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Co-axial monodromy

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Abstract. For Riemannian metrics of constant positive curvature on a punctured sphere with conic singularities at the punctures and co-axial monodromy of the developing map, possible angles at the singularities are completely described. This completes the recent result of Mondello and Panov.

The related problem of describing possible multiplicities of critical points of logarithmic potentials of finitely many charges is also solved.

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1. Introduction

Existence of a Riemannian metric of constant positive curvature in a given conformal class on a punctured sphere with conic singularities at the punctures and prescribed angles at the singularities is an important problem, but at present the solution in this generality seems to be out of reach, see for example the surveys in the introductions of [6,10].

Mondello and Panov [10] proposed a reduced problem: to describe possible angles at the singularities for such metrics. (The conformal class is not prescribed.) They solved this problem for generic angles. In this paper their solution is completed by the study of the remaining case which was excluded in [10].

The results in [10] are the following. Let

$$\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}, \quad \alpha_j > 0, \quad \alpha_j \neq 1 \tag{1.1}$$

be the set of angles. *We measure all angles in turns:* 1 *turn is* 2π *radians*. Strictly speaking, $\boldsymbol{\alpha}$ is an unordered *multiset*; some elements can be repeated. Alternatively, $\boldsymbol{\alpha}$ is an element of the n-th symmetric power of $R_{>0}$. Whenever it is convenient, we list non-integer α_j's first, followed by integer α_j's.

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The restrictions are: the Gauss-Bonnet theorem,

$$\sum_{j=1}^{n}(\alpha_j - 1) + 2 > 0, \tag{1.2}$$

and

$$d_1(\mathbf{Z}_o^n, \boldsymbol{\alpha} - \mathbf{1}) \geq 1, \tag{1.3}$$

where $\mathbf{1} = (1, \ldots, 1)$, and $\mathbf{Z}_o^n$ is the subset of the integer lattice consisting of vectors with odd sums of coordinates, and d_1 is the ℓ_1 distance.

Mondello and Panov proved that these two conditions are always necessary, and if one replaces (1.3) by the strict inequality, they also become sufficient. So to complete their description, it remains to investigate the case of equality in (1.3):

$$d_1(\mathbf{Z}_o^n, \boldsymbol{\alpha} - \mathbf{1}) = 1. \tag{1.4}$$

Moreover, they proved that every metric satisfying (1.4) is *co-axial*, which means that the monodromy group of the developing map is a subgroup of the unit circle. This gives a motivation for study of metrics with co-axial monodromy. Here and in what follows, S is the Riemann sphere (compact simply-connected Riemann surface). The sphere with the standard spherical metric will be denoted by $\overline{\mathbf{C}}$. In general, co-axial monodromy does not imply (1.4).

We say that a multiset (1.1) is *admissible* if there exists a metric of constant curvature 1 on $S\backslash\{n \text{ points}\}$ with conic singularities at these n points with angles α_j, and the developing map of this metric has co-axial monodromy. Our main result is:

Theorem 1.1. *For a multiset* (1.1) *assume that* $\alpha_{m+1}, \ldots, \alpha_n$ *are integers while* $\alpha_1, \ldots, \alpha_m$ *are not integers. For* $\boldsymbol{\alpha}$ *to be admissible it is necessary that there exist a choice of signs* $\epsilon_j \in \{\pm 1\}$ *and a non-negative integer* k' *such that*

$$\sum_{j=1}^{m} \epsilon_j \alpha_j = k', \tag{1.5}$$

and the number

$$k'' := \sum_{j=m+1}^{n} \alpha_j - n - k' + 2 \quad \textit{is non-negative and even.} \tag{1.6}$$

If the coordinates of the vector

$$\boldsymbol{c} := (\alpha_1, \ldots, \alpha_m, \underbrace{1, \ldots, 1}_{k'+k'' \text{ times}}) \tag{1.7}$$

are incommensurable, then (1.5) *and* (1.6) *are also sufficient.*

If $\boldsymbol{c} = \eta \boldsymbol{b}$, *where coordinates of* $\boldsymbol{b}$ *are integers whose greatest common factor is* 1, *then there is an additional necessary condition*

$$2 \max_{m+1 \le j \le n} \alpha_j \le \sum_{j=1}^{q} |b_j|, \quad \textit{where} \quad q := m + k' + k''. \tag{1.8}$$

Conditions (1.5), (1.6), (1.8) *are sufficient for* $\boldsymbol{\alpha}$ *to be admissible.*

As a corollary we mention that unless $n = 2$, a co-axial metric must have some integer angles whose sum is at least $n + k' - 2$ and has the same parity as $n + k'$, where k' is a number that satisfies (1.5) with some choice of signs.

The plan of the paper is the following. In Section 2 we explain some preliminaries, reduce Theorem 1.1 to a problem of potential theory (Question 2.3 below), state the answer to this question (Theorem 2.4), and give several examples. Section 3 contains the proof of the main technical result, Theorem 2.4. In Section 4, we complete the proof of Theorem 1.1 and end with a discussion of the results.

