR Weyl tractor, \mathcal{O} is real-valued. This also follows immediately from (2.19); in dimension three, this observation and (2.17) together recover the Bianchi-type identity for the Cartan tensor discovered by Cheng and Lee [21, Proposition 3.1]. Although \mathcal{O} is not in general a CR invariant, Proposition 8.1 implies that it is a CR invariant for three-dimensional CR manifolds. Indeed, by invariant theory, it must be a nonzero constant multiple of the obstruction function η in (1.10) (cf. [34]). In these ways, we can regard the pseudohermitian invariant \mathcal{O} as an extension of the obstruction function for three-dimensional CR manifolds to higher dimensions.

Proof. Recall that the tractor curvature $\Omega_{\alpha\bar{\beta}C\bar{E}} \in \mathcal{E}_{\alpha\bar{\beta}C\bar{E}}(0)$ is CR invariant and $\Omega_{\gamma C\bar{E}}^\gamma = 0$. The CR Weyl tractor is obtained from the embedding $\mathcal{E}_{(\alpha\bar{\beta})_0}(0) \hookrightarrow \mathcal{E}_{A\bar{B}}(-1)$, where $\mathcal{E}_{(\alpha\bar{\beta})_0}(0)$ is the space of CR invariant trace-free Hermitian $(1, 1)$ -forms of weight zero. One can directly check that

$$M_{A\bar{B}}^{\alpha\bar{\beta}} S_{\alpha\bar{\beta}} := (n + w - 1)(n + w' - 1)W_A^\alpha W_{\bar{B}}^{\bar{\beta}} S_{\alpha\bar{\beta}} - (n + w' - 1)W_A^\alpha Z_{\bar{B}} \nabla^{\bar{\sigma}} S_{\alpha\bar{\sigma}} \\ - (n + w - 1)Z_A W_{\bar{B}}^{\bar{\beta}} \nabla^\gamma S_{\gamma\bar{\beta}} \\ + Z_A Z_{\bar{B}} \left(\nabla^\gamma \nabla^{\bar{\sigma}} S_{\gamma\bar{\sigma}} + (n + w - 1)P^{\gamma\bar{\sigma}} S_{\gamma\bar{\sigma}} \right)$$

is a linear map $M_{A\bar{B}}^{\alpha\bar{\beta}} : \mathcal{E}_{(\alpha\bar{\beta})_0}(w, w') \rightarrow \mathcal{E}_{A\bar{B}}(w - 1, w' - 1)$.

Note that the above formula for $M_{A\bar{B}}^{\alpha\bar{\beta}}$ can be obtained by making the ansatz that

$$M_{A\bar{B}} = W_A^\alpha W_{\bar{B}}^{\bar{\beta}} S_{\alpha\bar{\beta}} + Z_A W_{\bar{B}}^{\bar{\beta}} \omega_{\bar{\beta}} + W_A^\alpha Z_{\bar{B}} \omega_\alpha + Z_A Z_{\bar{B}} \rho$$

is a tractor. Thus $\mathbb{D}^{\bar{B}} M_{A\bar{B}}$ is a tractor, and hence zero for generic values of w and w' . Computing $\mathbb{D}^{\bar{B}} M_{A\bar{B}}$ in components then yields the components for $M_{A\bar{B}}$ in terms of $S_{\alpha\bar{\beta}}$; multiplying by $(n + w - 1)(n + w' - 1)$ to cancel the poles of these components yields our expression for the operator $M_{A\bar{B}}^{\alpha\bar{\beta}}$. In fact, this argument can be modified to give a rigorous proof that $M_{A\bar{B}}^{\alpha\bar{\beta}}$ has the claimed mapping properties (cf. [31]).

Next, applying Lemma 2.2 to (8.2) yields

$$\begin{aligned} & \nabla^{\bar{\beta}} \Omega_{\alpha\bar{\beta}C\bar{E}} \\ &= -(n-1) \left[i W_C^\gamma W_{\bar{E}}^{\bar{\sigma}} V_{\gamma\bar{\sigma}\alpha} + i W_C^\gamma Z_{\bar{E}} Q_{\alpha\gamma} + Z_C W_{\bar{E}}^{\bar{\sigma}} U_{\alpha\bar{\sigma}} + i Z_C Z_{\bar{E}} Y_\alpha \right] \end{aligned}$$

and

$$\begin{aligned} \left(\nabla^\alpha \nabla^{\bar{\beta}} + (n-1) P^{\alpha\bar{\beta}} \right) \Omega_{\alpha\bar{\beta}C\bar{E}} &= (n-1)^2 W_C^\gamma W_{\bar{E}}^{\bar{\sigma}} U_{\gamma\bar{\sigma}} \\ &+ (n-1)^2 i W_C^\gamma Z_{\bar{E}} Y_\gamma - (n-1)^2 i Z_C W_{\bar{E}}^{\bar{\sigma}} Y_{\bar{\sigma}} \\ &+ (n-1) Z_C Z_{\bar{E}} \mathcal{O}. \end{aligned}$$

In particular, we see that $S_{A\bar{B}C\bar{E}} = \frac{1}{n-1} M_{A\bar{B}}^{\alpha\bar{\beta}} \Omega_{\alpha\bar{\beta}C\bar{E}}$ makes sense in all dimensions. Finally, direct inspection shows that the CR Weyl tractor has Weyl-type symmetries, as desired. \square

The relationship between the restriction of the ambient curvature and the CR Weyl tractor can be derived by considering the commutator $[D_A, D_{\bar{B}}]$ and its tractor analogue acting on homogeneous vectors of weight zero and sections of $\mathcal{E}_A(0)$, respectively. The following argument is an adaptation to the CR setting of the argument given from [29, page 369] for the analogous result in conformal geometry.

Lemma 8.3. *Let (M^{2n+1}, H) be an embeddable CR manifold with $n > 1$. Then*

$$S_{A\bar{B}C\bar{E}} = (n-1) R_{A\bar{B}C\bar{E}} \Big|_{\mathcal{F}}.$$

Proof. Let $V^C \in \mathcal{E}^C(0)$ and let $\tilde{V}^C \in \tilde{\mathcal{E}}^C$ be an extension of V^C which is homogeneous of degree zero. It follows from Proposition 6.3 and properties of the ambient connection that

$$\begin{aligned} [D_A, D_{\bar{B}}] \tilde{V}^C &= n(n-1) R_{A\bar{B}E}{}^C \tilde{V}^E \\ &- n Z_A R_{E\bar{B}F}{}^C W^{E\bar{\beta}} \nabla_{\bar{\beta}} \tilde{V}^F - n Z_{\bar{B}} R_{A\bar{E}F}{}^C W^{\bar{E}\alpha} \nabla_\alpha \tilde{V}^F. \end{aligned} \tag{8.3}$$

Since all the operators in (8.3) are tangential, we can restrict to M and regard this as a tractor formula for V^C . On the other hand, a straightforward computation using the definition of the tractor D -operator, the tractor curvature, and Proposition 8.1 yields

$$\begin{aligned}
 [\mathbb{D}_A, \mathbb{D}_{\bar{B}}]V^C &= nS_{A\bar{B}E}{}^C V^E \\
 &\quad - \frac{n}{n-1}Z_A S_{E\bar{B}F}{}^C W^{E\bar{\beta}} \nabla_{\bar{\beta}} V^F \\
 &\quad - \frac{n}{n-1}Z_{\bar{B}} S_{A\bar{E}F}{}^C W^{\bar{E}\alpha} \nabla_{\alpha} V^F.
 \end{aligned}
 \tag{8.4}$$

Multiplying both (8.3) and (8.4) by $Z_G Z_{\bar{H}}$ and skewing over the pairs (A, G) and (\bar{B}, \bar{H}) yields $(n-1)Z_{[G} Z_{\bar{H}} R_{A\bar{B}]E\bar{C}} = Z_{[G} Z_{\bar{H}} S_{A\bar{B}]E\bar{C}}$. Contracting $W^{G\bar{\beta}}$ into this yields $(n-1)Z_A Z_{[\bar{H}} R_{G\bar{B}]E\bar{C}} W^{G\bar{\beta}} = Z_A Z_{[\bar{H}} S_{G\bar{B}]E\bar{C}} W^{G\bar{\beta}}$, where our notation means skew over the pair (\bar{B}, \bar{H}) . Now multiplying both (8.3) and (8.4) by $Z_{\bar{H}}$ and skewing over the pair (\bar{B}, \bar{H}) yields $(n-1)Z_{[\bar{H}} R_{A\bar{B}]E\bar{C}} = Z_{[\bar{H}} S_{A\bar{B}]E\bar{C}}$. Contracting with $W^{\bar{H}\gamma}$ yields $(n-1)Z_{\bar{B}} R_{A\bar{H}E\bar{C}} W^{\bar{H}\gamma} = Z_{\bar{B}} S_{A\bar{H}E\bar{C}} W^{\bar{H}\gamma}$. Using this and its conjugate to compare (8.3) and (8.4) yields the desired result. \square

Considering Lemma 8.3 in the case of dimension five and using (8.1) yields the following tractor formulae for the sixth-order CR GJMS operator, the P' -operator, and the Q' -curvature.