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2. Reduction to a question of potential theory

We briefly recall the definition of the developing map of a metric of constant positive curvature. Start with a small region in $S\backslash\{\text{singularities}\}$. It is well-known that there is an isometry from this region to a region in the standard sphere $\overline{\mathbf{C}}$. This isometry is conformal and thus analytic, and it admits an analytic continuation along every curve which does not pass through the singularities. We obtain a multi-valued function $f : S\backslash\{\text{singularities}\} \to \overline{\mathbf{C}}$ (or a genuine function on the universal covering) which is called the developing map. Conic nature of the singularities means that $f(z) = f(a) + (c + o(1))z^{\alpha}$ near a singularity a with angle α, or $f(z) = (c + o(1))z^{-\alpha}$ if $f(a) = \infty$, where z is a local conformal coordinate which equals 0 at a, and α is the angle, $f(a)$ means the radial limit when $z \to 0$, and $c \neq 0$. The result f_γ of an analytic continuation of f along a closed path γ not passing through the singularities is related to the original germ of f by $f_\gamma = \phi \circ \gamma$, where ϕ is an isometry of $\overline{\mathbf{C}}$, so we obtain a representation of the fundamental group of $S\backslash\{\text{singularities}\}$ in the group of linear-fractional transformations. The image of this representation is called the monodromy group, and the developing map and the metric are called co-axial if this monodromy group is a subgroup of the unit circle. See [5,6,10].

Suppose now that $f : S \to \overline{\mathbf{C}}$ is the developing map of a metric with co-axial monodromy. Monodromy group consists of transformations $w \mapsto \lambda w,\ |\lambda| = 1.$

Then the meromorphic 1-form df/f is well defined on the sphere, so

$$R = f'/f \tag{2.1}$$

is a rational function. We assume without loss of generality that $\infty \in S$ is not a pole of $R(z)dz$, so that R has a zero of order two at ∞. From the local considerations we see that every pole of R is simple, the residue β is real, and every pole of R is a conic singularity of the metric with the angle $\alpha_j = |\beta|$, unless $\beta = \pm 1$. In the last case the pole of R is a non-singular point of the metric. Moreover, a zero of R in $\mathbf{C}$ of multiplicity r is a conic singularity with the angle $\alpha_j = r+1$. As R has a double zero at infinity, the number of zeros (counting multiplicity) in $\mathbf{C}$ is $q-2$, where q is the number of poles.

Thus

$$R(z) = \sum_{j=1}^{m} \frac{\epsilon_j \alpha_j}{z-a_j} + \sum_{j=1}^{k} \frac{\delta_j}{z-b_j}, \tag{2.2}$$

where $\epsilon_j \in \{\pm 1\}$, $\delta_j \in \{\pm 1\}$, and R has zeros in $\mathbf{C}$ whose sum of multiplicities is $q-2$, $q = m+k$, and these multiplicities are $\alpha_j - 1$ for $m+1 \leq j \leq n$. The developing map itself is thus given by

$$f(z) = \prod_{j=1}^{q} (z-z_j)^{\beta_j}, \tag{2.3}$$

where $\beta_j = \epsilon_j \alpha_j$ for $1 \leq j \leq m$ and $\beta_j = \delta_{j-m}$ for $m+1 \leq j \leq q$. The condition

$$\sum_{j=1}^{m} \epsilon_j \alpha_j + \sum_{j=1}^{k} \delta_j = 0 \tag{2.4}$$

holds by the residue theorem. As the number of zeros of R in $\overline{\mathbf{C}}$ must be $m+k-2$, and each zero is a singularity of the metric, we obtain

$$\sum_{j=m+1}^{n} (\alpha_j - 1) = m+k-2. \tag{2.5}$$

So a necessary condition for $\boldsymbol{\alpha}$ to be admissible is:

Condition 2.1. There exists a partition $\boldsymbol{\alpha} = A \cup B$ into two sub-multisets $A = \{\alpha_1, \ldots, \alpha_m\}$ and $B = \{\alpha_{m+1}, \ldots, \alpha_n\}$ so that:

- All elements of B are integers;
- There exist an integer k, and a choice of signs $\epsilon_j \in \{\pm 1\}$, $1 \leq j \leq m$ and $\delta_j \in \{\pm 1\}$, $1 \leq j \leq k$, such that (2.4) and (2.5) hold.

Proposition 2.2. *Condition* 2.1 *implies* (1.5) *and* (1.6).

Proof. Condition 2.1 coincides with (1.5) and (1.6) when A contains no integers. In this case we have $k = k' + k''$.

If for some $\boldsymbol{\alpha}$ a partition A, B and numbers k, ϵ_j, δ_j as required by Condition 2.1 exist, then there exists another partition A', B' with the same properties and with the additional property that A' contains no integers.