Proposition 8.4. *Let (M, H) be a five-dimensional CR manifold. Then*

$$\frac{1}{8}Z_{\bar{C}}Z_A Z_{\bar{B}} P_6 f = \mathbb{D}_{\bar{C}}\mathbb{D}_A \mathbb{D}_{\bar{B}} f - Z_{\bar{C}} S_{A\bar{B}E}{}^F \mathbb{D}_F \mathbb{D}^E f
 \tag{8.5}$$

for all $f \in \mathcal{E}(0)$. Moreover, given a choice of contact form $\theta = (\sigma \bar{\sigma})^{-1} \theta$, it holds that

$$\frac{1}{8}Z_{\bar{C}}Z_A Z_{\bar{B}} P' u = \mathbb{D}_{\bar{C}} K_{A\bar{B}}(u) - Z_{\bar{C}} S_{A\bar{B}E}{}^F K_F{}^E(u),
 \tag{8.6}$$

$$\frac{1}{8}Z_{\bar{C}}Z_A Z_{\bar{B}} Q' = \mathbb{D}_{\bar{C}} I_{A\bar{B}} - Z_{\bar{C}} S_{A\bar{B}E}{}^F I_F{}^E
 \tag{8.7}$$

for all $u \in \mathcal{P}$, where $K_{A\bar{B}}(u)$ and $I_{A\bar{B}}$ are as in Lemma 4.3 and Lemma 4.5, respectively, and we require that θ is pseudo-Einstein in (8.7).

Remark 8.5. Since all formulas in Proposition 8.4 depend only on finite jets of the geometry, we can approximate our CR manifold by a real analytic manifold with the same result. The latter manifolds are always embeddable, and thus we need not assume the original CR manifold is itself embeddable in this result.

Using Proposition 8.1 and the formula for the tractor D -operator, one could derive local formulae for the sixth-order CR GJMS operator P_6 in general dimensions as well as the P' -operator and the Q' -curvature for five-dimensional CR manifolds. Here we derive local formulae for the P' -operator and Q' -curvature of a pseudo-Einstein five-manifold. First, we consider the Q' -curvature.

Corollary 8.6. *Let (M, H, θ) be a five-dimensional pseudo-Einstein manifold. Then*

$$\begin{aligned} \frac{1}{8} Q' &= \frac{1}{2} \Delta_b^2 P + \frac{1}{2} \Delta_b |A_{\alpha\beta}|^2 - 2 \operatorname{Im} \nabla^\gamma (A_{\beta\gamma} \nabla^\beta P) \\ &\quad - 2 \Delta_b P^2 - 4P |A_{\alpha\beta}|^2 + 4P^3 - 2\mathcal{O}. \end{aligned} \tag{8.8}$$

In particular, the total Q' -curvature of a compact pseudo-Einstein five-manifold is

$$\begin{aligned} &\int_{M^5} Q' \theta \wedge (d\theta)^2 \\ &= 16 \int_{M^5} \left(2P^3 - 2P |A_{\alpha\beta}|^2 - S_{\alpha\bar{\beta}\gamma\bar{\delta}} A^{\alpha\gamma} A^{\bar{\beta}\bar{\delta}} - |V_{\alpha\bar{\beta}\gamma}|^2 \right) \theta \wedge (d\theta)^2. \end{aligned} \tag{8.9}$$

Proof. The local formula (8.8) follows from a straightforward computation using Proposition 8.1 and Proposition 8.4. Lemma 2.2 and the definition of \mathcal{O} imply that

$$\int_{M^5} \mathcal{O} \theta \wedge (d\theta)^2 = \int_{M^5} \left(S_{\alpha\bar{\beta}\gamma\bar{\delta}} A^{\alpha\gamma} A^{\bar{\beta}\bar{\delta}} + |V_{\alpha\bar{\beta}\gamma}|^2 \right) \theta \wedge (d\theta)^2,$$

from which (8.9) readily follows. □

Second, we consider the P' -operator on pseudo-Einstein manifolds. Note that while the formula below can be derived from Proposition 8.1 and Proposition 8.4, the derivation is simplified using (8.8) and the transformation formula (1.2) for the Q' -curvature.

Corollary 8.7. *Let (M, H, θ) be a five-dimensional pseudo-Einstein manifold. Then*

$$\begin{aligned} P' \Upsilon &= -2 \Delta_b^3 \Upsilon + 24 \operatorname{Re} \Delta_b \nabla^\gamma (P \nabla_\gamma \Upsilon) + 24 \operatorname{Re} \nabla^\gamma (P \nabla_\gamma \Delta_b \Upsilon) \\ &\quad + 8 \operatorname{Im} \Delta_b \nabla^\gamma (A_{\beta\gamma} \nabla^\beta \Upsilon) + 8 \operatorname{Im} \nabla^\gamma (A_{\beta\gamma} \nabla^\beta \Delta_b \Upsilon) \\ &\quad - 16 \operatorname{Re} \nabla^\beta \nabla^\gamma (P \nabla_\gamma \nabla_\beta \Upsilon) \\ &\quad - 16 \operatorname{Re} \nabla^\gamma \left[\left(2U_\gamma{}^\beta + 2A_{\gamma\mu} A^{\mu\beta} + \left(\Delta_b P - 4P^2 - |A_{\alpha\delta}|^2 \right) h_{\gamma}{}^\beta \right) \nabla_\beta \Upsilon \right] \\ &\quad - 64 \operatorname{Im} \nabla^\gamma (P A_{\beta\gamma} \nabla^\beta \Upsilon) \end{aligned}$$

for all $\Upsilon \in \mathcal{P}$.

Remark 8.8. Note that this formula for the P' -operator is manifestly formally self-adjoint. In particular, Corollary 8.7 and the transformation formula (1.2) give an intrinsic proof of the fact that the total Q' -curvature is a global invariant in dimension five.

Proof. Since $\Upsilon \in \mathcal{P}$ and $U_{\gamma\bar{\beta}} + A_{\gamma\mu}A^{\mu\bar{\beta}} - \frac{1}{2}|A_{\alpha\bar{\sigma}}|^2h_{\gamma\bar{\beta}}$ is trace-free, Lemma 2.2 implies that

$$\begin{aligned} & \nabla^\gamma \left(\left(U_{\gamma\bar{\beta}} + A_{\gamma\mu}A^{\mu\bar{\beta}} - \frac{1}{2}|A_{\alpha\bar{\sigma}}|^2h_{\gamma\bar{\beta}} \right) \nabla_\beta \Upsilon \right) \\ &= \left(iY^\gamma - iA^{\beta\gamma} \nabla_\beta P + \frac{1}{2} \nabla^\gamma |A_{\alpha\bar{\sigma}}|^2 \right) \nabla_\gamma \Upsilon. \end{aligned} \tag{8.10}$$

Using the commutator identities [44, Lemma 2.2] and the assumption $\Upsilon \in \mathcal{P}$, we compute that

$$\nabla^\gamma \nabla_\gamma \nabla_\beta \Upsilon = \frac{3}{2} \nabla_\beta \Delta_b \Upsilon + 3i A_{\beta\gamma} \nabla^\gamma \Upsilon + 3P \nabla_\beta \Upsilon. \tag{8.11}$$

Consider now the family $\widehat{\theta}_t = e^{t\Upsilon}\theta$ of pseudo-Einstein contact forms. In the following, we shall use hats to denote pseudohermitian invariants defined in terms of $\widehat{\theta}_t$ and suppress the dependence on t in our notation. It follows from (1.2) that

$$P'(\Upsilon) = \frac{\partial}{\partial t} \Big|_{t=0} e^{3t\Upsilon} \widehat{Q}'. \tag{8.12}$$

The right-hand side of (8.12) is readily expanded using the identities

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} e^{t\Upsilon} \widehat{P} &= -\frac{1}{2} \Delta_b \Upsilon, \\ \frac{\partial}{\partial t} \Big|_{t=0} \widehat{A}_{\alpha\beta} &= i \nabla_\alpha \nabla_\beta \Upsilon, \\ \frac{\partial}{\partial t} \Big|_{t=0} e^{2t\Upsilon} \widehat{O} &= -4 \operatorname{Im} Y^\gamma \nabla_\gamma \Upsilon, \\ \frac{\partial}{\partial t} \Big|_{t=0} e^{(1-w)t\Upsilon} \nabla^\gamma (e^{wt\Upsilon} \omega_\gamma) &= (w+2)\omega_\gamma \nabla^\gamma \Upsilon \end{aligned}$$

for all $\omega_\gamma \in \mathcal{E}_\gamma$ and all $w \in \mathbb{R}$ (cf. [28, 44]). Using (8.10) and (8.11) to simplify the resulting expansion yields the desired formula. \square