Indeed, suppose that $A = \{\alpha_1, \dots, \alpha_m\}$ and α_m is an integer. Then define $A' = \{\alpha_1, \dots, \alpha_{m-1}\}$ and $B' = \{\alpha_m, \dots, \alpha_n\}$. To restore (2.4) we must add α_m of δ_j's equal to $-\epsilon_m$; this increases k to $k^* = k + \alpha_m$ and decreases m to $m^* = m - 1$, so the total increases in the right- and left-hand sides of (2.5) are equal, so this condition (2.5) is satisfied for the new partition A', B'. We repeat this procedure until all integer angles are removed from A. In the special case when all angles are integers, A will be empty. This proves the proposition and necessity of conditions (1.5) and (1.6) in Theorem 1.1. □

We will call such partitions where A consists of all non-integer angles of $\boldsymbol{\alpha}$ *reduced*. In a reduced partition of $\boldsymbol{\alpha}$, m is the number of non-integer angles, and the only reason why a reduced partition may be non-unique is that different choices of signs ϵ_j in (2.2) may be possible.

When the number of non-integer angles $m \leq 3$, conditions equivalent to (1.5) and (1.6) were obtained in [5], [6, Theorem 4.1], [7], and for $m \leq 3$ they are also sufficient.

Formula (2.2) for a reduced partition can be written as

$$\frac{f'}{f} = \sum_{j=1}^{m} \frac{\epsilon_j \alpha_j}{z - a_j} - \sum_{j=1}^{k'} \frac{1}{z - b_j} + \sum_{j=k'+1}^{k'+k''} \frac{(-1)^j}{z - b_j},$$

and the residue theorem combined with (1.5) shows that k'' must be even, as stated in (1.6).

We will see that for co-axial metrics, (1.5) and (1.6) are also sufficient in the generic situation, when the coordinates of the vector $\boldsymbol{c}$ in (1.7) are incommensurable. When coordinates of $\boldsymbol{c}$ are commensurable, there are additional restrictions.

For a given multiset $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ satisfying Condition 2.1 we call a quadruple $(A, B, k, \{\epsilon_j\}, \{\delta_j\})$ of parameters in (2.2) an *arrangement* for $\boldsymbol{\alpha}$. For an admissible $\boldsymbol{\alpha}$, different arrangements may give different metrics. We do not require that all $\alpha_j \in A$ are non-integers. If they are non-integers, the arrangement is called *reduced*. If $\boldsymbol{\alpha}$ is admissible, there exist finitely many arrangements, at least one of them is reduced. This reduced arrangement may be non-unique: various choices of ϵ_j in (1.5) are sometimes possible. A priori, we have to deal with non-reduced arrangements because we have further conditions besides conditions (1.5) and (1.6); it is possible that some of the arrangements satisfy them, others do not.

The geometric meaning of a reduced arrangement is the following. The developing map f as in (2.3) is a multi-valued function, but the preimage $f^{-1}(\{0, \infty\})$ is well defined (as radial limits). Metrics corresponding to reduced arrangements are exactly those for which the developing map does not take the values $0, \infty$ at the singularities with integer angles.

From now on, we assume that Condition 2.1 is satisfied, an arrangement (perhaps not reduced) is fixed, and the logarithmic derivative of the developing map is written as in (2.2). We denote for simplicity

$$\{\epsilon_1\alpha_1, \ldots, \epsilon_m\alpha_m, \delta_1, \ldots, \delta_k\} = \{c_1, \ldots, c_q\}, \tag{2.6}$$

where $q = m + k$. In the case of reduced arrangement $k = k' + k''$. The question is what multiplicities of zeros of R are possible for a given vector of residues c:

Question 2.3. Suppose that real non-zero numbers $\{c_1, \ldots, c_q\}$ are given, and

$$\sum_{j=1}^{q} c_j = 0. \tag{2.7}$$

Which partitions $\{\ell_1, \ldots, \ell_k\}$ of $q - 2$ can be realized as multiplicities of zeros in $\mathbf{C}$ of the function

$$g(z) = \sum_{j=1}^{q} \frac{c_j}{z - z_j}, \tag{2.8}$$

where z_j are pairwise distinct complex numbers?
Notice that zeros of g are critical points of the potential

$$u(z) = \sum_{j=1}^{q} c_j \log|z - z_j|, \tag{2.9}$$

and Question 2.3 seems to be of independent interest.

The trivial but important property is the following:

If all c_j are multiplied by a constant, the multiplicities of zeros of R do not change.

So we introduce the real projective space $\mathbf{RP}^{q-2}$ which consists of non-zero q-tuples $(c_1, \ldots, c_q)$ satisfying (2.7), modulo proportionality. A point $\boldsymbol{c} \in \mathbf{RP}^{q-2}$ is called *rational* if its equivalence class contains a q-tuple with all c_j rational. Let Z be the union of coordinate hyperplanes $Z_j = \{c : c_j = 0\}$, $1 \leq j \leq q$. Let P be a partition of $q - 2$. A point $\boldsymbol{c} \in \mathbf{RP}^{q-2}$ is called *P-admissible*, if there exist pairwise distinct $z_j \in \mathbf{C}$ such that the function g in (2.8) has zeros of multiplicities P. A point $\boldsymbol{c} \in \mathbf{RP}^{q-2}\backslash Z$ which is not P-admissible is called *P-exceptional*.