It is interesting to compare the total Q' -curvature (8.9) to the other known and interesting global secondary invariant in dimension five, namely the Burns-Epstein invariant [11]. Marugame [47] computed the Burns-Epstein invariant $\mu(M)$ of the boundary M of a strictly pseudoconvex bounded domain $X \subset \mathbb{C}^3$, showing that

$$\begin{aligned} & \mu(M) \\ &= -\frac{1}{16\pi^3} \int_M \left(2P^3 - 2P|A_{\alpha\beta}|^2 - S_{\alpha\bar{\beta}\gamma\bar{\delta}} A^{\alpha\gamma} A^{\bar{\beta}\bar{\delta}} + \frac{1}{2} P |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2 \right) \theta \wedge (d\theta)^2, \end{aligned} \tag{8.13}$$

while the Burns-Epstein invariant is related to the Euler characteristic of X via the formula

$$\chi(X) = \int_X \left(c_3 - \frac{1}{2} c_1 c_2 + \frac{1}{8} c_1^3 \right) + \mu(M). \tag{8.14}$$

Indeed, one can regard the formula (8.13) as defining a global pseudohermitian invariant $\mu(M^5)$. By realizing M as the boundary of a complex manifold, Marugame gave an extrinsic proof that $\mu(M)$ is a global secondary invariant [47].

Direct comparison of (8.9) and (8.13) implies both Theorem 1.3 and the fact that $\int |V_{\alpha\bar{\beta}\gamma}|^2 + \frac{1}{2}P|S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2$ is a secondary invariant. We here give an intrinsic proof of the latter fact under the additional assumption that $c_2(H^{1,0})$ vanishes in $H^4(M; \mathbb{R})$ by studying properties of the pseudohermitian invariant \mathcal{I}' .

Proposition 8.9. *Let (M, H, θ) be a five-dimensional pseudohermitian manifold and define*

$$\mathcal{I}' = -\frac{1}{8}\Delta_b |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2 + |V_{\alpha\bar{\beta}\gamma}|^2 + \frac{1}{2}P |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2, \tag{8.15}$$

$$X_\alpha = -iS_{\alpha\bar{\rho}\gamma\bar{\sigma}}V^{\bar{\rho}\gamma\bar{\sigma}} + \frac{1}{4}\nabla_\alpha |S_{\gamma\bar{\sigma}\delta\bar{\rho}}|^2. \tag{8.16}$$

Suppose $\widehat{\theta} = e^\Upsilon\theta$. Then

$$e^{3\Upsilon}\widehat{\mathcal{I}}' = \mathcal{I}' + 2\operatorname{Re} X^\Upsilon\nabla_\Upsilon\Upsilon. \tag{8.17}$$

Moreover, if M is compact, $c_2(H^{1,0})$ vanishes in $H^4(M; \mathbb{R})$, and both θ and $\widehat{\theta}$ are pseudo-Einstein, then

$$\int_M \widehat{\mathcal{I}}'\widehat{\theta} \wedge (d\widehat{\theta})^2 = \int_M \mathcal{I}'\theta \wedge (d\theta)^2.$$

The proof of Proposition 8.9 depends on an explicit realisation of the real Chern class $c_2(H^{1,0}) \in H^4(M; \mathbb{R})$ and the observation that

$$\mathcal{S}_{\alpha\bar{\beta}} := S_{\alpha\bar{\rho}\gamma\bar{\sigma}}S^{\bar{\rho}}_{\bar{\beta}}{}^{\bar{\sigma}\gamma} - \frac{1}{n} |S_{\gamma\bar{\sigma}\delta\bar{\rho}}|^2 h_{\alpha\bar{\beta}} \tag{8.18}$$

vanishes in dimension five. Indeed, if $\dim_{\mathbb{C}} H^{1,0} = 2$, then

$$0 = h_{[\alpha\bar{\beta}}S_{\gamma}{}^\Upsilon{}_{|\mu|}{}^\nu S_{\rho]}{}^\rho{}_\nu{}^\mu = 2\mathcal{S}_{\alpha\bar{\beta}},$$

where our notation means that we skew over the lower indices α, γ, ρ .

Define

$$\xi = X_\alpha\theta \wedge d\theta \wedge \theta^\alpha + X_{\bar{\beta}}\theta \wedge d\theta \wedge \theta^{\bar{\beta}}. \tag{8.19}$$

The vanishing of $\mathcal{S}_{\alpha\bar{\beta}}$ enables us to identify ξ as an element of $4\pi^2c_2(H^{1,0})$ on any five-dimensional pseudo-Einstein manifold.

Lemma 8.10. *Let (M, H, θ) be a five-dimensional pseudo-Einstein manifold and let ξ be as in (8.19). Then ξ is a representative of $c_2(H^{1,0}) \in H^4(M; \mathbb{R})$. In particular, if the real Chern class $c_2(H^{1,0})$ vanishes, then*

$$\operatorname{Re} \int_M X^\gamma \nabla_\gamma v \theta \wedge (d\theta)^2 = 0$$

for all $v \in \mathcal{P}$.

Proof. Observe that $\xi \wedge \theta = 0 = \xi \wedge d\theta$. Suppose that ξ is exact. As observed by Rumin [48], we obtain a three-form α such that $d\alpha = \xi$ and $\alpha \wedge \theta = 0 = \alpha \wedge d\theta$. Denote $d_b^c v = -i\nabla_\alpha v \theta^\alpha + i\nabla_{\bar{\beta}} v \theta^{\bar{\beta}}$ and observe that $d_b^c v \wedge \xi = 2\operatorname{Re} X^\gamma \nabla_\gamma v$. Since $v \in \mathcal{P}$ if and only if $dd_b^c v = 0 \pmod{d\theta}$ (cf. [44]), we conclude that if $v \in \mathcal{P}$, then

$$2\operatorname{Re} X^\gamma \nabla_\gamma v = d_b^c v \wedge \xi = -d(d_b^c v \wedge \alpha).$$

In particular, $\operatorname{Re} \int X^\gamma \nabla_\gamma v = 0$.

We now show that ξ is exact. It suffices to show that $\xi \in 4\pi^2 c_2(H^{1,0})$.

Since θ is pseudo-Einstein, $c_1(H^{1,0})$ vanishes in $H^2(M; \mathbb{R})$ (cf. [44]). It follows that $8\pi^2 c_2(H^{1,0}) = [\Pi_\mu{}^\nu \wedge \Pi_\nu{}^\mu]$ for $\Pi_\alpha{}^\beta$ the curvature forms (2.2). Since $\dim_{\mathbb{C}} H^{1,0} = 2$, we compute that

$$\begin{aligned} \Pi_\mu{}^\nu \wedge \Pi_\nu{}^\mu &= R_{\alpha\bar{\beta}\mu}{}^\nu R_{\gamma\bar{\sigma}\nu}{}^\mu \theta^\alpha \wedge \theta^{\bar{\beta}} \wedge \theta^\gamma \wedge \theta^{\bar{\sigma}} - 2R_{\alpha\bar{\beta}\mu}{}^\nu \nabla^\mu A_{\gamma\nu} \theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}} \wedge \theta^\gamma \\ &\quad + 2R_{\alpha\bar{\beta}\mu}{}^\nu \nabla_\nu A_{\bar{\sigma}}{}^\mu \theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}} \wedge \theta^{\bar{\sigma}} - 2i A_{\bar{\sigma}}{}^\nu \nabla_{\bar{\beta}} A_{\gamma\nu} \theta \wedge \theta^\gamma \wedge \theta^{\bar{\beta}} \wedge \theta^{\bar{\sigma}} \\ &\quad - 2i A_{\alpha\mu} \nabla_\gamma A_{\bar{\sigma}}{}^\mu \theta \wedge \theta^\alpha \wedge \theta^\gamma \wedge \theta^{\bar{\sigma}} - 2A_{\alpha\mu} A_{\bar{\rho}}{}^\mu h_{\nu\bar{\sigma}} \theta^\alpha \wedge \theta^{\bar{\sigma}} \wedge \theta^\nu \wedge \theta^{\bar{\rho}}. \end{aligned}$$