Theorem 2.4. *Let $P = \{\ell_1, \ldots, \ell_s\}$ be a partition of $q - 2$. Every irrational point $\boldsymbol{c} \in \mathbf{RP}^{q-2}\backslash Z$ is P-admissible. A rational point if P-admissible if and only if*

$$2\left(1 + \max_{1 \leq j \leq s} \ell_j\right) \leq \sum_{j=1}^{q} |b_j|, \tag{2.10}$$

where

$$\{b_1, \ldots, b_q\} = \{\eta c_1, \ldots, \eta c_q\}, \quad \eta \neq 0, \tag{2.11}$$

is a vector with mutually prime integer coordinates proportional to $\boldsymbol{c}$.

When all residues in (2.8) are mutually prime integers the developing map $f = \exp \int g$ is a rational function, for which positive residues are multiplicities of zeros, negative residues are multiplicities of poles, and $k_j := \ell_j + 1,\ 1 \leq j \leq s$ are the multiplicities of f at other critical points, different from zeros and poles. So our Question 2.3 with mutually prime integer residues is a special case of the Hurwitz problem [2,9]:

Question 2.5. Given two partitions $\{n_1, \dots, n_r\}$ and $\{m_1, \dots, m_t\}$ of the same number $d > 1$ and a multiset of integers $\{k_1, \dots, k_s\},\ k_j \geq 2$, such that

$$\sum_{j=1}^{r}(n_j - 1) + \sum_{j=1}^{t}(m_j - 1) + \sum_{j=1}^{s}(k_j - 1) = 2d - 2, \tag{2.12}$$

does there exist a rational function f of degree d with zeros of multiplicities m_j and poles of multiplicities n_j and other critical points where multiplicities of f are k_j?

Here m_j are positive residues in (2.8) and n_j are negative residues in (2.8).

By a simple perturbation argument, it is sufficient to consider the case when the values of f at these "other critical points" are all distinct: critical points with the same critical value other than 0, ∞ can be perturbed so that all these critical points will have different critical values, and the multiplicities of zeros and poles are not affected.

The answer to Question 2.5 was recently obtained by Song and Xu [12]. The necessary and sufficient condition of existence of f is

$$k_j \leq d = \frac{1}{2}\sum_{i=1}^{q}|b_i|, \quad 1 \leq j \leq s. \tag{2.13}$$

The necessity of this condition is evident because the right-hand side is the degree d of f. Sufficiency was proved by Song and Yu who generalized the result of Boccara [2] for $s = 1$. This solves Question 2.5 and proves Theorem 2.4 for the case of rational vector $\boldsymbol{c}$.

We state a trivial but important fact:

Remark 2.6. For each P there are finitely many integer vectors $(b_1, \dots, b_q)$ which do not satisfy (2.13), thus there are finitely many rational points in $\mathbf{RP}^{q-2}\backslash Z$ which are P-exceptional.

Theorem 2.4 gives an algorithm which determines whether a given multiset $\boldsymbol{\alpha}$ is admissible. The algorithm works as follows. Starting with a multiset $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ we check conditions (1.5) and (1.6). If they are not satisfied, then $\boldsymbol{\alpha}$ is not admissible. If these conditions are satisfied, we consider all arrangements for $\boldsymbol{\alpha}$ and vectors $\boldsymbol{c}$ corresponding to them as in (2.6). If one of these vectors is irrational, then $\boldsymbol{\alpha}$ is admissible. If all are rational, we construct integer vectors $\boldsymbol{b}$ as in (2.11). If one of these vectors $\boldsymbol{b}$ satisfies (2.10) with $\ell_j = \alpha_{j+m} - 1$, then $\boldsymbol{\alpha}$ is admissible, if none, then not.

Most of Theorem 1.1 is a corollary of Theorem 2.4, except the statement that it is enough to check condition (1.8) only for one reduced arrangement. This will be addressed in the formal proof of Theorem 1.1 in the end of the paper.

Example 2.7. For arbitrary non-integer $\beta > 0$ the multiset $\{\beta, \beta, \beta, \beta, 3\}$ is not admissible. Conditions (1.5) and (1.6) are satisfied. The only reduced arrangement is $A = \{\beta, \beta, \beta, \beta\},\ B = \{3\},\ k = 0$. So $q = 4, b = (1, 1, -1, -1)$, and condition (1.8) is violated.

Example 2.8. For arbitrary $\beta > 0$ the multiset $\{\beta, \beta, 2\beta, 2\beta, 3\}$ is admissible. Conditions (1.5) and (1.6) are satisfied. Take the arrangement $A = \{\beta, \beta, 2\beta, 2\beta\}$, $B = \{3\},\ k = 0$. We have $q = 4, b = (1, -1, 2, -2)$. Inequality (1.8) is satisfied. Let us write a developing map explicitly for this case:

$$f(z) = \left(\frac{(z-1)^2(2z+1)}{(z+1)^2(2z-1)} \right)^{\beta} = h^{\beta}(z).$$

The corresponding metric has angles β at $\pm 1/2$ and angles 2β at ± 1. In addition to this, there is angle 3 at 0, because h has a triple point at 0 with critical value -1.