To simplify this, observe that, since $\dim_{\mathbb{C}} H^{1,0} = 2$ and $\nabla_\rho A_{\alpha\gamma} = \nabla_\gamma A_{\alpha\rho}$ (cf. [44]),

$$\begin{aligned} d(A_{\alpha\mu} A_{\bar{\beta}}{}^\mu \theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}}) &= A_{\alpha\mu} \nabla_\gamma A_{\bar{\beta}}{}^\mu \theta \wedge \theta^\alpha \wedge \theta^\gamma \wedge \theta^{\bar{\beta}} \\ &\quad + A_{\bar{\sigma}}{}^\mu \nabla_{\bar{\beta}} A_{\alpha\mu} \theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}} \wedge \theta^{\bar{\sigma}} + A_{\alpha\mu} A_{\bar{\beta}}{}^\mu d\theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}}. \end{aligned} \tag{8.20}$$

Furthermore, since $\dim_{\mathbb{C}} H^{1,0} = 2$ and θ is pseudo-Einstein,

$$R_{\alpha\bar{\beta}\mu}{}^\nu R_{\gamma\bar{\sigma}\nu}{}^\mu \theta^\alpha \wedge \theta^{\bar{\beta}} \wedge \theta^\gamma \wedge \theta^{\bar{\sigma}} = \left(\frac{1}{2} |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2 - 3P^2 \right) d\theta \wedge d\theta, \tag{8.21}$$

$$R_{\alpha\bar{\beta}\mu}{}^\nu \nabla^\mu A_{\gamma\nu} \theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}} \wedge \theta^\gamma = \left(i S_{\gamma\bar{\sigma}\mu\bar{\nu}} V^{\bar{\sigma}\mu\bar{\nu}} - \frac{3}{2} \nabla_\gamma (P^2) \right) \theta \wedge d\theta \wedge \theta^\gamma. \tag{8.22}$$

Using (8.20), (8.21) and (8.22) to simplify the expression for $\Pi_\mu^\nu \wedge \Pi_\nu^\mu$ yields

$$\Pi_\mu^\nu \wedge \Pi_\nu^\mu = 2\xi + d \left[\left(\frac{1}{2} |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2 - 3P^2 \right) \theta \wedge d\theta - 2i A_{\alpha\mu} A^\mu_{\bar{\beta}} \theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}} \right].$$

In particular, we see that $4\pi^2 c_2(H^{1,0}) = [\xi]$, as desired. □

Proof of Proposition 8.9. Let (M^{2n+1}, θ) be a pseudohermitian manifold and define X_α by

$$X_\alpha = -i S_{\alpha\bar{\rho}\gamma\bar{\sigma}} V^{\bar{\rho}\gamma\bar{\sigma}} + \frac{1}{2n} \nabla_\alpha |S_{\gamma\bar{\sigma}\delta\bar{\rho}}|^2.$$

A straightforward computation using the transformation formulae in [28,44] yields that if $\hat{\theta} = e^\Upsilon \theta$, then

$$e^{2\Upsilon} \hat{X}_\alpha = X_\alpha + S_{\alpha\bar{\beta}} \nabla^\beta \Upsilon.$$

It follows immediately that X_α is a CR invariant when $n = 2$. This observation and the transformation formulae in [28,44] readily yield (8.17). The final conclusion now follows from Lemma 8.10. □

One important class of CR manifolds are those which embed as boundaries of Stein domains. As pointed out to us by Taiji Marugame, five-dimensional CR manifolds in this class always have vanishing Chern classes.

Proposition 8.11. *Let (M, H, θ) be a five-dimensional pseudo-Einstein manifold which is the boundary of a Stein manifold V . Then $c_2(H^{1,0}) = 0$.*

Proof. Note that the holomorphic normal bundle $N^{1,0}$ of $M \subset V$ is trivial and $H^{1,0} \oplus N^{1,0} = i^* T^{1,0} V$, the pullback of $T^{1,0} V$ with respect to the inclusion $i: M \rightarrow V$. Thus

$$c_2(H^{1,0}) = c_2(H^{1,0} \oplus N^{1,0}) = c_2(i^* T^{1,0} V).$$

By naturality of the Chern classes,

$$c_2(i^* T^{1,0} V) = i^* c_2(T^{1,0} V).$$

Since $\dim_{\mathbb{C}} V = 3$ and V is Stein, it holds that $H^4(V; \mathbb{C}) = 0$ (see [49]). In particular, $c_2(T^{1,0} V) = 0$. □

We conclude this article with two remarks about possible interpretations of the pseudohermitian invariant \mathcal{I}' .

Remark 8.12. \mathcal{I}' can be informally regarded as the “primed analogue” of the conformal invariant $|\nabla_A R_{BC EF}|^2$ discovered by Fefferman and Graham [24], in the same spirit as the P' -operator is the “primed analogue” of the Paneitz operator. Indeed, on a pseudohermitian manifold (M, H, θ) , define

$$\begin{aligned} \mathcal{I} = \nabla^\gamma \left(i S_{\gamma\bar{\sigma}\alpha\bar{\beta}} V^{\bar{\sigma}\alpha\bar{\beta}} - \frac{1}{2n} \nabla_\gamma |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2 \right) - S^{\alpha\bar{\beta}} P_{(\alpha\bar{\beta})_0} \\ + (n-2) \left(\frac{1}{2n(n-4)} \Delta_b |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2 + |V_{\alpha\bar{\beta}\gamma}|^2 - \frac{2}{n(n-4)} P |S_{\alpha\bar{\beta}\gamma\bar{\sigma}}|^2 \right). \end{aligned} \tag{8.23}$$

Set

$$K_{C\bar{E}} = \Omega_{\alpha\bar{\beta}C\bar{F}} \Omega^{\bar{\beta}\alpha\bar{F}}_{\bar{E}} \in \mathcal{E}_{C\bar{E}}(-2).$$

It is straightforward to compute that

$$\frac{1}{n-3} D^C K_{C\bar{E}} - \frac{1}{2(n-4)} D_{\bar{E}} \left| S_{\alpha\bar{\beta}\gamma\bar{\sigma}} \right|^2 = \mathcal{I} Z_{\bar{E}} \in \mathcal{E}_{\bar{E}}(-2, -3). \tag{8.24}$$

In particular, it follows that (8.23) defines a CR invariant of weight $(-3, -3)$. (Strictly speaking, it is $(n-4)$ -times the right-hand side of (8.23) which is a CR invariant in all dimensions, and the verification that this is a CR invariant when $n = 3$ – and hence the tractor expression (8.24) has a pole – follows from either direct computation or analytic continuation in the dimension.) The invariant \mathcal{I} is defined via a tractor expression equivalent to the tractor expression giving the Fefferman-Graham invariant (cf. [13]). Restricting to pseudo-Einstein metrics, one observes that, modulo divergences, $\mathcal{I} = (n-2)\mathcal{I}' + O((n-2)^2)$. This motivates the definition of \mathcal{I}' .

Remark 8.13. Alexakis [1] proved that any local Riemannian invariant $I(g)$ for which $\int I(g) \text{dvol}_g$ is a conformal invariant admits a decomposition

$$I(g) = cQ_g + (\text{local conformal invariant}) + (\text{divergence}).$$

Hirachi conjectured [39, page 242] that any local pseudohermitian invariant $I(\theta)$ for which $\int I(\theta) \theta \wedge d\theta^n$ is a secondary invariant should admit a similar decomposition in terms of a constant multiple of the Q' -curvature, a local CR invariant, and a divergence. It seems to us that the \mathcal{I}' -invariant, through the following two questions, provides a new insight into this conjecture.

First, is there a five-dimensional pseudo-Einstein manifold for which the CR invariant one-form X_γ is nonzero? If so, then \mathcal{I}' is not a local secondary invariant, and thus provides a counterexample to Hirachi’s conjecture. If not, then Hirachi’s conjecture seems correct, at least in dimension five and after modifying it to allow local secondary invariants.

Second, how can one understand the transformation formula (8.17)? Specifically, observe that the proof of Lemma 8.10 shows that if $\xi = d\alpha$ for $\alpha = i\Omega_{\alpha\bar{\beta}}\theta \wedge \theta^\alpha \wedge \theta^{\bar{\beta}}$ and $\Omega_{\alpha\bar{\beta}} \in \mathcal{E}_{(\alpha\beta)_0}$ – that is, if α can be chosen to be an element of $F^{2,1}$ in the graded Rumin complex [26] – then the map $\mathcal{P} \ni v \mapsto \operatorname{Re} X^\gamma \nabla_\gamma v$ is formally self-adjoint on the space of CR pluriharmonic functions. Thus, one might suspect that the transformation formula for \mathcal{L}' is governed by a formally self-adjoint operator on CR pluriharmonic functions, a property shared by the Q' -curvature.

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