3. Proof of Theorem 2.4

Sketch of the proof. First we notice that the problem of constructing a surface of constant positive curvature, with co-axial monodromy and with prescribed angles at conic singularities is equivalent to a similar construction problem for a surface with a flat metric. Trying to construct this flat surface by gluing cylinders, we discover the general nature of obstructions: the given angles must satisfy some systems of inequalities. These inequalities are too complicated to write explicitly, but we determine their general nature: they are inequalities between some *linear forms* in the residues c_j with *integer* coefficients. Therefore, for each partition P, the set of P-exceptional points is a rational polyhedron in the space $\mathbf{RP}^{q-2}$. If this polyhedron consists of infinitely many points, then it must also contain infinitely many rational points. But we know from Theorem 2.4 that the number of exceptional rational points is finite for given P, see Remark 2.6. Therefore the polyhedron of P-exceptional points consists of finitely many points and *thus* all exceptional points must be rational.

Proof of Theorem 2.4.

1. *From spherical to flat and back.* Let $f : S \to \overline{\mathbf{C}}$ be the developing map as in (2.3). Let $\Omega = S \backslash \{z_1, \ldots, z_q\}$. Then we have the restricted map $f^* : \Omega \to \mathbf{C}^*$.

We equip $\mathbf{C}^*$ with the flat metric whose length element is $|dz/z|$. This metric makes $\mathbf{C}^*$ into an open cylinder infinite in two directions whose *girth* (the length of the shortest non-trivial geodesic, a. k. a. the systole) is 2π. We pull back this flat metric to Ω via f^* and obtain a flat surface which is conformally equivalent to a sphere with q punctures, and some neighborhoods of the punctures are semi-infinite cylinders of girths $2\pi|\beta_j|$. We call this surface (Ω, ρ), where ρ is the flat metric. The developing map f^* of (Ω, ρ) has two special features: it maps Ω to $\mathbf{C}^*$ (rather then $\mathbf{C}$) and it tends to 0 or to ∞ at the punctures in the sense of radial limits.

Conversely, suppose that Ω is a Riemann surface conformally equivalent to a punctured sphere, equipped with a flat Riemannian metric ρ such that some neighborhoods of the punctures are semi-infinite cylinders of girth $2\pi\mu_j$. Moreover, suppose that the developing map h maps Ω to $\mathbf{C}^*$ and tends at each puncture either to 0 or to ∞. By filling the punctures, we can extend h to a (multivalued) map $f : S \to \overline{\mathbf{C}}$ and pull back the spherical metric to S. The resulting surface has constant curvature 1, and in addition to conic singularities in Ω has conic singularities at the punctures $S\backslash\Omega$. The angles at these additional singularities z_j are μ_j.

2. *From flat surface to a system of linear inequalities*. Now we study this auxiliary flat surface (Ω, ρ) and its developing map h. The level sets

$$L_t = \{z \in \Omega : \log|h(z)| = t\}, \quad -\infty < t < \infty,$$

make a foliation of Ω. This means that Ω is a disjoint union of *leaves* and finitely many *critical points* of $\log|h|$. Leafs are the curves on which $|h(z)|$ is constant; these curves are either simple closed curves (ordinary leaves) or simple open curves with both ends at singular points (singular leaves). Foliations are considered here as topological objects: up to homeomorphisms which respect leaves.

Notice that unlike the developing map h, the function $u = \log|h|$ is a well-defined (single-valued) harmonic function. Level sets L_t which contain singular points are called *critical level sets*. A non-critical level set consists of finitely many ordinary leaves, while a critical level set may contain both ordinary and singular leaves and some critical points.

The region Ω is a disjoint union of open *foliated cylinders* and critical level sets. A model foliated cylinder is obtained by taking a rectangle in the plane foliated into horizontal segments and identifying its vertical sides in the natural way. An open foliated surface homeomorphic to such a cylinder, by a homeomorphism respecting the foliation is called a foliated cylinder.

Every singular point in Ω is a saddle point of u and it has an index: a positive integer k such that the singular leaves in a neighborhood of this point look like the $2(k+1)$ intervals of the set $\{z : |z| < 1, \operatorname{Re} z^{k+1} = 0\}$ meeting at 0. This is because our function u is harmonic.

Our foliation has an additional structure: there are two functions on the set of leaves: one is the *height* t, another is the length of a leaf with respect to the intrinsic metric ρ. For a leaf $\gamma \subset L_t$ the height is t. The length of a leaf γ is a positive number which can be computed by the formula

$$|\gamma| = \int_\gamma \left|\frac{\partial u}{\partial n}\right| |dz|, \quad u(z) = \log|h(z)|,$$

where n is the unit normal to γ. The same formula defines the length of any arc of a leaf.

Suppose that an interval (t', t'') contains no critical values of u. For $t \in (t', t'')$, let $\gamma_t \subset L_t$ be a leaf which depends continuously on t. (Convergence of leaves which is used here is uniform, using some parametrization).

Then *the length* $|\gamma_t|$ *does not depend on* t.

This follows from Green's formula applied to u in the ring between γ_{t_1} and γ_{t_2} where t_1, t_2 are any numbers between t' and t''.

When t passes through a critical value, some leaves break into singular leaves and then these singular leaves re-assemble into new ordinary leaves.

More precisely, let (t', t'') be as above, and suppose that t' is a singular value. Choose a leaf $\gamma_t \in L_t$ which depends continuously on t for $t \in (t', t'')$. Then as $t \to t'+$, some parametrization of γ_t converges uniformly to a closed curve, which can be an ordinary leaf, or a finite union of singular leaves $\gamma^j \subset L_{t'}$ and singular points. Moreover, we have

$$|\gamma_t| = \sum_j |\gamma^j|, \tag{3.1}$$

where the summation is over all those leaves which form the limit of γ_t, and all summands in the right-hand side are strictly positive.

Relations (3.1) form a system of linear equations which the lengths of leaves of a given topological foliation must satisfy, assuming that the lengths of ordinary leaves do not change with height.

3. *From foliations with height and length back to flat surfaces.* Suppose now that Ω is a topological punctured sphere with a topological foliation whose leaves are level sets of some smooth function $v : \Omega \to \mathbf{R}$ with finitely many critical points, and $v(z) \to \pm\infty$ when z tends to a puncture, and

a) In a neighborhood of each critical point v is topologically equivalent to a harmonic function.

Suppose further that a strictly positive function ϕ on the set of leaves is given which has the formal properties of the length function, namely:

b) If $\gamma_t \subset L_t$ is a family of ordinary leaves continuously depending on $t \in (t', t'')$ on an interval containing no critical values, then $\phi(\gamma_t)$ is constant on (t', t'');
c) If t' is a singular value, and γ_t is the same as in b), and γ_t tends to the union of singular leaves $\cup_j \gamma^j$ as $t \to t'$, we have

$$\phi(\gamma_t) = \sum_j \phi\big(\gamma^j\big). \tag{3.2}$$

We claim that whenever such a foliation and functions t and ϕ on the leaves are given, one can introduce a flat metric on Ω whose developing map h has the property that the level sets of $v = \log|h|$ define our given foliation, and the function ϕ is the length of the leaves of this foliation. The flat metric defines on Ω the conformal structure of a punctured sphere.

To prove the claim, we consider the partition of Ω into foliated cylinders C_j and critical level sets as described in part 2 of the proof. Each cylinder is mapped by v into a maximal interval (t', t'') free of critical values of v. In the *trivial case* when there are no critical points at all, we have $(t', t'') = (-\infty, \infty)$. In all other cases there are two such semi-infinite intervals and finitely many finite intervals.

Each foliated cylinder C_j is homeomorphic to the product $\gamma_j \times (t', t'')$, where γ_j is an ordinary leaf in L_t for some $t \in (t', t'')$. We pull back to C_j the standard Euclidean metric from this product, so that $|\gamma_j| = \phi(\gamma_j)$. This defines the flat metric ρ on the cylinders of the foliation. Some of them are of finite height, others semi-infinite, except the trivial case when there is only one doubly-infinite cylinder.

Let $\overline{C_j}$ be completions of the C_j with respect to their metrics. The boundary circles of C_j correspond to some leaves of the foliation on the singular level sets, and some finite sets of points on each boundary circle must be glued together into singular points. So we break every boundary circle into arcs which will correspond to the singular leaves. The lengths of these arcs are determined by our function ϕ, and this is where relation (3.2) is used. Then we glue together our cylinders along these arcs respecting the length. To perform this gluing we use the theorem of Aleksandrov and Zalgaller, see, for example [11, Theorem 8.3.2], about gluing two surfaces along a geodesic arc. It guarantees that we obtain a "surface of bounded curvature" in the sense of Aleksandrov, with a flat metric and finitely many conic singular points. (This is a surface of special kind which is called a polyhedral surface in [11]). The total angle at a singularity is equal to one half of the number of boundary points of cylinders which are glued together at this point.

That the resulting surface is connected and of genus 0 is guaranteed by the topology of the foliation. For a given foliation, the only condition for the possibility of this gluing is the linear relations (3.2) between the lengths of the leaves.

4. *Conclusion of the proof of Theorem* 2.4. Suppose that the vector c has p positive and r negative coordinates, $p + r = q$. Take a q-punctured sphere Ω, and construct a function $v : \Omega \to \mathbf{R}$ which tends to $-\infty$ at p punctures of Ω and to $+\infty$ at the remaining r punctures. Moreover, we require that all critical points of v in Ω are saddle points of the topological types which are possible for harmonic functions, and the multiplicities of these critical points are the parts of the partition P.

Lemma 3.1. *For every p, r and P there exists a function v with these properties.*

Postponing the proof of the lemma, we complete the proof of Theorem 2.4.

Consider *all* foliations defined by functions v satisfying our conditions with fixed p, r, P. To assign a length function ϕ consistent with a foliation, we have to solve the system of equations (3.2) which is determined by the foliation. In this system, the given numbers are the girths of the semi-infinite cylinders (these are our $|c_j|$), and the unknown variables are the girths of all finite height cylinders and the lengths of the singular leaves.

In addition to (3.2), we have the restriction that all girths and lengths must be strictly positive.

If this linear system has a strictly positive solution, we can construct our metric by performing steps described in parts 3 and 1. If not, a metric corresponding to this particular foliation does not exist.

We give an illustrating example. Suppose we want to construct a function g as in (2.8) with two positive residues a, b, two negative residues $-c, -d$ and a single critical point where the local degree of h is 3. Then the critical level set must have the form as in Fig. 1, where the regions represent semi-infinite cylinders, the dots labeled a, b are the poles of h, and the dots labeled c, d are zeros of h. The girths of the four cylinders are $2\pi a, 2\pi b, 2\pi c, 2\pi d$ and the length of the three singular leaves are $2\pi b, 2\pi d$, and $2\pi x$ (see Figure 3.1). Non-singular leaves of the foliation are not shown in the picture, each of them is a Jordan curve that surrounds the puncture in its cylinder. Equations (3.2) for this case are

$$a = x + d, \quad c = x + b,$$

which are consistent if and only if $a-d+b-c = 0$. Now x must be strictly positive, so we obtain necessary and sufficient conditions of existence of such g: $a > d$ and $c > b$, in other words, the positive residues must be unequal and negative residues must be unequal. This explains examples 2.7, 2.8 above.

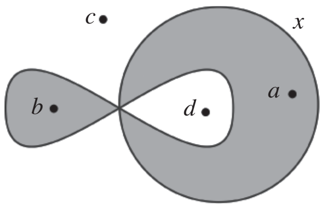

Figure 3.1. A critical level set.

In any case, the condition that a vector c is P-admissible is stated in terms of linear equations and linear inequalities with integer coefficients and Boolean operations. So P-exceptional vectors c form a rational polyhedron in $\mathbf{RP}^{q-2}$. If this polyhedron is infinite, then it contains infinitely many rational points [4], which is not the case: we have seen that there are only finitely many P-exceptional rational vectors for each given P (see Remark 2.6). So the polyhedron is finite. So it consists of only rational points.

This completes the proof of Theorem 2.4. □

5. *Proof of Lemma* 3.1. It is sufficient to prove the lemma for the special case when there is only one critical point. Then it can be broken into pieces according to partition P by a perturbation, as shown in the first three lines of Figure 3.2. In lines 1-3, on the left-hand side we have a critical point of multiplicity 4. In lines 1 and 2 it is broken to two critical points of multiplicity 2, in line 3 it is broken into one critical point of multiplicity 2 and two critical points of multiplicity 1. A foliation with one critical point is defined by its critical level set, say $v(z) = 0$, and by assigning a black or white color to the components of the complement according to

(5,1)
(4,2)
(3,3)
(4,1)
(6,1)
(3,2)
(5,2)

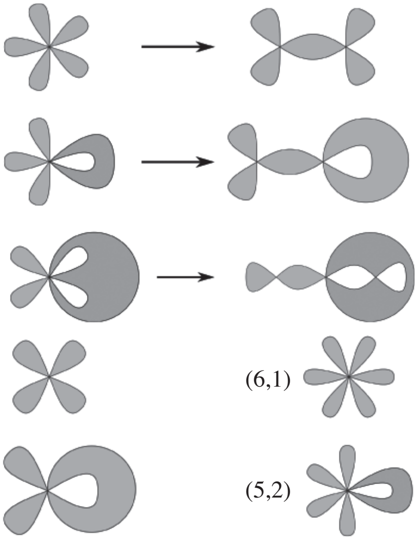

Figure 3.2. Black and white regions represent semi-infinite cylinders, numbers on the left are (p, r), and the arrows arrows show breaking a high multiplicity critical point into critical points of the lower multiplicity.

the sign of v. Instead of describing these foliations in words we just present pictures of their critical level sets in Figure 3.2. Each black or white region represents a semi-infinite cylinder with one puncture inside. Numbers (p, r) are written on the left. We start with $(p, 1)$, a flower with p petals, and then pass to $(p - 1, 2)$, $(p - 2, 3)$ etc., as shown in the picture. □

4. Completion of the proof of Theorem 1.1

Necessity of conditions (1.5), (1.6) has been already explained, and sufficiency of (1.5), (1.6) and (1.8) follows from Theorem 2.4. It remains to prove the necessity of (1.8). It is necessary that (1.8) is satisfied for some arrangement. We have to prove that it is enough to check it only for one reduced arrangement.

Suppose that $\boldsymbol{\alpha}$ is a multiset satisfying Condition 2.1. We claim that if it is admissible, then there exists a metric with these angles corresponding to some reduced arrangement. Indeed if there are singular points with integer angles for which the developing map takes the values 0 or ∞, then one can find another metric with co-axial monodromy with the same angles for which the developing map does not take the values 0 or ∞ at the singular points with integer angles. This follows from a general argument which permits to "move around" a singular point with integer angle.

Let $f : S \to \overline{\mathbf{C}}$ be the developing map of a surface of curvature 1 with conic singularities, and suppose that $a \in S$ is a singular point with integer angle α. Let

$r > 0$ be smaller than the distance from a to other singularities, and such that the closed intrinsic disk D of radius r centered at a is homeomorphic to a closed disk in the plane.

We will remove the interior of D from S, and paste $S\backslash\text{int}\, D$ with a new surface C homeomorphic to a closed disc in the plane, equipped with a metric of the same constant curvature, having one singularity in the interior with the same angle α. This can be so arranged that the distance in C from the singularity to ∂C is any positive number less than r, and the closest point to the singularity on ∂C is any given point of ∂C. So we have a continuous family of deformations. Moreover, the resulting surface $S' = C \cup (S\backslash\text{int}\, D)$ is smooth, and has constant curvature except at the conic singularities in $S\backslash D$ and in C.

Consider the disk $U = \{z : |z| < R\},\ R = \tan(r/2)\}$ equipped with the standard spherical metric ρ. (The spherical radius of this disk is r.) Let $C = \{z : |z| \leq R^{1/\alpha}\}$ equipped with the metric $\rho_1 = f^*\rho$, where $f(z) = z^\alpha$. Let $D = \{z : |z| \leq R^{1/\alpha}\}$ equipped with the metric $\rho_2 = g^*\rho$, where

$$g(z) = R\frac{z^\alpha + aR}{1 + \overline{a}z^\alpha},$$

where $|a| < 1$. Consider the annulus $B \subset U$, $B = \{z : t < |z| < R\}$ where $t \in (R|a|, R)$, and let denote $A_1 = f^{-1}(B)$, $A_2 = g^{-1}(B)$. Then the metric spaces $(A_1, \rho_1) \subset D$ and $A_2 \subset C$ are isometric, because they are both isometric to the covering of B of degree α. So we can remove from our surface S a disk isometric to D and glue in C instead. The parameter of deformation is a. Notice that this deformation is *isomonodromic*, does not change the monodromy group.

This explains why the necessary condition (1.8) in Theorem 1.1 is enough to verify for reduced arrangements: we can always perturb a co-axial metric and obtain another co-axial metric with the same angles and reduced arrangement.

Now we notice, that if some reduced arrangement satisfies (1.8) of Theorem 1.1, then all other reduced arrangements for the same multiset of angles will also satisfy (1.8), because q and $\sum |b_j|$ are the same for all reduced arrangements. Indeed, m and

$$q = m + k' + k'' = \sum_{j=m+1}^{n} (\alpha_j - 1) + 2,$$

depend only on $\boldsymbol{\alpha}$, and b_j depend only on non-integer angles in $\boldsymbol{\alpha}$ and on $k = k' + k''$. This proves necessity of condition (1.8) and completes the proof of Theorem 1.1. ☐

Remark 4.1. A similar deformation of a singularity with non-integer angle is impossible. Consider, for example a "football", the sphere with a metric of curvature 1 and two conic singularities. The singularities of such surface must have equal angles, and for each angle there is such a surface. But if the angle is non-integer, then a football is unique, while with an integer angle there is a 1-parametric family of footballs [13]. What was used in our argument is that the developing map is single-valued in a neighborhood of a singularity with integer angle.

Remark 4.2. If $\boldsymbol{\alpha}$ is an admissible multiset, there exists a metric of positive curvature with angles $\boldsymbol{\alpha}$. But the conformal class of this metric cannot be arbitrarily assigned. Take for example $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_n = n - 2$, the rest of the angles are not integers and

$$\sum_{j=1}^{n-1} \alpha_j = 0.$$

Compare Example 2.8 above. The developing map satisfies

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n-1} \frac{\pm\alpha_j}{z - z_j},$$

and the right-hand side must have a zero of multiplicity $n - 3$. This imposes $n - 4$ conditions on the poles z_j. Indeed, we may assume without loss of generality that $z_1 = 0,\ z_2 = 1$, so we obtain $n-4$ conditions on $n-3$ variables z_j which suggests that there is only a one-dimensional family of conformal classes of such metrics.

Similar phenomenon may occur when all angles are non-integer. Lin and Wang [8] studied a problem which is equivalent to description of metrics of positive curvature on the sphere with four singularities with angles $(1/2, 1/2, 1/2, 3/2)$. The conformal type of these metrics depends on one complex parameter, and it turns out that the moduli space of quadruply punctured spheres is split into two parts, each with non-empty interior, such that for one part a metric with these angles exists and for the other part it does not. In all these examples the angles are very special. The results in [1,3] suggest that perhaps for generic angles satisfying (1.2) and (1.3) a metric of curvature 1 exists in prescribed conformal class of the punctured sphere.

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